# Fundamentals of Artificial Intelligence
# Exercise 7: Probability theory and Bayesian networks

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# Problem 7.1

Consider the following probabilistic inference problem with random variables for driver expertise, road conditions and accidents. The joint probability distribution is provided in the table below (values are guessed). The first variable $R \in \{dry, wet, snow/ice\}$ is a discrete random variable and represents the considered road conditions. The rest of the variables are Boolean: $E$ is associated to the event that the driver is experienced or not and $A$ to the event that an accident happens or not.

Table: Joint probability distribution $\mathbf{P}(R, E, A)$.

| $E$ | $A$ | $\mathbf{P}(R = dry, E, A)$ | $\mathbf{P}(R = wet, E, A)$ | $\mathbf{P}(R = snow/ice, E, A)$ |
|---|---|---|---|---|
| t | t | 0.0607 | 0.0449 | 0.0084 |
| t | f | 0.3605 | ? | 0.0240 |
| f | t | 0.0851 | 0.0654 | 0.0152 |
| f | f | 0.1435 | 0.0400 | 0.0022 |

# Problem 7.1 a

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a. Calculate the value of the missing probability in the table.

$$\sum_{\omega \in \Omega} P(\omega) = 1 \qquad P_{missing} = P(wet, e, \neg a) = 1 - \sum_{\omega \in \Omega \setminus \{(wet, e, \neg a)\}} P(\omega) = 0.1501$$

# Problem 7.1 b

| $E$ | $A$ | $\mathbf{P}(R = dry, E, A)$ | $\mathbf{P}(R = wet, E, A)$ | $\mathbf{P}(R = snow/ice, E, A)$ |
|---|---|---|---|---|
| t | t | 0.0607 | 0.0449 | 0.0084 |
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b. What is the prior probability distribution of the random variables $R$ and $E$?

$$P(R=x) = \sum_{E \times A} P(R=x \mid E, A) \rightarrow P(R=wet) = P(wet)$$

$$= 0{,}0449 + 0{,}1501 + 0{,}0654 + 0{,}04$$

$$= 0{,}3$$

$$P(dry) = 0{,}65 \quad ; \quad P(ice) = 1 - P(wet) - P(dry) = 0{,}05$$

# Problem 7.1 c

| $E$ | $A$ | $\mathbf{P}(R = dry, E, A)$ | $\mathbf{P}(R = wet, E, A)$ | $\mathbf{P}(R = snow/ice, E, A)$ |
|---|---|---|---|---|
| t | t | 0.0607 | 0.0449 | 0.0084 |
| t | f | 0.3605 | 0.1501 | 0.0240 |
| f | t | 0.0851 | 0.0654 | 0.0152 |
| f | f | 0.1435 | 0.0400 | 0.0022 |

c. What is the probability that the driver is not experienced given that there is an accident and the road is wet?

$$P(\neg e \mid a, wet) = \frac{P(\neg e, a, wet)}{P(a, wet, \neg e) + P(a, wet, e)} = \frac{0{,}0654}{0{,}0654 + 0{,}0449} \approx 0{,}6$$

$$\underline{P(\neg e, a, wet) = P(a, wet) \cdot P(\neg e \mid a, wet)}$$

# Problem 7.1 d

| $E$ | $A$ | $\mathbf{P}(R = dry, E, A)$ | $\mathbf{P}(R = wet, E, A)$ | $\mathbf{P}(R = snow/ice, E, A)$ |
|---|---|---|---|---|
| t | t | 0.0607 | 0.0449 | 0.0084 |
| t | f | 0.3605 | 0.1501 | 0.0240 |
| f | t | 0.0851 | 0.0654 | 0.0152 |
| f | f | 0.1435 | 0.0400 | 0.0022 |

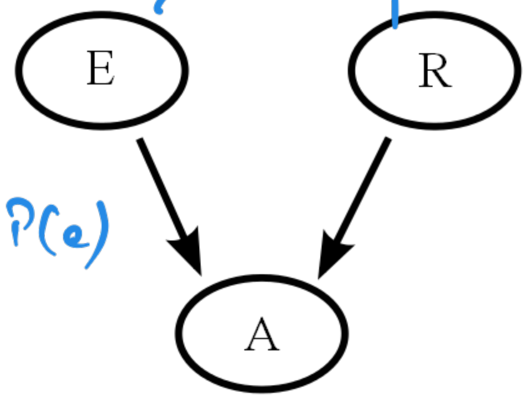
d. Construct a corresponding Bayesian network with the conditional probability tables (calculate the required table entries).

