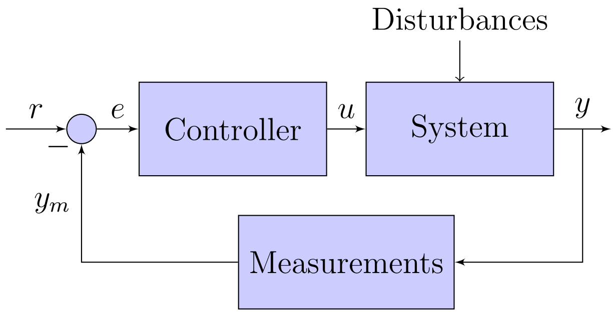
Recapitulation: Mass-Spring-Systems

Being able to establish the dynamics equations of a robot is still not enough to make it carry out a desired trajectory. There are several disturbances in physical systems that cannot be modeled, which means that a controlling scheme is required to alleviate the influence of those disturbances. The following schematic gives an overview of a typical control architecture:



The value r would be the desired state of the system. Computing the difference $e = y_m - r$ yields the error of the current state y_m with respect to the desired state. The controller takes e into account when generating control signals u, in order to correct any error introduced by disturbances.

As an prerequisite to understanding the function of a PD controller, mass-spring-systems have been studied in the lecture. The importance of these systems lies in their connection to the behaviour of the error e in control problems. We will later see that the error e behaves like a mass-spring-system itself.

Such a system can be described informally as follows: We observe an object with mass m that is affected by a spring force. The corresponding spring constant is denoted by k. Furthermore, the object is located on some surface, and there is friction between the surface and the object, which also affects the movement of the object. The friction constant is denoted by b. The deflection of the object from its resting position is usually denoted by x, thus $\dot{x}$ and $\ddot{x}$ are speed and acceleration of the object. The force corrsponding to friction is $b \cdot \dot{x}$, and thus proportional to the speed of the object. The force exerted by the spring is $k \cdot x$ and is thus proportional to the deflection of the object from the resting position.

Since we want to control physical systems, we assume that we are also able to apply a force f to the object. Overall, we can thus state the following dynamics equation:

$$m\ddot{x} + b\dot{x} + kx = f$$

This equation is, in the context of control, also called the *open-loop-equation*, since it does not take into account any feedback from the system (like, e.g., in the case of a robot, joint position measurements from sensors or such).

But usually, robots are equipped with sensors measuring joint positions and speeds. Thus, for the simpler situation of mass-spring-systems, we will also assume that there are sensors measuring of $\ddot{x}, \dot{x}, x$.

What we want to achieve now is a so-called critical damping of the mass-spring system, by exerting an additional force f on the system. We distinguish three possible cases: Under-damping leads to oscillation of the system (spring stiffness dominates), while over-damping leads to slow convergence to the resting position (friction dominates). Critical damping leads to the fastest possible transition of the system to its resting configuration, avoiding both overdamping and underdamping (friction and stiffness are in balance).

How can we achieve critical damping? The force f that we exert on the object must depend on friction and spring forces, and we can assume that it's of the form

$$f = -k_p x - k_v \dot{x}.$$

This yields the closed-loop equation

$$m\ddot{x} + b\dot{x} + kx = -k_p x - k_v \dot{x}.$$

We are interested in controlling the movement of the body, so we are interested in determining the function $x(t)$, and we need to solve the following differential equation:

$$m\ddot{x} + (b + k_v)\dot{x} + (k + k_p)x = 0$$

Note that this equation still looks like that of a normal mass-spring system. This means that we cannot achieve arbitrary movements of the object, but we can influence its oscillation behaviour.

