

**Figure 1:** 4R Robot (Problem 1)

## Problem 1

In this problem, we are working with the same robot as in problem 10 of the first problem sheet, which is shown again in Figure 1. This time, the robot is equipped with a force-torque-sensor that has system  $\{4\}$  as frame of reference. An external force is applied to the robot, such that its force-torque sensor reports a measurement of

$$\begin{pmatrix} {}^4f \\ {}^4n \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 0 \\ 7 \\ 0 \\ 8 \end{pmatrix}.$$

- Determine the joint torques that are required to cancel out the external influences and thus keep the robot static.
- Assume now that there is a screwdriver attached to the last link, and the tip of the screwdriver is translated along the  $z$ -axis about 9 length units, so  ${}^4P_{\text{tip}} = (0, 0, 9)^T$ . With the same force-torque measurement reported by the sensor in system 4, which forces and torques are present at the screwdriver tip? Which forces and torques are caused by the robot in direction of the screw driver (i.e., in  $Z$  direction)?

$${}^0_1T = \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^1_2T = \begin{pmatrix} c_2 & -s_2 & 0 & 1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^2_3T = \begin{pmatrix} c_3 & -s_3 & 0 & 0 \\ \frac{\sqrt{2}}{2}s_3 & \frac{\sqrt{2}}{2}c_3 & -\frac{\sqrt{2}}{2} & -1 \\ \frac{\sqrt{2}}{2}s_3 & \frac{\sqrt{2}}{2}c_3 & \frac{\sqrt{2}}{2} & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^3_4T = \begin{pmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$i$	$a_{i-1}$	$\alpha_{i-1}$	$d_i$	$\theta_i$
1	0	$0^\circ$	0	$\theta_1$
2	1	$0^\circ$	0	$\theta_2$
3	0	$45^\circ$	$\sqrt{2}$	$\theta_3$
4	$\sqrt{2}$	$0^\circ$	0	0

$${}^0_2T = \begin{pmatrix} c_{12} & -s_{12} & 0 & c_1 \\ c_1(c_2 - s_1 s_2) & -c_1 s_2 - s_1 c_2 & 0 & s_1 \\ s_1(c_2 + c_1 s_2 s_{12}) & -s_1 s_2 + c_1 c_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^0_3T = \begin{pmatrix} c_{12}c_3 - \frac{\sqrt{2}}{2}s_{12}s_3 & -c_{12}s_3 - \frac{\sqrt{2}}{2}s_{12}c_3 & \frac{\sqrt{2}}{2}s_{12} & s_{12} + c_1 \\ s_{12}c_3 + \frac{\sqrt{2}}{2}c_{12}s_3 & -s_{12}s_3 + \frac{\sqrt{2}}{2}c_{12}c_3 & -\frac{\sqrt{2}}{2}c_{12} & -c_{12} + s_1 \\ \frac{\sqrt{2}}{2}s_3 & \frac{\sqrt{2}}{2}c_3 & \frac{\sqrt{2}}{2} & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^0_4T = \begin{pmatrix} s_{12} + c_1 + \sqrt{2}(c_{12}c_3 - s_{12}s_3) & -c_{12} + s_1 + \sqrt{2}(s_{12}c_3 + c_{12}s_3) & s_3 + 1 & 1 \end{pmatrix}$$

$$\angle + 1 = 1.707$$

$${}^4f_4 = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} \quad {}^4n_4 = \begin{pmatrix} 7 \\ 0 \\ 8 \end{pmatrix} \quad \text{静力迭代}$$

$${}^3f_3 = {}^3R_4 \cdot {}^4f_4 = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}$$

$${}^2f_2 = {}^2R_3 \cdot {}^3f_3 = \begin{pmatrix} -6s_3 \\ 3\sqrt{2}c_3 \\ 3\sqrt{2}c_3 \end{pmatrix}$$

$${}^1f_1 = {}^1R_2 \cdot {}^2f_2 = \begin{pmatrix} -6s_3c_2 - 3\sqrt{2}c_3s_2 \\ -6s_3s_2 + 3\sqrt{2}c_3c_2 \\ 3\sqrt{2}c_3 \end{pmatrix}$$

