Tutorial Robotics IN2067

Exercise Sheet 03

P01

Problem 1

In this problem, we are working with the same robot as in problem 10 of the first problem sheet, which is shown again in Figure 1. This time, the robot is equipped with a force-torque-sensor that has system $\{4\}$ as frame of reference. An external force is applied to the robot, such that its force-torque sensor reports a measurement of

$$\begin{pmatrix} {}^4f \\ {}^4n \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 0 \\ 7 \\ 0 \\ 8 \end{pmatrix} .$$

a) Determine the joint torques that are required to cancel out the external influences and thus keep the robot static.

b) Assume now that there is a screwdriver attached to the last link, and the tip of the screwdriver is translated along the z-axis about 9 length units, so ${}^4P_{\text{tip}} = (0, 0, 9)^T$. With the same force-torque measurement reported by the sensor in system 4, which forces and torques are present at the screwdriver tip? Which forces and torques are caused by the robot in direction of the screw driver (i.e., in Z direction)?

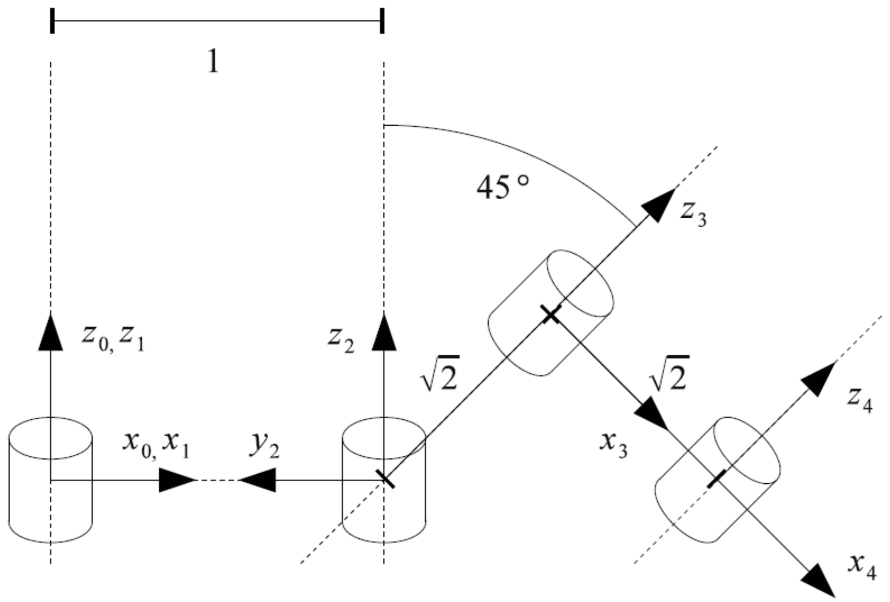

Figure 1: *4R Robot (Problem 1)*

P01

- We propagate the force and torques sensed at joint 4 to the remaining joints of the robot.
  Then we compute how much counter-force or counter-torque the robot must apply to each joint for it to be static.

$$ {}^{i}f_{i} = {}^{i}_{i+1}R\, {}^{i+1}f_{i+1} $$

$$ {}^{i}n_{i} = {}^{i}_{i+1}R\, {}^{i+1}n_{i+1} + {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R\, {}^{i+1}f_{i+1} \right) $$

$$ {}^{i+1}f_{i+1} = {}^{i+1}_{i}R\, {}^{i}f_{i} $$

$$ {}^{i+1}n_{i+1} = {}^{i+1}_{i}R \left({}^{i}n_{i} - {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R\, {}^{i+1}f_{i+1} \right) \right) $$

P01

- Recap from T01-P10:

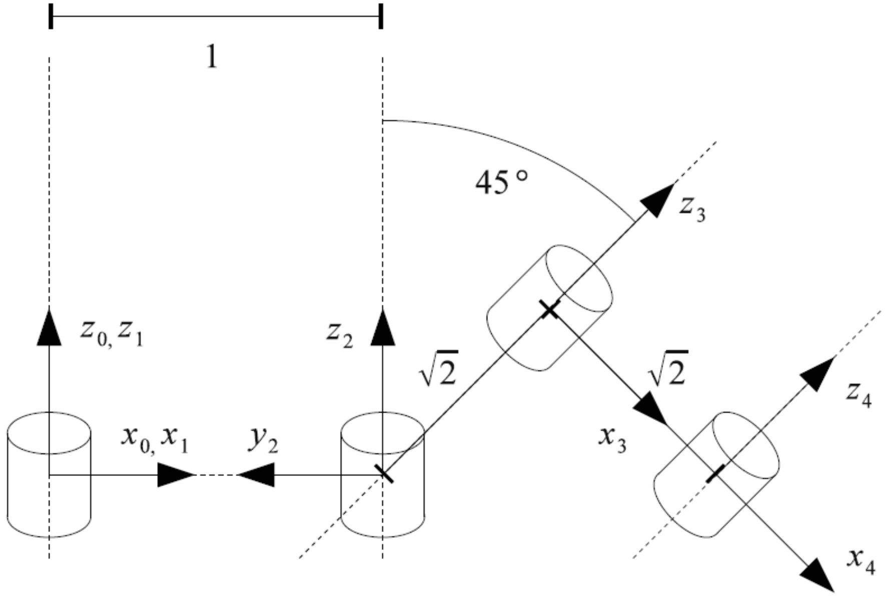

Figure 1: *4R Robot (Problem 1)*

DH-CF	α	a	d	θ
1	0°	0	0	θ_1
2	0°	1	0	θ_2
3	45°	0	$\sqrt{2}$	θ_3
4	0°	$\sqrt{2}$	0	0

$$ {}^0_1T = \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad {}^1_2T = \begin{pmatrix} c_2 & -s_2 & 0 & 1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^2_3T = \begin{pmatrix} c_3 & -s_3 & 0 & 0 \\ s_3\frac{\sqrt{2}}{2} & c_3\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -1 \\ s_3\frac{\sqrt{2}}{2} & c_3\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad {}^3_4T = \begin{pmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

