Dynamics Examples

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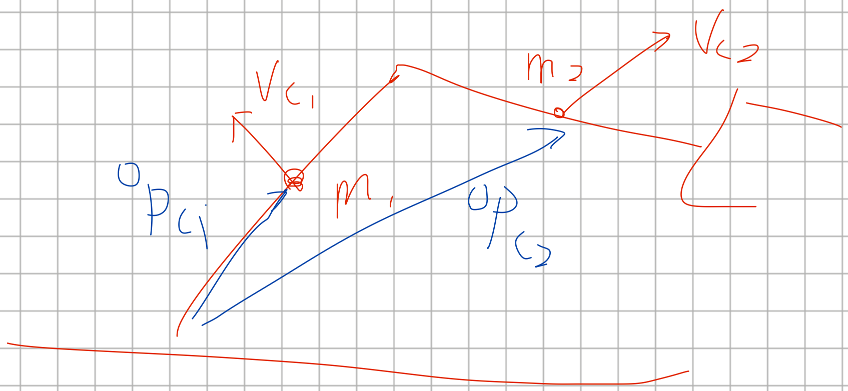
$K = \sum_i K_i \qquad P = \sum_i P_i$

$K_i = \frac{1}{2} m_i \, v_{c_i}^2$

$P_i = -m \vec{g} \cdot {}^{0}p_{c_i}$



Lagrange Equation

- Lagrange equations:

$$\begin{cases} L = K - P \\ \sum_\mu F_\mu \dfrac{\partial x_\mu}{\partial q_i} = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right) - \dfrac{\partial L}{\partial q_i} \end{cases} \qquad x_\mu = x_\mu(q_1 \cdots q_N, t)$$

External forces
(no potential)

$$K = \frac{1}{2} m \mathbf{v}^T \mathbf{v} + \frac{1}{2} \boldsymbol{\omega}^T I \boldsymbol{\omega}$$

$${}^{c}\tilde{I}_i = \begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{pmatrix}$$

point-mass $\tilde{I} = \tilde{0}$

${}^0P_{c_i} = {}^0_i\tilde{T}\,{}^iP_{c_i}$

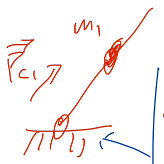
$P_i = -m\vec{g}^T \cdot {}^0\vec{P}_{c_i}$ $\qquad P = \sum_i P_i$

$K_i = \frac{1}{2} m_i \vec{v}_i^T \vec{v}_i + \frac{1}{2} \vec{\omega}^T \tilde{I} \cdot \vec{\omega}$ $\qquad K = \sum_i K_i$

$\tau = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}$

$\dot{q}_i = \begin{cases} \dot{\theta}_i & \text{Rotation} \\ \dot{d}_i & \text{Prismatic} \end{cases}$

$\tau_i = \begin{cases} \tau_i & \text{rotate} \\ f_i & \text{prismatic} \end{cases}$



$$\begin{aligned} {}^0\vec{\omega} &= \tilde{0} \\ {}^0\dot{\vec{\omega}} &= \tilde{0} \\ {}^0\vec{J} &= \tilde{0} \\ {}^0\dot{\vec{v}} &= \vec{g} \end{aligned}$$

Lagrange Analysis

Remember:

$$\mathbf{I}_c = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

$$I_{xx} = \int_{\mathcal{B}} (y^2 + z^2)\,dm \qquad I_{xy} = \int_{\mathcal{B}} xy\,dm$$

$$I_{yy} = \int_{\mathcal{B}} (x^2 + z^2)\,dm \qquad I_{xz} = \int_{\mathcal{B}} xz\,dm$$

$$I_{zz} = \int_{\mathcal{B}} (x^2 + y^2)\,dm \qquad I_{yz} = \int_{\mathcal{B}} yz\,dm$$

The Lagrangian dynamics formulation is another method for determining the dynamics of a robot which is derived from energy considerations. For each link of the robot, the kinetic energy can be computed as:

$$k_i = \frac{1}{2} m_i v_{C_i}^{\mathrm{T}} \cdot v_{C_i} + \frac{1}{2} {}^i\omega_i^{\mathrm{T}} \cdot {}^{C_i}I_i \cdot {}^i\omega_i$$

