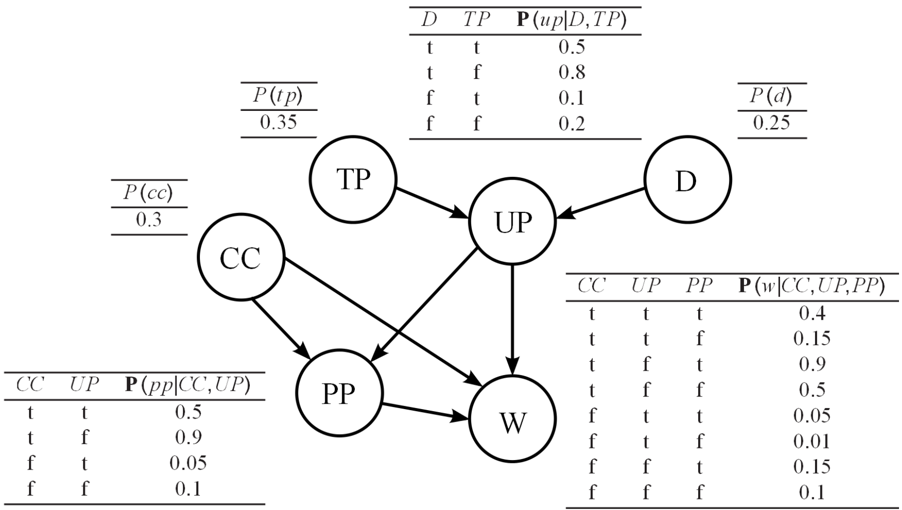
Exercise 8: Bayesian networks – Problems Winter Semester 2023/24
Laura Lützow

1 Presented Problems

Problem 8.1:



$P(tp)$
0.35

$P(d)$
0.25

$P(cc)$
0.3

D	TP	$\mathbf{P}(up\|D,TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

CC	UP	$\mathbf{P}(pp\|CC,UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w\|CC,UP,PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

Figure 1: A Bayesian Network with Boolean random variables: $D = DemotivatedPilot$, $TP = TalentedPilot$, $CC = CompetitiveCar$, $UP = UnderPerformance$, $PP = PolePosition$, $W = Wins$ (the values in the tables are guessed).

Consider the Bayesian network about a race in Fig.1. The true and false event of the random variables are abbreviated with the lowercase and the negated lowercase respectively (e.g. $D = true$ is d and $D = false$ is $\neg d$).

a. Which of these statements are true?

i. TP, UP and D are independent.

ii. TP and D are independent.

iii. PP and D are conditionally independent given UP.

iv. TP and D are conditionally independent given CC.

v. $\mathbf{P}(D|TP,UP) = \mathbf{P}(D|TP,CC,UP,PP,W)$.

b. Write the formula for computing the joint probability distribution in terms of the conditional probabilities exploiting the conditional independencies in the considered network.

c. Calculate $P(\neg d, tp, cc, \neg up, pp, w)$ and $P(\neg d, tp, cc, \neg up, \neg pp, w)$.

d. Calculate the probability that the pilot wins given that she/he is talented, motivated and starts from the pole position on the grid.

Problem 8.1:

$P(tp)$
0.35

D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
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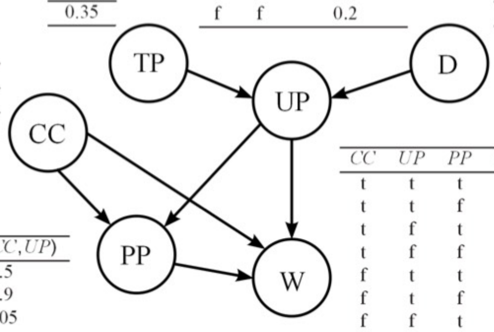
$P(d)$
0.25

