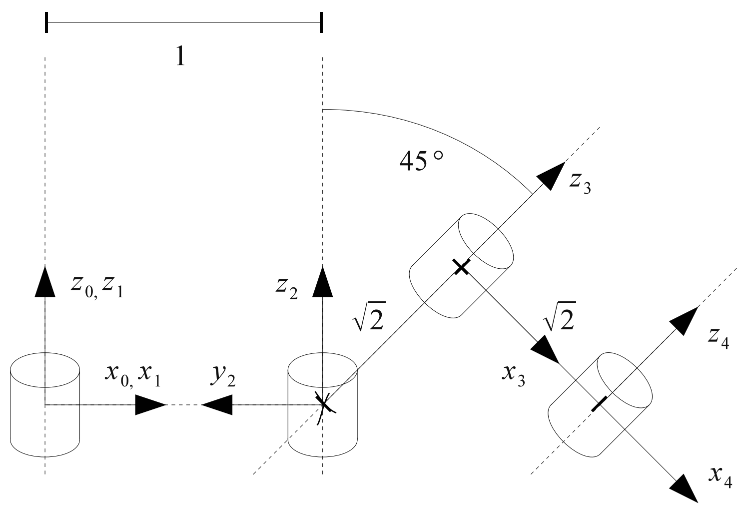

Figure 1: *4R Robot (Problem 1)*

Problem 1

In this problem, we are working with the same robot as in problem 10 of the first problem sheet, which is shown again in Figure 1. This time, the robot is equipped with a force-torque-sensor that has system $\{4\}$ as frame of reference. An external force is applied to the robot, such that its force-torque sensor reports a measurement of

$$\begin{pmatrix} {}^4f \\ {}^4n \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 0 \\ 7 \\ 0 \\ 8 \end{pmatrix}.$$

a) Determine the joint torques that are required to cancel out the external influences and thus keep the robot static.

b) Assume now that there is a screwdriver attached to the last link, and the tip of the screwdriver is translated along the z-axis about 9 length units, so ${}^4P_{\text{tip}} = (0, 0, 9)^T$. With the same force-torque measurement ~~reported by the~~ sensor in system 4, which forces and torques are present at the screwdriver tip? Which forces and torques are caused by the robot in direction of the screw driver (i.e., in Z direction)?

$$\begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}$$

$${}^4f = \begin{pmatrix}0\\6\\0\end{pmatrix} \quad {}^4n \begin{pmatrix}7\\0\\8\end{pmatrix}$$

J1

static Robotic

$${}^3f_3 = {}^3_4R\,{}^4f_4$$

$${}^3f_3 = \begin{pmatrix}1&&0\\&1&\\0&&1\end{pmatrix}\begin{pmatrix}0\\6\\0\end{pmatrix} = \begin{pmatrix}0\\6\\0\end{pmatrix}$$

$${}^3n_3 = {}^3_4R\,{}^4n_4 + {}^3P_4 \times {}^3f_3$$

$${}^3n_3 = \begin{pmatrix}1&&0\\&1&\\0&&1\end{pmatrix}\cdot\begin{pmatrix}7\\0\\8\end{pmatrix} + \begin{pmatrix}\sqrt{2}\\0\\0\end{pmatrix}\times\begin{pmatrix}0\\6\\0\end{pmatrix}$$

$$= \begin{pmatrix}7\\0\\8\end{pmatrix} + \begin{pmatrix}0\\0\\6\sqrt{2}\end{pmatrix}$$

$$= \begin{pmatrix}7\\0\\8+6\sqrt{2}\end{pmatrix}$$

$${}^{i-1}_{\ \ i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0_1T = \begin{pmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^1_2T = \begin{pmatrix} c\theta_2 & -s\theta_2 & 0 & 1 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^2_3T = \begin{pmatrix} c\theta_3 & -s\theta_3 & 0 & 0 \\ \frac{\sqrt{2}}{2}s\theta_3 & \frac{\sqrt{2}}{2}c\theta_3 & -\frac{\sqrt{2}}{2} & -1 \\ \frac{\sqrt{2}}{2}s\theta_3 & \frac{\sqrt{2}}{2}c\theta_3 & \frac{\sqrt{2}}{2} & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^3_4T = \begin{pmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{\begin{aligned} {}^if_i &= {}^i_{i+1}R\,{}^{i+1}f_{i+1} \quad (5\text{-}80) \\ {}^in_i &= {}^i_{i+1}R\,{}^{i+1}n_{i+1} + {}^iP_{i+1}\times{}^if_i \quad (5\text{-}81) \end{aligned}}$$

