Multiple View Geometry: Exercise 1

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Math Background

1. Show for each of the following sets (1) whether they are linearly independent, (2) whether they span $\mathbb{R}^3$ and (3) whether they form a basis of $\mathbb{R}^3$:

   (a) $B_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

   (b) $B_2 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

   (c) $B_3 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

2. Which of the following sets forms a group (with matrix-multiplication)? Prove or disprove!

   (a) $G_1 := \left\{ A \in \mathbb{R}^{n \times n} \middle| \det(A) \neq 0 \wedge A^\top = A \right\}$

   (b) $G_2 := \{ A \in \mathbb{R}^{n \times n} | \det(A) = -1 \}$

   (c) $G_3 := \{ A \in \mathbb{R}^{n \times n} | \det(A) > 0 \}$

3. Prove or disprove: There exist vectors $\mathbf{v}_1, ..., \mathbf{v}_5 \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$, which are pairwise orthogonal, i.e.

$$\forall i, j = 1, ..., 5 : \quad i \neq j \implies \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$$

4. Which groups have you seen in the lecture? Write down the names and the correct inclusions! (e.g.: group A $\subset$ group B)

5. Let A be a symmetric matrix, and λ_a, λ_b eigenvalues with eigenvectors v_a and v_b. Prove: if v_a and v_b are not orthogonal, it follows: $\lambda_a = \lambda_b$.

   Hint: What can you say about $\langle A v_a, v_b \rangle$?

6. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix with the orthonormal basis of eigenvectors $v_1, \ldots, v_n$ and eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. Find all vectors x, that minimize the following term:

$$\min_{||x||=1} x^\top A x$$

1\. 1. Show for each of the following sets (1) whether they are linearly independent, (2) whether they span $\mathbb{R}^3$ and (3) whether they form a basis of $\mathbb{R}^3$:

(a) $B_1 = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$

(b) $B_2 = \left\{ \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\}$

(c) $B_3 = \left\{ \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}3\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$

(a) All independent
Span R^3 $\Leftarrow$ B中 vector 的 combination 能表述 R^3中的向量
Basis $\Rightarrow$ 线性无关; 能够生成向量空间

(b) All independent
not span R^3 $\Leftarrow$ Z axis is always zero on X-Y plane
not basis

(c) Not all independent
span R^3
not basis $\Leftarrow$ not more than 3 independent vector

2\. 2. Which of the following sets forms a group (with matrix-multiplication)? Prove or disprove!

(a) $G_1 := \{A \in \mathbb{R}^{n\times n} \mid \det(A) \neq 0 \wedge A^\top = A\}$
(b) $G_2 := \{A \in \mathbb{R}^{n\times n} \mid \det(A) = -1\}$
(c) $G_3 := \{A \in \mathbb{R}^{n\times n} \mid \det(A) > 0\}$

$G(A, \cdot)$

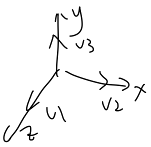
(a) 可逆 对称
不是 Group，举反例
$A, B \in G_1 // (A \cdot B)^T = AB$ If G_1 is Group
$B^T \cdot A^T = B \cdot A \neq AB$

(b) 没有实数
可逆 $G_3 \in GL(n)$
$\det(AB^{-1}) = \det(A) \cdot [\det(B)]^{-1} > 0$
$AB^{-1} \in G_3$ why ?
G_3 is a subgroup of Line group

3\. 3. Prove or disprove: There exist vectors $\mathbf{v}_1, \ldots, \mathbf{v}_5 \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$, which are pairwise orthogonal, i.e.

$$\forall i, j = 1, \ldots, 5: \quad i \neq j \implies \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$$

We assume first three vector is along the axis since not 0 vector



That mean in this 3D space, there exist no other vector orthogonal with these first three vectors in the same time

4\. Which groups have you seen in the lecture? Write down the names and the correct inclusions!
(e.g.: group A $\subset$ group B)

special orthogonal group special Euclidean group

5\. Let A be a symmetric matrix, and λ_a, λ_b eigenvalues with eigenvectors v_a and v_b. Prove: if v_a and v_b are not orthogonal, it follows: $\lambda_a = \lambda_b$.
Hint: What can you say about $\langle Av_a, v_b \rangle$?

