Multiple View Geometry: Exercise 4

Dr. Haoang Li, Daniil Sinitsyn, Sergei Solonets, Viktoria Ehm
Computer Vision Group, TU Munich
`Wednesdays 16:00-18:15 at Hörsaal 2, "Interims I" (5620.01.102), and on RBG Live`

Exercise: May 31, 2023

The Lucas-Kanade method
The weighted Lucas-Kanade energy $E(\mathbf{v})$ is defined as

$$E(\mathbf{v}) = \int_{W(\mathbf{x})} G(\mathbf{x}-\mathbf{x}') \left\| \nabla I(\mathbf{x}',t)^\top \mathbf{v} + \partial_t I(\mathbf{x}',t) \right\|^2 \mathrm{d}\mathbf{x}' \,.$$

Assume that the weighting function G is chosen such that $G(\mathbf{x}-\mathbf{x}') = 0$ for any $\mathbf{x}' \notin W(\mathbf{x})$.
In the following, we note $I_t = \partial_t I$ and $(I_{x_1}, I_{x_2})^\top = \nabla I$.

1. Prove that the minimizer $\mathbf{b}$ of $E(\mathbf{v})$ can be written as

   $$\mathbf{b} = -M^{-1}\mathbf{q}$$

   where the entries of M and $\mathbf{q}$ are given by

   $$m_{ij} = G * \left(I_{x_i} \cdot I_{x_j}\right) \quad \text{and} \quad q_i = G * (I_{x_i} \cdot I_t)$$

2. Show that if the gradient direction is constant in $W(\mathbf{x})$, i.e. $\nabla I(\mathbf{x}',t) = \alpha(\mathbf{x}',t)\mathbf{u}$ for a scalar function α and a 2D vector $\mathbf{u}$, M is not invertible.

   Explain how this observation is related to the aperture problem.

   Note: In the formalism of Lucas and Kanade, one cannot always estimate a translational motion. This problem is often referred to as the aperture problem. It arises for example, if the region in the window $W(x)$ around the point x has entirely constant intensity (for example a white wall), because then $\delta I(x) = 0$ and $I_t(x) = 0$ for all points in the window.

3. Write down explicit expressions for the two components b_1 and b_2 of the minimizer in terms of m_{ij} and q_i.

Note: $G * A$ denotes the convolution of image A with a kernel $G : \mathbb{R}^2 \to \mathbb{R}$ and is defined as

$$G * A = \int_{\mathbb{R}^2} G(\mathbf{x}-\mathbf{x}')A(\mathbf{x}')\mathrm{d}\mathbf{x}' \,.$$

The Lucas-Kanade method

The weighted Lucas-Kanade energy $E(\mathbf{v})$ is defined as

$$E(\mathbf{v}) = \int_{W(\mathbf{x})} G(\mathbf{x} - \mathbf{x}') \left\| \nabla I(\mathbf{x}', t)^\top \mathbf{v} + \partial_t I(\mathbf{x}', t) \right\|^2 \mathrm{d}\mathbf{x}' .$$

Assume that the weighting function G is chosen such that $G(\mathbf{x} - \mathbf{x}') = 0$ for any $\mathbf{x}' \notin W(\mathbf{x})$. In the following, we note $I_t = \partial_t I$ and $(I_{x_1}, I_{x_2})^\top = \nabla I$.

1. Prove that the minimizer $\mathbf{b}$ of $E(\mathbf{v})$ can be written as

$$\mathbf{b} = -M^{-1}\mathbf{q}$$

where the entries of M and $\mathbf{q}$ are given by

$$m_{ij} = G * \left(I_{x_i} \cdot I_{x_j}\right) \quad \text{and} \quad q_i = G * (I_{x_i} \cdot I_t)$$

$$E(v) = \int_{w(x)} G(x - x') (\nabla I(x', t)^T v)^2 \, dx'$$

$$+ \int_{w(x)} G(x - x') (2 \nabla I(x', t)^T v \cdot \partial_t I(x', t)) \, dx'$$

$$\cancel{+ \int_{w(x)} G(x - x') (\partial_t I(x', t)^2) \, dx'}$$

$$\frac{\partial E(v)}{\partial v} = \int_{w(x)} G(x - x') \left[2 \nabla I(x', t)^T v\right] \left[\nabla I(x', t)\right] dx$$

$$+ \int_{w(x)} G(x - x)' \left[2 \nabla I(x', t)^T \partial_t I(x', t)\right] dx$$

$$= 2G \nabla I^T v \nabla I + 2G \nabla I^T \partial_t I$$

$$= 2G * (I_{x_1}, I_{x_2})^T \begin{pmatrix} I_{x_1} \\ I_{x_2} \end{pmatrix}^T v + 2G (I_{x_1}, I_{x_2}) \cdot I_t$$

$$= 2G * \begin{pmatrix} I_{x_1} I_{x_1} & I_{x_1} I_{x_2} \\ I_{x_2} I_{x_1} & I_{x_2} I_{x_2} \end{pmatrix} v + 2G * (I_{x_1} I_t, I_{x_2} I_t)$$

$$= 2 M \cdot v + 2 q = 0$$

$$b = -M^{-1} q$$

2. Show that if the gradient direction is constant in $W(\mathbf{x})$, i.e. $\nabla I(\mathbf{x}', t) = \alpha(\mathbf{x}', t)\mathbf{u}$ for a scalar function α and a 2D vector $\mathbf{u}$, M is not invertible.

Explain how this observation is related to the aperture problem.

Note: In the formalism of Lucas and Kanade, one cannot always estimate a translational motion. This problem is often referred to as the aperture problem. It arises for example, if the region in the window $W(x)$ around the point x has entirely constant intensity (for example a white wall), because then $\delta I(x) = 0$ and $I_t(x) = 0$ for all points in the window.

$$= 2G\,\nabla I^T v \nabla I + 2G\,\nabla I^T \partial_t I$$

$$= 2G\,\alpha^2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} (u_1\ u_2) + 2G\,\alpha\,(u_1 I_t,\ u_2 I_t)$$

$$= 2G\,\alpha^2$$

$$\begin{vmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{vmatrix} = 0$$

3. Write down explicit expressions for the two components b_1 and b_2 of the minimizer in terms of m_{ij} and q_i.

Note: $G * A$ denotes the convolution of image A with a kernel $G : \mathbb{R}^2 \to \mathbb{R}$ and is defined as

$$G * A = \int_{\mathbb{R}^2} G(\mathbf{x} - \mathbf{x}')A(\mathbf{x}')\mathrm{d}\mathbf{x}' \,.$$

$$b = -m^{-1} q$$

$$= -\frac{1}{\det m} \begin{bmatrix} m_{22} & -m_{21} \\ -m_{12} & m_{11} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$