Chair of Robotics, Arificial Intelligence and Real-Time Systems
Department of Informatics
Technical University of Munich

Ecorrection
Place student sticker here

Note:

- During the attendance check a sticker containing a unique code will be put on this exam.
- This code contains a unique number that associates this exam with your registration number.
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Robotics

Exam:	IN2067 / Retake	**Date:**	Friday 22nd April, 2022
Examiner:	Prof. Dr.-Ing. Darius Burschka	**Time:**	08:00 – 09:30

Working instructions

- This exam consists of **14 pages** with a total of **4 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 125 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources:
  - one **non-programmable pocket calculator**
  - one **analog dictionary** English ↔ native language
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.
- Physically turn off all electronic devices, put them into your bag and close the bag.

Problem 1 Authentication (0 credits)

The process is meant to help in disputes about possible fraud accusations later. You are supposed to sit in the room alone. Enable the BBB session with the link provided in TUMexam.

0
1

Write in your own handwriting (using mouse or pen in your PDF editor) the following text hand-written by you: *Hereby, I confirm that I prepared the solution without any help.*
Also write your zodiac sign.

Problem 2 Kinematics (33 credits)

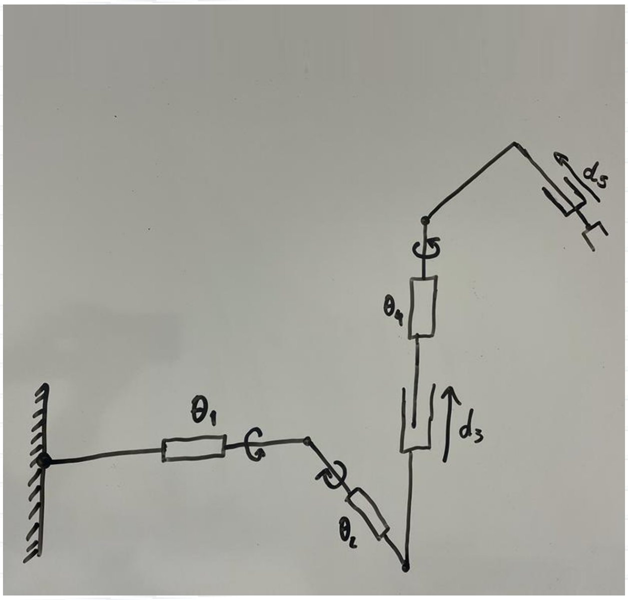


Figure 2.1: 3D view of the robot

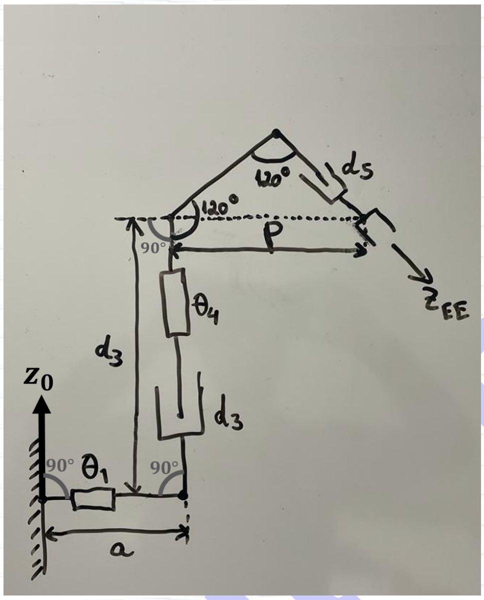


Figure 2.2: Side view of the robot

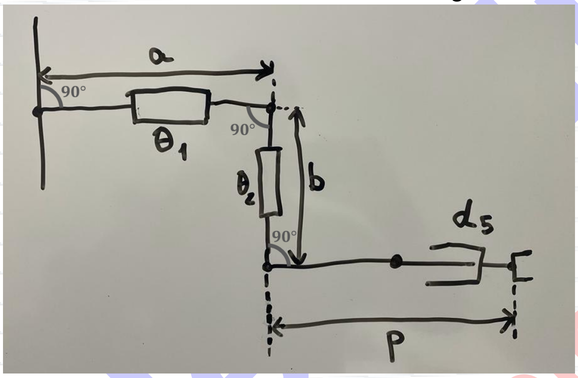


Figure 2.3: Top view of the robot

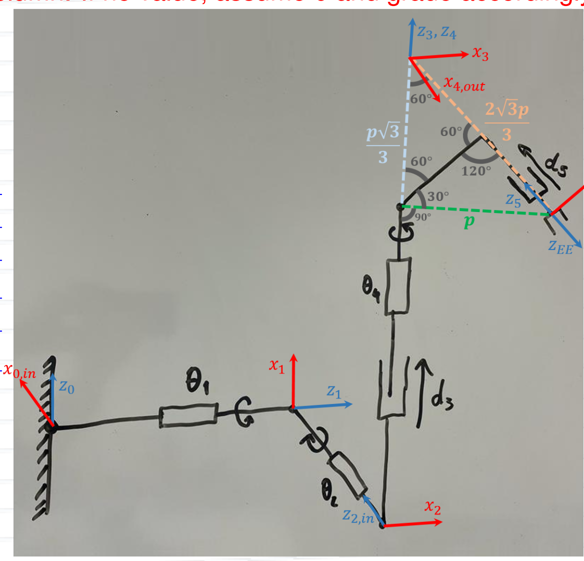
Fig. 2.1 shows the **RRPRP** robot's positive joint rotation and movement directions. Figures 2.2 and 2.3 show a side- and top-view of the robot for better 3D understanding. In this configuration, only the second link is out of the drawing plane. The z-axis of the end effector is pointing along the last link. Black angles are rigid link connections.

a)* Write the robot's Denavit-Hartenberg table. Give values for **each** joint parameter according to the figures. Draw coordinate frames for your answer to be understandable. If possible, make sure your α_{i-1}-values are non-negative.

0 1 2 3 4 5 6

✓ per completely correct line INCLUDING the *value* column. If no value, assume 0 and grade accordingly...

CF	α_{i-1}	a_{i-1}	d_i	θ_i	value
1	90°	0	a	θ_1	90°
2	90°	0	$-b$	θ_2	90°
3	90°	0	$d_3 + \frac{p}{\sqrt{3}}$	0°//−90°	∀ value >0m
4	0°	0	0	θ_4	−90°//0°
5	60°	0	$d_5 - \frac{2p}{\sqrt{3}}$	0°	0m
EE	180°	0	0	0°	–



0 1 2 3 4 5 6 7

b) Determine the (general) coordinates of this robot's y_{EE}-axis relative to coordinate frame 2. Show your work.

