Problem 2 **Kinematics (37 credits)**

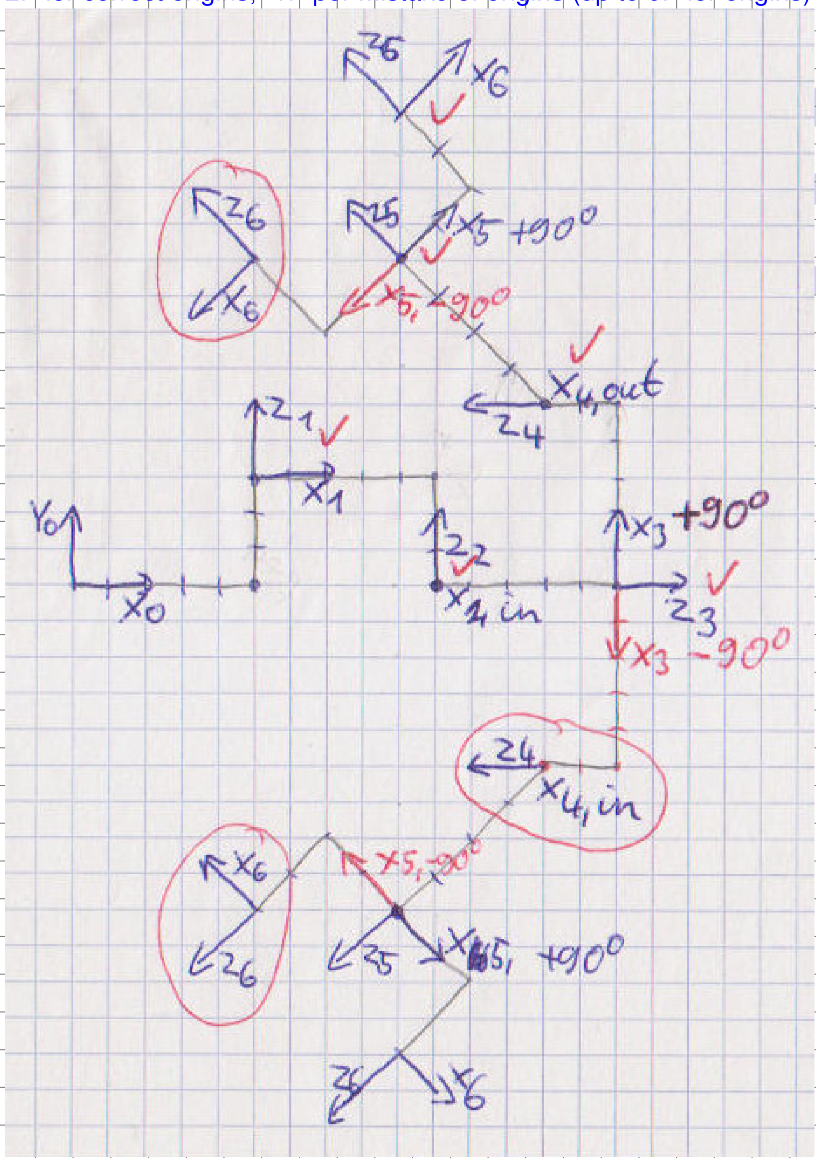
a)*

Joint	a_{i-1}	α_i	d_i	Θ_i
1	5	$-90°$	3	$0°$
2	5	$0°$	-3	$90°$
3	0	$90°$	5	
4	5	$180°$	2	$-90°$
5	0	$-45°$	$4\sqrt{2}$	
6	$2\sqrt{2}$	$0°$	$2\sqrt{2}$	$0°$

