Kinematics

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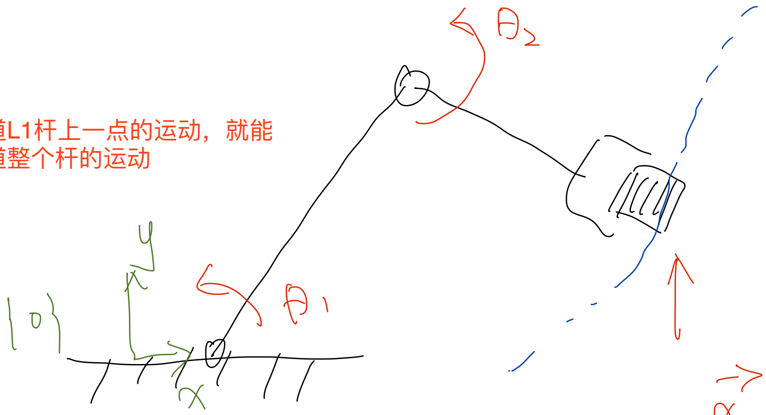
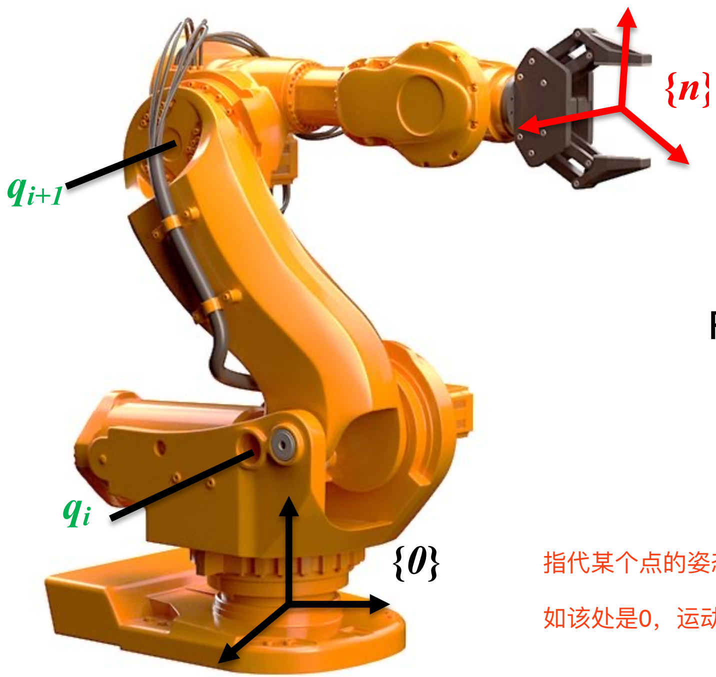
知道L1杆上一点的运动，就能
知道整个杆的运动



$$\vec{x} = f(\vec{\theta})$$

forward kinematics

Kinematics of Robot Manipulators



1. Usually, variables written in uppercase represent vectors or matrices. Lower-case variables are scalars.
2. Leading subscripts and superscripts identify which coordinate system a quantity is written in. For example, ${}^{A}P$ represents a position vector written in coordinate system $\{A\}$, and ${}^{A}_{B}R$ is a rotation matrix[3] that specifies the relationship between coordinate systems $\{A\}$ and $\{B\}$.
3. Trailing superscripts are used (as widely accepted) for indicating the inverse or transpose of a matrix (e.g., R^{-1}, R^{T}).
4. Trailing subscripts are not subject to any strict convention but may indicate a vector component (e.g., x, y, or z) or may be used as a description—as in P_{bolt}, the position of a bolt.
5. We will use many trigonometric functions. Our notation for the cosine of an angle θ_1 may take any of the following forms: $\cos\theta_1 = c\theta_1 = c_1$.

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Relation between joints (q_i) and the pose (position/orientation) of some point (e.g.: frame $\{n\}$)

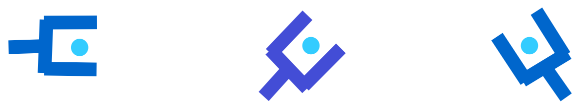
指代某个点的姿态编号编号

如该处是0，运动完成一次之后为1

Workspace

- Primary Workspace (*reachable*): WS_1

  Positions that can be reached with <u>at least one orientation</u>



Each point can be reached
(orientation "does not matter")

主要工作范围：该机械臂在至少一种情况下才能完成。

也就是图中的半圆

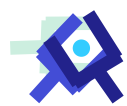
  - Out of WS_1 there is no solution to the problem
  - For all $\mathbf{p}$ WS_1 (using a proper orientation), there is at least one solution

- Secondary Workspace (*dexterous*): WS_2

次要工作范围：机械臂可以以任意一种姿态完成

也就是图中除了虚线，半圆其他的部分

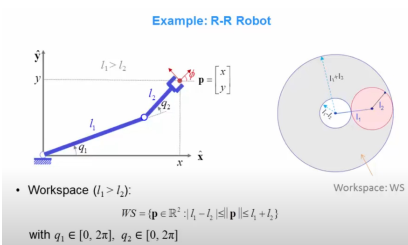
  Positions can be reached with <u>any orientation</u>



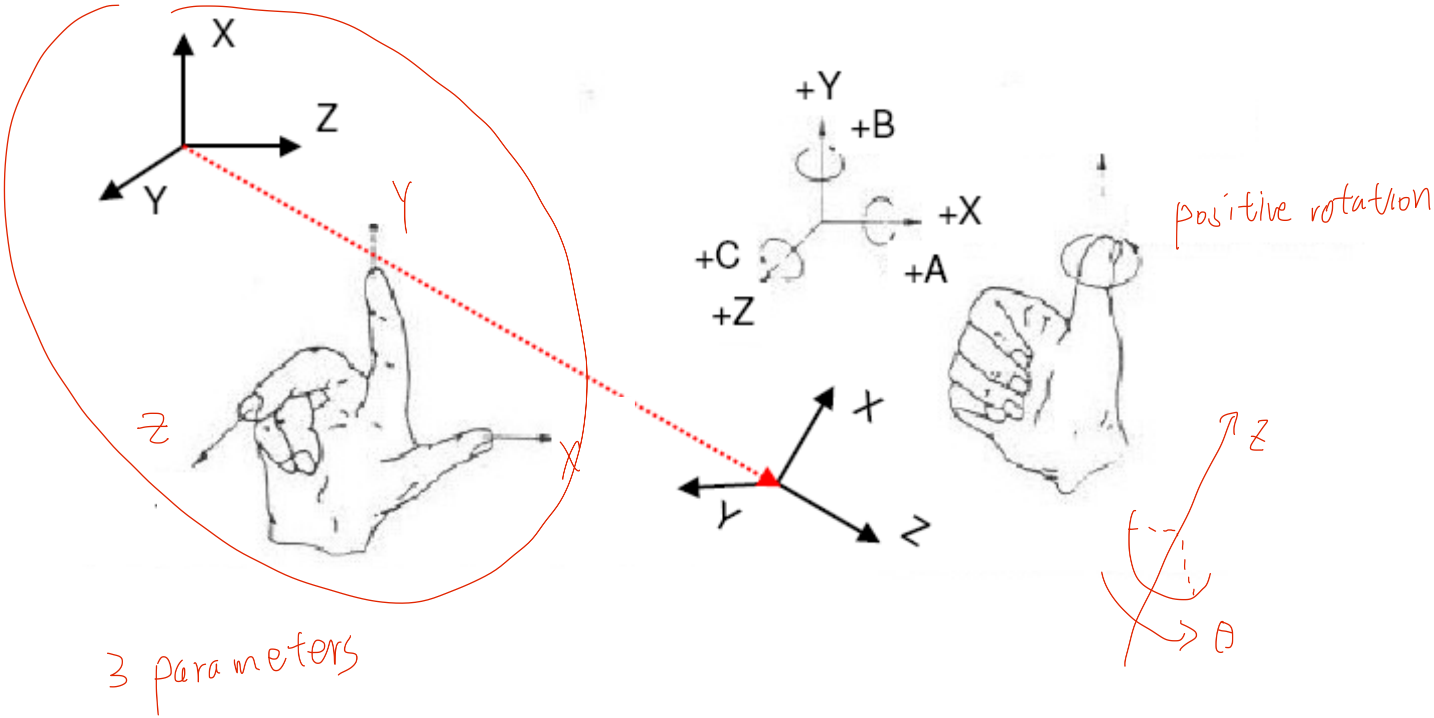
Reach every poing with all
possible orientations

  - For all $\mathbf{p}$ WS_2 there is (at least) one solution for every orientation

