FUNDAMENTALS OF ARTIFICIAL INTELLIGENCE
Exercise 9: Hidden Markov Models – Problems | Winter Semester 2023/24
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1 Presented Problem

Problem 9.1:

(Adapted from [1] Ex. 15.13, 15.14 and 15.15) A professor wants to know if students are getting enough sleep. Each day, the professor observes whether the students sleep in class, and whether they have red eyes. The professor has the following domain theory:

- The prior probability of getting enough sleep, with no observations, is 0.7.
- The probability of getting enough sleep on night t is 0.8 given that the student got enough sleep the previous night, and 0.3 if not.
- The probability of having red eyes is 0.2 if the student got enough sleep, and 0.7 if not.
- The probability of sleeping in class is 0.1 if the student got enough sleep, and 0.3 if not.

a. Formulate this information as an Hidden Markov Model that has only one observation variable. Give the complete probability tables for the model.

Consider the following evidence values:

- $\mathbf{e}_1$ = not red eyes, not sleeping in class;
- $\mathbf{e}_2$ = red eyes, not sleeping in class;
- $\mathbf{e}_3$ = red eyes, sleeping in class.

b. State estimation: Compute $P(EnoughSleep_t|\mathbf{e}_{1:t})$ for each $t = 1,2,3$.
c. Smoothing: Compute $P(EnoughSleep_t|\mathbf{e}_{1:3})$ for each $t = 1,2,3$.
d. Find the most likely state sequence.

Consider that a student shows up with red eyes and sleeps every day in the class.

e. The probability that the student had enough sleep does not go to zero but converges to a fixed point. What does the point represent?
f. Calculate analytically the fixed point.

1 Presented Problem

Problem 9.1:

ES

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SC RE

- The prior probability of getting enough sleep, with no observations, is 0.7. $p(es) = 0.7$
- The probability of getting enough sleep on night t is 0.8 given that the student got enough sleep the previous night, and 0.3 if not.
- The probability of having red eyes is 0.2 if the student got enough sleep, and 0.7 if not.
- The probability of sleeping in class is 0.1 if the student got enough sleep, and 0.3 if not.

$p(es_t \mid es_{t-1}) = 0.8$

$p(es_t \mid \neg es_{t-1}) = 0.3$

$p(re \mid es) = 0.2$

$p(re \mid \neg es) = 0.7$

$p(sc \mid es) = 0.1$

$p(sc \mid \neg es) = 0.3$

a. Formulate this information as an Hidden Markov Model that has only one observation variable. Give the complete probability tables for the model.

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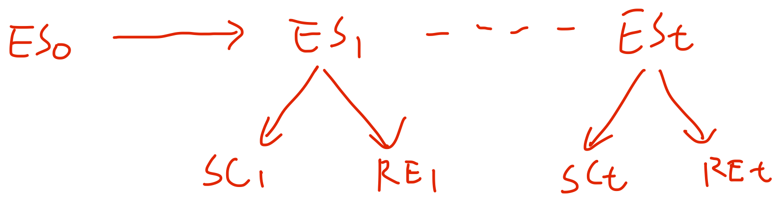
a.

$p(ES_0)$
0.7

ES_{t-1}	$p(es_t \mid ES_{t-1})$
t	0.8
f	0.3



$E_t = SC_t \wedge RE_t$

ES_t	$P(sc_t \wedge re_t \mid ES_t)$	$P(sc_t \wedge \neg re_t \mid ES_t)$	$P(\neg sc_t \wedge re_t \mid ES_t)$	$P(\neg sc_t \wedge \neg re_t \mid ES_t)$
t	$0.2 \cdot 0.1 = 0.02$	0.08	0.18	0.72
f	$0.7 \cdot 0.3 = 0.21$	0.04	0.49	0.21

Consider the following evidence values:

- $\mathbf{e}_1$ = not red eyes, not sleeping in class;
- $\mathbf{e}_2$ = red eyes, not sleeping in class;
- $\mathbf{e}_3$ = red eyes, sleeping in class.