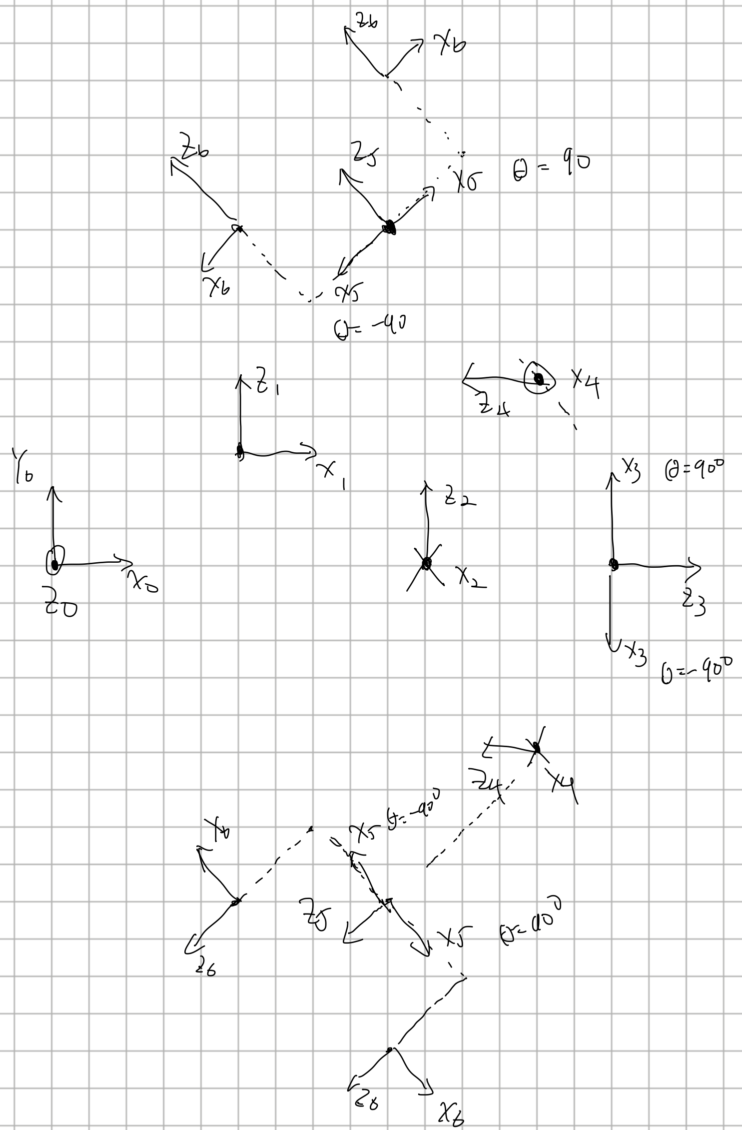
The table on the left is a description of a robot according to Denavit-Hartenberg-Parameters. Draw all coordinate systems into the grid below. One cell in the grid correspond to one length unit in DH. Choose the values of Θ_3 and Θ_5 such that all coordinate system origins lie in the drawing plane. You need only draw the x and z axes of the coordinate systems, possibly pointing in or out.

Write down your choices of Θ_3 and Θ_5. Start with the origin on the left in the middle with Y_0 pointing up and X_0 to the right.

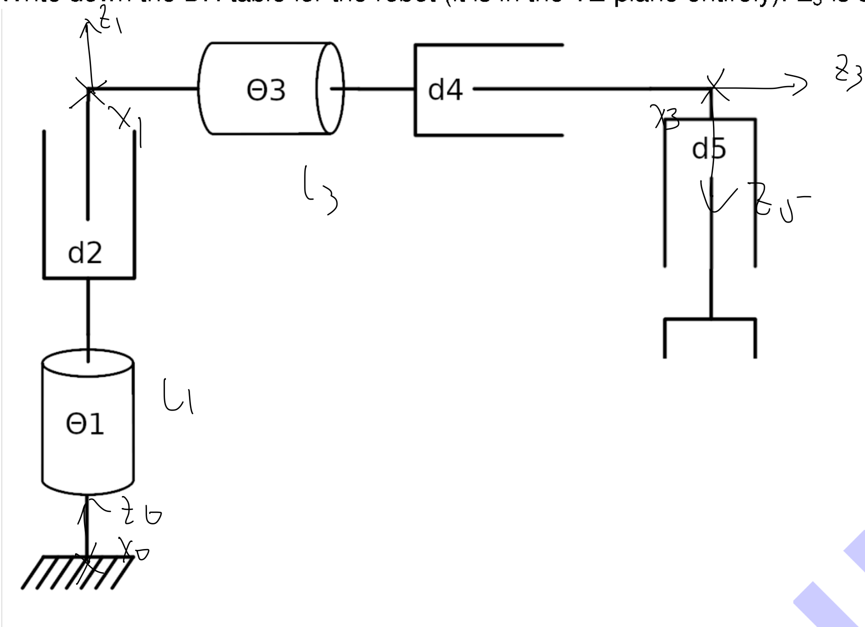
$\Theta_3 = \pm 90°$ and $\Theta_5 = \pm 90°$.
1P per correct orientation of frames.
2P for correct origins, -1P per mistake of origins (up to 0P for origins)