# Problem 7.1 d

$p(a,b) = p(a)\,p(b|a) = p(a)\,p(b)$

independence

| $\mathbf{P}(e)$ |
|---|
| 0.6486 |




| P(R) |
|---|
| 0.65 0.30 0.05 |

$P(\neg e) = 1 - P(e)$

| $R$ | $E$ | $\mathbf{P}(a \mid R, E)$ |
|---|---|---|
| dry | t | $x_1$ |
| | f | $x_2$ |
| wet | t | $x_3$ |
| | f | $x_4$ |
| snow/ice | t | $x_5$ |
| | f | $x_6$ |

$$P(A|R,E) = \frac{P(A,R,E)}{P(R,E)} = \frac{P(A,R,E)}{P(R)\cdot P(E)}$$

$$x_1 = P(a|e,dry) = \frac{P(a,e,dry)}{P(e)\cdot P(dry)} = 0{,}144 \mathrel{\hat=} 14\%$$

$$x_4 = P(a|\neg e,wet) = \frac{P(a,\neg e,wet)}{P(\neg e)\,P(wet)} = 0{,}6204 \mathrel{\hat=} 62\%$$

$$P(\neg a|\neg e,wet) = 1 - P(a|\neg e,wet)$$

## Reminder: Bayes' Rule and its interpretation

Y: "assumption" / "hypothesis"

X: "evidence"

likelihood

posterior    prob. evid., given hypothesis    prior distribution

**Bayes Rule**

$$P\big((Y=y)|(X=x)\big) = \frac{P\big((X=x)|(Y=y)\big)P(Y=y)}{P(X=x)}$$

marginal prob.

# Reminder: Bayes' Rule and its interpretation

N : Sun has gone nova

D : Device says that sun has gone nova

$p(d \mid \neg n) = \text{small} = \frac{1}{36}$

$\rightarrow$ n must be true

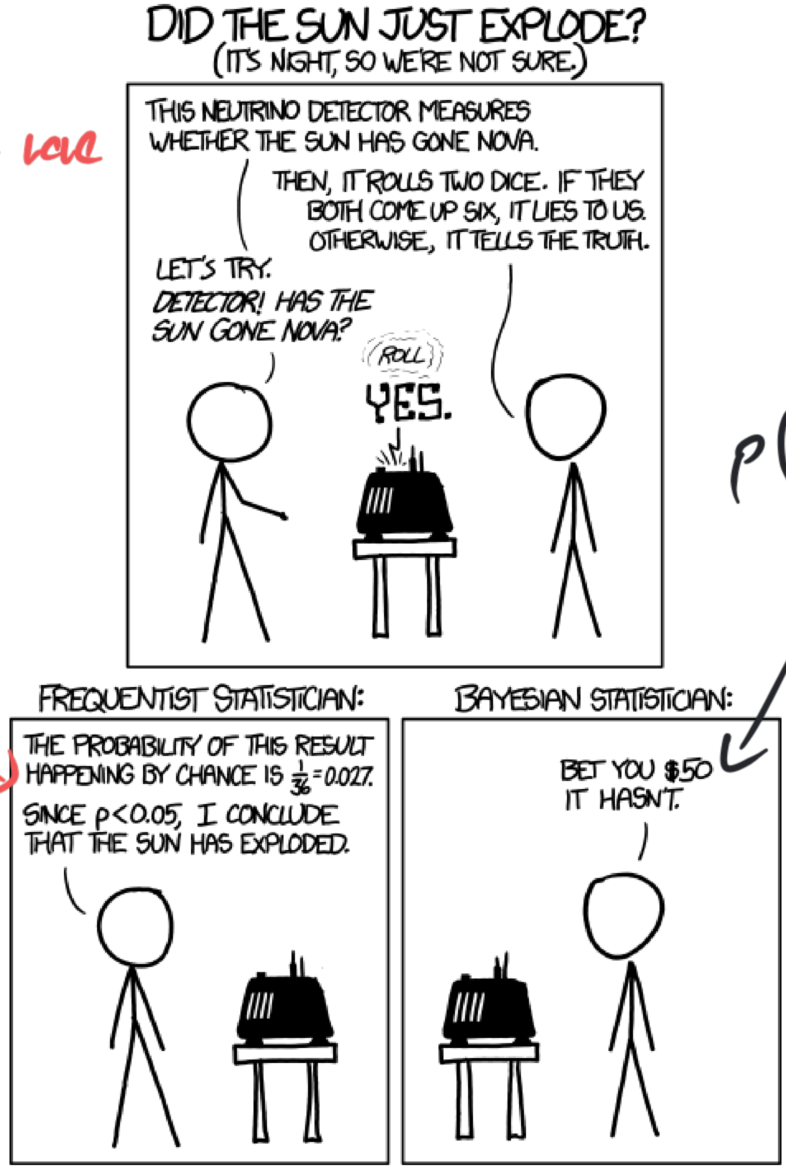



Figure: ©xkcd, https://xkcd.com/1132/

$p(d \mid \neg n)$ might be small

$p(n \mid d) = \frac{p(d \mid n)}{p(d)} p(n)$, $p(d \mid n) = \frac{35}{36} \sim 1$

