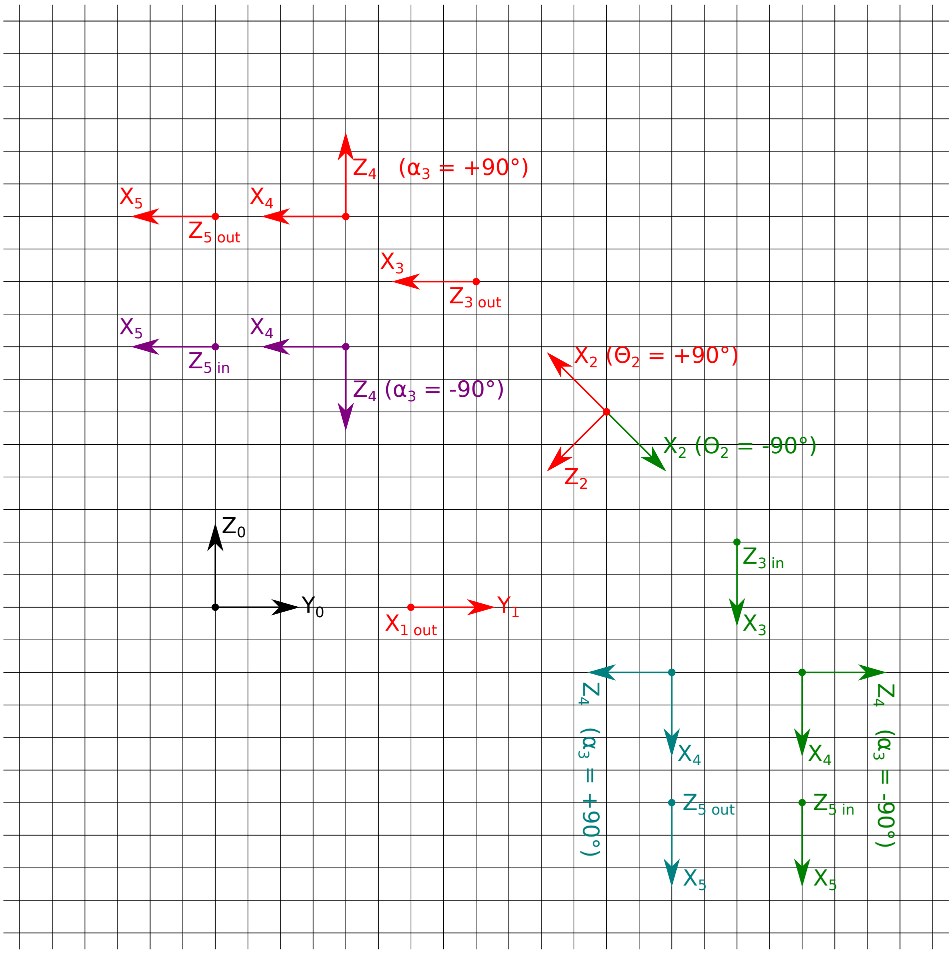
Problem 1

a)

The following table is a description of a robot according to modified Denavit-Hartenberg-Parameters (as used by Craig):

i	a_{i-1}	α_{i-1}	d_i	Θ_i
1	0	$-90°$	3	$0°$
2	0	$-135°$	$-3\sqrt{2}$	$\pm 90°$
3	$2\sqrt{2}$	$90°$	0	$45°$
4	2	$\pm 90°$	1	$0°$
(5)	2	$-90°$	0	$0°$

The Robot has 4 joints, and the last transformation leads to coordinate system of the end effector. Draw all coordinate systems (joints and end effector) into the grid below. Two cells in the grid correspond to one length unit in the DH parameter table. Choose the values of the parameters Θ_2 and α_3 (blank entries in the table) such that all coordinate system origins lie in the $y_0 - z_0$ plane. You need only draw the x and z axes of the coordinate systems. Where applicable, enter your choice of joint variables into above table.



