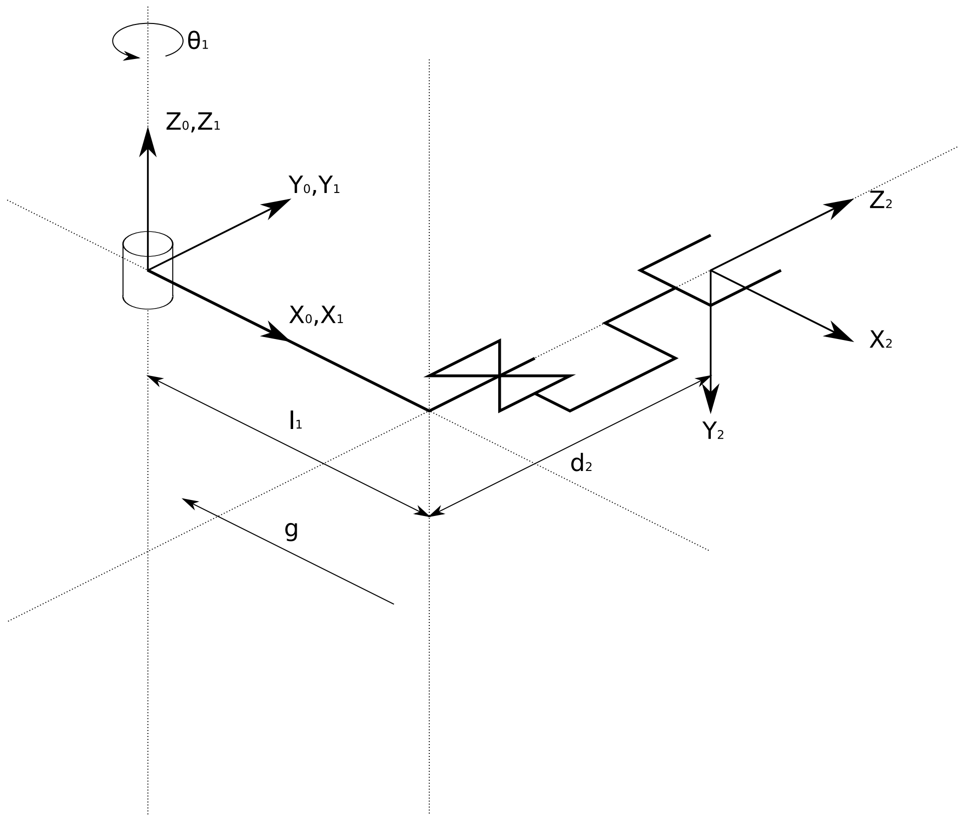


Figure 1: *RP Robot (Problem 1)*

Problem 1

Figure 1 shows a robot with one rotational joint and one prismatic joint. The DH parameters for this robot are

i	a_{i-1}	α_{i-1}	d_i	Θ_i
1	0	0	0	Θ_1
2	l_1	-90°	d_2	0

The manipulator is shown for configuration $\Theta_1 = 0, d_2 \neq 0$. Gravitational force applies in negative X_0-direction, as shown. The inertia tensors are:

$$^{C_1}I_1 = \begin{pmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{pmatrix} \quad ^{C_2}I_2 = \begin{pmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{pmatrix}$$

The masses of the robot's links are m_1 and m_2, and the centers of mass of the links are located at

$$^1P_{C_1} = \left(\frac{l_1}{2}, 0, 0\right)^{\mathrm{T}}$$
$$^2P_{C_2} = (0, 0, l_2)^{\mathrm{T}}.$$

a) Determine the dynamics equations using the Newton-Euler method

b) Formulate the equations in state space (M-V-G) form

P_1

$${}^{i-1}_{\ i}T=\begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	a_{i-1}	α_{i-1}	d_i	Θ_i
1	0	0	0	Θ_1 .= 0.
2	l_1	$-90°$	d_2	0
3	0	0	0	0

$${}^0_1T=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \qquad {}^0_3T=\begin{pmatrix}1&0&0&L_1\\0&0&1&d_2\\0&-1&0&0\\0&0&0&1\end{pmatrix}$$

$${}^1_2T=\begin{pmatrix}1&0&0&l_1\\0&0&1&d_2\\0&-1&0&0\\0&0&0&1\end{pmatrix}$$

$${}^2_3T=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$w_0 \, . \, \dot{w}_0 \, v_0 \, . f_3 \, . n_3 = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$${}^0\dot{v}_0 = g\hat{x}_0$$

$${}^{i+1}\omega_{i+1}={}^{i+1}_{\ i}R\,{}^i\omega_i+\dot{\theta}_{i+1}{}^{i+1}Z_{i+1}$$

$${}^{i+1}\dot{\omega}_{i+1}={}^{i+1}_{\ i}R\,{}^i\dot{\omega}_i+{}^{i+1}_{\ i}R\,{}^i\omega_i\times\dot{\theta}_{i+1}{}^{i+1}\hat{Z}_{i+1}+\ddot{\theta}_{i+1}{}^{i+1}\hat{Z}_{i+1} \xrightarrow{P} {}^{i+1}\dot{\omega}_{i+1}={}^{i+1}_{\ i}R\,{}^i\dot{\omega}_i$$

$${}^{i+1}\dot{v}_{i+1}={}^{i+1}_{\ i}R({}^i\dot{\omega}_i\times{}^iP_{i+1}+{}^i\omega_i\times({}^i\omega_i\times{}^iP_{i+1})+{}^i\dot{v}_i) \rightarrow {}^{i+1}\dot{v}_{i+1}={}^{i+1}_{\ i}R({}^i\dot{\omega}_i\times{}^iP_{i+1}+{}^i\omega_i\times({}^i\omega_i\times{}^iP_{i+1})+{}^i\dot{v}_i)+2\,{}^{i+1}\omega_{i+1}\times\dot{d}_{i+1}{}^{i+1}\hat{Z}_{i+1}+\ddot{d}_{i+1}{}^{i+1}\hat{Z}_{i+1}$$

$${}^{i+1}\dot{v}_{C_{i+1}}={}^{i+1}\dot{\omega}_{i+1}\times{}^{i+1}P_{C_{i+1}}+{}^{i+1}\omega_{i+1}\times({}^{i+1}\omega_{i+1}\times{}^{i+1}P_{C_{i+1}})+{}^{i+1}\dot{v}_{i+1}$$

$${}^{i+1}F_{i+1}=m_{i+1}{}^{i+1}\dot{v}_{C_{i+1}}$$

$${}^{i+1}N_{i+1}={}^{C_{i+1}}I_{i+1}{}^{i+1}\dot{\omega}_{i+1}+{}^{i+1}\omega_{i+1}\times{}^{C_{i+1}}I_{i+1}{}^{i+1}\omega_{i+1}$$

$${}^1w_1={}^1_0R\,{}^0w_0+\dot{\Theta}_1{}^1\hat{Z}_1 = \begin{pmatrix}0\\0\\\dot{\Theta}_1\end{pmatrix}$$

$${}^1\dot{V}_1={}^1_0R(\cancel{{}^0\dot{w}_0}\times{}^0P_1+\cancel{{}^0w_0}\times({}^0w_0\times{}^0P_1)+{}^0\dot{v}_0) = \begin{pmatrix}g\\0\\0\end{pmatrix}$$