P01

$$ {}^4f_4 = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} \qquad {}^4n_4 = \begin{pmatrix} 7 \\ 0 \\ 8 \end{pmatrix} $$

$$ {}^if_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} $$

$$ {}^in_i = {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R\,{}^{i+1}f_{i+1}\right) $$

P01

$$ {}^{4}f_{4} = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} \qquad {}^{4}n_{4} = \begin{pmatrix} 7 \\ 0 \\ 8 \end{pmatrix} $$

$$ {}^{3}f_{3} = \begin{pmatrix} -6S_4 \\ 6C_4 \\ 0 \end{pmatrix} \qquad {}^{3}n_{3} = \begin{pmatrix} 7C_4 \\ 7S_4 \\ 8 + 6\sqrt{2}\,C_4 \end{pmatrix} $$

$$ {}^{i}f_{i} = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} $$

$$ {}^{i}n_{i} = {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R\,{}^{i+1}f_{i+1}\right) $$

$$ {}^{3}_{4}T = \begin{pmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

P01

$$ {}^{i}f_{i} = {}^{i}_{i+1}R \, {}^{i+1}f_{i+1} $$

$$ {}^{i}n_{i} = {}^{i}_{i+1}R \, {}^{i+1}n_{i+1} + {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R \, {}^{i+1}f_{i+1} \right) $$

$$ {}^{4}f_{4} = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} \qquad {}^{4}n_{4} = \begin{pmatrix} 7 \\ 0 \\ 8 \end{pmatrix} $$

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$$ {}^{3}_{4}T = \begin{pmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{2}f_{2} = \begin{pmatrix} -6S_{34} \\ 3\sqrt{2}\,C_{34} \\ 3\sqrt{2}\,C_{34} \end{pmatrix} \qquad {}^{2}n_{2} = \begin{pmatrix} (7-6\sqrt{2})\,C_{34} \\ \left(7\frac{\sqrt{2}}{2}-6\right) S_{34} - 6C_4 - 4\sqrt{2} \\ \left(7\frac{\sqrt{2}}{2}-6\right) S_{34} + 6C_4 + 4\sqrt{2} \end{pmatrix} $$

$$ {}^{2}_{3}T = \begin{pmatrix} C_3 & -S_3 & 0 & 0 \\ S_3\frac{\sqrt{2}}{2} & C_3\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -1 \\ S_3\frac{\sqrt{2}}{2} & C_3\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

P01

$$^{i}f_{i} = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}$$

$$^{i}n_{i} = {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R\,{}^{i+1}f_{i+1}\right)$$

$$^{4}f_{4} = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} \qquad ^{4}n_{4} = \begin{pmatrix} 7 \\ 0 \\ 8 \end{pmatrix}$$

$$^{3}_{4}T = \begin{pmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$^{3}f_{3} = \begin{pmatrix} -6S_4 \\ 6C_4 \\ 0 \end{pmatrix} \qquad ^{3}n_{3} = \begin{pmatrix} 7C_4 \\ 7S_4 \\ 8 + 6\sqrt{2}\,C_4 \end{pmatrix}$$

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$$^{2}f_{2} = \begin{pmatrix} -6S_{34} \\ 3\sqrt{2}\,C_{34} \\ 3\sqrt{2}\,C_{34} \end{pmatrix} \qquad ^{2}n_{2} = \begin{pmatrix} (7-6\sqrt{2})\,C_{34} \\ \left(7\frac{\sqrt{2}}{2}-6\right)S_{34} - 6C_4 - 4\sqrt{2} \\ \left(7\frac{\sqrt{2}}{2}-6\right)S_{34} + 6C_4 + 4\sqrt{2} \end{pmatrix}$$

$$^{1}_{2}T = \begin{pmatrix} C_2 & -S_2 & 0 & 1 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$^{1}f_{1} = \begin{pmatrix} -3\sqrt{2}\,S_2C_{34} - 6\,C_2S_{34} \\ 3\sqrt{2}\,C_2C_{34} - 6\,S_2S_{34} \\ 3\sqrt{2}\,C_{34} \end{pmatrix}$$

$$^{1}n_{1} = \begin{pmatrix} \left(\left(-7\frac{\sqrt{2}}{2}+6\right)S_{34} + 6C_4 + 4\sqrt{2}\right)S_2 + (7-6\sqrt{2})\,C_2C_{34} \\ -\left(\left(-7\frac{\sqrt{2}}{2}+6\right)S_{34} + 6C_4 + 4\sqrt{2}\right)C_2 + (7-6\sqrt{2})\,S_2C_{34} - 3\sqrt{2}\,C_{34} \\ -6S_2S_{34} + \left(7\frac{\sqrt{2}}{2}-6\right)S_{34} + 3\sqrt{2}\,C_2C_{34} + 6C_4 + 4\sqrt{2} \end{pmatrix}$$

P01

- For determining the vector τ of the robot joint torques, we first look at the type of each joint:
  - Prismatic joint $i \Rightarrow$ joint torque $\tau_i = {}^i f_{iz}$ (the z-component of the force vector)
  - Rotational joint $i \Rightarrow$ joint torque $\tau_i = {}^i n_{iz}$ (the z-component of the torque vector)
- Then we collect all joint torque values into the vector τ

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  - Rotational joint $i \Rightarrow$ joint torque $\tau_i = {}^{i}n_{iz}$ (the z-component of the torque vector)
- Then we collect all joint torque values into the vector τ

$$\tau = \begin{pmatrix} {}^{1}n_{1z} \\ {}^{2}n_{2z} \\ {}^{3}n_{3z} \\ {}^{4}n_{4z} \end{pmatrix} = \begin{pmatrix} -6S_2S_{34} + \left(7\frac{\sqrt{2}}{2} - 6\right)S_{34} + 3\sqrt{2}C_2C_{34} + 6C_4 + 4\sqrt{2} \\ \left(7\frac{\sqrt{2}}{2} - 6\right)S_{34} + 6C_4 + 4\sqrt{2} \\ 8 + 6\sqrt{2}C_4 \\ 8 \end{pmatrix}$$