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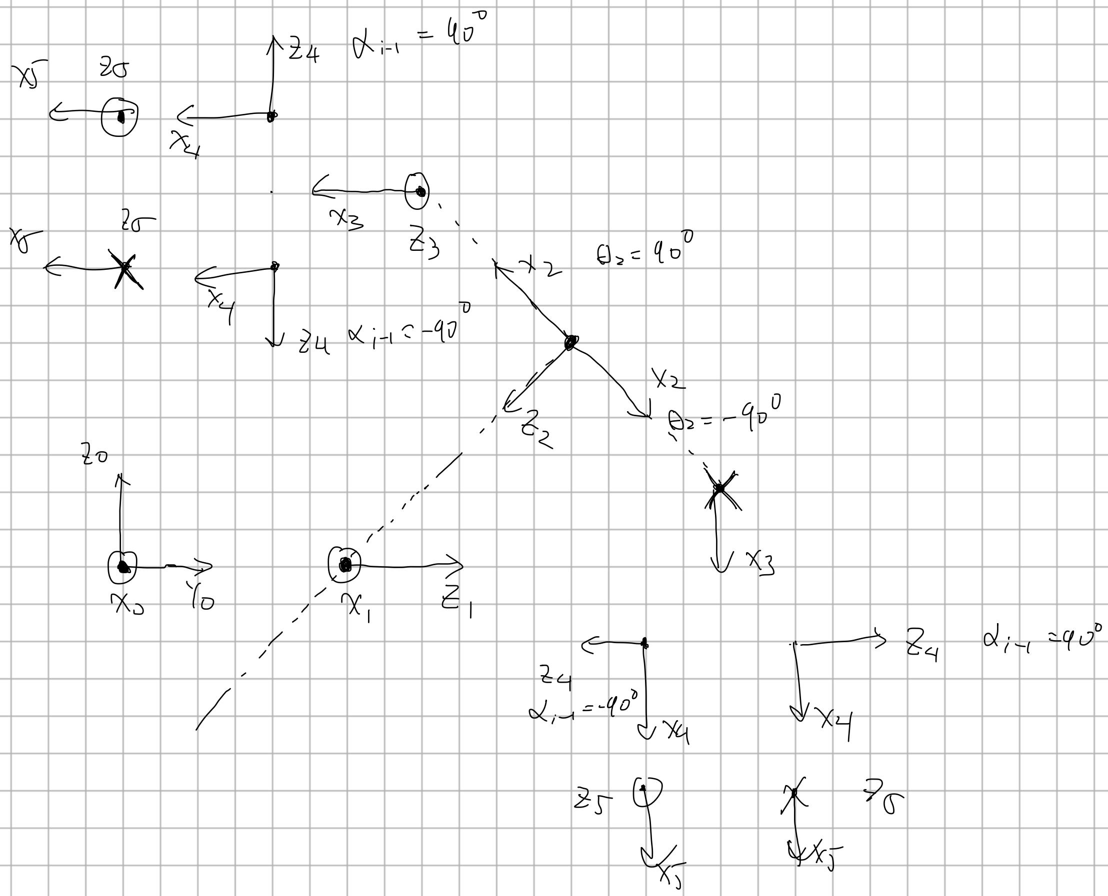
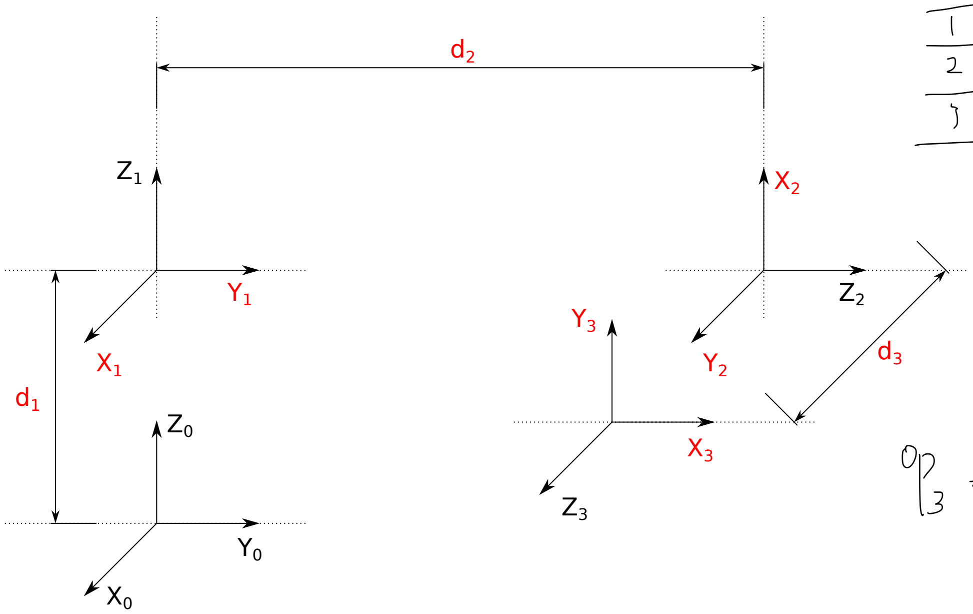


b)

An especially simple type of robot is a so-called Cartesian manipulator, which maps robot configuration parameters directly to Cartesian coordinates. The coordinate systems for such a manipulator could look like this:

	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0	d_1	0
2	0	$-90°$	d_2	$-90°$
3	0	$-90°$	d_3	$-90°$



$${}^0P_3 = \begin{pmatrix} d_3 \\ d_2 \\ d_1 \end{pmatrix}$$

As can be seen from the drawing, the manipulator has the following characteristics:

$${}^0Z_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, {}^0Z_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, {}^0Z_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Draw the X-axes that are missing in the coordinate frames into above figure. Also annotate the quantities marked as ? with the corresponding DH parameters. Determine Denavit-Hartenberg parameters for the Cartesian manipulator. Enter the values into the following table:

i	a_{i-1}	α_{i-1}	d_i	Θ_i
1	0	$0°$	d_1	$0°$
2	0	$-90°$	d_2	$-90°$
3	0	$-90°$	d_3	$-90°$

What is the position ${}^0P_3$ of the origin of frame 3 with respect to the base frame?

$${}^0P_3 = \begin{pmatrix} d_3 \\ d_2 \\ d_1 \end{pmatrix}$$

What is the 3×3 Jacobian of above manipulator with respect to Cartesian coordinates? Specify the Jacobian relative to the base frame $\{0\}$.

$${}^0J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial d_3}{\partial d_1} & \frac{\partial d_3}{\partial d_2} & \frac{\partial d_3}{\partial d_3} \\ & - & - \\ & - & - \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

c) Yes $J^T \cdot J = \lambda \cdot I$ All configuration are isotropic

- Does the Cartesian manipulator from the previous problem have any isotropic configurations?

