Chair of Robotics, Arificial Intelligence and Real-Time Systems
Department of Informatics
Technical University of Munich

Esolution
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Note:

- During the attendance check a sticker containing a unique code will be put on this exam.
- This code contains a unique number that associates this exam with your registration number.
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Robotics

Exam:	IN2067 / Endterm	**Date:**	Friday 18^{th} February, 2022
Examiner:	Prof. Dr.-Ing. Darius Burschka	**Time:**	08:00 – 09:30

Working instructions

- This exam consists of **14 pages** with a total of **4 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 108 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources:
  - one **non-networked pocket calculator**
  - one **analog dictionary** English ↔ native language
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.
- Provide solution only within the specified solution box.
- **Copies of printed text must not (!!) be inserted into the solution box, only handwritten text or text added with the PDF editing tool.**

Problem 1 Authentication (0 credits)

The process is meant to help in disputes about possible fraud accusations later. You are supposed to sit in the room alone. Enable the BBB session with the link provided in TUMexam.

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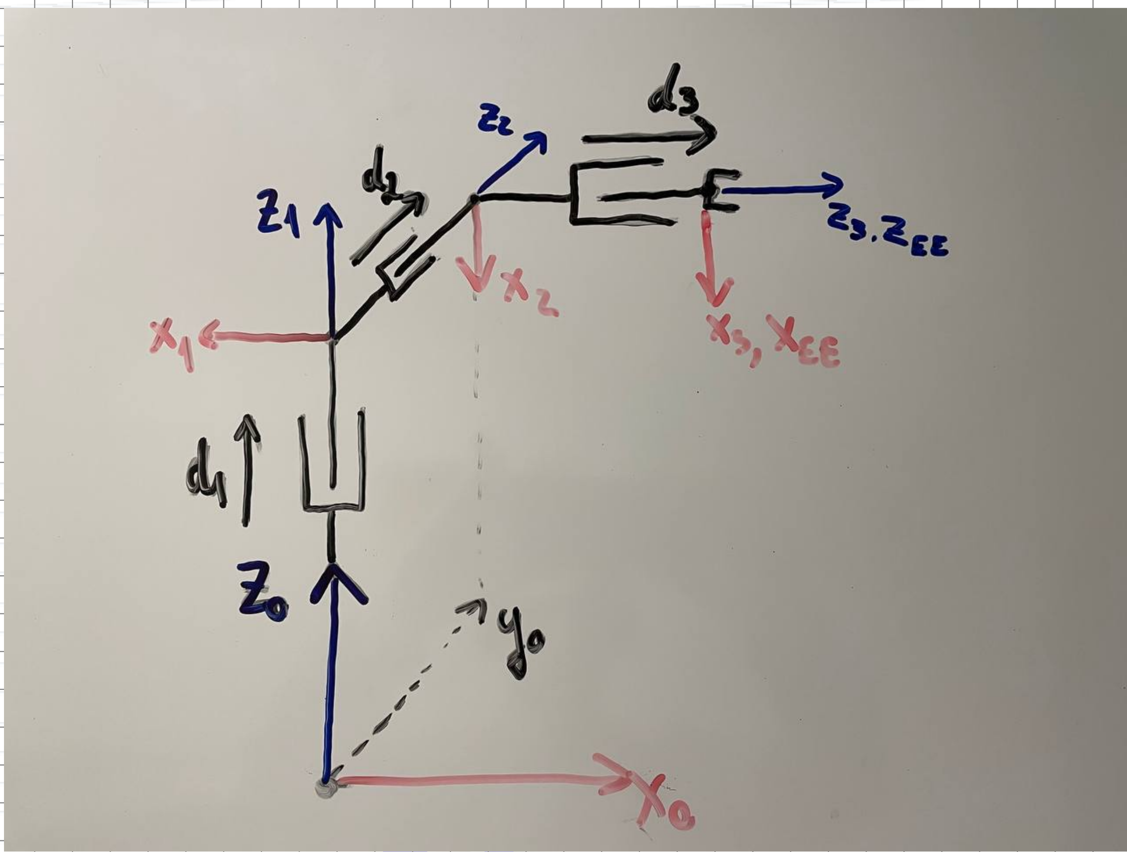
1

Write in your own handwriting (using mouse or pen in your PDF editor) the following text hand-written by you: *Hereby, I confirm that I prepared the solution without any help.*
Also write the city in which you were born.

Problem 2 Kinematics (35 credits)

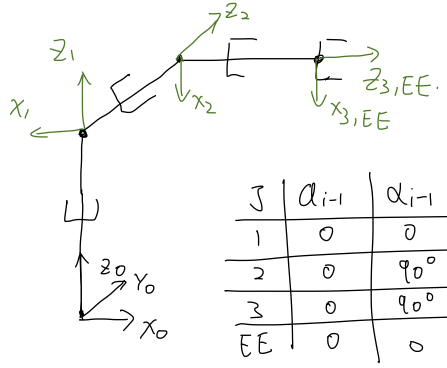
a)* Draw a 3-jointed robot such that the end-effector will only move in the global z-direction upon actuation of joint 1, will only move in the global y-direction upon actuation of joint 2 and will only move in the global x-direction upon actuation of joint 3. Draw the (right-handed) global coordinate axes. Write the Denavit-Hartenberg table for this robot while also giving values for the joint parameters.
Your drawing should be in 3D! Use the diagonal of a square to represent moving outside or inside of the drawing plane. Draw the positive direction of motion or rotation and name the joint parameters. Also draw the coordinate frames of every joint $\{1\}$, $\{2\}$ and $\{3\}$ and the coordinate frame of the end-effector $\{EE\}$. Two squares are 1cm for lengths that are in the drawing plane. One square diagonal equals 1cm for lengths that are not in the drawing plane.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14



CF	α_i	a_{i-1}	d_i	θ_i	value
1	$*°$	$*$	d_1	$(180°//0°) \pm \{x_0\text{-dependent-constant}\}°$	$*cm$
2	$90°//-90°$	$*$	d_2	$\pm 90°//\pm 90°$	$*cm$
3	$\mp 90°//\pm 90°$	$*$	d_3	$*°$	$*cm$
EE	$*°$	$*$	0	$*°$	–

a)



J	a_{i-1}	α_{i-1}	d_i	θ_i	Value
1	0	0	d_1	180°	≥ 0
2	0	90°	d_2	-90°	≥ 0
3	0	90°	d_3	0	≥ 0
EE	0	0	0	0	

b)

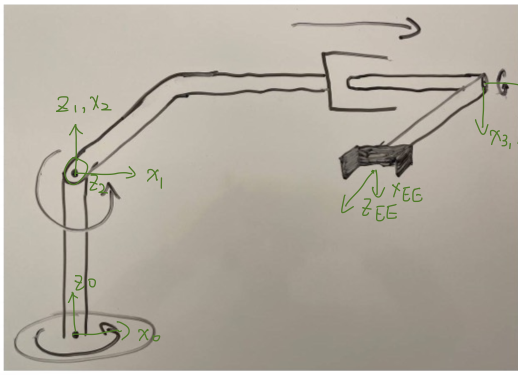
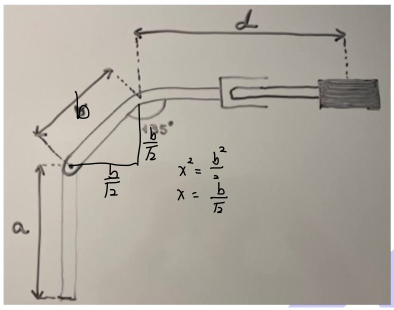
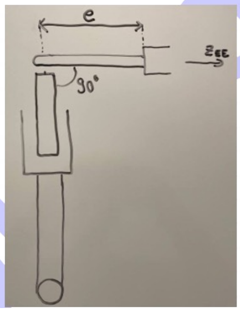


