Problem 2 Kinematics (37 credits)

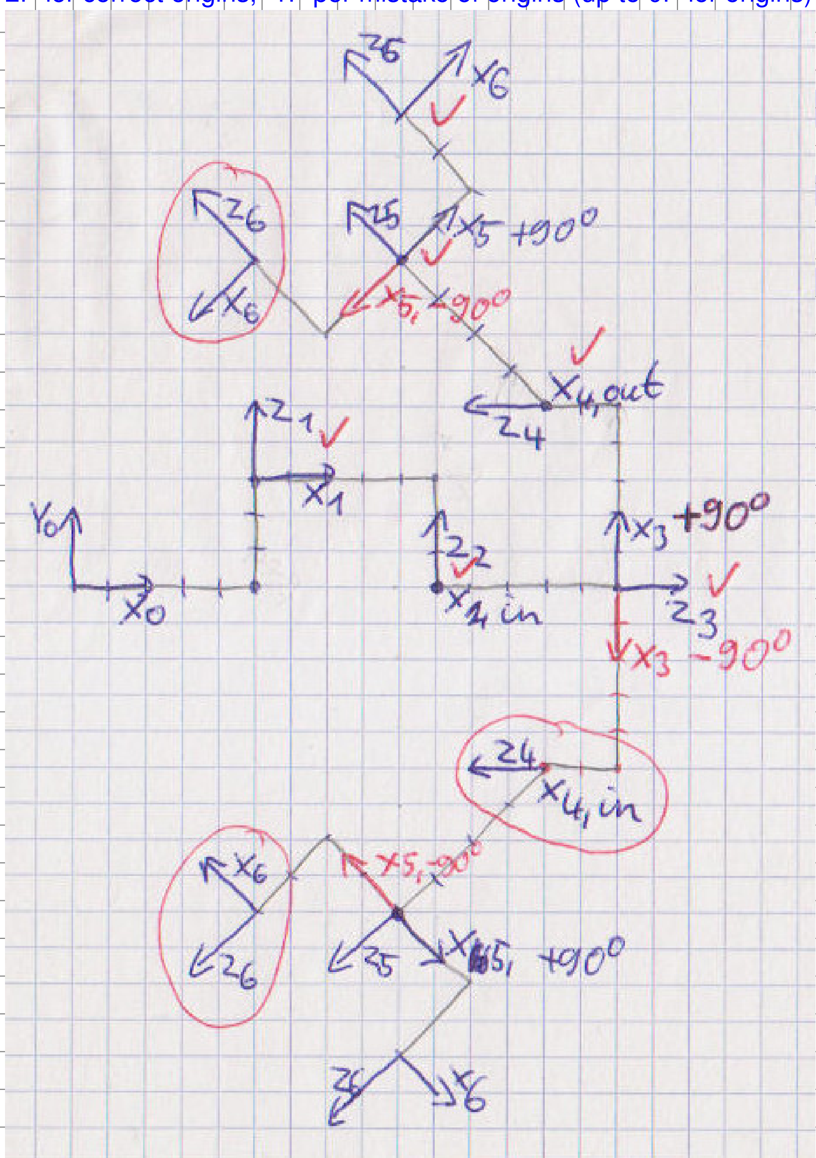
a)*

Joint	a_{i-1}	α_i	d_i	Θ_i
1	5	$-90°$	3	$0°$
2	5	$0°$	-3	$90°$
3	0	$90°$	5	
4	5	$180°$	2	$-90°$
5	0	$-45°$	$4\sqrt{2}$	
6	$2\sqrt{2}$	$0°$	$2\sqrt{2}$	$0°$

