Fundamentals of Artificial Intelligence
Exercise 9: Hidden Markov Models

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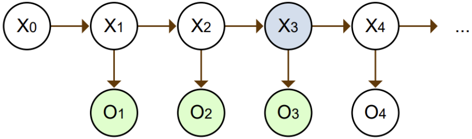
Recap Hidden Markov Models

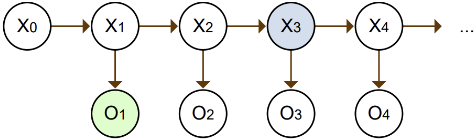
- Discrete underlying real-world states X_t that changes over time t
- Observations E_t at each timestep that can be used to guess at the state with some level of confidence
- Idea is to use the observation E at time t to guess at the state X_t, but then refine our guess using the observations before and after t

Recap Inference Tasks

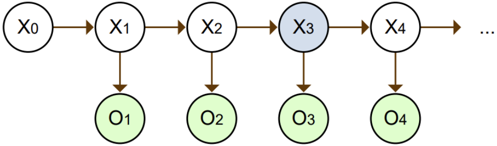
Filtering $P(X_t|o_{1:t})$



Prediction $P(X_k|o_{1:t}), k > t$



Smoothing $P(X_k|o_{1:t}), 0 \leq k < t$



Problem 9.1

$ES: \{es, \neg es\}$

A professor wants to know if students are getting enough sleep. Each day, the professor observes whether the students sleep in class, and whether they have red eyes. The professor has the following domain theory:

$SC\ \{sc, \neg sc\}$ $RE\ \{re, \neg re\}$

- The prior probability of getting enough sleep, with no observations, is 0.7. $p(es) = 0.7$
- The probability of getting enough sleep on night t is 0.8 given that the student got enough sleep the previous night, and 0.3 if not.
  $p(es_t \mid es_{t-1}) = 0.8$ / $p(es_t \mid \neg es_{t-1}) = 0.3$
- The probability of having red eyes is 0.2 if the student got enough sleep, and 0.7 if not.
  $p(re \mid es) = 0.2$ $p(re \mid \neg es) = 0.7$
- The probability of sleeping in class is 0.1 if the student got enough sleep, and 0.3 if not.
  $p(sc \mid es) = 0.1$ $p(sc \mid \neg es) = 0.3$

a. Formulate this information as an Hidden Markov Model that has only one observation variable. Give the complete probability tables for the model.

$P(es) = 0.7$, and $P(\neg es) = 0.3$;
$P(es_t | es_{t-1}) = 0.8$, and $P(es_t | \neg es_{t-1}) = 0.3$;
$P(re | es) = 0.2$, and $P(re | \neg es) = 0.7$;
$P(sc | es) = 0.1$, and $P(sc | \neg es) = 0.3$.

es_{t-1}	$p(es_t \mid es_{t-1})$
t	0.8
f	0.3

$$
\begin{tikzcd}
ES_0 \arrow[r] & \cdots & ES_{t-1} \arrow[r] & ES_t \arrow[d] \arrow[dr] & \\
 & & & RE_t & SC_t
\end{tikzcd}
$$

	$p(es_0)$
t	0.7
f	0.3

$E_t = RE_t \wedge SC_t$

$\in \{re_t \wedge sc_t,\ re_t \wedge \neg sc_t,\ \neg re_t \wedge sc_t,\ \neg re_t \wedge \neg sc_t\}$

$P(es) = 0.7$, and $P(\neg es) = 0.3$;
$P(es_t | es_{t-1}) = 0.8$, and $P(es_t | \neg es_{t-1}) = 0.3$;
$P(re | es) = 0.2$, and $P(re | \neg es) = 0.7$;
$P(sc | es) = 0.1$, and $P(sc | \neg es) = 0.3$.

$$p(RE_t \wedge SC_t | ES_t) = P(RE_t | SC_t, ES_t) \cdot p(SC_t | ES_t)$$
$$= P(RE_t | ES_t) \cdot p(SC_t | ES_t)$$

ES_t	$P(E_t = RE_t \wedge SC_t \mid ES_t)$	$\neg re_t \wedge sc_t$	$re_t \wedge \neg sc_t$	$\neg re_t \wedge \neg sc_t$
t	0.02	0.08	0.18	0.72
f	0.21	0.09	0.49	0.21

Problem 9.1 b

Consider the following evidence values:

- $\mathbf{e}_1$ = not red eyes, not sleeping in class;
- $\mathbf{e}_2$ = red eyes, not sleeping in class;
- $\mathbf{e}_3$ = red eyes, sleeping in class.