Figure 2.1: 3D view of the robot





RRPR

J	a_{i-1}	α_{i-1}	d_i	θ_i	value
1	0	0	a	θ_1	0
2	0	90°	0	θ_2	90°
3	$\frac{b}{\sqrt{2}}$	90°	$\frac{b}{\sqrt{2}}+d_3$	180°	≥ 0m
4	0	0	0	θ_4	0°
EE	0	90°	e	0	

c)

$$ {}^0_1R = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad {}^1_2R = \begin{bmatrix} c_2 & -s_2 & 0 \\ 0 & 0 & -1 \\ s_2 & c_2 & 0 \end{bmatrix} $$

3R → XYZ space → no 0

2R → XY space → 1 0

R → X space → 4 0

$$ {}^0_2R = \begin{bmatrix} c_1c_2 & -c_1s_2 & s_1 \\ s_1c_2 & s_1s_2 & -c_1 \\ s_2 & c_2 & 0 \end{bmatrix} $$

d)

$$ {}^2_3T = \begin{bmatrix} -1 & 0 & 0 & \frac{\sqrt{2}}{2}b \\ 0 & 0 & -1 & -\frac{\sqrt{2}}{2}b-d_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^3_4T = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^4_{EE}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -e \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^2_4T = \begin{bmatrix} -c_4 & s_4 & 0 & \frac{\sqrt{2}}{2}b \\ 0 & 0 & -1 & -\frac{\sqrt{2}}{2}b-d_3 \\ -s_4 & -c_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^2_{EE}T = \begin{bmatrix} -c_4 & 0 & -s_4 & -es_4+\frac{\sqrt{2}}{2}b \\ 0 & -1 & 0 & -\frac{\sqrt{2}}{2}b-d_3 \\ -s_4 & 0 & c_4 & ec_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^2_{EE}P = \begin{pmatrix} \frac{\sqrt{2}}{2}b-es_4 \\ -\frac{\sqrt{2}}{2}b-d_3 \\ ec_4 \end{pmatrix} $$

e) $ {}^0J_w = \left({}^0r_1 \,\vdots\, {}^0r_2 \; \cdots \; \vdots\, {}^0r_n \right) $

$$ {}^0_1R = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad {}^1_2R = \begin{bmatrix} c_2 & -s_2 & 0 \\ 0 & 0 & -1 \\ s_2 & c_2 & 0 \end{bmatrix} \quad {}^2_3R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \quad {}^3_4R = \begin{bmatrix} c_4 & -s_4 & 0 \\ s_4 & c_4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad {}^4_{EE}R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} $$

$$ {}^0_2R = \begin{bmatrix} c_1c_2 & -c_1s_2 & s_1 \\ s_1c_2 & s_1s_2 & -c_1 \\ s_2 & c_2 & 0 \end{bmatrix} \quad {}^0_3R = \begin{bmatrix} -c_1c_2 & -s_1 & c_1s_2 \\ -s_1c_2 & c_1 & -s_1s_2 \\ -s_2 & 0 & c_2 \end{bmatrix} \quad {}^0_4R = \begin{bmatrix} -c_1c_2c_4 - s_1s_4 & c_1c_2s_4 - s_1c_4 & c_1s_2 \\ -s_1c_2c_4 + c_1s_4 & s_1c_2s_4 + c_1c_4 & -s_1s_2 \\ -s_2c_4 & s_2s_4 & c_2 \end{bmatrix} $$

$$ {}^0J_w = \begin{pmatrix} 0 & s_1 & 0 & c_1s_2 \\ 0 & -c_1 & 0 & -s_1s_2 \\ 1 & 0 & 0 & c_2 \end{pmatrix} $$

$$ \cancel{ {}^0_{EE}R = \begin{bmatrix} -c_1c_2c_4 - s_1s_4 & c_1s_2 & s_1c_4 - c_1c_2s_4 \\ -s_1c_2c_4 + c_1s_4 & -s_1s_2 & -c_1c_4 - s_1c_2s_4 \\ -s_2c_4 & c_2 & -s_2s_4 \end{bmatrix} } $$

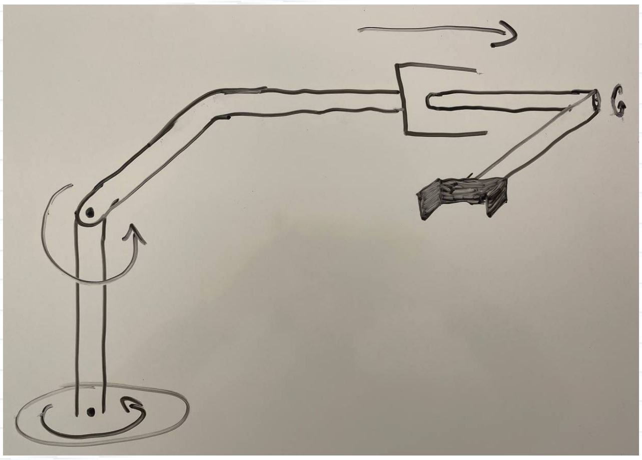


Figure 2.1: 3D view of the robot

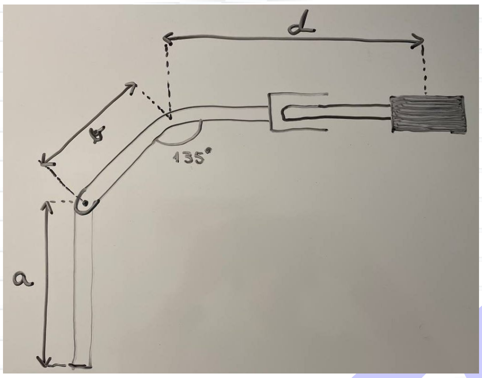


Figure 2.2: Side view of the robot

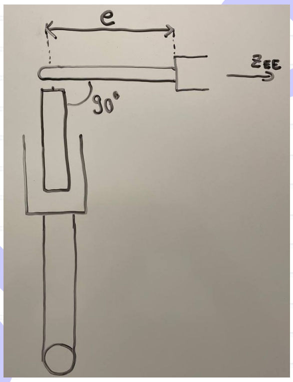


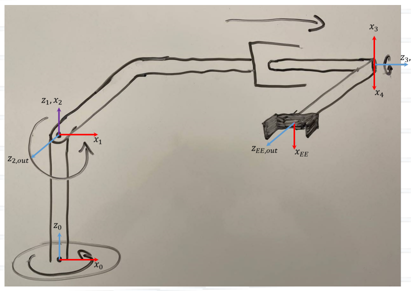
Figure 2.3: Top view of the robot

Fig. 2.1 shows the RRPR robot's positive joint rotation and movement directions. In this configuration, only the last link sticks out of the drawing plane (see Fig. 2.2 and Fig. 2.3) and the prismatic joint is parallel to the ground. The prismatic joint is perpendicular to the last link and the z-axis of the end effector is pointing along this last link.

0 1 2 3 4 5

b)* Write the robot's Denavit-Hartenberg table. Estimate values for each joint parameter according to the figures.

CF	α_i	a_{i-1}	d_i	θ_i	value
1	0°	0	a	θ_1	0°
2	90°	0	0	θ_2	90°
3	90°	$b\frac{\sqrt{2}}{2}$	$d_3 + b\frac{\sqrt{2}}{2}$	0°	any value >0m ok
4	0°	0	0	θ_4	180° or 0°
EE	90° or −90°	0	e	0°	–



0 1 2 3 4

c)* How many 0-entries will the (general form of the) rotation matrix ${}^{0}_{EE}R$ have? Why?