$P(cc)$
0.3



CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
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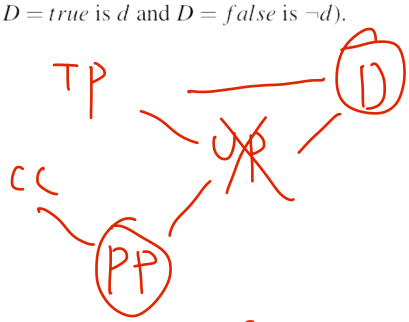
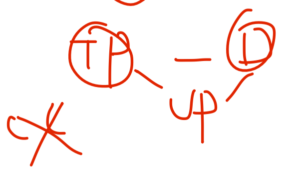
Consider the Bayesian network about a race in Fig. 1. The true and false event of the random variables are abbreviated with the lowercase and the negated lowercase respectively (e.g. $D = true$ is d and $D = false$ is $\neg d$).

a. Which of these statements are true?

i. TP, UP and D are independent. ✗
ii. TP and D are independent. ✓
iii. PP and D are conditionally independent given UP. ✓
iv. TP and D are conditionally independent given CC. ✗
v. $\mathbf{P}(D \mid TP, UP) = \mathbf{P}(D \mid TP, CC, UP, PP, W)$. ✓

iii


iv


b. Write the formula for computing the joint probability distribution in terms of the conditional probabilities exploiting the conditional independencies in the considered network.

$$P(TP, D, UP, CC, PP, W) = P(TP) \cdot P(D) \cdot P(UP \mid TP, D) \cdot P(CC) \cdot P(PP \mid CC, UP) \cdot P(W \mid PP, CC, UP)$$

c. Calculate $P(\neg d, tp, cc, \neg up, pp, w)$ and $P(\neg d, tp, cc, \neg up, \neg pp, w)$.

$$= P(\neg d)\, P(tp) \cdot P(\neg up \mid tp, \neg d) \cdot P(cc) \cdot P(pp \mid cc, \neg up) \cdot P(w \mid pp, cc, \neg up)$$
$$= (1 - 0.25) \cdot 0.35 \cdot (1 - 0.1) \cdot 0.3 \cdot 0.9 \cdot 0.9$$
$$= 0.0574$$

$$= P(\neg d)\, P(tp) \cdot P(\neg up \mid tp, \neg d) \cdot P(cc) \cdot P(\neg pp \mid cc, \neg up) \cdot P(w \mid \neg pp, cc, \neg up)$$
$$= (1 - 0.25) \cdot 0.35 \cdot (1 - 0.1) \cdot 0.3 \cdot (1 - 0.9) \cdot 0.5$$
$$= 0.0035$$

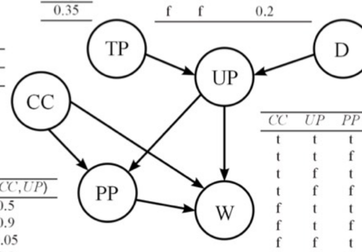
d. Calculate the probability that the pilot wins given that she/he is talented, motivated and starts from the pole position on the grid.

$P(tp)$
0.35

D	TP	$\mathbf{P}(up\mid D,TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(d)$
0.25

$P(cc)$
0.3



CC	UP	$\mathbf{P}(pp\mid CC,UP)$
t	t	0.5
t	f	0.9
f	t	0.05
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CC	UP	PP	$\mathbf{P}(w\mid CC,UP,PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

$$p(w \mid tp, \neg d, pp)$$

$$= \frac{p(w, tp, \neg d, pp)}{p(\neg w, tp, \neg d, pp) + p(w, tp, \neg d, pp)}$$

$$= \sum_{up}\sum_{cc} p(w, tp, \neg d, pp, UP, CC)$$

$$= \sum_{up}\sum_{cc} p(tp)\cdot p(\neg d)\cdot p(up \mid tp, \neg d)\cdot p(cc)\cdot p(pp \mid cc, up)\cdot p(w \mid pp, cc, up)$$