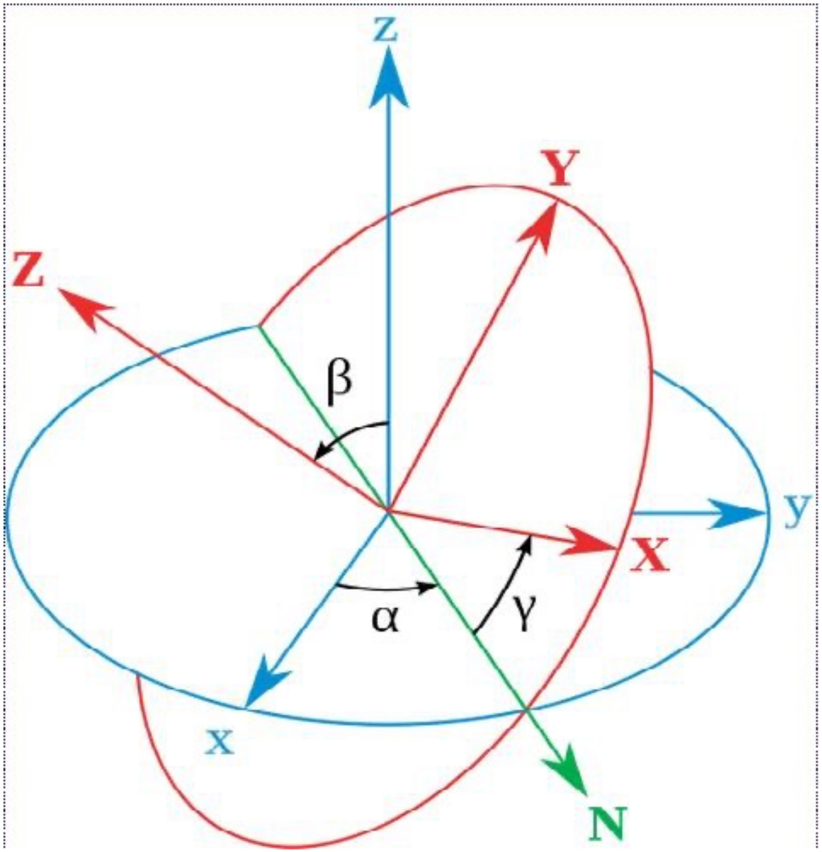
- Relation between WS_1 y WS_2: $$WS_2 \subseteq WS_1$$



Degrees of Freedom N – number of independent motion parameters of a body in space



Rotation matrix



Pose of an object in space

${}^0\vec{P} \to {}^n\vec{P}$.

通过该公式，p点可以在借助任意坐
标系表达，这对于有摄像头的机械
臂更容易进行距离测量

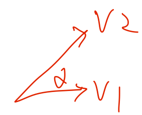
${}^A\vec{P} = {}^A_B\vec{R} \cdot {}^B\vec{P} + {}^A\vec{T}$.

- q = (position, orientation) = (x, y, z, **???**)

- Parametrization of orientations by matrix:

  $q = (r_{11}, r_{12}, \ldots, r_{33}, r_{33})$ where $r_{11}, r_{12}, \ldots, r_{33}$ are the elements of rotation matrix

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$



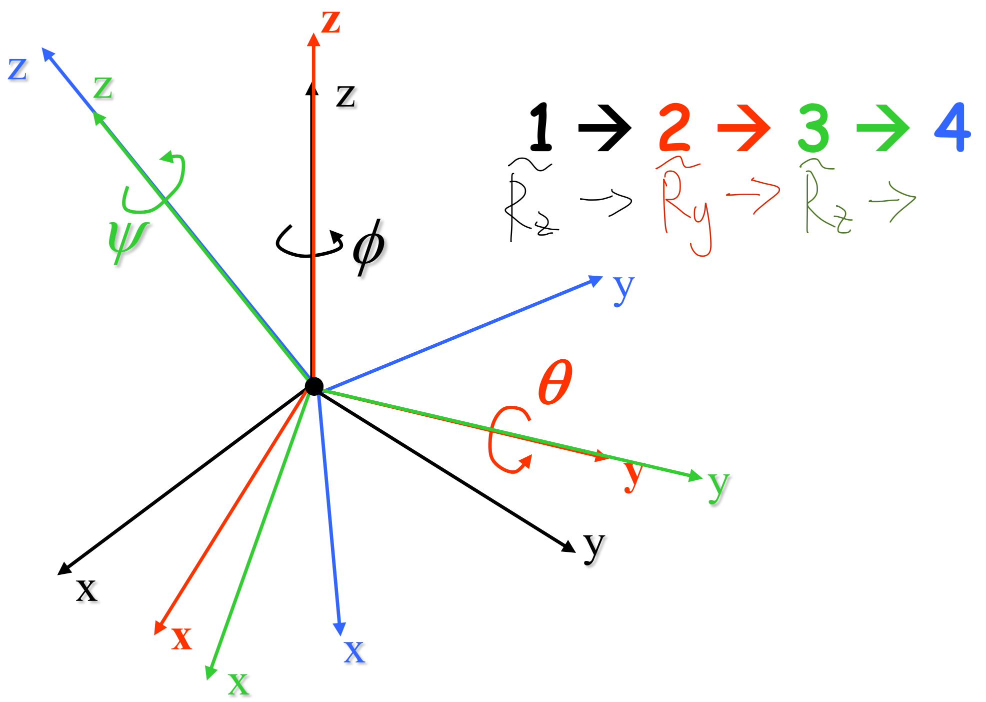
$V_1^T \cdot V_2 = ||V_1|| \cdot ||V_2|| \cdot \cos\alpha$

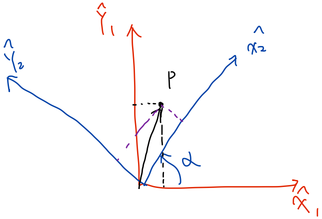
  with

  - $r_{1i}^2 + r_{2i}^2 + r_{3i}^2 = 1$ for all i ,   为什么不能选择9个元素，因为他们是互相关的
  - $r_{1i}\, r_{1j} + r_{2i}\, r_{2j} + r_{3i}\, r_{3j} = 0$ for all $i \neq j$,   每一列与其他列都是正交的 ⇒ 正交矩阵
  - $\det(R) = +1$ 总距离是一样的，只是改变了角度

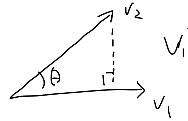
Example: rigid robot in 3-D workspace

- Parametrization of orientations by Euler angles: (ϕ, θ, ψ)





$$^1\vec{p} = \begin{pmatrix} \hat{x}_1^T \cdot {}^1\vec{p} \\ \hat{y}_1^T \cdot {}^1\vec{p} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$



$$V_1^T \cdot V_2 = \|V_1\| \|V_2\| \cdot \cos\theta$$

$$^2\vec{p} = \begin{pmatrix} \hat{x}_2^T \cdot {}^1\vec{p} \\ \hat{y}_2^T \cdot {}^1\vec{p} \end{pmatrix}$$

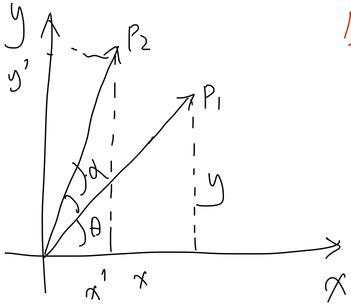
eg: $^1\hat{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad$ $^1\hat{x}_2 = \begin{pmatrix} c\alpha \\ s\alpha \end{pmatrix}$

$^1\hat{y}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad$ $^1\hat{y}_2 = \begin{pmatrix} -s\alpha \\ c\alpha \end{pmatrix}$

$$\tilde{R} = \begin{pmatrix} c\alpha & -s\alpha \\ s\alpha & c\alpha \end{pmatrix} \xRightarrow{3D} \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = \hat{R}_z(\alpha)$$

$$\begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix} = \hat{R}_x(\beta)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{pmatrix} = \hat{R}_z(\gamma)$$



令 $P_1 = P_2 = r$

$P_1 \, (r\cdot\cos\theta, \; r\cdot\sin\theta)$

$P_2 \, (r\cdot\cos(\alpha+\theta), \; r\cdot\sin(\alpha+\theta))$

$r[\sin\alpha\cdot\cos\theta + \cos\alpha\cdot\sin\theta]$

$= x_1\cdot\sin\alpha + y_1\cdot\cos\alpha$

$= x_1\cdot\cos\alpha - y_1\cdot\sin\alpha$

$x_2 = x_1\cdot\cos\alpha - y_1\cdot\sin\alpha$

$y_2 = x_1\cdot\sin\alpha + y_1\cdot\cos\alpha$

$$\Rightarrow {}^2\vec{p} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & +\cos\alpha \end{pmatrix} \cdot {}^1\vec{p}$$

$${}^3p \Rightarrow R = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example of a singularity in Euler representation

为什么R是从后到前

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) $$

$$ {}^{3}\vec{p} = {}^{3}_{2}\tilde{R}\left({}^{2}_{1}\tilde{R}\left({}^{1}_{0}\tilde{R}\,{}^{0}\vec{p}\right)\right) $$

$$ = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}, $$

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

$$ = \sqrt{c^2\alpha c^2\beta + s^2\alpha c^2\beta} = \sqrt{c^2\beta(c^2\alpha + s^2\alpha)} = c\beta\sqrt{c^2\alpha + s^2\alpha} = c\beta $$

$= s\beta$

$$ \beta = \mathrm{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}), \quad = \frac{s\beta}{c\beta} = \tan\beta $$

$s\alpha c\beta \quad c\alpha c\beta$

$$ \alpha = \mathrm{Atan2}(r_{21}/c\beta, r_{11}/c\beta), \quad = \frac{s\alpha}{c\alpha} = \tan\alpha $$