  $\det(J(\theta)) = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0 - 1 \mid -0 - 0 = -1$

- Are there any singular configurations? NO.
- How can singular configurations be found in general? If $\det(J(\theta)) = 0$ than has Singular condiv;
- Name one problem that appears if the robot is near to a singular configuration.

  Lose one dim. , i.e. can't move in one directi

  The Velocity of the EE will be indifinitly $\to \infty$

- All configurations are isotropic.
- There are no singular configurations.
- Singular configurations can be found by solving the equation $\det(J) = 0$.
- Incremental inverse kinematics: Joint rates might become extremely large. Also: Cartesian control. Torques can approach infinity.

$$ {}^0_1R = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad {}^1_2R = \begin{bmatrix} c_2 & -s_2 & 0 \\ 0 & 0 & 1 \\ -s_2 & -c_2 & 0 \end{bmatrix} \quad {}^2_3R = \begin{pmatrix} c_3 & -s_3 & 0 \\ 0 & 0 & 1 \\ -s_3 & -c_3 & 0 \end{pmatrix} $$

d)

The following table describes a three-link robot with only rotational joints:

$$ {}^0_2R = \begin{bmatrix} c_1c_2 & -c_1s_2 & -s_1 \\ s_1c_2 & -s_1s_2 & c_1 \\ -s_2 & -c_2 & 0 \end{bmatrix} $$

i	a_{i-1}	α_{i-1}	d_i	Θ_i
1	2	$0°$	0	Θ_1
2	2	$-90°$	0	Θ_2
3	2	$-90°$	0	Θ_3

Fill in the missing values of the 3×3 Jacobian for the robot with respect to angular coordinates.

$$ J = \begin{pmatrix} 0 & -s_1 & -c_1s_2 \\ 0 & c_1 & -s_1s_2 \\ 1 & 0 & -c_2 \end{pmatrix} $$

Find the singular configurations of the robot based on that Jacobian.

Problem 2

The following table describes a robot with 3 joints:

i	a_{i-1}	α_{i-1}	d_i	Θ_i
1	l	$0°$	d_1	$90°$
2	0	$90°$	0	Θ_2
3	l	$0°$	d_3	$90°$

← 平移 不旋转!

All of the robot's links have the same mass m. The inertia tensors are

$$^{C_1}I_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{l^2}{2} & 0 \\ 0 & 0 & \frac{l^2}{2} \end{pmatrix}, \quad ^{C_2}I_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{l^2}{2} & 0 \\ 0 & 0 & \frac{l^2}{2} \end{pmatrix}, \quad ^{C_3}I_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{l^2}{2} & 0 \\ 0 & 0 & \frac{l^2}{2} \end{pmatrix}.$$

The positions of the origins of the joint coordinate systems are:

$$^0P_1 = \begin{pmatrix} l \\ 0 \\ d_1 \end{pmatrix}, {}^1P_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, {}^2P_3 = \begin{pmatrix} l \\ 0 \\ d_3 \end{pmatrix}$$

Positions of the center of mass for each link are:

$$^1P_{C1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, {}^2P_{C2} = \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix}, {}^3P_{C3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Rotation matrices between systems are:

$$^0_1R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, {}^1_2R = \begin{pmatrix} \mathrm{c}_2 & -\mathrm{s}_2 & 0 \\ 0 & 0 & -1 \\ \mathrm{s}_2 & \mathrm{c}_2 & 0 \end{pmatrix}, {}^2_3R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To account for gravity, you should use $^0\dot{v}_0 = (0, g, 0)^T$ in your computations.

a)

Compute the rotational velocities and accelerations $^i\omega_i$, $^i\dot{\omega}_i$ for $i = 1, 2, 3$, and the linear accelerations $^i\dot{v}_i$ and $^i\dot{v}_{Ci}$ for $i = 1, 2$.

$$^1\omega_1 = {}^1\dot{\omega}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$^2\omega_2 = \begin{pmatrix} 0 \\ 0 \\ \dot{\Theta}_2 \end{pmatrix}, {}^2\dot{\omega}_2 = \begin{pmatrix} 0 \\ 0 \\ \ddot{\Theta}_2 \end{pmatrix}$$

$$^3\omega_3 = \begin{pmatrix} 0 \\ 0 \\ \dot{\Theta}_2 \end{pmatrix}, {}^3\dot{\omega}_3 = \begin{pmatrix} 0 \\ 0 \\ \ddot{\Theta}_2 \end{pmatrix}$$

a)