✓ for representing the y-axis correctly (with 0 as fourth component in homogeneous coordinates). ✓ for each rotation matrix, four max (ignore errors in translational part of the transformation matrix, if written...). One could merge ${}^{2}_{3}T$, ${}^{3}_{4}T$ into one matrix since the z-axes of CF 3 and 4 are the same; if merged, still four points.
Remaining two ✓ for the correct result (if one component wrong max. 1 ✓ obtainable).
Consider the DH-table from point b) when correcting.
If no explanation/justification for the result was given, maximal points reachable = 3 out of 7 and give the three ✓ per correct x-, y- and z-component.

$$
{}^{2}_{3}T \checkmark = \begin{pmatrix} 1 & 0 & 0 & b\frac{\sqrt{2}}{2} \\ 0 & 0 & -1 & -d_3 - p\sqrt{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^{3}_{4}T \checkmark = \begin{pmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^{4}_{5}T \checkmark = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2}(d_5 - 2p) \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{1}{2}(d_5 - 2p) \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^{5}_{EE}T \checkmark = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^{2}y_{EE} = {}^{2}_{3}T \cdot {}^{3}_{4}T \cdot {}^{4}_{5}T \cdot {}^{5}_{EE}T \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \checkmark = {}^{2}_{3}R \cdot {}^{3}_{4}R \cdot {}^{4}_{5}R \cdot {}^{5}_{EE}R \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\sin(\theta_4)}{2} \\ \frac{\sqrt{3}}{2} \\ -\frac{\cos(\theta_4)}{2} \end{pmatrix} \checkmark\checkmark
$$

0 1 2 3 4 5 6

c) Determine the orientation Jacobian (in general configuration) with respect to the global coordinate frame. Show your work.

✓ for the Jacobian's formula (the one at the bottom). No credit if the z-axes meant were correct (not the last z-axis of the end-effector!). 1 ✓ per correct column of the Jacobian considering the DH-table from point b.

$$
z_1 = {}^{0}_{1}R^{1}z_1 = {}^{0}_{1}R \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = {}^{0}z_1 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}
$$

$$
z_2 = {}^{0}_{2}R^{2}z_2 = {}^{0}_{2}R \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = {}^{0}z_2 = \begin{pmatrix} \sin(\theta_1) \\ 0 \\ -\cos(\theta_1) \end{pmatrix}
$$

$z_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ because joint 3 is prismatic

$$
z_4 = {}^{0}_{4}R^{4}z_4 = {}^{0}_{4}R \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = {}^{0}z_4 = \begin{pmatrix} \cos(\theta_1)\sin(\theta_2) \\ \cos(\theta_2) \\ \sin(\theta_1)\sin(\theta_2) \end{pmatrix}
$$

$z_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ because joint 5 is prismatic

$$
{}^{0}J_{\omega} = \left(z_1 \mid z_2 \mid z_3 \mid z_4 \mid z_5 \right) = \begin{pmatrix} 0 & \sin(\theta_1) & 0 & \cos(\theta_1)\sin(\theta_2) & 0 \\ -1 & 0 & 0 & \cos(\theta_2) & 0 \\ 0 & -\cos(\theta_1) & 0 & \sin(\theta_1)\sin(\theta_2) & 0 \end{pmatrix}
$$

d)* Write the characteristics of a wrist configuration. Does this robot have one? Can the robot achieve any orientation? Justify your answers!