$c\beta s\gamma \quad c\beta c\gamma$

$$ \gamma = \mathrm{Atan2}(r_{32}/c\beta, r_{33}/c\beta), \quad = \frac{s\gamma}{c\gamma} = \tan\gamma $$

$$ \beta = 90.0^\circ, $$

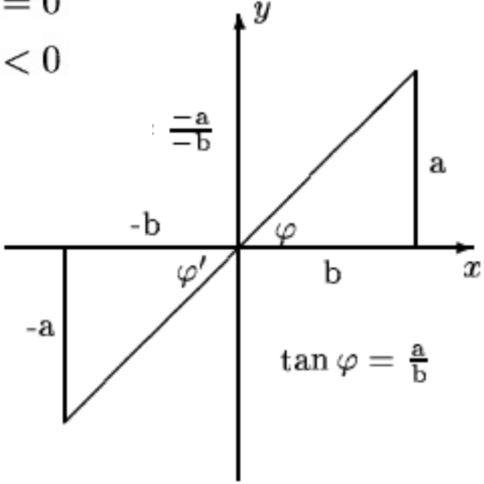
$$ \alpha = 0.0. $$

$$ \gamma = \mathrm{Atan2}(r_{12}, r_{22}), $$

ATAN2 Function

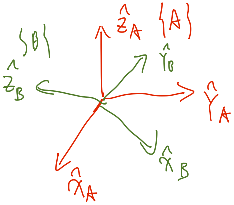
一个返回方位角的函数

$$\mathrm{ATAN2}(a,b) = \begin{cases} arctan(\frac{a}{b}) & \text{falls } b > 0 \\ \frac{\pi}{2} & \text{falls } b = 0, a > 0 \\ \text{undefiniert} & \text{falls } b = 0, a = 0 \\ -\frac{\pi}{2} & \text{falls } b = 0, a < 0 \\ arctan(\frac{a}{b}) + \pi & \text{falls } b < 0 \end{cases}$$



Euler representations

Proper Euler angles
$X_1 Z_2 X_3 = \begin{bmatrix} c_2 & -c_3 s_2 & s_2 s_3 \\ c_1 s_2 & c_1 c_2 c_3 - s_1 s_3 & -c_3 s_1 - c_1 c_2 s_3 \\ s_1 s_2 & c_1 s_3 + c_2 c_3 s_1 & c_1 c_3 - c_2 s_1 s_3 \end{bmatrix}$
$X_1 Y_2 X_3 = \begin{bmatrix} c_2 & s_2 s_3 & c_3 s_2 \\ s_1 s_2 & c_1 c_3 - c_2 s_1 s_3 & -c_1 s_3 - c_2 c_3 s_1 \\ -c_1 s_2 & c_3 s_1 + c_1 c_2 s_3 & c_1 c_2 c_3 - s_1 s_3 \end{bmatrix}$
$Y_1 X_2 Y_3 = \begin{bmatrix} c_1 c_3 - c_2 s_1 s_3 & s_1 s_2 & c_1 s_3 + c_2 c_3 s_1 \\ s_2 s_3 & c_2 & -c_3 s_2 \\ -c_3 s_1 - c_1 c_2 s_3 & c_1 s_2 & c_1 c_2 c_3 - s_1 s_3 \end{bmatrix}$
$Y_1 Z_2 Y_3 = \begin{bmatrix} c_1 c_2 c_3 - s_1 s_3 & -c_1 s_2 & c_3 s_1 + c_1 c_2 s_3 \\ c_3 s_2 & c_2 & s_2 s_3 \\ -c_1 s_3 - c_2 c_3 s_1 & s_1 s_2 & c_1 c_3 - c_2 s_1 s_3 \end{bmatrix}$
$Z_1 Y_2 Z_3 = \begin{bmatrix} c_1 c_2 c_3 - s_1 s_3 & -c_3 s_1 - c_1 c_2 s_3 & c_1 s_2 \\ c_1 s_3 + c_2 c_3 s_1 & c_1 c_3 - c_2 s_1 s_3 & s_1 s_2 \\ -c_3 s_2 & s_2 s_3 & c_2 \end{bmatrix}$
$Z_1 X_2 Z_3 = \begin{bmatrix} c_1 c_3 - c_2 s_1 s_3 & -c_1 s_3 - c_2 c_3 s_1 & s_1 s_2 \\ c_3 s_1 + c_1 c_2 s_3 & c_1 c_2 c_3 - s_1 s_3 & -c_1 s_2 \\ s_2 s_3 & c_3 s_2 & c_2 \end{bmatrix}$



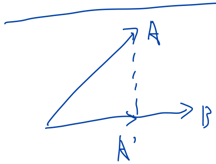
空间矩阵 $V_A = \{x_A, y_A, z_A\}^T \quad V_B = \{x_B, y_B, z_B\}^T$

$$\Rightarrow V_B = R \cdot V_A.$$

令 $x_A = \{1,0,0\}^T \quad y_A = \{0,1,0\}^T \quad z_A = \{0,0,1\}^T$

$$\Rightarrow \hat{x}_B = R\cdot\begin{Bmatrix}1\\0\\0\end{Bmatrix} \quad \hat{y}_B = R\cdot\begin{Bmatrix}0\\1\\0\end{Bmatrix} \quad \hat{z}_B = R\cdot\begin{Bmatrix}0\\0\\1\end{Bmatrix}$$

$$\Rightarrow R = \{\hat{x}_B, \hat{y}_B, \hat{z}_B\}$$



$A\cdot B = \|A\|\cdot\|B\|\cdot\cos\alpha.$

当 $\|B\| = 1$ 时

$A\cdot B$ 即为 A 在 B 上的投影

$$\Rightarrow \text{易得 } {}^A\hat{x}_B = \begin{bmatrix}\vec{x}_B\cdot\vec{x}_A\\ \vec{x}_B\cdot\vec{y}_A\\ \vec{x}_B\cdot\vec{z}_A\end{bmatrix}$$

$${}^A_BR = \left[{}^A\hat{x}_B,\ {}^A\hat{y}_B,\ {}^A\hat{z}_B\right]$$

$$= \begin{bmatrix}\boxed{\vec{x}_B\cdot\vec{x}_A \quad \vec{y}_B\cdot\vec{x}_A \quad \vec{z}_B\cdot\vec{x}_A}\\ \vec{x}_B\cdot\vec{y}_A & \vec{y}_B\cdot\vec{y}_A & \vec{z}_B\cdot\vec{y}_A\\ \vec{x}_B\cdot\vec{z}_A & \vec{y}_B\cdot\vec{z}_A & \vec{z}_B\cdot\vec{z}_A\end{bmatrix}$$

$$\Rightarrow {}^B\vec{x}_A{}^T = \begin{bmatrix}\vec{x}_A\cdot\vec{x}_B\\ \vec{x}_A\cdot\vec{y}_B\\ \vec{x}_A\cdot\vec{z}_B\end{bmatrix}$$

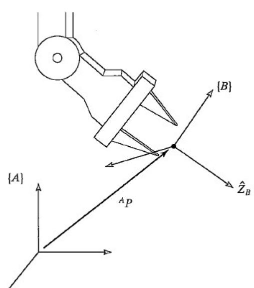
$$= \begin{bmatrix}{}^B\vec{x}_A{}^T\\ {}^B\vec{y}_A{}^T\\ {}^B\vec{z}_A{}^T\end{bmatrix} = \left[{}^B\vec{x}_A \quad {}^B\vec{y}_A \quad {}^B\vec{z}_A\right] = {}^B_AR^T$$

$$\therefore {}^A_BR = {}^B_AR^T = {}^B_AR^{-1}$$

书p22

We denote the unit vectors giving the principal directions of coordinate system $\{B\}$ as $\hat{X}_B$, $\hat{Y}_B$, and $\hat{Z}_B$. When written in terms of coordinate system $\{A\}$, they are called ${}^A\hat{X}_B$, ${}^A\hat{Y}_B$, and ${}^A\hat{Z}_B$. It will be convenient if we stack these three unit vectors together as the columns of a 3 × 3 matrix, in the order ${}^A\hat{X}_B$, ${}^A\hat{Y}_B$, ${}^A\hat{Z}_B$. We will call this matrix a **rotation matrix**, and, because this particular rotation matrix describes $\{B\}$ relative to $\{A\}$, we name it with the notation ${}^A_BR$ (the choice of leading sub- and superscripts in the definition of rotation matrices will become clear in following sections):

$$ {}^A_BR = \left[\,{}^A\hat{X}_B \quad {}^A\hat{Y}_B \quad {}^A\hat{Z}_B\,\right] = \begin{bmatrix} r_{11} & r_{12} & r_{13}\\ r_{21} & r_{22} & r_{23}\\ r_{31} & r_{32} & r_{33}\end{bmatrix}. \tag{2.2}$$



$${}^1_0\vec{R}^{-1}\left({}^1\vec{p} - {}^1\vec{T}\right) = \underbrace{{}^1_0\vec{R}^{-1}\,{}^1_0\vec{R}}_{I}\cdot{}^0\vec{p}$$

$${}^0\vec{p} = {}^1_0\vec{R}^{-1}\left({}^1\vec{p} - {}^1\vec{T}\right) \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{R} = \begin{pmatrix} {}^0\hat{x}^T \\ {}^0\hat{y}^T \\ {}^0\hat{z}^T \end{pmatrix}$$