P01

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- Then we collect all joint torque values into the vector τ

$$\tau = \begin{pmatrix} {}^{1}n_{1z} \\ {}^{2}n_{2z} \\ {}^{3}n_{3z} \\ {}^{4}n_{4z} \end{pmatrix} = \begin{pmatrix} -6S_2S_{34} + \left(7\frac{\sqrt{2}}{2}-6\right)S_{34} + 3\sqrt{2}C_2C_{34} + 6C_4 + 4\sqrt{2} \\ \left(7\frac{\sqrt{2}}{2}-6\right)S_{34} + 6C_4 + 4\sqrt{2} \\ 8 + 6\sqrt{2}\,C_4 \\ 8 \end{pmatrix}$$

$$\text{for } \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 90^\circ \\ -90^\circ \\ 0 \end{pmatrix} \Rightarrow \tau = \begin{pmatrix} 18{,}707 \\ 12{,}707 \\ 16{,}485 \\ 8 \end{pmatrix}$$

P01

- From {4} to {5}, we assume no rotation (to make things easier)
- We know ${}^{4}_{5}t = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}$

P01

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- We know ${}^{4}_{5}t = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}$

$$ {}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} $$

$$ {}^{i}n_i = {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R\,{}^{i+1}f_{i+1}\right) $$

$$ \Rightarrow $$

$$ {}^{i+1}f_{i+1} = {}^{i+1}_{i}R\,{}^{i}f_i $$

$$ {}^{i+1}n_{i+1} = {}^{i+1}_{i}R\left({}^{i}n_i - {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R\,{}^{i+1}f_{i+1}\right)\right) $$

P01

- From {4} to {5}, we assume no rotation (to make things easier)
- We know ${}^4_5t = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}$

$$ {}^if_i = {}^i_{i+1}R\,{}^{i+1}f_{i+1} $$

$$ {}^in_i = {}^i_{i+1}R\,{}^{i+1}n_{i+1} + {}^i_{i+1}t \times \left({}^i_{i+1}R\,{}^{i+1}f_{i+1}\right) $$

$\Rightarrow$

$$ {}^{i+1}f_{i+1} = {}^{i+1}_iR\,{}^if_i $$

$$ {}^{i+1}n_{i+1} = {}^{i+1}_iR\left({}^in_i - {}^i_{i+1}t \times \left({}^i_{i+1}R\,{}^{i+1}f_{i+1}\right)\right) $$

$$ {}^5f_5 = {}^5_4R\,{}^4f_4 = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} $$

$$ {}^5n_5 = {}^5_4R\left({}^4n_4 - {}^4_5t \times {}^4f_4\right) = I_3\left(\begin{pmatrix} 7 \\ 0 \\ 8 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix} \times \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 7+54 \\ 0 \\ 8 \end{pmatrix} = \begin{pmatrix} 61 \\ 0 \\ 8 \end{pmatrix} $$

P01

- From {4} to {5}, we assume no rotation (to make things easier)
- We know ${}^{4}_{5}t = \begin{pmatrix}0\\0\\9\end{pmatrix}$

$$ {}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} $$

$$ {}^{i}n_i = {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R\,{}^{i+1}f_{i+1}\right) $$

$\Rightarrow$

$$ {}^{i+1}f_{i+1} = {}^{i+1}_{i}R\,{}^{i}f_i $$

$$ {}^{i+1}n_{i+1} = {}^{i+1}_{i}R\left({}^{i}n_i - {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R\,{}^{i+1}f_{i+1}\right)\right) $$

$$ {}^{5}f_5 = {}^{5}_{4}R\,{}^{4}f_4 = \begin{pmatrix}0\\6\\0\end{pmatrix} $$

$$ {}^{5}n_5 = {}^{5}_{4}R\left({}^{4}n_4 - {}^{4}_{5}t \times {}^{4}f_4\right) = I_3\left(\begin{pmatrix}7\\0\\8\end{pmatrix} - \begin{pmatrix}0\\0\\9\end{pmatrix} \times \begin{pmatrix}0\\6\\0\end{pmatrix}\right) = \begin{pmatrix}7+54\\0\\8\end{pmatrix} = \begin{pmatrix}61\\0\\8\end{pmatrix} $$