The first term corresponds to the kinetic energy caused by the linear motion of the link, and the second term corresponds to the kinetic energy caused by the rotational velocity of the link. To determine these energies, we need to compute the linear and rotational velocities of the joints. The overall kinetic energy computes then as sum of the kinetic energies of all links:

$$k = \sum_{i=1}^{n} k_i$$

$$k_i = \frac{1}{2} m_i \cdot \vec{v}_i^{T} \cdot \vec{v}_i + \frac{1}{2} \vec{w}_i^{T} \cdot I_i \cdot \vec{w}_i$$

$$= \frac{1}{2} \left(\vec{v}_i^{T} \cdot m \vec{v}_i + \vec{w}_i^{T} \cdot \tilde{I}_i \cdot \vec{w}_i \right)$$

$$\left((\vec{w}_i \times \vec{L}_i)^{T} \cdot m \cdot (\vec{w}_i \times L) + \vec{w}_i^{T} \cdot I_i \cdot \vec{w}_i \right)$$

Another way to compute kinetic energy is

$$k(\Theta, \dot{\Theta}) = \frac{1}{2} \dot{\Theta}^{\mathrm{T}} M(\Theta) \dot{\Theta},$$

$$= \frac{1}{2} \vec{w}_i^{T} \left(\vec{L}_i^{T} \cdot m \cdot \vec{L}_i + \tilde{I}_i \right) \cdot \vec{w}_i$$

$$= \frac{1}{2} \vec{\theta}_i^{T} \, M(\theta) \cdot \vec{\theta}_i$$

$$u_i = -m_i \cdot {}^0g^{\mathrm{T}} \cdot {}^0P_{C_i} + u_{\mathrm{ref}_i}$$

Here, g is the vector of gravity, ${}^0P_{C_i}$ denotes the center of mass of link i, and u_{ref_i} is an arbitrary constant (the constant is added because potential energy depends on height, and a certain base height can be chosen arbitrarily). In the further computations, this constant will not play a role, since only the derivatives of the potential energy are considered - and any constant will vanish when differentiated. The computation of τ is finally done through the following formula:

$$\tau = \frac{d}{dt}\frac{\partial k}{\partial \dot{\Theta}} - \frac{\partial k}{\partial \Theta} + \frac{\partial u}{\partial \Theta}$$

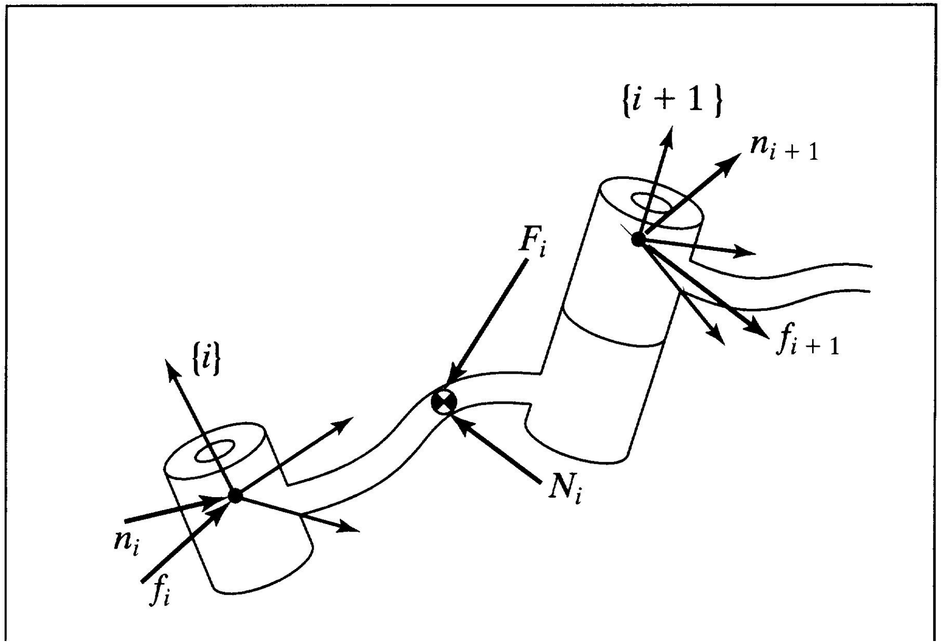
It is also possible to compute the joint torques τ_i on a per-joint basis, which is more practical in most cases. The formula then becomes:

$$\tau_i = \frac{d}{dt}\frac{\partial k}{\partial \dot{\Theta}_i} - \frac{\partial k}{\partial \Theta_i} + \frac{\partial u}{\partial \Theta_i}.$$