$${}^1\dot{w}_1={}^1_0R\,\dot{w}_0+{}^1_0R\,{}^0w_0\times\dot{\Theta}_1{}^1\hat{Z}_1+\ddot{\Theta}_1{}^1\hat{Z}_1 = \begin{pmatrix}0\\0\\\ddot{\Theta}_1\end{pmatrix}$$

$${}^1\dot{V}_{C_1}={}^1\dot{W}_1\times{}^1P_{C_1}+{}^1W_1\times({}^1W_1\times{}^1P_{C_1})+{}^1\dot{V}_1$$

$$=\begin{pmatrix}0\\0\\\ddot{\Theta}_1\end{pmatrix}\times\begin{pmatrix}\frac{L_1}{2}\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\\dot{\Theta}_1\end{pmatrix}\times\left[\begin{pmatrix}0\\0\\\dot{\Theta}\end{pmatrix}\times\begin{pmatrix}\frac{L_1}{2}\\0\\0\end{pmatrix}\right]+\begin{pmatrix}g\\0\\0\end{pmatrix}$$

$$=\begin{pmatrix}0\\\ddot{\Theta}_1\cdot\frac{L_1}{2}\\0\end{pmatrix}+\begin{pmatrix}0\\0\\\dot{\Theta}_1\end{pmatrix}\times\begin{pmatrix}0\\\dot{\Theta}_1\cdot\frac{L_1}{2}\\0\end{pmatrix}+\begin{pmatrix}g\\0\\0\end{pmatrix}$$

$$=\begin{pmatrix}g-\dot{\Theta}_1^2\cdot\frac{L_1}{2}\\\ddot{\Theta}_1\cdot\frac{L_1}{2}\\0\end{pmatrix}$$

$${}^1F_1=m_1\cdot{}^1\dot{V}_{C_1} = \begin{pmatrix}m_1\cdot g-m_1\cdot\dot{\Theta}_1^2\cdot\frac{L_1}{2}\\m_1\cdot\ddot{\Theta}_1\cdot\frac{L_1}{2}\\0\end{pmatrix}$$

$${}^1N_1={}^{C_1}I_1\,{}^1\dot{W}_1+{}^1W_1\times{}^{C_1}I_1\,{}^1W_1$$

$$=\begin{pmatrix}I_{xx1}&0&0\\0&I_{yy1}&0\\0&0&I_{zz1}\end{pmatrix}\cdot\begin{pmatrix}0\\0\\\ddot{\Theta}_1\end{pmatrix}+\begin{pmatrix}0\\0\\\dot{\Theta}_1\end{pmatrix}\times\begin{pmatrix}I_{xx1}&0&0\\0&I_{yy1}&0\\0&0&I_{zz1}\end{pmatrix}\begin{pmatrix}0\\0\\\dot{\Theta}_1\end{pmatrix}$$

$$=\begin{pmatrix}0\\0\\\ddot{\Theta}_1\cdot I_{zz1}\end{pmatrix}+\underbrace{\begin{pmatrix}0\\0\\\dot{\Theta}_1\end{pmatrix}\times\begin{pmatrix}0\\0\\\dot{\Theta}_1\cdot I_{zz1}\end{pmatrix}}_{0}=\begin{pmatrix}0\\0\\\ddot{\Theta}_1\cdot I_{zz1}\end{pmatrix}$$

$$^2w_2 = {}^2_1R\,{}^1w_1 + \dot\theta_2\,{}^2Z_2 \qquad {}^2\dot w_2 = {}^2_1R\,{}^1\dot w_1$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \dot\theta_1 \end{pmatrix} \qquad = \begin{pmatrix} 0 \\ -\ddot\theta_1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -\dot\theta_1 \\ 0 \end{pmatrix}$$

质心集中

$$^2\dot v_2 = {}^2_1R\left({}^1\dot w_1 \times {}^1P_2 + {}^1w_1 \times ({}^1w_1 \times {}^1P_2) + {}^1\dot v_1\right) + 2\,{}^2w_2 \times \dot d_2\,{}^2\hat Z_2 + \ddot d_2\,{}^2\hat Z_2$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\left[\begin{pmatrix} 0 \\ 0 \\ \ddot\theta_1 \end{pmatrix} \times \begin{pmatrix} l_1 \\ d_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot\theta_1 \end{pmatrix} \times \left[\underbrace{\begin{pmatrix} 0 \\ 0 \\ \dot\theta_1 \end{pmatrix} \times \begin{pmatrix} l_1 \\ d_2 \\ 0 \end{pmatrix}}_{\begin{pmatrix} -\dot\theta_1 d_2 \\ \dot\theta_1 l_1 \\ 0 \end{pmatrix}}\right] + \begin{pmatrix} g \\ 0 \\ 0 \end{pmatrix}\right] + \cdots$$

$$= R\left[\begin{pmatrix} -\ddot\theta_1 \cdot d_2 \\ \ddot\theta_1 l_1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\dot\theta_1^2 l_1 \\ -\dot\theta_1^2 d_2 \\ 0 \end{pmatrix} + \begin{pmatrix} g \\ 0 \\ 0 \end{pmatrix}\right]$$

$$R\begin{pmatrix} -\ddot\theta_1 d_2 - \dot\theta_1^2 l_1 + g \\ \ddot\theta_1 l_1 - \dot\theta_1^2 d_2 \\ 0 \end{pmatrix} + 2\cdot\begin{pmatrix} 0 \\ -\dot\theta_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \dot d_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot d_2 \end{pmatrix}$$

$$= \begin{pmatrix} -\ddot\theta_1 d_2 - \dot\theta_1^2 l_1 + g \\ 0 \\ \ddot\theta_1 l_1 - \dot\theta_1^2 d_2 \end{pmatrix} + \begin{pmatrix} -2\dot\theta_1 \cdot \dot d_2 \\ 0 \\ \ddot d_2 \end{pmatrix} \qquad \begin{pmatrix} -2\dot\theta_1 \cdot \dot d_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot d_2 \end{pmatrix}$$

$$= \begin{pmatrix} -\ddot\theta_1 d_2 - 2\dot\theta_1 \dot d_2 - \dot\theta_1^2 l_1 + g \\ 0 \\ -\dot\theta_1^2 d_2 + \ddot d_2 + \ddot\theta_1 l_1 \end{pmatrix}$$

$$^2\dot v_{c2} = {}^2\dot w_2 \times {}^2P_{c2} + {}^2w_2 \times ({}^2w_2 \times {}^2P_{c2}) + {}^2\dot v_2$$

$$= \underbrace{\begin{pmatrix} 0 \\ -\ddot\theta_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ l_2 \end{pmatrix}}_{\begin{pmatrix} -\ddot\theta_1 l_2 \\ 0 \\ 0 \end{pmatrix}} + \underbrace{\begin{pmatrix} 0 \\ -\dot\theta_1 \\ 0 \end{pmatrix} \times \left[\underbrace{\begin{pmatrix} 0 \\ -\dot\theta_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ l_2 \end{pmatrix}}_{}\right]}_{\begin{pmatrix} 0 \\ -\dot\theta_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -\dot\theta_1 l_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\dot\theta_1^2 l_2 \end{pmatrix}} + \begin{pmatrix} -\ddot\theta_1 d_2 - 2\dot\theta_1 \dot d_2 - \dot\theta_1^2 l_1 + g \\ 0 \\ -\dot\theta_1^2 d_2 + \ddot d_2 + \ddot\theta_1 l_1 \end{pmatrix}$$