$$ {}^{5}f_{5z} = 0 $$

$$ {}^{5}n_{5z} = 8 $$

P02

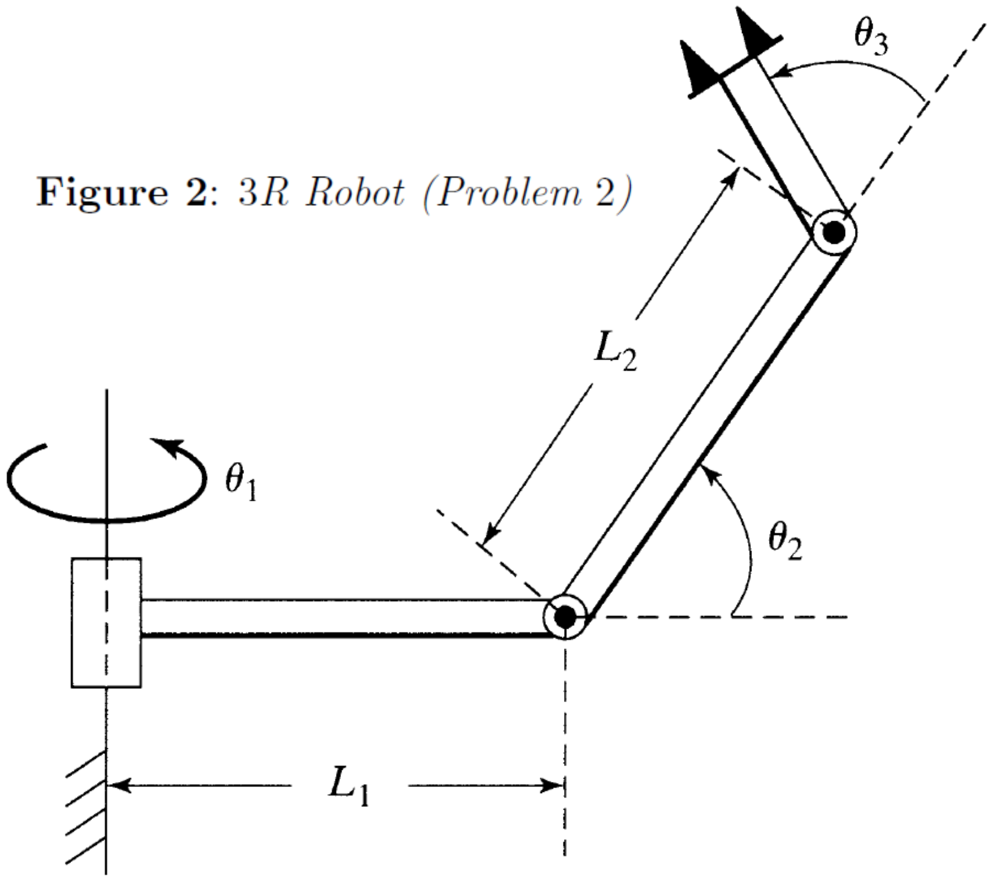


Figure 2: *3R Robot (Problem 2)*

Problem 2

For the robot shown in Figure 2, determine the Jacobian w.r.t. reference frame {4} using three different approaches:

a) Compute velocities in system 4, and derive the Jacobian

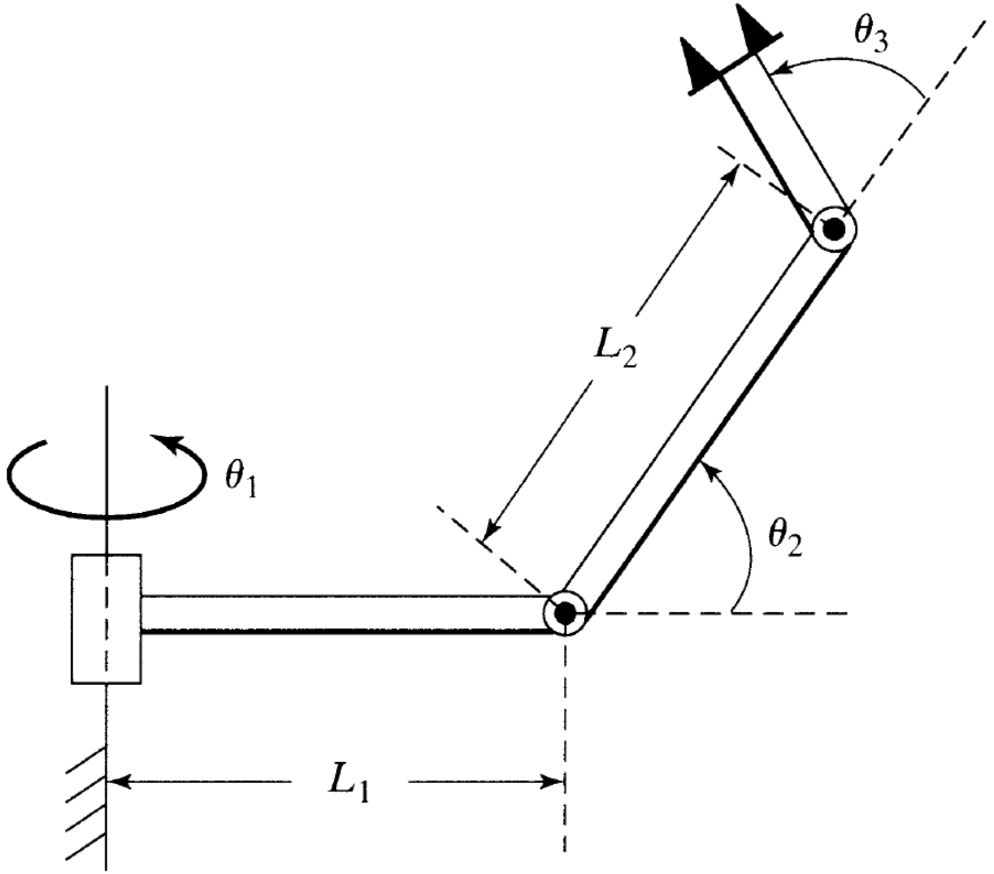
b) Compute force-torque relations for system 4, derive the Jacobian

c) Geometric observations

P02

DH	α	a	d	θ
1	0°	0	0	θ_1
2	90°	l_1	0	θ_2
3	0°	l_2	0	θ_3
4	0°	l_3	0	0°

$$\Rightarrow {}^0_4T = \begin{pmatrix} C_1C_{23} & -C_1S_{23} & S_1 & (l_1+l_2C_2+l_3C_{23})C_1 \\ S_1C_{23} & -S_1S_{23} & -C_1 & (l_1+l_2C_2+l_3C_{23})S_1 \\ S_{23} & C_{23} & 0 & l_2S_2+l_3S_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



P02

a) $\begin{pmatrix} {}^{i}v_{EE} \\ {}^{i}\omega_{EE} \end{pmatrix} = {}^{i}J \cdot \dot{\theta}$, EE is the robot's end-effector

b) $\tau = {}^{i}J^{T} \cdot {}^{i}\mathcal{F} = {}^{i}J^{T} \cdot \begin{pmatrix} {}^{i}f_{EE} \\ {}^{i}n_{EE} \end{pmatrix}$, EE is the robot's end-effector

c) $${}^{0}\dot{p}_{EE} = \dot{\mathrm{fkm}}(\theta) = {}^{0}J \cdot \dot{\theta} = \begin{pmatrix} {}^{0}J_{v} \\ {}^{0}J_{\omega} \end{pmatrix} \cdot \dot{\theta},\ {}^{0}p_{EE} \in \mathbb{R}^{6}$$

$${}^{i}J = \begin{pmatrix} {}^{i}_{0}R & 0_3 \\ 0_3 & {}^{i}_{0}R \end{pmatrix} \cdot {}^{0}J$$

