Exam “Robotics” Wintersemester 2009/2010

Please provide accurate personal information below and keep your personal ID and student ID ready for inspection during the examination.

Make sure to put your Name and Immatrikulations-Nummer on each of the papers you use.

Return **all** sheets of paper (including the empty ones!) at the end of the exam.

You should have received this problem sheet, and three sheets to write on. Please write your answers for each of the problems 1, 2, 3 on a **seperate** sheet!

Good luck!

First name, Last name	
Matr.-Nr.:	
Branch of Study : (Fachrichtung)	

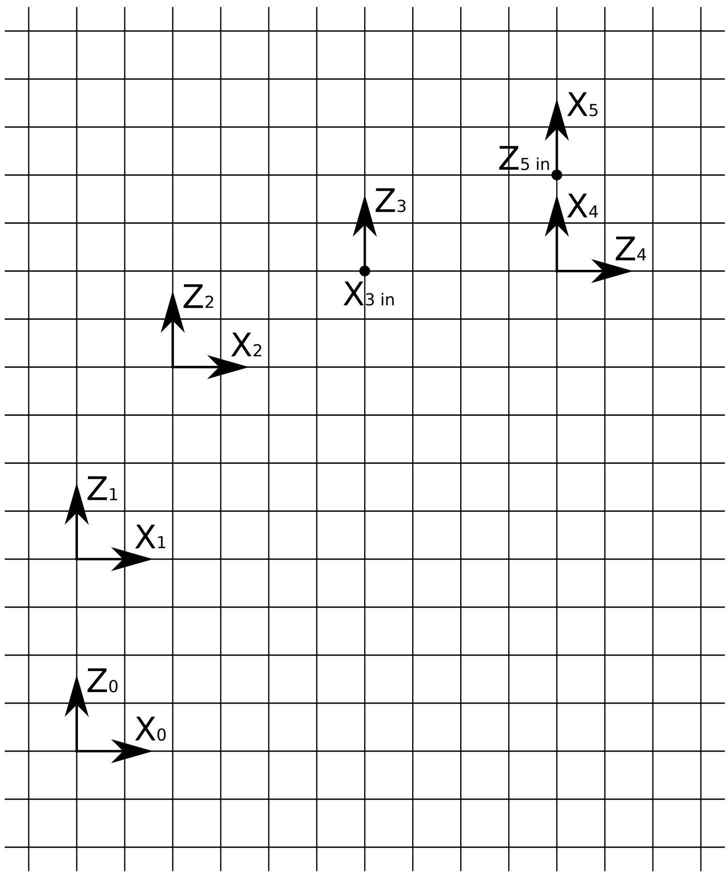
Problem 1

a)

The following table is a description of a robot according to modified Denavit-Hartenberg-Parameters (as used by Craig):

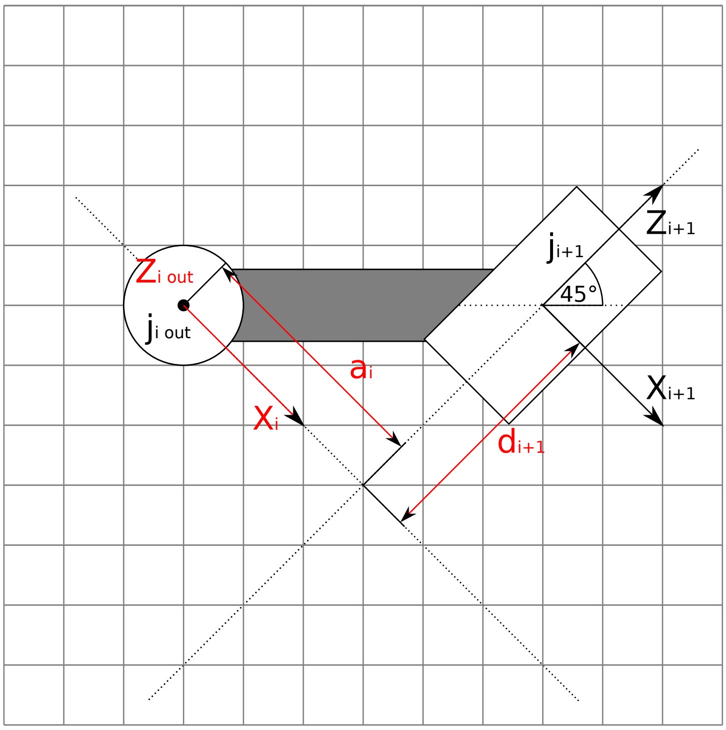
i	α_{i-1}	a_{i-1}	Θ_i	d_i	value of robot parameter
1	$0°$	0	Θ_1	4	$\Theta_1 = 0°$
2	$0°$	2	Θ_2	4	$\Theta_2 = 0°$
3	$0°$	4	Θ_3	2	$\Theta_3 = 90°$
4	$90°$	0	Θ_4	4	$\Theta_4 = -90°$
(5)	$90°$	2	0	0	