Newton Euler Method (force and torque balance)

$$^{i}F_{i} = {}^{i}f_{i} - {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}.$$

$$^{i}N_{i} = {}^{i}n_{i} - {}^{i}n_{i+1} + (-{}^{i}P_{C_i}) \times {}^{i}f_{i} - ({}^{i}P_{i+1} - {}^{i}P_{C_i}) \times {}^{i}f_{i+1}.$$



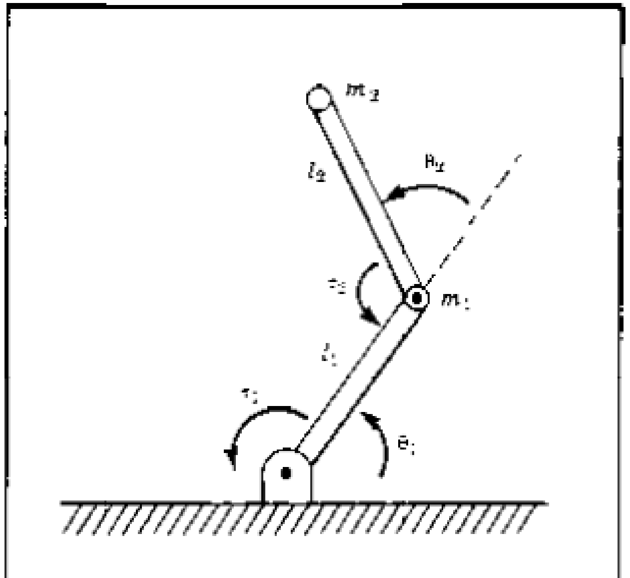
Iterative Closed Loop Form (Newton-Euler)

Outward iterations: $i : 0 \rightarrow 5$

$$
\begin{aligned}
{}^{i+1}\omega_{i+1} &= {}^{i+1}_{i}R\, {}^{i}\omega_i + \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}, \\
{}^{i+1}\dot{\omega}_{i+1} &= {}^{i+1}_{i}R\, {}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\, {}^{i}\omega_i \times \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}, \\
{}^{i+1}\dot{v}_{i+1} &= {}^{i+1}_{i}R({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_i), \\
{}^{i+1}\dot{v}_{C_{i+1}} &= {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} \\
&\quad + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}) + {}^{i+1}\dot{v}_{i+1}, \\
{}^{i+1}F_{i+1} &= m_{i+1}\, {}^{i+1}\dot{v}_{C_{i+1}}, \\
{}^{i+1}N_{i+1} &= {}^{C_{i+1}}I_{i+1}\, {}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\, {}^{i+1}\omega_{i+1}.
\end{aligned}
$$

Inward iterations: $i : 6 \rightarrow 1$

$$
\begin{aligned}
{}^{i}f_i &= {}^{i}_{i+1}R\, {}^{i+1}f_{i+1} + {}^{i}F_i, \\
{}^{i}n_i &= {}^{i}N_i + {}^{i}_{i+1}R\, {}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i \\
&\quad + {}^{i}P_{i+1} \times {}^{i}_{i+1}R\, {}^{i+1}f_{i+1}, \\
\tau_i &= {}^{i}n_i^T\, {}^{i}\hat{Z}_i.
\end{aligned}
$$