Compute the rotational velocities and accelerations ${}^i\omega_i$, ${}^i\dot\omega_i$ for $i = 1, 2, 3$, and the linear accelerations ${}^i\dot v_i$ and ${}^i\dot v_{Ci}$ for $i = 1, 2$.

$$\omega_0 = \dot\omega_0 = v_0 = 0 \qquad {}^0\dot v_0 = \begin{pmatrix}0\\g\\0\end{pmatrix}$$

$${}^1\omega_1 = {}^1_0R\,{}^0\omega_0 = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$${}^1\dot\omega_1 = {}^1_0R\,{}^0\dot\omega_0 = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$${}^1\dot v_1 = {}^1_0R\left(\cancel{{}^0\dot\omega_0\times{}^0P_1} + \cancel{{}^0\omega_0\times({}^0\omega_0\times{}^0P_1)} + {}^0\dot v_0\right) + \cancel{2\,{}^1\omega_1\times\dot d_1\,{}^1\hat Z_1} + \ddot d_1\,{}^1\hat Z_1$$

$$= \begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0\\g\\0\end{pmatrix} + \begin{pmatrix}0\\0\\\ddot d_1\end{pmatrix}$$

$$= \begin{pmatrix}g\\0\\\ddot d_1\end{pmatrix}$$

$${}^1\dot v_{C1} = \cancel{{}^1\dot\omega_1\times{}^1P_{C1}} + \cancel{{}^1\omega_1\times({}^1\omega_1\times{}^1P_{C1})} + {}^1\dot v_1$$

$$= \begin{pmatrix}g\\0\\\ddot d_1\end{pmatrix}$$

$${}^2\omega_2 = \cancel{{}^2_1R\,{}^1\omega_1} + \dot\theta_2\,{}^2\hat Z_2 = \begin{pmatrix}0\\0\\\dot\theta_2\end{pmatrix}$$

$${}^2\dot\omega_2 = \cancel{{}^2_1R\,{}^1\dot\omega_1} + \cancel{{}^2_1R\,{}^1\omega_1\times\dot\theta_2\,{}^2\hat Z_2} + \ddot\theta_2\,{}^2\hat Z_2 = \begin{pmatrix}0\\0\\\ddot\theta_2\end{pmatrix}$$

$${}^2\dot v_2 = {}^2_1R\left(\cancel{{}^1\dot\omega_1\times{}^1P_2} + \cancel{{}^1\omega_1\times({}^1\omega_1\times{}^1P_2)} + {}^1\dot v_1\right) = \begin{pmatrix}c_2&0&s_2\\-s_2&0&c_2\\0&-1&0\end{pmatrix}\begin{pmatrix}g\\0\\\ddot d_1\end{pmatrix} = \begin{pmatrix}c_2 g + s_2\ddot d_1\\-s_2 g + c_2\ddot d_1\\0\end{pmatrix}$$

$${}^2\dot v_{C2} = {}^2\dot\omega_2\times{}^2P_{C2} + {}^2\omega_2\times({}^2\omega_2\times{}^2P_{C2}) + {}^2\dot v_2$$

$$= \begin{pmatrix}0\\0\\\ddot\theta_2\end{pmatrix}\times\begin{pmatrix}L\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\\dot\theta_2\end{pmatrix}\times\left(\begin{pmatrix}0\\0\\\dot\theta_2\end{pmatrix}\times\begin{pmatrix}L\\0\\0\end{pmatrix}\right) + \begin{pmatrix}c_2 g + s_2\ddot d_1\\-s_2 g + c_2\ddot d_1\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\L\ddot\theta_2\\0\end{pmatrix} + \begin{pmatrix}-L\dot\theta_2^2\\0\\0\end{pmatrix} + \begin{pmatrix}c_2 g + s_2\ddot d_1\\-s_2 g + c_2\ddot d_1\\0\end{pmatrix}$$

$$= \begin{pmatrix}c_2 g + s_2\ddot d_1 - L\dot\theta_2^2\\-s_2 g + c_2\ddot d_1 + L\ddot\theta_2\\0\end{pmatrix}$$

$$\begin{pmatrix}-l\dot\Theta_2^2 + c_2 g + s_2\ddot d_1\\l\ddot\Theta_2 - s_2 g + c_2\ddot d_1\\0\end{pmatrix}$$

$${}^3\omega_3 = {}^3_2R\,{}^2\omega_2 = \begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0\\0\\\dot\theta_2\end{pmatrix} = \begin{pmatrix}0\\0\\\dot\theta_2\end{pmatrix}$$

$${}^3\dot\omega_3 = {}^3_2R\,{}^2\dot\omega_2 = \begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0\\0\\\ddot\theta_2\end{pmatrix} = \begin{pmatrix}0\\0\\\ddot\theta_2\end{pmatrix}$$