$$= \begin{pmatrix} -\ddot\theta_1 d_2 - 2\dot\theta_1 \dot d_2 - \dot\theta_1^2 l_1 + g - \ddot\theta_1 l_2 \\ 0 \\ -\dot\theta_1^2 d_2 + \ddot d_2 + \ddot\theta_1 l_1 - \dot\theta_1^2 l_2 \end{pmatrix}$$

$$^2N_2 = {}^{C_2}I_2\,{}^2\dot w_2 + {}^2w_2 \times {}^{C_2}I_2\,{}^2w_2$$

$$= \begin{pmatrix} I_{xx_2} & 0 & 0 \\ 0 & I_{yy_2} & 0 \\ 0 & 0 & I_{zz_2} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -\ddot\theta_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\dot\theta_1 \\ 0 \end{pmatrix} \times \underbrace{\begin{pmatrix} I_{xx_2} & 0 & 0 \\ 0 & I_{yy_2} & 0 \\ 0 & 0 & I_{zz_2} \end{pmatrix}\begin{pmatrix} 0 \\ -\dot\theta_1 \\ 0 \end{pmatrix}}_{\begin{pmatrix} 0 \\ -\dot\theta_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -\dot\theta_1 I_{yy_2} \\ 0 \end{pmatrix} = 0}$$

$$= \begin{pmatrix} 0 \\ -\ddot\theta_1 \cdot I_{yy_2} \\ 0 \end{pmatrix}$$

$$^2F_2 = m_2 \cdot {}^2\dot v_{c2}$$

$$= \begin{pmatrix} (-\ddot\theta_1 d_2 - 2\dot\theta_1 \dot d_2 - \dot\theta_1^2 l_1 + g - \ddot\theta_1 l_2)\,m_2 \\ 0 \\ (-\dot\theta_1^2 d_2 + \ddot d_2 + \ddot\theta_1 l_1 - \dot\theta_1^2 l_2)\,m_2 \end{pmatrix}$$

$${}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i$$
$${}^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i}\times {}^{i}F_i + {}^{i}P_{i+1}\times {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}$$
$$\tau_i = {}^{i}n_i^{T}\,{}^{i}\hat{Z}_i$$

对于移动关节i，有

$$\tau_i = {}^{i}f_i^{T}\,{}^{i}\hat{Z}_i$$

$${}^2f_2 = {}^2_3R\,{}^3f_3 + {}^2F_2 = {}^2F_2 = \begin{pmatrix} (-\ddot\theta_1 d_2 - 2\dot\theta_1\dot d_2 - \dot\theta_1^2 l_1 + g - \ddot\theta_1 l_2)m_2 \\ 0 \\ (-\dot\theta_1^2 d_2 + \ddot d_2 + \ddot\theta_1 l_1 - \dot\theta_1^2 l_2)m_2 \end{pmatrix}$$

$${}^2n_2 = {}^2N_2 + \cancel{{}^2_3R\,{}^3n_3} + {}^2P_{C_2}\times{}^2F_2 + \cancel{{}^2P_3\times{}^2_3R\,{}^3f_3}$$
$$= \begin{pmatrix} 0 \\ -\ddot\theta_1\cdot I_{yy2} \\ 0 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ l_2 \end{pmatrix} \times \begin{pmatrix} (-\ddot\theta_1 d_2 - 2\dot\theta_1\dot d_2 - \dot\theta_1^2 l_1 + g - \ddot\theta_1 l_2)m_2 \\ 0 \\ (-\dot\theta_1^2 d_2 + \ddot d_2 + \ddot\theta_1 l_1 - \dot\theta_1^2 l_2)m_2 \end{pmatrix}}_{\begin{pmatrix} 0 \\ (-\ddot\theta_1 d_2 - 2\dot\theta_1\dot d_2 - \dot\theta_1^2 l_1 + g - \ddot\theta_1 l_2)\,l_2\cdot m_2 \\ 0 \end{pmatrix}}$$
$$= \begin{pmatrix} 0 \\ -\ddot\theta_1\cdot I_{yy2} + (-\ddot\theta_1 d_2 - 2\dot\theta_1\dot d_2 - \dot\theta_1^2 l_1 + g - \ddot\theta_1 l_2)\,l_2\cdot m_2 \\ 0 \end{pmatrix}$$

$$\tau_2 = {}^2f_2^{T}\cdot{}^2\hat Z_2 = (-\dot\theta_1^2 d_2 + \ddot d_2 + \ddot\theta_1 l_1 - \dot\theta_1^2 l_2)\,m_2$$

$${}^1f_1 = {}^1_2R\,{}^2f_2 + {}^1F_1$$
$$= \begin{pmatrix} 1&0&0\\0&0&1\\0&-1&0 \end{pmatrix}\cdot \begin{pmatrix} (-\ddot\theta_1 d_2 - 2\dot\theta_1\dot d_2 - \dot\theta_1^2 l_1 + g - \ddot\theta_1 l_2)m_2 \\ 0 \\ (-\dot\theta_1^2 d_2 + \ddot d_2 + \ddot\theta_1 l_1 - \dot\theta_1^2 l_2)m_2 \end{pmatrix} + \begin{pmatrix} m_1\cdot g - m_1\cdot\dot\theta_1^2\cdot\frac{l_1}{2} \\ m_1\cdot\ddot\theta_1\cdot\frac{l_1}{2} \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} (-\ddot\theta_1 d_2 - 2\dot\theta_1\dot d_2 - \dot\theta_1^2 l_1 + g - \ddot\theta_1 l_2)\cdot m_2 \\ (-\dot\theta_1^2 d_2 + \ddot d_2 + \ddot\theta_1 l_1 - \dot\theta_1^2 l_2)\,m_2 \\ 0 \end{pmatrix} + \begin{pmatrix} (g - \dot\theta_1^2\cdot\frac{l_1}{2})\,m_1 \\ (\ddot\theta_1\cdot\frac{l_1}{2})\,m_1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} (-\ddot\theta_1 d_2 - 2\dot\theta_1\dot d_2 - \dot\theta_1^2 l_1 + g - \ddot\theta_1 l_2)\cdot m_2 + (-\dot\theta_1^2\cdot\frac{l_1}{2} + g)\,m_1 \\ (-\dot\theta_1^2 d_2 + \ddot d_2 + \ddot\theta_1 l_1 - \dot\theta_1^2 l_2)\,m_2 + (\ddot\theta_1\cdot\frac{l_1}{2})\,m_1 \\ 0 \end{pmatrix}$$

$${}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i$$
$${}^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i}\times {}^{i}F_i + {}^{i}P_{i+1}\times {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}$$
$$\tau_i = {}^{i}n_i^{T}\,{}^{i}\hat{Z}_i$$

