Recapitulation: Forces and Torques for static manipulators

For propagation of forces and torques in a non-moving manipulator, the following equations hold:

$$
\begin{aligned}
{}^{i}f_i &= {}^{i}_{i+1}R \cdot {}^{i+1}f_{i+1} \\
{}^{i}n_i &= {}^{i}_{i+1}R \cdot {}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i
\end{aligned}
$$

The force that affects a link is denoted by f (a three-dimensional vector), and the torque on that link is denoted by n (also three-dimensional). Note that some parts of the forces and torques apply directly to the corresponding joint, and some parts are absorbed by the mechanics of the robot. The relation between these quantities is:

$$
\tau_i = {}^{i}n_i^{\mathrm{T}}\,{}^{i}Z_i = {}^{i}n_i^{\mathrm{T}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

$$
\tau_i = {}^{i}f_i^{\mathrm{T}}\,{}^{i}Z_i = {}^{i}f_i^{\mathrm{T}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

The first equation is used for rotational joints, the second equation for prismatic joints. The quantities τ_i thus specify the amount of torque resp. force that is affecting the joint, and thus the amount of torque resp. force that the robot should counteract in order to remain static. The joint torques/forces τ_i are 1-dimensional quantities. The Jacobian plays a very important role here as well: It relates joint torques/forces τ_i to endeffector forces and torques f, n:

$$
\begin{pmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_n \end{pmatrix} = \tau = {}^{A}J^{T}\,{}^{A}\mathcal{F} = {}^{A}J^{T} \begin{pmatrix} {}^{A}f \\ {}^{A}n \end{pmatrix}
$$

Where $\mathcal{F}$ is a 6-dimensional Vector of force and torque containing 3D force and 3D torque vectors stacked above each other. It is important that the force-torque vector has the same frame of reference as the Jacobian.

Rotational velocities and the Jacobian

In the previous problem sheet, entries of the Jacobian concerning rotations have been computed based on simplified position descriptions. We have not explained yet how the Jacobian can be computed efficiently in the general case where arbitrary rotations are possible.

Let $p : \mathbb{R}^n \to \mathbb{R}^3$ be a function that computes the coordinates of the origin of the end effector with respect to system $\{0\}$, then the full Jacobian looks like this:

$$
{}^{0}J = \begin{pmatrix}
\frac{\partial p_1}{\partial x_1} & \frac{\partial p_1}{\partial x_2} & \cdots & \frac{\partial p_1}{\partial x_n} \\
\frac{\partial p_2}{\partial x_1} & \frac{\partial p_2}{\partial x_2} & \cdots & \frac{\partial p_2}{\partial x_n} \\
\frac{\partial p_3}{\partial x_1} & \frac{\partial p_3}{\partial x_2} & \cdots & \frac{\partial p_3}{\partial x_n} \\
{}^{0}\hat{Z}_1 & {}^{0}\hat{Z}_2 & \cdots & {}^{0}\hat{Z}_n
\end{pmatrix}
$$

We are familiar with the first three rows of this Jacobian, but the final row is something new. First of all, we need to explain the meaning of ${}^{j}\hat{Z}_i$. This denotes the z-axis of system i, expressed relative

