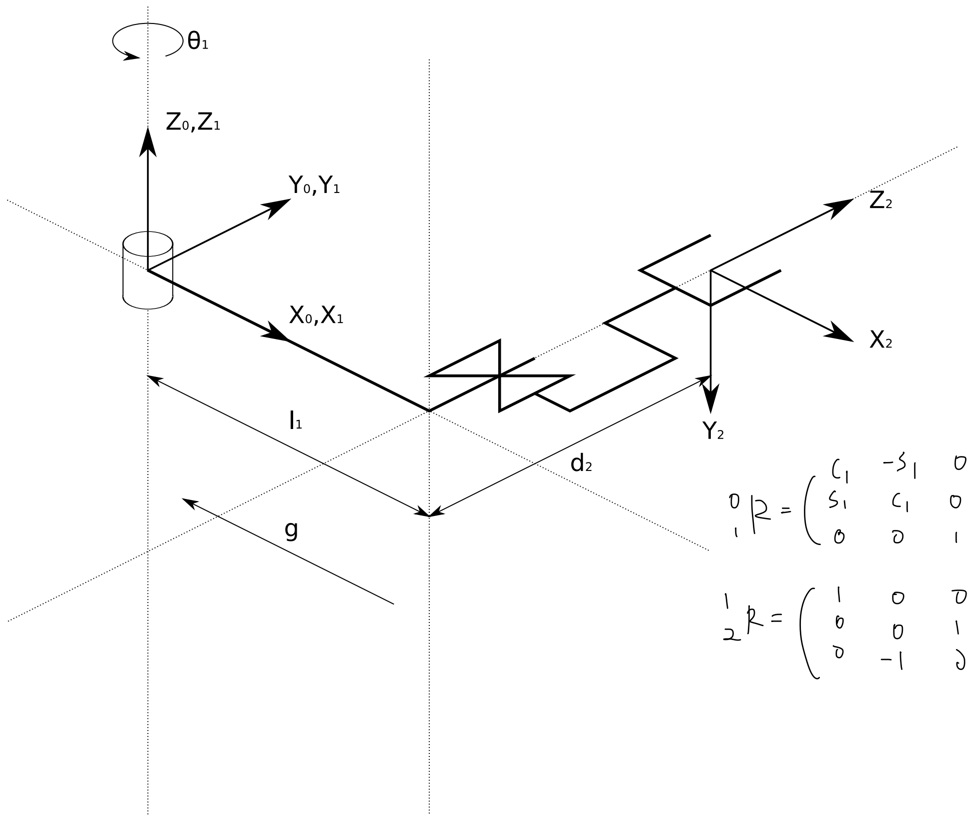

$${}^0_1R = \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$${}^1_2R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Figure 1: *RP Robot (Problem 1)*

Problem 1

Figure 1 shows a robot with one rotational joint and one prismatic joint. The DH parameters for this robot are

i	a_{i-1}	α_{i-1}	d_i	Θ_i
1	0	0	0	Θ_1
2	l_1	$-90°$	d_2	0

$${}^0\dot{V}_0 = \begin{pmatrix} g \\ 0 \\ 0 \end{pmatrix}$$ ← 都是正的

The manipulator is shown for configuration $\Theta_1 = 0, d_2 \neq 0$. Gravitational force applies in negative X_0-direction, as shown. The inertia tensors are:

$${}^{C_1}I_1 = \begin{pmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{pmatrix} \quad {}^{C_2}I_2 = \begin{pmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{pmatrix}$$

The masses of the robot's links are m_1 and m_2, and the centers of mass of the links are located at

$${}^1P_{C_1} = \left(\frac{l_1}{2}, 0, 0\right)^{\mathrm{T}}$$
$${}^2P_{C_2} = (0, 0, l_2)^{\mathrm{T}}.$$

a) Determine the dynamics equations using the Newton-Euler method

b) Formulate the equations in state space (M-V-G) form

$$\omega_0 = \dot\omega_0 = v_0 = 0 \qquad {}^0\dot v_0 = (g,0,0)^T$$

$${}^1\omega_1 = \cancel{{}^1_0R\,{}^0\omega_0} + \dot\theta_1\,{}^1\hat Z_1 = \begin{pmatrix}0\\0\\\dot\theta_1\end{pmatrix}$$