Joint	a_{i-1}	α_i	d_i	Θ_i	
1	5	-90°	3	0°	
2	5	0°	-3	90°	
3	0	90°	5		-90° 90°
4	5	180°	2	-90°	
5	0	-45°	$4\sqrt{2}$		-90° 90°
6	$2\sqrt{2}$	0°	$2\sqrt{2}$	0°	



b) Given the RPRPP pick and place robot in the figure below. The robot is in zero configuration. Assume Z_0 up, X_0 in, origin at the bottom. Unknown lengths are l_1, l_3 for the rotational joints. What is special about joint pairs 1,2 and 3,4? Use that to get less coordinate frames/shorter DH table. Write down the DH table for the robot (it is in the YZ-plane entirely). Z_5 is down.



Joints 1,2 and 3,4 have coaxial Z-axes(or similar explanation), thus they can be merged into one coordinate system per pair.

short version, +2P

Joint	a_{i-1}	α_i	d_i	Θ_i
1	0	0°	$l_1 + d2$	Θ_1
2	0	90°	$l_3 + d4$	Θ_3
3	0	90°	$d5$	0°

long version, only 5P for full DH table

Joint	a_{i-1}	α_i	d_i	Θ_i
1	0	0°	l_1	Θ_1
2	0	0°	d_2	0°
3	0	90°	l_3	Θ_3
4	0	0°	d_4	0°
5	0	90°	d_5	0°

These two joint can be expressed as one joint

They have co-axis z-axis

margeed into one co-ordinate

	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0	$L_1 + d_2$	0 θ_1
3	0	90°	$L_3 + d_4$	0 θ_3
5	0	90°	d_5	0

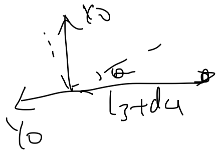
c) Given that $\Theta_3 = 0°$ is **fixed** for the robot from the previous problem as a **simplified** robot.
What kind of workspace does it reach? Imagine joints 1,2,4,5 moving and describe the resulting shape shortly.
Which joints influence which dimension in the base frame? (e.g. joint 1 is X_0 direction, joint 2 is radius l_2 etc)
Which joints determine orientation of the end effector?
Write down the Kinematics (position only) of the end effector for the **simplified** robot.

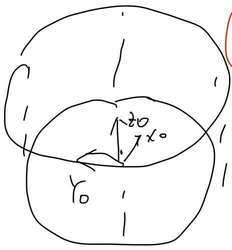
0 1 2 3 4 5 6 7 8 9

The workspace of the robot is a cylindrical shape.
Joint 1 determines the angle in the XY-plane.
Joint 2 and Joint 5 determine the height in Z_0.(only one point if said separately)
(Joint 3 is fixed as defined in the problem.)
Joint 4 determines the radius of the cylinder $l_3 + d_4$.

Only joint 1 determines the orientation of the end effector.

$$^0p_0(\Theta) = \begin{pmatrix} s_1 * (l_3 + d_4) \\ -c_1 * (l_3 + d_4) \\ l_1 + d_2 - d_5 \end{pmatrix}$$





- $\boxed{\text{Cylindrical shape}}$

- Joint 1 is for $l_3 + d_4$ radius   determines Rotation in X-Y plane

Joint 2 is for Z_0

Joint 4 is ~~for Y_0~~ determine the radius of cylinder l_3+d_4

Joint 5 is for Z_0.

- Only Joint 1

$$P_{EE} = \begin{pmatrix} s_1 \cdot (l_3 + d_4) \\ -c_1 (l_3 + d_4) \\ l_1 + d_2 - d_5 \end{pmatrix}$$

d) Derive the Jacobian for the position for above (simplified) robot. Determine singularities by reasoning.
There is no regular determinant to calculate for a $m \times n$ matrix.