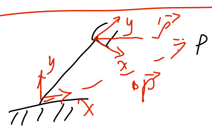
$$\because R = \begin{pmatrix} c\alpha & s\alpha \\ -s\alpha & c\alpha \end{pmatrix} \Rightarrow \begin{pmatrix} c\alpha & -s\alpha \\ s\alpha & c\alpha \end{pmatrix}$$

$$\therefore \hat{R}^{-1} = \left({}^0\hat{x} \quad {}^0\hat{y} \quad {}^0\hat{z}\right)$$

在什么投影系下的矩阵

$$\Rightarrow \boxed{{}^1_0\vec{R}^{-1} = {}^1_0\vec{R}^T = {}^0_1\vec{R}}$$

$$\Rightarrow \boxed{{}^0\vec{p} = {}^1_0\vec{R}^T\,{}^1\vec{p} + \left(-{}^1_0\vec{R}^T\cdot{}^1\vec{T}\right)}$$

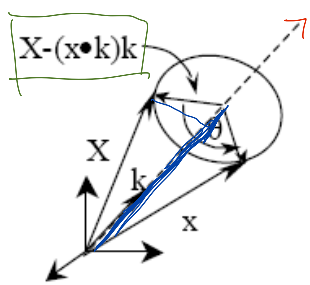


$${}^1\vec{p} = {}^1_0\vec{R}\cdot{}^0\vec{p} + {}^1\vec{T}$$

$${}^1_0\vec{R} = \left({}^0\vec{x}^1,\ {}^0\vec{y}^1\right)$$

Axis-angle representation of Rotation

Let's start from a geometric view point. Imagine a coordinate with a vector $\vec{X}$ where $\vec{k}$ is the unit vector representing the axis of rotation. Let the vector $\vec{x}$ be the result of rotating $\vec{X}$ by an angle θ about $\vec{k}$. You can imagine a circle created by $\vec{X}$ and $\vec{x}$ with the axis of rotation going through its center (see Figure 1).



$\vec{k}$ → 用向量计算太复杂了，所以要用矩阵 → $\tilde{k}$

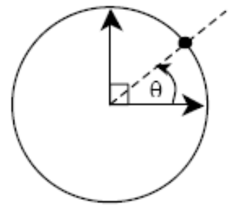


Figure 1: Axis and angle of rotation

Hence $\vec{x} = \boxed{(\vec{X}\bullet\vec{k})\vec{k}} + \boxed{(\vec{X} - (\vec{X}\bullet\vec{k})\vec{k})}\cos\theta + (\vec{k}\times\vec{X})\sin\theta$ (1)

(Also a good exercise to prove that $\vec{k}\times\vec{X}$ is perpendicular to $\vec{X}-(\vec{X}\bullet\vec{k})\vec{k}$)

Let define a skew symmetric matrix K such that $K = \mathrm{J}(\vec{k})$. This means

$$K = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} \text{ and we know that } K\vec{v} = \vec{k}\times\vec{v}$$

$$\vec{k} \times \vec{x} = \tilde{k}\vec{x}$$

$$= \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} k_y z - k_z y \\ k_z x - k_x z \\ k_x y - k_y x \end{pmatrix} = \begin{pmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\vec{k} \to \tilde{k}$$

Now we can write $\vec{x}$ as

$$\begin{aligned}\vec{x} &= \vec{X} - \vec{X} + (\vec{X} \bullet \vec{k})\vec{k} + (\vec{X} - (\vec{X} \bullet \vec{k})\vec{k})\cos\theta + (K\vec{X})\sin\theta \\ &= \vec{X} - (\vec{X} - (\vec{X} \bullet \vec{k})\vec{k}) + (\vec{X} - (\vec{X} \bullet \vec{k})\vec{k})\cos\theta + (K\vec{X})\sin\theta \\ &= \vec{X} - (1 - \cos\theta)(\vec{X} - (\vec{X} \bullet \vec{k})\vec{k}) + (K\vec{X})\sin\theta\end{aligned} \tag{2}$$

There exists an identity that $a \times (a \times b) = (a \bullet a)b - (a \bullet b)a$. You can also try to prove this for exercise as well. Now we can rewrite $(\vec{X} - (\vec{X} \bullet \vec{k})\vec{k})$ using this identity as

$$(\vec{X} - (\vec{X} \bullet \vec{k})\vec{k}) = (\vec{k} \bullet \vec{k})\vec{X} - (\vec{X} \bullet \vec{k})\vec{k} = \vec{k} \times (\vec{X} \times \vec{k}) = -\vec{k} \times (\vec{k} \times \vec{X}) \tag{3}$$

Note here that $(\vec{k} \bullet \vec{k})$ is just 1, so this doesn't change anything. Then rewrite the result using the property of the skew symmetric matrix K, we get

$$-\vec{k}\times(\vec{k}\times\vec{X}) = -\vec{k}\times K\vec{X} = -(K(K\vec{X})) = -K^2\vec{X} \tag{4}$$

Substitute (4) in (2), we get

$q = (x, y, z, \alpha, \beta, \gamma) \leftarrow$ Euler.

$$\begin{aligned}\vec{x} &= \vec{X} - (1-\cos\theta)(K^2\vec{X}) + (K\vec{X})\sin\theta \\ &= (I + (1-\cos\theta)K^2 + \sin\theta K)\vec{X}\end{aligned} \tag{5}$$

$q = (x, y, z, \underbrace{\vec{k}, \theta}_{\vec{r}})$

Since $\vec{x} = R\vec{X}$, therefore, the rotation matrix is described by

$\vec{r} = \vec{k}\cdot\theta$

$$\boxed{R = (I + (1-\cos\theta)K^2 + \sin\theta K)} \tag{6}$$

$\vec{k} = \frac{\vec{r}}{\|\vec{r}\|}, = \theta$

Rodrigues formula.

Now we can use this formula to find back $\vec{k}$ and θ. Knowing that $R^T(\vec{k},\theta) = R(\vec{k},-\theta)$ applying Rodrigues formula for both sides, we will get

$$\boxed{\begin{aligned} R - R^T &= 2\sin\theta K \\ K &= \frac{R - R^T}{2\sin\theta}\end{aligned}} \tag{7}$$

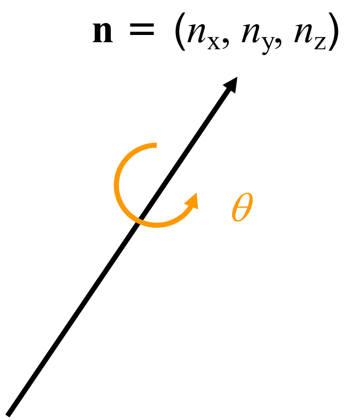
$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & I \end{pmatrix} - \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & I \end{pmatrix} = \begin{pmatrix} 0 & - & \cdot \\ \cdot & 0 & \cdot \\ \cdot & - & 0 \end{pmatrix}$$

Hence, $\vec{k} = \frac{1}{2\sin\theta}vect(K)$ and θ can be determined by solving $2\sin\theta = \|vect(R - R^T)\|$

Note: Problems arise when θ is small since the axis of rotation is ill-defined and that $(\vec{k},\theta)$ and $(-\vec{k},-\theta)$ result in the same orientation.

Example: rigid obot in 3-D workspace

- Parametrization of orientations by **unit quaternion**: $u = (u_1, u_2, u_{3,} u_4)$ with $u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1$.
  - Note $(u_1, u_2, u_{3,} u_4) = (\cos\theta/2, n_x\sin\theta/2, n_y\sin\theta/2, n_z\sin\theta/2)$ with $n_x^2 + n_y^2 + n_z^2 = 1$.
  - Compare with representation of orientation in 2-D: $(u_1, u_2) = (\cos\theta, \sin\theta)$



Example: rigid robot in 3-D workspace

- Advantage of unit quaternion representation
  - Compact

    没有那么多限制
  - No singularity

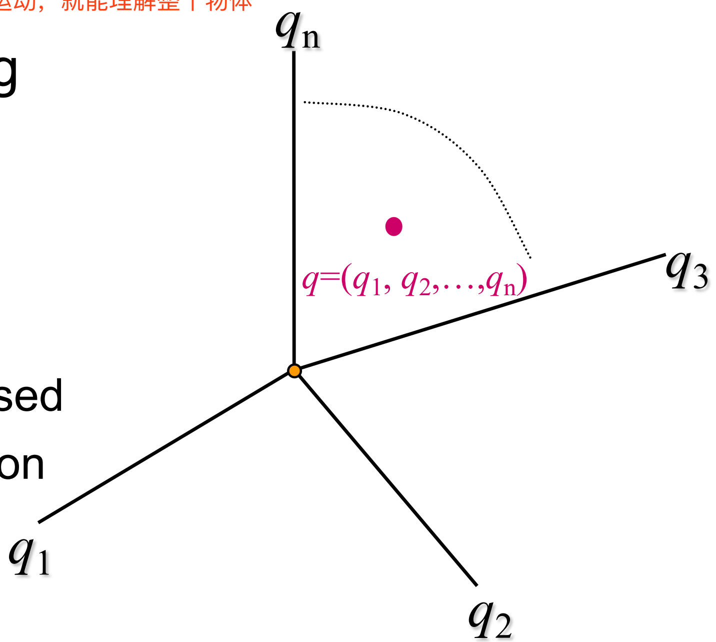
    可逆　　行列式非0
  - Naturally reflect the topology of the space of orientations