$$\begin{pmatrix} l_1 \\ d_2 \\ 0 \end{pmatrix}$$

$${}^1n_1 = {}^1N_1 + {}^1_2R\,{}^2n_2 + {}^1P_{C_1}\times{}^1F_1 + {}^1P_2\times{}^1_2R\,{}^2f_2$$
$$= \begin{pmatrix} 0 \\ 0 \\ \ddot\theta_1 I_{zz1} \end{pmatrix} + \begin{pmatrix} 1&0&0\\0&0&1\\0&-1&0 \end{pmatrix}{}^2n_2 + \begin{pmatrix} \frac{l_1}{2} \\ 0 \\ 0 \end{pmatrix}\times\begin{pmatrix} m_1\cdot g - m_1\cdot\dot\theta_1^2\cdot\frac{l_1}{2} \\ m_1\cdot\ddot\theta_1\cdot\frac{l_1}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} l_1 \\ d_2 \\ 0 \end{pmatrix}\times\underbrace{\begin{pmatrix} 1&0&0\\0&0&1\\0&-1&0 \end{pmatrix}\begin{pmatrix} (-\ddot\theta_1 d_2 - 2\dot\theta_1\dot d_2 - \dot\theta_1^2 l_1 + g - \ddot\theta_1 l_2)m_2 \\ 0 \\ (-\dot\theta_1^2 d_2 + \ddot d_2 + \ddot\theta_1 l_1 - \dot\theta_1^2 l_2)m_2 \end{pmatrix}}$$
$$= \begin{pmatrix} 0 \\ 0 \\ \ddot\theta_1 I_{zz1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot\theta_1 I_{yy2} + (\ddot\theta_1 d_2 + 2\dot\theta_1\dot d_2 + \dot\theta_1^2 l_1 + \ddot\theta_1 l_2 - g)\,l_2\cdot m_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ m_1\cdot\ddot\theta_1\cdot\frac{l_1^2}{4} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ (-\dot\theta_1^2 d_2 + \ddot d_2 + \ddot\theta_1 l_1 - \dot\theta_1^2 l_2)\cdot l_1\cdot m_2 - (-\ddot\theta_1 d_2 - 2\dot\theta_1\dot d_2 - \dot\theta_1^2 l_1 + g - \ddot\theta_1 l_2)\cdot d_2\cdot m_2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \\ \ddot\theta_1 I_{zz1} + \ddot\theta_1 I_{yy2} + (\ddot\theta_1 d_2 + 2\dot\theta_1\dot d_2 + \dot\theta_1^2 l_1 + \ddot\theta_1 l_2 - g)\,l_2\cdot m_2 + m_1\cdot\ddot\theta_1\frac{l_1^2}{4} + (-\dot\theta_1^2\cdot d_2 + \ddot d_2 + \ddot\theta_1 l_1 - \dot\theta_1^2\cdot l_2)\cdot l_1\cdot m_2 + (\ddot\theta_1 d_2 + 2\dot\theta_1\dot d_2 + \dot\theta_1^2 l_1 - g + \ddot\theta_1 l_2)\cdot d_2\cdot m_2 \end{pmatrix}$$

$$\tau_1 = (I_{zz1} + I_{yy2} + (\cancel{d_2} + \cancel{l_2})\,l_2\cdot m_2 + \frac{l_1^2}{4}\cdot m_1 + \cancel{l_1^2 m_2} + d_2^2 m_2 + \cancel{l_2\cdot d_2 m_2})$$

$$l_1\cdot m_2$$

$$= (I_{zz1} + I_{yy2} + (2d_2\cdot l_2 + l_2^2 + l_1^2 + d_2^2)\,m_2 + \frac{l_1^2}{4}m_1)\cdot\ddot\theta_1 + l_1\cdot m_2\cdot\ddot d_2 +$$
$$+ 2\dot\theta_1\dot d_2\cdot l_2\cdot m_2 + \cancel{\dot\theta_1^2 l_1 l_2 m_2} - \cancel{\dot\theta_1^2 d_2 l_1 m_2} - \cancel{\dot\theta_1^2 l_2 l_1 m_2} + 2\dot\theta_1\dot d_2 d_2 m_2 + \cancel{\dot\theta_1^2 l_1 d_2 m_2}$$
$$= (l_2 + d_2)\,2\dot\theta_1\dot d_2 m_2$$
$$- g l_2 m_2 - g d_2 m_2 = -g m_2 (l_2 + d_2)$$

P2

$$\tau_1 = \left[(I_{zz1} + I_{yy2}) + \frac{l_1^2}{4} m_1 + (2 d_2 l_2 + d_2^2 + l_1^2 + l_2^2)\, m_2 \right] \ddot{\theta}_1 + l_1 \cdot m_2 \cdot \ddot{d}_2$$
$$+ (l_2 + d_2) \cdot 2\, \dot{\theta}_1 \dot{d}_2 m_2 - (l_2 + d_2) \cdot m_2 g$$

$$\tau_2 = l_1 m_2 \cdot \ddot{\theta}_1 + m_2 \cdot \ddot{d}_2 - \left[(d_2 + l_2) \cdot \dot{\theta}_1^2 \cdot m_2 \right]$$

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta)$$

$$M(\theta) = \begin{pmatrix} (I_{zz1} + I_{yy2}) + \frac{l_1^2}{4} m_1 + (2 d_2 l_2 + d_2^2 + l_1^2 + l_2^2)\, m_2 & l_1 \cdot m_2 \\ l_1 \cdot m_2 & m_2 \end{pmatrix}$$

$$V(\theta,\dot{\theta}) = \begin{pmatrix} 2 \cdot (l_2 + d_2) \cdot \dot{\theta}_1 \dot{d}_2 m_2 \\ -(d_2 + l_2) \cdot \dot{\theta}_1^2 \cdot m_2 \end{pmatrix}$$

$$G(\theta) = \begin{pmatrix} -(l_2 + d_2) \cdot m_2 \cdot g \\ 0 \end{pmatrix}$$

$$M(\theta) = \begin{pmatrix} I_{zz1} + I_{yy2} + m_2 (d_2^2 + 2 d_2 l_2 + l_2^2 + l_1^2) + \frac{m_1}{4} l_1^2 & l_1 m_2 \\ l_1 m_2 & m_2 \end{pmatrix}$$

$$V(\theta,\dot{\theta}) = \begin{pmatrix} 2 m_2 \dot{d}_2 \dot{\theta}_1 (d_2 + l_2) \\ - m_2 \dot{\theta}_1^2 (d_2 + l_2) \end{pmatrix}$$

$$G(\theta) = \begin{pmatrix} g\left(-m_2 c_1 (d_2 + l_2) - l_1 s_1 \left(\frac{m_1}{2} + m_2\right)\right) \\ -g\, m_2 s_1 \end{pmatrix}$$