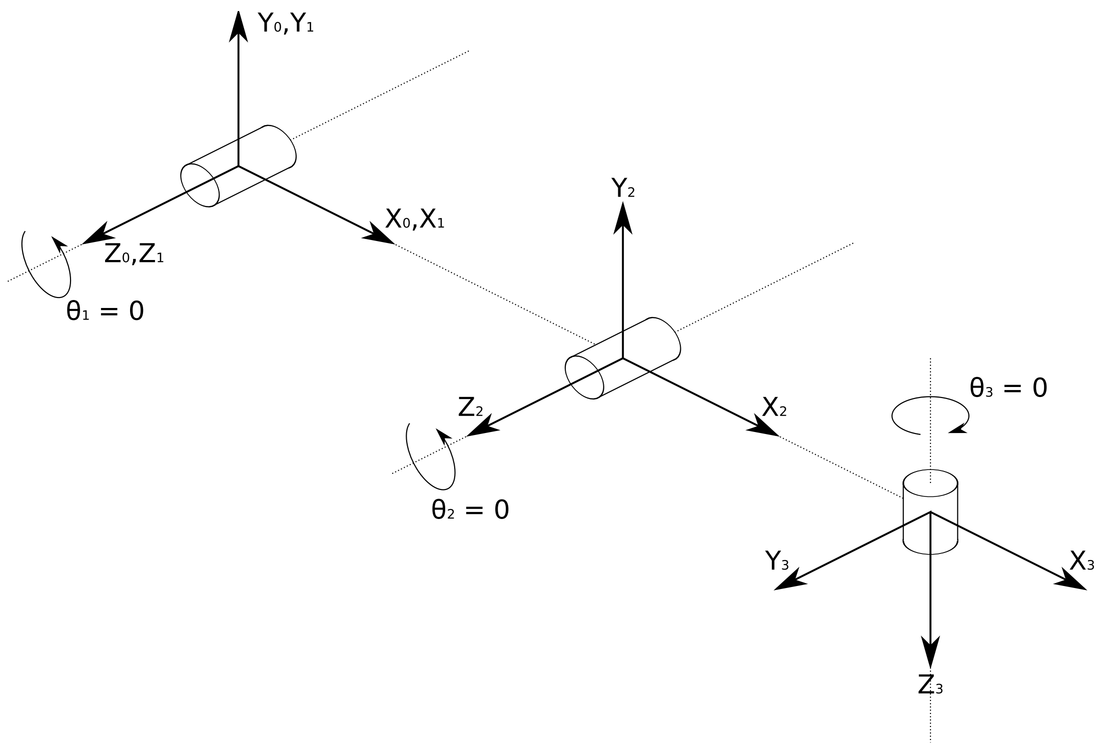

Figure 1: *3R-Robot*

to frame $\{j\}$. Thus, it is a three-dimensional unit vector, and we have ${}^{i}\hat{Z}_i = (0,0,1)^T$. So we see that the last row in the above matrix really stands for three rows. An example: For the well-known planar 3R-manipulator from the previous exercises, the last three rows of the complete Jacobian would look like this:

$$J = \begin{pmatrix} {}^{0}\hat{Z}_1 & {}^{0}\hat{Z}_2 & {}^{0}\hat{Z}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

This is obvious, since all joint axes are parallel and represent rotations around ${}^{0}\hat{Z}_0$, and it also corresponds to the observation that joint rates simply add up. Now what if the third axis were rotated about 90° instead of being parallel to the second axis? The corresponding robot is shown in Figure 1. According to above rules, the lower part of the Jacobian would look like this:

$$J = \begin{pmatrix} {}^{0}\hat{Z}_1 & {}^{0}\hat{Z}_2 & {}^{0}\hat{Z}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \mathrm{s}_{12} \\ 0 & 0 & -\mathrm{c}_{12} \\ 1 & 1 & 0 \end{pmatrix}$$

This observation often simplifies the computation of the Jacobian: All you have to do is to determine the direction of the z-axes of the coordinate systems. Note however, that a prismatic joint will never generate a rotational velocity, thus the entry in the column corresponding to the rotational velocity of a prismatic joint is set to $(0,0,0)^T$ instead of ${}^{0}\hat{Z}_i$.

If determining the directions of the joint axes becomes too complicated, you can still apply the formulas for computing rotational velocities ${}^{i}\omega_i$ and derive the Jacobian from that. If you do that, always keep in mind that you need to be in the right frame of reference, i.e., if the position description is in frame 0, you need to transform ${}^{n}\omega_n$ to frame 0 as well if you want to compute ${}^{0}J$.

The mathematical justification for above rule is as follows: For the computation of ${}^{1}\omega_1, {}^{2}\omega_2, \ldots$ there is a recursive formula

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R \cdot {}^{i}\omega_i + \dot{\Theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}.$$

If you apply this formula generally to ${}^n\omega_n$, a certain pattern becomes visible:

$$\begin{aligned}
{}^n\omega_n &= {}^{n}_{n-1}R \cdot {}^{n-1}\omega_{n-1} + \dot{\Theta}_n \cdot {}^n\hat{Z}_n \\
&= {}^{n}_{n-1}R \cdot ({}^{n-1}_{n-2}R \cdot {}^{n-2}\omega_{n-2} + \dot{\Theta}_{n-1} \cdot {}^{n-1}\hat{Z}_{n-1}) + \dot{\Theta}_n \cdot {}^n\hat{Z}_n \\
&= {}^{n}_{n-1}R \cdot ({}^{n-1}_{n-2}R \cdot ({}^{n-2}_{n-3}R \cdot {}^{n-3}\omega_{n-3} + \dot{\Theta}_{n-2} \cdot {}^{n-2}\hat{Z}_{n-2}) + \dot{\Theta}_{n-1} \cdot {}^{n-1}\hat{Z}_{n-1}) + \dot{\Theta}_n \cdot {}^n\hat{Z}_n \\
&= {}^{n}_{1}R \cdot \dot{\Theta}_1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + {}^{n}_{2}R \cdot \dot{\Theta}_2 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + {}^{n}_{3}R \cdot \dot{\Theta}_3 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \ldots + {}^{n}_{n}R \cdot \dot{\Theta}_n \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}$$