The Robot has 4 joints, and the last transformation leads to coordinate system of the end effector. Draw all coordinate systems (joints and end effector) into the grid below. The size of a grid cell is 1×1. Choose the values of the joint parameters such that the whole robot lies in the $x_0 - z_0$ plane. You need only draw the x and z axes of the coordinate systems. Where applicable, enter your choice of joint variables into above table.



b)

The drawing below shows a small part of a robot. That part of the robot consists, as shown, of two rotational joints connected through one link. How is the coordinate frame i determined according to Denavit-Hartenberg convention? Draw the axes x_i and z_i into below figure. What are the values of $\alpha_i, a_i, \Theta_{i+1}, d_{i+1}$? Label a_i and d_{i+1} in the drawing below. The size of a grid cell is again 1×1.



The DH parameter values are:

$$\alpha_i = -90°, a_i = 3\sqrt{2}, \Theta_{i+1} = 0°, d_{i+1} = 3\sqrt{2}$$

c)

Now we will be dealing with a part of the robot from subproblem a). We will consider the movement of the third frame of the robot, thus we will ignore everything that comes after the third joint. The Cartesian coordinates for the third coordinate system are

$$^0P_3 = \begin{pmatrix} 4\cos(\Theta_2 + \Theta_1) + 2\cos\Theta_1 \\ 4\sin(\Theta_2 + \Theta_1) + 2\sin\Theta_1 \\ 10 \end{pmatrix}.$$

Compute the corresponding 6×3 Jacobian for that section of the robot. In how many degrees of freedom can the third coordinate system move?

The Jacobian is:

$$J = \begin{pmatrix} -4\sin(\Theta_1+\Theta_2) - 2\sin(\Theta_1) & -4\sin(\Theta_1+\Theta_2) & 0 \\ 4\cos(\Theta_1+\Theta_2) + 2\cos(\Theta_1) & -4\cos(\Theta_1+\Theta_2) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

The robot has 3 degrees of freedom, as can be seen from the rank of the Jacobian.

d)

What are the singular configurations for the movement of frame 3? For answering this question, please use the Jacobian matrix J':

$$J' = \begin{pmatrix} 2\ \cos(\Theta_2+\Theta_1) + \cos\Theta_1 & 2\ \cos(\Theta_2+\Theta_1) & 0 \\ -2\ \sin(\Theta_2+\Theta_1) - \sin\Theta_1 & -2\ \sin(\Theta_2+\Theta_1) & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Note that this matrix has been computed according to a different representation of the end effector position, and thus will not be the same as the matrix you have computed in question c).

$$\begin{aligned} & \det J' = 0 \\ \Leftrightarrow\quad & -4(c_{12}+c_1)s_{12} + 4c_{12}(s_{12}+s_1) = 0 \\ \Leftrightarrow\quad & -c_1 s_{12} + c_{12} s_1 = 0 \\ \Leftrightarrow\quad & -c_1(s_1 c_2 + s_2 c_1) + s_1(c_1 c_2 - s_1 s_2) = 0 \\ \Leftrightarrow\quad & -c_1^2 s_2 - s_1^2 s_2 = 0 \\ \Leftrightarrow\quad & s_2 = 0 \Leftrightarrow \Theta_2 \in \{0^\circ, 180^\circ, \ldots\} \end{aligned}$$