$${}^2f_2 = {}^2_3R\,{}^3f_3$$

$$= \begin{pmatrix} c_3 & -s_3 & 0 \\ \frac{\sqrt{2}}{2}s_3 & \frac{\sqrt{2}}{2}c_3 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2}s_3 & \frac{\sqrt{2}}{2}c_3 & \frac{\sqrt{2}}{2} \end{pmatrix} \cdot \begin{pmatrix}0\\6\\0\end{pmatrix}$$

$$= \begin{pmatrix} -6s_3 \\ 3\sqrt{2}c_3 \\ 3\sqrt{2}s_3 \end{pmatrix}$$

$${}^2n_2 = {}^2_3R\,{}^3n_3 + {}^2P_3 \times {}^2f_2$$

$$= \begin{pmatrix} c_3 & -s_3 & 0 \\ \frac{\sqrt{2}}{2}s_3 & \frac{\sqrt{2}}{2}c_3 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2}s_3 & \frac{\sqrt{2}}{2}c_3 & \frac{\sqrt{2}}{2} \end{pmatrix} \cdot \begin{pmatrix}7\\0\\8+6\sqrt{2}\end{pmatrix} + \begin{pmatrix}0\\ -\sqrt{2}\cdot\frac{\sqrt{2}}{2}=-1 \\ \sqrt{2}\cdot\frac{\sqrt{2}}{2}=1\end{pmatrix}\begin{pmatrix}-6s_3\\ 3\sqrt{2}c_3 \\ 3\sqrt{2}s_3\end{pmatrix}$$

$$= \begin{pmatrix} 7c_3 \\ \frac{7}{2}\sqrt{2}s_3 - 4\sqrt{2} - 6 \\ \frac{7}{2}\sqrt{2}s_3 + 4\sqrt{2} + 6 \end{pmatrix} + \begin{pmatrix} -6\sqrt{2}c_3 \\ +6c_3 \\ -6s_3 \end{pmatrix}$$

$$= \begin{pmatrix} 7c_3 - 6\sqrt{2}c_3 \\ \frac{7}{2}\sqrt{2}s_3 + 6c_3 - 4\sqrt{2} - 6 \\ \frac{7}{2}\sqrt{2}s_3 - 6s_3 + 4\sqrt{2} + 6 \end{pmatrix}$$

$${}^1f_1 = {}^1_2R\,{}^2f_2$$

$$= \begin{pmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -6s_3 \\ 3\sqrt{2}c_3 \\ 3\sqrt{2}c_3 \end{pmatrix}$$

$$= \begin{pmatrix} -6c_2s_3 - 3\sqrt{2}s_2c_3 \\ -6s_2s_3 + 3\sqrt{2}c_2c_3 \\ 3\sqrt{2}c_3 \end{pmatrix}$$

$${}^1n_1 = {}^1_2R\,{}^2n_2 + {}^1P_2 \times {}^1f_1$$

$$= \begin{pmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 7c_3 - 6\sqrt{2}c_3 \\ \frac{7}{2}\sqrt{2}s_3 - 6c_3 - 4\sqrt{2} - 6 \\ \frac{7}{2}\sqrt{2}s_3 + 6s_3 + 4\sqrt{2} + 6 \end{pmatrix} + \begin{pmatrix}1\\0\\0\end{pmatrix} \times {}^1f_1$$