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7

A wrist configuration is characterized by three (consecutive) rotational-joints' z-axes that intersect in one point. ✓. No pair of consecutive axes are allowed to be parallel ✓.
This robot does not have a wrist configuration ✓ and another ✓ for one of the justifications.

- z_3, z_4 and z_5 are intersecting in one point, but z_3 and z_4 are parallel.
- There are no three (consecutive) rotational axes intersecting in one point.
- Other explanation why there is no wrist configuration in this robot (according to the above mentioned wrist-configuration-criteria).

The robot has three rotational joints each with a different rotation axis. ✓ This means, the robot's end effector can reach any orientation ✓, despite having no wrist configuration. ✓.

e) What happens to the robot when $\theta_2 = 180°$? Why? For what other joint values does this phenomenon happen?

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7

Singularity ✓ : z_1 and z_4 are parallel ✓, and a degree of freedom is lost. ✓

Compute the determinant of the modified orientation Jacobian $J^*_\omega = \begin{pmatrix} 0 & sin(\theta_1) & cos(\theta_1)sin(\theta_2) \\ -1 & 0 & cos(\theta_2) \\ 0 & -cos(\theta_1) & sin(\theta_1)sin(\theta_2) \end{pmatrix}$ ✓.

$|J^*_\omega| = sin(\theta_2)$ ✓ ; $|J^*_\omega| = 0 \Leftrightarrow sin(\theta_2) = 0 \Leftrightarrow \theta_2 \in \{0°, 180°\}$ ✓.
Singularity also happens for $\theta_2 = 0°$ ✓.

Problem 3 Dynamics (46 credits)

Fig. 3.3 shows the top-view and Fig. 3.2 shows the side-view of an **RPR** robot. Gravity acts in the positive z_0 direction. The Denavit-Hartenberg table for the robot is partially given. The centers of mass of each link are at ${}^1P_{c_1} = \begin{pmatrix} -\frac{1}{2}a \\ 0 \\ -\frac{1}{2}b \end{pmatrix}$, ${}^2P_{c_2} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2}d_2 \end{pmatrix}$ and ${}^3P_{c_3} = \begin{pmatrix} 0 \\ 0 \\ -\frac{2}{3}e \end{pmatrix}$ respectively. The inertia matrices are ${}^{c_1}I_1 = \begin{pmatrix} I_{1x} & 0 & 0 \\ 0 & I_{1y} & 0 \\ 0 & 0 & I_{1z} \end{pmatrix}$, ${}^{c_2}I_2 = \begin{pmatrix} I_{2x} & 0 & 0 \\ 0 & I_{2y} & 0 \\ 0 & 0 & I_{2z} \end{pmatrix}$, ${}^{c_3}I_3 = \begin{pmatrix} I_{3x} & 0 & 0 \\ 0 & I_{3x} & 0 \\ 0 & 0 & I_{3z} \end{pmatrix}$, with $I_{3xx} = I_{3yy}$ and the links' masses are m_1, m_2 and m_3.