There are three rotational joins each with a different rotation axis.
This means, the robot's end effector can reach any orientation.
This also means there is no 0-entry in the general form of the rotation matrix ${}^{0}_{EE}R$.

d) Determine the position of the end effector relative to coordinate frame 2. Show your work.

0 1 2 3 4 5 6

$$ {}^2_3T = \begin{pmatrix} 1 & 0 & 0 & b\frac{\sqrt{2}}{2} \\ 0 & 0 & -1 & -d_3 - b\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^3_4T = \begin{pmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^4_{EE}T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & e \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^2p_{EE} = {}^2_3T \cdot {}^3_4T \cdot {}^4_{EE}T \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b\frac{\sqrt{2}}{2} - es_4 \\ -d_3 - b\frac{\sqrt{2}}{2} \\ ec_4 \end{pmatrix} $$

e) Determine the orientation Jacobian with respect to the global coordinate frame. Show your work.

0 1 2 3 4 5 6

$$ z_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = {}^0_1R\,{}^1z_1 = {}^0_1R \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = {}^0z_1 $$

$$ z_2 = \begin{pmatrix} sin(\theta_1) \\ -cos(\theta_1) \\ 0 \end{pmatrix} = {}^0_2R\,{}^2z_2 = {}^0_2R \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = {}^0z_2 $$

$$ z_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ because joint 3 is prismatic} $$

$$ z_4 = \begin{pmatrix} cos(\theta_1)sin(\theta_2) \\ sin(\theta_1)sin(\theta_2) \\ -cos(\theta_2) \end{pmatrix} = {}^0_4R\,{}^4z_4 = {}^0_4R \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = {}^0z_4 $$

$$ {}^0J_\omega = \left(z_1 \mid z_2 \mid z_3 \mid z_4 \right) = \begin{pmatrix} 0 & sin(\theta_1) & 0 & cos(\theta_1)sin(\theta_2) \\ 0 & -cos(\theta_1) & 0 & sin(\theta_1)sin(\theta_2) \\ 1 & 0 & 0 & -cos(\theta_2) \end{pmatrix} $$

Problem 3 Dynamics (36 credits)

Fig. 3.3 shows the top-view and Fig. 3.2 shows the side-view of a PRP robot. Gravity acts in the negative z_0 direction. The Denavit-Hartenberg table for the robot is also given. The centers of mass of each link are at $^0P_{c_1} = \begin{pmatrix} 0 \\ -\frac{2}{3}d_1 \\ -\frac{1}{3}l_1 \end{pmatrix}$, $^0P_{c_2} = \begin{pmatrix} \frac{1}{2}l_2 cos(\theta_2) \\ -d_1 - \frac{1}{2}l_2 sin(\theta_2) \\ -l_1 \end{pmatrix}$ and $^0P_{c_3} = \begin{pmatrix} l_2 cos(\theta_2) \\ -d_1 - l_2 sin(\theta_2) \\ -l_1 - \frac{1}{2}d_3 \end{pmatrix}$ respectively.

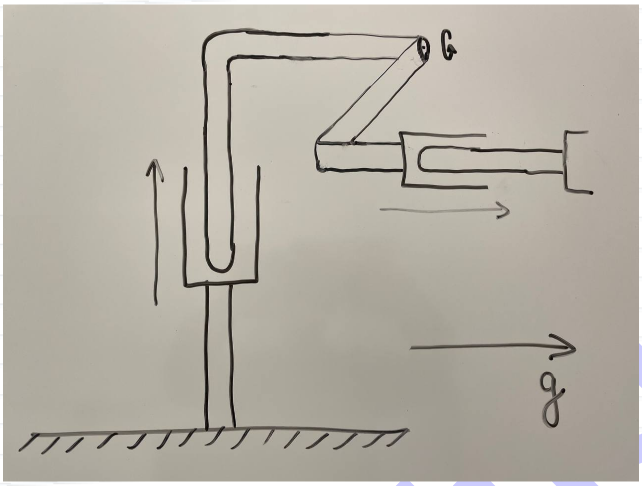


Figure 3.1: 3D view of the robot

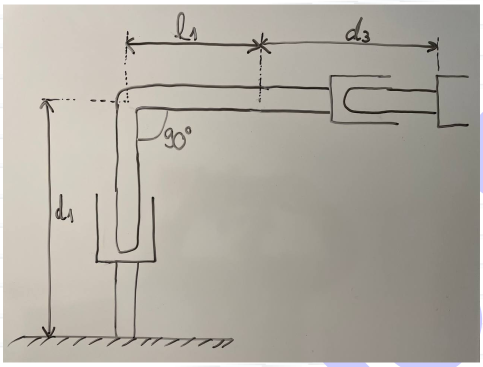


Figure 3.2: Side view of the robot

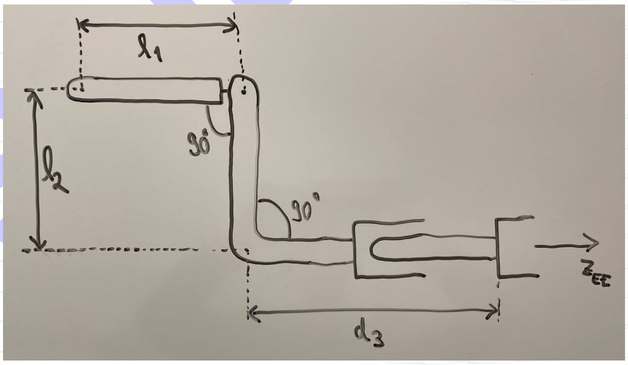


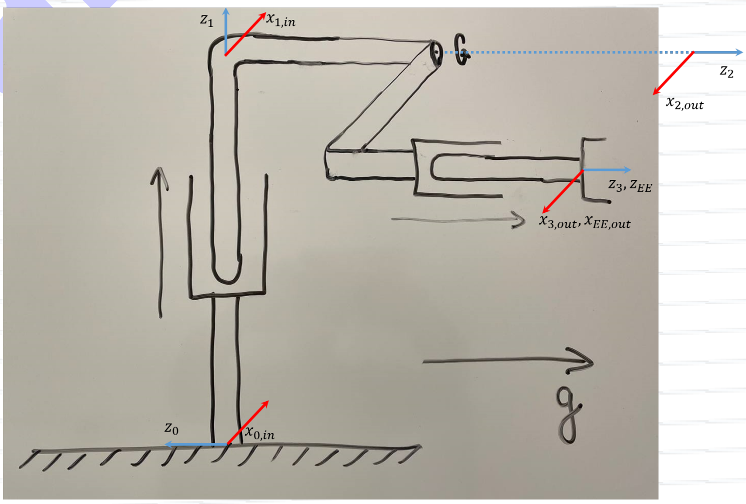
Figure 3.3: Top view of the robot

CF	α_{i-1}	a_{i-1}	d_i	θ_i
1	90°	0	d_1	0°
2	90°	0	$d_3 + l_1$	θ_2

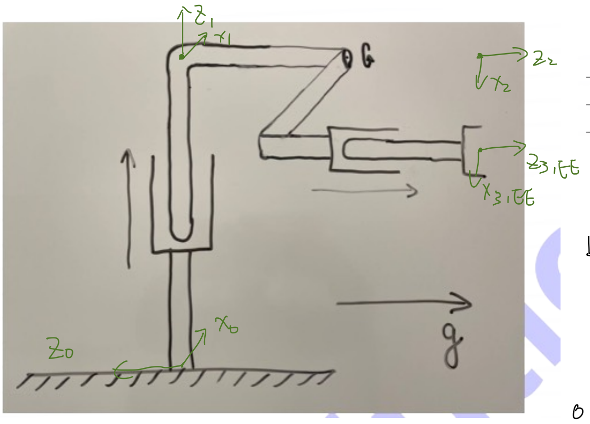
CF	α_{i-1}	a_{i-1}	d_i	θ_i
3	0°	l_2	0	0°
EE	0°	0	0	0°

0 1 2 3 4 5

a)* Which value has θ_2 for the drawn robot configuration? Show your work by indicating/drawing coordinate frames.