Problem 2

The following table describes a robot with 3 joints:

i	α_{i-1}	a_{i-1}	Θ_i	d_i
1	0	l	$\frac{\pi}{2}$	d_1
2	$\frac{\pi}{2}$	l	0	d_2
3	0	l	Θ_3	0
(4)	0	l	0	0

The masses of the robot's links are m_1, m_2, m_3. The inertia tensor for the third link is

$$^{C_3}I_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{l^2}{12} & 0 \\ 0 & 0 & \frac{l^2}{12} \end{pmatrix}.$$

The positions of the origins of the joint coordinate systems are:

$$^0P_1 = \begin{pmatrix} l \\ 0 \\ d_1 \end{pmatrix}, {}^1P_2 = \begin{pmatrix} l \\ -d_2 \\ 0 \end{pmatrix}, {}^2P_3 = \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix}$$

Positions of the center of mass for each link are:

$$^1P_{C1} = \begin{pmatrix} \frac{l}{2} \\ -\frac{d_2}{2} \\ 0 \end{pmatrix}, {}^2P_{C2} = \begin{pmatrix} \frac{l}{2} \\ 0 \\ 0 \end{pmatrix}, {}^3P_{C3} = \begin{pmatrix} \frac{l}{2} \\ 0 \\ 0 \end{pmatrix}$$

Rotation matrices between systems are:

$$^0_1R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, {}^1_2R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, {}^2_3R = \begin{pmatrix} \cos\Theta_3 & -\sin\Theta_3 & 0 \\ \sin\Theta_3 & \cos\Theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To account for gravity, you should use $^0\dot{v}_0 = (0, -g, 0)^T$ in your computations.

a)

Compute the velocities and accelerations $^i\omega_i$, $^i\dot{\omega}_i$, $^i\dot{v}_i$ and $^i\dot{v}_{Ci}$ in the outwards iterations of the Newton-Euler-Computation for $i = 1, 2, 3$.

$$^1\omega_1 = \vec{0},$$
$$^1\dot{\omega}_1 = \vec{0},$$
$$^1\dot{v}_1 = {}^1_0R(\vec{0} + \vec{0} + {}^0\dot{v}_0) + \begin{pmatrix} 0 \\ 0 \\ \ddot{d}_1 \end{pmatrix} = \begin{pmatrix} -g \\ 0 \\ \ddot{d}_1 \end{pmatrix},$$
$$^1\dot{v}_{c_1} = \vec{0} + \vec{0} + {}^1\dot{v}_1 = \begin{pmatrix} -g \\ 0 \\ \ddot{d}_1 \end{pmatrix},$$

$$
\begin{aligned}
{}^2\omega_2 &= \vec{0},\\
{}^2\dot{\omega}_2 &= \vec{0},\\
{}^2\dot{v}_2 &= {}^2_1R(\vec{0}+\vec{0}+{}^1\dot{v}_1) + \begin{pmatrix}0\\0\\\ddot{d}_2\end{pmatrix} = \begin{pmatrix}-g\\\ddot{d}_1\\\ddot{d}_2\end{pmatrix},\\
{}^2\dot{v}_{c_2} &= \vec{0}+\vec{0}+{}^2\dot{v}_2 = \begin{pmatrix}-g\\\ddot{d}_1\\\ddot{d}_2\end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
{}^3\omega_3 &= \begin{pmatrix}0\\0\\\dot{\Theta}_3\end{pmatrix},\\
{}^3\dot{\omega}_3 &= \begin{pmatrix}0\\0\\\ddot{\Theta}_3\end{pmatrix},\\
{}^3\dot{v}_3 &= {}^3_2R(\vec{0}+\vec{0}+{}^2\dot{v}_2) = \begin{pmatrix}-g\mathrm{c}_3+\ddot{d}_1\mathrm{s}_3\\ g\mathrm{s}_3+\ddot{d}_1\mathrm{c}_3\\\ddot{d}_2\end{pmatrix},\\
{}^3\dot{v}_{c_3} &= {}^3\dot{\omega}_3\times{}^3P_{C_3} + {}^3\omega_3\times\left({}^3\omega_3\times{}^3P_{C_3}\right)+{}^3\dot{v}_3\\
&= \begin{pmatrix}0\\0\\\ddot{\Theta}_3\end{pmatrix}\times\begin{pmatrix}\frac{l}{2}\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\\dot{\Theta}_3\end{pmatrix}\times\left(\begin{pmatrix}0\\0\\\dot{\Theta}_3\end{pmatrix}\times\begin{pmatrix}\frac{l}{2}\\0\\0\end{pmatrix}\right)+{}^3\dot{v}_3 = \begin{pmatrix}-g\mathrm{c}_3+\ddot{d}_1\mathrm{s}_3-\frac{l}{2}\dot{\Theta}_3^2\\ g\mathrm{s}_3+\ddot{d}_1\mathrm{c}_3+\frac{l}{2}\ddot{\Theta}_3\\\ddot{d}_2\end{pmatrix}
\end{aligned}
$$

