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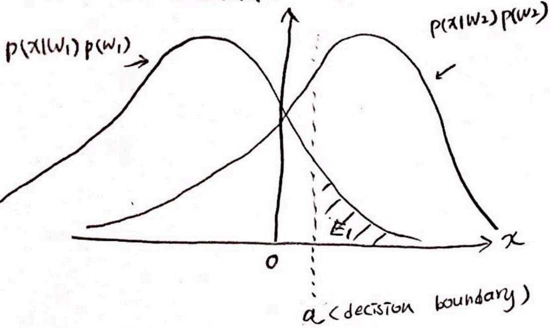
120210211 이지환 #HW 1

6. (a) Draw two univariate normal distribution, $p(x|w_i)p(w_i)$



$$\therefore \int_{R_2} p(x|w_1)p(w_1)dx = \frac{1}{2}\int_{a}^{\infty} N(\mu_1, \sigma_1^2)\,dx$$

$$\le E1$$

(a is a function of E1)

(b) $$E_2 = \int_{R_1} p(x|w_2)p(w_2)dx = \frac{1}{2}\int_{-\infty}^{a} N(\mu_2, \sigma_2^2)dx$$

(c) overall error rate

$$E = E_1 + E_2$$

$$= \frac{1}{2}\int_{a}^{\infty} N(\mu_1, \sigma_1^2)dx + \frac{1}{2}\int_{-\infty}^{a} N(\mu_2, \sigma_2^2)\,dx$$

(d) $\begin{cases} p(x|w_1) \sim N(-1, 1) \\ p(x|w_2) \sim N(1, 1) \end{cases}$ $E1 = 0.05$

and Assume that $a = 0.2815$

$$E = E_1 + E_2$$

$$= 0.05 + \frac{1}{2}\int_{-\infty}^{0.2815} N(\mu_2, \sigma_2^2)\,dx$$

$$= 0.05 + \frac{1}{2}\int_{-\infty}^{0.2815} \frac{1}{\sqrt{2\pi}0.05} \exp\left[-\frac{1}{2}\left(\frac{(x-0.5)^2}{(0.5)^2}\right)\right]dx$$

$$= 0.168$$

(e) If $a = 0$, the bayes error is

$$E_B = 2\int_{0}^{\infty} \frac{1}{2} N(\mu_1, \sigma_1^2)\,dx$$

$$= \int_{0}^{\infty} N(1, 1)\,dx = 0.159$$

lower than the Neyman-pearson conditions. And if E_B is lower than 2×0.05, use the Bayes decision point for the Neyman-pearson case.

9. (a) In problem 7,

$$p(x|w_i) = \frac{1}{\pi b} \cdot \frac{1}{1 + \left(\frac{x - a_i}{b}\right)^2} \quad (i = 1, 2)$$

(where $a_2 > a_1$, and equal priors)

decision boundary : $\frac{(a_1 + a_2)}{2}$

$$p(\text{error}) = \int_{-\infty}^{\frac{(a_1+a_2)}{2}} p(x|w_2)p(w_2)dx + \int_{\frac{(a_1+a_2)}{2}}^{\infty} p(x|w_1)p(w_1)\,dx$$

$$= \frac{1}{\pi b}\int_{-\infty}^{\frac{(a_1+a_2)}{2}} \frac{1/2}{1 + \left(\frac{x - a_2}{b}\right)^2}dx$$

$$+ \frac{1}{\pi b}\int_{\frac{(a_1+a_2)}{2}}^{\infty} \frac{1/2}{1 + \left(\frac{x - a_1}{b}\right)^2}dx$$

$$= \frac{1}{\pi b}\int_{-\infty}^{(a_1-a_2)/2} \frac{1}{1 + \left(\frac{x - a_2}{b}\right)^2}dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{(a_1-a_2)/2} \frac{1}{1 + y^2}\,dy$$

$\frac{x - a_2}{b} = y$ 치환

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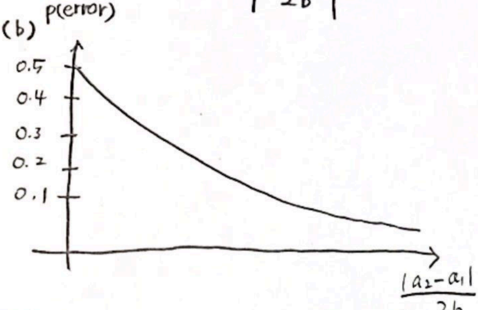
$$\therefore P(\text{error}) = \frac{1}{\pi}\left[\tan^{-1}\left|\frac{a_1-a_2}{2b}\right| - \underbrace{\tan^{-1}[-\infty]}_{\frac{1}{2}}\right]$$

$$= \frac{1}{2} - \frac{1}{\pi}\tan^{-1}\left|\frac{a_2-a_1}{2b}\right|$$

(b)



(c)

$$P(\text{error}) = \frac{1}{2} - \frac{1}{\pi}\tan^{-1}\left|\frac{a_2-a_1}{2b}\right| \le \frac{1}{2}$$

$\therefore \left|\frac{a_2-a_1}{2b}\right| = 0$ 일때

P(error)가 maximum 이다.

