Machine Learning for Graphs and Sequential Data Exercise Sheet 03

Temporal Point Processes

Problem 1: Consider a temporal point process, where all the inter-event times $\tau_i = t_i - t_{i-1}$ are sampled i.i.d. from the distribution with the survival function

$$S(\tau) = \exp\Big(-(e^{b\tau} - 1)\Big)$$

with a parameter $b > 0$.

a) Write down the closed-form expression for the conditional intensity function $\lambda^*(t)$ of this TPP. Simplify as far as you can.

b) Write down the closed-form expression for the log-likelihood of a sequence $\{t_1, ..., t_N\}$ generated from this TPP on the interval $[0, T]$. Simplify as far as you can.

Problem 2: Consider an inhomogeneous Poisson process (IPP) on $[0, 1]$ with the intensity function $\lambda(t) = 2t$. We simulate a sample from this IPP using thinning. For this, we first simulate a *homogeneous* Poisson process (HPP) with intensity $\mu = 4$ and apply the thinning procedure described in the lecture. What is the expected number of events from the HPP that will be rejected when using this procedure?

Problem 3: Consider an inhomogeneous Poisson process on $[0, 4]$ with the intensity function $\lambda(t) = \beta t$, where $\beta > 0$ is a parameter that has to be estimated. You have observed a single sequence $\{1, 2.1, 3.3, 3.8\}$ generated from this IPP. What is the maximum likelihood estimate of the parameter β?

Problem 4: Consider a *neural* temporal point process where the conditional intensity function is defined with a neural network. In particular, for a time point t_i, we represent the history $\{t_1, t_2, \dots, t_{i-1}\}$ with a fixed-sized vector $\boldsymbol{h}_i \in \mathbb{R}^d$. The conditional intensity function $\lambda^*(t)$ is defined as a function of $\boldsymbol{h}_i$. We will use the transformer architecture (see previous lecture). We propose the following implementation.

Given the full sequence $\{t_1, t_2, \dots, t_n\}$, we calculate all $\{\boldsymbol{h}_1, \boldsymbol{h}_2, \dots, \boldsymbol{h}_n\}$ in parallel. We first calculate vectors $\boldsymbol{q}_i, \boldsymbol{k}_i, \boldsymbol{v}_i \in \mathbb{R}^d$ as a function of t_i. We stack these vectors into matrices $\boldsymbol{Q}, \boldsymbol{K}, \boldsymbol{V} \in \mathbb{R}^{n \times d}$. The output of the transformer is: $\boldsymbol{H} = \text{softmax}(\boldsymbol{Q}\boldsymbol{K}^T)\boldsymbol{V}$, then $\boldsymbol{h}_i$ is the ith row of $\boldsymbol{H}$.

Identify the errors in this implementation compared to the original definition of $\boldsymbol{h}_i$. Propose a solution.

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(a)

$$F^*(\tau) = 1 - S(\tau) = 1 - \exp(-(e^{b\tau} - 1))$$

$$p^*(\tau) = \frac{dF^*(\tau)}{d\tau}$$

$$= -\exp(-(e^{b\tau}-1)) \cdot (-e^{b\tau}) \cdot b$$

$$\lambda^*(\tau) = \frac{p^*(\tau)}{S^*(\tau)} = \frac{-\cancel{\exp(-(e^{b\tau}-1))} \cdot (-e^{b\tau}) \cdot b}{\cancel{\exp(-(e^{b\tau}-1))}}$$

$$= b \cdot e^{b\tau}$$

$$\lambda^*(t) = b \cdot \exp[b \cdot (t - t_{i-1})]$$

(b)

$$p(\{\lambda^*(t_1) \dots \lambda^*(t_N)) = \left(\prod_{i=1}^{N} \lambda^*(t_i) \cdot S^*(t_i)\right) \cdot S^*(T)$$

$$\log p = \left(\sum_{i=1}^{N} \log \lambda^*(t_i) + \log S^*(t_i)\right) + \log S^*(T)$$

$$= \sum_{i=1}^{N} (\log b + b(t_i - t_{i-1}) + 1 - \exp b(t_i - t_{i-1})) + 1 - e^{b(T - t_N)}$$

$$= N \log b + b(t_N - 0) + N - \sum_{i=1}^{N} \exp[b(t_i - t_{i-1})] +$$

$$\Rightarrow N \log b + b t_N + N + 1 - \sum_{i=1}^{N+1} \exp[b(t_i - t_{i-1})]$$

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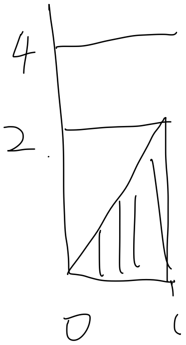

\# expected IPP $= S_0 = \int_0^1 \lambda(t)\, dt = t^2\big|_0^1 = 1$

\# expected HPP $= 4 \cdot 1 = 4$

$\therefore\ 4 - 1 = 3$

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$$\beta = \max \log \underbrace{P(\{s\} \mid \beta)}$$

$$= \sum_{i=1}^{N} \log \lambda^*(t_i) - \int_0^T \lambda^*(u)\, du$$

$$= \sum_{i=1}^{N} \log \beta + \log t - \int_0^T \lambda^*(u)\, du$$

$$= N \log \beta + \sum_{i=1}^{N} \log t_i - \int_0^T \lambda^*(u)\, du$$

$$= N \log \beta - \frac{1}{2} \beta u^2 \Big|_0^T + \sum_{i=1}^{N} \log t_i$$

$$= N \log \beta - \frac{T^2}{2} \beta + \sum_{i=1}^{N} \log t_i$$

$$\partial \beta = 0$$

$$\beta^* = \frac{N}{\beta} - \frac{T^2}{2} = 0$$

$$\frac{2N}{\boxed{T}^2} = \beta^* = \frac{1}{2}$$

4.

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Identify the errors in this implementation compared to the original definition of $\boldsymbol{h}_i$. Propose a solution.

$\lambda^*(t) = f(h_i)$

only consider the past point

Consider future points