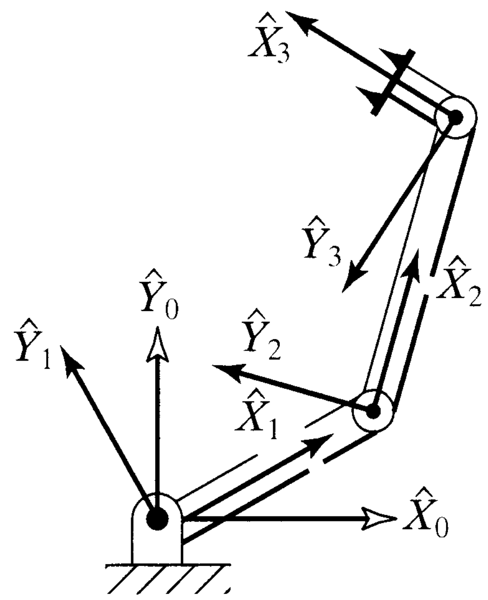


Figure 1: *3R Robot (Problem 1)*

Problem 1

For the 3R manipulator shown in Figure 1, solve the following problems:

a) Compute the forward kinematics, i.e., the position and orientation, of the end effector, for this manipulator. Note that the manipulator has an especially simple configuration, because all rotation axes are parallel. The robot endeffector position can be described by specifying a planar position x, y and the rotation angle Θ_{tip}. The three roboter parameters are denoted by $\Theta_1, \Theta_2, \Theta_3$, the lengths of the robot links are given by l_1, l_2, l_3.

b) Determine the Jacobian of the manipulator.

c) Express $\dot{p}$ as a function of

$$\Theta_1, \Theta_2, \Theta_3, \dot{\Theta}_1, \dot{\Theta}_2, \dot{\Theta}_3$$

d) Determine the singularities of the manipulator.

e) For each singularity, determine which degrees of freedom are lost, and try to give an intuitive explanation for that.

Problem 2

A manipulator may have special configurations, called "isotropic points," that are characterized by the Jacobi matrix having orthogonal columns of equal length, thus $J^T J = \delta I$

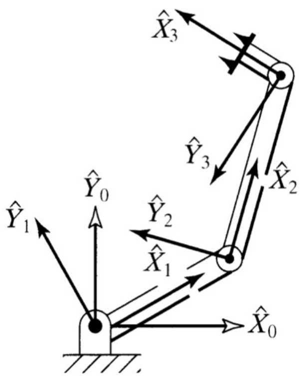


Figure 1: *3R Robot (Problem 1)*

	a_{i-1}	α_{i-1}	d_i	θ_i	Value.
1	0	0	0	θ_1	
2	L_1	0	0	θ_2	
3	L_2	0	0	θ_3	
4	L_3	0	0	$0°$	

$$ {}^0_1T = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_2T = \begin{bmatrix} \underbrace{C_1C_2 - S_1S_2}_{C_{12}} & \underbrace{-C_1S_2 - S_1C_2}_{-S_{12}} & 0 & L_1 \cdot C_1 \\ \underbrace{S_1C_2 + C_1S_2}_{S_{12}} & \underbrace{-S_1S_2 + C_1C_2}_{C_{12}} & 0 & L_1 \cdot S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_2T = \begin{bmatrix} C_2 & -S_2 & 0 & L_1 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_3T = \begin{bmatrix} \underbrace{C_{12}C_3 - S_{12}S_3}_{C_{123}} & \underbrace{-C_{12}S_3 - S_{12}C_3}_{-S_{123}} & 0 & L_1C_1 + L_2C_{12} \\ \underbrace{S_{12}C_3 + C_{12}S_3}_{S_{123}} & \underbrace{-S_{12}S_3 + C_{12}C_3}_{C_{123}} & 0 & L_1S_1 + L_2S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^2_3T = \begin{bmatrix} C_3 & -S_3 & 0 & L_2 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_4T = \begin{bmatrix} C_{123} & -S_{123} & 0 & L_1C_1 + L_2C_{12} + L_3C_{123} \\ S_{123} & C_{123} & 0 & L_1S_1 + L_2S_{12} + L_3S_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^3_4T = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0J = \begin{bmatrix} {}^0J_v \\ {}^0J_w \end{bmatrix} \qquad {}^0\dot{P}_{EE} = {}^0J_v(\Theta) \cdot \dot{\Theta} $$

$$ {}^0P_{EE} = \begin{bmatrix} L_1C_1 + L_2C_{12} + L_3C_{123} \\ L_1S_1 + L_2S_{12} + L_3S_{123} \\ \theta_1 + \theta_2 + \theta_3 \end{bmatrix} \leftarrow \text{部分 } J_w(\theta) $$

$$ {}^0J_v(\theta) = \begin{bmatrix} -L_1S_1 - L_2S_{12} - L_3S_{123} & -L_2S_{12} - L_3S_{123} & -L_3S_{123} \\ L_1C_1 + L_2C_{12} + L_3C_{123} & L_2C_{12} + L_3C_{123} & L_3C_{123} \\ 1 & 1 & 1 \end{bmatrix} $$