A standard method for solving such equations makes use of the solutions of the so-called characteristic equation

$$ms^2 + (b + k_v)s + (k + k_p) = 0.$$

The solutions to this quadratic equation are given by the well-known formula

$$s_{1,2} = \frac{-(b + k_v) \pm \sqrt{(b + k_v)^2 - 4m(k + k_p)}}{2m} = \frac{-b' \pm \sqrt{b'^2 - 4mk'}}{2m}.$$

It is then common to abbreviate $(b + k_v)$ as b' and $(k + k_p)$ as k'. The solutions s_1, s_2 determine the trajectory $x(t)$ as follows:

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

A special case is $s_1 = s_2 \in \mathbb{R}$. This case corresponds to critical damping of the system. Note that this can be achieved by choosing b' and k' such that $b'^2 - 4mk' = 0$, or $b' = 2\sqrt{mk'}$ (we can assume b' and k' to be positive).

In the case of an oscillating system, the equations can also be stated in a different form:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

If the complex solutions to the characteristic equation are $\lambda \pm i\mu$, there is a relation to the parameters of the modified form:

$$\begin{aligned} \lambda &= -\zeta\omega_n \\ \mu &= \omega_n \sqrt{1 - \zeta^2}. \end{aligned}$$

Here, ζ is called "damping ratio" and ω_n "natural frequency." For a simple linear system those values can be determined as follows:

$$\begin{aligned} \zeta &= \frac{b'}{2\sqrt{k'm}} \\ \omega_n &= \sqrt{\frac{k'}{m}} \end{aligned}$$

To be able to solve the first problem, we also need to explain the concept of the "resonant frequency." Until now, we assumed that all parts of the mechanic systems we observed do not deform at all. But in reality, components of robots have a finite stiffness, which means that they deform minimally under stress. This can lead to the undesired effect of resonance, which leads to deformations adding up until finally components may be damaged.

A simple possibility of taking the effect of resonance into account is through enforcing the following inquality:

$$\omega_n \leq \frac{1}{2}\omega_{\text{res}}$$

Here, ω_{res} is the so-called resonant frequency of the object. If this condition holds, we can be sure that no undesired resonances occur.

Solution 1

The following quantities are specified:

$$m = 1, \quad b = 4, \quad k = 5, \quad \omega_{\text{res}} = 6$$

We are supposed to compute k_p and k_v, assuring that the natural frequency ω_n is chosen such that resonance is avoided. Using above formulas, we obtain:

$$\omega_n = \sqrt{k'} \leq 3$$

Since we know k, we can already deduce a condition on k_p:

$$\sqrt{k + k_p} \leq 3 \Rightarrow k_p \leq 9 - k = 4$$

Apart from that, we can choose the value of k_p arbitrarily, so let's say $k_p = 4$. A relation between k' and b' can be derived from the characteristic equation

$$s^2 + b's + k' = 0.$$

The solutions of that equation are

$$s_{1/2} = \frac{-b' \pm \sqrt{b'^2 - 4k'}}{2}.$$

Since we want to achieve critical damping, $b'^2 - 4k' = 0$ must hold, from which we can deduce

$$b' = 2\sqrt{k'} = 2\sqrt{5+4} = 6 \quad \Rightarrow \quad k_v = 2.$$

Note that we can always assume that k' and b' are positive values. Negative values would lead to an unstable system.

Recapitulation: Control Law Partitioning

Until now, we have used the following simple rule to influence the behaviour of a mass-spring system:

$$f = -k_p x - k_v \dot{x}.$$

Using this rule, we were able to achieve critical damping of a mass-spring system, choosing parameters k' and b' according to $b' = 2\sqrt{mk'}$. However, the choice of b' depends on m, which is acceptable for a simple mass-spring system, but makes things very complicated in more complex systems. Thus, we want to decouple the mass-dependent part from the equation, using an extended rule that reads as follows:

$$f = \alpha f' + \beta$$

Through this rule, we want to achieve the following: Factors α and β should be chosen such that the system, considering only f' as input, behaves like a unit mass, governed by the equation

$$\ddot{x} = f'.$$

Writing down the complete equation for the system, we obtain

$$m\ddot{x} + b\dot{x} + kx = \alpha f' + \beta.$$

Now comes the interesting part: How should we choose α and β in order to achieve the desired effect? Obviously, $\beta = b\dot{x} + kx$ and $\alpha = m$ must hold. Note that we are now able to influence the system directly through f'. This is the decoupled open-loop equation. Again, be aware that for the simple case of a mass-spring system there is not much gained through rewriting the system like this, but for manipulating more complicated systems, the advantage is substantial.