$$= \begin{pmatrix} c_2(7c_3 - 6\sqrt{2}c_3) - s_2\left(\frac{7}{2}\sqrt{2}s_3 - 6c_3 - 4\sqrt{2} - 6\right) \\ s_2(7c_3 - 6\sqrt{2}c_3) + c_2\left(\frac{7}{2}\sqrt{2}s_3 - 6c_3 - 4\sqrt{2} - 6\right) \\ \frac{7}{2}\sqrt{2}s_3 - 6s_3 + 4\sqrt{2} + 6 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ -3\sqrt{2}c_3 \\ -6s_2s_3 + 3\sqrt{2}c_2c_3 \end{pmatrix}$$

$$\boxed{\tau_i = {}^i n_i^T \, {}^i\hat{Z}_i} \leftarrow \text{Rotation}$$

算出关节驱动力为

$$\boxed{\tau_i = {}^i f_i^T \, {}^i\hat{Z}_i}$$

$$\tau_3 = {}^3 n_3^T \, {}^3\hat{Z}_3$$
$$= \begin{pmatrix} 7 & 0 & 8+6\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$= 8+6\sqrt{2}$$

$$\tau_2 = {}^2 n_2^T \, {}^2\hat{Z}_2$$
$$= \frac{7}{2}\sqrt{2}s_3 \cancel{+} - 6s_3 + 4\sqrt{2} + 6$$

$$\tau_1 = \frac{7}{2}\sqrt{2}s_3 \cancel{+} - 6s_3 + 4\sqrt{2} + 6 - 6s_2s_3 + 3\sqrt{2}c_2c_3$$

$$\tau_4 = 8$$

$$\tau = \begin{pmatrix} \frac{7}{2}\sqrt{2}s_3 \cancel{+} - 6s_3 + 4\sqrt{2} + 6 - 6s_2s_3 + 3\sqrt{2}c_2c_3 \\ \frac{7}{2}\sqrt{2}s_3 \cancel{+} - 6s_3 + 4\sqrt{2} + 6 \\ 8+6\sqrt{2} \\ 8 \end{pmatrix}$$

$$\Theta = \begin{bmatrix} 0 & 90^\circ & -90^\circ & 0^\circ \end{bmatrix}$$
$$\quad 1 \quad\quad 2 \quad\quad 3 \quad\quad 4.$$

$$\tau = \begin{pmatrix} -\frac{7}{2}\sqrt{2} \cancel{-} + \cancel{6} + 4\sqrt{2} \cancel{+6} + 6 \\ -\frac{7}{2}\sqrt{2} \cancel{-} + \cancel{6} + 4\sqrt{2} \cancel{+6} \\ 8+6\sqrt{2} \\ \cancel{8} \end{pmatrix} = \begin{pmatrix} -\frac{7}{2}\sqrt{2} + 4\sqrt{2} + \cancel{6}\,18 \\ -\frac{7}{2}\sqrt{2} + 4\sqrt{2} + 12 \\ 8+6\sqrt{2} \\ 8 \end{pmatrix} \begin{matrix} 18.707 \\ 12.707 \\ 16.485 \\ \ \end{matrix}$$

(b) $${}^4_5 t = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}$$ No Rotation

$${}^4_5 R = \tilde{I}$$

$$\boxed{\begin{aligned} {}^i f_i &= {}^i_{i+1}R \, {}^{i+1}f_{i+1} \quad (5\text{-}80) \\ {}^i n_i &= {}^i_{i+1}R \, {}^{i+1}n_{i+1} + {}^iP_{i+1} \times {}^i f_i \quad (5\text{-}81) \end{aligned}}$$

$${}^4 f_4 = {}^4_5 R \, {}^5 f_5$$

$${}^5 f_5 = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}$$

$${}^4 n_4 = {}^4_5 R \, {}^5 n_5 + {}^4 P_5 \times {}^4 f_4$$
$$\begin{pmatrix} 7 \\ 0 \\ 8 \end{pmatrix} = {}^5 n_5 + \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix} \times \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}$$
$${}^5 n_5 = \begin{pmatrix} 7 \\ 0 \\ 8 \end{pmatrix} - \begin{pmatrix} -54 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 61 \\ 0 \\ 8 \end{pmatrix}$$