$${}^1N_1 = \cancel{{}^{C_1}I_1\,{}^1\dot\omega_1} + \cancel{{}^1\omega_1\times{}^{C_1}I_1\,{}^1\omega_1} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$${}^2N_2 = {}^{C_2}I_2\,{}^2\dot\omega_2 + {}^2\omega_2\times{}^{C_2}I_2\cdot{}^2\omega_2$$

$$= \begin{pmatrix}0&0&0\\0&\frac{L^2}{2}&0\\0&0&\frac{L^2}{2}\end{pmatrix}\cdot\begin{pmatrix}0\\0\\\ddot\theta_2\end{pmatrix} + \begin{pmatrix}0\\0\\\dot\theta_2\end{pmatrix}\times\underbrace{\begin{pmatrix}0&0&0\\0&\frac{L^2}{2}&0\\0&0&\frac{L^2}{2}\end{pmatrix}\cdot\begin{pmatrix}0\\0\\\dot\theta_2\end{pmatrix}} = \begin{pmatrix}0\\0\\\frac{L^2}{2}\dot\theta_2\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\\frac{L^2}{2}\ddot\theta_2\end{pmatrix} + \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\\frac{L^2}{2}\ddot\theta_2\end{pmatrix} = {}^3N_3$$

$${}^3N_3 = {}^{C_3}I_3\,{}^3\dot\omega_3 + {}^3\omega_3\times{}^{C_3}I_3\cdot{}^3\omega_3$$

$$= \begin{pmatrix}0&0&0\\0&\frac{L^2}{2}&0\\0&0&\frac{L^2}{2}\end{pmatrix}\cdot\Big($$

${}^iF_i$ and ${}^iN_i$ are the required Force and torque act on the center of mass of link i which make the robot move as desired

$$^1\dot{v}_1 = {}^1_0R(0+0+G)+0+\begin{pmatrix}0\\0\\\ddot{d}_1\end{pmatrix} = \begin{pmatrix}g\\0\\\ddot{d}_1\end{pmatrix}$$

$$^1\dot{v}_{C1} = 0 + {}^1\dot{v}_1$$

$$^2\dot{v}_2 = {}^2_1R(0+0+{}^1\dot{v}_1) = \begin{pmatrix}c_2 g + s_2\ddot{d}_1\\-s_2 g + c_2\ddot{d}_1\\0\end{pmatrix}$$

$$^2\dot{v}_{C2} = \begin{pmatrix}0\\0\\\ddot{\Theta}_2\end{pmatrix}\times\begin{pmatrix}l\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\\dot{\Theta}_2\end{pmatrix}\times\left(\begin{pmatrix}0\\0\\\dot{\Theta}_2\end{pmatrix}\times\begin{pmatrix}l\\0\\0\end{pmatrix}\right)+{}^2\dot{v}_2 = \begin{pmatrix}-l\dot{\Theta}_2^2 + c_2 g + s_2\ddot{d}_1\\l\ddot{\Theta}_2 - s_2 g + c_2\ddot{d}_1\\0\end{pmatrix}$$

b)

Compute the values of $^1N_1, {}^2N_2, {}^2N_2$. Briefly explain the physical meaning of the values of ^iN_i and ^iF_i, as they have been discussed in lecture and exercises.

$$^1N_1 = \begin{pmatrix}0\\0\\0\end{pmatrix} \; {}^2N_2 = \begin{pmatrix}0\\0\\\frac{\ddot{\Theta}_2 l^2}{2}\end{pmatrix} \; {}^3N_3 = \begin{pmatrix}0\\0\\\frac{\ddot{\Theta}_2 l^2}{2}\end{pmatrix}$$

The values ^iN_i and ^iF_i are the torques and forces that are required to act on the center of mass of link i in order to make the robot move as desired.

c)

Compute the values of τ_3, ${}^3n_3$, and τ_2. You can use the following results:

$$ {}^3\dot{v}_{C3} = \begin{pmatrix} l\ddot{\Theta}_2 - \mathrm{s}_2 g + \mathrm{c}_2\ddot{d}_1 \\ l\dot{\Theta}_2^2 - \mathrm{c}_2 g - \mathrm{s}_2\ddot{d}_1 \\ \ddot{d}_3 \end{pmatrix}, $$

and

$$ ({}^2P_3 \times {}^2f_3)^T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = ml(l\ddot{\Theta}_2 + \ddot{d}_1\mathrm{c}_2 - \mathrm{s}_2 g). $$

For τ_3, the computation looks as follows:

$$ \begin{aligned} \tau_3 &= (0,0,1)\cdot {}^3f_3 \\ {}^3f_3 &= {}^3F_3 = m\cdot {}^3\dot{v}_{C3} \quad \Rightarrow \quad \tau_3 = m\ddot{d}_3 \end{aligned} $$

For τ_2, we have $\tau_2 = (0,0,1)\cdot {}^2n_2$. We are only interested in the third component of ${}^2n_2$, which computes according to the formula

$$ {}^2n_2 = {}^2N_2 + {}^2_3R\,{}^3n_3 + {}^2P_{C2} \times {}^2F_2 + {}^2P_3 \times ({}^2_3R\,{}^3f_3) $$