$\left|\frac{a_2-a_1}{2b}\right| = 0$ 인경우

- case 1) $a_2 = a_1 \rightarrow$ two distributions are same
- case 2) $b = \infty \rightarrow$ two distributions are flat

13. $* \; R(\alpha_i|x) = \sum_{j=1}^{c} \lambda(\alpha_i|\omega_j) P(\omega_j|x)$

$$= \lambda_s \sum_{i \ne j} P(\omega_j|x)$$

$$= \lambda_s (1 - P(\omega_i|x))$$ 로 정의됨

i) $P(\omega_i|x) \ge P(\omega_j|x)$

$$-P(\omega_i|x) \le -P(\omega_j|x)$$

$$1 - P(\omega_i|x) \le 1 - P(\omega_j|x)$$

$$\lambda_s(1-P(\omega_i|x)) \le \lambda_s(1-P(\omega_j|x))$$

$$\therefore R(\alpha_i|x) \le R(\alpha_j|x)$$

ii) $P(\omega_i|x) \ge 1 - \frac{\lambda_r}{\lambda_s}$ 이면

$$R(\alpha_i|x) = \lambda_s(1 - P(\omega_i|x)) \le \lambda_s\left(1 - \left(1 - \frac{\lambda_r}{\lambda_s}\right)\right) = \lambda_r = R(\alpha_{c+1}|x)$$

$\therefore R(\alpha_i|x)$이 minimum risk

$*$ $\lambda_r = 0$ 이면,

$R(\alpha_i|x) \ge 0 = \lambda_r$ 이므로 always reject 이다

$*$ $\lambda_r > \lambda_s$ 이면

$$R(\alpha_i|x) = \lambda_s(1 - P(\omega_i|x)) \le \lambda_s < \lambda_r$$

$\therefore R(\alpha_i|x) < \lambda_r$ 이므로 always not reject 이다

31. (a) $p(x|\omega_i) \sim N(\mu_i, \sigma^2)$

$P(\omega_1) = P(\omega_2) = \frac{1}{2}$

x^* : decision region

$$p(x^*|\omega_2)P(\omega_2) = p(x^*|\omega_1)P(\omega_1)$$

$$\cancel{\ln\frac{1}{2}} - \cancel{\ln(\sqrt{2\pi}\sigma)} - \frac{1}{2}\left(\frac{x^*-\mu_2}{\sigma}\right)^2 = \cancel{\ln\frac{1}{2}} - \cancel{\ln(\sqrt{2\pi}\sigma)} - \frac{1}{2}\left(\frac{x^*-\mu_1}{\sigma}\right)^2$$

$$\left(\frac{x^*-\mu_2}{\sigma}\right)^2 = \left(\frac{x^*-\mu_1}{\sigma}\right)^2$$

$$|x^*-\mu_2| = |x^*-\mu_1|$$

$$x^* = \frac{\mu_1+\mu_2}{2}$$

$$P(\text{error}) = P(\omega_2)\int_{-\infty}^{x^*} p(x|\omega_2)\,dx + P(\omega_1)\int_{x^*}^{\infty} p(x|\omega_1)\,dx$$

$$= 2 \times \frac{1}{2}\int_{x^*}^{\infty} p(x|\omega_1)\,dx = \int_{x^*}^{\infty} p(x|\omega_1)\,dx$$

case1) $\mu_1 < \mu_2$

$$p(error) = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{x^*}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma}\right)^2}dx$$

$$\left(y = \frac{x-\mu_1}{\sigma} \Rightarrow \sigma dy = dx\right)$$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}}\int_a^{\infty} e^{-\frac{1}{2}y^2}\cancel{\sigma}dy = \frac{1}{\sqrt{2\pi}}\int_a^{\infty} e^{-\frac{1}{2}y^2}dy$$

case) $\mu_1 > \mu_2$

$$p(error) = \frac{1}{\sqrt{2\pi}}\int_a^{\infty} e^{-\frac{1}{2}y^2}dy$$

$$\therefore p(error) = \frac{1}{\sqrt{2\pi}}\int_a^{\infty} e^{-\frac{y^2}{2}}dy$$