$${}^5 f_{5Z} = 0 \qquad {}^5 n_{5Z} = 8$$

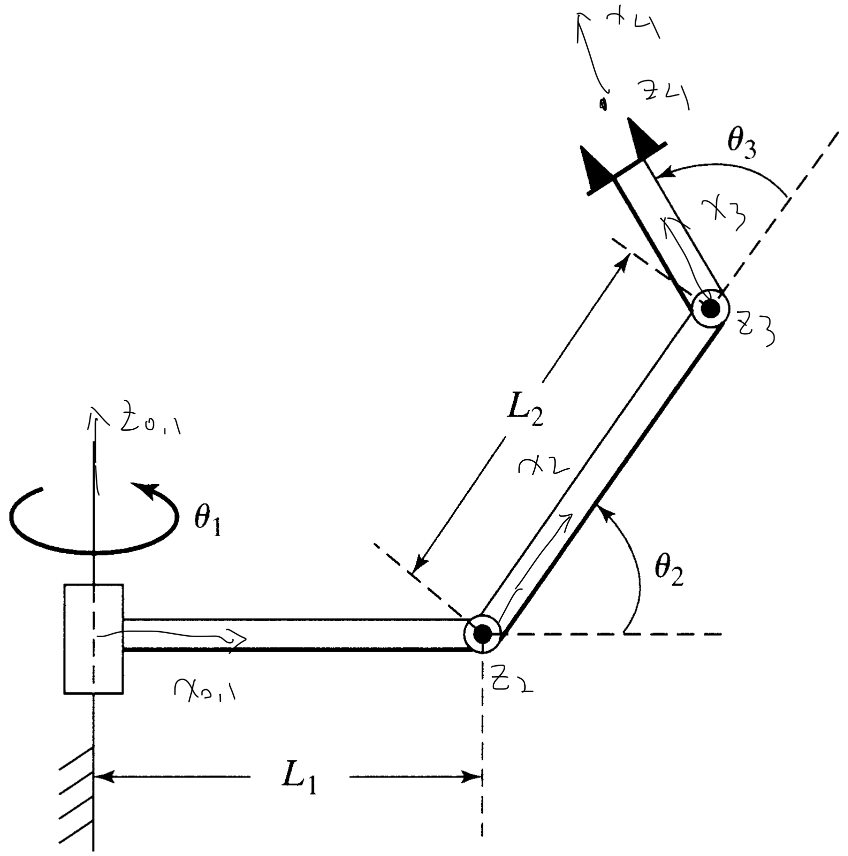

Figure 2: *3R Robot (Problem 2)*

Problem 2

For the robot shown in Figure 2, determine the Jacobian w.r.t. reference frame {4} using three different approaches:

a) Compute velocities in system 4, and derive the Jacobian

b) Compute force-torque relations for system 4, derive the Jacobian

c) Geometric observations

	a_{i-1}	d_i	α_{i-1}	θ_i
1	0	0	0	θ_1
2	L_1	0	90°	θ_2
3	l_2	0	0	θ_3
4	l_3	0	0	0

$$ {}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{2}T = \begin{pmatrix} c_1c_2 & -c_1s_2 & s_1 & L_1c_1 \\ s_1c_2 & -s_1s_2 & -c_1 & L_1s_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{0}_{1}T = \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{0}_{3}T = \begin{pmatrix} c_1c_2c_3 - c_1s_2c_3 & -c_1c_2s_3 - c_1s_2c_3 & s_1 & l_1c_1 + l_2c_1c_2 \\ s_1c_2c_3 - s_1s_2s_3 & -s_1c_2s_3 - s_1s_2c_3 & -c_1 & l_1s_1 + l_2s_1c_2 \\ s_2c_3 + c_2s_3 & -s_2s_3 + c_2c_3 & 0 & l_2s_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{1}_{2}T = \begin{pmatrix} c_2 & -s_2 & 0 & L_1 \\ 0 & 0 & -1 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{2}_{3}T = \begin{pmatrix} c_3 & -s_3 & 0 & l_2 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{3}_{4}T = \begin{pmatrix} 1 & 0 & 0 & l_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{0}_{4}T = \begin{pmatrix} c_1c_2c_3 - c_1s_2c_3 & -c_1c_2s_3 - c_1s_2s_3 & s_1 & l_1c_1 + l_2c_1c_2 + l_3[c_1c_2c_3 - c_1s_2c_3] \\ s_1c_2c_3 - s_1s_2s_3 & -s_1c_2s_3 - s_1s_2c_3 & -c_1 & l_1s_1 + l_2s_1c_2 + l_3[s_1c_2c_3 - s_1s_2s_3] \\ s_2c_3 + c_2s_3 & -s_2s_3 + c_2c_3 & 0 & l_2s_2 + l_3(s_2c_3 + c_2s_3) \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$\boxed{{}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot\theta_{i+1}\,{}^{i+1}\hat{Z}_{i+1}}$$