First we determine the value of the various quantities which will appear in the recursive Newton-Euler equations. The vectors which locate the center of mass for each link are

$$^{1}P_{C_1} = l_1 \hat{X}_1,$$

$$^{2}P_{C_2} = l_2 \hat{X}_2.$$

Because of the point mass assumption, the inertia tensor written at the center of mass for each link is the zero matrix:

$$^{C_1}I_1 = 0,$$

$$^{C_2}I_2 = 0.$$

There are no forces acting on the end-effector, and so we have

$$f_3 = 0,$$

$$n_3 = 0.$$

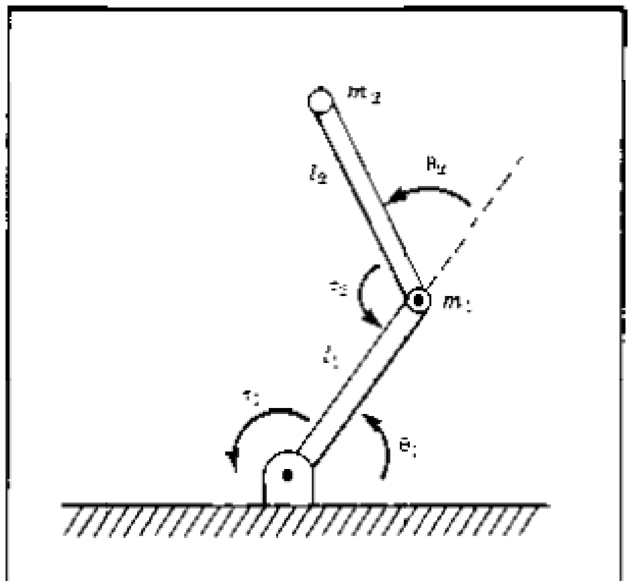
The base of the robot is not rotating, and hence we have

$$\omega_0 = 0,$$

$$\dot{\omega}_0 = 0.$$

To include gravity forces we will use

$${}^0\dot{v}_0 = g\hat{Y}_0.$$



$$ {}^{1}P_{C_1} = l_1 \hat{X}_1, $$
$$ {}^{2}P_{C_2} = l_2 \hat{X}_2. $$

Because of the point-mass assumption, the inertia tensor written at the center of mass for each link is the zero matrix:

$$ {}^{C_1}I_1 = 0, $$
$$ {}^{C_2}I_2 = 0. $$

There are no forces acting on the end-effector, so we have

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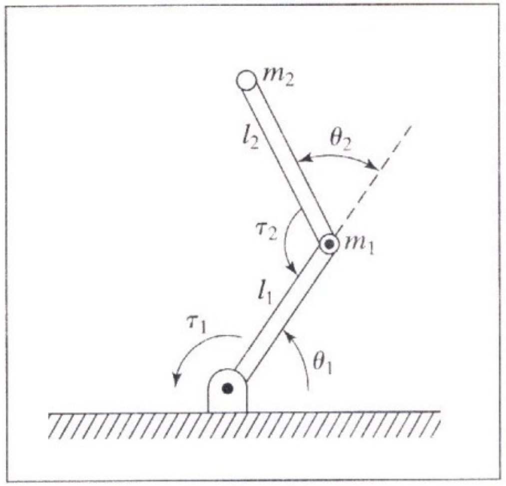
$$ n_3 = 0. $$

The base of the robot is not rotating; hence, we have

$$ \omega_0 = 0, $$
$$ \dot{\omega}_0 = 0. $$

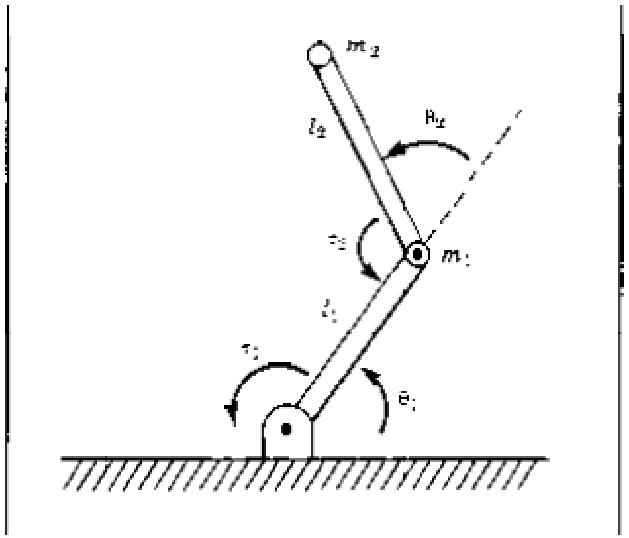


$$^0\dot{v}_0 = g\hat{Y}_0.$$

The rotation between successive link frames is given by

$$^{i}_{i+1}R = \begin{bmatrix} c_{i+1} & -s_{i+1} & 0.0 \\ s_{i+1} & c_{i+1} & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix},$$

$$^{i+1}_{i}R = \begin{bmatrix} c_{i+1} & s_{i+1} & 0.0 \\ -s_{i+1} & c_{i+1} & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}.$$



We now apply equations (6.46) through (6.53).