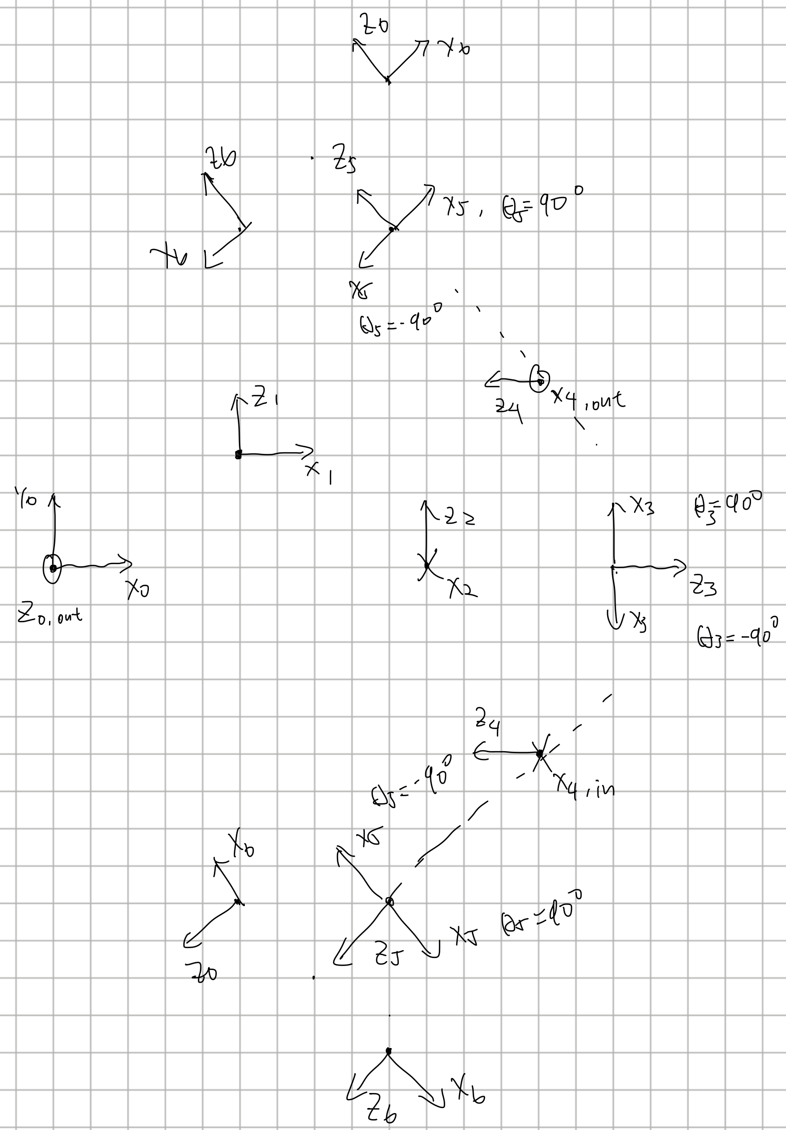
The table on the left is a description of a robot according to Denavit-Hartenberg-Parameters.
Draw all coordinate systems into the grid below. One cell in the grid correspond to one length unit in DH.
Choose the values of Θ_3 and Θ_5 such that all coordinate system origins lie in the drawing plane. You need only draw the x and z axes of the coordinate systems, possibly pointing in or out.

Write down your choices of Θ_3 and Θ_5. Start with the origin on the left in the middle with Y_0 pointing up and X_0 to the right.

$\Theta_3 = \pm 90°$ and $\Theta_5 = \pm 90°$.
1P per correct orientation of frames.
2P for correct origins, -1P per mistake of origins (up to 0P for origins)

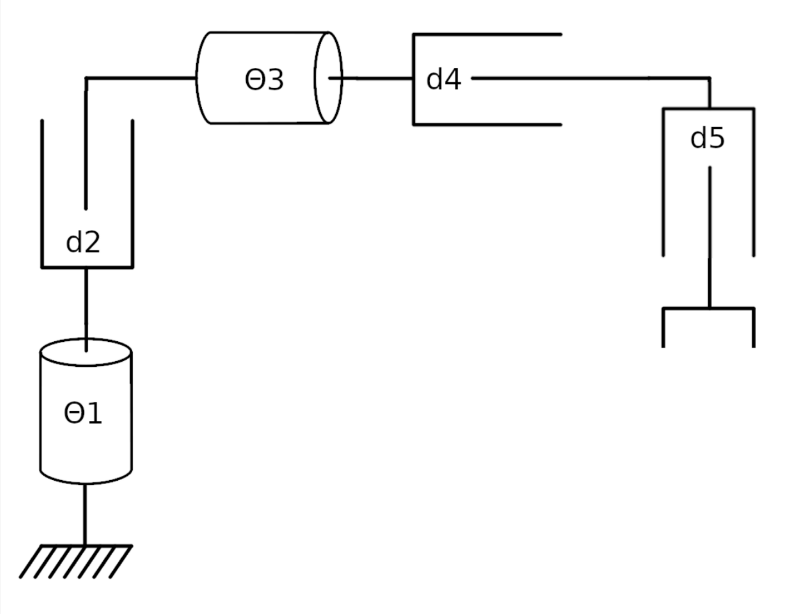


Joint	a_{i-1}	α_i	d_i	Θ_i
1	5	-90°	3	0°
2	5	0°	-3	90°
3	0	90°	5	
4	5	180°	2	-90°
5	0	-45°	$4\sqrt{2}$	
6	$2\sqrt{2}$	0°	$2\sqrt{2}$	0°



0 1 2 3 4 5 6 7 8 9

b) Given the RPRPP pick and place robot in the figure below. The robot is in zero configuration. Assume Z_0 up, X_0 in, origin at the bottom. Unknown lengths are l_1, l_3 for the rotational joints. What is special about joint pairs 1,2 and 3,4? Use that to get less coordinate frames/shorter DH table. Write down the DH table for the robot (it is in the YZ-plane entirely). Z_5 is down.