$$= p(tp)\,p(\neg d)\cdot \sum_{up} p(up \mid tp, \neg d)\cdot \sum_{cc} p(cc)\cdot p(pp \mid cc, up)\cdot p(w \mid pp, cc, up)$$

$$= 0.35\cdot(1-0.25)\Big[\, up \rightarrow 0.1\cdot\big(cc\rightarrow 0.3\cdot 0.5\cdot 0.4 + \neg cc \rightarrow 0.7\cdot 0.05\cdot 0.05\big)$$
$$\neg up \rightarrow 0.9\cdot\big(cc \rightarrow 0.3\cdot 0.9\cdot 0.9 + \neg cc \rightarrow 0.7\cdot 0.1\cdot 0.15\big)\Big]$$

$$= 0.35\cdot 0.75\cdot\big[6.175\times10^{-3} + 0.22815\big]$$

$$= 0.0615$$

$P(tp)$
0.35

D	TP	$\mathbf{P}(up\mid D,TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(d)$
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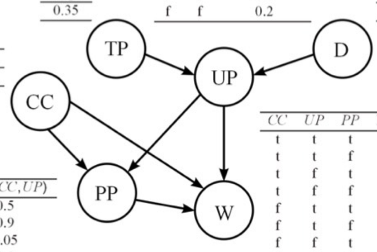
$P(cc)$
0.3



CC	UP	$\mathbf{P}(pp\mid CC,UP)$
t	t	0.5
t	f	0.9
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t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

$$= \sum_{up}\sum_{cc} p(\neg w, tp, \neg d, pp, UP, CC)$$

$$= \sum_{up}\sum_{cc} p(tp)\cdot p(\neg d)\cdot p(up \mid tp, \neg d)\cdot p(cc)\cdot p(pp \mid cc, up)\cdot p(\neg w \mid pp, cc, up)$$

$$= p(tp)\,p(\neg d)\cdot \sum_{up} p(up \mid tp, \neg d)\cdot \sum_{cc} p(cc)\cdot p(pp \mid cc, up)\cdot p(\neg w \mid pp, cc, up)$$

$$= 0.35\cdot(1-0.25)\Big[\, up \rightarrow 0.1\cdot\big(cc\rightarrow 0.3\cdot 0.5\cdot 0.6 + \neg cc \rightarrow 0.7\cdot 0.05\cdot 0.95\big)$$
$$\neg up \rightarrow 0.9\cdot\big(cc \rightarrow 0.3\cdot 0.9\cdot 0.1 + \neg cc \rightarrow 0.7\cdot 0.1\cdot 0.85\big)\Big]$$

$$= 0.35\cdot 0.75\cdot\big[0.012325 + 0.07785\big]$$

$$= 0.0237$$

$$\approx \langle \underline{0.7218}, 0.2782 \rangle$$

Problem 8.2:

Consider a driving situation that contains the Boolean random variables $H = Hurry$, $CD = CarefulDriver$, $DF = DriveFast$, $A = Accident$ and $GF = GetFined$. In order to have a more compact notation, abbreviate the true and false event of the random variables with the lowercase and the negated lowercase respectively (e.g. $H = true$ is h and $H = false$ is $\neg h$).

a. Draw the Bayesian network corresponding to:

$$\mathbf{P}(H,CD,DF,A,GF) = \mathbf{P}(GF|DF,A)\,\mathbf{P}(A|DF)\,\mathbf{P}(DF|CD,H)\,\mathbf{P}(CD)\,\mathbf{P}(H).$$

$P(h)$
0.5

$P(cd)$
0.6

DF	$\mathbf{P}(a|DF)$
t	0.7
f	0.25

CD	H	$\mathbf{P}(df|CD,H)$
t	t	0.15
t	f	0.01
f	t	0.99
f	f	0.1

DF	A	$\mathbf{P}(gf|DF,A)$
t	t	0.99
t	f	0.4
f	t	0.5
f	f	0.05

b. Calculate $\mathbf{P}(CD|\neg a, gf)$ using enumeration.

c. Calculate $\mathbf{P}(CD|\neg a, gf)$ using variable elimination.

d. Compare the number of operations required to compute the result in **b.** and **c.** .