令 $z = V_a^T A \cdot V_b$ z是标量 $\Rightarrow z^T = z$. $\Rightarrow V_a^T A V_b = V_b^T A V_a$

对称矩阵 $A^T = A$ $\Rightarrow V_a^T A V_b = V_a^T A^T V_b = V_b^T A \cdot V_a$

$\Downarrow$ $\Downarrow$ $\Downarrow$

$V_a^T \lambda_b V_b$ $\lambda_a V_a^T V_b$ $V_b^T \lambda_a V_a$

$\Downarrow$ $\Downarrow$

$\lambda_b V_a^T V_b$ $\lambda_a V_b^T V_a$

$\Rightarrow (\lambda_b - \lambda_a) V_a^T V_b = 0$

If V_a and V_b not orthongonal $V_a^T \cdot V_b \neq 0$.

$\Rightarrow \lambda_b = \lambda_a$

6\. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with the orthonormal basis of eigenvectors $v_1, \ldots, v_n$ and eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. Find all vectors x, that minimize the following term:

$$\min_{||x||=1} x^\top A x$$

How many solutions exist? How can the term be maximized?

Hint: Use the expression $x = \sum_{i=1}^{n} \alpha_i v_i$ with coefficients $\alpha_i \in \mathbb{R}$ and compute appropriate coefficients!

$$\min_{||x||=1} x^T A x = \min_{||x||=1} \left(\sum_{i=1}^{N} \alpha_i v_i\right)^T A \left(\sum_{j=1}^{N} \alpha_j v_j\right)$$

$$= \min \sum_{i=1}^{N} \sum_{j=1}^{M} \alpha_i \alpha_j v_i^T A v_j$$

$$= \min \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j v_i^T \lambda_j v_j$$

$$\sum_{i=1}^{N} \sum_{j=1}^{M} 2\alpha_i \alpha_j \lambda_j v_j = 0.$$

How many solutions exist? How can the term be maximized?

Hint: Use the expression $x = \sum_{i=1}^{n} \alpha_i v_i$ with coefficients $\alpha_i \in \mathbb{R}$ and compute appropriate coefficients!

7. Let $A \in \mathbb{R}^{m\times n}$. Prove that kernel$(A) =$ kernel$(A^\top A)$.

   Hint: Consider a) $x \in$ kernel$(A) \Rightarrow x \in$ kernel$(A^\top A)$
   and b) $x \in$ kernel$(A^\top A) \Rightarrow x \in$ kernel(A).

8. Singular Value Decomposition (SVD)

   Let $A = USV^\top$ be the SVD of A.

   (a) Write down possible dimensions for A, U, S and V.

   (b) What are the similarities and differences between the SVD and the eigenvalue decomposition?

   (c) What do you know about the relationship between U, S, V and the eigenvalues and eigenvectors of $A^\top A$ and $AA^\top$?

   (d) What is the interpretation of the entries in S and what do the entries of S tell us about A?

7. If $x \in$ kernel$(A) \Rightarrow Ax = 0 \Rightarrow A^T A x = 0$ so x belong to kerne$(A^T A)$

8 a) assume $A \in R^{m\times n}$ $U \in R^{m\times m}$ $S \in R^{m\times r}$ $V \in R^{n\times r}$

b) A must be square matrix.
also expand by matrix multiplication

$Av = \lambda v$ $\quad v^T A^T = \lambda v^T$ $\qquad A = USV^T$

$v^T A^T A v = \lambda^2 v^T v$ $\qquad A^T = VS^T U^T$

$A^T A = VS^T U^T USV^T$

$v^T V S^T U^T U S V^T v = \lambda^2 v^T v$