- Number of dofs = 6
- Topology: R^3 x $\mathrm{SO}(3)$

Configuration space

理解一个点的运动，就能理解整个物体的运动

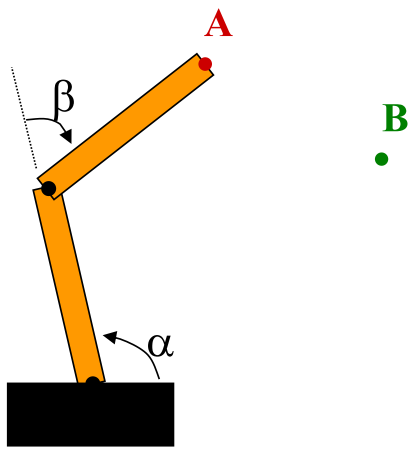
- The **configuration** of a moving object is a specification of the position of **every** point on the object.
  - Usually a configuration is expressed as a vector of position & orientation parameters: $q = (q_1, q_2, \ldots, q_n)$



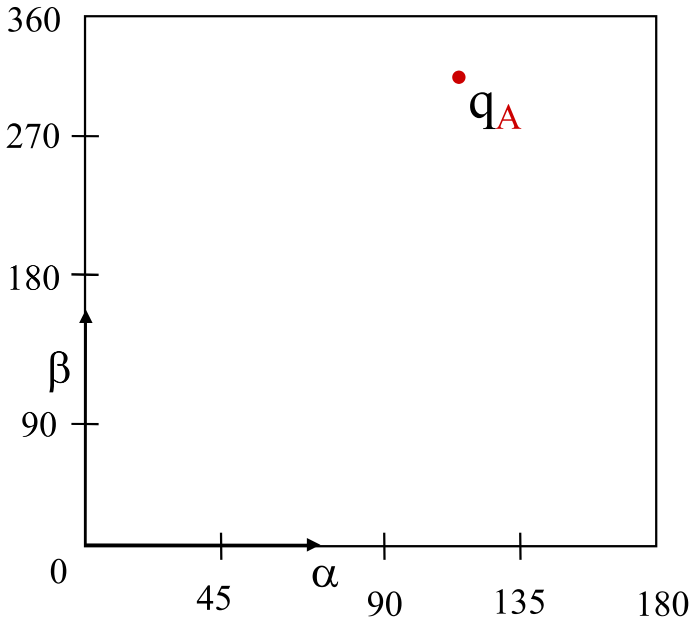
- The **configuration space** C is the set of all possible configurations.
  - A configuration is a point in C.

Configuration Space

Where can we put $\bullet\ q_B$?



An obstacle in the robot's workspace

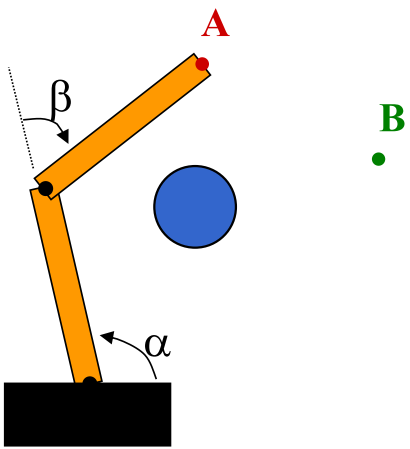


Torus

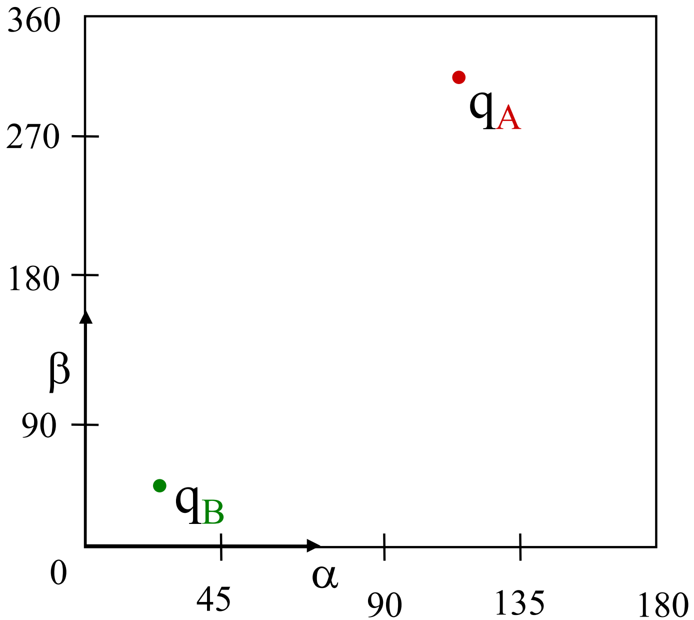
(wraps horizontally and vertically)

Configuration Space "Quiz"

Where do we put ?



An obstacle in the robot's workspace

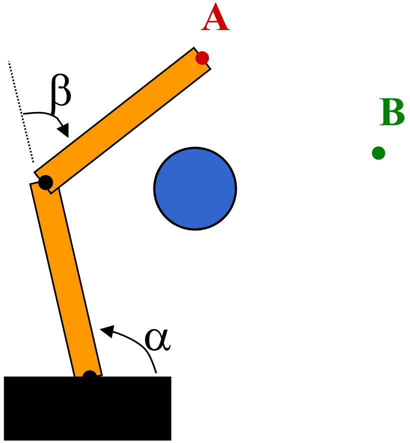


Torus

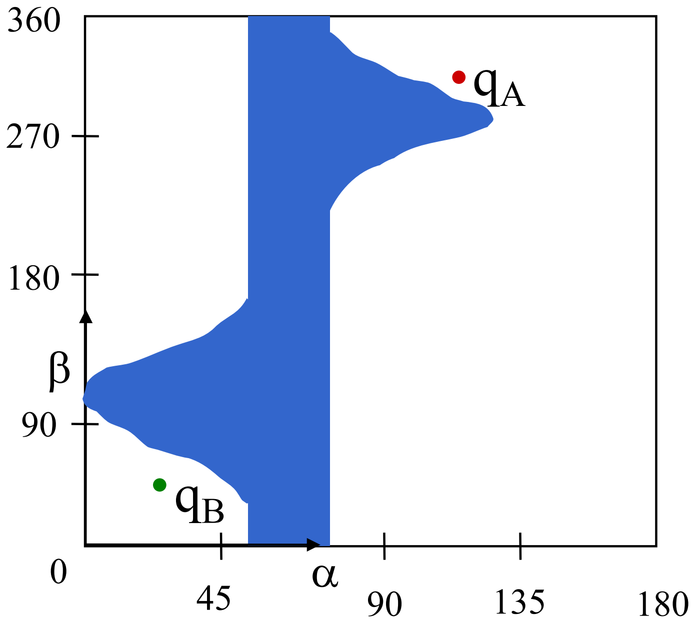
(wraps horizontally and vertically)

Configuration Space

How do we get from A to B ?



An obstacle in the robot's workspace

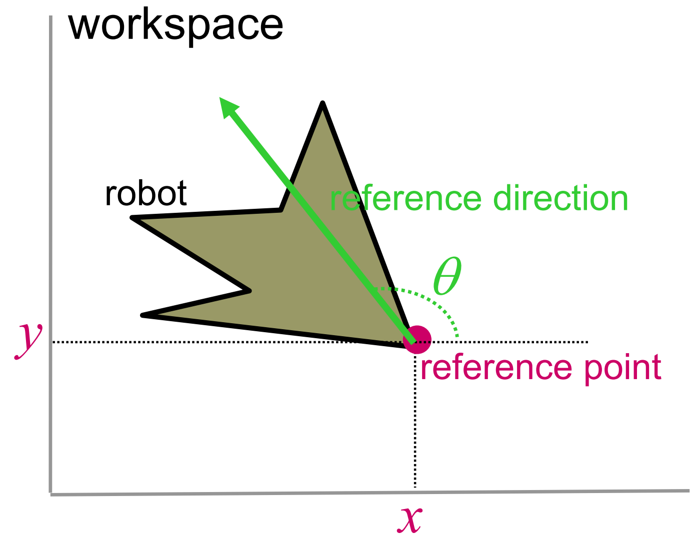


The C-space representation of this obstacle…

Dimension of configuration space

- The **dimension of a configuration space** is the **minimum** number of parameters needed to specify the configuration of the object completely.
- It is also called the **number of degrees of freedom** (dofs) of a moving object.

Example: rigid robot in 2-D workspace



- 3-parameter specification: $q = (x, y, \theta)$ with $\theta \in [0, 2\pi)$.
  - 3-D configuration space

Example: rigid robot in 2-D workspace

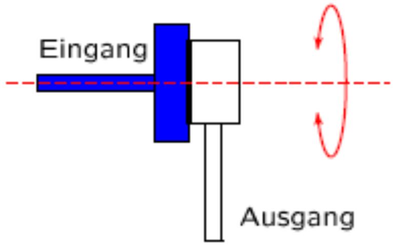
- 4-parameter specification: $q = (x, y, u, v)$ with $u^2+v^2 = 1$. Note $u = \cos\theta$ and $v = \sin\theta$. u和v是相关的

- dim of configuration space = **???** **3**
  - Does the dimension of the configuration space (number of dofs) depend on the parametrization?