$f_{1:t} = \alpha \cdot O_t \; T \cdot f_{1:t-1}$

b. State estimation: Compute $P(EnoughSleep_t|\mathbf{e}_{1:t})$ for each $t = 1, 2, 3$.

$$P(ES_t | e_{1:t}) = P(ES_t / e_{1:t-1}, e_t)$$
$$= \alpha\, p(e_t / ES_t, \cancel{e_{1:t-1}}) \cdot p(ES_t / e_{1:t-1})$$
$$= \alpha\, p(e_t | ES_t) \cdot \sum_{ES_{t-1}} p(ES_t, ES_{t-1} | e_{1:t-1})$$
$$= \alpha\, p(e_t | ES_t) \cdot \sum_{ES_{t-1}} p(ES_t | ES_{t-1}, \cancel{e_{1:t-1}}) \cdot p(ES_{t-1} | e_{1:t-1})$$
$$= \alpha\, \underbrace{p(e_t | ES_t)}_{O_t} \cdot \sum_{ES_{t-1}} \underbrace{p(ES_t | ES_{t-1})}_{T} \cdot \underbrace{p(ES_{t-1} | e_{1:t-1})}_{f_{1:t-1}}$$

$$T = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \quad O_1 = \begin{bmatrix} 0.72 & 0 \\ 0 & 0.21 \end{bmatrix} \quad f_0 = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$$

$$O_2 = \begin{bmatrix} 0.18 & 0 \\ 0 & 0.49 \end{bmatrix}$$

$$O_3 = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.21 \end{bmatrix}$$

$$f_{1:1} = \alpha \cdot \begin{bmatrix} 0.72 & 0 \\ 0 & 0.21 \end{bmatrix} \cdot \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = \alpha \cdot \begin{bmatrix} 0.468 \\ 0.0735 \end{bmatrix}$$
$$\approx \begin{bmatrix} 0.8643 \\ 0.1357 \end{bmatrix}$$

$$f_{1:2} = \alpha \begin{bmatrix} 0.18 & 0 \\ 0 & 0.49 \end{bmatrix} \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.8643 \\ 0.1357 \end{bmatrix} = \alpha \begin{bmatrix} 0.1318 \\ 0.1313 \end{bmatrix}$$
$$\approx \begin{bmatrix} 0.5010 \\ 0.4990 \end{bmatrix}$$

$$f_{1:3} = \alpha \begin{bmatrix} 0.02 & 0 \\ 0 & 0.21 \end{bmatrix} \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.5010 \\ 0.4990 \end{bmatrix} \approx \alpha \begin{bmatrix} 0.0110 \\ 0.0944 \end{bmatrix}$$
$$\approx \begin{bmatrix} 0.1044 \\ 0.8956 \end{bmatrix}$$

Consider the following evidence values:

- $\mathbf{e}_1$ = not red eyes, not sleeping in class;
- $\mathbf{e}_2$ = red eyes, not sleeping in class;
- $\mathbf{e}_3$ = red eyes, sleeping in class.

$$b_{k+1:t} = T^T O_{k+1} \cdot b_{k+2:t}$$

b. State estimation: Compute $P(EnoughSleep_t | \mathbf{e}_{1:t})$ for each $t = 1, 2, 3$.

c. Smoothing: Compute $P(EnoughSleep_t | \mathbf{e}_{1:3})$ for each $t = 1, 2, 3$.

c. $P(ES_k \mid e_{1:t}) = P(ES_k \mid e_{1:k}, e_{k+1:t})$

$$= \alpha\, p(e_{k+1:t}, ES_k, e_{1:k})$$

$$= \alpha \cdot \underbrace{p(e_{k+1:t} \mid ES_k)}_{b_{k+1:t}}\, \underbrace{p(ES_k \mid e_{1:k})}_{f_{1:k}} \cdot \cancel{p(e_{1:k})}$$

$$\sum_{ES_{k+1}} p(e_{k+1:t}, ES_{k+1} \mid ES_k)$$

$$= \sum_{ES_{k+1}} p(e_{k+1:t} \mid ES_{k+1}, \cancel{ES_k}) \cdot p(ES_{k+1} \mid ES_k)$$

$$= \sum_{ES_{k+1}} \underbrace{p(e_{k+1} \mid ES_{k+1})}_{O_{k+1}}\, \underbrace{p(e_{k+2:t} \mid ES_{k+1})}_{b_{k+2:t}} \cdot \underbrace{p(ES_{k+1} \mid ES_k)}_{T}$$

$$b_{k+1:t} = T^T O_{k+1} \cdot b_{k+2:t}$$

$t=3.\quad k=2$ $\qquad b_{4:3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$b_{3:3} = T^T O_3 \cdot b_{4:3}$$

$$= \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.02 & 0 \\ 0 & 0.21 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.058 \\ 0.153 \end{bmatrix}$$