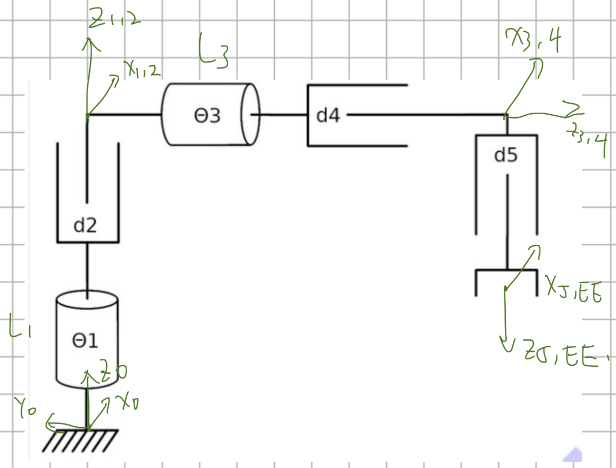
Joints 1,2 and 3,4 have coaxial Z-axes(or similar explanation), thus they can be merged into one coordinate system per pair.

short version, +2P

Joint	a_{i-1}	α_i	d_i	Θ_i
1	0	0°	$l_1 + d2$	Θ_1
2	0	90°	$l_3 + d4$	Θ_3
3	0	90°	$d5$	0°

long version, only 5P for full DH table

Joint	a_{i-1}	α_i	d_i	Θ_i
1	0	0°	l_1	Θ_1
2	0	0°	d_2	0°
3	0	90°	l_3	Θ_3
4	0	0°	d_4	0°
5	0	90°	d_5	0°



RPRPP

Joint 1 和 Joint 2 can be seen as a joint.

J	a_{i-1}	α_{i-1}	d_i	θ_i	value
1	0	0	L_1+d_2	θ_1	
2	0	0	0	0	
3	0	90^0	L_3+d_4	θ_3	
4	0	0	0	0	
5	0	90^0	d_5	0	
EE	0	0	0	0	

J	a_{i-1}	α_{i-1}	d_i	θ_i
1+2	0	0	L_1+d_2	θ_1
3+4	0	90^0	L_3+d_4	θ_3
5	0	90^0	d_5	0

$\theta_3 = 0$

- cylinedrical space
- $\theta_1 \rightarrow$ ~~angular~~ angle in XY.
  - $d_2 \rightarrow Z_0$
  - $d_4 \rightarrow Y_0 \rightarrow$ radius in Y_0 plane
  - $d_5 \rightarrow Z_0$

- Joint 1

$$ {}^0P_{EE} = \begin{pmatrix} (l_3+d_4)s_1 \\ -(l_3+d_4)c_1 \\ l_1+d_2-d_5 \end{pmatrix} $$

$$ {}^0J = \begin{array}{c} \begin{array}{cccc} 1\;\;2 & & 4\;\;5 & \end{array} \\ \begin{pmatrix} (l_3+d_4)c_1 & 0 & s_1 & 0 \\ (l_3+d_4)s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \end{array} $$

- Joint 2 and 5 coupled $\Longleftrightarrow$ constant singularity
- $\theta_1 = \{0, 180^0\} \wedge (l_3+d_4)=0 \Rightarrow row_1 = 0$
- $\theta_1 = \{90, 270^0\} \wedge (l_3+d_4)=0 \Rightarrow row_2 = 0.$

c) Given that $\Theta_3 = 0^\circ$ is **fixed** for the robot from the previous problem as a **simplified** robot.
What kind of workspace does it reach? Imagine joints 1,2,4,5 moving and describe the resulting shape shortly.
Which joints influence which dimension in the base frame? (e.g. joint 1 is X_0 direction, joint 2 is radius l_2 etc)
Which joints determine orientation of the end effector?
Write down the Kinematics (position only) of the end effector for the **simplified** robot.

The workspace of the robot is a cylindrical shape.
Joint 1 determines the angle in the XY-plane.
Joint 2 and Joint 5 determine the height in Z_0.(only one point if said separately)
(Joint 3 is fixed as defined in the problem.)
Joint 4 determines the radius of the cylinder $l_3 + d_4$.