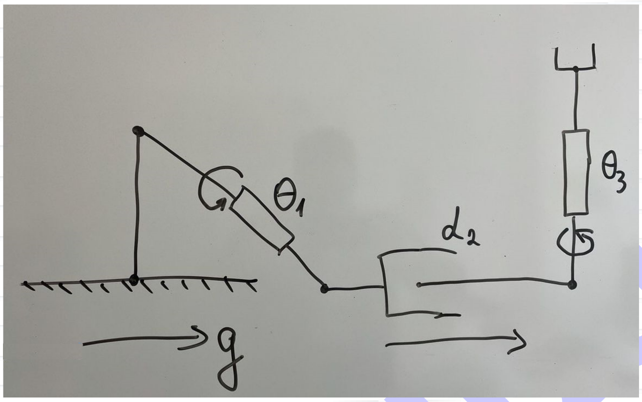


Figure 3.1: 3D view of the robot

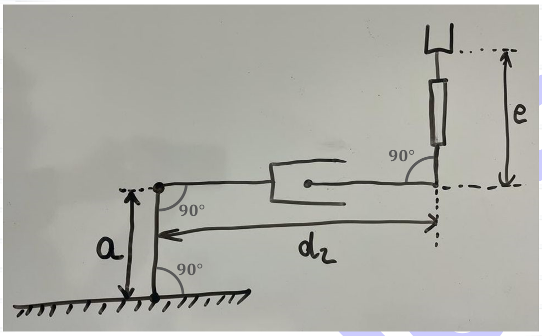


Figure 3.2: Side view of the robot

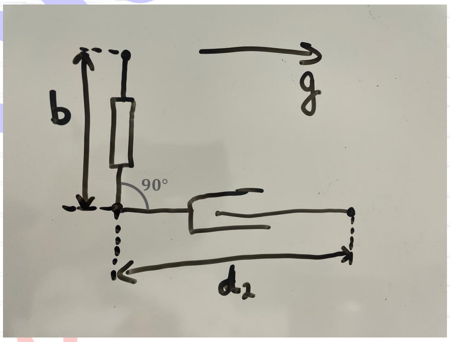


Figure 3.3: Top view of the robot

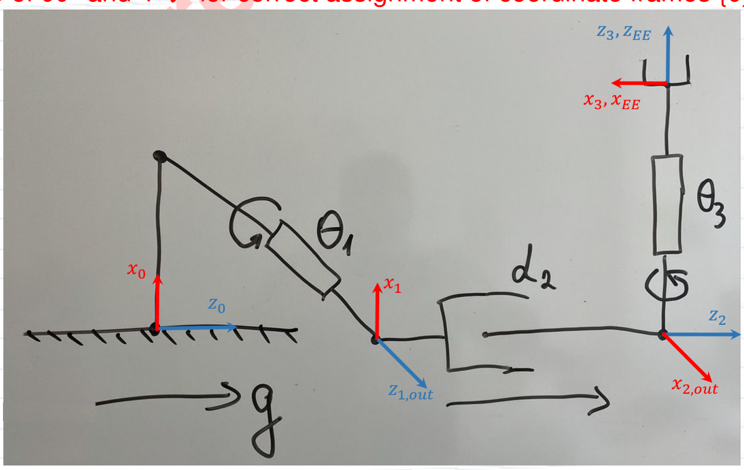
CF	α_{i-1}	a_{i-1}	d_i	θ_i	value
1	$-90°$	a	b	θ_1	$0°$
2	$90°$	0	d_2	$90°$	10cm

CF	α_{i-1}	a_{i-1}	d_i	θ_i	value
3	$90°$	0	e	θ_3	$-90°$
EE	$0°$	0	0	$0°$	–

0 1 2 3 4 5

a)* Fill the missing entry of the DH table. Show your work by drawing the coordinate frames {0}, {1}, {2} and {3}.

✓ for correct value of 90° and 4 ✓ for correct assignment of coordinate frames {0}, {1}, {2}, and {3}.



Answer: 90°

b) Compute the kinetic and potential energies for each link. Be sure to indicate all your intermediate results and the values for your variables (e.g. rotation matrices, velocities and other) for full points.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19

1 ✓ per correct ${}^{i-1}_{i}T$ OR ${}^{0}_{i}T$ for $i \in \{1,2,3\}$

1 ✓ per correct ${}^0P_{c_i}$ for $i \in \{1,2,3\}$

1 ✓ per correct ${}^0v_{c_i}$ for $i \in \{1,2,3\}$

1 ✓ per correct ${}^i\omega_i$ for $i \in \{1,2,3\}$

1 ✓ per correct k_i for $i \in \{1,2,3\}$

1 ✓ per correct u_i for $i \in \{1,2,3\}$

1 ✓ for correct G-vector

$${}^0_1T = \begin{pmatrix} c_1 & -s_1 & 0 & a \\ 0 & 0 & 1 & b \\ -s_1 & -c_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ✓, {}^1_2T = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -d_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ✓, {}^2_3T = \begin{pmatrix} c_3 & -s_3 & 0 & 0 \\ 0 & 0 & -1 & -e \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ✓, {}^3_{EE}T = I_4$$

$${}^0_2T = \begin{pmatrix} 0 & -c_1 & s_1 & a + d_2s_1 \\ 1 & 0 & 0 & b \\ 0 & s_1 & c_1 & d_2c_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^0_3T = \begin{pmatrix} s_1s_3 & s_1c_3 & c_1 & a + d_2s_1 + ec_1 \\ c_3 & -s_3 & 0 & b \\ c_1s_3 & c_1c_3 & -s_1 & d_2c_1 - es_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^0_{EE}T = {}^0_3T$$

$${}^0P_{c_1} = {}^0_1T\,{}^1P_{c_1} = \begin{pmatrix} a - \frac{1}{2}ac_1 \\ \frac{1}{2}b \\ \frac{1}{2}as_1 \end{pmatrix} ✓ \rightarrow {}^0v_{c_1} = \frac{d}{dt}{}^0P_{c_1} = \begin{pmatrix} \frac{1}{2}a\dot{\theta}_1s_1 \\ 0 \\ \frac{1}{2}a\dot{\theta}_1c_1 \end{pmatrix} ✓$$

$${}^0P_{c_2} = {}^0_2T\,{}^2P_{c_2} = \begin{pmatrix} a + \frac{1}{2}d_2s_1 \\ b \\ \frac{1}{2}d_2c_1 \end{pmatrix} ✓ \rightarrow {}^0v_{c_2} = \frac{d}{d}{}^0P_{c_2} = \begin{pmatrix} \frac{1}{2}\dot{d}_2s_1 + \frac{1}{2}d_2c_1\dot{\theta}_1 \\ 0 \\ \frac{1}{2}\dot{d}_2c_1 - \frac{1}{2}d_2s_1\dot{\theta}_1 \end{pmatrix} ✓$$