The outward iterations for link 1 are as follows:

$$^1\omega_1 = \dot{\theta}_1\, ^1\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix},$$

$$^1\dot{\omega}_1 = \ddot{\theta}_1\, ^1\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix},$$

Outward iterations: $i : 0 \rightarrow 5$

$$^{i+1}\omega_{i+1} = ^{i+1}_{i}R\, ^i\omega_i + \dot{\theta}_{i+1}\, ^{i+1}\hat{Z}_{i+1},$$

$$^{i+1}\dot{\omega}_{i+1} = ^{i+1}_{i}R\, ^i\dot{\omega}_i + ^{i+1}_{i}R\, ^i\omega_i \times \dot{\theta}_{i+1}\, ^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\, ^{i+1}\hat{Z}_{i+1},$$

$$^1\dot{v}_1 = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} = \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix},$$

$$^1\dot{v}_{C_1} = \begin{bmatrix} 0 \\ l_1\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -l_1\dot{\theta}_1^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1\dot{\theta}_1^2 + gs_1 \\ l_1\ddot{\theta}_1 + gc_1 \\ 0 \end{bmatrix},$$

$$^1F_1 = \begin{bmatrix} -m_1 l_1\dot{\theta}_1^2 + m_1 g s_1 \\ m_1 l_1\ddot{\theta}_1 + m_1 g c_1 \\ 0 \end{bmatrix}$$

$$^1N_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The outward iterations for link 2 are as fo

$$^2\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix},$$

$$^2\dot{\omega}_2 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix},$$

Outward iterations: $i : 0 \rightarrow 5$

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\, {}^i\omega_i + \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1},$$

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\, {}^i\dot{\omega}_i + {}^{i+1}_{i}R\, {}^i\omega_i \times \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}$$

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R({}^i\dot{\omega}_i \times {}^iP_{i+1} + {}^i\omega_i \times ({}^i\omega_i \times {}^iP_{i+1}) + {}^i\dot{v}_i),$$

$$^{i+1}\dot{v}_{C_{i+1}} = {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}) + {}^{i+1}\dot{v}_{i+1},$$

$$^{i+1}F_{i+1} = m_{i+1}\, {}^{i+1}\dot{v}_{C_{i+1}},$$

$$^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1}\, {}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\, {}^{i+1}\omega_{i+1}.$$

$$
{}^{2}\dot{v}_{2} = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -l_1\dot{\theta}_1^2 + gs_1 \\ l_1\ddot{\theta}_1 + gc_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1\ddot{\theta}_1 s_2 - l_1\dot{\theta}_1^2 c_2 + gs_{12} \\ l_1\ddot{\theta}_1 c_2 + l_1\dot{\theta}_1^2 s_2 + gc_{12} \\ 0 \end{bmatrix},
$$

$$
\begin{aligned}
{}^{2}\dot{v}_{C_2} &= \begin{bmatrix} 0 \\ l_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix} + \begin{bmatrix} -l_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ 0 \\ 0 \end{bmatrix} \\
&+ \begin{bmatrix} l_1\ddot{\theta}_1 s_2 - l_1\dot{\theta}_1^2 c_2 + gs_{12} \\ l_1\ddot{\theta}_1 c_2 + l_1\dot{\theta}_1^2 s_2 + gc_{12} \\ 0 \end{bmatrix},
\end{aligned} \tag{6.55}
$$

$$
{}^{2}F_{2} = \begin{bmatrix} m_2 l_1\ddot{\theta}_1 s_2 - m_2 l_1\dot{\theta}_1^2 c_2 + m_2 g s_{12} - m_2 l_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ m_2 l_1\ddot{\theta}_1 c_2 + m_2 l_1\dot{\theta}_1^2 s_2 + m_2 g c_{12} + m_2 l_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix},
$$