$k=1$

$$b_{2:3} = T^T O_2\, b_{3:3}$$

$$= \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.18 & 0 \\ 0 & 0.49 \end{bmatrix} \begin{bmatrix} 0.058 \\ 0.153 \end{bmatrix} = \begin{bmatrix} 0.0233 \\ 0.0556 \end{bmatrix}$$

$$p(ES_2 \mid e_{1:3}) = \alpha\, b_{3:3} \times f_{1:2} = \alpha \cdot \begin{bmatrix} 0.058 \\ 0.153 \end{bmatrix} \times \begin{bmatrix} 0.5010 \\ 0.4990 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.0291 \\ 0.0763 \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.2761 \\ 0.7239 \end{bmatrix}$$

$$P(ES_1 | e_{1:3}) = \alpha\, b_{2:3} \times f_{1:1} = \alpha \begin{bmatrix} 0.0233 \\ 0.0556 \end{bmatrix} \begin{bmatrix} 0.8643 \\ 0.1357 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0.0201 \\ 0.0075 \end{bmatrix}$$

$$T = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \qquad \approx \begin{bmatrix} 0.7283 \\ 0.2717 \end{bmatrix}$$

d. Find the most likely state sequence.

$$\max P(x_1 \cdots x_t, x_{t+1} | e_{1:t+1}) = \alpha\, p(e_{t+1} | x_{t+1}) \cdot \max \left[P(x_{t+1} | x_t) \cdot \underline{\max P(x_1 \cdots x_t | e_{1:t})} \right]$$

$$\mu_1(ES_1) = P(ES_1 | e_1) = f_{1:1} = \begin{bmatrix} 0.8643 \\ 0.1357 \end{bmatrix}$$

$$\mu_2(ES_2) = \cdot\, p(e_2 | ES_2) \cdot \max \left[P(ES_2 | ES_1) \cdot \mu_1(ES_1) \right]$$

$$= \cdot \begin{bmatrix} 0.18 & 0 \\ 0 & 0.49 \end{bmatrix} \max \left\langle \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} \cdot 0.8643, \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} \cdot 0.1357 \right\rangle$$

$$\max \left\langle \underline{\begin{bmatrix} 0.6914 \\ 0.1729 \end{bmatrix}}, \begin{bmatrix} 0.0407 \\ 0.0950 \end{bmatrix} \right\rangle$$

$$= \begin{bmatrix} 0.1245 \\ 0.0847 \end{bmatrix}$$

$$\mu_3(ES_3) = p(e_3 | ES_3) \cdot \max \left[p(ES_3 | ES_2) \cdot \mu_2(ES_2) \right]$$

$$= \begin{bmatrix} 0.02 & 0 \\ 0 & 0.21 \end{bmatrix} \cdot \max \left\langle \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} \cdot 0.1245, \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} \cdot 0.0847 \right\rangle$$

$$\max \left\langle \begin{bmatrix} \underline{0.0996} \\ 0.0249 \end{bmatrix}, \begin{bmatrix} 0.0254 \\ \underline{0.0593} \end{bmatrix} \right\rangle$$

$$= \begin{bmatrix} 0.0020 \\ 0.0125 \end{bmatrix}$$

Consider that a student shows up with red eyes and sleeps every day in the class.

steady state

e. The probability that the student had enough sleep does not go to zero but converges to a fixed point. What does the point represent?

f. Calculate analytically the fixed point.

2 Additional Problem

Problem 9.2:

Consider a simplified localization problem with a mobile robot that is randomly placed in an environment with a known map (see Fig.1). The robot has a sensing system composed of a compass and four proximity sensors. Each sensor aims at detecting the presence of a wall (solid lines in the map) in a different direction: North (N), South (S), West (W) and East (E).