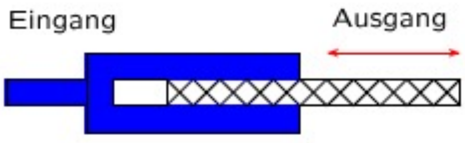
- Topology: a 3-D cylinder $C = \mathrm{R}^2 \times \mathrm{S}^1$

  - Does the topology depend on the parametrization?

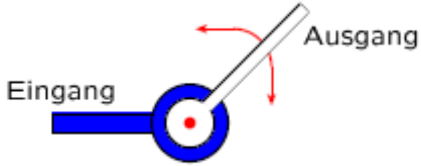
Motion of the robot is generated by its joints



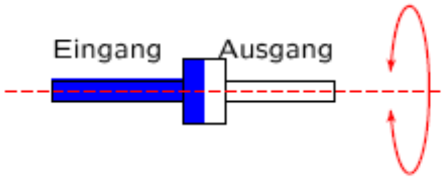
revolving joint



linear joint



rotational joint

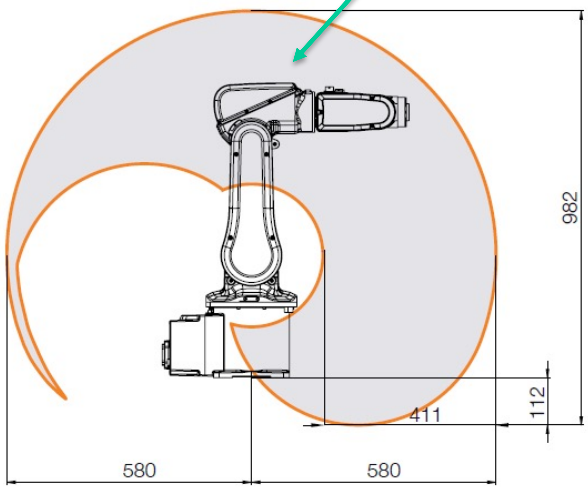


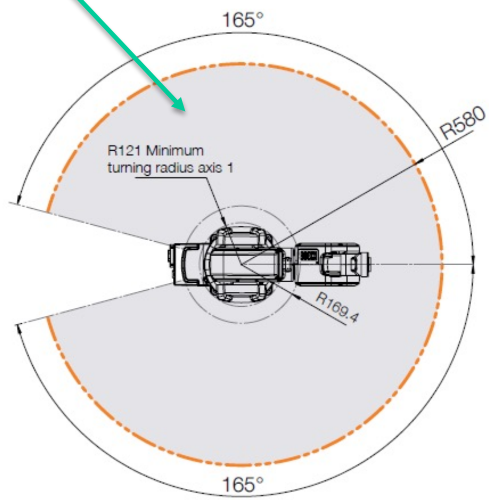
twisting joint

Workspace

- Example: ABB's IRB 120 robot

WS_1 (=WS_2 for spherical wrist)



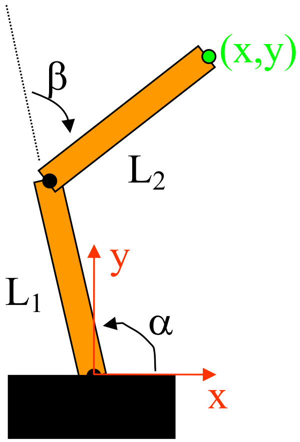


It is used to evaluate the robot for a specific application

[ABB IRB 120 Industrial Robot – Datasheet]

Robot Manipulators

What are this arm's forward kinematics?

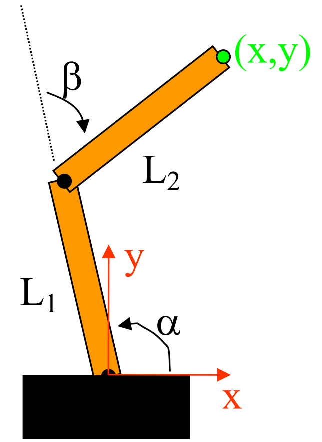


(How does its position depend on its joint angles?)

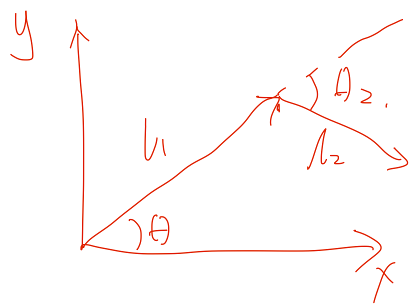
$\vec{x} = f(\vec{\Theta})$

Robot Manipulators

What are this arm's forward kinematics?



Find (x,y) in terms of α and β ...



$$\vec{x} = \begin{pmatrix} \cos\theta_1 \cdot l_1 \\ \sin\theta_1 \cdot l_1 \end{pmatrix} + \begin{pmatrix} \cos(\theta_1+\theta_2) \cdot l_2 \\ \sin(\theta_1+\theta_2) \cdot l_2 \end{pmatrix}$$

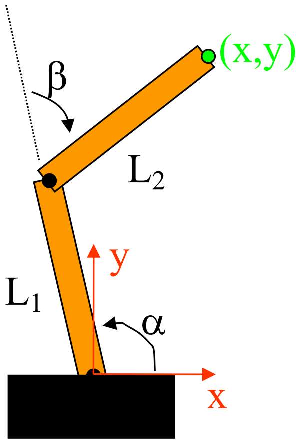
Keeping it "simple"

$c_\alpha = \cos(\alpha)$, $s_\alpha = \sin(\alpha)$

$c_\beta = \cos(\beta)$, $s_\beta = \sin(\beta)$

$c_+ = \cos(\alpha+\beta)$, $s_+ = \sin(\alpha+\beta)$

Manipulator kinematics



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_1 c_\alpha \\ L_1 s_\alpha \end{pmatrix} + \begin{pmatrix} L_2 c_+ \\ L_2 s_+ \end{pmatrix}$$ Position

Keeping it "simple"

$c_\alpha = \cos(\alpha)$, $s_\alpha = \sin(\alpha)$

$c_\beta = \cos(\beta)$, $s_\beta = \sin(\beta)$

$c_+ = \cos(\alpha+\beta)$, $s_+ = \sin(\alpha+\beta)$

In general, a point in n-D space transforms by

P' = rotate(point) + translate(point)

In 2-D space, this can be written as a matrix equation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} Cos(\theta) & -Sin(\theta) \\ Sin(\theta) & Cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} tx \\ ty \end{pmatrix}$$

In 3-D space (or n-D), this can generalized as a matrix equation:

$$p' = R\,p + T \quad \text{or} \quad p = R^t\,(p' - T)$$

$${}^{n}p = \boxed{{}^{n}_{n-1}R \cdots {}^{1}_{0}R \cdot {}^{0}p + \cdots\cdots}$$

向量计算过于复杂，因此将引入矩阵计算

Now, using the idea of homogeneous transforms,
we can write:

$$p' = \begin{pmatrix} & R & & T \\ 0 & 0 & 0 & 1 \end{pmatrix} p$$

The group of rigid body rotations SO(3) $\times$ $\Re(3)$ is denoted SE(3) (for special Euclidean group)

What does the inverse transformation look like?

$$p' = \begin{pmatrix} & R & & T \\ 0 & 0 & 0 & 1 \end{pmatrix} p$$

$$^1p_H = \begin{pmatrix} R & T_x \\ & T_y \\ & T_z \\ 0 & 1 \end{pmatrix} {}^0\begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} \Leftarrow$$

Cartesion homagenous

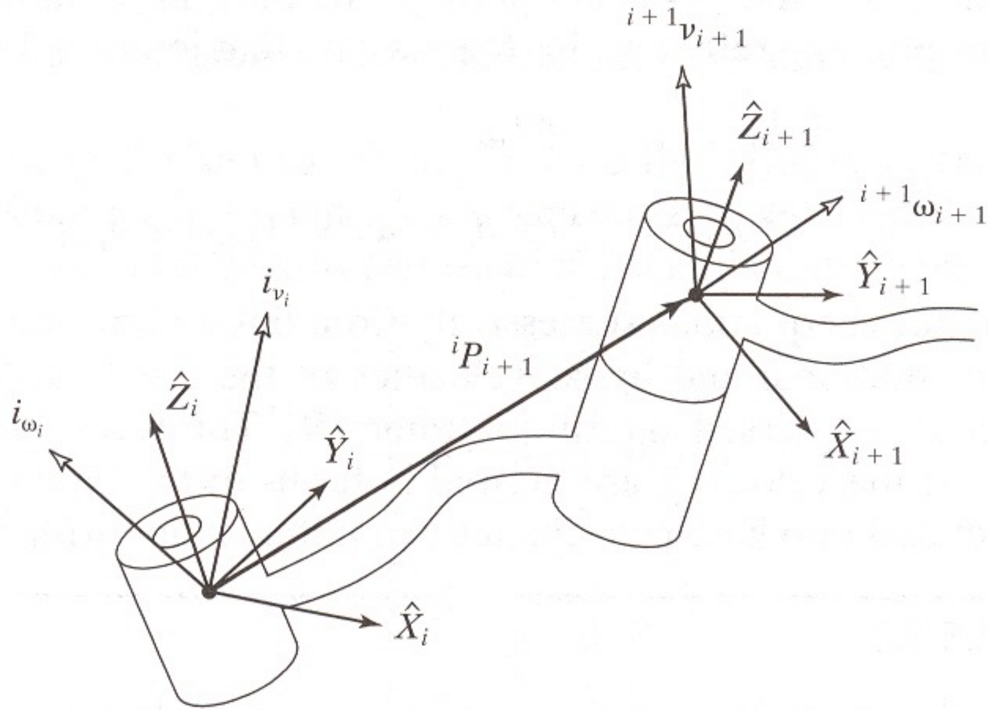
$$p = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow p_H = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \quad \vec{V} = \begin{pmatrix} V_x \\ V_y \\ V_z \\ \alpha \end{pmatrix}$$