$p(\neg re \wedge \neg sc)$

$p(re \wedge \neg sc)$

$p(re \wedge sc)$

b. State estimation: Compute $P(EnoughSleep_t | \mathbf{e}_{1:t})$ for each $t = 1, 2, 3$.

$$\underbrace{p(ES_t | e_{1:t})}_{f_{1:t}} = p(ES_t | e_{1:t-1}, e_t)$$

$$= \alpha\, p(e_t | ES_t, \cancel{e_{1:t-1}}) \cdot p(ES_t | e_{1:t-1})$$

$$= \alpha\, \underbrace{p(e_t | ES_t)}_{O_t} \cdot \sum_{es_{t-1}} p(ES_t, es_{t-1} | e_{1:t-1})$$

$$= \underbrace{p(ES_t | es_{t-1}, \cancel{e_{1:t-1}})}_{T} \cdot \underbrace{p(es_{t-1} | e_{1:t-1})}_{f_{1:t-1}}$$

$$f_{1:t} = \alpha\, O_t T \cdot f_{1:t-1}$$

$\mathbf{e}_1 = \neg re \wedge \neg sc,$
$\mathbf{e}_2 = re \wedge \neg sc,$
$\mathbf{e}_3 = re \wedge sc$

ES_{t-1}	$P(es_t \mid ES_{t-1})$
true	0.8
false	0.3

ES_t	$P(re \wedge sc \mid ES_t)$	$P(\neg re \wedge sc \mid ES_t)$	$P(re \wedge \neg sc \mid ES_t)$	$P(\neg re \wedge \neg sc \mid ES_t)$
true	0.02	0.08	0.18	0.72
false	0.21	0.09	0.49	0.21

$$T_{ij} = P(ES_t = x_i \mid ES_{t-1} = x_j) = \begin{bmatrix} p(es_t \mid es_{t-1}) & p(es_t \mid \neg es_{t-1}) \\ p(\neg es_t \mid es_{t-1}) & p(\neg es_t \mid \neg es_{t-1}) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

$$(O_{ij})_T = \begin{cases} p(E_t \mid ES_t = x_i), & j = i \\ 0 \end{cases}$$

$$O_1 = \begin{bmatrix} 0.72 & 0 \\ 0 & 0.21 \end{bmatrix}$$

$$O_2 = \begin{bmatrix} 0.18 & 0 \\ 0 & 0.49 \end{bmatrix}$$

$$O_3 = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.21 \end{bmatrix}$$

$$\mathbf{f}_{1:t} = \alpha \mathbf{O}_t \mathbf{T} \mathbf{f}_{1:t-1}, \quad \mathbf{T} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}, \quad \mathbf{O}_1 = \begin{bmatrix} 0.72 & 0 \\ 0 & 0.21 \end{bmatrix}, \quad \mathbf{O}_2 = \begin{bmatrix} 0.18 & 0 \\ 0 & 0.49 \end{bmatrix}, \quad \mathbf{O}_3 = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.21 \end{bmatrix}$$

$$f_0 = \begin{bmatrix} P(es_0) \\ P(\neg es_0) \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$$

$$\frac{1}{0.468+0.0735} = \frac{2000}{1083}$$

$$f_{1:1} = \alpha O_1 \cdot T \cdot f_0 = \alpha \begin{bmatrix} 0.72 & 0 \\ 0 & 0.21 \end{bmatrix} \cdot \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \cdot \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = \alpha \begin{bmatrix} 0.468 \\ 0.0735 \end{bmatrix} = \begin{bmatrix} 0.8643 \\ 0.1357 \end{bmatrix}$$

$$f_{1:2} = \alpha O_2 \cdot T \cdot f_{0:1} = \alpha \begin{bmatrix} 0.18 & 0 \\ 0 & 0.49 \end{bmatrix} \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.8643 \\ 0.1357 \end{bmatrix} = \alpha \begin{bmatrix} 0.1318 \\ 0.1312 \end{bmatrix} = \begin{bmatrix} 0.5010 \\ 0.4990 \end{bmatrix}$$

$$f_{1:3} = \alpha O_3 T f_{0:2} = \alpha \begin{bmatrix} 0.02 & 0 \\ 0 & 0.21 \end{bmatrix} \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.5010 \\ 0.4990 \end{bmatrix} = \alpha \begin{bmatrix} 0.0110 \\ 0.0944 \end{bmatrix} = \begin{bmatrix} 0.1045 \\ 0.8955 \end{bmatrix}$$