$\theta_2 = 180°$



$\theta_2 = 180°$

CF	α_{i-1}	a_{i-1}	d_i	θ_i
1	$90°$	0	d_1	$0°$
2	$90°$	0	$d_3 + l_1$	θ_2

CF	α_{i-1}	a_{i-1}	d_i	θ_i
3	$0°$	l_2	0	$0°$
EE	$0°$	0	0	$0°$

b)

$$ {}^0P_{c_1} = \begin{pmatrix} 0 \\ -\frac{2}{3}d_1 \\ -\frac{1}{3}l_1 \end{pmatrix}, {}^0P_{c_2} = \begin{pmatrix} \frac{1}{2}l_2\cos(\theta_2) \\ -d_1 - \frac{1}{2}l_2\sin(\theta_2) \\ -l_1 \end{pmatrix} \text{ and } {}^0P_{c_3} = \begin{pmatrix} l_2\cos(\theta_2) \\ -d_1 - l_2\sin(\theta_2) \\ -l_1 - \frac{1}{2}d_3 \end{pmatrix} $$

PRP

$$ {}^0_1T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & -1 & -d_3-l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^2_3T = \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^3_4T = I $$

$$ {}^1_0R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad {}^2_1R = \begin{bmatrix} c_2 & 0 & s_2 \\ -s_2 & 0 & c_2 \\ 0 & -1 & 0 \end{bmatrix} $$

$$ \begin{bmatrix} {}^2P_{c_2} \\ 1 \end{bmatrix} = {}^2_0T \begin{bmatrix} {}^0P_{c_2} \\ 1 \end{bmatrix} $$

$$ {}^0_2T = \left[\begin{array}{ccc|c} c_2 & -s_2 & 0 & 0 \\ -s_2 & -c_2 & 0 & -d_1 \\ 0 & 0 & -1 & -d_3-L_1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ {}^2_0T = \left[\begin{array}{ccc|c} c_2 & -s_2 & 0 & -s_2 d_1 \\ -s_2 & -c_2 & 0 & -c_2 d_1 \\ 0 & 0 & -1 & -d_3-L_1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} \frac{1}{2}L_2C_2 \\ -d_1 - \frac{1}{2}L_2S_2 \\ -L_1 \\ 1 \end{bmatrix} $$

$$ \begin{matrix} -c_2 & s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{matrix} \cdot \begin{matrix} 0 \\ -d_1 \\ -d_3-L_1 \end{matrix} $$

$$ = \begin{bmatrix} \frac{1}{2}L_2C_2^2 + \cancel{d_1S_2} + \frac{1}{2}L_2S_2^2 - \cancel{S_2d_1} \\ \cancel{-\frac{1}{2}L_2C_2S_2} + \cancel{d_1C_2} + \cancel{\frac{1}{2}L_2S_2C_2} - \cancel{C_2d_1} \\ \cancel{L_1} - d_3 - \cancel{L_1} \\ 1 \end{bmatrix} $$

$$ {}^2P_{c_2} = \begin{bmatrix} \frac{1}{2}L_2 \\ 0 \\ -d_3 \end{bmatrix} $$

$$ {}^2_1R = \begin{bmatrix} c_2 & 0 & s_2 \\ -s_2 & 0 & c_2 \\ 0 & -1 & 0 \end{bmatrix} $$

PRP

$$ {}^0\dot{v}_0 = -G = \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix} \quad {}^0w_0 = {}^0\dot{w}_0 = {}^0V_0 = 0 $$

$$ {}^1w_1 = \cancel{{}^1_0R\,{}^0w_0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} $$

$$ {}^1V_1 = \cancel{{}^1_0R({}^0V_0 + {}^0w_0 \times {}^0P_1)} + \begin{pmatrix} 0 \\ 0 \\ \dot{d}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{d}_1 \end{pmatrix} $$

$$ {}^1\dot{w}_1 = \cancel{{}^1_0R\,{}^0\dot{w}_0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} $$

$$ {}^1\dot{V}_1 = {}^1_0R(\cancel{{}^0\dot{w}_0 \times {}^0P_1} + \cancel{{}^0w_0 \times ({}^0w_0 \times {}^0P_1)} + {}^0\dot{v}_0) + \cancel{2 \cdot {}^1w_1 \times \begin{pmatrix} 0 \\ 0 \\ \dot{d}_1 \end{pmatrix}} + \begin{pmatrix} 0 \\ 0 \\ \ddot{d}_1 \end{pmatrix} $$

$$ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot{d}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ g \\ \ddot{d}_1 \end{pmatrix} $$

$$ {}^1\dot{V}_{c_1} = \cancel{{}^1\dot{w}_1 \times {}^1P_{c_1}} + \cancel{{}^1w_1 \times ({}^1w_1 \times {}^1P_{c_1})} + {}^1\dot{V}_1 = \begin{pmatrix} 0 \\ g \\ \ddot{d}_1 \end{pmatrix} $$

$$ {}^2w_2 = \cancel{{}^2_1R\,{}^1w_1} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix} $$

$$ {}^2V_2 = {}^2_1R({}^1V_1 + \cancel{{}^1w_1 \times {}^1P_2}) = \begin{pmatrix} c_2 & 0 & s_2 \\ -s_2 & 0 & c_2 \\ 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \dot{d}_1 \end{pmatrix} = \begin{pmatrix} s_2\dot{d}_1 \\ c_2\dot{d}_1 \\ 0 \end{pmatrix} $$

$$ {}^2\dot{w}_2 = \cancel{{}^2_1R\,{}^1\dot{w}_1} + \cancel{{}^2_1R\,{}^1w_1 \times \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix}} + \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{pmatrix} $$

$$ {}^2\dot{V}_2 = {}^2_1R(\cancel{{}^1\dot{w}_1 \times {}^1P_2} + \cancel{{}^1w_1 \times ({}^1w_1 \times {}^1P_2)} + {}^1\dot{V}_1) $$

$$ = \begin{pmatrix} c_2 & 0 & s_2 \\ -s_2 & 0 & c_2 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ g \\ \ddot{d}_1 \end{pmatrix} = \begin{pmatrix} s_2\ddot{d}_1 \\ c_2\ddot{d}_1 \\ -g \end{pmatrix} $$

$$ {}^2\dot{V}_{c_2} = {}^2\dot{w}_2 \times {}^2P_{c_2} + {}^2w_2 \times ({}^2w_2 \times {}^2P_{c_2}) + {}^2\dot{V}_2 $$

$$ = \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} \frac{1}{2}L_2 \\ 0 \\ d_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix} \times \left[\begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} \frac{1}{2}L_2 \\ 0 \\ d_3 \end{pmatrix} \right] + \begin{pmatrix} s_2\ddot{d}_1 \\ c_2\ddot{d}_1 \\ -g \end{pmatrix} $$

$$ = \begin{pmatrix} 0 \\ \frac{1}{2}L_2\ddot{\theta}_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} 0 \\ \frac{1}{2}L_2\dot{\theta}_2 \\ 0 \end{pmatrix} + \begin{pmatrix} s_2\ddot{d}_1 \\ c_2\ddot{d}_1 \\ -g \end{pmatrix} $$

$$ = \begin{pmatrix} -\frac{1}{2}L_2\dot{\theta}_2^2 + s_2\ddot{d}_1 \\ \frac{1}{2}L_2\ddot{\theta}_2 + c_2\ddot{d}_1 \\ -g \end{pmatrix} $$