$${}^0P_{c_3} = {}^0_3T\,{}^3P_{c_3} = \begin{pmatrix} a + d_2s_1 + \frac{1}{3}ec_1 \\ b \\ d_2c_1 - \frac{1}{3}es_1 \end{pmatrix} ✓ \rightarrow {}^0v_{c_3} = \frac{d}{dt}{}^0P_{c_3} = \begin{pmatrix} -\frac{1}{3}es_1\dot{\theta}_1 + \dot{d}_2s_1 + d_2c_1\dot{\theta}_1 \\ 0 \\ -\frac{1}{3}ec_1\dot{\theta}_1 + \dot{d}_2c_1 - d_2s_1\dot{\theta}_1 \end{pmatrix} ✓$$

$${}^0\omega_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};\ {}^1\omega_1 = {}^1_0R\,{}^0\omega_0 + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix};✓\ {}^2\omega_2 = \begin{pmatrix} \dot{\theta}_1 \\ 0 \\ 0 \end{pmatrix} ✓,\ {}^3\omega_3 = \begin{pmatrix} \dot{\theta}_1c_3 \\ -\dot{\theta}_1s_3 \\ \dot{\theta}_3 \end{pmatrix} ✓$$

$k_i = \frac{1}{2}m_i{}^0v_{c_i}^{T\,0}v_{c_i} + \frac{1}{2}{}^i\omega_i{}^{c_i}I_i{}^i\omega_i$

$k_1 = \frac{1}{2}m_1{}^0v_{c_1}^{T\,0}v_{c_1} + \frac{1}{2}{}^1\omega_1{}^{c_1}I_1{}^1\omega_1 = \frac{1}{8}a^2m_1\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_1^2I_{1z}$ ✓

$k_2 = \frac{1}{8}m_2\left(\dot{d}_2^2 + d_2^2\dot{\theta}_1^2\right) + \frac{1}{2}\dot{\theta}_1^2I_{2x}$ ✓

$k_3 = \frac{1}{2}m_3\left(\dot{d}_2^2 + d_2^2\dot{\theta}_1^2 + \frac{1}{9}e^2\dot{\theta}_1^2 - \frac{2}{3}e\dot{\theta}_1\dot{d}_2\right) + \frac{1}{2}\dot{\theta}_1^2I_{3x} + \frac{1}{2}\dot{\theta}_3^2I_{3z}$ ✓

$G = \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}$ ✓

$u_i = -m_iG^T \cdot {}^0P_{c_i}$

$u_1 = -\frac{1}{2}m_1gas_1$ ✓

$u_2 = -\frac{1}{2}m_2gd_2c_1$ ✓

$u_3 = m_3g\left(\frac{1}{3}es_1 - d_2c_1\right)$ ✓

0 1 2 3 4 5 6 7 8 9

c) Compute the needed derivatives for completing the Lagrange analysis.

1 ✓ for each correct $\frac{\partial u}{\partial \theta_i}$ for $i \in \{1, 2, 3\}$ (max. 3)

1 ✓ for each correct $\frac{\partial k}{\partial \theta_i}$ for $i \in \{1, 2, 3\}$ (max. 3)

1 ✓ for each correct $\frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_i}$ for $i \in \{1, 2, 3\}$ (max. 3)

$\frac{\partial u}{\partial \theta_1} = g\left(-\frac{1}{2}m_1 a c_1 + \frac{1}{2}m_2 d_2 s_1 + m_3 d_2 s_1 + \frac{1}{3}m_3 e c_1\right)$ ✓

$\frac{\partial u}{\partial d_2} = -g\left(\frac{1}{2}m_2 c_1 + m_3 c_1\right)$ ✓

$\frac{\partial u}{\partial \theta_3} = 0$ ✓

$\frac{\partial k}{\partial \theta_1} = 0$ ✓

$\frac{\partial k}{\partial d_2} = d_2\dot{\theta}_1^2\left(\frac{1}{4}m_2 + m_3\right)$ ✓

$\frac{\partial k}{\partial \theta_3} = 0$ ✓

$\frac{\partial k}{\partial \dot{\theta}_1} = \dot{\theta}_1\left(I_{1z} + I_{2x} + I_{3x}\right) + \frac{1}{4}\dot{\theta}_1(a^2 m_1 + d_2^2 m_2) + m_3\left(d_2^2\dot{\theta}_1 - \frac{1}{3}\dot{d}_2 * e + \frac{1}{9}e^2\dot{\theta}_1\right)$

$\frac{\partial k}{\partial \dot{d}_2} = \frac{1}{4}\dot{d}_2 m_2 + m_3\left(\dot{d}_2 - \frac{1}{3}e\dot{\theta}_1\right)$