(b) $$P_e \le \frac{1}{\sqrt{2\pi}a}e^{-a^2/2} \quad \left(\text{where } a = \frac{|\mu_2-\mu_1|}{2\sigma}\right)$$

$$\frac{|\mu_2-\mu_1|}{\sigma} = 2a \rightarrow \infty \text{ 이면;}$$

$$P_e \le \lim_{a\to\infty}\frac{1}{\sqrt{2\pi}a}e^{-\frac{a^2}{2}} \rightarrow 0 \text{ 이다.}$$

38. (a) Assume $a \ge b$ (where a and b are nonnegative numbers),

$$a \cdot b \ge b^2 \iff \sqrt{ab} \ge b \rightsquigarrow \min[a,b]$$

$$\therefore \min[a b] \le \sqrt{ab}$$

(b) Bayes classifier 공식

$p(x|w_1)p(w_1) \ge p(x|w_2)p(w_2)$ 이면 w_1 선택.

$$p(error) = \int \min[p(x|w_1)p(w_1), p(x|w_2)p(w_2)]$$

$p(w_1|x), p(w_2|x) \ge 0$ 이고,

$$p(error) \le \int\sqrt{p(x|w_1)p(w_1)p(x|w_2)p(w_2)}\,dx$$

$$= \sqrt{p(w_1)p(w_2)}\,\rho$$

$$p(w_1)+p(w_2) = 1$$

$$p(w_1)p(w_2) = p(w_1)(1-p(w_1))$$

$\therefore p(w_1) = \frac{1}{2}$ 일 때 $p(w_1)p(w_2)$최대

$$\therefore p(error) \le \sqrt{p(w_1)p(w_2)}\,\rho \le \sqrt{\frac{1}{4}}\,\rho$$

$$= \frac{1}{2}\rho$$

2. (a) $\Sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \Sigma_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$$g_i(x) = p(x|w_i)p(w_i)$$

$$= \ln p(x|w_i) + \ln p(w_i)$$

$$= -\frac{1}{2}(x-\mu_i)^t\Sigma_i^{-1}(x-\mu_i)$$

$$-\frac{d}{2}\ln 2\pi - \frac{1}{2}\ln|\Sigma_i| + \ln p(w_i)$$

$$= -\frac{1}{2}(x-\mu_i)^t\Sigma_i^{-1}(x-\mu_i)$$

$$-\frac{1}{2}\ln|\Sigma_i| + \ln p(w_i)$$

$*$ $g(x) = g_1(x) - g_2(x)$

$$= -\frac{1}{2}\begin{pmatrix} x_1-0 \\ x_2-1 \end{pmatrix}^t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1-0 \\ x_2-1 \end{pmatrix}$$

$$\cancel{-\frac{1}{2}\ln 1} \cancel{+\ln\frac{1}{2}} + \frac{1}{2}\begin{pmatrix} x_1-2 \\ x_2-0 \end{pmatrix}^t\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1-2 \\ x_2-0 \end{pmatrix}$$

$$\cancel{+\frac{1}{2}\ln 1} \cancel{-\ln\frac{1}{2}}$$

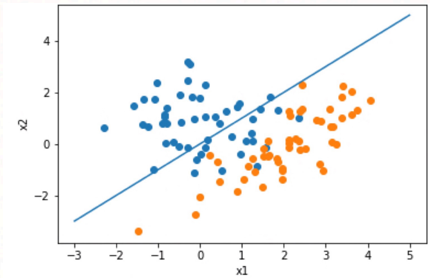
$$= -\frac{1}{2}(x_1^2 + \cancel{x_2^2} - 2x_2 + 1)$$

$$+\frac{1}{2}(2x_1^2 - 4x_1 - 4x_1 + 8 - x_1x_2 + 2x_2$$

$$-x_1x_2 + 2x_2 + \cancel{x_2^2})$$

$$= \frac{1}{2}x^2 - 4x_1 + 3x_2 - x_1x_2 + \frac{17}{2}$$

(b)



(c) $p(error) \le \sqrt{P(w_1)P(w_2)}\; e^{-k(\frac{1}{2})}$

$$k(\beta) = \frac{\beta(1-\beta)}{2}(\mu_1-\mu_2)^t\left[(1-\beta)\Sigma_1+\beta\Sigma_2\right]^{-1}(\mu_1-\mu_2) + \frac{1}{2}\ln\frac{|(1-\beta)\Sigma_1+\beta\Sigma_2|}{|\Sigma_1|^{1-\beta}|\Sigma_2|^{\beta}}$$

$$k\left(\frac{1}{2}\right) = \frac{1}{8}\begin{pmatrix}-2\\1\end{pmatrix}^t\begin{pmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & \frac{3}{2}\end{pmatrix}^{-1}\begin{pmatrix}-2\\1\end{pmatrix} + \frac{1}{2}\ln\frac{\frac{5}{4}}{\sqrt{1}\sqrt{1}}$$