b)

Compute the moments ${}^iN_i$ and forces ${}^iF_i$ acting on the centers of masses for $i=1,2,3$.

$$
F_1 = \begin{pmatrix}-m_1\,g\\0\\\ddot{d}_1\,m_1\end{pmatrix},\, N_1 = \begin{pmatrix}0\\0\\0\end{pmatrix}
$$

$$
F_2 = \begin{pmatrix}-m_2\,g\\\ddot{d}_1\,m_2\\\ddot{d}_2\,m_2\end{pmatrix},\, N_2 = \begin{pmatrix}0\\0\\0\end{pmatrix}
$$

$$
F_3 = \begin{pmatrix}-\frac{m_3\,\dot{\Theta}_3^2\,l+2\,m_3\,\cos\Theta_3\,g-2\,\ddot{d}_1\,m_3\,\sin\Theta_3}{2}\\\frac{m_3\,\ddot{\Theta}_3\,l+2\,m_3\,\sin\Theta_3\,g+2\,\ddot{d}_1\,m_3\,\cos\Theta_3}{2}\\\ddot{d}_2\,m_3\end{pmatrix},\, N_3 = \begin{pmatrix}0\\0\\\frac{\ddot{\Theta}_3\,l^2}{12}\end{pmatrix}
$$

c)

Compute the joint moments τ_i for $i = 3, 2$.

$$
\begin{aligned}
{}^3n_3 &= {}^3N_3 + \vec{0} + {}^3P_{C_3} \times {}^3f_3 + {}^3P_4 \times \vec{0} \\
&= {}^3N_3 + \begin{pmatrix} \frac{l}{2} \\ 0 \\ 0 \end{pmatrix} \times {}^3F_3 \\
&= \begin{pmatrix} \bullet \\ \bullet \\ \frac{\ddot{\Theta}_3 l^2}{12} \end{pmatrix} + \begin{pmatrix} \bullet \\ \bullet \\ \frac{l^2}{4} m_3 \ddot{\Theta}_3 + \frac{1}{2}\mathrm{s}_3 l g + \frac{1}{2} l m_3 \ddot{d}_1 \mathrm{c}_3 \end{pmatrix} \\
&= \begin{pmatrix} \bullet \\ \bullet \\ \frac{l^2}{12}\ddot{\Theta}_3 + \frac{l^2}{4} m_3 \ddot{\Theta}_3 + \frac{1}{2} m_3 \mathrm{s}_3 l g + \frac{1}{2} l m_3 \ddot{d}_1 \mathrm{c}_3 \end{pmatrix} \\
{}^2f_2 &= {}^2_3R\, {}^3f_3 + {}^2F_2 \\
&= \begin{pmatrix} \bullet \\ \bullet \\ \ddot{d}_2 m_2 + \ddot{d}_2 m_3 \end{pmatrix}
\end{aligned}
$$

d)