$\frac{\partial k}{\partial \dot{\theta}_3} = I_{3z}\dot{\theta}_3$

$\frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_1} = \ddot{\theta}_1\left(I_{1z} + I_{2x} + I_{3x}\right) + \frac{1}{4}\ddot{\theta}_1(a^2 m_1 + d_2^2 m_2) + \frac{1}{2}d_2\dot{d}_2 m_2\dot{\theta}_1 + m_3\left(2d_2\dot{d}_2\dot{\theta}_1 + d_2^2\ddot{\theta}_1 - \frac{1}{3}\ddot{d}_2 e + \frac{1}{9}e^2\ddot{\theta}_1\right)$ ✓

$\frac{d}{dt}\frac{\partial k}{\partial \dot{d}_2} = \frac{1}{4}\ddot{d}_2 m_2 + m_3\left(\ddot{d}_2 - \frac{1}{3}e\ddot{\theta}_1\right)$ ✓

$\frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_3} = I_{3z}\ddot{\theta}_3$ ✓

d) Finally, write the joint torques and write the joint torques equation in M-V-G form.

0 1 2 3 4 5 6

1 ✓ per correct τ_i for $i \in \{1,2,3\}$ (max. 3) and 1 ✓ per correct M, V, and G terms.

$$\tau_i = \frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_i} - \frac{\partial k}{\partial \theta_i} + \frac{\partial u}{\partial \theta_i}$$

$$\tau_1 = \ddot{\theta}_1 \left(I_{1z} + I_{2x} + I_{3x}\right) + \tfrac{1}{4}\ddot{\theta}_1(a^2 m_1 + d_2^2 m_2) + \tfrac{1}{2} d_2 \dot{d}_2 m_2 \dot{\theta}_1 + m_3 \left(2 d_2 \dot{d}_2 \dot{\theta}_1 + d_2^2 \ddot{\theta}_1 - \tfrac{1}{3}\ddot{d}_2 e + \tfrac{1}{9} e^2 \ddot{\theta}_1\right) +$$
$$+ g\left(-\tfrac{1}{2} m_1 a c_1 + \tfrac{1}{2} m_2 d_2 s_1 + m_3 d_2 s_1 + \tfrac{1}{3} m_3 e c_1\right) ✓$$

$$\tau_2 = \tfrac{1}{4}\ddot{d}_2 m_2 + m_3\left(\ddot{d}_2 - \tfrac{1}{3} e \ddot{\theta}_1\right) - d_2 \dot{\theta}_1^2 \left(\tfrac{1}{4} m_2 + m_3\right) - g\left(\tfrac{1}{2} m_2 c_1 + m_3 c_1\right) ✓$$

$$\tau_3 = I_{3z}\ddot{\theta}_3 ✓$$

$$\tau = M(\theta)\begin{pmatrix}\ddot{\theta}_1\\ \ddot{d}_2\\ \ddot{\theta}_3\end{pmatrix} + V\left(\theta, \dot{\theta}\right) + G(\theta)$$

$$M(\theta) = \begin{pmatrix} I_{1z} + I_{2x} + I_{3x} + \frac{1}{4}\left(a^2 m_1 + d_2^2 m_2\right) + m_3 d_2^2 + \frac{1}{9} m_3 e^2 & -\frac{1}{3} m_3 e & 0 \\ \frac{1}{3} m_3 e & \frac{1}{4} m_2 + m_3 & 0 \\ 0 & 0 & I_{3z} \end{pmatrix} ✓$$

$$V\left(\theta, \dot{\theta}\right) = \begin{pmatrix} d_2 \dot{d}_2 \dot{\theta}_1 \left(\frac{1}{2} m_2 + 2 m_3\right) \\ -d_2 \dot{\theta}_1^2 \left(\frac{1}{4} m_2 + m_3\right) \\ 0 \end{pmatrix} ✓$$

$$G(\theta) = \begin{pmatrix} g\left(-\frac{1}{2} m_1 a c_1 + \frac{1}{2} m_2 d_2 s_1 + m_3 d_2 s_1 + \frac{1}{3} m_3 e c_1\right) \\ -g\left(\frac{1}{2} m_2 + m_3\right) c_1 \\ 0 \end{pmatrix} ✓$$

e)* Convert the following joint torques equation to M-B-C-G form. Show your work and clearly mark M, B, C and G. Under which circumstances is it advantageous to use this form of the joint torques equation?