Only joint 1 determines the orientation of the end effector.

$$^0p_0(\Theta) = \begin{pmatrix} s_1 * (l_3 + d_4) \\ -c_1 * (l_3 + d_4) \\ l_1 + d_2 - d_5 \end{pmatrix}$$

d) Derive the Jacobian for the position for above (simplified) robot. Determine singularities by reasoning.
There is no regular determinant to calculate for a $m \times n$ matrix.

$$^0J(\Theta) = \begin{pmatrix} c_1 * (l_3 + d_4) & 0 & 0 & s_1 & 0 \\ s_1 * (l_3 + d_4) & 0 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

Joint 5 is directly opposing joint 2, so they are in constant singularity.

Note: No credit for joint 3, as it is optional (it is being treated as just a piece with length l_3 now.
It may be missing in the Jacobian entirely.

0
1
2
3
4

e)* A fellow student drew link frames for a robot but forgot the base frame X_0,Z_0.
"You may put an arbitrary frame in as we can rotate it to the first link anyway".
Is that true? Refer to/add the relevant formula and point out and explain why/why not.

$$
{}^{i-1}_{i}\mathrm{T} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_{i-1} \\ \sin\theta_i \cos\alpha_{i-1} & \cos\theta_i \cos\alpha_{i-1} & -\sin\alpha_{i-1} & -\sin\alpha_{i-1} d_i \\ \sin\theta_i \sin\alpha_{i-1} & \cos\theta_i \sin\alpha_{i-1} & \cos\alpha_{i-1} & \cos\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Mention/add above formula: Transformation between consecutive frames with DH-convention
No, arbitrary movement is not possible
Entry in row 1, column 3 is zero
No rotation around Y-Axis is possibleOR only rotation around X and Z is possible.

0
1

f)* Additional space for **P1 only** if you need it. Refer to here if you used it!

Problem 3 Lagrange (37 credits)

The motion $(\Theta_1(t), \Theta_2(t))$ of a double-pendulum depicted in Fig. 3.1 is supposed to be analysed using a **Lagrangian method** discussed in the lecture. The two rotational joints are at the origin and at m_1. The links (l_1, l_2) have no mass or inertia and the entire mass can be considered as point masses at (m_1, m_2). The direction of Earth acceleration g is shown in the figure.

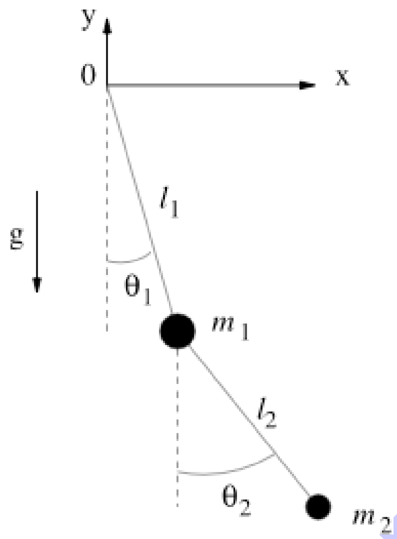


Figure 3.1: Double-pendulum

a)* Calculate the positions (x_i, y_i) and velocities $(\dot{x}_i.\dot{y}_i)$ of (m_1, m_2).

0 1 2 3 4

$$x_1 = l_1 \sin(\theta_1) \qquad \dot{x}_1 = l_1\dot{\theta}_1 \cos(\theta_1)$$

$$y_1 = -l_1 \cos(\theta_1) \qquad \dot{y}_1 = l_1\dot{\theta}_1 \sin(\theta_1)$$

$$x_2 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2) \qquad \dot{x}_2 = l_1\dot{\theta}_1 \cos(\theta_1) + l_2\dot{\theta}_2 \cos(\theta_2)$$

$$y_2 = -l_1 \cos(\theta_1) - l_2 \cos(\theta_2), \qquad \dot{y}_2 = l_1\dot{\theta}_1 \sin(\theta_1) + l_2\dot{\theta}_2 \sin(\theta_2).$$

① a) ${}^0P_1 = \begin{pmatrix} L_1S_1 \\ -L_1C_1 \\ 0 \end{pmatrix}$ ${}^0P_2 = \begin{pmatrix} L_1S_1 + L_2S_2 \\ -L_1C_1 - L_2C_2 \\ 0 \end{pmatrix}$