$${}^3n_3 = {}^3R_4 \cdot {}^4n_4 + {}^3P_4 \times {}^5f_3$$

$$= \begin{pmatrix} 7 \\ 0 \\ 8 \end{pmatrix} + \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 0 \\ 8+6\sqrt{2} \end{pmatrix}$$

$${}^2n_2 = {}^2R_3 \cdot {}^3n_3 + {}^2P_3 \times {}^2f_2$$

$$= \begin{pmatrix} 7c_3 \\ \frac{7\sqrt{2}}{2}s_3 - \frac{\sqrt{2}}{2}(8+6\sqrt{2}) \\ \frac{7\sqrt{2}}{2}s_3 + \frac{\sqrt{2}}{2}(8+6\sqrt{2}) \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -6s_3 \\ 3\sqrt{2}c_3 \\ 3\sqrt{2}c_3 \end{pmatrix}$$

$${}^i f_i = {}^i R_{i+1} \cdot {}^{i+1} f_{i+1}$$

$${}^i n_i = {}^i R_{i+1} \cdot {}^{i+1} n_{i+1} + {}^i P_{i+1} \times {}^i f_i$$

$$+ \begin{pmatrix} -6\sqrt{2}c_3 \\ -6s_3 \\ -6s_3 \end{pmatrix}$$

$$= \begin{pmatrix} 7c_3 - 6\sqrt{2}c_3 \\ \frac{7\sqrt{2}}{2}s_3 - \frac{\sqrt{2}}{2}(8+6\sqrt{2}) - 6s_3 \\ \frac{7\sqrt{2}}{2}s_3 + \frac{\sqrt{2}}{2}(8+6\sqrt{2}) - 6s_3 \end{pmatrix}$$

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$${}^1n_1 = {}^1R_2 \cdot {}^2n_2 + {}^1P_2 \times {}^1f_1$$

$$= \begin{pmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot {}^2n_2 + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times {}^1f_1$$

$$= \begin{pmatrix} (7-6\sqrt{2})c_3 \cdot c_2 - (\frac{7\sqrt{2}}{2}-6)s_3s_2 + \frac{\sqrt{2}}{2}(8+6\sqrt{2})s_2 \\ (7-6\sqrt{2})c_3 \cdot s_2 + (\frac{7\sqrt{2}}{2}-6)s_3c_2 - \frac{\sqrt{2}}{2}(8+6\sqrt{2})c_2 \\ (\frac{7\sqrt{2}}{2}-6)s_3 + \frac{\sqrt{2}}{2}(8+6\sqrt{2}) \end{pmatrix} + \begin{pmatrix} 0 \\ -3\sqrt{2}c_3 \\ -6s_3s_2 + 3\sqrt{2}c_3c_2 \end{pmatrix}$$

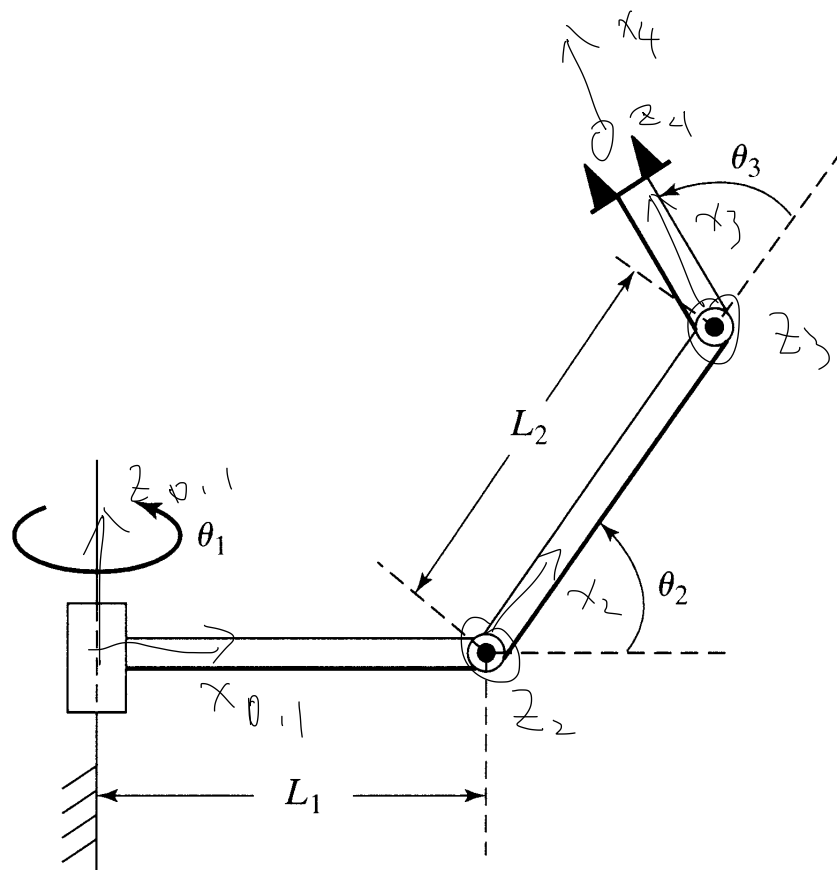
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$$\Theta = [0^\circ, 90^\circ, -90^\circ, 0^\circ].$$