0 1 2 3 4 5 6 7

$$\tau = \begin{pmatrix} m_1(l_1^2 c_1^2 + 2) & c_1 & 0 \\ m_2(1 + c_2) & m_2 l_2^2 & m_2 + m_3 \\ (l_3 + 1/2)^2 (m_1 + m_2) & m_2 & m_3 l_3^2 \end{pmatrix} \begin{pmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{d}_3\end{pmatrix} + \begin{pmatrix} \dot{\theta}_1\dot{\theta}_2 m_1(l_1 - 2) - d_3 m_3 g + \dot{\theta}_1 \dot{d}_3 d_3 c_2 + m_3 g l_3 c_1 c_2 \\ -m_3 g l_3 s_1 c_2 + \dot{\theta}_2^2(2m_2 + 3) - \dot{\theta}_1(\dot{d}_3 + l_2 c_2 \dot{\theta}_1) \\ \dot{d}_3^2 - \ddot{\theta}_2 + \dot{\theta}_1\dot{\theta}_2 m_1 l_2 + m_2 g l_2 s_2 - \dot{\theta}_2^2 d_3 c_{12} \end{pmatrix}$$

✓ for correct M-B-C-G equation WITH θ-vectors expanded! No explicit θ-vectors, no credit here.

$$\tau = M(\theta)\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\\ \ddot{d}_3\end{pmatrix} + B(\theta)\begin{pmatrix}\dot{\theta}_1\dot{\theta}_2\\ \dot{\theta}_1\dot{d}_3\\ \dot{\theta}_2\dot{d}_3\end{pmatrix} + C(\theta)\begin{pmatrix}\dot{\theta}_1^2\\ \dot{\theta}_2^2\\ \dot{d}_3^2\end{pmatrix} + G(\theta)$$

$$M = \begin{pmatrix} m_1(l_1^2 c_1^2 + 2) & c_1 & 0 \\ m_2(1 + c_2) & m_2 l_2^2 & m_2 + m_3 \\ (l_3 + 1/2)^2 (m_1 + m_2) & m_2 - 1 & m_3 l_3^2 \end{pmatrix}, B = \begin{pmatrix} m_1(l_1 - 2) & d_3 c_2 & 0 \\ 0 & -1 & 0 \\ m_1 l_2 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 0 \\ -l_2 c_2 & 2m_2 + 3 & 0 \\ 0 & -d_3 c_{12} & 1 \end{pmatrix},$$

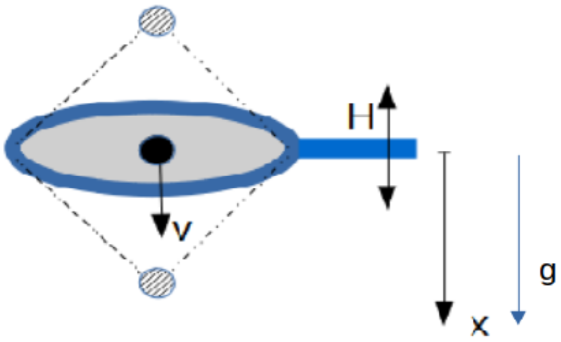
$$G = \begin{pmatrix} m_3 g(l_3 c_1 c_2 - d_3) \\ -m_3 g l_3 s_1 c_2 \\ m_2 g l_2 s_2 \end{pmatrix}$$

1 ✓ per completely correct term (M, B, C and G).

In the case where we lack computing power ✓, we could further split the M-V-G equation in this M-B-C-G form, where the matrices only depend on the joint values θ, not also on the joint velocities $\dot{\theta}$ ✓.

Problem 4 Control (46 credits)

A gray membrane with a black ball with mass m=3 (see Figure below) rigidly attached to it can be moved along the x-direction by moving the handle H with position x_H. The motion of the membrane causes a velocity dependent damping force $f_d = \nu \cdot (\dot{x}_H - \dot{x}_{ball})$ with the damping factor $\nu = 4$. The position of the ball x_{ball} is measured by an external sensor. A stretch of the membrane in x-direction (by moving the handle along x) causes a force $f_p = \mu \cdot (x_H - x_{ball})$ with $\mu = 2$ that resists the stretch of the membrane (the two possible extreme stretch positions of the membrane are shown with dashed lines in the figure below). We assume that the mass of the membrane and of the handle can be neglected.



0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18

a)*
At first, assume that the handle H is static at the middle position x=0 and the ball is moving with the velocity $\dot{x}_{ball} = 2m/s$ downwards at time t=0. There is no gravity (g=0) for now. The ball passes the $x_{ball} = 0$ position at time t=0. Write the corresponding differential equation for the ball motion x(t) as solution of the equation of force balance (which forces act on the ball)? What type of motion will the ball perform with the given parameters (μ, ν)? Sketch, how the ball position evolves over time t. What is the maximum deflection of the membrane (position of the ball) given the boundary conditions for t=0?

a) $m \cdot \ddot{x}_{ball} = f_d + f_p$ $\quad \dot{x}_H = x_H = 0$

$m \cdot \ddot{x}_{ball} + \nu \dot{x}_{ball} + \mu x_{ball} = 0$

$ms^2 + \nu s + \mu = 0$

$3s^2 + 4s + 2 = 0$

$s_{1,2} = \frac{-4 \pm \sqrt{16-24}}{6} = -\frac{2}{3} \pm i \frac{\sqrt{2}}{3}$