$$^3\vec{p}_H = \underbrace{{}^3_2T \cdot {}^2_1T \cdot {}^1_0T}_{{}^3_0T} \cdot {}^0\vec{p}_H$$

$$^3_0T^{-1} = {}^0_3T = \begin{pmatrix} {}^3_0R^T & -{}^3_0R^T \cdot {}^3\vec{T} \\ 0\,0\,0 & 1 \end{pmatrix}$$

$$\vec{p}(\vec{\theta}) = \vec{f}(\vec{\theta}) = \begin{pmatrix} f_x \\ f_y \\ f_z \\ f_\alpha \\ f_\beta \\ f_\gamma \end{pmatrix} \Rightarrow {}^0\vec{T} = \begin{pmatrix} f_x(\vec{\theta}) \\ f_y(\vec{\theta}) \\ f_z(\vec{\theta}) \end{pmatrix}$$

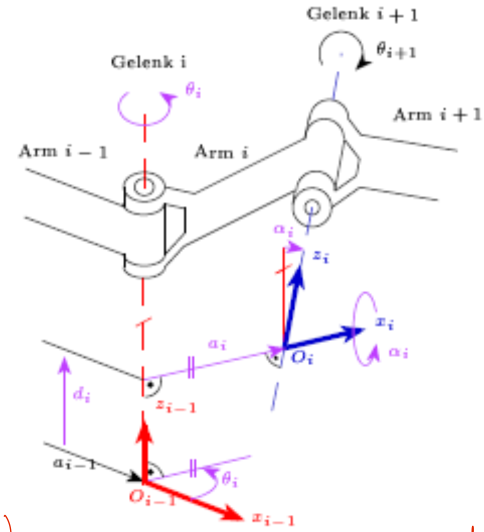
Coordinate frames

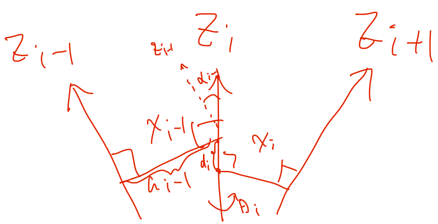


z-axis along the axis of motion
x-axis perpendicular on the two consecutive motion axes
y in a direction defined by a right hand system

沿着运动轴的Z轴 右手螺旋，正向随旋转方向
X轴垂直于两个连续的运动轴 两轴间的最短距离
y在右手系统定义的方向上

① Find z_i
② Define x_i





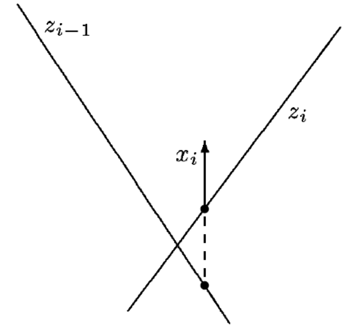
两 Z_i 轴间角度

(negative)

	a_{i-1}	α_{i-1}	d_i	θ_i
i				

对于平移组件，d_i 是变量，θ_i 是定值，
而对于旋转组件，d_i 是定值，θ_i 是变量。

DH Rules



3D position
3D orientation } → 6D

${}^{i+1}p = R_z(\theta) \cdot T_z(d)\, T_x(a)\, R_x(\alpha)\, {}^{i}p$

→ Rotation θ_i
→ prismatic d_i

1. Translation along z_i
2. Rotation around z_i
3. Translation along to the origin of the next frame
4. Rotation between the coordinate frames

$\boldsymbol{T}(0, 0, d_i)$
$\boldsymbol{R}(z, \theta_i)$
$\boldsymbol{T}(a_i, 0, 0)$
$\boldsymbol{R}(x, \alpha_i)$

${}^{i}_{0}\vec{T} \quad \tilde{T}_z(\theta_i)\ \tilde{T}_z(d_i)\ \tilde{T}\ \ T_{x_{i-1}}(a_{i-1})$

DH rules

$${}^{i-1}\boldsymbol{A}_i = \boldsymbol{T}(0, 0, d_i) \cdot \boldsymbol{R}(z, \theta_i) \cdot \boldsymbol{T}(a_i, 0, 0) \cdot \boldsymbol{R}(x, \alpha_i)$$

modified DH-rules : ${}^{i-1}p = T_x R_x R_z T_z\, {}^{i}p$

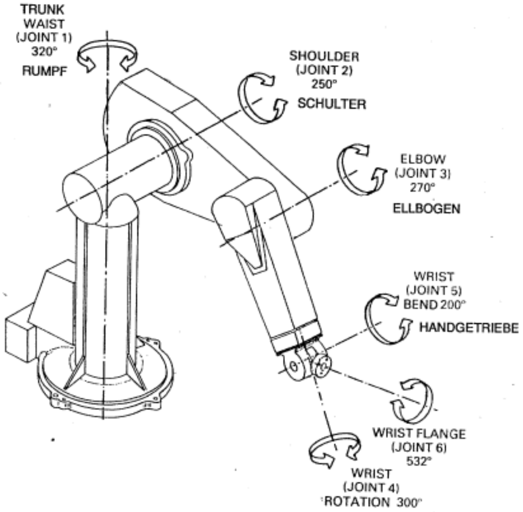
$${}^{i-1}\boldsymbol{A}_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} C\theta_i & -S\theta_i & 0 & 0 \\ S\theta_i & C\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_i & -S\alpha_i & 0 \\ 0 & S\alpha_i & C\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

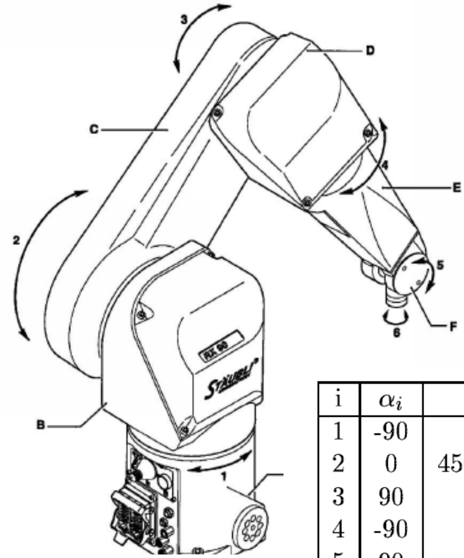
.. results in

$$
{}^{i-1}\boldsymbol{A}_i = \begin{pmatrix} C\theta_i & -C\alpha_i \cdot S\theta_i & S\alpha_i \cdot S\theta_i & a_i \cdot C\theta_i \\ S\theta_i & C\alpha_i \cdot C\theta_i & -S\alpha_i \cdot C\theta_i & a_i \cdot S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

… the inverse direction can be calculated to

$$
{}^{i-1}\boldsymbol{A}_i^{-1} = {}^{i}\boldsymbol{A}_{i-1} = \begin{pmatrix} C\theta_i & S\theta_i & 0 & -a_i \\ -C\alpha_i \cdot S\theta_i & C\alpha_i \cdot C\theta_i & -S\alpha_i & -d_i \cdot S\alpha_i \\ S\alpha_i \cdot S\theta_i & -S\alpha_i \cdot C\theta_i & C\alpha_i & -d_i \cdot C\alpha_i \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$





i	α_i	a_i	d_i	θ_i-Bereich
1	-90	0	0	$-160° \cdots +160°$
2	0	450mm	0	$-227.5° \cdots +47.5°$
3	90	0	0	$-52.5° \cdots +232.5°$
4	-90	0	$450mm$	$-270° \cdots +270°$
5	90	0	0	$-105° \cdots +120°$
6	0	0	$85mm$	$-270° \cdots +270°$

$$^0\boldsymbol{A}_6 = {}^0\boldsymbol{A}_1 \cdot {}^1\boldsymbol{A}_2 \cdot {}^2\boldsymbol{A}_3 \cdot {}^3\boldsymbol{A}_4 \cdot {}^4\boldsymbol{A}_5 \cdot {}^5\boldsymbol{A}_6$$

$$^0\boldsymbol{A}_3 = {}^0\boldsymbol{A}_1 \cdot {}^1\boldsymbol{A}_2 \cdot {}^2\boldsymbol{A}_3 = \begin{pmatrix} \cos\theta_1 \cdot \cos(\theta_2+\theta_3) & -\sin\theta_1 & \cos\theta_1 \cdot \cos(\theta_2+\theta_3) & a_2 \cdot \cos\theta_1 \cdot \cos\theta_2 \\ \sin\theta_1 \cdot \cos(\theta_2+\theta_3) & \cos\theta_1 & \sin\theta_1 \cdot \sin(\theta_2+\theta_3) & a_2 \cdot \sin\theta_1 \cdot \cos\theta_2 \\ -\sin(\theta_2+\theta_3) & 0 & \cos(\theta_2+\theta_3) & -a_2\sin\theta_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Forward Kinematics