$${}^1\dot\omega_1 = \cancel{{}^1_0R\,{}^0\dot\omega_0} + \cancel{{}^1_0R\,{}^0\omega_0\times\dot\theta_1\,{}^1\hat Z_1} + \ddot\theta_1\,{}^1\hat Z_1 = \begin{pmatrix}0\\0\\\ddot\theta_1\end{pmatrix}$$

$${}^1\dot v_1 = {}^1_0R\left(\cancel{{}^0\dot\omega_0\times{}^0P_1} + \cancel{{}^0\omega_0\times({}^0\omega_0\times{}^0P_1)} + {}^0\dot v_0\right) = \begin{pmatrix}c_1 & s_1 & 0\\-s_1 & c_1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}g\\0\\0\end{pmatrix} = \begin{pmatrix}c_1\cdot g\\-s_1\cdot g\\0\end{pmatrix}$$

$${}^1\dot v_{C1} = {}^1\dot\omega_1\times{}^1P_{C1} + {}^1\omega_1\times({}^1\omega_1\times{}^1P_{C1}) + {}^1\dot v_1 = \underbrace{\begin{pmatrix}0\\0\\\ddot\theta_1\end{pmatrix}\times\begin{pmatrix}\frac{L_1}{2}\\0\\0\end{pmatrix}} + \begin{pmatrix}0\\0\\\dot\theta_1\end{pmatrix}\times\Big[\underbrace{\begin{pmatrix}0\\0\\\dot\theta_1\end{pmatrix}\times\begin{pmatrix}\frac{L_1}{2}\\0\\0\end{pmatrix}}\Big] + \begin{pmatrix}c_1 g\\-s_1 g\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\\frac{L_1}{2}\cdot\ddot\theta_1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\\dot\theta_1\end{pmatrix}\times\begin{pmatrix}0\\\frac{L_1}{2}\cdot\dot\theta_1\\0\end{pmatrix} + \begin{pmatrix}c_1 g\\-s_1 g\\0\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{L_1}{2}\cdot\dot\theta_1^2 + c_1 g\\\frac{L_1}{2}\cdot\ddot\theta_1 - s_1 g\\0\end{pmatrix}$$

$${}^1F_1 = m\cdot{}^1\dot v_{C_1} = \begin{pmatrix}(-\frac{L_1}{2}\dot\theta_1^2 + c_1 g)\cdot m\\(\frac{L_1}{2}\ddot\theta_1 - s_1 g)\cdot m\\0\end{pmatrix}$$

$${}^1N_1 = {}^{C_1}I_1\cdot{}^1\dot\omega_1 + {}^1\omega_1\times{}^{C_1}I_1\cdot{}^1\omega_1$$

$$= \begin{pmatrix}I_{xx_1} & 0 & 0\\0 & I_{yy_1} & 0\\0 & 0 & I_{zz_1}\end{pmatrix}\cdot\begin{pmatrix}0\\0\\\ddot\theta_1\end{pmatrix} + \begin{pmatrix}0\\0\\\dot\theta_1\end{pmatrix}\times\underbrace{\begin{pmatrix}I_{xx_1} & 0 & 0\\0 & I_{yy_1} & 0\\0 & 0 & I_{zz_1}\end{pmatrix}\begin{pmatrix}0\\0\\\dot\theta_1\end{pmatrix}}$$

$$\begin{pmatrix}0\\0\\\dot\theta_1\end{pmatrix}\times\begin{pmatrix}0\\0\\I_{zz1}\cdot\dot\theta_1\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\I_{zz_1}\cdot\ddot\theta_1\end{pmatrix} + \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\I_{zz_1}\cdot\ddot\theta_1\end{pmatrix}$$

Joint 2 is persmatic

$${}^2\omega_2 = {}^2_1R\,{}^1\omega_1 = \begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}\cdot\begin{pmatrix}0\\0\\\dot\theta_1\end{pmatrix} = \begin{pmatrix}0\\-\dot\theta_1\\0\end{pmatrix}$$

$${}^2\dot\omega_2 = {}^2_1R\,{}^1\dot\omega_1 = \begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}\cdot\begin{pmatrix}0\\0\\\ddot\theta_1\end{pmatrix} = \begin{pmatrix}0\\-\ddot\theta_1\\0\end{pmatrix}$$