P02

a) $\begin{pmatrix} {}^{i}v_{EE} \\ {}^{i}\omega_{EE} \end{pmatrix} = {}^{i}J \cdot \dot{\theta}$, EE is the robot's end-effector

b) $\tau = {}^{i}J^{T} \cdot {}^{i}\mathcal{F} = {}^{i}J^{T} \cdot \begin{pmatrix} {}^{i}f_{EE} \\ {}^{i}n_{EE} \end{pmatrix}$, EE is the robot's end-effector

c) $${}^{0}\dot{p}_{EE} = \dot{\mathrm{fkm}}(\theta) = {}^{0}J \cdot \dot{\theta} = \begin{pmatrix} {}^{0}J_{v} \\ {}^{0}J_{\omega} \end{pmatrix} \cdot \dot{\theta},\ {}^{0}p_{EE} \in \mathbb{R}^{6}$$

$${}^{i}J = \begin{pmatrix} {}_{0}^{i}R & 0_3 \\ 0_3 & {}_{0}^{i}R \end{pmatrix} \cdot {}^{0}J$$

P02

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{\ \ i}R\,{}^{i}\omega_i + \begin{pmatrix}0\\0\\\dot{\theta}_{i+1}\end{pmatrix}$$

$${}^{i+1}v_{i+1} = {}^{i+1}_{\ \ i}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{\ \ i}_{i+1}t\right) + \begin{pmatrix}0\\0\\\dot{d}_{i+1}\end{pmatrix}$$

- For the prismatic joint $i+1$: $\dot{\theta}_{i+1} = 0$
- For the rotational joint $i+1$: $\dot{d}_{i+1} = 0$

P02

$$
{}^{i+1}\omega_{i+1} = {}^{i+1}_{\ \ \ i}R\,{}^{i}\omega_{i} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{pmatrix}
$$

$$
{}^{i+1}v_{i+1} = {}^{i+1}_{\ \ \ i}R\left({}^{i}v_{i} + {}^{i}\omega_{i} \times {}^{\ \ \ i}_{i+1}t\right) + \begin{pmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{pmatrix}
$$

$$
{}^{\ \ \ i}_{i+1}T = \begin{pmatrix} {}^{\ \ \ i}_{i+1}R & {}^{\ \ \ i}_{i+1}t \\ 000 & 1 \end{pmatrix} = \begin{pmatrix} {}^{i+1}_{\ \ \ i}R^{T} & {}^{\ \ \ i}_{i+1}t \\ 000 & 1 \end{pmatrix}
$$

- For the prismatic joint $i + 1$: $\dot{\theta}_{i+1} = 0$
- For the rotational joint $i + 1$: $\dot{d}_{i+1} = 0$

P02

$${}^{0}\omega_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad {}^{0}v_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{\ \ i}R\, {}^{i}\omega_i + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{pmatrix}$$

$${}^{i+1}v_{i+1} = {}^{i+1}_{\ \ i}R \left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}_{i+1}t \right) + \begin{pmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{pmatrix}$$

P02

$$
{}^{0}\omega_{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad {}^{0}v_{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
{}^{1}\omega_{1} = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{1} \end{pmatrix} \qquad {}^{1}v_{1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
{}^{i+1}\omega_{i+1} = {}^{i+1}_{\ \ i}R\,{}^{i}\omega_{i} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{pmatrix}
$$

$$
{}^{i+1}v_{i+1} = {}^{i+1}_{\ \ i}R\left({}^{i}v_{i} + {}^{i}\omega_{i} \times {}^{\ \ i}_{i+1}t\right) + \begin{pmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{pmatrix}
$$

P02

$$^{i+1}\omega_{i+1} = {}^{i+1}_{\;\;i}R\,{}^{i}\omega_{i} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{pmatrix}$$

$$^{i+1}v_{i+1} = {}^{i+1}_{\;\;i}R\left({}^{i}v_{i} + {}^{i}\omega_{i} \times {}^{\;\;i}_{i+1}t\right) + \begin{pmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{pmatrix}$$

$$^{0}\omega_{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad {}^{0}v_{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$^{1}\omega_{1} = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{1} \end{pmatrix} \qquad {}^{1}v_{1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$^{2}\omega_{2} = \begin{pmatrix} \dot{\theta}_{1} s_{2} \\ \dot{\theta}_{1} c_{2} \\ \dot{\theta}_{2} \end{pmatrix} \qquad {}^{2}v_{2} = \begin{pmatrix} 0 \\ 0 \\ -l_{1}\dot{\theta}_{1} \end{pmatrix}$$

P02

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{\ \ i}R\,{}^{i}\omega_i + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{pmatrix} $$

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{\ \ i}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{\ \ i}_{i+1}t\right) + \begin{pmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{pmatrix} $$

$$ {}^{0}\omega_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad {}^{0}v_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} $$

$$ {}^{1}\omega_1 = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} \qquad {}^{1}v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} $$

$$ {}^{2}\omega_2 = \begin{pmatrix} \dot{\theta}_1 s_2 \\ \dot{\theta}_1 c_2 \\ \dot{\theta}_2 \end{pmatrix} \qquad {}^{2}v_2 = \begin{pmatrix} 0 \\ 0 \\ -l_1 \dot{\theta}_1 \end{pmatrix} $$

$$ {}^{3}\omega_3 = \begin{pmatrix} \dot{\theta}_1 s_{23} \\ \dot{\theta}_1 c_{23} \\ \dot{\theta}_2 + \dot{\theta}_3 \end{pmatrix} \qquad {}^{3}v_3 = \begin{pmatrix} l_2 \dot{\theta}_2 s_3 \\ l_2 \dot{\theta}_2 c_3 \\ -\dot{\theta}_1 (l_1 + l_2 c_2) \end{pmatrix} $$