We are only interested in ${}^0\omega_n$, thus we further compute:

$$\begin{aligned}
{}^0\omega_n &= {}^{0}_{n}R^n\omega_n \\
&= {}^{0}_{n}R^n_1R \cdot \dot{\Theta}_1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + {}^{0}_{n}R^n_2R \cdot \dot{\Theta}_2 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + {}^{0}_{n}R^n_3R \cdot \dot{\Theta}_3 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \ldots + {}^{0}_{n}R^n_nR \cdot \dot{\Theta}_n \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= {}^{0}_{1}R \cdot \dot{\Theta}_1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + {}^{0}_{2}R \cdot \dot{\Theta}_2 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + {}^{0}_{3}R \cdot \dot{\Theta}_3 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \ldots + {}^{0}_{n}R \cdot \dot{\Theta}_n \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \dot{\Theta}_1{}^0\hat{Z}_1 + \dot{\Theta}_2{}^0\hat{Z}_2 + \dot{\Theta}_3{}^0\hat{Z}_3 + \ldots + \dot{\Theta}_n{}^0\hat{Z}_n
\end{aligned}$$

Collecting the factors of $\dot{\Theta}_1, \dot{\Theta}_2, \dot{\Theta}_3, \ldots$, we see that the entries correspond to ${}^0\hat{Z}_1$, ${}^0\hat{Z}_2$ and so on, and we see that the scheme for computation of the Jacobian is correct!

Solution 1

a)

The Jacobian relates joint torques to external forces as follows:

$$\vec{\tau} = {}^4J^{\mathrm{T}4}\vec{f}$$

Here, we have 4 joint parameters, and the formula amounts to

$$\begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} = ({}^4J^{\mathrm{T}})^4\vec{f} = ({}^0J^{\mathrm{T}})^0\vec{f}.$$

To determine the Jacobian (according to the recipe in the recap.), we need to compute ${}^0P_{4\mathrm{ORG}}$. The transformation matrices are:

$${}^0_1\mathrm{T} = \begin{pmatrix} \cos\Theta_1 & -\sin\Theta_1 & 0 & 0 \\ \sin\Theta_1 & \cos\Theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^1_2\mathrm{T} = \begin{pmatrix} \cos\Theta_2 & -\sin\Theta_2 & 0 & 1 \\ \sin\Theta_2 & \cos\Theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^2_3\mathrm{T} = \begin{pmatrix} \cos\Theta_3 & -\sin\Theta_3 & 0 & 0 \\ \frac{\sin\Theta_3}{\sqrt{2}} & \frac{\cos\Theta_3}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -1 \\ \frac{\sin\Theta_3}{\sqrt{2}} & \frac{\cos\Theta_3}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^3_4\mathrm{T} = \begin{pmatrix} \cos\Theta_4 & -\sin\Theta_4 & 0 & \sqrt{2} \\ \sin\Theta_4 & \cos\Theta_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Based on that we can compute ${}^0_1\mathrm{T}^1_2\mathrm{T}^2_3\mathrm{T}^3_4\mathrm{T} \cdot (0,0,0,1)^\mathrm{T}$ step by step, which results in:

$$p(\Theta) = \begin{pmatrix} -\sin(\Theta_2+\Theta_1)\sin\Theta_3 + \sqrt{2}\cos(\Theta_2+\Theta_1)\cos\Theta_3 + \sin(\Theta_2+\Theta_1) + \cos\Theta_1 \\ \cos(\Theta_2+\Theta_1)\sin\Theta_3 + \sqrt{2}\sin(\Theta_2+\Theta_1)\cos\Theta_3 - \cos(\Theta_2+\Theta_1) + \sin\Theta_1 \\ \sin\Theta_3 + 1 \\ 1 \end{pmatrix}$$