$${}^2\dot v_2 = {}^2_1R({}^1\dot\omega_1\times{}^1P_2 + {}^1\omega_1\times({}^1\omega_1\times{}^1P_2) + {}^1\dot v_1) + 2\,{}^2\omega_2\times\dot d_2\,{}^2\hat Z_2 + \ddot d_2\,{}^2\hat Z_2$$

$$= \begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}\left[\underbrace{\begin{pmatrix}0\\0\\\ddot\theta_1\end{pmatrix}\times\begin{pmatrix}L_1\\0\ d_2\\0\end{pmatrix}} + \begin{pmatrix}0\\0\\\dot\theta_1\end{pmatrix}\times\Big[\underbrace{\begin{pmatrix}0\\0\\\dot\theta_1\end{pmatrix}\times\begin{pmatrix}L_1\\0\\0\end{pmatrix}}\Big] + \begin{pmatrix}c_1 g\\-s_1 g\\0\end{pmatrix}\right] + 2\cdot\begin{pmatrix}0\\-\dot\theta_1\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\\dot d_2\end{pmatrix} + \begin{pmatrix}0\\0\\\ddot d_2\end{pmatrix}$$

$$= \left[\begin{pmatrix}0\\L_1\cdot\ddot\theta_1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\\dot\theta_1\end{pmatrix}\times\begin{pmatrix}0\\L_1\cdot\dot\theta_1\\0\end{pmatrix} + \begin{pmatrix}c_1 g\\-s_1 g\\0\end{pmatrix}\right] + \begin{pmatrix}-2\dot\theta_1\cdot\dot d_2\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\\ddot d_2\end{pmatrix}$$

$$= \begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}\begin{pmatrix}-L_1\dot\theta_1^2 + c_1 g\\L_1\ddot\theta_1 - s_1 g\\0\end{pmatrix} + \begin{pmatrix}-2\dot\theta_1\dot d_2\\0\\\ddot d_2\end{pmatrix}$$

$$= \begin{pmatrix}-L_1\dot\theta_1^2 + c_1 g - 2\dot\theta_1\dot d_2\\0\\L_1\ddot\theta_1 - s_1 g + \ddot d_2\end{pmatrix}$$

$$
{}^2\dot{v}_{C_2} = {}^2\dot{\omega}_2 \times {}^2P_{C_2} + {}^2W_2 \times ({}^2W_2 \times {}^2P_{C_2}) + {}^2\dot{v}_2
$$

$$
= \begin{pmatrix} 0 \\ -\ddot{\theta}_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ l_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -\dot{\theta}_1 \\ 0 \end{pmatrix} \times \left[\begin{pmatrix} 0 \\ -\dot{\theta}_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ l_2 \end{pmatrix} \right] + \begin{pmatrix} -L_1\dot{\theta}_1^2 + C_1 g - 2\dot{\theta}_1 \dot{d}_2 \\ 0 \\ L_1\ddot{\theta}_1 - S_1 g + \ddot{d}_2 \end{pmatrix}
$$

$$
= \begin{pmatrix} -l_2\ddot{\theta}_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\dot{\theta}_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -l_2\dot{\theta}_1 \\ 0 \\ 0 \end{pmatrix} + \cdots
$$

$$
= \begin{pmatrix} -l_2\ddot{\theta}_1 - L_1\dot{\theta}_1^2 + C_1 g - 2\dot{\theta}_1\dot{d}_2 \\ 0 \\ -l_2\dot{\theta}_1^2 + L_1\ddot{\theta}_1 - S_1 g + \ddot{d}_2 \end{pmatrix}
$$

$$
{}^2F_2 = m \cdot {}^2\dot{v}_{C_2} = \begin{pmatrix} (-l_2\ddot{\theta}_1 - L_1\dot{\theta}_1^2 + C_1 g - 2\dot{\theta}_1\dot{d}_2)m \\ 0 \\ (-l_2\dot{\theta}_1^2 + L_1\ddot{\theta}_1 - S_1 g + \ddot{d}_2)m \end{pmatrix}
$$

$$
{}^2N_2 = {}^{C_2}I_2 \cdot {}^2\dot{\omega}_2 + {}^2\omega_2 \times {}^{C_2}I_2 \cdot {}^2\omega_2
$$