P02

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{\;\;i}R \; {}^{i}\omega_{i} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{pmatrix} $$

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{\;\;i}R \left({}^{i}v_{i} + {}^{i}\omega_{i} \times {}^{\;\;i}_{i+1}t \right) + \begin{pmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{pmatrix} $$

$$ {}^{0}\omega_{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad {}^{0}v_{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} $$

$$ {}^{1}\omega_{1} = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{1} \end{pmatrix} \qquad {}^{1}v_{1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} $$

$$ {}^{2}\omega_{2} = \begin{pmatrix} \dot{\theta}_{1} s_{2} \\ \dot{\theta}_{1} c_{2} \\ \dot{\theta}_{2} \end{pmatrix} \qquad {}^{2}v_{2} = \begin{pmatrix} 0 \\ 0 \\ -l_{1} \dot{\theta}_{1} \end{pmatrix} $$

$$ {}^{3}\omega_{3} = \begin{pmatrix} \dot{\theta}_{1} s_{23} \\ \dot{\theta}_{1} c_{23} \\ \dot{\theta}_{2} + \dot{\theta}_{3} \end{pmatrix} \qquad {}^{3}v_{3} = \begin{pmatrix} l_{2} \dot{\theta}_{2} s_{3} \\ l_{2} \dot{\theta}_{2} c_{3} \\ -\dot{\theta}_{1} (l_{1} + l_{2} c_{2}) \end{pmatrix} $$

$$ {}^{4}\omega_{4} = \begin{pmatrix} \dot{\theta}_{1} s_{23} \\ \dot{\theta}_{1} c_{23} \\ \dot{\theta}_{2} + \dot{\theta}_{3} \end{pmatrix} \qquad {}^{4}v_{4} = \begin{pmatrix} l_{2} \dot{\theta}_{2} s_{3} \\ \dot{\theta}_{2} (l_{2} c_{3} + l_{3}) + l_{3} \dot{\theta}_{3} \\ -\dot{\theta}_{1} (l_{1} + l_{2} c_{2} + l_{3} c_{23}) \end{pmatrix} $$

P02

$$\begin{pmatrix} {}^{4}v_{4} \\ {}^{4}\omega_{4} \end{pmatrix} = {}^{4}J\,\dot{\Theta}$$

P02

$$\begin{pmatrix} {}^{4}v_{4} \\ {}^{4}\omega_{4} \end{pmatrix} = {}^{4}J\,\dot{\Theta} \iff \begin{pmatrix} l_2 s_3\,\dot{\theta}_2 \\ (l_2 c_3 + l_3)\dot{\theta}_2 + l_3\dot{\theta}_3 \\ -(l_1 + l_2 c_2 + l_3 c_{23})\dot{\theta}_1 \\ \dot{\theta}_1 s_{23} \\ \dot{\theta}_1 c_{23} \\ \dot{\theta}_2 + \dot{\theta}_3 \end{pmatrix} = {}^{4}J\,\dot{\Theta}$$

P02

$$\begin{pmatrix} {}^4v_4 \\ {}^4\omega_4 \end{pmatrix} = {}^4J\,\dot{\Theta} \iff \begin{pmatrix} l_2 s_3 \dot{\theta}_2 \\ (l_2 c_3 + l_3)\dot{\theta}_2 + l_3 \dot{\theta}_3 \\ -(l_1 + l_2 c_2 + l_3 c_{23})\dot{\theta}_1 \\ \dot{\theta}_1 s_{23} \\ \dot{\theta}_1 c_{23} \\ \dot{\theta}_2 + \dot{\theta}_3 \end{pmatrix} = {}^4J\,\dot{\Theta} \Rightarrow$$

$$\Rightarrow {}^4J = \begin{pmatrix} 0 & l_2 s_3 & 0 \\ 0 & l_2 c_3 + l_3 & l_3 \\ -(l_1 + l_2 c_2 + l_3 c_{23}) & 0 & 0 \\ s_{23} & 0 & 0 \\ c_{23} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

P02

a) $\begin{pmatrix} {}^{i}v_{EE} \\ {}^{i}\omega_{EE} \end{pmatrix} = {}^{i}J \cdot \dot{\theta}$, EE is the robot's end-effector

b) $\tau = {}^{i}J^{T} \cdot {}^{i}\mathcal{F} = {}^{i}J^{T} \cdot \begin{pmatrix} {}^{i}f_{EE} \\ {}^{i}n_{EE} \end{pmatrix}$, EE is the robot's end-effector

c) ${}^{0}\dot{p}_{EE} = \dot{\mathrm{fkm}}(\theta) = {}^{0}J \cdot \dot{\theta} = \begin{pmatrix} {}^{0}J_{v} \\ {}^{0}J_{\omega} \end{pmatrix} \cdot \dot{\theta}$, ${}^{0}p_{EE} \in \mathbb{R}^{6}$

$${}^{i}J = \begin{pmatrix} {}^{i}_{0}R & 0_3 \\ 0_3 & {}^{i}_{0}R \end{pmatrix} \cdot {}^{0}J$$

P02

$${}^{i}f_{i} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \qquad {}^{i}n_{i} = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}$$

$${}^{i}f_{i} = {}^{i}_{i+1}R\, {}^{i+1}f_{i+1}$$

$${}^{i}n_{i} = {}^{i}_{i+1}R\, {}^{i+1}n_{i+1} + {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R\, {}^{i+1}f_{i+1}\right)$$

P02

$$ {}^{i}f_{i} = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} $$

$$ {}^{i}n_{i} = {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R\,{}^{i+1}f_{i+1}\right) $$

$$ {}^{4}f_{4} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \qquad {}^{4}n_{4} = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} $$

$$ {}^{3}f_{3} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \qquad {}^{3}n_{3} = \begin{pmatrix} N_1 \\ N_2 - F_3 l_3 \\ N_3 + F_2 l_3 \end{pmatrix} $$