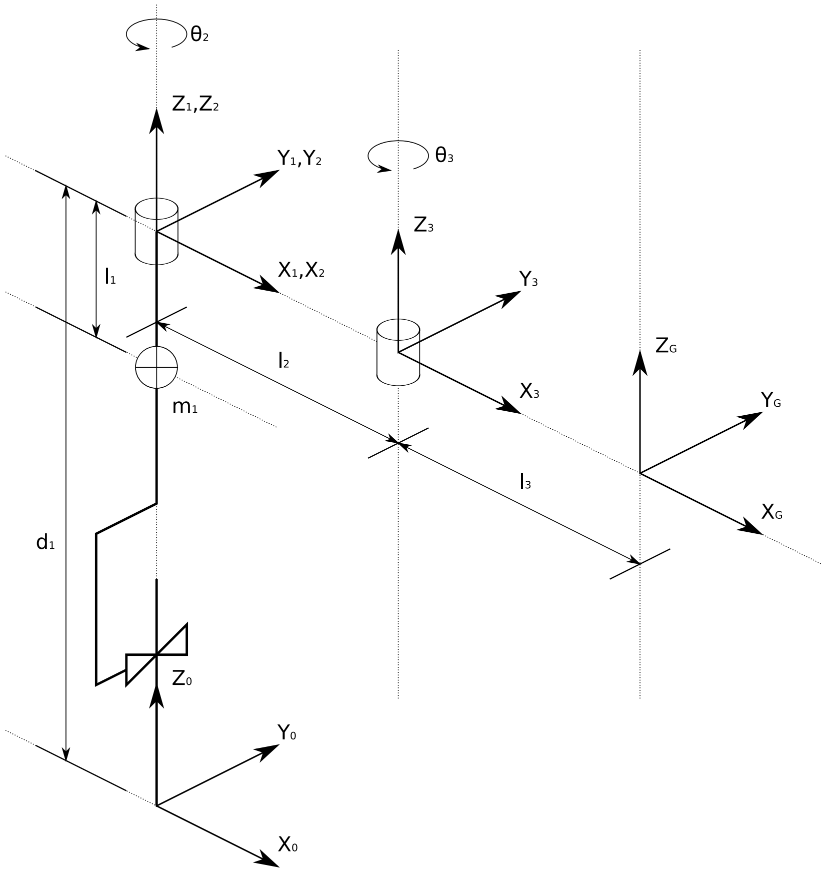


Figure 2: *PRR Robot (Problem 2)*

Problem 2

The manipulator shown in Figure 2 has the following properties:

- Masses of the three links are m_1, m_2, m_3
- Inertia tensors are

$$^{C_1}I_1 = \begin{pmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{pmatrix} \quad ^{C_2}I_2 = \begin{pmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{pmatrix} \quad ^{C_3}I_3 = \begin{pmatrix} I_{xx3} & 0 & 0 \\ 0 & I_{yy3} & 0 \\ 0 & 0 & I_{zz3} \end{pmatrix}$$

- The positions of the centers of mass for the three links are:

$$^0P_{C_1} = \begin{pmatrix} 0 \\ 0 \\ d_1 - l_1 \end{pmatrix}, \quad ^0P_{C_2} = \begin{pmatrix} \frac{l_2}{2}\mathrm{c}_2 \\ \frac{l_2}{2}\mathrm{s}_2 \\ d_1 \end{pmatrix}, \quad ^0P_{C_3} = \begin{pmatrix} l_2\mathrm{c}_2 + \frac{l_3}{2}\mathrm{c}_{23} \\ l_2\mathrm{s}_2 + \frac{l_3}{2}\mathrm{s}_{23} \\ d_1 \end{pmatrix}$$

Furthermore, gravity applies in negative Z_0-direction, as shown. Using the Lagrangian approach to robot dynamics, determine the manipulator dynamic equations in state space (M-V-G) and configuration space (M-B-C-G) form.

$$k_i = \frac{1}{2} m_i\, v_{C_i}^T v_{C_i} + \frac{1}{2}\, {}^i\omega_i^T\, {}^{C_i}I_i\, {}^i\omega_i$$

$$u_i = -m_i\, {}^0g^T\, {}^0P_{C_i} + u_{ref_i}$$

$$\frac{L_2^2}{4} S_2^2 \dot\theta_2^2 + \frac{L_2^2}{4} C_2^2 \dot\theta_2^2 + \dot d_1^2$$

$$= \frac{L_2^2}{4}\dot\theta_2^2 + \dot d_1^2$$

$${}^0V_{C_1} = \frac{d\,{}^0P_{C_1}}{dt} = \begin{pmatrix} 0 \\ 0 \\ \dot d_1 \end{pmatrix}$$

对t求导

$${}^0V_{C_2} = \frac{d\,{}^0P_{C_2}}{dt} = \begin{pmatrix} -\frac{L_2}{2} S_2 \\ \frac{L_2}{2} C_2 \\ \dot d_1 \end{pmatrix} = \begin{pmatrix} -\frac{L_2}{2} S_2 \cdot \dot\theta_2 \\ \frac{L_2}{2} C_2 \cdot \dot\theta_2 \\ \dot d_1 \end{pmatrix}$$

$${}^0V_{C_3} = \frac{d\,{}^0P_{C_3}}{dt} = \begin{pmatrix} -L_2S_2 - \frac{L_3}{2} S_{23} \\ L_2C_2 + \frac{L_3}{2} C_{23} \\ \dot d_1 \end{pmatrix} = \begin{pmatrix} -L_2S_2\cdot\dot\theta_2 - \frac{L_3}{2} S_{23}\cdot(\dot\theta_2+\dot\theta_3) \\ L_2C_2\cdot\dot\theta_2 + \frac{L_3}{2} C_{23}\,(\dot\theta_2+\dot\theta_3) \\ \dot d_1 \end{pmatrix}$$

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\, {}^i\omega_i + \dot\theta_{i+1}\, {}^{i+1}Z_{i+1}$$

$${}^0w_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad {}^1w_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad {}^2w_2 = \begin{pmatrix} 0 \\ 0 \\ \dot\theta_2 \end{pmatrix} \quad {}^3w_3 = \begin{pmatrix} 0 \\ 0 \\ \dot\theta_2+\dot\theta_3 \end{pmatrix}$$

$$k_1 = \frac{1}{2} m_1 (0\ 0\ \dot d_1)\begin{pmatrix} 0 \\ 0 \\ \dot d_1 \end{pmatrix} = \frac{1}{2} m_1 \cdot \dot d_1^2 \qquad 1\times3\ \ 3\times3\ \ 3\times1$$

$$u_1 = -m_1 \cdot (0\ 0\ {-g}) \cdot \begin{pmatrix} 0 \\ 0 \\ d_1 - L_1 \end{pmatrix} + u_{ref_1} = m_1 g\,(d_1 - L_1)$$

$$k_2 = \frac{1}{2} m_2 \left(\frac{L_2^2}{4}\dot\theta_2^2 + \dot d_1^2\right) + \frac{1}{2}\dot\theta_2^2 \cdot I_{zz2}$$

$$u_2 = -m_2(0\ 0\ {-g})\begin{pmatrix} \frac{L_2}{2} C_2 \\ \frac{L_2}{2} S_2 \\ d_1 \end{pmatrix} + u_{ref_1} = m_2 \cdot g \cdot d_1$$

$$k_3 = \frac{1}{2} m_3 \Big[L_2^2 \cancel{S_2^2}\dot\theta_2^2 + L_2S_2\dot\theta_2 \cdot L_3S_{23}(\dot\theta_2+\dot\theta_3) + \frac{L_3^2}{4}\cancel{S_{23}^2\,(\dot\theta_2+\dot\theta_3)^2}$$
$$+ L_2^2\cancel{C_2^2}\dot\theta_2^2 + L_2C_2\dot\theta_2 \cdot L_3C_{23}(\dot\theta_2+\dot\theta_3) + \frac{L_3^2}{4}\cancel{C_{23}\,(\dot\theta_1+\dot\theta_2)^2}$$
$$+ \dot d_1^2 \Big]$$