$$
= \begin{pmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -\ddot{\theta}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\dot{\theta}_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -\dot{\theta}_1 \\ 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 \\ -I_{yy2}\cdot\ddot{\theta}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\dot{\theta}_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -I_{yy2}\cdot\dot{\theta}_1 \\ 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 \\ -I_{yy2}\cdot\ddot{\theta}_1 \\ 0 \end{pmatrix}
$$

$$
{}^2f_2 = \cancel{{}^2_3R\,{}^3f_3} + {}^2F_2 = {}^2F_2 = \begin{pmatrix} (-l_2\ddot{\theta}_1 - L_1\dot{\theta}_1^2 + C_1 g - 2\dot{\theta}_1\dot{d}_2)m_2 \\ 0 \\ (-l_2\dot{\theta}_1^2 + L_1\ddot{\theta}_1 - S_1 g + \ddot{d}_2)m_2 \end{pmatrix}
$$

$$
{}^2n_2 = {}^2N_2 + \cancel{{}^2_3R\,{}^3n_3} + {}^2P_{C_2} \times {}^2F_2 + \cancel{{}^2P_3 \times {}^2_3R\,{}^3f_3}
$$

$$
= \begin{pmatrix} 0 \\ -I_{yy2}\cdot\ddot{\theta}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ l_2 \end{pmatrix} \times \begin{pmatrix} (-l_2\ddot{\theta}_1 - L_1\dot{\theta}_1^2 + C_1 g - 2\dot{\theta}_1\dot{d}_2)m \\ 0 \\ (-l_2\dot{\theta}_1^2 + L_1\ddot{\theta}_1 - S_1 g + \ddot{d}_2)m \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 \\ -I_{yy2}\cdot\ddot{\theta}_1 + l_2 m_2(-l_2\ddot{\theta}_1 - L_1\dot{\theta}_1^2 + C_1 g - 2\dot{\theta}_1\dot{d}_2) \\ 0 \end{pmatrix}
$$

$$
{}^1f_1 = {}^1_2R\,{}^2f_2 + {}^1F_1
$$

$$
= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} (-l_2\ddot{\theta}_1 - L_1\dot{\theta}_1^2 + C_1 g - 2\dot{\theta}_1\dot{d}_2)m_2 \\ 0 \\ (-l_2\dot{\theta}_1^2 + L_1\ddot{\theta}_1 - S_1 g + \ddot{d}_2)m_2 \end{pmatrix} + \begin{pmatrix} (-\frac{L_1}{2}\dot{\theta}_1^2 + C_1 g)\cdot m_1 \\ (\frac{L_1}{2}\ddot{\theta}_1 - S_1 g)\cdot m_1 \\ 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} (-l_2\ddot{\theta}_1 - L_1\dot{\theta}_1^2 + C_1 g - 2\dot{\theta}_1\dot{d}_2)\cdot m_2 + (-\frac{L_1}{2}\dot{\theta}_1^2 + C_1 g)\cdot m_1 \\ (-l_2\dot{\theta}_1^2 + L_1\ddot{\theta}_1 - S_1 g + \ddot{d}_2)\cdot m_2 + (\frac{L_1}{2}\ddot{\theta}_1 - S_1 g)\cdot m_1 \\ 0 \end{pmatrix}
$$

$$
{}^1n_1 = {}^1N_1 + {}^1_2R\,{}^2n_2 + {}^1P_{C_1} \times {}^1F_1 + {}^1P_2 \times {}^1_2R\,{}^2f_2
$$

$$
= \begin{pmatrix} 0 \\ 0 \\ I_{zz1}\cdot\ddot{\theta}_1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -I_{yy2}\cdot\ddot{\theta}_1 + l_2 m_2(-l_2\ddot{\theta}_1 - L_1\dot{\theta}_1^2 + C_1 g - 2\dot{\theta}_1\dot{d}_2) \\ 0 \end{pmatrix}
$$

$$
+ \begin{pmatrix} \frac{L_1}{2} \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} (-\frac{L_1}{2}\dot{\theta}_1^2 + C_1 g)\cdot m_1 \\ (\frac{L_1}{2}\ddot{\theta}_1 - S_1 g)\cdot m_1 \\ 0 \end{pmatrix} + \begin{pmatrix} L_1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} (-l_2\ddot{\theta}_1 - L_1\dot{\theta}_1^2 + C_1 g - 2\dot{\theta}_1\dot{d}_2)m_2 \\ 0 \\ (-l_2\dot{\theta}_1^2 + L_1\ddot{\theta}_1 - S_1 g + \ddot{d}_2)m_2 \end{pmatrix}
$$