${}^0V_{C_1} = \begin{pmatrix} L_1C_1\dot\theta_1 \\ L_1S_1\dot\theta_1 \\ 0 \end{pmatrix}$ ${}^0V_{C_2} = \begin{pmatrix} L_1C_1\dot\theta_1 + L_2C_2\dot\theta_2 \\ L_1S_1\dot\theta_1 + L_2S_2\dot\theta_2 \\ 0 \end{pmatrix}$

b) $k_i = \frac{1}{2} m_i \, {}^0V_{C_i}^T \cdot {}^0V_{C_i} + \frac{1}{2} {}^i\omega_i^T \, {}^{C_i}I_i \, {}^i\omega_i$ $I=0$

$k_1 = \frac{1}{2} m_1 \left[L_1^2C_1^2\dot\theta_1^2 + L_1^2S_1^2\dot\theta_1^2 \right] = \frac{1}{2} m_1 L_1^2 \dot\theta_1^2$

$k_2 = \frac{1}{2} m_2 \left[L_1^2\cancel{C_1^2}\dot\theta_1^2 + 2L_1L_2C_1C_2\dot\theta_1\dot\theta_2 + \cancel{L_2^2C_2^2\dot\theta_2^2} + L_1^2\cancel{S_1^2}\dot\theta_1^2 + 2L_1L_2S_1S_2\dot\theta_1\dot\theta_2 + \cancel{L_2^2S_2^2\dot\theta_2^2} \right]$

$= \frac{1}{2} m_2 \left[L_1^2\dot\theta_1^2 + L_2^2\dot\theta_2^2 + 2L_1L_2\dot\theta_1\dot\theta_2 C_{1-2} \right]$

$u_i = -m_i \, {}^0g^T \, {}^0P_{C_i} + u_{ref_i}$ ${}^0g = \begin{pmatrix} 0 \\ -g \\ 0 \end{pmatrix}$

$u_1 = -m_1 \cdot (0\ -g\ 0) \begin{pmatrix} L_1S_1 \\ -L_1C_1 \\ 0 \end{pmatrix} + u_{ref_1}$

$= -m_1 \cdot (L_1C_1g) + u_{ref_1}$

$= -m_1L_1C_1g + m_1L_1g$

$u_2 = -m_2 \cdot (0\ -g\ 0) \begin{pmatrix} L_1S_1 + L_2S_2 \\ -L_1C_1 - L_2C_2 \\ 0 \end{pmatrix} + u_{ref_1}$

$= -m_2 \cdot (L_1C_1 + L_2C_2) g + u_{ref_1}$

$= -m_2 g (L_1C_1 + L_2C_2) + (L_1 + L_2) m_2 g$

$k = \frac{1}{2} m_1 L_1^2 \dot\theta_1^2 + \frac{1}{2} m_2 \left[L_1^2\dot\theta_1^2 + L_2^2\dot\theta_2^2 + 2L_1L_2\dot\theta_1\dot\theta_2 C_{1-2} \right]$

$u = (-m_1 - m_2)L_1C_1g - m_2L_2C_2g + m_1L_1g + m_2(L_1+L_2)g$

$L = k - p = \frac{1}{2} m_1L_1^2\dot\theta_1^2 + \frac{1}{2} m_2 \left[L_1^2\dot\theta_1^2 + L_2^2\dot\theta_2^2 + 2L_1L_2\dot\theta_1\dot\theta_2 C_{1-2} \right] + (m_1+m_2)L_1C_1g + m_2L_2C_2g - m_1L_1g - m_2(L_1+L_2)g$

$\tau_i = \frac{d}{dt}\frac{\partial L}{\partial \dot\theta_i} - \frac{\partial L}{\partial \theta_i}$

$\frac{\partial L}{\partial \dot\theta_1} = m_1L_1^2\dot\theta_1 + m_2L_1^2\dot\theta_1 + m_2L_1L_2\dot\theta_2C_{1-2}$

$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_1} = m_1L_1^2\ddot\theta_1 + m_2L_1^2\ddot\theta_1 + m_2L_1L_2\left(\ddot\theta_2C_{1-2} - \dot\theta_2 \cdot S_{1-2}(\dot\theta_1 - \dot\theta_2)\right)$

$\frac{\partial L}{\partial \dot\theta_2} = m_2L_2^2\dot\theta_2 + m_2L_1L_2\dot\theta_1C_{1-2}$

$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_2} = m_2L_2^2\ddot\theta_2 + m_2L_1L_2\left[\ddot\theta_1C_{1-2} - \dot\theta_1S_{1-2}(\dot\theta_1 - \dot\theta_2)\right]$