Assuming that

$$
\tau_1 = -\frac{m_3\, \dot{\Theta}_3^2\, \sin\Theta_3\, l - m_3\, \ddot{\Theta}_3\, \cos\Theta_3\, l - 2\, \ddot{d}_1\, m_3 - 2\, \ddot{d}_1\, m_2 - 2\, \ddot{d}_1\, m_1}{2}
$$

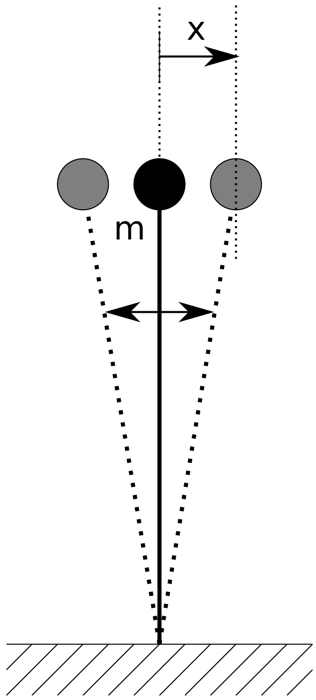
write down the dynamic equations in state-space (M-V-G) form.

$$
M = \begin{pmatrix} m_3 + m_2 + m_1 & 0 & \frac{m_3\, \cos\Theta_3\, l}{2} \\ 0 & m_3 + m_2 & 0 \\ \frac{m_3\, \cos\Theta_3\, l}{2} & 0 & \frac{(3\, m_3 + 1)\, l^2}{12} \end{pmatrix}
$$

$$
V = \begin{pmatrix} -\frac{m_3\, \dot{\Theta}_3^2\, \sin\Theta_3\, l}{2} \\ 0 \\ 0 \end{pmatrix}, G = \begin{pmatrix} 0 \\ 0 \\ \frac{m_3\, \sin\Theta_3\, g\, l}{2} \end{pmatrix}
$$

Problem 3

Consider the situation shown in the Figure on the right. Shown there is a rod with a finite, but very high stiffness k, that is planted firmly into the ground. Since the rod is not infinitely rigid, it will deform minimally under stress. The rod has a mass m attached to its distal end. The rod is assumed to be infinitely thin, and it's mass can be neglected. Since the deformations are very small, we will estimate them using small angle approximations. All forces and deflections are assumed to be effective in the x-direction indicated in the drawing, so the movement of the mass is only one-dimensional. Using this simplification, the mass m will be affected **solely** by the following forces:



- $f_s = -kx$ is the force caused by the rod's stiffness, which counteracts a deforming force and tries to return the rod to its normal shape.
- $f_r = -b\dot{x}$ is a damping force. Some of the kinetic energy that is causing the deformation will be converted into thermal energy, which means that this energy is lost.

The effect of gravity is neglected.

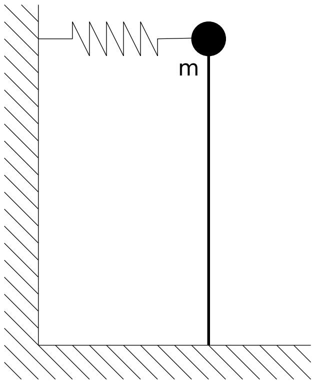
a)

- How can the behaviour of the rod be described using a differential equation?
- What kind of behaviour would you expect from that system?
- Let $m = 10$, $k = 900$. For which value of b is the system is critically damped?
- Compute the natural frequency ω_n and the damping ratio ζ of the system in the case of critical damping.

- A differential equation describing the behaviour of the rod would be:
$$m\ddot{x} + b\dot{x} + kx = 0$$
- The mass on top of the rod behaves like a mass-spring system.
- Critical damping is achieved for:
$$b = 2\sqrt{mk} = 2\sqrt{9000} = 60\sqrt{10}$$
- The natural frequency is:
$$\omega_n = \sqrt{k/m} = 3\sqrt{10}$$
The damping ratio in the case of critical damping is 1:
$$\zeta = \frac{b}{2\sqrt{mk}} = \frac{2\sqrt{mk}}{2\sqrt{mk}} = 1$$

b)

In order to stabilize the system, it has been modified as shown on the right: A spring is attached to the mass on top of the rod. We ignore the radius of the mass object at the end of the rod, and assume that the spring is connected to the center of the mass. Furthermore, since small angle approximations are effective, the force exterted by that new spring is estimated as $f_c = -k_c \cdot x$, where $k_c > 0$ is the spring constant.