P02

$$ {}^{i}f_{i} = {}^{i}_{i+1}R \, {}^{i+1}f_{i+1} $$

$$ {}^{i}n_{i} = {}^{i}_{i+1}R \, {}^{i+1}n_{i+1} + {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R \, {}^{i+1}f_{i+1} \right) $$

$$ {}^{4}f_{4} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \qquad {}^{4}n_{4} = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} $$

$$ {}^{3}f_{3} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \qquad {}^{3}n_{3} = \begin{pmatrix} N_1 \\ N_2 - F_3 l_3 \\ N_3 + F_2 l_3 \end{pmatrix} $$

$$ {}^{2}f_{2} = \begin{pmatrix} F_1 c_3 - F_2 s_3 \\ F_1 s_3 + F_2 c_3 \\ F_3 \end{pmatrix} \qquad {}^{2}n_{2} = \begin{pmatrix} N_1 c_3 + (F_3 l_3 - N_2) s_3 \\ -F_3 l_2 + N_1 s_3 + (-F_3 l_3 + N_2) c_3 \\ F_2 l_3 + N_3 + l_2 (F_1 s_3 + F_2 c_3) \end{pmatrix} $$

P02

$$
{}^{i}f_{i} = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}
$$

$$
{}^{i}n_{i} = {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R\,{}^{i+1}f_{i+1}\right)
$$

$$
{}^{4}f_{4} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \qquad {}^{4}n_{4} = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}
$$

$$
{}^{3}f_{3} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \qquad {}^{3}n_{3} = \begin{pmatrix} N_1 \\ N_2 - F_3 l_3 \\ N_3 + F_2 l_3 \end{pmatrix}
$$

$$
{}^{2}f_{2} = \begin{pmatrix} F_1 c_3 - F_2 s_3 \\ F_1 s_3 + F_2 c_3 \\ F_3 \end{pmatrix} \qquad {}^{2}n_{2} = \begin{pmatrix} N_1 c_3 + (F_3 l_3 - N_2) s_3 \\ -F_3 l_2 + N_1 s_3 + (-F_3 l_3 + N_2) c_3 \\ F_2 l_3 + N_3 + l_2 (F_1 s_3 + F_2 c_3) \end{pmatrix}
$$

$$
{}^{1}f_{1} = \begin{pmatrix} F_1 c_{23} - F_2 s_{23} \\ -F_3 \\ F_1 s_{23} + F_2 c_{23} \end{pmatrix} \qquad {}^{1}n_{1} = \begin{pmatrix} (F_3 l_3 - N_2) s_{23} + N_1 c_{23} + F_3 l_2 s_2 \\ F_1 l_1 s_{23} - F_1 l_2 s_3 - F_2 l_1 c_{23} - F_2 l_2 c_3 - F_2 l_3 - N_3 \\ (N_2 - F_3 l_3) c_{23} + N_1 s_{23} - F_3 l_2 c_2 - F_3 l_1 \end{pmatrix}
$$

P02

- All joints are rotational $\Rightarrow \tau_i = {}^{i}n_{iz}$

$$\tau = \begin{pmatrix} {}^{1}n_{1z} \\ {}^{2}n_{2z} \\ {}^{3}n_{3z} \end{pmatrix} = \begin{pmatrix} -F_3(l_1 + l_2 c_2 + l_3 c_{23}) + N_1 s_{23} + N_2 c_{23} \\ F_1 l_2 s_3 + F_2 (l_2 c_3 + l_3) + N_3 \\ F_2 l_3 + N_3 \end{pmatrix}$$

P02

- All joints are rotational $\Rightarrow \tau_i = {}^{i}n_{iz}$

$$\tau = \begin{pmatrix} {}^{1}n_{1z} \\ {}^{2}n_{2z} \\ {}^{3}n_{3z} \end{pmatrix} = \begin{pmatrix} -F_3(l_1 + l_2 c_2 + l_3 c_{23}) + N_1 s_{23} + N_2 c_{23} \\ F_1 l_2 s_3 + F_2 (l_2 c_3 + l_3) + N_3 \\ F_2 l_3 + N_3 \end{pmatrix} = {}^{4}J^{T} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ N_1 \\ N_2 \\ N_3 \end{pmatrix}$$

P02

- All joints are rotational $\Rightarrow \tau_i = {}^{i}n_{iz}$

$$\tau = \begin{pmatrix} {}^{1}n_{1z} \\ {}^{2}n_{2z} \\ {}^{3}n_{3z} \end{pmatrix} = \begin{pmatrix} -F_3(l_1 + l_2 c_2 + l_3 c_{23}) + N_1 s_{23} + N_2 c_{23} \\ F_1 l_2 s_3 + F_2 (l_2 c_3 + l_3) + N_3 \\ F_2 l_3 + N_3 \end{pmatrix} = {}^{4}J^{T} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ N_1 \\ N_2 \\ N_3 \end{pmatrix} \Rightarrow$$

$$\Rightarrow {}^{4}J = \begin{pmatrix} 0 & l_2 s_3 & 0 \\ 0 & l_2 c_3 + l_3 & l_3 \\ -(l_1 + l_2 c_2 + l_3 c_{23}) & 0 & 0 \\ s_{23} & 0 & 0 \\ c_{23} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

P02

a) $\begin{pmatrix} {}^{i}v_{EE} \\ {}^{i}\omega_{EE} \end{pmatrix} = {}^{i}J \cdot \dot{\theta}$, EE is the robot's end-effector

b) $\tau = {}^{i}J^{T} \cdot {}^{i}\mathcal{F} = {}^{i}J^{T} \cdot \begin{pmatrix} {}^{i}f_{EE} \\ {}^{i}n_{EE} \end{pmatrix}$, EE is the robot's end-effector

c) $${}^{0}\dot{p}_{EE} = \dot{\mathrm{fkm}}(\theta) = {}^{0}J \cdot \dot{\theta} = \begin{pmatrix} {}^{0}J_{v} \\ {}^{0}J_{\omega} \end{pmatrix} \cdot \dot{\theta},\ {}^{0}p_{EE} \in \mathbb{R}^{6}$$
$${}^{i}J = \begin{pmatrix} {}^{i}_{0}R & 0_{3} \\ 0_{3} & {}^{i}_{0}R \end{pmatrix} \cdot {}^{0}J$$