$$\boxed{{}^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1})}$$

$${}^0w_0 = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad {}^0V_0 = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$${}^1w_1 = {}^1_0R\,{}^0w_0 + \dot\theta_1\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$${}^1w_1 = \begin{pmatrix}0\\0\\\dot\theta_1\end{pmatrix}$$

$${}^1V_1 = {}^1_0R({}^0V_0 + {}^0W_0 \times {}^0P_1)$$

$${}^1V_1 = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$${}^2_1R = {}^1_2R^T = \begin{pmatrix}c_2 & 0 & s_2\\ -s_2 & 0 & c_2\\ 0 & -1 & 0\end{pmatrix}$$

$${}^2W_2 = {}^2_1R\,{}^1W_1 + \dot\theta_2\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$${}^2W_2 = \begin{pmatrix}c_2 & 0 & s_2\\ -s_2 & 0 & c_2\\ 0 & -1 & 0\end{pmatrix}\cdot\begin{pmatrix}0\\0\\\dot\theta_1\end{pmatrix} + \begin{pmatrix}0\\0\\\dot\theta_2\end{pmatrix}$$

$$= \begin{pmatrix}s_2\cdot\dot\theta_1\\ c_2\cdot\dot\theta_1\\ 0\end{pmatrix} + \begin{pmatrix}0\\0\\\dot\theta_2\end{pmatrix}$$

$$= \begin{pmatrix}s_2\dot\theta_1\\ c_2\dot\theta_1\\ \dot\theta_2\end{pmatrix}$$

$${}^2V_2 = {}^2_1R({}^1V_1 + {}^1W_1 \times {}^1P_2)$$

$$= \begin{pmatrix}c_2 & 0 & s_2\\ -s_2 & 0 & c_2\\ 0 & -1 & 0\end{pmatrix}\left(\begin{pmatrix}0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\\dot\theta_1\end{pmatrix} \times \begin{pmatrix}l_1\\0\\0\end{pmatrix}\right)$$

$$= R\begin{pmatrix}0\\\dot\theta_1 l_1\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\-\dot\theta_1 l_1\end{pmatrix}$$

$${}^3_2R = {}^2_3R^T = \begin{pmatrix}c_3 & s_3 & 0\\ -s_3 & c_3 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$${}^3W_3 = {}^3_2R\,{}^2W_2 + \dot\theta_3\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}c_3 & s_3 & 0\\ -s_3 & c_3 & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}s_2\dot\theta_1\\ c_2\dot\theta_1\\ \dot\theta_2\end{pmatrix} + \begin{pmatrix}0\\0\\\dot\theta_3\end{pmatrix}$$

$$= \begin{pmatrix}s_2c_3\cdot\dot\theta_1 + c_2s_3\dot\theta_1\\ -s_2s_3\cdot\dot\theta_1 + c_2c_3\dot\theta_1\\ \dot\theta_2\end{pmatrix} + \begin{pmatrix}0\\0\\\dot\theta_3\end{pmatrix}$$

$$= \begin{pmatrix}s_{2+3}\dot\theta_1\\ c_{2+3}\dot\theta_1\\ \dot\theta_2+\dot\theta_3\end{pmatrix}$$