$$= \frac{1}{2} m_3 \Big[L_2^2\dot\theta_2^2 + \frac{L_3^2}{4}(\dot\theta_2^2 + 2\dot\theta_2\cdot\dot\theta_3 + \dot\theta_3^2) + L_2S_2L_3S_{23}(\dot\theta_2^2 + \dot\theta_2\dot\theta_3) + L_2C_2L_3S_{23}(\dot\theta_2^2+\dot\theta_2\dot\theta_3) + \dot d_1^2\Big]$$
$$+ \frac{1}{2} I_{zz3} \cdot (\dot\theta_2+\dot\theta_3)^2$$

$$u_3 = -m_3 \cdot (0\ 0\ {-g})\begin{pmatrix} \vdots \end{pmatrix} + \cdots = m_3 g d_1$$

$$\begin{pmatrix} -L_2 S_2 \cdot \dot\theta_2 - \frac{L_3}{2} S_{23} \cdot (\dot\theta_2 + \dot\theta_3) \\ L_2 C_2 \cdot \dot\theta_2 + \frac{L_3}{2} C_{23} (\dot\theta_2 + \dot\theta_3) \\ \dot d_1 \end{pmatrix}$$

$$\frac{1}{2} m_3 \Big[\cancel{L_2^2 S_2^2} \dot\theta_2^2 + L_2 S_2 \dot\theta_2 \cdot L_3 S_{23} (\dot\theta_2 + \dot\theta_3) + \cancel{\frac{L_3^2}{4} S_{23}^2 (\dot\theta_2 + \dot\theta_3)^2}$$
$$+ \cancel{L_2^2 C_2^2} \dot\theta_2^2 + L_2 C_2 \dot\theta_2 L_3 C_{23} (\dot\theta_2 + \dot\theta_3) + \cancel{\frac{L_3^2}{4} C_{23}^2 (\dot\theta_2 + \dot\theta_3)^2}$$
$$+ \dot d_1^2 \Big] + \frac{1}{2} I_{zz3} (\dot\theta_2 + \dot\theta_3)^2$$

$$\frac{1}{2} m_3 \Big[L_2^2 \dot\theta_2^2 + \frac{L_3^2}{4} (\dot\theta_2^2 + 2\dot\theta_2 \dot\theta_3 + \dot\theta_3^2) \boxed{+ L_2 S_2 \dot\theta_2^2 \cdot L_3 S_{23} + L_2 C_2 \dot\theta_2^2 L_3 S_{23}} \boxed{+ L_2 S_2 \dot\theta_2 \dot\theta_3 L_3 S_{23} + L_2 C_2 \dot\theta_2 \dot\theta_3 L_3 C_{23}}$$
$$+ \dot d_1^2 \Big] + \frac{1}{2} I_{zz3} (\dot\theta_2 + \dot\theta_3)^2$$

$$L_2 L_3 \dot\theta_2^2 \left[S_2 \cdot S_{23} + C_2 \cdot C_{23} \right]$$
$$= S_2 \cdot S_2 C_3 + \cancel{S_2 \cdot C_2 \cdot S_3} + C_2 \cdot C_2 \cdot C_3 - \cancel{C_2 \cdot S_2 S_3}$$
$$= S_2^2 C_3 + C_2^2 \cdot C_3$$
$$= C_3$$

$$L_2 L_3 \cdot \dot\theta_2 \dot\theta_3 \cdot \left[S_2 \cdot S_{23} + C_2 \cdot C_{23} \right]$$
$$= S_2 \cdot S_2 C_3 + \cancel{S_2 \cdot C_2 S_3} + C_2 \cdot C_2 \cdot C_3 - \cancel{C_2 S_2 S_3}$$
$$= S_2^2 C_3 + C_2^2 C_3$$
$$= C_3$$

$$\frac{m_3}{2} \Big[\dot d_1^2 + \left(L_2^2 + \frac{L_3^2}{4} + L_2 L_3 C_3 \right) \dot\theta_2^2 + \left(\frac{L_3^2}{2} + L_2 L_3 C_3 \right) \dot\theta_2 \dot\theta_3 + \frac{L_3^2}{4} \cdot \dot\theta_3^2 \Big]$$
$$+ \frac{I_{zz3}}{2} \left(\dot\theta_2^2 + 2 \dot\theta_2 \cdot \dot\theta_3 + \dot\theta_3^2 \right)$$

k_3

$$= \frac{m_3}{2} \dot d_1^2 + \left[\frac{m_3}{2} \left(L_2^2 + \frac{L_3^2}{4} + L_2 L_3 C_3 \right) + \frac{I_{zz3}}{2} \right] \dot\theta_2^2 + \left[\frac{m_3}{2} \left(\frac{L_3^2}{2} + L_2 L_3 C_3 \right) + I_{zz3} \right] \cdot \dot\theta_2 \dot\theta_3$$
$$+ \left(\frac{m_3}{8} L_3^2 + \frac{I_{zz3}}{2} \right) \cdot \dot\theta_3^2$$

$$k = \frac{1}{2} m_1 \dot{d}_1^2 + \frac{1}{2} m_2 \cdot \frac{L_2^2}{4} \dot{\theta}_2^2 + \frac{1}{2} m_2 \cdot \dot{d}_1^2$$

$$\frac{m_3}{2} \dot{d}_1^2 + \left[\frac{m_3}{2} \left(L_2^2 + \frac{L_3^2}{4} + L_2 L_3 C_3 \right) + \frac{I_{zz3}}{2} \right] \dot{\theta}_2^2 + \left[\frac{m_3}{2} \left(\frac{L_3^2}{2} + L_2 L_3 C_3 \right) + I_{zz3} \right] \cdot \dot{\theta}_2 \dot{\theta}_3$$

$$\frac{1}{2} \dot{\theta}_2^2 \cdot I_{zz2} \qquad + \left(\frac{m_3}{8} L_3^2 + \frac{I_{zz3}}{2} \right) \cdot \dot{\theta}_3^2$$

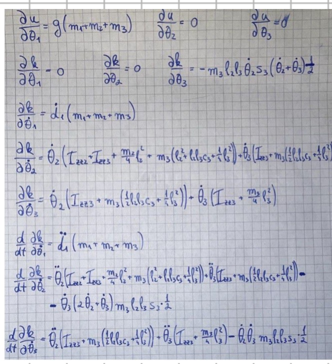
$$= \left(\frac{m_1}{2} + \frac{m_2}{2} + \frac{m_3}{2} \right) \dot{d}_1^2 + \left[\frac{m_3}{2} \left(L_2^2 + \frac{L_3^2}{4} + L_2 L_3 C_3 \right) + \frac{I_{zz2} + I_{zz3}}{2} + \frac{m_2}{2} \frac{L_2^2}{4} \right] \dot{\theta}_2^2$$

$$+ \left[\frac{m_3}{2} \left(\frac{L_3^2}{2} + L_2 L_3 C_3 \right) + I_{zz3} \right] \cdot \dot{\theta}_2 \dot{\theta}_3 + \left(\frac{m_3}{8} L_3^2 + \frac{I_{zz3}}{2} \right) \cdot \dot{\theta}_3^2 + \frac{m_2}{8} \cdot L_2^2$$