Now, we move again from the open-loop form to the closed-loop form. Thus, let f' again depend on spring force and friction force, such that

$$f' = -k_p x - k_v \dot{x},$$

with some new constants k_p and k_v. The equation of motion becomes

$$\ddot{x} + k_p x + k_v \dot{x} = 0.$$

In particular, we see that now k_v and k_p are now independent of system parameters. The system will always be critically damped if $k_v = 2\sqrt{k_p}$.

Recapitulation: Multi-dimensional Systems

The decoupling method might not seem very useful at first sight, because it does not really help much with the simple case of a mass-spring-system with only one object. The main advantage of using this partitioning scheme is that it greatly simplifies the control problem for multi-dimensional problems. Assuming that x is a multi-dimensional quantity, the equations of motion without partitioning will look like this:

$$M\ddot{x} + B\dot{x} + Kx = f$$

Here we can as well adopt the approach of using system feedback, in hope of achieving critical damping for that system. We set

$$f = -K_v \dot{x} - K_p x$$

with some matrices K_v and K_p, and we end up with:

$$M\ddot{x} + (B + K_v)\dot{x} + (K + K_p)x = 0$$

This is a multi-dimensional linear differential equation, and it is quite difficult to handle and to analyze. Determining K_v and K_p in order to achieve critical damping becomes a **major** problem when using this approach!

Now let's find out how this works when the partitioning scheme is in use. We set $f = \alpha f' + \beta$, where $\alpha = M$, and $\beta = B\dot{x} + Kx$. We end up with:

$$M\ddot{x} + B\dot{x} + Kx = Mf' + B\dot{x} + Kx$$

Assuming that M is invertible, this transforms into

$$\ddot{x} = f',$$

just as it did in the one-dimensional case. Further setting $f' = -K_v \dot{x} - K_p x$, we end up with the following system:

$$\ddot{x} + K_v \dot{x} + K_p x = 0.$$

Choosing K_p and K_v as **diagonal** matrices with entries k_{pi}, k_{v_i}, this becomes a series of decoupled differential equations:

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_n \end{pmatrix} + \begin{pmatrix} k_{v_1} & 0 & \dots & 0 \\ 0 & k_{v_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & k_{v_n} \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} + \begin{pmatrix} k_{p_1} & 0 & \dots & 0 \\ 0 & k_{p_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & k_{p_n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_n \end{pmatrix} + \begin{pmatrix} k_{v_1}\dot{x}_1 \\ k_{v_2}\dot{x}_2 \\ \vdots \\ k_{v_n}\dot{x}_n \end{pmatrix} + \begin{pmatrix} k_{p_1}x_1 \\ k_{p_2}x_2 \\ \vdots \\ k_{p_n}x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Achieving critical damping of this system is extremely easy: Set $k_{vi} = 2\sqrt{k_{pi}}$, and we're done!

Recapitulation: Trajectory Following

Now we no longer assume that we are simply interested in achieving critical damping of a mass-spring system, but instead we want the system to carry out a certain trajectory. The computed trajectory shall be denoted by $x_d(t)$, and we assume that $x_d(t)$ is a twice continuously differentiable function. Let $e(t) = x_d(t) - x(t)$ denote the difference between the actual position and the desired position.