X

$$ {}^0J_w(\theta) = \begin{bmatrix} {}^0r_1 & \vdots & {}^0r_2 & \vdots & {}^0r_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} $$

$$ {}^{0}\dot{P}_{EE} = \begin{bmatrix} -l_1 S_1 - l_2 S_{12} - l_3 S_{123} & -l_2 S_{12} - l_3 S_{123} & -l_3 S_{123} \\ l_1 C_1 + l_2 C_{12} + l_3 & l_2 C_{12} + l_3 C_{123} & l_3 C_{123} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} $$

Singularity $\quad \det(J(\theta)) = 0$

$$ \begin{vmatrix} -l_1 S_1 & -l_2 S_{12} & -l_3 S_{123} \\ l_1 C_1 & l_2 C_{12} & l_3 C_{123} \\ 0 & 0 & 1 \end{vmatrix} = 0 $$

$$ -\cancel{l_1} S_1 \cdot \cancel{l_2} C_{12} + \cancel{l_1} C_1 \cdot \cancel{l_2} S_{12} = 0 $$

$$ \cancel{C_1 S_1 \cdot C_2} + C_1 \cdot C_1 S_2 = \cancel{S_1 \cdot C_1 C_2} - S_1 \cdot S_1 S_2 $$

$$ C_1^2 S_2 + S_1^2 S_2 = 0 $$

例

$$ S_2 = 0 $$

$$ S_2 = \begin{cases} 0^\circ \\ 180^\circ \end{cases} $$

计算 DOF 的丢失，雅可比矩阵换算到末端

$$ {}^3J = \begin{pmatrix} l_1 S_{23} + l_2 S_3 & l_2 S_3 & 0 \\ l_1 C_{23} + l_2 C_3 + l_3 & l_2 C_3 + l_3 & l_3 \\ 1 & 1 & 1 \end{pmatrix} $$

Two Singularity

① Working Space boundary Singularit $\Longleftarrow$ row $= 0$

① $\theta_1 = \theta_2 = \theta_3 = 0 / 180$

$$ {}^3J = \begin{pmatrix} 0 & 0 & 0 \\ l_1 + l_2 + l_3 & l_2 + l_3 & l_3 \\ 1 & 1 & 1 \end{pmatrix} $$

$\Longleftarrow$ No motion in X-dim

② $l_3 = 0 \;\wedge\; \theta_3 = \{90^\circ, 270^\circ\} \quad \theta_2 = 0$

$$ {}^3J = \begin{pmatrix} l_1 + l_2 & l_2 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} $$

$\Longleftarrow$ No Motion in Y Axis

② Linear dependent

① Row 1, Row 2 dependent

$l_3 = 0 \qquad \theta_3 = \{45^\circ, 225^\circ\}$

x, y are coupled

② Row 2, Row 3

$l_3 = 1 \qquad \theta_3 = \{90^\circ, 270^\circ\}$

$\underbrace{y, z}_{\text{round } w}$ are coupled

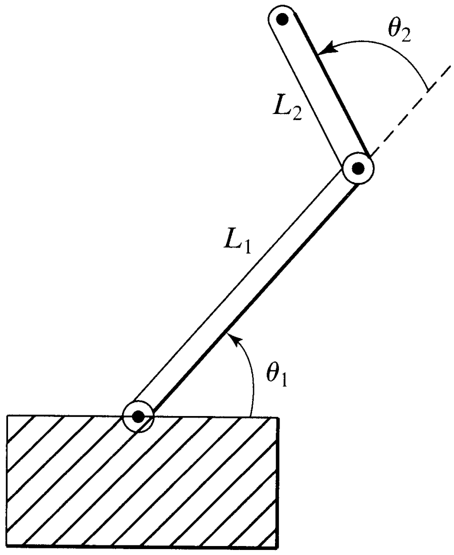


Figure 2: 2*R Robot (Problem* 2*)*

for some $\delta \in \mathbb{R}$. Now consider the 2R manipulator shown in Figure 2. It's Jacobian (with respect to gripper position only, ignoring gripper orientation) looks like this:

$$^{3}J(\Theta) = \begin{pmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{pmatrix}$$

Determine the manipulator's isotropic points. Draw the manipulator in the corresponding configuration(s). Can you give an interpretation of the special role that the isotropic configuration plays?