The entries of the Jacobian, evaluated for $\Theta = (0, 90°, -90°)$, are then:

$$\frac{\partial p_1}{\partial \Theta_1} = -\sqrt{2}\mathrm{s}_{12}\mathrm{c}_3 - \mathrm{c}_{12}\mathrm{s}_3 + \mathrm{c}_{12} - \mathrm{s}_1 = -0 - 0 + 0 - 0 = 0$$
$$\frac{\partial p_1}{\partial \Theta_2} = -\sqrt{2}\mathrm{s}_{12}\mathrm{c}_3 - \mathrm{c}_{12}\mathrm{s}_3 + \mathrm{c}_{12} = -0 - 0 + 0 = 0$$
$$\frac{\partial p_1}{\partial \Theta_3} = -\sqrt{2}\mathrm{c}_{12}\mathrm{s}_3 - \mathrm{s}_{12}\mathrm{c}_3 = -0 - 0 = 0$$
$$\frac{\partial p_2}{\partial \Theta_1} = \sqrt{2}\mathrm{c}_{12}\mathrm{c}_3 - \mathrm{s}_{12}\mathrm{s}_3 + \mathrm{s}_{12} + \mathrm{c}_1 = 0 - (-1) + 1 + 1 = 3$$
$$\frac{\partial p_2}{\partial \Theta_2} = \sqrt{2}\mathrm{c}_{12}\mathrm{c}_3 - \mathrm{s}_{12}\mathrm{s}_3 + \mathrm{s}_{12} = 0 + 1 + 1 = 2$$
$$\frac{\partial p_2}{\partial \Theta_3} = -\sqrt{2}\mathrm{s}_{12}\mathrm{s}_3 + \mathrm{c}_{12}\mathrm{c}_3 = \sqrt{2} + 0 = \sqrt{2}$$
$$\frac{\partial p_3}{\partial \Theta_1} = 0$$
$$\frac{\partial p_3}{\partial \Theta_2} = 0$$
$$\frac{\partial p_3}{\partial \Theta_3} = \mathrm{c}_3 = 0$$

Thus we have computed the first three rows of the Jacobian. The lower 3 rows can be determined according to above considerations as ${}^0\hat{Z}_1, {}^0\hat{Z}_2, {}^0\hat{Z}_3, {}^0\hat{Z}_4$: $(0,0,1)^\mathrm{T}, (0,0,1)^\mathrm{T}, (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})^\mathrm{T}, (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})^\mathrm{T}$. All in all, the transposed Jacobian looks like this:

$$J^\mathrm{T} = \begin{pmatrix} 0 & 3 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Since the force-torque-vector is specified relative to system 4, we need to transform it to system 0. This is done by multiplying with the corresponding rotation matrix:

$${}^0f = \begin{pmatrix} {}^0_4R & 0_{3\times3} \\ 0_{3\times3} & {}^0_4R \end{pmatrix} {}^4f = \begin{pmatrix} {}^0_4R & 0_{3\times3} \\ 0_{3\times3} & {}^0_4R \end{pmatrix} \begin{pmatrix} 0 \\ 6 \\ 0 \\ 7 \\ 0 \\ 8 \end{pmatrix}$$

The rotation matrix ${}^0_4R$ in the current configuration $\Theta = (0, 90, -90, 0)^\mathrm{T}$ is

$${}^0_1\mathrm{R}^1_2\mathrm{R}^2_3\mathrm{R}^3_4\mathrm{R} = \mathrm{I} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix} \mathrm{I} = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

Since we already have determined the direction of the z-axis of system $\{4\}$, we can also derive the rotation matrix ${}^0_4R$ by figuring out ${}^0\hat{X}_4, {}^0\hat{Y}_4$ in addition to ${}^0\hat{Z}_4$:

$$
{}^0\hat{X}_4 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}, {}^0\hat{Y}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, {}^0\hat{Z}_4 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}
$$

The vectors are the columns of the rotation matrix.

All in all, we obtain:

$$
{}^0f = \begin{pmatrix} 0 \\ 6 \\ 0 \\ \frac{15}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}
$$

Finally, we can apply the formula for computing joint moments:

$$
\tau = \begin{pmatrix} 0 & 3 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 6 \\ 0 \\ \frac{15}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 18.707 \\ 12.707 \\ 16.485 \\ 8 \end{pmatrix}
$$

b)