$x(t) = e^{-\frac{2}{3}t}\left(c_1 \cos\left(\frac{\sqrt{2}}{3}t\right) + c_2 \sin\left(\frac{\sqrt{2}}{3}t\right)\right)$

$\dot{x}(t) = -\frac{2}{3} e^{-\frac{2}{3}t}\left(c_1 \cos\left(\frac{\sqrt{2}}{3}t\right) + c_2 \sin\left(\frac{\sqrt{2}}{3}t\right)\right) + \frac{\sqrt{2}}{3} e^{-\frac{2}{3}t}\left(-c_1 \sin\left(\frac{\sqrt{2}}{3}t\right) + c_2 \cos\left(\frac{\sqrt{2}}{3}t\right)\right)$

$$\left.\begin{array}{l} x(0) = 0 \rightarrow c_1 = 0 \\ \dot{x}(0) = 2 \rightarrow c_2 = \frac{6}{\sqrt{2}} \end{array}\right\} x(t) = \frac{6}{\sqrt{2}} \cdot e^{-\frac{2}{3}t} \sin\left(\frac{\sqrt{2}}{3}t\right)$$

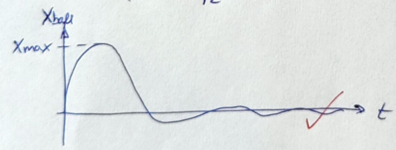
Oscillating motion with $\omega = \frac{\sqrt{2}}{3}$ and damping $\lambda = -\frac{2}{3}$

max. Amplitude where $\dot{x}(t) \overset{!}{=} 0$

$\rightarrow -\frac{\cancel{2}}{\cancel{3}} \cancel{e^{-\frac{2}{3}t}} \cdot \frac{\cancel{6}}{\sqrt{2}} \sin\left(\frac{\sqrt{2}}{3}t\right) + \frac{\sqrt{2}}{\cancel{3}} \cancel{e^{-\frac{2}{3}t}} \frac{\cancel{6}}{\sqrt{2}} \cos\left(\frac{\sqrt{2}}{3}t\right) = 0$

$\sqrt{2} \sin\left(\frac{\sqrt{2}}{3}t\right) = \cos\left(\frac{\sqrt{2}}{3}t\right) \rightarrow t_{max} = \frac{3}{\sqrt{2}} \operatorname{atan}\left(\frac{1}{\sqrt{2}}\right)$

$x_{max}(t_{max}) = \frac{6}{\sqrt{2}} \cdot e^{-\sqrt{2} \operatorname{atan}\left(\frac{1}{\sqrt{2}}\right)} \cdot \sin\left(\operatorname{atan}\left(\frac{1}{\sqrt{2}}\right)\right)$



b)
A controller needs to be developed to control the position, velocity and acceleration of the ball. What are the parameters in the control law partitioning for the above example? How is the physical system abstracted through the application of the control law partitioning and why?

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6
7
8

b) $m\ddot{x}_b + \nu(\dot{x}_b - \dot{x}_H) + \mu(x_b - x_H) = \alpha f' + \beta$

$\beta = \nu(\dot{x}_b - \dot{x}_H) + \mu(x_b - x_H)$

$\alpha = m$

Abstraction of physical system to a point mass
damping and non-linearity compensated in model part of control.

Why: no need to pass physical parameters to user

0 1 2 3 4 5 6 7 8 9 10

c)* Assume that the controlling force f_c that you calculate based on your chosen control law is generated by moving the membrane to a position $x_H(t)$ (through motion of the handle). How do you choose the parameters in your control law to get the fastest compensation of any errors after control law partitioning in the system after the initial disturbance? Give the corresponding equations and the numerical values for the resulting PD controller.

c) $f_c = \ddot{x}_d + k_v \dot{e} + k_p e \qquad \ddot{x}_d = \dot{x}_d = 0$

$$\rightarrow f_c = 0 - k_v \dot{x}_b + k_p (x_d - x_b)$$

$$= \mu (x_H - x_b)$$

optimal for critical damping

$$\Rightarrow k_v = \sqrt{k_p} \cdot 2$$

$$\mu (x_H - x_b) = -2\sqrt{k_p}\, \dot{x}_b + k_p (x_d - x_b)$$

$$x_H = x_b + \frac{k_p (x_d - x_b) - 2\sqrt{k_p}\, \dot{x}_b}{\mu}$$

d)*

Which additional component needs to be added to the control law to compensate the influence of a possible additional constant gravitational force f_G (vertical down) treated as disturbance here? Write the form of the resulting control law f_c that compensates this disturbance. What is compensated through this extension in general?

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d) $f_c = \ddot{x}_d + k_v \dot{e} + k_p e + k_I \int e \, dt$

Integration part brings steady-state error to 0. k_I needs to be chosen small compared to k_v and k_p the resulting additional root for in negative

Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.