$$ {}^3w_3 = {}^3_2R\,{}^2w_2 = \begin{pmatrix} 0 \\ 0 \\ \dot\theta_2 \end{pmatrix} $$

$$ {}^3\dot w_3 = {}^3_2R\,{}^2\dot w_2 = \begin{pmatrix} 0 \\ 0 \\ \ddot\theta_2 \end{pmatrix} $$

$$ {}^3v_3 = {}^3_2R\left({}^2v_2 + {}^2w_2 \times {}^2P_3\right) + \begin{pmatrix} 0 \\ 0 \\ \dot d_3 \end{pmatrix} = \begin{pmatrix} S_2\dot d_1 \\ C_2\dot d_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot\theta_2 \end{pmatrix} \times \begin{pmatrix} L_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot d_3 \end{pmatrix} = \begin{pmatrix} S_2\dot d_1 \\ C_2\dot d_1 + L_2\dot\theta_2 \\ \dot d_3 \end{pmatrix} $$

$$ {}^3\dot v_3 = {}^3_2R\left({}^2\dot w_2 \times {}^2P_3 + {}^2w_2 \times ({}^2w_2 \times {}^2P_3) + {}^2\dot v_2\right) + 2\,{}^3w_3 \times \begin{pmatrix} 0 \\ 0 \\ \dot d_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot d_3 \end{pmatrix} $$

$$ = \begin{pmatrix} 0 \\ 0 \\ \ddot\theta_2 \end{pmatrix} \times \begin{pmatrix} L_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot\theta_2 \end{pmatrix} \times \left[\begin{pmatrix} 0 \\ 0 \\ \dot\theta_2 \end{pmatrix} \times \begin{pmatrix} L_2 \\ 0 \\ 0 \end{pmatrix}\right] + \begin{pmatrix} S_2\ddot d_1 \\ C_2\ddot d_1 \\ -g \end{pmatrix} + \cancel{\begin{pmatrix} 0 \\ 0 \\ 2\dot\theta_2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \dot d_3 \end{pmatrix}} + \begin{pmatrix} 0 \\ 0 \\ \ddot d_3 \end{pmatrix} $$

$$ = \begin{pmatrix} 0 \\ L_2\ddot\theta_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot\theta_2 \end{pmatrix} \times \begin{pmatrix} 0 \\ L_2\dot\theta_2 \\ 0 \end{pmatrix} + \begin{pmatrix} S_2\ddot d_1 \\ C_2\ddot d_1 \\ -g \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot d_3 \end{pmatrix} $$

$$ = \begin{pmatrix} -L_2\dot\theta_2^2 + S_2\ddot d_1 \\ L_2\ddot\theta_2 + C_2\ddot d_1 \\ -g + \ddot d_3 \end{pmatrix} $$

$$ {}^3P_{C_3} = {}^3_0T\,{}^0P_{C_3} $$

$$ {}^0_2T = \begin{bmatrix} C_2 & -S_2 & 0 & 0 \\ -S_2 & -C_2 & 0 & -d_1 \\ 0 & 0 & -1 & -d_3 - L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_3T = \begin{bmatrix} C_2 & -S_2 & 0 & L_2C_2 \\ -S_2 & -C_2 & 0 & -S_2L_2 - d_1 \\ 0 & 0 & -1 & -d_3 - L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^3_0T = \begin{bmatrix} C_2 & -S_2 & 0 & -L_2C_2^2 - S_2^2L_2 - S_2d_1 \\ -S_2 & -C_2 & 0 & L_2C_2S_2 - L_2S_2C_2 - C_2d_1 \\ 0 & 0 & -1 & -d_3 - L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left[\begin{pmatrix} l_2\cos(\theta_2) \\ -d_1 - l_2\sin(\theta_2) \\ -l_1 - \frac{1}{2}d_3 \end{pmatrix} \right. $$

$$ 1 $$

$$ \begin{matrix} -C_2 & S_2 & 0 & L_2C_2 \\ S_2 & C_2 & 0 & -S_2L_2 - d_1 \\ 0 & 0 & 1 & -d_3 - L_1 \end{matrix} $$

$$ \cancel{L_2C_2^2} + \cancel{S_2d_1} + \cancel{L_2S_2^2} - \cancel{L_2C_2^2} - \cancel{S_2^2L_2} - \cancel{S_2d_1} $$

$$ -\cancel{L_2C_2S_2} + \cancel{C_2d_1} + \cancel{C_2L_2S_2} + \cancel{L_2C_2S_2} - \cancel{L_2S_2C_2} - \cancel{C_2d_1} $$

$$ \cancel{L_1} + \frac{1}{2}d_3 - d_3 - \cancel{L_1} $$

$$ {}^3P_{C_3} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2}d_3 \end{pmatrix} $$

$$ {}^3\dot v_{C_3} = {}^3\dot w_3 \times {}^3P_{C_3} + {}^3w_3 \times ({}^3w_3 \times {}^3P_{C_3}) + {}^3\dot v_3 $$

$$ = \cancel{\begin{pmatrix} 0 \\ 0 \\ \ddot\theta_2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2}d_3 \end{pmatrix}} + \cancel{\begin{pmatrix} 0 \\ 0 \\ \dot\theta_2 \end{pmatrix} \times \left[\begin{pmatrix} 0 \\ 0 \\ \dot\theta_3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2}d_3 \end{pmatrix}\right]} + {}^3\dot v_3 $$

$$ = \begin{pmatrix} -L_2\dot\theta_2^2 + S_2\ddot d_1 \\ L_2\ddot\theta_2 + C_2\ddot d_1 \\ -g + \ddot d_3 \end{pmatrix} $$

b) Perform the forward propagation of velocities and accelerations for each joint and compute the acceleration of the centers of mass for each link. Be sure to indicate all your intermediate results and the values for your variables (e.g. rotation matrices and other) for full points.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

$$
{}^0_1T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^1_2T = \begin{pmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & -1 & -d_3 - l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^2_3T = \begin{pmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^3_{EE}T = I_4
$$

$$
{}^0_2T = \begin{pmatrix} c_2 & -s_2 & 0 & 0 \\ -s_2 & c_2 & 0 & -d_1 \\ 0 & 0 & -1 & -d_3 - l_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^0_3T = \begin{pmatrix} c_2 & -s_2 & 0 & l_2c_2 \\ -s_2 & -c_2 & 0 & -l_2s_2 - d_1 \\ 0 & 0 & -1 & -d_3 - l_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^0_{EE}T = {}^0_3T
$$

$$
{}^1_0T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^2_0T = \begin{pmatrix} c_2 & -s_2 & 0 & -d_1s_2 \\ -s_2 & c_2 & 0 & -d_1c_2 \\ 0 & 0 & -1 & -d_3 - l_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^3_0T = \begin{pmatrix} c_2 & -s_2 & 0 & -d_1s_2 - l_2 \\ -s_2 & c_2 & 0 & -d_1c_2 \\ 0 & 0 & -1 & -d_3 - l_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = {}^{EE}_0T
$$

$$
{}^1P_{c_1} = {}^1_0T\,{}^0P_{c_1} = \begin{pmatrix} 0 \\ -l_1/3 \\ -d_1/3 \end{pmatrix}, {}^2P_{c_2} = {}^2_0T\,{}^0P_{c_2} = \begin{pmatrix} l_2/2 \\ 0 \\ d_3 \end{pmatrix}, {}^3P_{c_3} = {}^3_0T\,{}^0P_{c_3} = \begin{pmatrix} 0 \\ 0 \\ -d_3/2 \end{pmatrix}
$$

$$
{}^0\omega_0 = {}^0\dot{\omega}_0 = {}^0v_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};\ {}^0\dot{v}_0 = \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}
$$