Now we want to compute the forces and torques at the tip of a screwdriver that is attached to the robot. If we denote the system of the screw driver's tip with $\{5\}$, we have:

$$
{}^4f_4 = {}^4_5R\,{}^5f_5 \Rightarrow {}^5f_5 = {}^4f_4
$$

$$
{}^4n_4 = {}^4_5R\,{}^5n_5 + {}^4_5P \times {}^4f_4 \Rightarrow {}^5n_5 = {}^4n_4 - {}^4_5P \times {}^4f_4 = \begin{pmatrix} 7 \\ 0 \\ 8 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix} \times \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 61 \\ 0 \\ 8 \end{pmatrix}
$$

The force that the robot causes in direction of the screw driver (along ${}^4\hat{Z}_4 = {}^5\hat{Z}_5$) is equal to the third component of ${}^5f_5$, because:

$$
{}^4\hat{Z}_4^{\mathrm{T}} \cdot \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} = 0
$$

Thus, the robot does not apply any force at all in that direction. Furthermore, along the same axis, a torque of

$$
{}^4\hat{Z}_4^{\mathrm{T}} \cdot \begin{pmatrix} 61 \\ 0 \\ 8 \end{pmatrix} = 8
$$

is caused by the robot.

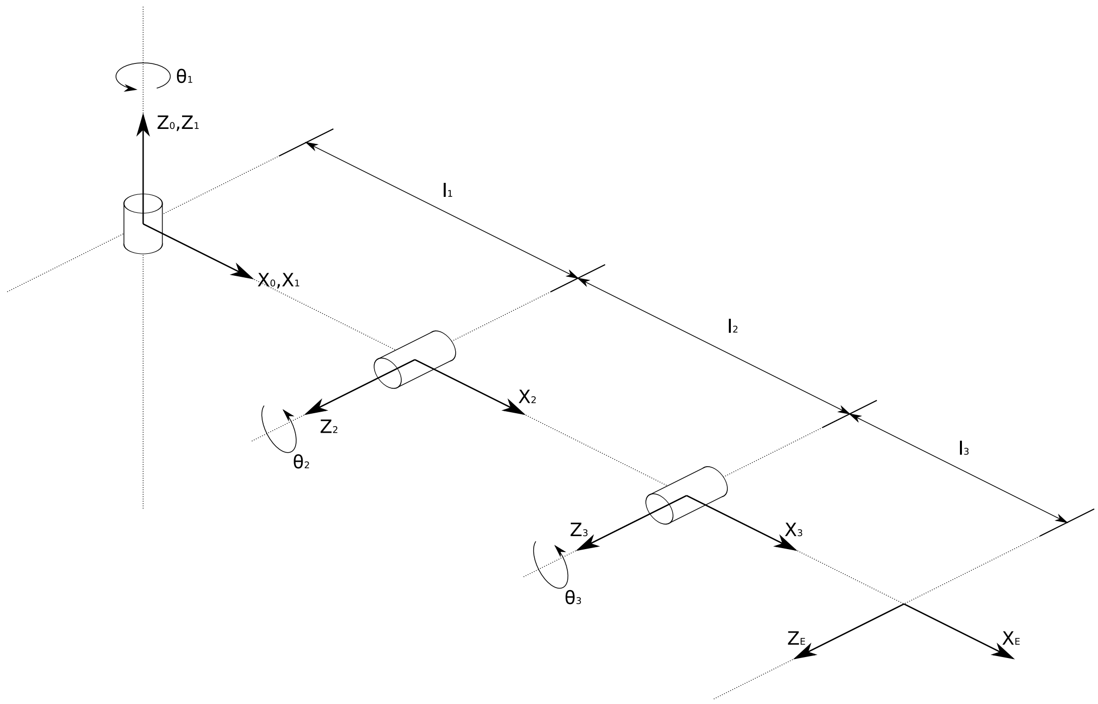

Figure 2: *Choice of coordinate systems*

Solution 2

Let's start by computing the Jacobian based on velocities. The formulas needed for the computation have been summarized in solution 2. For application of these formulas, we need to determine the rotation matrices between the systems. To determine these rotation matrices, we first need to apply the DH convention to determine coordinate systems and DH parameters. Figure 2 shows the coordinate systems. The robot is shown in configuration $\Theta_1 = \Theta_2 = \Theta_3 = 0$, and the DH parameters are now:

i	a_{i-1}	α_{i-1}	d_i	Θ_i
1	0	0°	0	Θ_1
2	l_1	90°	0	Θ_2
3	l_2	0°	0	Θ_3
(4)	l_3	0°	0	0°

