Recapitulation: Manipulator Control

On the previous problem sheet, we discussed some control theory, and we already learned quite well to control mass-spring models. We were able to critically damp mass-spring systems themselves, as well as the error when following a trajectory. We have also seen the usefulness of the approach of partitioning a controller into a servo portion and a model-based portion.

In the case of problem 2 of sheet 5, the equations of motion turned out to be multidimensional linear differential equations of type

$$M\ddot{x} + B\dot{x} + Kx = f,$$

where f is an external force applied to the objects in the system. Choosing $\alpha = M, \beta = B\dot{x} + Kx$, the model-based portion was then

$$f = \alpha f' + \beta,$$

while the servo portion for trajectory following was

$$f' = -K_v\dot{e} - K_p e.$$

We have seen that the decoupling scheme greatly simplifies controlling such multi-dimensional linear systems. How could it be applied to a robot?

By using either the Newton-Euler or the Lagrange method, we are already able to determine equations of motion of a robot. The equations, formulated in state-space form, generally look like this:

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

This is a multi-dimensional system, but it's not even linear, unlike the system we have encountered last time. Even though we have not discussed this case yet, it turns out that dealing with a nonlinear system is no problem at all when applying the partitioning scheme. Partitioning works exactly like before (let $\alpha = M(\Theta)$, $\beta =$"everything else") and leads to an easily controllable system. Applying the well-known partitioning scheme

$$\tau = \alpha\tau' + \beta$$

with $\alpha = M(\Theta)$, $\beta = V(\Theta, \dot{\Theta}) + G(\Theta)$ to our system, we can use the servo control law

$$\tau' = \ddot{\Theta}_d + K_v(\dot{\Theta}_d - \dot{\Theta}) + K_p(\Theta_d - \Theta)$$

to control our robot. Here, Θ_d is the vector of desired joint positions. The expression $(\Theta_d - \Theta)$ will now be abbreviated as E. Inserting above values into the state space equation, we obtain:

$$\begin{aligned}
M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta) &= M(\Theta)(\ddot{\Theta}_d + K_v\dot{E} + K_pE) + V(\Theta, \dot{\Theta}) + G(\Theta)\\
0 &= M(\Theta)(\ddot{\Theta}_d - \ddot{\Theta} + K_v\dot{E} + K_pE)\\
0 &= \ddot{E} + K_v\dot{E} + K_pE
\end{aligned}$$

This is the so-called error equation. Again, we choose K_v and K_p to be diagonal:

$$K_v = \begin{pmatrix} k_{v1} & 0 & 0 & \cdots & 0\\ 0 & k_{v2} & 0 & \cdots & 0\\ 0 & 0 & \ddots & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & k_{vn} \end{pmatrix}, \quad K_p = \begin{pmatrix} k_{p1} & 0 & 0 & \cdots & 0\\ 0 & k_{p2} & 0 & \cdots & 0\\ 0 & 0 & \ddots & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & k_{pn} \end{pmatrix}$$

As before, we end up with a couple of independent error equations, and each one of those can be seen as a mass-spring model that we whish to damp critically. The equations would be

$$\ddot{e}_i + k_{v_i}\dot{e}_i + k_{p_i}e_i = 0.$$

When talking about natural frequencies in the context of manipulator control, we are referring to the natural frequency associated with those error equations. This means that the following simple relationship holds:

$$\omega_{ni} = \sqrt{k_{pi}}.$$

Finally, critical damping can be achieved as usual by letting

$$k_{vi} = 2\sqrt{k_{pi}}.$$

Solution 1

a)

The first subproblem is a well-known problem: Determining the dynamic equations using Lagrange's method. Since the corresponding computations have already been shown in detail, we are not going to do an exhaustive computation, but instead we will show only the final results and some important intermediate steps.

$${}^0P_{C_1} = \begin{pmatrix} \mathrm{s}_1 \cdot l_1 \\ -\mathrm{c}_1 \cdot l_1 \\ 0 \end{pmatrix}, \quad {}^0P_{C_2} = \begin{pmatrix} \mathrm{s}_1 \cdot d_2 \\ -\mathrm{c}_1 \cdot d_2 \\ 0 \end{pmatrix}, \quad {}^1\omega_1 = \begin{pmatrix} 0 \\ 0 \\ \dot{\Theta}_1 \end{pmatrix}, \quad {}^2\omega_2 = \begin{pmatrix} 0 \\ \dot{\Theta}_1 \\ 0 \end{pmatrix}$$