$$
{}^1\omega_1 = {}^1_0R\,{}^0\omega_0 + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};\ {}^1\dot{\omega}_1 = {}^1_0R\,{}^0\dot{\omega}_0 + \left({}^1_0R\,{}^0\omega_0\right) \times \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
{}^1v_1 = {}^1_0R\left({}^0v_0 + {}^0\omega_0 \times {}^0_1t\right) + \begin{pmatrix} 0 \\ 0 \\ \dot{d}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{d}_1 \end{pmatrix}
$$

$$
{}^1\dot{v}_1 = {}^1_0R\left({}^0\dot{v}_0 + {}^0\dot{\omega}_0 \times {}^0_1t + {}^0\omega_0 \times \left({}^0\omega_0 \times {}^0_1t\right)\right) + 2\left[{}^1\dot{\omega}_1 \times \begin{pmatrix} 0 \\ 0 \\ \dot{d}_1 \end{pmatrix}\right] + \begin{pmatrix} 0 \\ 0 \\ \ddot{d}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ g \\ \ddot{d}_1 \end{pmatrix}
$$

$$
{}^1\dot{v}_{c_1} = {}^1\dot{v}_1 + {}^1\dot{\omega}_1 \times {}^1P_{c_1} + {}^1\omega_1 \times \left({}^1\omega_1 \times {}^1P_{c_1}\right) = \begin{pmatrix} 0 \\ g \\ \ddot{d}_1 \end{pmatrix}
$$

$$
{}^2\omega_2 = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix}, {}^2\dot{\omega}_2 = \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{pmatrix}, {}^2v_2 = \begin{pmatrix} \dot{d}_1s_2 \\ \dot{d}_1c_2 \\ 0 \end{pmatrix}, {}^2\dot{v}_2 = \begin{pmatrix} \ddot{d}_1s_2 \\ \ddot{d}_1c_2 \\ -g \end{pmatrix}, {}^2\dot{v}_{c_2} = \begin{pmatrix} \ddot{d}_1s_2 - \frac{l_2}{2}\dot{\theta}_2^2 \\ \ddot{d}_1c_2 + \frac{l_2}{2}\ddot{\theta}_2 \\ -g \end{pmatrix}
$$

$$
{}^3\omega_3 = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix}, {}^3\dot{\omega}_3 = \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{pmatrix}, {}^3v_3 = \begin{pmatrix} \dot{d}_1s_2 \\ \dot{d}_1c_2 + l_2\dot{\theta}_2 \\ \dot{d}_3 \end{pmatrix}, {}^3\dot{v}_3 = \begin{pmatrix} \ddot{d}_1s_2 \\ \ddot{d}_1c_2 \\ \ddot{d}_3 - g \end{pmatrix}, {}^3\dot{v}_{c_3} = \begin{pmatrix} \ddot{d}_1s_2 - l_2\dot{\theta}_2^2 \\ \ddot{d}_1c_2 + l_2\ddot{\theta}_2 \\ \ddot{d}_3 - g \end{pmatrix}
$$

$${}^1F_1 = m_1 \cdot {}^1\dot{v}_{c_1} = m_1 \begin{pmatrix} 0 \\ g \\ \ddot{d}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ m_1 g \\ m_1 \ddot{d}_1 \end{pmatrix}$$

$${}^1N_1 = \cancel{{}^{c_1}I_1\, {}^1\dot{w}_1 + {}^1w_1 \times {}^{c_1}I_1\, {}^1w_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$${}^2F_2 = m_2\, {}^2\dot{v}_{c_2} = m_2 \begin{pmatrix} -\frac{1}{2}L_2\dot{\theta}_2^2 + S_2\ddot{d}_1 \\ \frac{1}{2}L_2\ddot{\theta}_2 + C_2\ddot{d}_1 \\ -g \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}m_2 L_2\dot{\theta}_2^2 + m_2 S_2\ddot{d}_1 \\ \frac{1}{2}m_2 L_2\ddot{\theta}_2 + m_2 C_2\ddot{d}_1 \\ -m_2 g \end{pmatrix}$$

$${}^2N_2 = {}^{c_2}I_2 \cdot {}^2\dot{w}_2 + {}^2w_2 \times {}^{c_2}I_2 \cdot {}^2w_2 =$$

$$= \begin{pmatrix} I_{2xx} & 0 & 0 \\ 0 & I_{2yy} & 0 \\ 0 & 0 & I_{2zz} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{pmatrix} + \cancel{\begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} I_{2xx} & 0 & 0 \\ 0 & I_{2yy} & 0 \\ 0 & 0 & I_{2zz} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix}}$$

$$= \begin{pmatrix} 0 \\ 0 \\ I_{2zz} \cdot \ddot{\theta}_2 \end{pmatrix}$$

$${}^3F_3 = m_3 \cdot {}^3\dot{v}_{c_3} = m_3 \begin{pmatrix} -L_2\dot{\theta}_2^2 + S_2\ddot{d}_1 \\ L_2\ddot{\theta}_2 + C_2\ddot{d}_1 \\ -g + \ddot{d}_3 \end{pmatrix} = \begin{pmatrix} -L_2 m_3\dot{\theta}_2^2 + S_2 m_3\ddot{d}_1 \\ L_2 m_3\ddot{\theta}_2 + C_2 m_3\ddot{d}_1 \\ -m_3 g + m_3\ddot{d}_3 \end{pmatrix}$$

$${}^3N_3 = {}^{c_3}I_3 \cdot {}^3\dot{w}_3 + {}^3w_3 \times {}^{c_3}I_3 \cdot {}^3w_3$$

$$= \begin{pmatrix} I_{3xx} & 0 & 0 \\ 0 & I_{3yy} & 0 \\ 0 & 0 & I_{3zz} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{pmatrix} + \cancel{\begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} I_{3xx} & 0 & 0 \\ 0 & I_{1yy} & 0 \\ 0 & 0 & I_{3zz} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix}}$$

$$= \begin{pmatrix} 0 \\ 0 \\ I_{3zz} \cdot \ddot{\theta}_2 \end{pmatrix}$$

0 1 2 3 4 5 6

c) ${}^{c_1}I_1 = \begin{pmatrix} I_{1xx} & 0 & 0 \\ 0 & I_{1yy} & 0 \\ 0 & 0 & I_{1zz} \end{pmatrix}$, ${}^{c_2}I_2 = \begin{pmatrix} I_{2xx} & 0 & 0 \\ 0 & I_{2yy} & 0 \\ 0 & 0 & I_{2zz} \end{pmatrix}$, ${}^{c_3}I_3 = \begin{pmatrix} I_{3xx} & 0 & 0 \\ 0 & I_{3yy} & 0 \\ 0 & 0 & I_{3zz} \end{pmatrix}$ and the links' masses are m_1, m_2 and m_3. Compute the inertial forces ${}^iF_i$ and torques ${}^iN_i$ for each link.