- Given joint variables $q = (q_1, q_2, \cdots q_n)$
- End-effector position & orientation $Y = (x, y, z, \phi, \theta, \psi)$

Homogeneous matrix T_0^n

- specifies the location of the ith coordinate frame w.r.t. the base coordinate system
- chain product of successive coordinate transformation matrices of T_{i-1}^i

$$T_0^n = T_0^1 T_1^2 \ldots T_{n-1}^n$$

$$= \begin{bmatrix} \boxed{R_0^n} & \boxed{P_0^n} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} n & s & a & P_0^n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Orientation matrix

Position vector

Yaw-Pitch-Roll representation for orientation

$$T_0^n = \begin{bmatrix} C\phi C\vartheta & C\phi S\theta S\psi - S\phi C\psi & C\phi S\theta C\psi + S\phi S\psi & p_x \\ S\phi C\theta & S\phi S\theta S\psi + C\phi C\psi & S\phi S\theta C\psi - C\phi S\psi & p_y \\ -S\theta & C\theta S\psi & C\theta C\psi & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

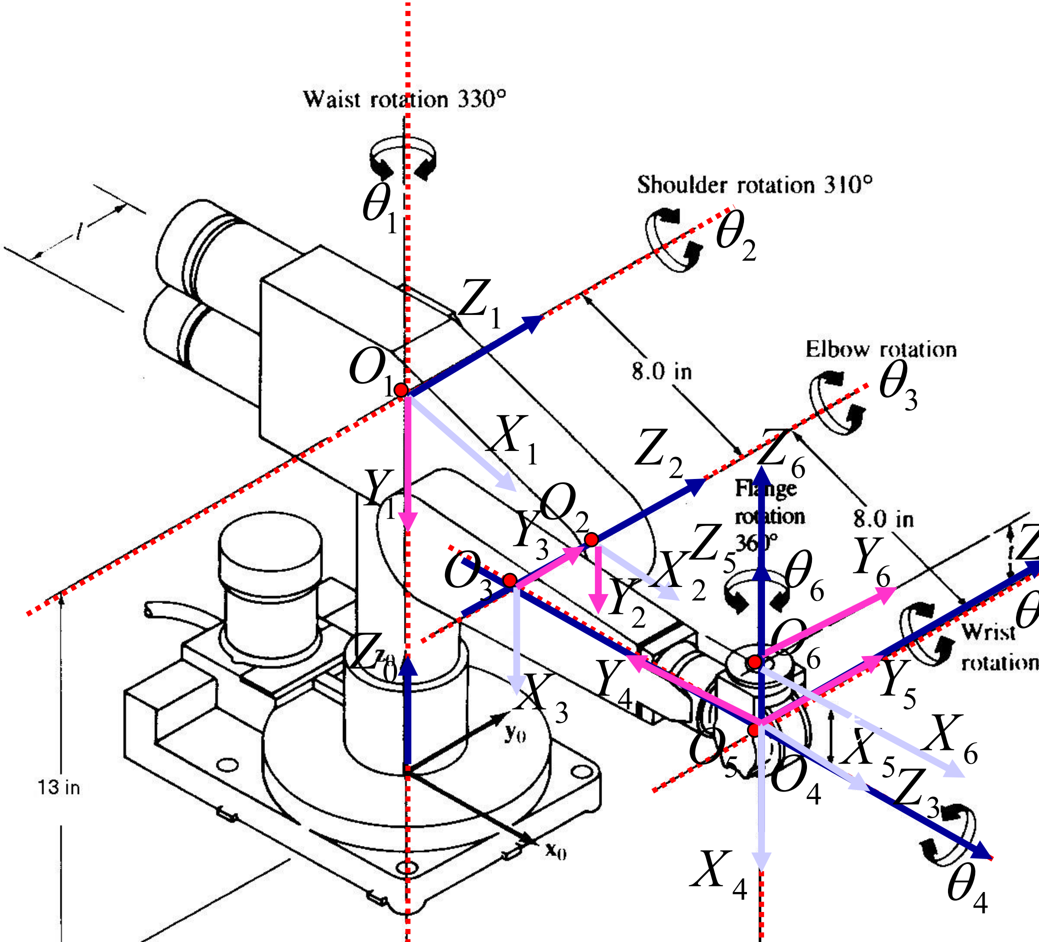
$$T_0^n = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\theta = \sin^{-1}(-n_z)$$

$$\psi = \cos^{-1}\left(\frac{a_z}{\cos\theta}\right)$$

$$\phi = \cos^{-1}\left(\frac{n_x}{\cos\theta}\right)$$

Example: PUMA 260



PUMA 260

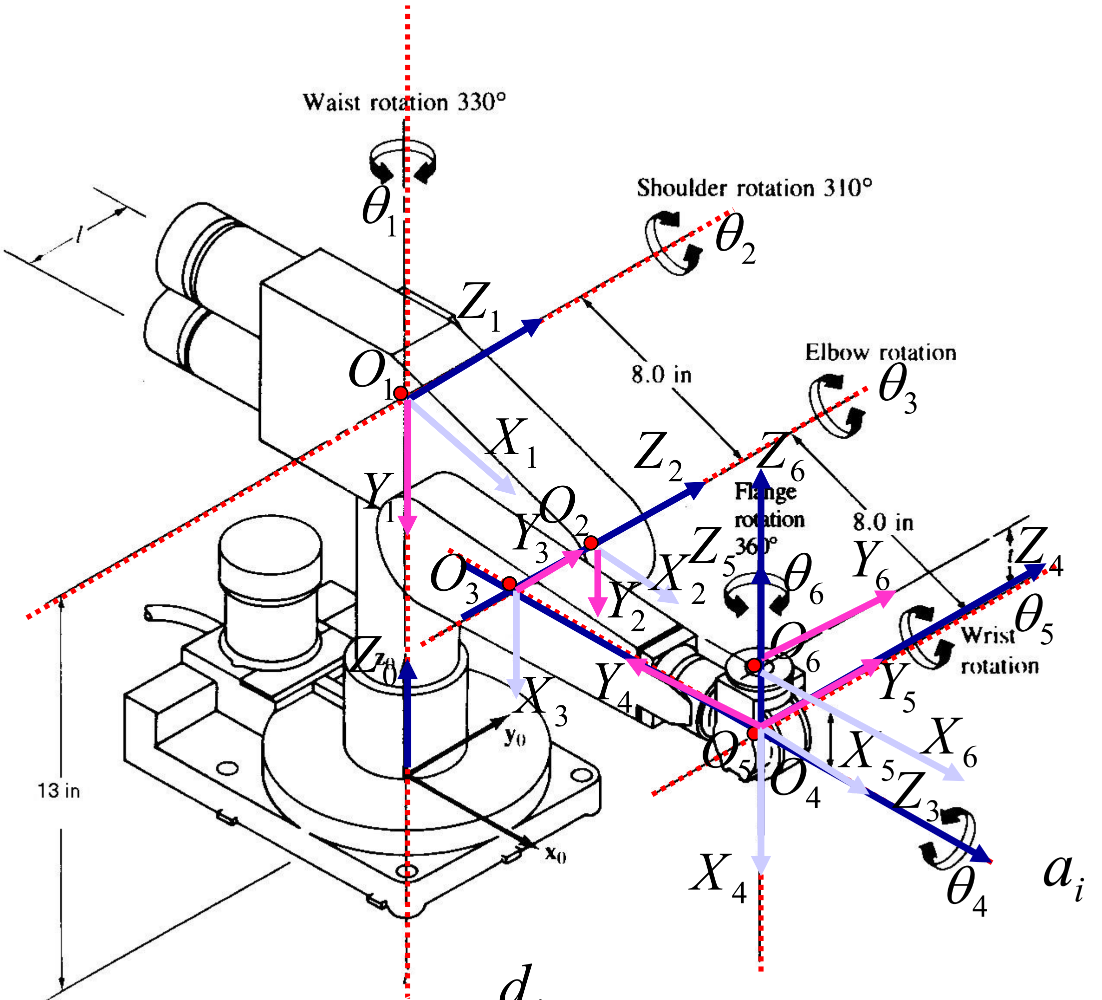
1. Number the joints
2. Establish base frame
3. Establish joint axis Z_i
4. Locate origin, (intersect. of Z_i & Z_{i-1}) OR (intersect of common normal & Z_i)

$$X_i = \pm(Z_{i-1} \times Z_i)/\|Z_{i-1} \times Z_i\|$$

$$Y_i = +(Z_i \times X_i)/\|Z_i \times X_i\|$$

5. Establish X_i,Y_i

Link Parameters



J	θ_i	α_i	a_i	d_i
1	θ_1	-90	0	13
2	θ_2	0	8	0
3	θ_3	90	0	
4	θ_4	-90	0	8
5	θ_5	90	0	0
6	θ_6	0	0	t

θ_i

α_i