$\frac{\partial L}{\partial \theta_1} = -m_2L_1L_2\dot\theta_1\dot\theta_2S_{1-2} - (m_1+m_2)L_1S_1g$

$\frac{\partial L}{\partial \theta_2} = m_2L_1L_2\dot\theta_1\dot\theta_2S_{1-2} - m_2L_2S_2g$

$\tau_1 = (m_1+m_2)L_1^2\ddot\theta_1 + m_2L_1L_2\left(\ddot\theta_2C_{1-2} - \cancel{\dot\theta_2} \cdot S_{1-2}(\cancel{\dot\theta_1} - \dot\theta_2)\right) + m_2L_1\cancel{L_2\dot\theta_1\dot\theta_2}S_{1-2} + (m_1+m_2)L_1S_1g$

$\tau_2 = m_2L_2^2\ddot\theta_2 + m_2L_1L_2\left[\ddot\theta_1C_{1-2} - \cancel{\dot\theta_1}S_{1-2}(\dot\theta_1 - \cancel{\dot\theta_2})\right] - m_2L_1\cancel{L_2\dot\theta_1\dot\theta_2}S_{1-2} + m_2L_2S_2g$

$\tau_1 = (m_1+m_2)L_1\ddot\theta_1 + m_2L_1L_2C_{1-2}\ddot\theta_2 + m_2L_1L_2S_{1-2}\dot\theta_2^2 + (m_1+m_2)L_1S_1g$

$\tau_2 = m_2L_1L_2C_{1-2}\ddot\theta_1 + m_2L_2^2\ddot\theta_2 - m_2L_1L_2S_{1-2}\dot\theta_1^2 + m_2L_2S_2g$

$M(\theta) = \begin{bmatrix} (m_1+m_2)L_1^2 & m_2L_1L_2C_{1-2} \\ m_2L_1L_2C_{1-2} & m_2L_2^2 \end{bmatrix}$ $G(\theta) = \begin{bmatrix} (m_1+m_2)L_1S_1g \\ m_2L_2S_2g \end{bmatrix}$

$V(\theta, \dot\theta) = \begin{bmatrix} m_2L_1L_2S_{1-2}\dot\theta_2^2 \\ -m_2L_1L_2S_{1-2}\dot\theta_1^2 \end{bmatrix}$

0
1
2
3
4
5
6
7
8

b) Calculate the potential energy U and kinetic energy K for the entire double-pendulum. Use values estimated in a).

$$T = \frac{1}{2}\Big[m_1 l_1^2 \dot{\theta}_1^2 + m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1\dot{\theta}_2\left\{\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right\}\right)\Big]$$

$$= \frac{1}{2}\Big[m_1 l_1^2 \dot{\theta}_1^2 + m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)\right)\Big]$$

$$U = -(m_1 + m_2) g l_1 \cos(\theta_1) - m_2 g l_2 \cos(\theta_2),$$

0
1
2
3
4

c) Calculate the expression for the Lagrangian L from the above values. Simpify the sine and cosine products using trigonometric identities.

$$L = T - U$$

$$= \frac{1}{2}\Big[m_1 l_1^2 \dot{\theta}_1^2 + m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)\right)\Big]$$

$$+ (m_1 + m_2) g l_1 \cos(\theta_1) + m_2 g l_2 \cos(\theta_2).$$

d)* Calculate the necessary derivatives to estimate the torques in the rotational joints (τ_1, τ_2) required for the Lagrangian method. In case that you were not able to calculate the Lagrangian, assume a non-trivial trigonometric expression for L for partial points.

$$\frac{\partial L}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 \sin(\theta_1)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \left[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)\right]$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin(\theta_2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = m_2 \left(l_2^2 \ddot{\theta}_2 + l_1 l_2 \left[\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)\right]\right).$$

e) Write the expressions for (τ_1, τ_2) from above calculations giving also the symbolic equation, which terms from above are combined.

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \tau_j$$

$$(m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \left[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)\right] + (m_1 + m_2) g l_1 \sin(\theta_1) = \tau_1$$

$$m_2 \left(l_2^2 \ddot{\theta}_2 + l_1 l_2 \left[\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)\right]\right) + m_2 g l_2 \sin(\theta_2) = \tau_2$$

0 1 2 3 4 5 6

f) In case of free swinging pendulum, both torques (τ_1, τ_2) are equal to zero. Write the corresponding two equations relating angular parameters of the double pendulum (angle, angular velocities, angular accelerations. Identify the M,V,G expressions used to design a stable control.