$$\tau_i = {}^i n_i^T \cdot {}^i \hat{z}_1$$

$$\tau = \begin{pmatrix} (\frac{7\sqrt{2}}{2}-6)s_3 + (3\sqrt{2}-6)c_{23} + \frac{\sqrt{2}}{2}(8+6\sqrt{2}) \\ (\frac{7\sqrt{2}}{2}-6)s_3 + \frac{\sqrt{2}}{2}(8+6\sqrt{2}) \\ 8+6\sqrt{2} \\ 8 \end{pmatrix}$$

$$\tau = \begin{pmatrix} {}^1n_{1z} \\ {}^2n_{2z} \\ {}^3n_{3z} \\ {}^4n_{4z} \end{pmatrix} = \begin{pmatrix} -6s_2s_{31} + (\frac{7\sqrt{2}}{2}-6)s_{31} + 3\sqrt{2}c_2c_{31} + 6c_1 + 4\sqrt{2} \\ (\frac{7\sqrt{2}}{2}-6)s_{31} + 6c_1 + 4\sqrt{2} \\ 8+6\sqrt{2}c_1 \\ 8 \end{pmatrix}$$



**Figure 2:** 3R Robot (Problem 2)

## Problem 2

For the robot shown in Figure 2, determine the Jacobian w.r.t. reference frame  $\{4\}$  using three different approaches:

- Compute velocities in system 4, and derive the Jacobian
- Compute force-torque relations for system 4, derive the Jacobian
- Geometric observations

P2

j	$a_{i-1}$	$d_{i-1}$	$d_i$	$\theta_i$
1	0	0	0	$\theta_1$
2	$L_1$	$a_0^0$	0	$\theta_2$
3	$L_2$	0	0	$\theta_3$
4	$L_3$	0	0	0

$${}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & L_1 \\ 0 & 0 & -1 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2_3T = \begin{bmatrix} c_3 & -s_3 & 0 & L_2 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^3_4T = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{i+1}_i w_{i+1} = {}^{i+1}_i R {}^i w_i + \dot{\theta}_{i+1} \cdot {}^{i+1}_i \hat{z}_{i+1}$$