With these specifications, the rotation matrices are determined as:

$$
{}^0_1\mathrm{T} = \begin{pmatrix} \mathrm{c}_1 & -\mathrm{s}_1 & 0 & 0 \\ \mathrm{s}_1 & \mathrm{c}_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^1_2\mathrm{T} = \begin{pmatrix} \mathrm{c}_2 & -\mathrm{s}_2 & 0 & l_1 \\ 0 & 0 & -1 & 0 \\ \mathrm{s}_2 & \mathrm{c}_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^2_3\mathrm{T} = \begin{pmatrix} \mathrm{c}_3 & -\mathrm{s}_3 & 0 & l_2 \\ \mathrm{s}_3 & \mathrm{c}_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^3_4\mathrm{T} = \begin{pmatrix} 1 & 0 & 0 & l_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Note the additional transformation ${}^3_4\mathrm{T}$ that is only a translation along ${}^4x$ to the endeffector system. The rotation matrices ${}^{i+1}_{i}R$ are equal to the transpose of the 3×3 rotation part of the transformation matrices. The velocities compute as follows:

$$
\begin{aligned}
{}^1\omega_1 &= {}^1_0R \cdot \vec{0} + \dot{\Theta}_1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\Theta}_1 \end{pmatrix} \\
{}^1v_1 &= {}^1_0R \cdot (\vec{0} + \vec{0} \times {}^0P_1) = 0
\end{aligned}
$$

$$
\begin{aligned}
{}^2\omega_2 &= {}^2_1R \cdot \begin{pmatrix} 0 \\ 0 \\ \dot{\Theta}_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\Theta}_2 \end{pmatrix} = \begin{pmatrix} \mathrm{s}_2\dot{\Theta}_1 \\ \mathrm{c}_2\dot{\Theta}_1 \\ \dot{\Theta}_2 \end{pmatrix} \\
{}^2v_2 &= {}^2_1R \cdot \left(\vec{0} + \begin{pmatrix} 0 \\ 0 \\ \dot{\Theta}_1 \end{pmatrix} \times \begin{pmatrix} l_1 \\ 0 \\ 0 \end{pmatrix} \right) = {}^2_1R \cdot \begin{pmatrix} 0 \\ l_1\dot{\Theta}_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -l_1\dot{\Theta}_1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^3\omega_3 &= {}^3_2R \cdot \begin{pmatrix} \mathrm{s}_2\dot{\Theta}_1 \\ \mathrm{c}_2\dot{\Theta}_1 \\ \dot{\Theta}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\Theta}_3 \end{pmatrix} = \begin{pmatrix} \mathrm{s}_2\dot{\Theta}_1\mathrm{c}_3 + \mathrm{c}_2\dot{\Theta}_1\mathrm{s}_3 \\ -\mathrm{s}_2\dot{\Theta}_1\mathrm{s}_3 + \mathrm{c}_2\dot{\Theta}_1\mathrm{c}_3 \\ \dot{\Theta}_2 + \dot{\Theta}_3 \end{pmatrix} \\
{}^3v_3 &= {}^3_2R \cdot \left(\begin{pmatrix} 0 \\ 0 \\ -\dot{\Theta}_1 l_1 \end{pmatrix} + \begin{pmatrix} \mathrm{s}_2\dot{\Theta}_1 \\ \mathrm{c}_2\dot{\Theta}_1 \\ \dot{\Theta}_2 \end{pmatrix} \times \begin{pmatrix} l_2 \\ 0 \\ 0 \end{pmatrix} \right) = {}^3_2R \begin{pmatrix} 0 \\ \dot{\Theta}_2 l_2 \\ -l_2\mathrm{c}_2\dot{\Theta}_1 - \dot{\Theta}_1 l_1 \end{pmatrix} = \begin{pmatrix} \mathrm{s}_3\dot{\Theta}_2 l_2 \\ \mathrm{c}_3\dot{\Theta}_2 l_2 \\ -l_2\mathrm{c}_2\dot{\Theta}_1 - \dot{\Theta}_1 l_1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^4\omega_4 &= {}^3\omega_3 \\
{}^4v_4 &= \mathrm{I} \left({}^3v_3 + {}^3\omega_3 \times \begin{pmatrix} l_3 \\ 0 \\ 0 \end{pmatrix} \right) = \left({}^3v_3 + \begin{pmatrix} 0 \\ (\dot{\Theta}_2 + \dot{\Theta}_3) l_3 \\ -l_3(\mathrm{c}_2\dot{\Theta}_1\mathrm{c}_3 - \mathrm{s}_2\dot{\Theta}_1\mathrm{s}_3) \end{pmatrix} \right) \\
&= \begin{pmatrix} \mathrm{s}_3\dot{\Theta}_2 l_2 \\ \mathrm{c}_3\dot{\Theta}_2 l_2 + \dot{\Theta}_2 l_3 + \dot{\Theta}_3 l_3 \\ l_3\mathrm{s}_2\dot{\Theta}_1\mathrm{s}_3 - l_3\mathrm{c}_2\dot{\Theta}_1\mathrm{c}_3 - l_2\mathrm{c}_2\dot{\Theta}_1 - \dot{\Theta}_1 l_1 \end{pmatrix} = \begin{pmatrix} \dot{\Theta}_2(l_2\mathrm{s}_3) \\ \dot{\Theta}_2(l_2\mathrm{c}_3 + l_3) + l_3\dot{\Theta}_3 \\ \dot{\Theta}_1(l_3(\mathrm{s}_2\mathrm{s}_3 - \mathrm{c}_2\mathrm{c}_3) - l_2\mathrm{c}_2 - l_1) \end{pmatrix}
\end{aligned}
$$