Problem 9.1 c

c. Smoothing: Compute $P\left(EnoughSleep_t | \mathbf{e}_{1:3}\right)$ for each $t = 1, 2, 3$.

$$P(ES_t | e_{1:k}) = P(ES_t | e_{1:t}, e_{t+1:k})$$

$$= \alpha P(e_{t+1:k} | ES_t, \cancel{e_{1:t}}) \cdot P(ES_t | e_{1:t}) \cdot \cancel{P(e_{1:t})}$$

$$= \alpha \cdot \underbrace{P(ES_t | e_{1:t})}_{f_{1:t}} \cdot \underbrace{P(e_{t+1:k} | ES_t)}_{b_{t+1:k}}$$

$$= \alpha \cdot f_{1:t} \cdot b_{t+1:k}$$

$$b_{t:k} = P(e_{t:k} | ES_{t-1})$$

$$= \sum_{ES_t} P(e_{t:k}, ES_t | ES_{t-1})$$

$$= \sum_{ES_t} P(e_{t:k} | ES_{t-1}, ES_t)$$

$$= \sum_{ES_t} p(e_{t:k} | ES_t)\, p(ES_t | ES_{t-1})$$

$$= \sum_{ES_t} \underbrace{p(e_t | ES_t)}_{O_t} \cdot \underbrace{p(e_{t+1:k} | ES_t)}_{b_{t+1:k}} \underbrace{p(ES_t | ES_{t-1})}_{T}$$

$$\Rightarrow b_{t:k} = T^T \cdot O_t \cdot b_{t+1:k}$$

$\mathbf{b}_{t:r} = \mathbf{T}^\top \mathbf{O}_t \mathbf{b}_{t+1:r}$, $\mathbf{T} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$, $\mathbf{O}_1 = \begin{bmatrix} 0.72 & 0 \\ 0 & 0.21 \end{bmatrix}$, $\mathbf{O}_2 = \begin{bmatrix} 0.18 & 0 \\ 0 & 0.49 \end{bmatrix}$, $\mathbf{O}_3 = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.21 \end{bmatrix}$

$$b_{4:3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$b_{3:3} = T^T \cdot O_3\, b_{4:3} = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \cdot \begin{bmatrix} 0.02 & 0 \\ 0 & 0.21 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.058 \\ 0.153 \end{bmatrix}$$

$$b_{2:3} = T^T\, O_2\, b_{3:3} = \underline{\hspace{3em}} \begin{bmatrix} 0.18 & 0 \\ 0 & 0.49 \end{bmatrix} \begin{bmatrix} 0.058 \\ 0.153 \end{bmatrix} = \begin{bmatrix} 0.0233 \\ 0.0556 \end{bmatrix}$$

$$\Rightarrow \alpha \cdot f_{1:t} \times b_{t+1:k} \Rightarrow t=1 \quad \alpha \cdot f_{1:1} \times b_{2:3} = \alpha \begin{bmatrix} 0.8643 \\ 0.1357 \end{bmatrix} \times \begin{bmatrix} 0.0233 \\ 0.0556 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.0201 \\ 0.0075 \end{bmatrix} = \begin{bmatrix} 0.7283 \\ 0.2717 \end{bmatrix}$$

$$\Rightarrow t=2 \quad \alpha\, f_{1:2} \times b_{3:3} = \ldots \begin{bmatrix} 0.2757 \\ 0.7243 \end{bmatrix}$$

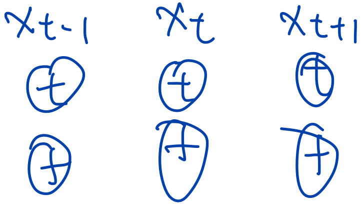
Problem 9.1 c

Problem 9.1 d

d. Find the most likely state sequence.