$$
\begin{bmatrix} (-l_2\ddot{\theta}_1 - L_1\dot{\theta}_1^2 + C_1 g - 2\dot{\theta}_1\dot{d}_2)m_2 \\ (-l_2\dot{\theta}_1^2 + L_1\ddot{\theta}_1 - S_1 g + \ddot{d}_2)m_2 \\ 0 \end{bmatrix}
$$

$$
= \begin{pmatrix} \\ \\ I_{zz1}\cdot\ddot{\theta}_1 + I_{yy2}\cdot\ddot{\theta}_1 - l_2 m_2(-l_2\ddot{\theta}_1 - L_1\dot{\theta}_1^2 + C_1 g - 2\dot{\theta}_1\dot{d}_2) + \frac{L_1}{2}m_1\cdot(\frac{L_1}{2}\ddot{\theta}_1 - S_1 g) + L_1\cdot m_2(-l_2\dot{\theta}_1^2 + L_1\ddot{\theta}_1 - S_1 g + \ddot{d}_2) \end{pmatrix}
$$

$$\tau_1 = I_{zz1}\cdot\ddot{\theta}_1 + I_{yy2}\cdot\ddot{\theta}_1 - L_2 m_2\left(-L_2\ddot{\theta}_1 - L_1\dot{\theta}_1^2 + C_1 g - 2\dot{\theta}_1\dot{d}_2\right) + \frac{L_1}{2}\cdot m_1\cdot\left(\frac{L_1}{2}\ddot{\theta}_1 - S_1 g\right) + L_1\cdot m_2\left(-L_2\dot{\theta}_1^2 + L_1\ddot{\theta}_1 - S_1 g + \ddot{d}_2\right)$$

$$= \left(I_{zz1} + I_{yy2}\right)\ddot{\theta}_1 + \Big(\cancel{L_2^2 m_2 \ddot{\theta}_1} + \cancel{L_1 L_2 m_2 \dot{\theta}_1^2} - \cancel{L_2 m_2 C_1 g} + \cancel{2 L_2 m_2 \cdot \dot{\theta}_1 \dot{d}_2} + \cancel{\frac{L_1}{4} m_1 \ddot{\theta}_1} - \cancel{\frac{L_1}{2} m_1 S_1 g}$$

$$- \cancel{L_1 L_2 m_2 \dot{\theta}_1^2} + \cancel{L_1^2 m_2 \cdot \ddot{\theta}_1} - \cancel{L_1 m_2 S_1 g} + \cancel{L_1 m_2 \ddot{d}_2}\Big)$$

$$= \left(I_{zz1} + I_{yy2} + L_1^2 m_2 + L_2^2 m_2 + \frac{L_1}{4} m_1\right)\ddot{\theta}_1 + L_1 m_2 \ddot{d}_2 + 2 L_2 m_2 \dot{\theta}_1 \dot{d}_2 + \left(-L_2 m_2 C_1 g - \frac{L_1}{2} m_1 S_1 g - L_1 m_2 S_1 g\right)$$

$$\tau_2 = L_1 m_2 \ddot{\theta}_1 + m_2 \cdot \ddot{d}_2 - L_2 m_2 \dot{\theta}_1^2 - S_1 m_2 g$$

$$M(\theta) = \begin{bmatrix} I_{zz1} + I_{yy2} + L_1^2 m_2 + L_2^2 m_2 + \frac{L_1}{4} m_1 & L_1 m_2 \\ L_1 m_2 & m_2 \end{bmatrix}$$

$$V(\theta, \dot{\theta}) = \begin{bmatrix} 2 L_2 m_2 \dot{\theta}_1 \dot{d}_2 \\ -L_2 m_2 \dot{\theta}_1^2 \end{bmatrix}$$

$$G(\theta) = \begin{bmatrix} -\left(L_2 m_2 C_1 g + \frac{L_1}{2} m_1 S_1 g + L_1 m_2 S_1 g\right) \\ -S_1 m_2 g \end{bmatrix}$$