The final values ${}^4v_4$ and ${}^4\omega_4$ can be simplified using the trigonometric identities ($\mathrm{s}_2\mathrm{s}_3 - \mathrm{c}_2\mathrm{c}_3 = -\mathrm{c}_{23}$ $\ldots$), and we can derive the complete Jacobian:

$$
{}^4J = \begin{pmatrix} 0 & l_2\mathrm{s}_3 & 0 \\ 0 & l_2\mathrm{c}_3 + l_3 & l_3 \\ -l_3\mathrm{c}_{23} - l_2\mathrm{c}_2 - l_1 & 0 & 0 \\ \mathrm{s}_{23} & 0 & 0 \\ \mathrm{c}_{23} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}
$$

Thus, we have determined the Jacobian based solely on velocities. For the following considerations, the computations will not be as exhaustive, we will more focus on the basic ideas.

For the force/torque-relations we have:

$$
\tau = {}^AJ^{T4}\mathcal{F}
$$

We denote the components of ${}^4\mathcal{F}$ as follows:

$$\mathcal{F} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ N_1 \\ N_2 \\ N_3 \end{pmatrix}$$

Note that:

$$\tau_i = {}^i n_i (0,0,1)^{\mathrm{T}}$$

This is true because our robot has only rotational joints. Applying the formulas for computing forces and torques yields:

$$ {}^4 f_4 = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \quad {}^4 n_4 = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}$$

$$ {}^3 f_3 = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \quad {}^3 n_3 = \begin{pmatrix} N_1 \\ N_2 - F_3\, l_3 \\ N_3 + F_2\, l_3 \end{pmatrix}$$

$$ {}^2 f_2 = \begin{pmatrix} F_1\, \cos\Theta_3 - F_2\, \sin\Theta_3 \\ F_1\, \sin\Theta_3 + F_2\, \cos\Theta_3 \\ F_3 \end{pmatrix} \quad {}^2 n_2 = \begin{pmatrix} N_1\, \cos\Theta_3 - (N_2 - F_3\, l_3)\, \sin\Theta_3 \\ N_1\, \sin\Theta_3 + (N_2 - F_3\, l_3)\, \cos\Theta_3 - l_2\, F_3 \\ l_2\, (F_1\, \sin\Theta_3 + F_2\, \cos\Theta_3) + N_3 + F_2\, l_3 \end{pmatrix}$$

$$ {}^1 f_1 = \begin{pmatrix} F_1\, \cos(\Theta_3 + \Theta_2) - F_2\, \sin(\Theta_3 + \Theta_2) \\ -F_3 \\ F_1\, \sin(\Theta_3 + \Theta_2) + F_2\, \cos(\Theta_3 + \Theta_2) \end{pmatrix}$$

$$ {}^1 n_1 = \begin{pmatrix} (F_3\, l_3 - N_2)\, \sin(\Theta_3 + \Theta_2) + N_1\, \cos(\Theta_3 + \Theta_2) + l_2\, F_3\, \sin\Theta_2 \\ -F_1\, l_1\, \sin(\Theta_3 + \Theta_2) - l_1\, F_2\, \cos(\Theta_3 + \Theta_2) - F_1\, l_2\, \sin\Theta_3 - F_2\, l_2\, \cos\Theta_3 - N_3 - F_2\, l_3 \\ N_1\, \sin(\Theta_3 + \Theta_2) + (N_2 - F_3\, l_3)\, \cos(\Theta_3 + \Theta_2) - l_2\, F_3\, \cos\Theta_2 - l_1\, F_3 \end{pmatrix}$$