$${}^3V_3 = {}^3_2R({}^2V_2 + {}^2W_2 \times {}^2P_3)$$

$$= \begin{pmatrix}c_3 & s_3 & 0\\ -s_3 & c_3 & 0\\ 0 & 0 & 1\end{pmatrix}\left(\begin{pmatrix}0\\0\\-l_1\dot\theta_1\end{pmatrix} + \begin{pmatrix}s_2\dot\theta_1\\ c_2\dot\theta_1\\ \dot\theta_2\end{pmatrix}\times\begin{pmatrix}l_2\\0\\0\end{pmatrix}\right)$$

$$= R\begin{pmatrix}0\\ l_2\cdot\dot\theta_2\\ -l_2c_2\cdot\dot\theta_1 - l_1\cdot\dot\theta_1\end{pmatrix}$$

$$= \begin{pmatrix}s_3l_2\dot\theta_2\\ c_3l_2\dot\theta_2\\ -l_1\cdot\dot\theta_1 - l_2\cdot c_2\cdot\dot\theta_1\end{pmatrix}$$

$${}^4_3R = {}^3_4R^T = I$$

$${}^4W_4 = {}^4_3\overset{I}{R}\,{}^3W_3 + \underset{0}{\dot\theta_4}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}s_{2+3}\dot\theta_1\\ c_{2+3}\dot\theta_1\\ \dot\theta_2+\dot\theta_3\end{pmatrix}$$

$${}^4V_4 = {}^4_3R({}^3V_3 + {}^3W_3 \times {}^3P_4)$$

$$= \begin{pmatrix}s_3l_2\cdot\dot\theta_2\\ c_3\cdot l_2\cdot\dot\theta_2\\ -l_1\cdot\dot\theta_1 - l_2\cdot c_2\cdot\dot\theta_1\end{pmatrix} + \begin{pmatrix}s_{2+3}\dot\theta_1\\ c_{2+3}\dot\theta_1\\ \dot\theta_2+\dot\theta_3\end{pmatrix} \times \begin{pmatrix}l_3\\0\\0\end{pmatrix}$$

$$= \begin{pmatrix}s_3l_2\cdot\dot\theta_2\\ c_3\cdot l_2\cdot\dot\theta_2 + l_3(\dot\theta_2+\dot\theta_3)\\ -l_1\cdot\dot\theta_1 - l_2\cdot c_2\cdot\dot\theta_1 - l_3c_{2+3}\dot\theta_1\end{pmatrix}$$

$$\underset{6\times1}{{}^4V_4} = \underset{6\times3}{{}^4J(\theta)}\cdot\underset{3\times1}{\dot\theta}$$

$$\begin{pmatrix}{}^4V_4\\ {}^4W_4\end{pmatrix} = {}^4J(\theta)\cdot\dot\theta$$

$${}^4J(\theta) = \begin{pmatrix}0 & l_2s_3 & 0\\ 0 & l_2c_3+l_3 & l_3\\ -l_1-l_2c_2-l_3c_{2+3} & 0 & 0\\ s_{2+3} & 0 & 0\\ c_{2+3} & 0 & 0\\ 0 & 1 & 1\end{pmatrix}$$

$$ {}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} $$
$$ {}^{i}n_i = {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i $$

$$ {}^4f_4 = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \qquad {}^4N_4 = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} $$

$$ {}^3f_3 = {}^3_4R \cdot {}^4f_4 = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} $$

$$ {}^3n_3 = {}^3_4R\,{}^4n_4 + {}^3P_4 \times {}^3f_3 = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} + \begin{pmatrix} L_3 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 - L_3 \cdot F_3 \\ N_3 + L_3 \cdot F_2 \end{pmatrix} $$

$$ {}^2f_2 = {}^2_3R \cdot {}^3f_3 = \begin{pmatrix} C_3 & -S_3 & 0 \\ S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} C_3F_1 - S_3F_2 \\ S_3F_1 + C_3F_2 \\ F_3 \end{pmatrix} $$