The map is discrete and the robot can be in one of the 6 sections at each time step. At each subsequent time step the robot changes section with a probability of 80% and moves randomly to a different free neighboring section. As soon as the robot enters a new section or decides to stay in the current one, it takes a measurement with its proximity sensors. Each sensor has a false positive rate of 10% and a false negative rate of 0%. The sensor values are conditionally independent given the position of the robot.
(Hint: A false positive rate of 10% in this context means that a wall is detected in a direction without a wall 1 out of 10 times; a false negative rate of 0% means that given that there is wall in a direction, the wall is certainly detected).

Suppose that we can not see the robot and the only information that it transmits are the sensed walls in the respective directions.

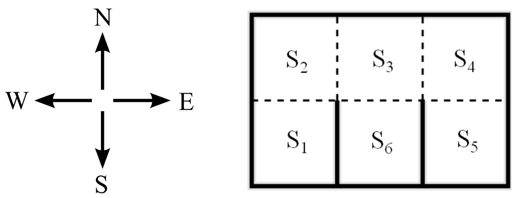


Figure 1: the map.

a. Formulate an Hidden Markov Model for this system. Provide the matrices for the state transition ($\mathbf{T}$) and for the observations ($\mathbf{O}_k$). For the purpose of this exercise, consider only the cases that the evidence indicates walls in the South-West-East (SWE) and North-East (NE).

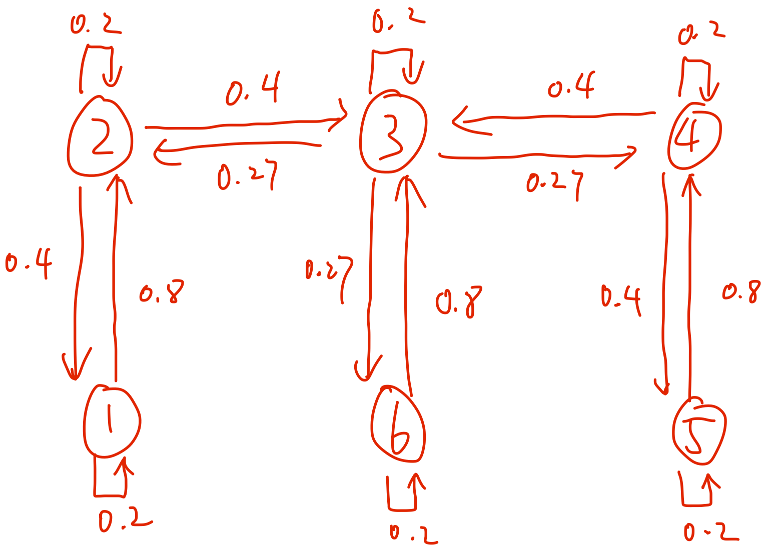
b. Suppose that we establish the communication with the robot after its first action (time step $k = 1$). The robot senses $\mathbf{e}_1 = SWE$ and sends us this information. Estimate the location of the robot with the corresponding probabilities.

c. Predict the location of the robot with the corresponding probabilities for $k = 2$.

d. Suppose that for $k = 2$ we receive the information: $\mathbf{e}_2 = NE$. Estimate the location of the robot with the corresponding probabilities.

References

[1] S. Russell and P. Norvig, *Artificial Intelligence: A Modern Approach*. Prentice Hall, 2010.



$$f_0 = p(X_0) = \left[\frac{1}{6} \; \frac{1}{6} \; \frac{1}{6} \; \frac{1}{6} \; \frac{1}{6} \; \frac{1}{6} \right]$$

$$T = \begin{bmatrix} 0.2 & 0.4 & 0 & 0 & 0 & 0 \\ 0.8 & 0.2 & 0.27 & 0 & 0 & 0 \\ 0 & 0.4 & 0.2 & 0.4 & 0 & 0.8 \\ 0 & 0 & 0.27 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0.4 & 0.2 & 0 \\ 0 & 0 & 0.27 & 0 & 0 & 0.2 \end{bmatrix}$$