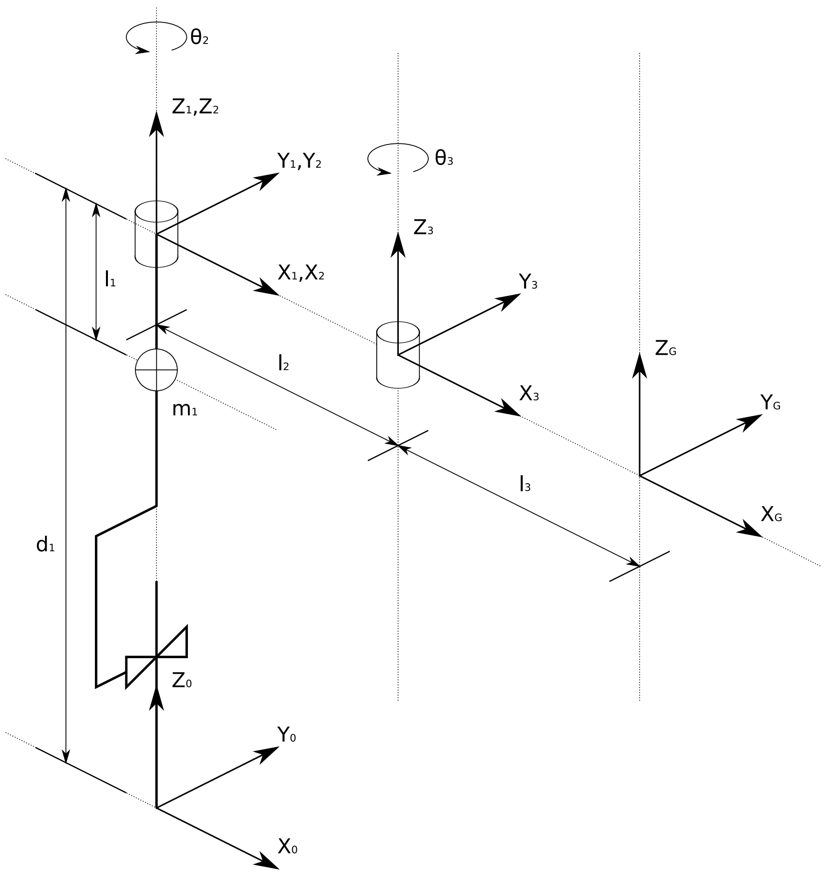

Figure 2: *PRR Robot (Problem 2)*

Problem 2

The manipulator shown in Figure 2 has the following properties:

- Masses of the three links are m_1, m_2, m_3
- Inertia tensors are

$$
{}^{C_1}I_1 = \begin{pmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{pmatrix} \quad {}^{C_2}I_2 = \begin{pmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{pmatrix} \quad {}^{C_3}I_3 = \begin{pmatrix} I_{xx3} & 0 & 0 \\ 0 & I_{yy3} & 0 \\ 0 & 0 & I_{zz3} \end{pmatrix}
$$

- The positions of the centers of mass for the three links are:

$$
{}^0P_{C_1} = \begin{pmatrix} 0 \\ 0 \\ d_1 - l_1 \end{pmatrix}, \quad {}^0P_{C_2} = \begin{pmatrix} \frac{l_2}{2}\mathrm{c}_2 \\ \frac{l_2}{2}\mathrm{s}_2 \\ d_1 \end{pmatrix}, \quad {}^0P_{C_3} = \begin{pmatrix} l_2\mathrm{c}_2 + \frac{l_3}{2}\mathrm{c}_{23} \\ l_2\mathrm{s}_2 + \frac{l_3}{2}\mathrm{s}_{23} \\ d_1 \end{pmatrix}
$$

Furthermore, gravity applies in negative Z_0-direction, as shown. Using the Lagrangian approach to robot dynamics, determine the manipulator dynamic equations in state space (M-V-G) and configuration space (M-B-C-G) form.