0 1 2 3 4 5

$$^0J(\Theta) = \begin{pmatrix} c_1 * (l_3 + d_4) & 0 & 0 & s_1 & 0 \\ s_1 * (l_3 + d_4) & 0 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

Joint 5 is directly opposing joint 2, so they are in constant singularity.

Note: No credit for joint 3, as it is optional (it is being treated as just a piece with length l_3 now.
It may be missing in the Jacobian entirely.

[illegible]似你们考虑 Θ_3

$$^0J_v = {}^0\dot{P}_{EE} = \begin{pmatrix} c_1(l_3+d_4) & 0 & 0 & s_1 & 0 \\ s_1(l_3+d_4) & 0 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

Column 2 and 5 · dependent

0
1
2
3
4

e)* A fellow student drew link frames for a robot but forgot the base frame X_0,Z_0.
"You may put an arbitrary frame in as we can rotate it to the first link anyway".
Is that true? Refer to/add the relevant formula and point out and explain why/why not.

$$
{}^{i-1}_{i}T = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_{i-1} \\ \sin\theta_i \cos\alpha_{i-1} & \cos\theta_i \cos\alpha_{i-1} & -\sin\alpha_{i-1} & -\sin\alpha_{i-1} d_i \\ \sin\theta_i \sin\alpha_{i-1} & \cos\theta_i \sin\alpha_{i-1} & \cos\alpha_{i-1} & \cos\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Mention/add above formula: Transformation between consecutive frames with DH-convention
No, arbitrary movement is not possible
Entry in row 1, column 3 is zero
No rotation around Y-Axis is possibleOR only rotation around X and Z is possible.

DH-Rules only use 4 parameters to descripe the joint space
So arbeirary frame may not rotate to the first link.

0
1

f)* Additional space for **P1 only** if you need it. Refer to here if you used it!

Problem 3 Lagrange (37 credits)

The motion $(\Theta_1(t), \Theta_2(t))$ of a double-pendulum depicted in Fig. 3.1 is supposed to be analysed using a **Lagrangian method** discussed in the lecture. The two rotational joints are at the origin and at m_1. The links (l_1, l_2) have no mass or inertia and the entire mass can be considered as point masses at (m_1, m_2). The direction of Earth acceleration g is shown in the figure.

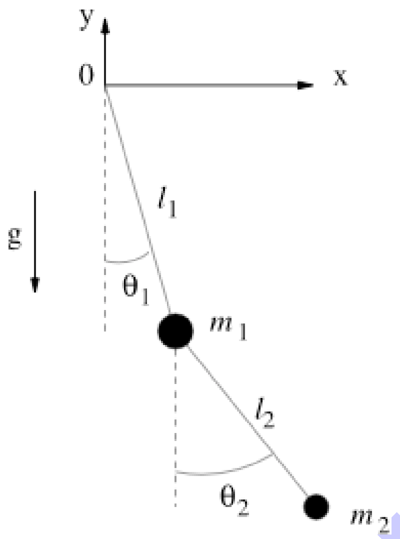


Figure 3.1: Double-pendulum

a)* Calculate the positions (x_i, y_i) and velocities $(\dot{x}_i . \dot{y}_i)$ of (m_1, m_2).

0 1 2 3 4

$$x_1 = l_1 \sin(\theta_1) \qquad \dot{x}_1 = l_1\dot{\theta}_1 \cos(\theta_1)$$
$$y_1 = -l_1 \cos(\theta_1) \qquad \dot{y}_1 = l_1\dot{\theta}_1 \sin(\theta_1)$$
$$x_2 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2) \qquad \dot{x}_2 = l_1\dot{\theta}_1 \cos(\theta_1) + l_2\dot{\theta}_2 \cos(\theta_2)$$
$$y_2 = -l_1 \cos(\theta_1) - l_2 \cos(\theta_2), \qquad \dot{y}_2 = l_1\dot{\theta}_1 \sin(\theta_1) + l_2\dot{\theta}_2 \sin(\theta_2).$$