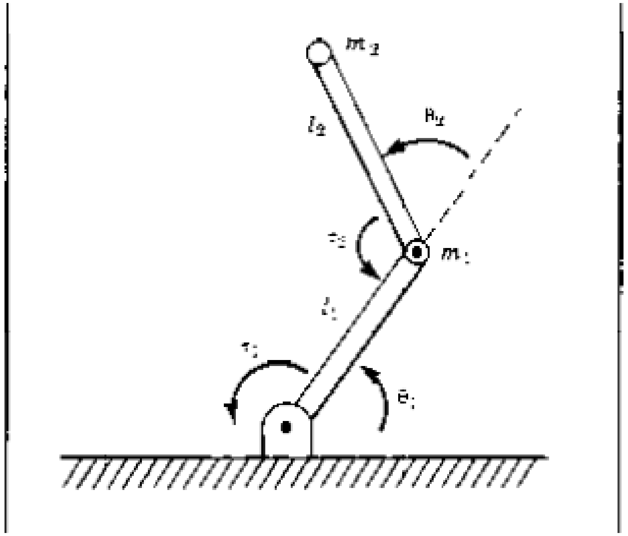
$$
{}^{2}N_{2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
$$

Outward iterations: $i : 0 \rightarrow 5$

$$
\begin{aligned}
{}^{i+1}\omega_{i+1} &= {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}, \\
{}^{i+1}\dot{\omega}_{i+1} &= {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}, \\
{}^{i+1}\dot{v}_{i+1} &= {}^{i+1}_{i}R({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_i), \\
{}^{i+1}\dot{v}_{C_{i+1}} &= {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} \\
&\quad + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}) + {}^{i+1}\dot{v}_{i+1},
\end{aligned}
$$

The inward iterations for link 2 are as follows:

$$
\begin{aligned}
{}^2f_2 &= {}^2F_2, \\
{}^2n_2 &= \begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) \end{bmatrix}.
\end{aligned} \tag{6.56}
$$



The inward iterations for link 1 are as follows:

$$
\begin{aligned}
{}^1f_1 = &\begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_2 l_1 s_2 \ddot{\theta}_1 - m_2 l_1 c_2 \dot{\theta}_1^2 + m_2 g s_{12} - m_2 l_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ m_2 l_1 c_2 \ddot{\theta}_1 + m_2 l_1 s_2 \dot{\theta}_1^2 + m_2 g c_{12} + m_2 l_2 (\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix} \\
&+ \begin{bmatrix} -m_1 l_1 \dot{\theta}_1^2 + m_1 g s_1 \\ m_1 l_1 \ddot{\theta}_1 + m_1 g c_1 \\ 0 \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
{}^1n_1 = &\begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ 0 \\ m_1 l_1^2 \ddot{\theta}_1 + m_1 l_1 g c_1 \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ 0 \\ m_2 l_1^2 \ddot{\theta}_1 - m_2 l_1 l_2 s_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 l_1 g s_2 s_{12} \\ + m_2 l_1 l_2 c_2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 g c_2 c_{12} \end{bmatrix}
\end{aligned}
$$

Inward iterations: $i : 6 \rightarrow 1$

$$
\begin{aligned}
{}^i f_i &= {}^{i}_{i+1}R \, {}^{i+1}f_{i+1} + {}^i F_i, \\
{}^i n_i &= {}^i N_i + {}^{i}_{i+1}R \, {}^{i+1}n_{i+1} + {}^i P_{C_i} \times {}^i F_i \\
&\quad + {}^i P_{i+1} \times {}^{i}_{i+1}R \, {}^{i+1}f_{i+1}, \\
\tau_i &= {}^i n_i^T \, {}^i \hat{Z}_i.
\end{aligned}
$$

Extracting the $\hat{Z}$ components of the ${}^{i}n_i$, we find the joint torques:

$$\begin{aligned}
\tau_1 = {} & m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 c_2 (2\ddot{\theta}_1 + \ddot{\theta}_2) + (m_1 + m_2) l_1^2 \ddot{\theta}_1 - m_2 l_1 l_2 s_2 \dot{\theta}_2^2 \\
& -2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1, \\
\tau_2 = {} & m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2).
\end{aligned} \tag{6.58}$$