$${}^{i+1}_i v_{i+1} = {}^{i+1}_i R ({}^i v_i + {}^i w_i \times {}^i p_{i+1})$$

$${}^0 v_0 = 0 \quad {}^0 w_0 = 0.$$

$${}^1 v_1 = {}^1_0 R ({}^0 v_0 + {}^0 w_0 \times {}^0 p_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$${}^1 w_1 = {}^1_0 R {}^0 w_0 + \dot{\theta}_1 \cdot {}^1 \hat{z}_1 = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} \times \begin{pmatrix} L_1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} {}^2 v_2 &= {}^2_1 R ({}^1 v_1 + {}^1 w_1 \times {}^1 p_2) \\ &= \begin{pmatrix} c_2 & 0 & s_2 \\ -s_2 & 0 & c_2 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ L_1 \cdot \dot{\theta}_1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ -L_1 \cdot \dot{\theta}_1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} {}^2 w_2 &= {}^2_1 R {}^1 w_1 + \dot{\theta}_2 \cdot {}^2 \hat{z}_2 \\ &= \begin{pmatrix} c_2 & 0 & s_2 \\ -s_2 & 0 & c_2 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix} \\ &= \begin{pmatrix} s_2 \cdot \dot{\theta}_1 \\ c_2 \cdot \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} {}^3 v_3 &= {}^3_2 R ({}^2 v_2 + {}^2 w_2 \times {}^2 p_3) \\ &= \begin{pmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left[ \begin{pmatrix} 0 \\ 0 \\ -L_1 \cdot \dot{\theta}_1 \end{pmatrix} + \begin{pmatrix} s_2 \cdot \dot{\theta}_1 \\ c_2 \cdot \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} L_2 \\ 0 \\ 0 \end{pmatrix} \right] \\ &= \begin{pmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ L_1 \cdot \dot{\theta}_2 \\ -L_1 \cdot \dot{\theta}_1 - L_2 \cdot c_2 \cdot \dot{\theta}_1 \end{pmatrix} \\ &= \begin{pmatrix} L_2 s_3 \dot{\theta}_2 \\ L_2 c_3 \dot{\theta}_2 \\ -L_1 \cdot \dot{\theta}_1 - L_2 \cdot c_2 \cdot \dot{\theta}_1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} {}^3 w_3 &= {}^3_2 R {}^2 w_2 + \dot{\theta}_3 \cdot {}^3 \hat{z}_3 \\ &= \begin{pmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_2 \cdot \dot{\theta}_1 \\ c_2 \cdot \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{pmatrix} \\ &= \begin{pmatrix} s_2 c_3 \dot{\theta}_1 + c_2 s_3 \dot{\theta}_1 \\ -s_2 s_3 \dot{\theta}_1 + c_2 c_3 \dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} {}^4 v_4 &= {}^4_3 R ({}^3 v_3 + {}^3 w_3 \times {}^3 p_4) \\ &= \begin{pmatrix} L_2 s_3 \dot{\theta}_2 \\ L_2 c_3 \dot{\theta}_2 \\ -L_1 \cdot \dot{\theta}_1 - L_2 \cdot c_2 \cdot \dot{\theta}_1 \end{pmatrix} + \begin{pmatrix} s_2 c_3 \dot{\theta}_1 + c_2 s_3 \dot{\theta}_1 \\ -s_2 s_3 \dot{\theta}_1 + c_2 c_3 \dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{pmatrix} \times \begin{pmatrix} L_3 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} L_2 s_3 \dot{\theta}_2 \\ L_2 c_3 \dot{\theta}_2 + L_3 (\dot{\theta}_2 + \dot{\theta}_3) \\ -L_1 \cdot \dot{\theta}_1 - L_2 \cdot c_2 \cdot \dot{\theta}_1 - L_3 (c_2 c_3 \dot{\theta}_1 - s_2 s_3 \dot{\theta}_1) \end{pmatrix} \end{aligned}$$

$$\begin{aligned} {}^4 w_4 &= {}^4_3 R {}^3 w_3 \\ &= \begin{pmatrix} s_2 c_3 \dot{\theta}_1 \\ c_2 c_3 \dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{pmatrix} \end{aligned}$$

$$V = \begin{pmatrix} L_2 S_3 \dot{\theta}_2 \\ (L_2 C_3 + L_3) \dot{\theta}_2 + L_3 \dot{\theta}_3 \\ (-L_1 - L_2 C_2 - L_3 C_{23}) \dot{\theta}_1 \\ S_{23} \dot{\theta}_1 \\ C_{23} \dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{pmatrix} = {}^4J \cdot \dot{\theta}$$

$${}^4J = \begin{pmatrix} 0 & L_2 S_3 & 0 \\ 0 & L_2 C_3 + L_3 & L_3 \\ -L_1 - L_2 C_2 - L_3 C_{23} & 0 & 0 \\ S_{23} & 0 & 0 \\ C_{23} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow {}^4J = \begin{pmatrix} 0 & L_2 S_3 & 0 \\ 0 & L_2 C_3 + L_3 & L_3 \\ -(L_1 + L_2 C_2 + L_3 C_{23}) & 0 & 0 \\ S_{23} & 0 & 0 \\ C_{23} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$