${}^iF_i = m_i {}^i\dot{v}_{c_i}$, ${}^iN_i = {}^{c_i}I_i {}^i\dot{\omega}_i + {}^i\omega_i \times \left({}^{c_i}I_i {}^i\omega_i\right)$

$${}^1F_1 = \begin{pmatrix} 0 \\ m_1 g \\ m_1 \ddot{d} \end{pmatrix}, {}^1N_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$${}^2F_2 = \begin{pmatrix} \frac{m_2}{2}\left(2\ddot{d}_1 s_2 - l_2\dot{\theta}_2^2\right) \\ \frac{m_2}{2}\left(2\ddot{d}_1 c_2 + l_2\ddot{\theta}_2\right) \\ -m_2 g \end{pmatrix}, {}^2N_2 = \begin{pmatrix} 0 \\ 0 \\ I_{2zz}\ddot{\theta}_2 \end{pmatrix}$$

$${}^3F_3 = \begin{pmatrix} m_3\left(\ddot{d}_1 s_2 - l_2\dot{\theta}_2^2\right) \\ m_3\left(\ddot{d}_1 c_2 + l_2\ddot{\theta}_2\right) \\ m_3\left(\ddot{d}_3 - g\right) \end{pmatrix}, {}^3N_3 = \begin{pmatrix} 0 \\ 0 \\ I_{3zz}\ddot{\theta}_2 \end{pmatrix}$$

d)* The robot has an ideal force-torque sensor inside joint 2 aligned with coordinate frame {2}. At joint configuration θ, it reads a force of $\begin{pmatrix}1 & 0 & -2\end{pmatrix}^T$ and a torque of $\begin{pmatrix}0 & -4 & 1\end{pmatrix}^T$. What is the force and torque acting on the end-effector? If you did not solve the previous exercises, perform the symbolic computation for partial points.

0 1 2 3 4 5 6 7 8 9 10

$$^2f_2 = \begin{pmatrix}1\\0\\-2\end{pmatrix} \qquad ^2\tau_2 = \begin{pmatrix}0\\-4\\1\end{pmatrix}$$

$$\begin{aligned}
{}^3f_3 &= {}^3_{EE}R\,{}^{EE}f_{EE} + {}^3F_3\\
{}^3n_3 &= {}^3_{EE}R\,{}^{EE}n_{EE} + {}^3_{EE}t \times \left({}^3_{EE}R\,{}^{EE}f_{EE}\right) + {}^3P_{c_3} \times {}^3F_3 + {}^3N_3\\
{}^2f_2 &= {}^2_3R\,{}^3f_3 + {}^2F_2\\
{}^2n_2 &= {}^2_3R\,{}^3n_3 + {}^2_3t \times \left({}^2_3R\,{}^3f_3\right) + {}^2P_{c_2} \times {}^2F_2 + {}^2N_2
\end{aligned}$$

$${}^2_3R = {}^3_{EE}R = I_3; \quad {}^3_{EE}t = \begin{pmatrix}0 & 0 & 0\end{pmatrix}^T; \quad {}^2_3t = \begin{pmatrix}l_2 & 0 & 0\end{pmatrix}^T$$

$$\begin{aligned}
{}^3f_3 &= {}^{EE}f_{EE} + {}^3F_3\\
{}^3n_3 &= {}^{EE}n_{EE} + {}^3P_{c_3} \times {}^3F_3 + {}^3N_3\\
{}^2f_2 &= {}^3f_3 + {}^2F_2 = {}^{EE}f_{EE} + {}^3F_3 + {}^2F_2\\
{}^2n_2 &= {}^3n_3 + {}^2_3t \times \left({}^{EE}f_{EE} + {}^3F_3\right) + {}^2P_{c_2} \times {}^2F_2 + {}^2N_2 = {}^{EE}n_{EE} + {}^3P_{c_3} \times {}^3F_3 + {}^3N_3 + {}^2_3t \times \left({}^{EE}f_{EE} + {}^3F_3\right) + {}^2P_{c_2} \times {}^2F_2 + {}^2N_2
\end{aligned}$$

$$\begin{aligned}
{}^{EE}f_{EE} &= {}^2f_2 - \left({}^3F_3 + {}^2F_2\right)\\
{}^{EE}n_{EE} &= {}^2n_2 - \left({}^2_3t \times {}^3F_3 + {}^3P_{c_3} \times {}^3F_3 + {}^2P_{c_2} \times {}^2F_2 + {}^3N_3 + {}^2N_2\right) - {}^2_3t \times {}^{EE}f_{EE}\\
{}^{EE}n_{EE} &= {}^2n_2 - {}^2_3t \times {}^2f_2 - \left({}^2_3t \times {}^3F_3 + {}^3P_{c_3} \times {}^3F_3 + {}^2P_{c_2} \times {}^2F_2 + {}^3N_3 + {}^2N_2\right) + {}^2_3t \times \left({}^3F_3 + {}^2F_2\right)\\
{}^{EE}n_{EE} &= {}^2n_2 - {}^2_3t \times {}^2f_2 + {}^2_3t \times {}^2F_2 - {}^3P_{c_3} \times {}^3F_3 - {}^2P_{c_2} \times {}^2F_2 - {}^3N_3 - {}^2N_2
\end{aligned}$$

$${}^{EE}f_{EE} = {}^2f_2 - \begin{pmatrix}\ddot{d}_1 s_2(m_2+m_3) - \dot{\theta}_2^2 l_2(\frac{m_2}{2}+m_3)\\ \ddot{d}_1 c_2(m_2+m_3) + \ddot{\theta}_2 l_2(\frac{m_2}{2}+m_3)\\ \ddot{d}_3 m_3 - g(m_2+m_3)\end{pmatrix}$$

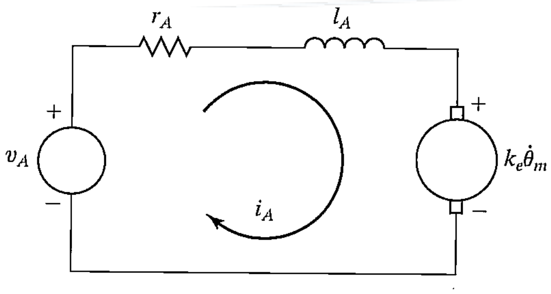
$${}^{EE}n_{EE} = {}^2n_2 - {}^2_3t \times {}^2f_2 + \begin{pmatrix}\ddot{d}_1 d_3 c_2(m_2+\frac{m_3}{2}) - \frac{1}{2}\ddot{\theta}_2 d_3 l_2(m_2+m_3)\\ \ddot{d}_1 d_3 s_2(m_2+\frac{m_3}{2}) - \frac{1}{2}\dot{\theta}_2^2 d_3 l_2(m_2+m_3) + \frac{1}{2}m_2 g l_2\\ -\ddot{\theta}_2(I_{2zz}+I_{3zz}) + \frac{1}{2}m_2\ddot{d}_1 l_2 c_2 + \frac{1}{4}\ddot{\theta}_2 m_2 l_2^2\end{pmatrix}$$

$${}^2f_2 = \begin{pmatrix}1\\0\\-2\end{pmatrix} \rightarrow {}^{EE}f_{EE} = \begin{pmatrix}1 - \ddot{d}_1 s_2(m_2+m_3) + \dot{\theta}_2^2 l_2(\frac{m_2}{2}+m_3)\\ -\ddot{d}_1 c_2(m_2+m_3) - \ddot{\theta}_2 l_2(\frac{m_2}{2}+m_3)\\ -2 - \ddot{d}_3 m_3 + g(m_2+m_3)\end{pmatrix}$$

$${}^2n_2 = \begin{pmatrix}0\\-4\\1\end{pmatrix}, {}^2_3t = \begin{pmatrix}l_2\\0\\0\end{pmatrix} \rightarrow {}^{EE}n_{EE} = \begin{pmatrix}\ddot{d}_1 d_3 c_2(m_2+\frac{m_3}{2}) - \frac{1}{2}\ddot{\theta}_2 d_3 l_2(m_2+m_3)\\ -4 - 2l_2 + \ddot{d}_1 d_3 s_2(m_2+\frac{m_3}{2}) - \frac{1}{2}\dot{\theta}_2^2 d_3 l_2(m_2+m_3) + \frac{1}{2}m_2 g l_2\\ 1 - \ddot{\theta}_2(I_{2zz}+I_{3zz}) + \frac{1}{2}m_2\ddot{d}_1 l_2 c_2 + \frac{1}{4}\ddot{\theta}_2 m_2 l_2^2\end{pmatrix}$$

Problem 4 **Control (37 credits)**

An electric motor depicted below responds to an input voltage v_A with some angular velocity $\dot{\Theta}_m = \omega_m$. We measure its angular velocity and angular acceleration $(\omega_m, \dot{\omega}_m)$ in dependence to the input voltage v_A.