Recognizing that ${}^2_3R^3f_3 = {}^2f_3$, we realize that all relevant values except ${}^3n_3$ are already known. We proceed by computing ${}^3n_3$ first:

$$ {}^3n_3 = {}^3N_3 + 0 + 0 + 0 = {}^3N_3 = \begin{pmatrix} 0 \\ 0 \\ \frac{\ddot{\Theta}_2 l^2}{2} \end{pmatrix} $$

Now we can continue computing the third component of ${}^2n_2$ (always keep in mind that we are not interested in the other two components).

$$ \begin{aligned} {}^2n_2 &= \begin{pmatrix} 0 \\ 0 \\ \frac{\ddot{\Theta}_2 l^2}{2} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{\ddot{\Theta}_2 l^2}{2} \end{pmatrix} + \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix} \times {}^2F_2 + {}^2P_3 \times {}^2f_3 \\ &= \begin{pmatrix} 0 \\ 0 \\ \ddot{\Theta}_2 l^2 \end{pmatrix} + \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix} \times m \begin{pmatrix} \bullet \\ l\ddot{\Theta}_2 - \mathrm{s}_2 g + \mathrm{c}_2\ddot{d}_1 \\ \bullet \end{pmatrix} + {}^2P_3 \times {}^2f_3 \\ &= \begin{pmatrix} 0 \\ 0 \\ \ddot{\Theta}_2 l^2 \end{pmatrix} + \begin{pmatrix} \bullet \\ \bullet \\ ml(l\ddot{\Theta}_2 - \mathrm{s}_2 g + \mathrm{c}_2\ddot{d}_1) \end{pmatrix} + \begin{pmatrix} \bullet \\ \bullet \\ ml(l\ddot{\Theta}_2 + \ddot{d}_1\mathrm{c}_2 - \mathrm{s}_2 g) \end{pmatrix} \end{aligned} $$

Finally, we are able to deduce

$$ \tau_2 = l^2\ddot{\Theta}_2 + 2ml(l\ddot{\Theta}_2 + \ddot{d}_1\mathrm{c}_2 - \mathrm{s}_2 g). $$

c)

Compute the values of τ_3, ${}^3n_3$, and τ_2. You can use the following results:

$$
{}^3\dot{v}_{C3} = \begin{pmatrix} l\ddot{\Theta}_2 - s_2 g + c_2\ddot{d}_1 \\ l\dot{\Theta}_2^2 - c_2 g - s_2\ddot{d}_1 \\ \ddot{d}_3 \end{pmatrix},
$$

and

$$
({}^2P_3 \times {}^2f_3)^T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = ml(l\ddot{\Theta}_2 + \ddot{d}_1 c_2 - s_2 g).
$$

Joint 3 is persmetic

$$\tau_3 = {}^3f_3^{T}\,{}^3\hat{Z}_3$$

$$
{}^3f_3 = \cancel{{}^3_4R\,{}^4f_4} + {}^3F_3 = {}^3F_3 = m\cdot{}^3\dot{V}_{C3} = \begin{pmatrix} (l\ddot{\theta}_2 - s_2 g + c_2\ddot{d}_1)\cdot m \\ (l\dot{\theta}_2^2 - c_2 g - s_2\ddot{d}_1)m \\ \ddot{d}_3\cdot m \end{pmatrix}
$$

$$\tau_3 = \ddot{d}_3\cdot m$$

$$
{}^3n_3 = {}^3N_3 \cancel{+ {}^3_4R\,{}^4n_4} + {}^3P_{C3}\times{}^3F_3 + \cancel{{}^3P_4 \times {}^3_4R\,{}^4f_4}
$$

$$
= {}^3N_3 + \cancel{{}^3P_{C3}\times{}^3F_3}
$$

$$
= \begin{pmatrix} 0 \\ 0 \\ \frac{l^2}{2}\ddot{\theta}_2 \end{pmatrix}
$$

Joint 2 is rotation.

$$\tau_2 = {}^2n_2^{T}\,{}^2\hat{Z}_2$$

$$
{}^2n_2 = {}^2N_2 + {}^2_3R\,{}^3n_3 + {}^2P_{C2}\times{}^2F_2 + {}^2P_3 \times {}^2_3R\,{}^3f_3
$$

$$
= \begin{pmatrix} 0 \\ 0 \\ \frac{l^2}{2}\ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \frac{l^2}{2}\ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} c_2 g + s_2\ddot{d}_1 - l\dot{\theta}_2^2 \\ -s_2 g + c_2\ddot{d}_1 + l\ddot{\theta}_2 \\ 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & * \\ 0 & * \\ l^2\ddot{\theta}_2 + (l\ddot{\theta}_2 - s_2 g + c_2\ddot{d}_1)\cdot lm + ml(l\ddot{\theta}_2 + \ddot{d}_1 c_2 - s_2 g) \end{pmatrix} + \begin{pmatrix} * \\ * \\ ml(l\cdot\ddot{\theta}_2 + \ddot{d}_1 c_2 - s_2 g) \end{pmatrix}
$$

$$
\tau_2 = l^2\ddot{\theta}_2 + 2(l\ddot{\theta}_2 - s_2 g + c_2\ddot{d}_1)ml
$$

d)