$$\ddot{\theta}_1 + \frac{m_2}{m_1 + m_2}\frac{l_2}{l_1}\left(\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2{}^2 \sin(\theta_1 - \theta_2)\right) + \frac{g}{l_1}\sin(\theta_1) = 0$$

$$\ddot{\theta}_2 + \frac{l_1}{l_2}\left(\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1{}^2 \sin(\theta_1 - \theta_2)\right) + \frac{g}{l_2}\sin(\theta_2) = 0.$$

$$M = \begin{pmatrix}(m_1 + m_2)l_1^2 & m_2 l_1 l_2 \cos(\Theta_1 - \Theta_2) m_2 l_1 l_2 \cos(\Theta_1 - \Theta_2) & m_1 l_2^2\end{pmatrix} \quad V = \begin{pmatrix} m_2 l_1 l_2 \sin(\Theta_1 - \Theta_2)\dot{\Theta}_2^2 \\ -l_1 l_2 m_2 \sin(\Theta_1 - \Theta_2)\dot{\Theta}_1^2 \end{pmatrix}$$

$$G = \begin{pmatrix}(m_1 + m_2) g l_1 \sin\Theta_1 \\ m_2 g l_2 \sin\Theta_1\end{pmatrix}$$

Problem 4 **Control (32 credits)**

A simple one link robot can be represented with the following simplified SMD system (Fig. 4.1).

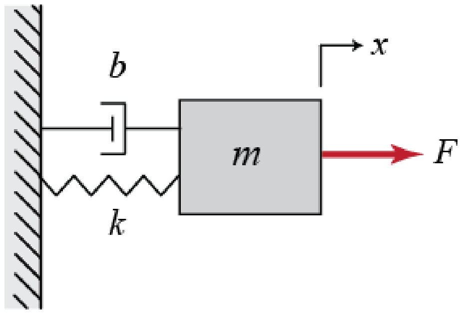


Figure 4.1: SMD system

a)* Write the equation balancing the forces and derive the characteristic equation for this system transforming this expression into the Laplace space with parameter **s** and using the identity $\delta^n x/\delta t^n = s^n \cdot X(s)$.

$$m\ddot{x} + b\dot{x} + kx = F$$

Taking the Laplace transform of the governing equation, we get

$$ms^2X(s) + bsX(s) + kX(s) = F(s)$$

The transfer function between the input force $F(s)$ and the output displacement $X(s)$ then becomes

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

b)* Convert the above characteristic equation into an expression using natural frequency ω_n and damping ξ. Explain how to calculate these values from the parameters in the figure above and name the value for a maximal frequency that can occur in such a system and how is it used to calculate absolute control values in the controler.

$$\zeta = \frac{b}{2\sqrt{km}},$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0, \qquad \omega_n = \sqrt{k/m}.$$

maximal frequency is $\omega_n$and this is used to move the system away from ω_{res}.

P4

$$m\ddot{x} + b\dot{x} + kx = F$$

$$ms^2 X(s) + bs\,X(s) + k\,X(s) = F(s)$$

$$\frac{F(s)}{X(s)} = ms^2 + bs + k$$

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

$$ms^2 + bs + k = 0 \Rightarrow s^2 + \frac{b}{m}s + \frac{k}{m} = 0$$

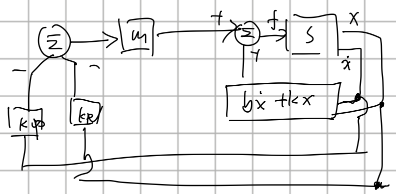
$$s^2 + 2\zeta w_n + w_n^2 = 0$$

$$w_n = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{b}{2\sqrt{km}}$$

$$w_n \le \frac{1}{2} W_{Resonance}$$

$$m\ddot{x} + b\dot{x} + kx = f = \alpha f' + \beta$$

$$\begin{cases} \alpha = m \\ \beta = b\dot{x} + ky \\ f' = \ddot{x} = -k_v\dot{x} + k_p x \end{cases}$$



$m=1 \quad b=1 \quad k=1$

$$s^2 + s + 1 = 0 \qquad x(0) = -1 \qquad s = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$b^2 < 4mk. \qquad \dot{x}(0) = 0 \qquad \lambda = -\frac{1}{2} \quad \mu = \frac{\sqrt{3}}{2}$$

$$x(t) = e^{-\frac{1}{2}t}\left(C_1 \cdot \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

$$x(0) = C_1 = -1$$

$$x'(0) = \cancel{e^{-\frac{1}{2}t}} \cdot \left(-\frac{1}{2}\right)\left(C_1\cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \cdot \cancel{\sin\left(\frac{\sqrt{3}}{2}t\right)}\right) + \cancel{e^{-\frac{1}{2}t}} \cdot \left(-C_1 \cancel{\sin\left(\frac{\sqrt{3}}{2}t\right)} \cdot \frac{\sqrt{3}}{2} + C_2\cos\left(\frac{\sqrt{3}}{2}t\right) \cdot \frac{\sqrt{3}}{2}\right)$$

$$= -\frac{1}{2}C_1 + \frac{\sqrt{3}}{2}C_2 = 0$$

$$\sqrt{3}C_2 = C_1$$

$$C_2 = -\frac{\sqrt{3}}{3}$$

$$x(t) = e^{-\frac{1}{2}t}\left(-\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