Equations (6.58) give expressions for the torque at the actuators as a function of joint position, velocity, and acceleration. Note that these rather complex functions arose from one of the simplest manipulators imaginable. Obviously, the closed-form equations for a manipulator with six degrees of freedom will be quite complex.

$$\boldsymbol{\tau} = M(\mathbf{q})\ddot{\mathbf{q}} + h(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + g(\mathbf{q})$$

External forces (control)

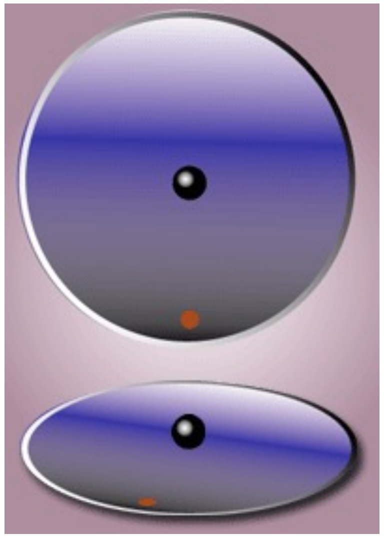
Inertia (generalized)

Coriolis, centrifugal effects

Gravity

$$\boldsymbol{F} - m\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \times \boldsymbol{r}' - 2m\boldsymbol{\omega} \times \boldsymbol{v}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}') = m\boldsymbol{a}'$$

where



$\boldsymbol{F}$ is the vector sum of the physical forces acting on the object
$\boldsymbol{\omega}$ is the angular velocity, of the rotating reference frame relative to the inertial frame
$\boldsymbol{v}'$ is the velocity relative to the rotating reference frame
$\boldsymbol{r}'$ is the position vector of the object relative to the rotating reference frame
$\boldsymbol{a}'$ is the acceleration relative to the rotating reference frame

- Euler force $-m\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \times \boldsymbol{r}'$
- Coriolis force $-2m(\boldsymbol{\omega} \times \boldsymbol{v}')$
- centrifugal force $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}')$

Cartesian state space equation

$$\mathcal{F} = M_x(\Theta)\ddot{\mathcal{X}} + V_x(\Theta, \dot{\Theta}) + G_x(\Theta),$$

$$J^{-T}\tau = J^{-T}M(\Theta)\ddot{\Theta} + J^{-T}V(\Theta, \dot{\Theta}) + J^{-T}G(\Theta),$$

$$\mathcal{F} = J^{-T}M(\Theta)\ddot{\Theta} + J^{-T}V(\Theta, \dot{\Theta}) + J^{-T}G(\Theta).$$

We next develop a relationship between joint space and Cartesian acceleration, starting with the definition of the Jacobian,

$$\dot{\mathcal{X}} = J\dot{\Theta}, \tag{6.95}$$

and differentiating to obtain

$$\ddot{\mathcal{X}} = \dot{J}\dot{\Theta} + J\ddot{\Theta}. \tag{6.96}$$

Solving (6.96) for joint space acceleration leads to

$$\ddot{\Theta} = J^{-1}\ddot{\mathcal{X}} - J^{-1}\dot{J}\dot{\Theta}. \tag{6.97}$$

Substituting (6.97) into (6.94) we have

$$\mathcal{F} = J^{-T}M(\Theta)J^{-1}\ddot{\mathcal{X}} - J^{-T}M(\Theta)J^{-1}\dot{J}\dot{\Theta} + J^{-T}V(\Theta,\dot{\Theta}) + J^{-T}G(\Theta), \tag{6.98}$$

from which we derive the expressions for the terms in the Cartesian dynamics as

$$\begin{aligned} M_x(\Theta) &= J^{-T}(\Theta)\ M(\Theta)\ J^{-1}(\Theta), \\ V_x(\Theta,\dot{\Theta}) &= J^{-T}(\Theta)\ \left(V(\Theta,\dot{\Theta}) - M(\Theta)\ J^{-1}(\Theta)\ \dot{J}(\Theta)\ \dot{\Theta}\right), \\ G_x(\Theta) &= J^{-T}(\Theta)\ G(\Theta). \end{aligned} \tag{6.99}$$