This yields (taking the chain rule into account):

$${}^0v_{C_1} = \frac{d}{dt}{}^0P_{C_1} = \begin{pmatrix} \mathrm{c}_1 \cdot \dot{\Theta}_1 \cdot l_1 \\ \mathrm{s}_1 \cdot \dot{\Theta}_1 \cdot l_1 \\ 0 \end{pmatrix}, \quad {}^0v_{C_2} = \frac{d}{dt}{}^0P_{C_2} = \begin{pmatrix} \mathrm{c}_1\dot{\Theta}_1 d_2 + \mathrm{s}_1\dot{d}_2 \\ \mathrm{s}_1\dot{\Theta}_1 d_2 - \mathrm{c}_1\dot{d}_2 \\ 0 \end{pmatrix}$$

Thus, the kinetic energies are determined as

$$k_1 = 0.07 \cdot \dot{\Theta}_1^2.$$

We have substituted the values m_1 and I_{zz1} directly. Analogously, we determine the kinetic energy for link 2 as follows:

$$k_2 = \frac{1}{2}m_2 \cdot d_2^2 \cdot \dot{\Theta}_1^2 + \frac{1}{2}m_2 \cdot \dot{d}_2^2 + \frac{1}{2}0.07 \cdot \dot{\Theta}_1^2$$

The potential energies are

$$\begin{aligned} u_1 &= g \cdot \mathrm{s}_1 \cdot 0.2, \\ u_2 &= m_2 \cdot g \cdot \mathrm{s}_1 \cdot d_2. \end{aligned}$$

We have omitted the reference energies u_{ref_i}, as usual. After computation of the partial derivatives, we obtain:

$$\begin{aligned} \tau_1 &= \ddot{\Theta}_1 \left(0.21 + m_2 \cdot d_2^2\right) + \dot{\Theta}_1 \cdot m_2 \cdot d_2 \cdot \dot{d}_2 \cdot 2 + g \cdot \mathrm{c}_1 \cdot 0.2 + m_2 \cdot g \cdot \mathrm{c}_1 \cdot d_2 \\ \tau_2 &= m_2 \cdot \ddot{d}_2 - m_2 \cdot d_2 \cdot \dot{\Theta}_1^2 + m_2 \cdot g \cdot \mathrm{s}_1 \end{aligned}$$

The M, V, G-form is then:

$$M(\Theta) = \begin{pmatrix} 0.21 + m_2 \cdot d_2^2 & 0 \\ 0 & m_2 \end{pmatrix}$$
$$V(\Theta, \dot{\Theta}) = \begin{pmatrix} 2\dot{\Theta}_1 \cdot m_2 \cdot d_2 \cdot \dot{d}_2 \\ -m_2 \cdot d_2 \cdot \dot{\Theta}_1^2 \end{pmatrix}$$
$$G(\Theta) = \begin{pmatrix} g \cdot (\mathrm{c}_1 \cdot 0.2 + m_2 \cdot \mathrm{c}_1 \cdot d_2) \\ g \cdot m_2 \cdot \mathrm{s}_1 \end{pmatrix}$$

b)

Now we are supposed to design a system controller based on a PD control law. To do this, we need the equations from the recapitulation - k_{v_i}-entries correspond to the differential constant, k_{p_i} correspond to the proportial constant. Then, we have $\alpha = M(\Theta)$, $\beta = V(\Theta, \dot{\Theta}) + G(\Theta)$, and

$$\tau' = \ddot{\Theta}_d + K_v(\dot{\Theta}_d - \dot{\Theta}) + K_p(\Theta_d - \Theta)$$

with diagonal matrices K_v and K_p with entries k_{v_i} and k_{p_i}, resp.

c)