$$u = m_1 g (d_1 - L_1) + m_2 g d_1 + m_3 g d_1$$

$$\frac{\partial k}{\partial \dot{\theta}} = \begin{pmatrix} (m_1 + m_2 + m_3) \cdot \dot{d}_1 \\ \left[m_3 \left(L_2^2 + \frac{L_3^2}{4} + L_2 L_3 C_3 \right) + I_{zz2} + I_{zz3} + \frac{m_2}{4} L_2^2 \right] \dot{\theta}_2 + \left[\frac{m_3}{2} \left(\frac{L_3^2}{2} + L_2 L_3 C_3 \right) + I_{zz3} \right] \dot{\theta}_3 \\ \left[\frac{m_3}{4} L_3^2 + I_{zz3} \right] \dot{\theta}_3 + \left[\frac{m_3}{2} \left(\frac{L_3^2}{2} + L_2 L_3 C_3 \right) + I_{zz3} \right] \dot{\theta}_2 \end{pmatrix}$$

$$\frac{\partial k}{\partial \theta} = \begin{pmatrix} 0 \\ 0 \\ -\frac{m_3}{2} L_2 L_3 S_3 \cdot \dot{\theta}_2^2 - \frac{m_3}{2} L_2 L_3 S_3 \cdot \dot{\theta}_2 \cdot \dot{\theta}_3 \end{pmatrix}$$



$$\frac{\partial u}{\partial \theta_1} = g(m_1 + m_2 + m_3) \quad \frac{\partial u}{\partial \theta_2} = 0 \quad \frac{\partial u}{\partial \theta_3} = 0$$

$$\frac{\partial k}{\partial \theta_1} = 0 \quad \frac{\partial k}{\partial \theta_2} = 0 \quad \frac{\partial k}{\partial \theta_3} = -m_3 l_2 l_3 \dot{\theta}_2 s_3 (\dot{\theta}_2 + \dot{\theta}_3) \frac{1}{2}$$

$$\frac{\partial k}{\partial \dot{\theta}_1} = \dot{d}_1 (m_1 + m_2 + m_3)$$

$$\frac{\partial k}{\partial \dot{\theta}_2} = \dot{\theta}_2 \left(I_{zz2} + I_{zz3} + \frac{m_2}{4} l_2^2 + m_3 \left(l_2^2 + l_2 l_3 c_3 + \frac{1}{4} l_3^2 \right) \right) + \dot{\theta}_3 \left(I_{zz3} + m_3 \left(\frac{1}{2} l_2 l_3 c_3 + \frac{1}{4} l_3^2 \right) \right)$$

$$\frac{\partial k}{\partial \dot{\theta}_3} = \dot{\theta}_2 \left(I_{zz3} + m_3 \left(\frac{1}{2} l_2 l_3 c_3 + \frac{1}{4} l_3^2 \right) \right) + \dot{\theta}_3 \left(I_{zz3} + \frac{m_3}{4} l_3^2 \right)$$

$$\frac{d}{dt} \frac{\partial k}{\partial \dot{\theta}_1} = \ddot{d}_1 (m_1 + m_2 + m_3)$$

$$\frac{d}{dt} \frac{\partial k}{\partial \dot{\theta}_2} = \ddot{\theta}_2 \left(I_{zz2} + I_{zz3} + \frac{m_2}{4} l_2^2 + m_3 \left(l_2^2 + l_2 l_3 c_3 + \frac{1}{4} l_3^2 \right) \right) + \ddot{\theta}_3 \left(I_{zz3} + m_3 \left(\frac{1}{2} l_2 l_3 c_3 + \frac{1}{4} l_3^2 \right) \right) - \dot{\theta}_3 (2 \dot{\theta}_2 + \dot{\theta}_3) m_3 l_2 l_3 s_3 \cdot \frac{1}{2}$$

$$\frac{d}{dt} \frac{\partial k}{\partial \dot{\theta}_3} = \ddot{\theta}_2 \left(I_{zz3} + m_3 \left(\frac{1}{2} l_2 l_3 c_3 + \frac{1}{4} l_3^2 \right) \right) + \ddot{\theta}_3 \left(I_{zz3} + \frac{m_3}{4} l_3^2 \right) - \dot{\theta}_2 \dot{\theta}_3 m_3 l_2 l_3 s_3 \cdot \frac{1}{2}$$

$$\frac{\partial u}{\partial \theta} = \begin{pmatrix} g (m_1 + m_2 + m_3) \\ 0 \\ 0 \end{pmatrix}$$

$$m_3 L_2 L_3 C_3 \cdot \dot{\theta}_2$$
$$= m_3 L_2 L_3 \left(-S_3 \cdot \dot{\theta}_3 \cdot \dot{\theta}_2 + \underline{C_3 \cdot \ddot{\theta}_2} \right)$$
$$- m_3 L_2 L_3 S_3 \, \dot{\theta}_3 \cdot \dot{\theta}_2$$

$$\frac{1}{2} m_3 L_2 L_3 C_3 \dot{\theta}_3$$
$$= \frac{1}{2} m_3 L_2 L_3 \left(-S_3 \cdot \dot{\theta}_3^2 + \underline{C_3 \cdot \ddot{\theta}_3} \right)$$
$$- \frac{1}{2} m_3 L_2 L_3 S_3 \dot{\theta}_3^2$$

$$\frac{d}{dt} \frac{\partial k}{\partial \dot{\Theta}} - \frac{\partial k}{\partial \Theta} + \frac{\partial u}{\partial \Theta} = \tau$$

$$\frac{d}{dt} \frac{\partial k}{\partial \dot{\theta}} = \begin{pmatrix} (m_1 + m_2 + m_3) \cdot \ddot{d}_1 \\ \left[\frac{m_2}{4} L_2^2 + m_3 \left(L_2^2 + \frac{L_3^2}{4} + L_2 L_3 C_3 \right) + I_{zz2} + I_{zz3} \right] \ddot{\theta}_2 + \left[\frac{m_3}{2} \left(\frac{L_3^2}{2} + L_2 L_3 C_3 \right) + I_{zz3} \right] \cdot \ddot{\theta}_3 - \dot{\theta}_3 \left(m_3 L_2 L_3 S_3 \left(\dot{\theta}_2 + \frac{\dot{\theta}_3}{2} \right) \right) \\ \left[\frac{m_3}{4} L_3^2 + I_{zz3} \right] \ddot{\theta}_3 + \left[\frac{m_3}{2} \left(\frac{L_3^2}{2} + L_2 L_3 C_3 \right) + I_{zz3} \right] \cdot \ddot{\theta}_2 - \frac{m_3}{2} L_2 L_3 S_3 \cdot \dot{\theta}_3 \cdot \dot{\theta}_2 \end{pmatrix}$$

$$\frac{m_3}{2} L_2 L_3 C_3 \dot{\theta}_2$$
$$= \frac{m_3}{2} L_2 L_3 \left(-S_3 \cdot \dot{\theta}_3 \cdot \dot{\theta}_2 + \underline{C_3 \cdot \ddot{\theta}_2} \right)$$
$$- \frac{m_3}{2} L_2 L_3 S_3 \cdot \dot{\theta}_3 \cdot \dot{\theta}_2$$