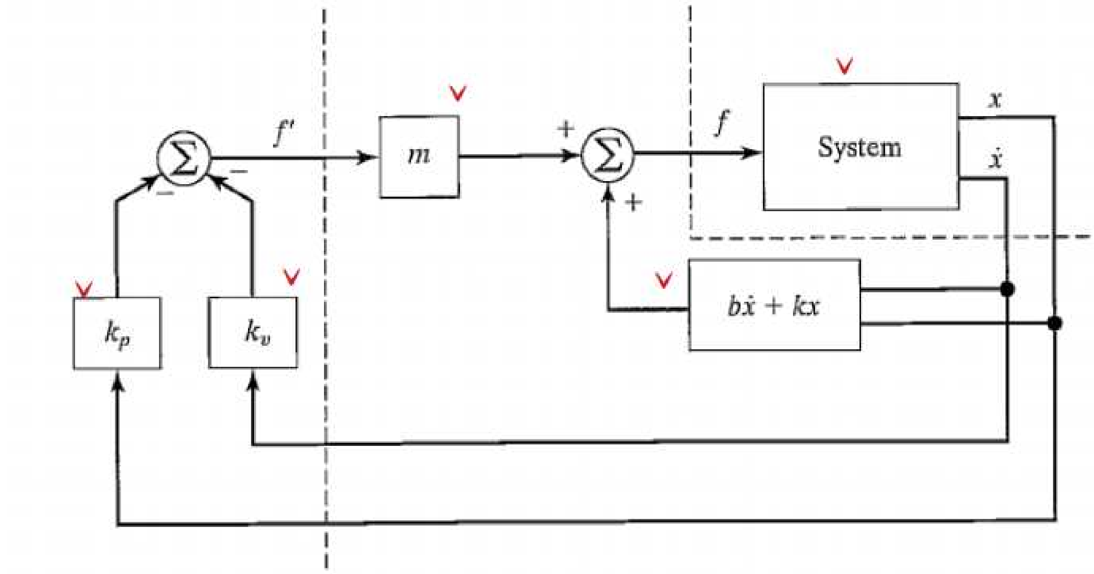
0 1 2 3 4 5 6 7 8 9 10 11

c)* Explain with the equation the control law partitioning applied to the force balance differential equation. Draw the corresponding control structure. How does the system appear to the controler?

$$m\ddot{x} + b\dot{x} + kx = \alpha f' + \beta.$$

Clearly, in order to make the system appear as a unit mass from the f' input, for this particular system we should choose α and β as follows:

$$\alpha = m,$$
$$\beta = b\dot{x} + kx.$$



0 1 2 3 4 5 6 7 8 9

d)* Find the motion of the system in Fig. 4.1 if the parameter values are m=1, b=1 and k=1 and the block (initially at rest) is released from position x=-1.

- write the characteristic equation with unknowns c_1, c_2
- estimate form of the x(t) equation with the unknowns (c_1, c_2)
- calculate c_1, c_2 from the inital conditions mentioned above
- write the resulting x(t)

$s^2 + s + 1 = 0$

which has the roots $s_i = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Hence, the response has the form

$$x(t) = e^{-\frac{t}{2}}\left(c_1 \cos\frac{\sqrt{3}}{2}t + c_2 \sin\frac{\sqrt{3}}{2}t\right).$$

We now use the given initial conditions, $x(0) = -1$ and $\dot{x}(0) = 0$, to compute c_1 and c_2. To satisfy these conditions at $t = 0$, we must have

$$c_1 = -1$$

and

$$-\frac{1}{2}c_1 - \frac{\sqrt{3}}{2}c_2 = 0,$$

which are satisfied by $c_1 = -1$ and $c_2 = \frac{\sqrt{3}}{3}$. So, the motion of the system for $t \geq 0$ is given by

$$x(t) = e^{-\frac{t}{2}}\left(-\cos\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}t\right).$$

Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.