$$k_i = \frac{1}{2} m_i V_{c_i}^T \cdot V_{c_i} + \frac{1}{2} {}^i w_i^T \cdot {}^{c_i}I_i \cdot {}^i w_i$$

$${}^0V_{c_i} = \frac{d\, {}^0P_{c_i}}{dt}$$

$${}^0V_{c_1} = \begin{pmatrix} 0 \\ 0 \\ \dot{d}_1 \end{pmatrix} \quad {}^0V_{c_2} = \begin{pmatrix} -\frac{L_2}{2} s_2 \cdot \dot{\theta}_2 \\ \frac{L_2}{2} c_2 \cdot \dot{\theta}_2 \\ \dot{d}_1 \end{pmatrix} \quad {}^0V_{c_3} = \begin{pmatrix} -L_2 s_2 \cdot \dot{\theta}_2 - \frac{L_3}{2} s_{23} (\dot{\theta}_2 + \dot{\theta}_3) \\ L_2 c_2 \dot{\theta}_2 + \frac{L_3}{2} c_{23} (\dot{\theta}_2 + \dot{\theta}_3) \\ \dot{d}_1 \end{pmatrix}$$

$${}^1w_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad {}^2w_2 = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix} \quad {}^3w_3 = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{pmatrix}$$

$$k_1 = \frac{1}{2} m_1 \cdot (0\ 0\ \dot{d}_1) \begin{pmatrix} 0 \\ 0 \\ \dot{d}_1 \end{pmatrix} = \frac{1}{2} m_1 \dot{d}_1^2$$

$$k_2 = \frac{1}{2} m_2 \cdot \left(\frac{L_2^2}{4} s_2^2 \cdot \dot{\theta}_2^2 + \frac{L_2^2}{4} c_2^2 \cdot \dot{\theta}_2^2 + \dot{d}_1^2 \right) + \frac{1}{2} (0\ 0\ \dot{\theta}_2) \begin{pmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix}$$

$$= \frac{1}{2} m_2 \left(\frac{L_2^2}{4} \dot{\theta}_2^2 + \dot{d}_1^2 \right) + \frac{1}{2} (0\ 0\ I_{zz2} \cdot \dot{\theta}_2) \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix}$$

$$= \frac{1}{2} m_2 \left(\frac{L_2^2}{4} \dot{\theta}_2^2 + \dot{d}_1^2 \right) + \frac{1}{2} I_{zz2} \cdot \dot{\theta}_2^2$$

$$k_3 = \frac{1}{2} m_3 \cdot \Big[L_2^2 s_2^2 \dot{\theta}_2^2 + L_2 L_3 s_2 s_{23} (\dot{\theta}_2^2 + \dot{\theta}_2 \dot{\theta}_3) + \frac{L_3^2}{4} s_{23}^2 (\dot{\theta}_2 + \dot{\theta}_3)^2$$
$$+ L_2^2 c_2^2 \cdot \dot{\theta}_2^2 + L_2 L_3 c_2 c_{23} (\dot{\theta}_2^2 + \dot{\theta}_2 \dot{\theta}_3) + \frac{L_3^2}{4} c_{23}^2 (\dot{\theta}_2 + \dot{\theta}_3)^2$$
$$+ \dot{d}_1^2 \Big] + \frac{1}{2} I_{zz3} (\dot{\theta}_2 + \dot{\theta}_3)^2$$

$$= \frac{1}{2} m_3 \left[L_2^2 \dot{\theta}_2^2 + \frac{L_3^2}{4} (\dot{\theta}_2 + \dot{\theta}_3)^2 + L_2 L_3 c_3 (\dot{\theta}_2^2 + \dot{\theta}_2 \cdot \dot{\theta}_3) + \dot{d}_1^2 \right] + \frac{1}{2} I_{zz3} (\dot{\theta}_2 + \dot{\theta}_3)^2$$

$$u_i = -m_i\, {}^0g^T\, {}^0P_{c_i} + u_{ref_i}$$

$$u_1 = -m_1 \cdot (0\ 0\ -g) \begin{pmatrix} 0 \\ 0 \\ d_1 - L_1 \end{pmatrix}$$

$$= m_1 \cdot g \cdot (d_1 - L_1)$$

$$u_2 = -m_2 (0\ 0\ -g) \begin{pmatrix} \frac{L_2}{2} c_2 \\ \frac{L_2}{2} s_2 \\ d_1 \end{pmatrix}$$

$$= m_2 g d_1$$

$$u_3 = m_3 g d_1$$

$$\tau_i = \frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_i} - \frac{\partial k}{\partial \theta_i} + \frac{\partial u}{\partial \theta_i}$$

$$\tau_1 = (m_1+m_2+m_3)(\ddot{d}_1+g)$$

$$\tau_2 = \frac{L_2^2}{4} m_2 \ddot{\theta}_2 + I_{zz2} \cdot \ddot{\theta}_2 + m_3 L_2^2 \ddot{\theta}_2$$