The value of τ_1 can be computed as:

$$\tau_1 = 2ml\left(\cos\Theta_2\,\ddot{\Theta}_2 - \sin\Theta_2\,\left(\dot{\Theta}_2\right)^2\right) + 3m\ddot{d}_1.$$

Write down the dynamic equations in state-space (M-V-G) form.

$$M = \begin{pmatrix} 3\,m & 2\,\cos\Theta_2\,l\,m & 0 \\ 2\,\cos\Theta_2\,l\,m & l^2\,(2\,m+1) & 0 \\ 0 & 0 & m \end{pmatrix}$$

$$V = \begin{pmatrix} -2\,\sin\Theta_2\,\left(\dot{\Theta}_2\right)^2\,l\,m \\ 0 \\ 0 \end{pmatrix}$$

$$G = \begin{pmatrix} 0 \\ -2\,\sin\Theta_2\,g\,l\,m \\ 0 \end{pmatrix}$$

$$\tau_2 = l^2\ddot{\theta}_2 + 2(l\ddot{\theta}_2 - S_2 g + C_2\ddot{d}_1)ml$$

$$\tau_3 = \ddot{d}_3 \cdot m$$

$$\tau = \begin{pmatrix} 2ml(\cos\theta_2 \cdot \ddot{\theta}_2 - \sin\theta_2 (\dot{\theta}_2)^2) + 3m\ddot{d}_1 \\ l^2\ddot{\theta}_2 + (l\ddot{\theta}_2 - S_2 g + C_2\ddot{d}_1)\,2ml \\ \ddot{d}_3 \cdot m \end{pmatrix}$$

$$M(\theta) = \begin{pmatrix} 3m & 2ml\cdot\cos\theta_2 & 0 \\ 2ml\cdot C_2 & 2ml^2 + l^2 & 0 \\ 0 & 0 & m \end{pmatrix}$$

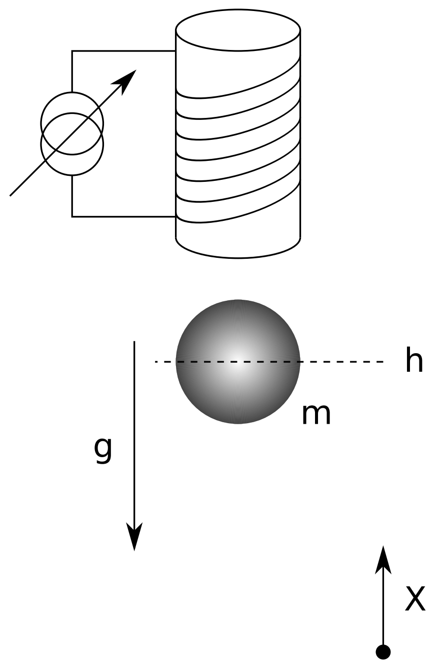
$$V(\theta, \dot{\theta}) = \begin{pmatrix} -2ml\cdot\sin\theta_2\cdot\dot{\theta}_2^2 \\ 0 \\ 0 \end{pmatrix}$$

$$G(\theta) = \begin{pmatrix} 0 \\ -2ml\cdot S_2 g \\ 0 \end{pmatrix}$$

Problem 3

Consider the situation shown in the figure on the right. You can see a coil wrapped around a ferrite core. Coil and core together act as a magnet, and the strength of the induced magnetic field is proportional to the current i that is passing through the coil. The object below the coil is an iron sphere whose height is the quantity to be controlled in this problem. The position h of the object, as well as all other quantities considered within this problem, are one-dimensional values. The X-axis along which the position is measured is pointing upwards, in opposite direction to g.
If the current applied to the coil is 0, no magnetic force at all influences the object. In that case, we assume that the object is affected exclusively by the following forces:

- Gravity force: $f_g = -mg$.
- Friction in the surrounding medium (which might be air, water, etc.): $f_b = -b\dot{h}^2$.



a)

Briefly explain the **general** approach for trajectory-following PD control of a robot manipulator. Write down two equations showing how the control torque vector τ should be computed by the controller. You can use the values of the desired trajectory $\Theta_d, \dot{\Theta}_d, \ddot{\Theta}_d$ as well as measurements $\Theta, \dot{\Theta}$ reported from the robot. You can assume that the dynamics equations of the robot in state-space form are known.
Typically, one would employ a controller that is partitioned into a system-dependent part and a servo part. The control torque is computed according to:

$$\tau = \alpha\tau' + \beta,$$

where $\alpha = M(\Theta)$ and $\beta = V(\Theta, \dot{\Theta}) + G(\Theta)$. The value of τ' is computed according to the servo rule

$$\tau' = \Theta_d + K_v\dot{E} + K_pE$$

for a trajectory-following controller, where $E = (\Theta_d - \Theta)$, and K_v and K_p are diagonal matrices.

When solving the following subproblems, you can assume that sensors measuring the height h and velocity $\dot{h}$ of the center of the sphere are present and can be used by your controller.