P02

$$ {}^{4}J = \begin{pmatrix} {}^{4}_{0}R & 0_3 \\ 0_3 & {}^{4}_{0}R \end{pmatrix} {}^{0}J = \begin{pmatrix} {}^{4}_{0}R & 0_3 \\ 0_3 & {}^{4}_{0}R \end{pmatrix} \begin{pmatrix} {}^{0}J_v \\ {}^{0}J_\omega \end{pmatrix} $$

P02

$$ {}^{0}J_{v}\,\dot{\theta} = {}^{0}\dot{p} $$

P02

$$
{}^0_4T = \begin{pmatrix} C_1C_{23} & -C_1S_{23} & S_1 & (l_1+l_2C_2+l_3C_{23})C_1 \\ S_1C_{23} & -S_1S_{23} & -C_1 & (l_1+l_2C_2+l_3C_{23})S_1 \\ S_{23} & C_{23} & 0 & l_2S_2+l_3S_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^0J_v\,\dot{\theta} = {}^0\dot{P} \overset{({}^0_4T)}{=} \begin{pmatrix} (l_1+l_2c_2+l_3c_{23})c_1 \\ (l_1+l_2c_2+l_3c_{23})s_1 \\ l_2s_2+l_3s_{23} \end{pmatrix}
$$

P02

$$
{}^{0}_{4}T = \begin{pmatrix} C_1 C_{23} & -C_1 S_{23} & S_1 & (l_1 + l_2 C_2 + l_3 C_{23}) C_1 \\ S_1 C_{23} & -S_1 S_{23} & -C_1 & (l_1 + l_2 C_2 + l_3 C_{23}) S_1 \\ S_{23} & C_{23} & 0 & l_2 S_2 + l_3 S_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^{0}J_v \dot{\theta} = {}^{0}\dot{P} \overset{({}^{0}_{4}T)}{=} \begin{pmatrix} (l_1 + l_2 c_2 + l_3 c_{23}) c_1 \\ (l_1 + l_2 c_2 + l_3 c_{23}) s_1 \\ l_2 s_2 + l_3 s_{23} \end{pmatrix}
$$

$$\Downarrow$$

$$
{}^{0}J_v = \begin{pmatrix} -(l_1 + l_2 c_2 + l_3 c_{23}) s_1 & -(l_2 s_2 + l_3 s_{23}) c_1 & -l_3 s_{23} c_1 \\ (l_1 + l_2 c_2 + l_3 c_{23}) c_1 & -(l_2 s_2 + l_3 s_{23}) s_1 & -l_3 s_{23} s_1 \\ 0 & l_2 c_2 + l_3 c_{23} & l_3 c_{23} \end{pmatrix}
$$

P02

For an n-jointed robot: ${}^{i}J_{\omega} = \left({}^{i}r_1 \quad \vdots \quad {}^{i}r_2 \quad \vdots \quad \cdots \quad \vdots \quad {}^{i}r_n\right) \in \mathbb{R}^{3\times n}$

${}^{i}r_j \in \mathbb{R}^3$ is the rotation axis of the j-th joint expressed in the coordinate frame $\{i\}$:

- If joint j is rotational: ${}^{i}r_j = {}^{i}z_j = {}^{i}_{j}R\,{}^{j}z_j = {}^{i}_{j}R \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
- If joint j is prismatic:

  there is no rotation axis $\Rightarrow$ ${}^{i}r_j = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

P02

$$ {}^{0}J_{\omega} = \left({}^{0}z_1 \;\middle|\; {}^{0}z_2 \;\middle|\; {}^{0}z_3 \right) $$

$$ {}^{0}z_i = {}^{0}_{i}R \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} $$

P02

DH	α	a	d	θ
1	0°	0	0	θ_1
2	90°	l_1	0	θ_2
3	0°	l_2	0	θ_3
4	0°	l_3	0	0°

$$^{0}J_{\omega} = \left({}^{0}Z_1 \;\vdots\; {}^{0}Z_2 \;\vdots\; {}^{0}Z_3 \right) = \begin{pmatrix} 0 & S_1 & S_1 \\ 0 & -C_1 & -C_1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$^{0}Z_i = {}^{0}_{i}R \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

P02

$$ {}^{0}_{4}T = \begin{pmatrix} C_1C_{23} & -C_1S_{23} & S_1 & (l_1+l_2C_2+l_3C_{23})C_1 \\ S_1C_{23} & -S_1S_{23} & -C_1 & (l_1+l_2C_2+l_3C_{23})S_1 \\ S_{23} & C_{23} & 0 & l_2S_2+l_3S_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{0}_{4}R = {}^{4}_{0}R^{T} $$

P02

$$ {}^{4}J_v = {}^{4}_{0}R\,{}^{0}J_v = \begin{pmatrix} 0 & l_2 s_3 & 0 \\ 0 & l_2 c_3 + l_3 & l_3 \\ -(l_1 + l_2 c_2 + l_3 c_{23}) & 0 & 0 \end{pmatrix} $$

$$ {}^{4}J_w = {}^{4}_{0}R\,{}^{0}J_w = \begin{pmatrix} s_{23} & 0 & 0 \\ c_{23} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} $$

$$ {}^{4}J = \begin{pmatrix} {}^{4}J_v \\ {}^{4}J_w \end{pmatrix} $$

$$ {}^{0}_{4}T = \begin{pmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 & (l_1 + l_2 c_2 + l_3 c_{23}) c_1 \\ s_1 c_{23} & -s_1 s_{23} & -c_1 & (l_1 + l_2 c_2 + l_3 c_{23}) s_1 \\ s_{23} & c_{23} & 0 & l_2 s_2 + l_3 s_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{0}_{4}R = {}^{4}_{0}R^{T} $$