We are interested in the values of τ_1, τ_2, τ_3 which are going to allow us to deduce the entries of the Jacobian. As stated above, they are the third components of ${}^i n_i$. Note that this means that we need only compute the third component of ${}^1 n_1$, which simplifies the computation. All in all, we have now computed the following values for τ:

$$\begin{aligned}
\tau_1 &= N_1\, \sin(\Theta_3 + \Theta_2) + (N_2 - F_3\, l_3)\, \cos(\Theta_3 + \Theta_2) - l_2\, F_3\, \cos\Theta_2 - l_1\, F_3 \\
\tau_2 &= l_2\, (F_1\, \sin\Theta_3 + F_2\, \cos\Theta_3) + N_3 + F_2\, l_3 \\
\tau_3 &= N_3 + F_2\, l_3
\end{aligned}$$

Now we collect the coefficients of $F_1, F_2, F_3, N_1, N_2, N_3$, which allows us to determine the shape of the transposed Jacobian:

$$ {}^4 J^{\mathrm{T}} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} N_1\, \sin(\Theta_3 + \Theta_2) + F_3\, (-l_3\, \cos(\Theta_3 + \Theta_2) - l_2\, \cos\Theta_2 - l_1) + N_2\, \cos(\Theta_3 + \Theta_2) \\ F_1\, l_2\, \sin\Theta_3 + F_2\, (l_2\, \cos\Theta_3 + l_3) + N_3 \\ N_3 + F_2\, l_3 \end{pmatrix}$$

$$\Rightarrow {}^4J^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & -l_3\,\mathrm{c}_{23} - l_2\,\mathrm{c}_2 - l_1 & \mathrm{s}_{23} & \mathrm{c}_{23} & 0 \\ l_2\,\mathrm{s}_3 & l_2\,\mathrm{c}_3 + l_3 & 0 & 0 & 0 & 1 \\ 0 & l_3 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The geometric method for computing the Jacobian would be the differentiation of the cartesian coordinates with respect to $\Theta_1, \Theta_2, \Theta_3$ for the upper 3 rows and computing the lower 3 rows as ${}^0Z_1, {}^0Z_2, {}^0Z_3$, as shown in Solution 1. We can thus determine ${}^0J$:

$$ {}^0J = \begin{pmatrix} \frac{\partial p_1}{\partial \Theta_1} & \frac{\partial p_1}{\partial \Theta_2} & \frac{\partial p_1}{\partial \Theta_3} \\ \frac{\partial p_2}{\partial \Theta_1} & \frac{\partial p_2}{\partial \Theta_2} & \frac{\partial p_2}{\partial \Theta_3} \\ \frac{\partial p_3}{\partial \Theta_1} & \frac{\partial p_3}{\partial \Theta_2} & \frac{\partial p_3}{\partial \Theta_3} \\ {}^0Z_1 & {}^0Z_2 & {}^0Z_3 \end{pmatrix}$$

The position of the origin of system 4 is computed as:

$${}^0_1\mathrm{T}{}^1_2\mathrm{T}{}^2_3\mathrm{T}{}^3_4\mathrm{T}(0,0,0,1)^T$$

The vectors ${}^0Z_1, {}^0Z_2, {}^0Z_3$ are:

$$\begin{aligned} {}^0Z_1 &= (0,0,1)^{\mathrm{T}} \\ {}^0Z_2 &= {}^0_1R(0,1,0)^{\mathrm{T}} = (-\mathrm{s}_1, \mathrm{c}_1, 0) \\ {}^0Z_3 &= {}^0Z_2 = (-\mathrm{s}_1, \mathrm{c}_1, 0) \end{aligned}$$

Then we can use the following relation:

$${}^0J = \begin{pmatrix} {}^0_4R & 0_{3\times3} \\ 0_{3\times3} & {}^0_4R \end{pmatrix} {}^4J \Leftrightarrow {}^4J = \begin{pmatrix} {}^4_0R & 0_{3\times3} \\ 0_{3\times3} & {}^4_0R \end{pmatrix} {}^0J$$

This relation has been established by multiplying both sides with the inverse of the compositional rotation matrices. The computation of the explicit value is pure computational work and will yield the same result as the other calculations before.