$$ {}^2n_2 = {}^2_3R \cdot {}^3n_3 + {}^2P_3 \times {}^2f_2 = \begin{pmatrix} C_3 & -S_3 & 0 \\ S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} N_1 \\ N_2 - L_3 \cdot F_3 \\ N_3 + L_3 \cdot F_2 \end{pmatrix} + \begin{pmatrix} L_2 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} C_3F_1 - S_3F_2 \\ S_3F_1 + C_3F_2 \\ F_3 \end{pmatrix} $$

$$ = \begin{pmatrix} C_3N_1 - S_3(N_2 - l_3 \cdot F_3) \\ S_3N_1 + C_3(N_2 - l_3F_3) - l_2 \cdot F_3 \\ N_3 + l_3 \cdot F_2 + l_2 \cdot (S_3F_1 + C_3F_2) \end{pmatrix} $$

$$ {}^1f_1 = {}^1_2R \cdot {}^2f_2 = \begin{pmatrix} C_2 & -S_2 & 0 \\ 0 & 0 & -1 \\ S_2 & C_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} C_3F_1 - S_3F_2 \\ S_3F_1 + C_3F_2 \\ F_3 \end{pmatrix} $$

$$ = \begin{pmatrix} C_2(C_3F_1 - S_3F_2) - S_2(S_3F_1 + C_3F_2) \\ -F_3 \\ S_2(C_3F_1 - S_3F_2) + C_2(S_3F_1 + C_3F_2) \end{pmatrix} = \begin{pmatrix} C_{2+3}F_1 - S_{2+3}F_2 \\ -F_3 \\ S_{2+3}F_1 + C_{2+3}F_2 \end{pmatrix} $$

$$ {}^1n_1 = {}^1_2R\,{}^2n_2 + {}^1P_2 \times {}^1f_1 = \begin{pmatrix} C_2 & -S_2 & 0 \\ 0 & 0 & -1 \\ S_2 & C_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} C_3N_1 - S_3(N_2 - l_3 \cdot F_3) \\ S_3N_1 + C_3(N_2 - l_3F_3) - l_2 \cdot F_3 \\ N_3 + l_3 \cdot F_2 + l_2(S_3 \cdot F_1 + C_3 \cdot F_2) \end{pmatrix} + \begin{pmatrix} L_1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} C_{2+3}F_1 - S_{2+3}F_2 \\ -F_3 \\ S_{2+3}F_1 + C_{2+3}F_2 \end{pmatrix} $$

$$ = \begin{pmatrix} C_2[C_3N_1 - S_3(N_2 - L_3F_3)] - S_2[S_3N_1 + C_3(N_2 - l_3F_3) - L_2F_3] \\ -N_3 - l_3F_2 - l_2(S_3F_1 + C_3F_2) - L_1(S_{2+3}F_1 + C_{2+3}F_2) \\ S_2[C_3N_1 - S_3(N_2 - L_3F_3)] + C_2[S_3N_1 + C_3(N_2 - l_3F_3) - L_2F_3] - L_1 \cdot F_3 \end{pmatrix} $$

$$ S_2\cancel{C_3}N_1 - S_2\cancel{S_3}N_2 + S_2\cancel{S_3 l_3}F_3 + C_2\cancel{S_3}N_1 + \cancel{C_2C_3}N_2 - C_2C_3\cancel{l_3F_3} - C_2l_2F_3 - L_1 \cdot F_3 $$

$$ \tau = \begin{pmatrix} S_{2+3}N_1 + C_{2+3}N_2 - (C_{2+3}l_3 + C_2l_2 + L_1)F_3 \\ N_3 + S_3l_2F_1 + (l_2 \cdot C_3 + l_3)F_2 \\ N_3 + l_3 \cdot F_2 \end{pmatrix} = \underset{3\times6}{J(\theta)} \cdot \underset{6\times1}{\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ N_1 \\ N_2 \\ N_3 \end{pmatrix}} $$