Viterbi's Algorithm

$$\max_{x_1 \dots x_t} \mathbf{P}\left(x_1, \dots x_t, X_{t+1} | \mathbf{e}_{1:t+1}\right) = \alpha \mathbf{P}\left(e_{t+1} | X_{t+1}\right) \max_{x_t} \left(\mathbf{P}\left(X_{t+1} | x_t\right) \max_{x_1 \dots x_{t-1}} P\left(x_1, \dots x_t | \mathbf{e}_{1:t}\right) \right).$$



Viterbi's Algorithm

$$\max_{x_1\dots x_t} \mathbf{P}\left(x_1,\dots x_t, X_{t+1}|\mathbf{e}_{1:t+1}\right) = \alpha \mathbf{P}\left(e_{t+1}|X_{t+1}\right) \max_{x_t}\left(\mathbf{P}\left(X_{t+1}|x_t\right) \underbrace{\max_{x_1\dots x_{t-1}} P\left(x_1,\dots x_t|\mathbf{e}_{1:t}\right)}_{\mu_1(ES_1)}\right).$$

$$\mu_1(ES_1) = p(ES_1|e_{1:1}) = f_{1:1} = \begin{bmatrix}0.8643\\0.1357\end{bmatrix}$$

$$T = \begin{bmatrix}0.8 & 0.3\\0.2 & 0.7\end{bmatrix}$$

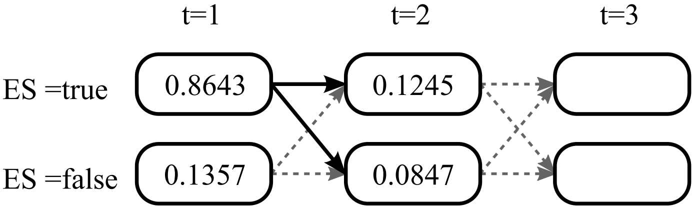
Step 1:

$$\mu_2\left(ES_2\right) = \mathbf{P}\left(e_2|ES_2\right) \max_{ES_1} \mathbf{P}\left(ES_2|ES_1\right)\mu_1\left(ES_1\right).$$

$$= \underbrace{p(e_2|ES_2)}_{O_2} \max\left\{\underbrace{p(ES_2|es_1)}_{T[i\ x]}\underbrace{\mu_1(ES_1)}_{0.8643}, \underbrace{p(ES_2|\neg es_1)}_{T[x\ i]}\underbrace{\mu_1(\neg ES_1)}_{0.1357}\right\}$$

$$= \begin{bmatrix}0.18\\0.49\end{bmatrix} \times \max\left\{\begin{bmatrix}0.6914\\0.1729\end{bmatrix}, \begin{bmatrix}0.1221\\0.0950\end{bmatrix}\right\}$$

$$= \begin{bmatrix}0.1245\\0.0847\end{bmatrix}$$


t=1
t=2
t=3
ES =true
ES =false
0.8643
0.1245
0.1357
0.0847

Viterbi's Algorithm

$$\max_{x_1 \ldots x_t} \mathbf{P}(x_1, \ldots x_t, X_{t+1} | \mathbf{e}_{1:t+1}) = \alpha \mathbf{P}(e_{t+1} | X_{t+1}) \max_{x_t} \Big(\mathbf{P}(X_{t+1} | x_t) \max_{x_1 \ldots x_{t-1}} P(x_1, \ldots x_t | \mathbf{e}_{1:t}) \Big).$$

$T = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$

$$\mu_2(ES_2) = \begin{bmatrix} 0.1245 \\ 0.0847 \end{bmatrix}$$

$T[:, x]$ $\quad$ $T[x, :]$

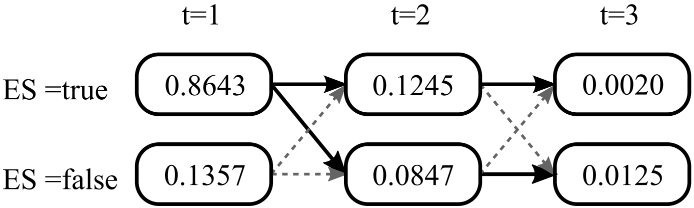
Step 2: O_3

$$\mu_3(ES_3) = \underline{\mathbf{P}(e_3 | ES_3)} \max_{ES_2} \mathbf{P}(ES_3 | ES_2) \, \mu_2(ES_2).$$

$\{ P(ES_3 | ES_2) \, \mu_2(ES_2), \; \underline{P(ES_3 | \neg ES_2)} \cdot \mu_2(\neg ES_2) \}$

$= \begin{bmatrix} 0.02 \\ 0.21 \end{bmatrix} \max \left\{ \begin{bmatrix} 0.0996 \\ 0.0249 \end{bmatrix}, \begin{bmatrix} 0.0254 \\ 0.0593 \end{bmatrix} \right\}$

$= \begin{bmatrix} 0.0020 \\ 0.0125 \end{bmatrix}$


t=1
t=2
t=3
ES =true
ES =false
0.8643
0.1245
0.0020
0.1357
0.0847
0.0125