$$= \frac{1}{8}[-2 \quad 1]\begin{pmatrix}\frac{6}{5} & -\frac{2}{5}\\ -\frac{2}{5} & \frac{4}{5}\end{pmatrix}(-2 \quad 1) + \frac{1}{2}\ln\frac{5}{4}$$

$$= \frac{9}{10} + \frac{1}{2}\ln\frac{5}{4} \approx 0.9 + 0.11 = 1.01$$

$$\therefore p(error) \le \frac{1}{2}\times e^{-1.01} \approx 0.16$$

3. (a) $\|X-\mu\| = \sqrt{(X-\mu)^t(X-\mu)}$

$$= \sqrt{(x_1-\mu_1)^2+(x_2-\mu_2)^2}$$

(b) $r^2 = (x-\mu)^t\Sigma^{-1}(x-\mu)$

$$= [x_1-\mu_1 \quad x_2-\mu_2]\begin{bmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{21} & \sigma_2^2\end{bmatrix}\begin{bmatrix}x_1-\mu_1\\ x_2-\mu_2\end{bmatrix}$$

$$= [x_1-\mu_1 \quad x_2-\mu_2]\frac{1}{\sigma_1^2\sigma_2^2-\sigma_{21}\sigma_{12}}\begin{bmatrix}\sigma_2^2 & -\sigma_{12}\\ -\sigma_{21} & \sigma_1^2\end{bmatrix}\begin{bmatrix}x_1-\mu_1\\ x_2-\mu_2\end{bmatrix}$$

$$= \frac{1}{\sigma_1^2\sigma_2^2-\sigma_{21}\sigma_{12}}\left[\sigma_2^2(x_1-\mu_1)-\sigma_{21}(x_2-\mu_2) \quad -\sigma_{12}(x_1-\mu_1)+\sigma_1^2(x_2-\mu_2)\right]\begin{bmatrix}x_1-\mu_1\\ x_2-\mu_2\end{bmatrix}$$

$$= \frac{1}{\sigma_1^2\sigma_2^2-\sigma_{21}\sigma_{12}}\Big(\sigma_2^2(x_1-\mu_1)^2-\sigma_{21}(x_2-\mu_2)(x_1-\mu_1) - \sigma_{12}(x_1-\mu_1)(x_2-\mu_2) + \sigma_1^2(x_2-\mu_2)^2\Big)$$

$$= \frac{1}{\sigma_1^2\sigma_2^2-\sigma_{21}\sigma_{12}}\Big(\sigma_2^2(x_1-\mu_1)^2-2\sigma_{12}(x_1-\mu_1)(x_2-\mu_2) + \sigma_1^2(x_2-\mu_2)^2\Big)$$

$$\therefore r = \sqrt{\frac{\sigma_2(x_1-\mu_1)^2-2\sigma_{12}(x_1-\mu)(x_2-\mu)+\sigma_1^2(x_2-\mu_2)^2}{\sigma_1^2\sigma_2^2-\sigma_{12}\sigma_{21}}}$$

(c) diagonal matrix $\rightarrow \sigma_{12}=\sigma_{21}=0$

$$\therefore r = \sqrt{\frac{\sigma_2(x_1-\mu_1)^2+\sigma_1^2(x_2-\mu_2)^2}{\sigma_1^2\sigma_2^2}}$$

$$= \sqrt{\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}}$$

(d) Euclidean : $\sqrt{(x_1-\mu_1)^2+(x_2-\mu_2)^2}$

Mahalanobis:

$$\sqrt{\frac{(x_1-\mu_1)^2}{\sigma_1^2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}}$$

If $\sigma_{12}=\sigma_{21}=1$, then

$$\sqrt{(x_1-\mu_1)^2+(x_2-\mu_2)^2}$$

∴ When covariance matrix is identical,

two distances are equal.

And Machalanobis distance encounter covariance between feature vector.

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* problem 2 (b) code (Implement as python)

```
import numpy as np
import matplotlib.pyplot as plt
a=np.random.multivariate_normal([0,1],[[1,0],[0,1]],50)
b=np.random.multivariate-normal([2,0],[[1,1],[1,2]],50)
f:lambda x:(1/2*x**2-4*x+7/2)/(x-3)
x1=np.linspace(-3,5)
x2=f(x1)
plt.scatter(a[:,0],a[:,1])
plt.scatter(b[:,0],b[:,1])
plt.plot(x1,x2)
plt.xlabel('x1')
plt.ylabel('x2')
plt.show()
```