$$ {}^4J(\theta) = \begin{pmatrix} 0 & 0 & -(L_1 + C_2l_2 + C_{2+3}l_3) & S_{2+3} & C_{2+3} & 0 \\ S_3l_2 & (l_2C_3 + l_3) & 0 & 0 & 0 & 1 \\ 0 & l_3 & 0 & 0 & 0 & 1 \end{pmatrix} $$

$${}^A J(\Theta) = \left[\begin{array}{c|c} {}^A_B R & 0 \\ \hline 0 & {}^A_B R \end{array}\right] {}^B J(\Theta).$$

$${}^0J_v \, \dot{\theta} = {}^0\dot{P} \overset{({}^0_4T)}{=} \begin{pmatrix} (l_1 + l_2 c_2 + l_3 c_{2+3}) c_1 \\ (l_1 + l_2 c_2 + l_3 c_{2+3}) s_1 \\ l_2 s_2 + l_3 s_{2+3} \end{pmatrix}'$$

$${}^4J(\Theta) = \begin{pmatrix} {}^4_0 R & \hat{0} \\ \hat{0} & {}^4_0 R \end{pmatrix} {}^0J(\Theta) = \begin{pmatrix} {}^0J_v(\Theta) \\ {}^0J_w(\Theta) \end{pmatrix}$$

$${}^0_4T = \begin{pmatrix} c_1c_2c_3 - c_1s_2c_3 & -c_1c_2s_3 - c_1s_2s_3 & s_1 \\ s_1c_2c_3 - s_1s_2s_3 & -s_1c_2s_3 - s_1s_2c_3 & -c_1 \\ s_2c_3 + c_2s_3 & -s_2s_3 + c_2c_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{matrix} l_1c_1 + l_2c_1c_2 + l_3[c_1c_2c_3 - c_1s_2s_3] \\ l_1s_1 + l_2s_1c_2 + l_3[s_1c_2c_3 - s_1s_2s_3] \\ l_2s_2 + l_3(s_2c_3 + c_2s_3) \end{matrix} \Bigg)$$

$${}^0J_v = \begin{pmatrix} -(l_1 + l_2c_2 + l_3c_{2+3})s_1 & -(l_2s_2 + l_3s_{2+3})c_1 & -l_3s_{2+3}c_1 \\ (l_1 + l_2c_2 + l_3c_{2+3})c_1 & -(l_2s_2 + l_3s_{2+3})s_1 & -l_3s_{2+3}s_1 \\ 0 & l_2c_2 + l_3c_{2+3} & l_3c_{2+3} \end{pmatrix}$$

$${}^iJ_w = ({}^ir_1 \,\vdots\, {}^ir_2 \cdots \vdots\, {}^ir_n)$$

For Rotation $\quad {}^ir_j = {}^iz_j = {}^i_jR\,{}^jz_j = {}^i_jR \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

For prismatic $\quad {}^ir_j = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$${}^0J_w = \left({}^0z_1 \,\vdots\, {}^0z_2 \,\vdots\, {}^0z_3 \right)$$

$$\left({}^0_1R \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \vdots \quad \cdots \right)$$

$$\begin{pmatrix} 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{pmatrix}$$

$${}^0J(\Theta) = \begin{pmatrix} -(l_1 + l_2c_2 + l_3c_{2+3})s_1 & -(l_2s_2 + l_3s_{2+3})c_1 & -l_3s_{2+3}c_1 \\ (l_1 + l_2c_2 + l_3c_{2+3})c_1 & -(l_2s_2 + l_3s_{2+3})s_1 & -l_3s_{2+3}s_1 \\ 0 & l_2c_2 + l_3c_{2+3} & l_3c_{2+3} \\ 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$^{4}J(\theta) = \left[\begin{array}{c|c} ^{4}_{0}R & 0 \\ \hline 0 & ^{4}_{0}R \end{array}\right] {}^{0}J(\theta)$$

$$^{4}J_v(\theta) = {}^{4}_{0}R \cdot {}^{0}J_v(\theta)$$

$$^{4}J_w(\theta) = {}^{4}_{0}R \cdot {}^{0}J_w(\theta)$$