Movement along the desired trajectory can now be achieved by employing the following control rule:

$$f' = \ddot{x}_d + k_v\dot{e} + k_p e$$

Substituting this rule into the partition scheme, we obtain:

$$\ddot{x} = \ddot{x}_d + k_v\dot{e} + k_p e \Leftrightarrow \ddot{e} + k_v\dot{e} + k_p e = 0$$

We see that the error e now behaves like a mass-spring system! This means that we can achieve critical damping of the error through appropriate choice of k_p and k_v. This means that the error will tend towards 0, and it will approach 0 with a speed depending on k_v and k_p - critical damping will thus provide the fastest possible convergence of the error towards 0. Note that this is only true if there are no further unmodeled effects present in the system - in practice, one usually employs a PID controller (PD controller with additional integral part) to assure that the error always approaches 0.

What we have discussed here is a mathematical treatment of the PD-controller. Above derivation explains the typical oscillating behaviour of PD-controllers. Furthermore, we see that it is possible to compute the proportional and differential factors directly.

Solution 2

Again, we are dealing with a mass-spring system, but now, there is not only one object, but two objects that are connected to each other via a spring. Thus, the differential equations governing the motion of the objects also have to look a little bit different.

First of all, let us try to figure out the force acting on the second, smaller object. Obviously, there is a spring force acting on that object. The length of the spring can be computed as $(x_2 - x_1)$. Thus, the deflection of the spring from its resting position will be $x_2 - x_1$, and the force exerted by that spring will be $-k(x_2 - x_1)$.

Furthermore, there is friction acting between both objects. With the usual model of friction, the equation of movement of the small object can be stated as:

$$m_2\ddot{x}_2 = -b_2(\dot{x}_2 - \dot{x}_1) - k(x_2 - x_1)$$

Note that you have to pay attention to the signs of the forces! The correct signs can be verified by inserting some values, and checking whether the forces point in the expected direction. For example, assume that $(x_2 - x_1) > 0$ holds. This means that the spring should exert a force on the second object that is pulling it to the left, towards the spring's resting position. Since k is a positive constant, the negative sign in above equation is correct.

Similarily, we can figure out the forces acting on the other object as:

$$m_1\ddot{x}_1 = b_2(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) - b_1\dot{x}_1$$

Note that the signs for spring force and friction force are reversed, since the forces affect the first object in the opposing direction than they affect the second object. The equations of motion can now be summarized as follows:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} b_2 + b_1 & -b_2 \\ -b_2 & b_2 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Where $f = (f_1, f_2)^{\mathrm{T}}$ is the controlling force in the system. The first matrix in above formula will now be abbreviated with M, the second matrix as B, and the third as K. Note that this is a rather complicated system of differential equations that we would not be able to control as it is. But, if we employ the principle of controller partitioning and the control law for trajectory following that we have discussed last time, we will succeed in controlling this system. Letting $e = x_d - x$ as usual, we can compute

$$f' = \ddot{x}_d + K_v\dot{e} + K_p e = \begin{pmatrix} \ddot{x}_{1d} \\ \ddot{x}_{2d} \end{pmatrix} + \begin{pmatrix} k_{v1} & 0 \\ 0 & k_{v2} \end{pmatrix} \begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} + \begin{pmatrix} k_{p1} & 0 \\ 0 & k_{p2} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

and

$$f = \alpha f' + \beta$$

where $\alpha = M$ and

$$\beta = B\dot{x} + Kx$$

All in all, we obtain

$$\begin{aligned} & M\ddot{x} + B\dot{x} + Kx = M(\ddot{x}_d + K_v\dot{e} + K_p e) + B\dot{x} + Kx \\ \Leftrightarrow \quad & \ddot{e} + K_v\dot{e} + K_p e = 0 \end{aligned}$$

The matrices K_v and K_p are diagonal, thus we can choose $k_{vi} = 2\sqrt{k_{pi}}$ to achieve critical damping. Thus, we have specified a PD controller meeting the requirements stated in this problem.