$$
\tau=\begin{pmatrix}
(m_1+m_2+m_3)\ddot{d}_1+(m_1+m_2+m_3)g \\
\left[\frac{m_3}{4}L_2^2+m_3\left(L_2^2+\frac{L_3^2}{4}+L_2L_3C_3\right)+I_{zz2}+I_{zz3}\right]\ddot{\Theta}_2+\left[\frac{m_3}{2}\left(\frac{L_3^2}{2}+L_2L_3C_3\right)+I_{zz3}\right]\cdot\ddot{\Theta}_3-\left[m_3L_2L_3S_3\left(\dot{\Theta}_2+\frac{\dot{\Theta}_3}{2}\right)\right]\dot{\Theta}_3 \\
\left[\frac{m_3}{2}\left(\frac{L_3^2}{2}+L_2L_3C_3\right)+I_{zz3}\right]\ddot{\Theta}_2+\left[\frac{m_3}{4}L_3^2+I_{zz3}\right]\ddot{\Theta}_3+\frac{m_3}{2}L_2L_3S_3\,\dot{\Theta}_2^2
\end{pmatrix}
$$

$$\tau_i=\frac{d}{dt}\frac{\partial k}{\partial\dot{\theta}_i}-\frac{\partial k}{\partial\theta_i}+\frac{\partial u}{\partial\theta_i}$$

$$\tau_1=\ddot{d}_1(m_1+m_2+m_3)+g(m_1+m_2+m_3)$$

$$\tau_2=\ddot{\theta}_2\left(I_{zz2}+I_{zz3}+\frac{m_2}{4}l_2^2+m_3\left(l_2^2+l_2l_3c_3+\frac{1}{4}l_3^2\right)\right)+\ddot{\theta}_3\left(I_{zz3}+\frac{m_3}{4}\left(2l_2l_3c_3+\frac{1}{4}l_3^2\right)\right)-$$
$$-\dot{\theta}_3\left(\dot{\theta}_2+\frac{1}{2}\dot{\theta}_3\right)m_3l_2l_3s_3$$

$$\tau_3=\ddot{\theta}_2\left(I_{zz3}+\frac{m_3}{4}\left(2l_2l_3c_3+l_3^2\right)\right)+\ddot{\theta}_3\left(I_{zz3}+\frac{m_3}{4}l_3^2\right)+\frac{1}{2}\dot{\theta}_2^2m_3l_2l_3s_3+$$

$$
M(\Theta)=\begin{pmatrix}
m_1+m_2+m_3 & 0 & 0 \\
0 & \left[\frac{m_3}{4}L_2^2+m_3\left(L_2^2+\frac{L_3^2}{4}+L_2L_3C_3\right)+I_{zz2}+I_{zz3}\right] & \left[\frac{m_3}{2}\left(\frac{L_3^2}{2}+L_2L_3C_3\right)+I_{zz3}\right] \\
0 & \left[\frac{m_3}{2}\left(\frac{L_3^2}{2}+L_2L_3C_3\right)+I_{zz3}\right] & \left[\frac{m_3}{4}L_3^2+I_{zz3}\right]
\end{pmatrix}
$$

$$
V(\Theta,\dot{\Theta})=\begin{pmatrix}
0 \\
-\left[m_3L_2L_3S_3\left(\dot{\Theta}_2+\frac{\dot{\Theta}_3}{2}\right)\right]\dot{\Theta}_3 \\
\frac{m_3}{2}L_2L_3S_3\,\dot{\Theta}_2^2
\end{pmatrix}
$$

$$
G(\Theta)=\begin{pmatrix}
(m_1+m_2+m_3)g \\ 0 \\ 0
\end{pmatrix}
$$

形位空间方程

将动力学方程中的速度项$V(\Theta,\dot{\Theta})$写成另外一种形式如下

$$\tau=M(\Theta)\ddot{\Theta}+B(\Theta)[\dot{\Theta}\dot{\Theta}]+C(\Theta)[\dot{\Theta}^2]+G(\Theta) \tag{6-63}$$

式中$B(\Theta)$是$n\times n(n-1)/2$阶的哥氏力系数矩阵，$[\dot{\Theta}\dot{\Theta}]$是$n(n-1)/2\times 1$阶的关节速度积矢量，即

$$[\dot{\Theta}\dot{\Theta}]=[\dot{\theta}_1\dot{\theta}_2\ \ \dot{\theta}_1\dot{\theta}_3\ \cdots\ \dot{\theta}_{n-1}\dot{\theta}_n]^T \tag{6-64}$$

$C(\Theta)$是$n\times n$阶离心力系数矩阵，而$[\dot{\Theta}^2]$是$n\times 1$阶矢量，即

$$[\dot{\theta}_1^2\ \ \dot{\theta}_2^2\ \cdots\ \dot{\theta}_n^2]^T \tag{6-65}$$

式（6-63）称为**形位空间方程**，因为它的系数矩阵仅是操作臂位置的函数[3]。

$$
M(\theta)=\begin{pmatrix}
m_1+m_2+m_3 & 0 & 0 \\
0 & I_{zz2}+I_{zz3}+\frac{m_2}{4}l_2^2+m_3\left(l_2^2+l_2l_3c_3+\frac{1}{4}l_3^2\right) & I_{zz3}+m_3\left(\frac{1}{2}l_2l_3c_3+\frac{1}{4}l_3^2\right) \\
0 & I_{zz3}+m_3\left(\frac{1}{2}l_2l_3c_3+\frac{1}{4}l_3^2\right) & I_{zz3}+\frac{m_3}{4}l_3^2
\end{pmatrix}
$$

$$
V(\theta,\dot{\theta})=\begin{pmatrix}
0 \\
-\dot{\theta}_3\left(\dot{\theta}_2+\frac{1}{2}\dot{\theta}_3\right)m_3l_2l_3s_3 \\
\frac{1}{2}\dot{\theta}_2^2\,m_3l_2l_3s_3
\end{pmatrix}
\qquad
G(\theta)=\begin{pmatrix}
g(m_1+m_2+m_3) \\ 0 \\ 0
\end{pmatrix}
$$

$$B(\Theta)\quad 3\times\frac{3(3-1)}{2}=3\times 3\qquad[\dot{\Theta}\dot{\Theta}]\quad 3\times 1$$

$$
B(\theta)=\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -m_3l_2l_3s_3 \\
0 & 0 & 0
\end{pmatrix}
\qquad
C(\theta)=\begin{pmatrix}
0 & 0 & -\frac{1}{2}m_3l_2l_3s_3 \\
0 & 0 & 0 \\
0 & \frac{1}{2}m_3l_2l_3s_3 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -m_3L_2L_3S_3 \\
0 & 0 & 0
\end{pmatrix}=B(\Theta)
\qquad
\begin{pmatrix}
\dot{\Theta}_1\dot{\Theta}_2 \\ \dot{\Theta}_1\dot{\Theta}_3 \\ \dot{\Theta}_2\dot{\Theta}_3
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -\frac{1}{2}m_3L_2L_3S_3 \\
0 & \frac{m_3}{2}L_2L_3S_3 & 0
\end{pmatrix}=C(\Theta)
\qquad
\begin{pmatrix}
\dot{\Theta}_1^2 \\ \dot{\Theta}_2^2 \\ \dot{\Theta}_3^2
\end{pmatrix}
$$