- How does this change the behaviour of the system? Write down a new differential equation.
- Is it possible to choose k_c such that the original system with values $k = 900$ and $b = 200$ is critically damped?
- What if $k = 900$ and $b = 20$?

- A differential equation describing the behaviour of the rod would be:

$$m\ddot{x} + b\dot{x} + (k + k_c)x = 0$$

- We can compute k_c for critical damping through:

$$(k + k_c) = \frac{b^2}{4m} \quad \Leftrightarrow \quad k_c = \frac{b^2}{4m} - k \quad \Rightarrow \quad k_c = \frac{40000}{40} - 900 = 1000 - 900 = 100$$

  So critical damping is achieved for $k_c = 100$.

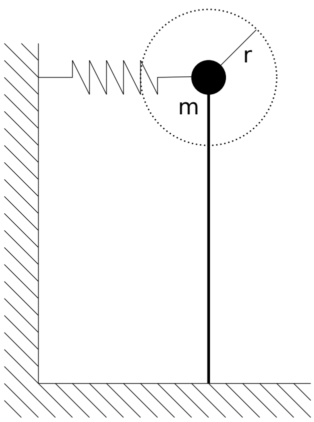
- For $b = 20$, we compute a value of k_c as:

$$k_c = \frac{400}{40} - 900 = 10 - 900 = -890$$

  So the spring would need to have a negative stiffness, which would not make sense. Thus it is not possible to achieve critical damping in this case.

c)

To further stabilize the system, a sphere-shaped object that acts as a sail is attached to the mass on top of the rod. That object is supposed to damp the system, causing a friction force $f_d = cr\dot{x}$ that applies to the original mass and is proportional to the radius r of the object. The value $c > 0$ is an arbitrary friction constant. The weight of the object is assumed to be very small and can thus be neglected. The only effect of the object is additional friction. For simplicity's sake, you can also assume that the spring that has been introduced in b) is still connected to the center of the mass.



- Write down the new equation of motion of the system.
- Is it possible to achieve critical damping by choosing the appropriate radius of the fricition-causing object?
- What advantage does this approach have over the one developed in subproblem b)?

- The new equation of motion would be

$$m\ddot{x} + (b - cr)\dot{x} + (k + k_c)x = 0$$

- For critical damping, we have

$$(b - cr) = 2\sqrt{m(k + k_c)} \quad \Leftrightarrow \quad r = -\frac{2\sqrt{m(k + k_c)} - b}{c}$$

- The advantage is that now we can choose k_c and r, so we would be able to achieve critical damping even for the case $b = 20$, $k = 900$, which was not possible with the approach from subproblem b).

d)

Explain the concepts of decoupling and control law partitioning in a general manner. Assuming that arbitrary forces can be exerted on the mass on the distal end of the rod, how would you decouple the original system that has been examined in subproblem a)? Which control law would you use to achieve following of some trajectory x_d?

Decoupling and control law partitioning are used for controlling physical systems using PD controllers. Through decoupling, the system is simplified to look like a system of unit masses:

$$f = \alpha f' + \beta$$

Using an appropriate control law, it is possible to achieve following of trajectories or steady-state control. For trajectory following, one would use a control law of the form

$$f' = \ddot{x}_d + k_v\dot{e} + k_p e$$

where $e = x - x_d$. For controlling the rod system that has been described above trough applying a force f, we would partition the system using $\alpha = m, \beta = b\dot{x} + kx$:

$$f = \alpha f' + \beta$$

The control law that is needed for trajectory following is then:

$$f' = \ddot{x}_d + k_v\dot{e} + k_p e$$