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a. Introducing $\ddot{\theta}_d$, $\dot{\theta}_d$ and θ_d and steady error $e = x_d - x$

$$\tau = \alpha \tau' + \beta$$

$$\tau' = \ddot{x} = \ddot{x}_d + k_v \dot{e} + k_p e$$

$$m\ddot{h} = -mg - b\dot{h}^2$$

$$m\ddot{h} + b\dot{h}^2 + mg = 0$$

$$m\ddot{h} + b\dot{h}^2 + mg = f = \alpha f' + \beta$$

$$\begin{cases} \alpha = m \\ f' = \ddot{h} = \ddot{h}_d + k_v \dot{e} + k_p e \\ \beta = b\dot{h}^2 + mg \end{cases} \qquad e = h_d - h.$$

$$m\ddot{h} + b\dot{h}^2 + mg = f_m = \frac{ik}{(h+h_0)^2}$$

$$\left(\frac{h+h_0}{k}\right)\left(m\ddot{h} + b\dot{h}^2 + mg\right) = i = \alpha i' + \beta$$

$$\alpha = \frac{h+h_0}{k} \cdot m$$

$$\beta = \frac{h+h_0}{k} \cdot (b\dot{h}^2 + mg)$$

$$i' = \ddot{h} = \ddot{h}_d + k_v \dot{e} + k_p e$$

$$w_n = \sqrt{k_p} \qquad w_n^2 = k_p$$

$$k_v = 2\sqrt{k_p}.$$

$$k_v = 2 w_n.$$

b)

Assuming that an arbitrary force f can be applied directly to the object shown in the figure above, how can the control scheme from subproblem a) be applied to the problem of controlling the position h of the iron sphere? Explicitly write down the two equations that are used to compute the total force f to be applied to the object. The values of the desired trajectory are denoted by $h_d, \dot{h}_d, \ddot{h}_d$.

The partitioning scheme can be applied as well to the described situation. The equation of motion is:

$$m\ddot{h} = f_g + f_b \Leftrightarrow m\ddot{h} + b\dot{h}^2 + mg = 0$$

Let $\alpha = m$, and set $\beta = b\dot{h}^2 + mg$. Compute

$$f = \alpha f' + \beta,$$

and

$$f' = \ddot{h}_d + k_v \dot{e} + k_p e.$$

c)

The control rule formulated in the previous subproblem computes a force f assuming that it can be exerted **directly** on the object. In our case however, this is not possible: We can influence the magnetic force exerted on the object only **indirectly** by controlling the electric current i that passes through the coil.

- Assuming that the dependency between magnetic force and electric current is

  $$f_m = \frac{i \cdot k}{(h + h_0)^2},$$

  where $k, h_0 \in \mathbb{R}$ are some constants, formulate the new equation of motion including the magnetic force.

- Devise a computation scheme for i that allows following of a trajectory h_d for the object. **Hint:** Start by reforming the equations of motion. You should be able to bring them in such a form that you can apply a partitioning scheme.

The new equation of motion is

$$m\ddot{h} + b\dot{h}^2 + mg = \frac{i \cdot k}{(h + h_0)^2}.$$

It can be reshaped to

$$\frac{(h + h_0)^2}{k}(m\ddot{h} + b\dot{h}^2 + mg) = i.$$

Now we can apply a partitioning scheme. Let

$$i = \alpha i' + \beta,$$

where $\alpha = \frac{(h+h_0)^2}{k}m$, and $\beta = \frac{(h+h_0)^2}{k}(b\dot{h}^2 + mg)$. This leads to the decoupled equation of motion

$$\ddot{h} = i'.$$

As usual, we can now set $i' = \ddot{h}_d + k_v\dot{e} + k_p e$ to achieve trajectory following.

d)

The previously designed controller is now tested using the following trivial trajectory:

$$h_d(t) = c, \dot{h}_d(t) = 0, \ddot{h}_d(t) = 0.$$

where $c \in \mathbb{R}$ is some constant vector. This should lead to the iron sphere hovering at a constant height c. You observe the following behavior:

1. When the controller is first tested, the object reaches the goal position, but only after oscillating around c for some time. The total time taken to reach the steady goal state is t_g.
2. The parameters of the controller are then tuned manually, and now the object reaches the goal state without oscillation, after the same time t_g as before.

Assuming that the physical model underlying the controller design is perfectly accurate, how can both observations be explained? What could the change in parameters have been between the first and the second observation? Is it possible to further improve the behavior of the system by adjusting the parameters?

1. The oscillating behavior of the PD controller must be the result of suboptimal controller parameters. The system is underdamped, which leads to an oscillating error behavior.
2. Since the same time t_g as before is taken to reach the goal state, we can conclude that the system is now overdamped. In the case of critical damping, the time taken would be smaller than t_g.

The change in behavior could have been achieved by increasing the differential constant k_v of the controller or by decreasing the proportional constant k_p. The behavior of the robot can be further improved by setting $k_v = 2\sqrt{k_p}$, thus achieving critical damping.