2 Additional Problems

Problem 8.3:

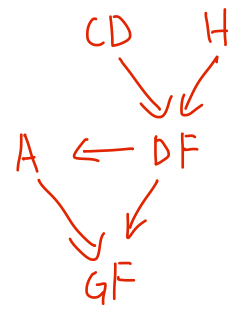
(Taken from **russel** Ex. 14.1) We have a bag of three biased coins a, b, and c with probabilities of coming up heads of 20%, 60%, and 80%, respectively. One coin is drawn randomly from the bag (with equal likelihood of drawing each of the three coins), and then the coin is flipped three times to generate the outcomes X1, X2, and X3.

a. Draw the Bayesian network corresponding to this setup and define the necessary conditional probability tables.

b. Calculate which coin was most likely to have been drawn from the bag if the observed flips come out heads twice and tails once.

Problem 8.2:

Consider a driving situation that contains the Boolean random variables $H = Hurry$, $CD = CarefulDriver$, $DF = DriveFast$, $A = Accident$ and $GF = GetFined$. In order to have a more compact notation, abbreviate the true and false event of the random variables with the lowercase and the negated lowercase respectively (e.g. $H = true$ is h and $H = false$ is $\neg h$).



a. Draw the Bayesian network corresponding to:
$\mathbf{P}(H, CD, DF, A, GF) = \mathbf{P}(GF|DF, A)\,\mathbf{P}(A|DF)\,\mathbf{P}(DF|CD, H)\,\mathbf{P}(CD)\,\mathbf{P}(H)$.

$P(h)$
0.5

$P(cd)$
0.6

DF	$\mathbf{P}(a\|DF)$
t	0.7
f	0.25

CD	H	$\mathbf{P}(df\|CD, H)$
t	t	0.15
t	f	0.01
f	t	0.99
f	f	0.1

DF	A	$\mathbf{P}(gf\|DF, A)$
t	t	0.99
t	f	0.4
f	t	0.5
f	f	0.05

b. Calculate $\mathbf{P}(CD|\neg a, gf)$ using enumeration.

c. Calculate $\mathbf{P}(CD|\neg a, gf)$ using variable elimination.

d. Compare the number of operations required to compute the result in **b.** and **c.** .

$$b.\ p(CD|\neg a, gf) = \alpha\, p(CD, \neg a, gf) = \alpha \sum_H \sum_{DF} p(H, CD, DF, \neg a, gf)$$

$$= p(CD)\cdot \sum_{DF} P(gf|DF, \neg a)\cdot p(\neg a|DF) \sum_H P(DF|CD, H)\cdot P(H)$$

$$= cd \rightarrow 0.6 \cdot \{ df \rightarrow 0.4\cdot(1-0.7)\cdot[h \rightarrow 0.15\cdot 0.5 + \neg h \rightarrow 0.01 \cdot 0.5] + \neg df \rightarrow 0.05\cdot(1-0.25)\cdot[h \rightarrow (1-0.15)\cdot 0.5 + \neg h \rightarrow (1-0.01)\cdot 0.5]\}$$

$$= 0.6 \cdot [0.4\cdot 0.3\cdot 0.08 + 0.05\cdot 0.75\cdot 0.92]$$

$$= 0.0265$$

$$= \neg cd \rightarrow 0.4 \cdot \{ df \rightarrow 0.4\cdot(1-0.7)\cdot[h \rightarrow 0.99\cdot 0.5 + \neg h \rightarrow 0.1 \cdot 0.5] + \neg df \rightarrow 0.05\cdot(1-0.25)\cdot[h \rightarrow (1-0.99)\cdot 0.5 + \neg h \rightarrow (1-0.1)\cdot 0.5]\}$$

$$= 0.4 \cdot [0.4\cdot 0.3\cdot 0.545 + 0.05\cdot 0.75\cdot 0.455]$$

$$= 0.0330$$

$$\langle 0.4454, 0.5546 \rangle$$

Problem 8.2:

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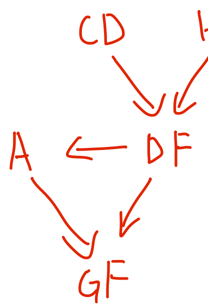
a. Draw the Bayesian network corresponding to:
$\mathbf{P}(H, CD, DF, A, GF) = \mathbf{P}(GF|DF, A)\mathbf{P}(A|DF)\mathbf{P}(DF|CD, H)\mathbf{P}(CD)\mathbf{P}(H)$.



$P(h)$
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$P(cd)$
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DF	$\mathbf{P}(a\|DF)$
t	0.7
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CD	H	$\mathbf{P}(df\|CD, H)$
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t	f	0.01
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DF	A	$\mathbf{P}(gf\|DF, A)$
t	t	0.99
t	f	0.4
f	t	0.5
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b. Calculate $\mathbf{P}(CD|\neg a, gf)$ using enumeration.

c. Calculate $\mathbf{P}(CD|\neg a, gf)$ using variable elimination.

d. Compare the number of operations required to compute the result in **b.** and **c.** .

c. $P(CD|\neg a, gf) = \underbrace{P(CD)}_{f_1(CD)} \cdot \sum_{DF} \underbrace{P(gf|DF, \neg a)}_{f_2(DF)} \cdot \underbrace{P(\neg a|DF)}_{f_3(DF)} \sum_{H} \underbrace{P(DF|CD, H)}_{f_4(DF, CD, H)} \cdot \underbrace{P(H)}_{f_5(H)}$

$f_4(DF, CD, H) = DF\downarrow \left\{ \overbrace{\begin{bmatrix} 0.15 & 0.99 \\ 0.85 & 0.01 \end{bmatrix}}^{CD} \begin{bmatrix} 0.01 & 0.1 \\ 0.99 & 0.9 \end{bmatrix} \right\}$ $\xrightarrow{H}$

$f_5(H) = [0.5 \quad 0.5]$

$f_4 \times f_5 = \begin{bmatrix} 0.08 & 0.545 \\ 0.92 & 0.455 \end{bmatrix} = f_7$

$f_2(DF) = \begin{bmatrix} 0.4 \\ 0.05 \end{bmatrix} \quad f_3(DF) = \begin{bmatrix} 0.3 \\ 0.75 \end{bmatrix}$

$f_2 \times f_3 = \begin{bmatrix} 0.12 \\ 0.6375 \end{bmatrix} \quad f_2 \times f_3 \times f_7 = [0.0441, 0.0825]$

$f_1(CD) = [0.6 \quad 0.4]$

$f_1 \times f_8 = [0.0265, 0.033]$

$\Rightarrow \langle 0.4454 \quad 0.5546 \rangle$

2 Additional Problems

Problem 8.3:

(Taken from **russel** Ex. 14.1) We have a bag of three biased coins a, b, and c with probabilities of coming up heads of 20%, 60%, and 80%, respectively. One coin is drawn randomly from the bag (with equal likelihood of drawing each of the three coins), and then the coin is flipped three times to generate the outcomes X1, X2, and X3.

a. Draw the Bayesian network corresponding to this setup and define the necessary conditional probability tables.

b. Calculate which coin was most likely to have been drawn from the bag if the observed flips come out heads twice and tails once.

$P(DC=a)$	$P(DC=b)$	$P(DC=c)$
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

	$P(X=\text{heads} \mid DC)$
a	0.2
b	0.6
c	0.8

$$p(DC \mid x_1=\text{heads}, x_2=\text{heads}, x_3=\text{tails})$$

$$= \alpha\, p(x_1=\text{heads} \mid DC)\, p(x_2=\text{heads} \mid DC) \cdot p(x_3=\text{head} \mid DC) \cdot p(DC)$$