$I_1 = I_2 = 0$

$x_1 = l_1 S_1 \qquad \dot{x}_1 = l_1 C_1 \cdot \dot{\theta}_1$

$y_1 = -l_1 C_1 \qquad \dot{y}_1 = l_1 S_1 \cdot \dot{\theta}_1$

$x_2 = l_1 S_1 + l_2 S_2 \qquad \dot{x}_2 = l_1 C_1 \cdot \dot{\theta}_1 + l_2 C_2 \cdot \dot{\theta}_2$

$y_2 = -l_1 C_1 - l_2 C_2 \qquad \dot{y}_2 = l_1 S_1 \cdot \dot{\theta}_1 + l_2 S_2 \cdot \dot{\theta}_2$

0 1 2 3 4 5 6 7 8

b) Calculate the potential energy U and kinetic energy K for the entire double-pendulum. Use values estimated in a).

$$T = \frac{1}{2}\Big[m_1 l_1^2\dot{\theta}_1^2 + m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\left\{\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right\}\right)\Big]$$

$$= \frac{1}{2}\Big[m_1 l_1^2\dot{\theta}_1^2 + m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right)\Big]$$

$$U = -(m_1+m_2)gl_1\cos(\theta_1) - m_2gl_2\cos(\theta_2),$$

$K = \Sigma$

$k_i = \frac{1}{2} m_i V_{c_i}^T V_{c_i} + \frac{1}{2} {}^i w_i \, {}^{c_i}I_i \, {}^i w_i$

$k_1 = \frac{1}{2} m_1 \cdot l_1^2 \dot{\theta}_1^2$

$k_2 = \frac{1}{2} m_2 \cdot (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 c_1 c_2 \dot{\theta}_1 \dot{\theta}_2 + 2 l_1 l_2 s_1 s_2 \dot{\theta}_1 \dot{\theta}_2)$

$= \frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 c_{1-2})$

$U_i = -m_i \, {}^0g^T \, {}^0P_{c_i} + u_{ref_i}$

$u_1 = -m_1 \cdot (0 \; -g \; 0) \begin{pmatrix} L_1 S_1 \\ -L_1 C_1 \\ 0 \end{pmatrix} = -m_1 \cdot g \cdot L_1 \cdot c_1 + m_1 g L_1$

$U_2 = -m_2 (0 \; -g \; 0) \begin{pmatrix} \cancel{x} \\ -L_1C_1 - l_2 C_2 \\ 0 \end{pmatrix} + u_{ref_i}$

$= -m_2 g (L_1 C_1 + l_2 C_2) + m_2 g (L_1 + L_2)$

$U = -(m_1 g \cdot L_1 + m_2 g L_1) \cdot C_1 - m_2 g l_2 C_2 + u_{ref_i}$

0 1 2 3 4

c) Calculate the expression for the Lagrangian L from the above values. Simpify the sine and cosine products using trigonometric identities.

$$L = T - U$$

$$= \frac{1}{2}\Big[m_1 l_1^2\dot{\theta}_1^2 + m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right)\Big] + (m_1+m_2)gl_1\cos(\theta_1) + m_2gl_2\cos(\theta_2).$$

$L = K - P$

$= \frac{1}{2} m_1 \cdot L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (L_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 L_1 l_2 \cdot \dot{\theta}_1 \dot{\theta}_2 C_{1-2})$

$+ (m_1 g L_1 + m_2 g L_1) C_1 + m_2 g L_2 C_2 - m_1 g L_1 - m_2 g (L_1 + l_2)$

d)* Calculate the necessary derivatives to estimate the torques in the rotational joints (τ_1, τ_2) required for the Lagrangian method. In case that you were not able to calculate the Lagrangian, assume a non-trivial trigonometric expression for L for partial points.

0 1 2 3 4 5 6 7 8 9

$$\frac{\partial L}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 \sin(\theta_1)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \left[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)\right]$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin(\theta_2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = m_2 \left(l_2^2 \ddot{\theta}_2 + l_1 l_2 \left[\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)\right]\right).$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 L_1^2 \dot{\theta}_1 + m_2 L_1^2 \dot{\theta}_1 + 2 L_1 L_2 m \dot{\theta}_2 C_{1-2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = m_1 L_1^2 \ddot{\theta}_1 + m_2 L_1^2 \ddot{\theta}_1 + L_1 L_2 \cdot m_2 \left[\ddot{\theta}_2 C_{12} - \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) S_{1-2}\right]$$

$$\frac{\partial L}{\partial \theta_1} = -L_1 L_2 m_2 \dot{\theta}_1 \dot{\theta}_2 S_{1-2} - (m_1 g L_1 + m_2 g L_1) S_1$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 L_2^2 \dot{\theta}_2 + m_2 L_1 L_2 \dot{\theta}_1 C_{12} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \left[\ddot{\theta}_1 C_{1-2} - \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) S_{1-2}\right]$$

e) Write the expressions for (τ_1, τ_2) from above calculations giving also the symbolic equation, which terms from above are combined.