Looking at the formulas from the recapitulation, we see that we only need to compute the values

$$\omega_{ni} = \sqrt{k_{pi}}.$$

This is done as follows:

$$\begin{aligned} \omega_{n_1} &= \sqrt{k_{p_1}} = 20 \quad \Rightarrow \quad k_{p_1} = 400 \\ \omega_{n_2} &= \sqrt{k_{p_2}} = 25 \quad \Rightarrow \quad k_{p_2} = 625 \end{aligned}$$

To achieve critical damping, we need to assure $k_{v_i} = 2\sqrt{k_{p_i}}$, so we compute:

$$\begin{aligned} k_{v_1} &= 2\sqrt{k_{p_1}} = 2\sqrt{400} = 40 \\ k_{v_2} &= 2\sqrt{k_{p_2}} = 2\sqrt{625} = 50 \end{aligned}$$

c)

The block diagram is shown in Figure 1.

Solution 2

Again we are supposed to design a controller, but this time the controlling task is a little bit different. Usually, we assume that values $\ddot{\Theta}_d, \dot{\Theta}_d, \Theta_d$ have been generated by a trajectory generator and are fed into the controller system. For this problem, there is only one fixed value Θ_d that is supposed to be maintained.

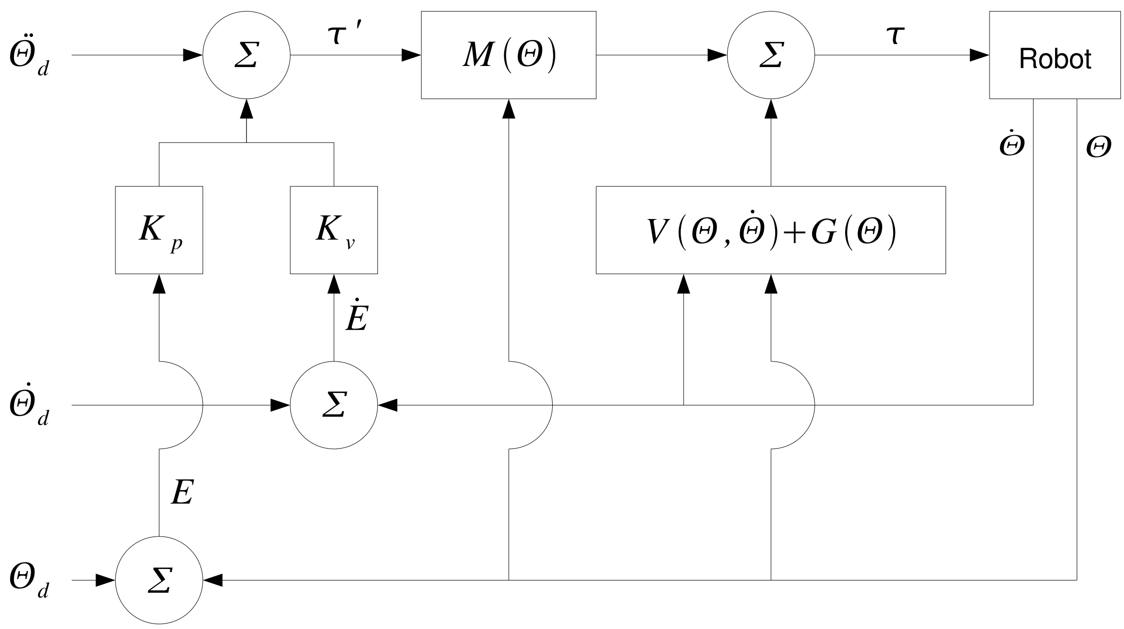

Figure 1: *Block diagram of the controller*

a)

First of all, we need to determine the dynamics equations of this system. Taking into account all the known values from the problem statement, we arrive at

$$\tau = ml^2\ddot{\Theta} + k_f\dot{\Theta} = I_{mzz}\ddot{\Theta} + k_f\dot{\Theta}.$$

This can be computed using the Newton-Euler-Method with the following inertia tensor:

$$I_m = \begin{pmatrix} 0 & 0 & 0 \\ 0 & l^2m & 0 \\ 0 & 0 & l^2m \end{pmatrix} \quad {}^1P_{C_1} = \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix}$$