The system above is described by the first-order differential equation

$$l_A \frac{d}{dt} i_A + r_A i_A = v_A - k_e \frac{d}{dt} \Theta_m$$

with i_A being the current flowing through the motor, l_A being the inductance of the motor windings, and the voltage generated in the motor due to its rotation being proportional to the angular velocity $\omega_m = \frac{d}{dt}\Theta_m$ with some proportionality factor k_e. The inertia of the motor around its axis is $I\frac{d^2}{dt^2}\Theta_m$. The rotation is damped by an angular velocity dependent damping torque $\tau_d = k_d\omega_m$. The torque generated by the motor is equal to $\tau = \kappa i_A$, which is entirely used to rotate the free spinning motor (overcoming the inertia and damping).

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

a)* You want to model the motor behavior for the case that v_A is set to zero, while the motor is rotating. Write the equation balancing all physical torques in the motor and solve express it in dependency to i_A. Replace i_A in that differential equation above with ω_m using the physical torques equation. What is the resulting differential equation?

$$I_c \cdot \ddot{\Theta}_m + k_d \dot{\Theta}_m = \tau = k \cdot i_A$$

$$l_A \dot{i}_A + r_A i_A + k_e \dot{\Theta}_m = v_A$$

$$\dot{i}_A = \frac{I_c}{k} \cdot \dddot{\Theta}_m + \frac{k_d}{k} \cdot \ddot{\Theta}_m$$

$$l_A \frac{I_c}{k} \dddot{\Theta}_m + l_A \frac{k_d}{k} \ddot{\Theta}_m + r_A \frac{I_c}{k} \ddot{\Theta}_m + r_A \frac{k_d}{k} \dot{\Theta}_m + k_e \dot{\Theta}_m = v_A$$

$$l_A \frac{I_c}{k} \ddot{\omega}_m + \left(l_A \frac{k_d}{k} + r_A \frac{I_c}{k}\right)\dot{\omega}_m + \left(r_A \frac{k_d}{k} + k_e\right)\omega_m = V_A$$

$V_A \quad \dot{\theta}_m = W_m$

$l_A \cdot \dot{i}_A + r_A \cdot i_A = V_A - k_e \dot{\theta}_m \qquad I_m = I \cdot \ddot{\theta}_m$

$\tau_d = k_d \cdot \dot{\theta}_m \qquad \tau = k i_A$

$\tau_m = I \cdot \ddot{\theta}_m + k_d \cdot \dot{\theta}_m = k \cdot i_A$

$i_A = \frac{I}{k} \cdot \ddot{\theta}_m + \frac{k_d}{k} \cdot \dot{\theta}_m$

$\dot{i}_A = \frac{I}{k} \cdot \dddot{\theta}_m + \frac{k_d}{k} \cdot \ddot{\theta}_m$

$l_A \left(\frac{I}{k} \dddot{\theta}_m + \frac{k_d}{k} \ddot{\theta}_m \right) + r_A \left(\frac{I}{k} \ddot{\theta}_m + \frac{k_d}{k} \dot{\theta}_m \right) + k_e \dot{\theta}_m = V_A$

$L_A \cdot \frac{I}{k} \dddot{\theta}_m + \left(L_A \cdot \frac{k_d}{k} + r_A \cdot \frac{I}{k} \right) \ddot{\theta}_m + \left(r_A \frac{k_d}{k} + k_e \right) \dot{\theta}_m = V_A$

$m\ddot{x} + b\dot{x} + kx = f = \alpha f' + \beta$

$\begin{cases} \alpha = m \\ \beta = b\dot{x} + kx \\ f' = \ddot{x} = -k_v \dot{x} - k_p x \end{cases}$

$L_A \cdot \frac{I}{k} \ddot{w}_m + \left(L_A \frac{k_d}{k} + r_A \frac{I}{k} \right) \dot{w}_m + \left(r_A \cdot \frac{k_d}{k} + k_e \right) \cdot w_m = \alpha \tau' + \beta$

$\alpha = L_A \cdot \frac{I}{k}$

$\tau' = \ddot{w}_m = -k_v \dot{w}_m - k_p w_m$

$\beta = \left(L_A \frac{k_d}{k} + r_A \frac{I}{k} \right) \dot{w}_m + \left(r_A \cdot \frac{k_d}{k} + k_e \right) w_m$

$\left(L_A \frac{k_d}{k} + \cancel{r_A \frac{I}{k}} \right)^2 - 4 \left(\cancel{L_A \frac{I}{k} \cdot V_A} \cdot \frac{k_d}{k} + L_A \frac{I}{k} \cdot k_e \right) = 0$

$V_A = 0 \qquad \Rightarrow \cancel{L_A} \frac{k_d}{\cancel{k}} = 2 \cancel{L_A} \frac{I}{\cancel{k}} k_e$

$k_d = 2 I k_e$

$I_m \quad \lambda \qquad \tau_m = \underbrace{\left(I_m + \frac{I}{\lambda^2} \right)}_{EI} \ddot{\theta}_m + \underbrace{\left(b_m + \frac{b}{\lambda^2} \right)}_{Ed} \dot{\theta}_m$

b) Explain briefly the concept of control law partitioning (equation example, what is the basic abstraction behind it) and apply it to the above equation. What are the expressions for the two partitioning blocks in the diagram for the given system for the case that $v_A = 0$ while the motor is spinning?

0 1 2 3 4 5 6 7 8 9

b) $\tau = \alpha \tau' + \beta$ ✓

The real physical ✓value is abstracted to an ✓ideal value with unit mass/force

Physical ✓parameters compensated in α, β ✓✓

$\alpha = l_A \cdot \frac{I_c}{k}$ ✓

$\beta = \left(l_A + \frac{k_d}{\cancel{k}} + r_A \frac{I_c}{k} \right) \dot{\omega}_n$ ✓

$+ \left(r_A \frac{k_d}{k} + k_2 \right) \omega_n$ ✓

c)* Check the time response of the angular velocity profile $\omega_m(t)$ for the case that the system switches from a constant spinning case at some $v_A \neq 0$ to v_A=0. Analyze for that the characteristic equation for ω_m (assume some constants in the equation if you did not solve a). For which relation of the above parameters is the motor system critically damped?

0 1 2 3 4 5

c) $k_1 \cdot \ddot{\omega}_m + k_2 \dot{\omega}_m + k_3 \cdot \omega_m = 0$ ✓

$s_{1/2} = -\frac{k_2}{2k_1} \pm \frac{\sqrt{k_2^2 - 4 k_1 k_3}}{2k_1}$ ✓

critically damped ✓

$k_2 = 2\sqrt{k_1 k_3}$ ✓

0 1 2 3 4 5 6 7 8

d)* A motor with own inertia I_m is connected to the robot link with a gear with a gear-ratio λ. The robot link exposes an inertia I on the link side of the gear. What is the equation describing the relation between the motor torque τ_m and the angular velocity of the motor axis ω_m? What does it mean for the required power of the motor?

$$\tau = \eta \, \tau_m$$

$$\dot{\theta} = (1/\eta)\, \dot{\theta}_m$$

$$\tau_m = \left(I_m + \frac{I}{\eta^2}\right)\dot{\omega}_m + \left(b_m + \frac{b}{\eta^2}\right)\omega_m$$

inertia "seen" by the motor $\left(I_m + \frac{I}{\eta^2}\right)$

The required power sinks with the square of the gear ratio

Additional space for solutions–clearly mark the (sub)problem your answers are related to and strike out invalid solutions.