0 1 2 3 4 5 6

$$\frac{\partial L}{\partial \theta_2} = -m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 S_{1-2} \; (-1)$$
$$- m_2 g L_2 S_2$$

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \tau_j$$

$$(m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \left[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)\right] + (m_1 + m_2) g l_1 \sin(\theta_1) = \tau_1$$

$$m_2 \left(l_2^2 \ddot{\theta}_2 + l_1 l_2 \left[\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)\right]\right) + m_2 g l_2 \sin(\theta_2) = \tau_2$$

$$\tau_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = (m_1 + m_2) L_1^2 \ddot{\theta}_1 + L_1 L_2 m_2 \left[\ddot{\theta}_2 C_{12} - \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) S_{1-2} + \dot{\theta}_1 \dot{\theta}_2 S_{1-2}\right] + (m_1 + m_2) g L_1 S_1$$

$$\tau_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \left[\ddot{\theta}_1 C_{1-2} - \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) S_{1-2}\right]$$

$$- \left[m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 S_{1-2} - m_2 g L_2 S_2 \right)$$

在自由摆动摆的情况下，两个力矩（τ 1, τ 2）都等于零。写出与双摆的角参数（角度、角速度、角加速度）有关的相应的两个方程。确定用于设计稳定控制的M、V、G表达式。

0 1 2 3 4 5 6

f) In case of free swinging pendulum, both torques (τ_1, τ_2) are equal to zero. Write the corresponding two equations relating angular parameters of the double pendulum (angle, angular velocities, angular accelerations. Identify the M,V,G expressions used to design a stable control.

$$\ddot{\theta}_1 + \frac{m_2}{m_1 + m_2}\frac{l_2}{l_1}\left(\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2{}^2 \sin(\theta_1 - \theta_2)\right) + \frac{g}{l_1}\sin(\theta_1) = 0$$

$$\ddot{\theta}_2 + \frac{l_1}{l_2}\left(\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1{}^2 \sin(\theta_1 - \theta_2)\right) + \frac{g}{l_2}\sin(\theta_2) = 0.$$

$$M = \begin{pmatrix} (m_1 + m_2)l_1^2 & m_2 l_1 l_2 \cos(\Theta_1 - \Theta_2) \\ m_2 l_1 l_2 \cos(\Theta_1 - \Theta_2) & m_1 l_2^2 \end{pmatrix} \quad V = \begin{pmatrix} m_2 l_1 l_2 \sin(\Theta_1 - \Theta_2)\dot{\Theta}_2^2 \\ -l_1 l_2 m_2 \sin(\Theta_1 - \Theta_2)\dot{\Theta}_1^2 \end{pmatrix}$$

$$G = \begin{pmatrix} (m_1 + m_2) g l_1 \sin\Theta_1 \\ m_2 g l_2 \sin\Theta_1 \end{pmatrix}$$

2

$$\tau_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = (m_1 + m_2) L_1^2 \ddot{\theta}_1 + L_1 L_2 m_2 \left[\ddot{\theta}_2 C_{12} - \dot{\theta}_2(\dot{\theta}_1 - \dot{\theta}_2) S_{1-2} + \dot{\theta}_1 \dot{\theta}_2 S_{1-2}\right] + (m_1 + m_2) g L_1 S_1$$

$$\tau_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \left[\ddot{\theta}_1 C_{1-2} - \dot{\theta}_1(\dot{\theta}_1 - \dot{\theta}_2) S_{1-2} - \dot{\theta}_1 \dot{\theta}_2 S_{1-2}\right] + m_2 g L_2 S_2$$

$$M(\theta) = \begin{bmatrix} (m_1 + m_2) L_1^2 & L_1 L_2 m_2 C_{1-2} \\ m_2 L_1 L_2 C_{1-2} & m_2 L_2^2 \end{bmatrix} \qquad V(\theta, \dot{\theta}) = \begin{bmatrix} L_1 L_2 m_2 \dot{\theta}_2^2 S_{1-2} \\ -L_1 L_2 m_2 \dot{\theta}_1^2 S_{1-2} \end{bmatrix}$$

$$G(\theta) = \begin{bmatrix} (m_1 + m_2) g L_1 S_1 \\ m_2 g L_2 S_2 \end{bmatrix}$$