As in the mass-spring system case, the friction force is computed with a term $k_f\dot{\Theta}$. The computation of the inertia tensor can be performed easily if the following formula for point-shaped masses is used:

$$I = \sum_i m_i \begin{pmatrix} y_i^2 + z_i^2 & -x_iy_i & -x_iz_i \\ -y_ix_i & x_i^2 + z_i^2 & -y_iz_i \\ -z_ix_i & -z_iy_i & x_i^2 + y_i^2 \end{pmatrix}$$

b)

As we have mentioned before, we are interested in developing a steady-state controller that tries to keep a desired state Θ_d. Thus, we can assume $\ddot{\Theta}_d = \dot{\Theta}_d = 0$. We intend to use the usual partitioning scheme. The control law is also the same as we have used before, but for the steady state problem it reduces to:

$$\tau' = \ddot{\Theta}_d + k_v\dot{e} + k_pe = -k_v\dot{\Theta} + k_p(\Theta_d - \Theta)$$

Inserted into the equations of motion, we obtain (using $\tau = \alpha\tau' + \beta$, as usual):

$$l^2m\ddot{\Theta} + k_f\dot{\Theta} = l^2m(-k_v\dot{\Theta} + k_p(\Theta_d - \Theta)) + k_f\dot{\Theta}$$

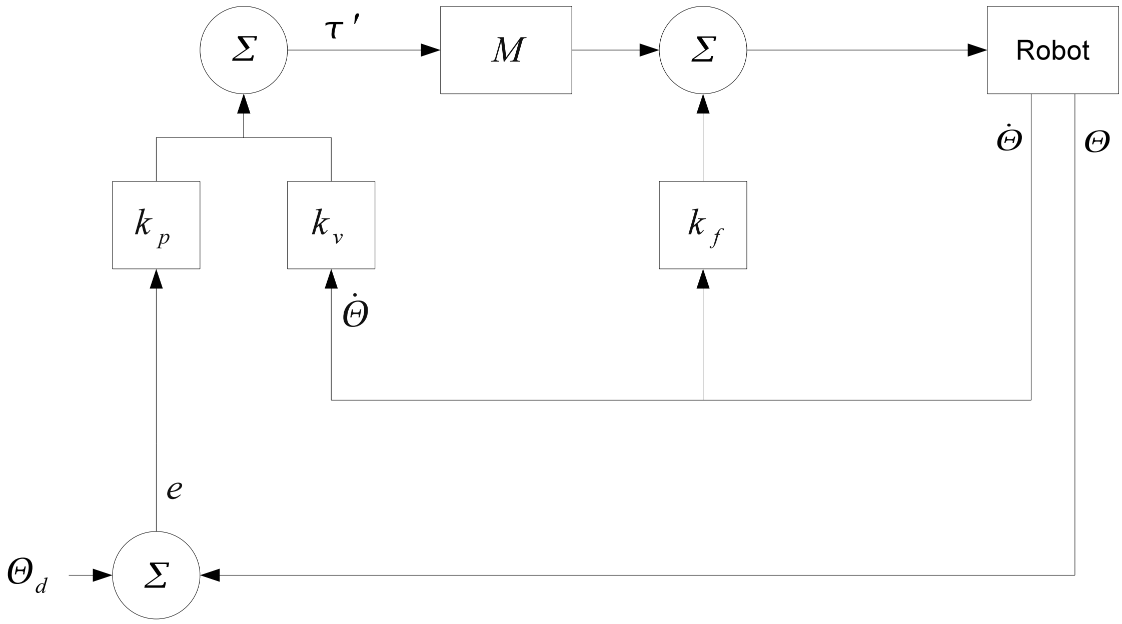

Figure 2: *Controller block diagram.*

With some rearrangements, this becomes:

$$l^2m(0 - \ddot{\Theta} + k_v(0 - \dot{\Theta}) + k_p(\Theta_d - \Theta)) = 0$$

And we see that the error equation holds as usual. Overall, we now have the following formula for computation of τ:

$$\tau = \alpha\tau' + \beta = l^2m(-k_v\dot{\Theta} + k_p(\Theta_d - \Theta)) + k_f\dot{\Theta}.$$

c)

The block diagram is shown in Figure 2.