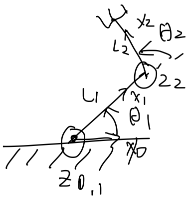
Problem 3

Show that determining singularities of the 2R robot from problem 2 is significantly easier based on the Jacobian relative to frame $\{3\}$ than based on the Jacobian relative to frame $\{0\}$. The Jacobian ^{0}J can be computed using two different methods:

- Direct computation (through differentiation of gripper position).
- Transforming ^{3}J into ^{0}J by exploiting the Jacobian transformation relation.

Perform the computation of ^{0}J using both methods. Determine singularities based on ^{0}J. How is it easier to compute singularities based on ^{3}J?

P3



J	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	L_1	0	0	θ_2
3	L_2	0	0	0

$$^0P_{EE} = \begin{pmatrix} L_1 \cdot C_1 + L_2 \cdot C_{12} \\ L_1 \cdot S_1 + L_2 \cdot S_{12} \\ 0 \\ 0 \\ 0 \\ \theta_1 + \theta_2 \end{pmatrix}$$

$$^0J(\theta) = \begin{pmatrix} \boxed{\begin{matrix} -L_1S_1 - L_2S_{12} & -L_2S_{12} \\ L_1C_1 + L_2C_{12} & L_2C_{12} \end{matrix}} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix} \end{pmatrix}$$

$$^0J(\theta) = \underbrace{{}^0_3R}_{2-1)} \; {}^3J(\theta),$$

$$= \begin{pmatrix} C_{12} & -S_{12} \\ S_{12} & C_{12} \end{pmatrix} \begin{pmatrix} L_1S_2 & 0 \\ L_1C_2 + L_2 & L_2 \end{pmatrix}$$

$$= \begin{pmatrix} \underline{L_1S_2C_{12} - S_{12}L_1 \cdot C_2} - S_{12} \cdot L_2 & -L_2S_{12} \\ \underline{L_1S_2S_{12} + L_1C_{12}C_2} + L_2C_{12} & L_2 \cdot C_{12} \end{pmatrix}$$

$$\det(^0J(\theta)) = \begin{vmatrix} -L_1S_1 & -L_2S_{12} \\ L_1C_1 & L_2C_{12} \end{vmatrix}$$

$$-L_1S_1L_2 \cdot C_{12} + L_1L_2C_1 \cdot S_{12} = 0$$

$$L_1L_2(\cancel{C_1S_1C_2} + C_1C_1S_2 - \cancel{S_1C_1C_2} + S_1S_1S_2) = 0$$

$$L_1L_2 \cdot S_2 = 0$$

$$S_2 = 0 \qquad \theta_2 = \{0, 180°\}$$

P_2

$$^3J^T(\theta) = \begin{pmatrix} l_1 s_2 & l_1 c_2 + l_2 \\ 0 & l_2 \end{pmatrix}$$

$$^3J^T(\theta) \cdot {}^3J(\theta) = \begin{pmatrix} l_1 s_2 & l_1 c_2 + l_2 \\ 0 & l_2 \end{pmatrix} \begin{pmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{pmatrix}$$

$$= \begin{pmatrix} l_1^2 s_2^2 + l_1^2 c_2^2 + 2 l_1 l_2 c_2 + l_2^2 & l_1 l_2 c_2 + l_2^2 \\ l_1 l_2 c_2 + l_2^2 & l_2^2 \end{pmatrix} = \delta I$$

$$l_2^2 = \delta \qquad l_2 = \sqrt{\delta}$$

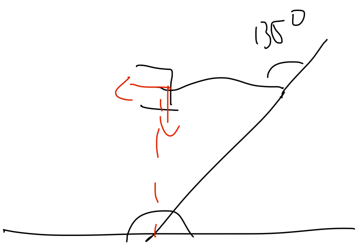
$$l_1 l_2 c_2 = -\delta$$

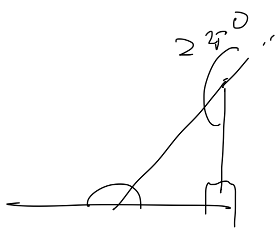
$$l_1^2 - 2\delta + \delta = \delta$$

$$l_1 = \sqrt{2\delta}$$

$$\sqrt{2}\,\delta c_2 + \cancel{\delta} = 0$$

$$c_2 = -\frac{\sqrt{2}}{2} \qquad \theta_2 = \{135^\circ, 225^\circ\}$$





V_1 map x_1-dim $\qquad$ V_2 map y dim