Problem 4 **Control (32 credits)**

A simple one link robot can be represented with the following simplified SMD system (Fig. 4.1).

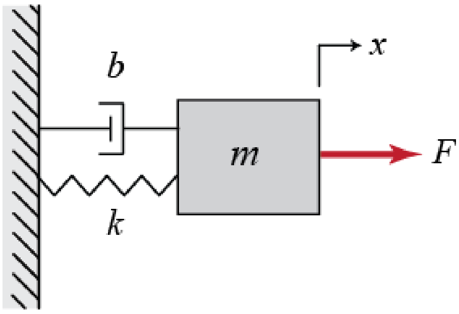


Figure 4.1: SMD system

a)* Write the equation balancing the forces and derive the characteristic equation for this system transforming this expression into the Laplace space with parameter **s** and using the identity $\delta^n x/\delta t^n = s^n \cdot X(s)$.

$$m\ddot{x} + b\dot{x} + kx = F$$

Taking the Laplace transform of the governing equation, we get

$$ms^2X(s) + bsX(s) + kX(s) = F(s)$$

The transfer function between the input force $F(s)$ and the output displacement $X(s)$ then becomes

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

b)* Convert the above characteristic equation into an expression using natural frequency ω_n and damping ξ. Explain how to calculate these values from the parameters in the figure above and name the value for a maximal frequency that can occur in such a system and how is it used to calculate absolute control values in the controler.

$$\zeta = \frac{b}{2\sqrt{km}},$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0, \qquad \omega_n = \sqrt{k/m}.$$

maximal frequency is $\omega_n$and this is used to move the system away from ω_{res}.

$$m\ddot{x} + b\dot{x} + kx = F$$

$$ms^2 X(s) + bsX(s) + kX(s) = F(s)$$

$$\frac{F(s)}{X(s)} = ms^2 + bs + k$$

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + 2\zeta w_n s + w_n^2}$$

$$s^2 + \frac{b}{m}s + \frac{k}{m}$$

$$w_n = \sqrt{\frac{k}{m}}$$

$$2\zeta w_n = \frac{b}{m}$$

$$\zeta\sqrt{\frac{k}{m}} = \frac{b}{2m}$$

$$\zeta = \sqrt{\frac{b^2}{4m^2} \cdot \frac{m}{k}}$$

$$= \frac{b}{2\sqrt{mk}}$$

$$w_n \leq \frac{1}{2} w_{rel_n}$$

0 1 2 3 4 5 6 7 8 9 10 11

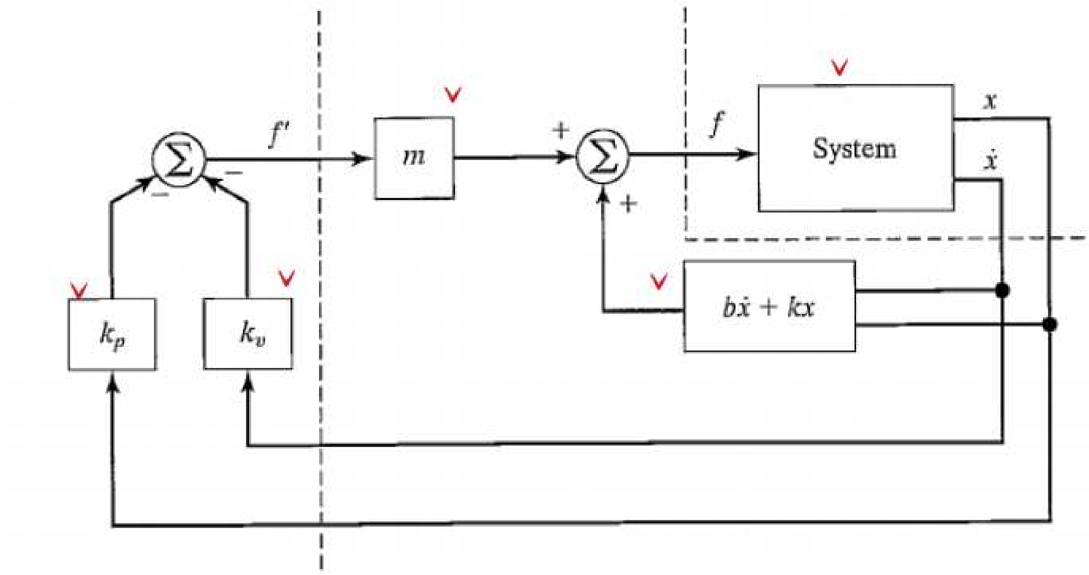
c)* Explain with the equation the control law partitioning applied to the force balance differential equation. Draw the corresponding control structure. How does the system appear to the controler?

$$m\ddot{x} + b\dot{x} + kx = \alpha f' + \beta.$$

Clearly, in order to make the system appear as a unit mass from the f' input, for this particular system we should choose α and β as follows:

$$\alpha = m,$$

$$\beta = b\dot{x} + kx.$$



0 1 2 3 4 5 6 7 8 9

d)* Find the motion of the system in Fig. 4.1 if the parameter values are m=1, b=1 and k=1 and the block (initially at rest) is released from position x=-1.

- write the characteristic equation with unknowns c_1, c_2
- estimate form of the x(t) equation with the unknowns (c_1, c_2)
- calculate c_1, c_2 from the inital conditions mentioned above
- write the resulting x(t)

$s^2 + s + 1 = 0$

which has the roots $s_i = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Hence, the response has the form

$$x(t) = e^{-\frac{t}{2}}\left(c_1 \cos\frac{\sqrt{3}}{2}t + c_2 \sin\frac{\sqrt{3}}{2}t\right).$$

We now use the given initial conditions, $x(0) = -1$ and $\dot{x}(0) = 0$, to compute c_1 and c_2. To satisfy these conditions at $t = 0$, we must have

$$c_1 = -1$$

and

$$-\frac{1}{2}c_1 - \frac{\sqrt{3}}{2}c_2 = 0,$$

which are satisfied by $c_1 = -1$ and $c_2 = \frac{\sqrt{3}}{3}$. So, the motion of the system for $t \geq 0$ is given by

$$x(t) = e^{-\frac{t}{2}}\left(-\cos\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}t\right).$$

$$m\ddot{x} + b\dot{x} + kx = f = \alpha f' + \beta$$

$$\begin{cases} \alpha = m \\ \beta = b\dot{x} + kx \\ f' = \ddot{x} \end{cases}$$

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$ms^2 + bs + k = 0$$

$$s^2 + s + 1 = 0$$

$$s_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$$s_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

$$x(t) = C_1 e^{\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)t} + C_2 e^{\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)t}$$

$$= e^{-\frac{1}{2}t}\left(C_1 \cdot \cos\left(\frac{\sqrt{3}}{2}t\right) - C_2 \cdot i \cdot \sin\left(\frac{\sqrt{3}}{2}t\right) \right)$$

$$x(0) = -1$$

$$C_1 = -1$$

$$\dot{x}(0) = 0$$

$$-\frac{1}{2}\left(C_1 \cdot \cos\left(\frac{\sqrt{3}}{2}t\right) - C_2 \cdot i \cdot \cancel{\sin\left(\frac{\sqrt{3}}{2}t\right)} \right) + \left(\cancel{-\frac{\sqrt{3}}{2} C_1 \cdot \sin\left(\frac{\sqrt{3}}{2}t\right)} - \frac{\sqrt{3}}{2} C_2 \cdot i \cdot \cos\left(\frac{\sqrt{3}}{2}t\right) \right)$$

$$-\frac{1}{2}C_1 - \frac{\sqrt{3}}{2} C_2 i = 0$$

$$C_1 + \sqrt{3} C_2 i = 0$$

$$C_1 = -1 \qquad C_2 = \frac{\sqrt{3}}{3}$$

Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.