$\approx \frac{p(n)}{p(d)} \ll 1$

# Problem 7.2 a

Suppose you are a witness to a nighttime hit-and-run accident involving a taxi in Athens. All taxis in Athens are blue or green. You swear, under oath, that the taxi was blue. Extensive testing shows that, under the dim lighting conditions, discrimination between blue and green is 75% reliable.

a. Is it possible to calculate the most likely color for the taxi?

# Problem 7.2 b

Suppose you are a witness to a nighttime hit-and-run accident involving a taxi in Athens. All taxis in Athens are blue or green. You swear, under oath, that the taxi was blue. Extensive testing shows that, under the dim lighting conditions, discrimination between blue and green is 75% reliable.

a. Is it possible to calculate the most likely color for the taxi?
b. What if you know that 9 out of 10 Athenian taxis are green?

# Problem 7.2 b

define $B$: true if the taxi is blue
$OB$: true if the observed color is blue

Bayes rule w. normalization

Previous results:
$P(ob|b) = 0.75$,
$P(\neg ob|b) = 0.25$,
$P(\neg ob|\neg b) = 0.75$,
$P(ob|\neg b) = 0.25$.
$P(b) = \frac{1}{10}$,
$P(\neg b) = \frac{9}{10}$.

$$P(b|ob) = \frac{P(ob|b)\cdot P(b)}{P(ob)} = \alpha \cdot P(ob|b)\,P(b)$$

$$= \alpha \cdot \frac{3}{4} \cdot \frac{1}{10} = \boxed{\alpha \cdot \frac{3}{40}} = 1/4$$

$$\alpha = \frac{1}{\frac{3}{4}\frac{1}{10} + \frac{9}{10}\cdot\frac{1}{4}} = \frac{10}{3}$$

$P(ob) = P(ob|b)\,P(b)$
$+ P(ob|\neg b)\,P(\neg b)$

$$P(\neg b|ob) = \frac{P(ob|\neg b)\cdot P(\neg b)}{P(ob)} = \alpha \cdot P(ob|\neg b)\cdot P(\neg b)$$

$$= \alpha \cdot \frac{1}{4}\cdot\frac{9}{10} = \frac{9}{40}\cdot\alpha$$

$$\frac{P(b|ob)}{P(\neg b|ob)} = \frac{\cancel{\alpha}\cdot 3/\cancel{40}}{\cancel{\alpha}\cdot 9/\cancel{40}} = \frac{1}{3}$$

# Problem 7.3 a

(Zero-knowledge proof) For Christmas, the siblings Josie and Andrew are each given a box of sweets from their grandmother. They look the same from the outside, but they know that one contains more chocolate while the other has more nuts. Both would like to have the box with more chocolate. Josie claims she can hear the difference when shaking the box with perfect accuracy, but Andrew doesn't believe her.

a. How can Josie convince Andrew she can indeed hear the difference without opening the present and without giving away which box has more chocolate?

# Problem 7.3 b

Based on experience, Andrew believes his sister lies 3 out of 10 times. How should he update his (conditional) belief about her statement after Josie has successfully mastered his test $n$ times?

T : true if Josie tells the truth (can hear the diff.)

S : true if Josie is successful in experiment

Information given:

$P(\neg t) = \frac{3}{10}$

$P(s \mid t) = 1$

$P(s \mid \neg t) = \frac{1}{2}$

$$P(\neg t \mid s_1, \dots, s_n) = \frac{P(s_1, \dots, s_n \mid \neg t)\, P(\neg t)}{P(s_1, \dots, s_n)} = \frac{\left(\frac{1}{2}\right)^n \frac{3}{10}}{\left(\frac{1}{2}\right)^n \frac{3}{10} + \frac{7}{10} \cdot 1^n}$$

$$= \frac{3}{3 + 7 \cdot 2^n}$$

$n = 1: p(\neg t \mid s^n) = \frac{3}{3+7 \cdot 2} = \frac{3}{17} \mathrel{\widehat{=}} 17\%$

$n = 0: p(\neg t \mid s^n) = \frac{3}{3+7 \cdot 2^0} = \frac{3}{10} = p(\neg t)$

$n = 10: p(\neg t \mid s^n) = \frac{3}{3+7 \cdot 2^{10}} \approx 0{,}04\%$

# Problem 7.3 c

translates to "and"

Given $n$ successful tests, what's the (unconditional) probability that Andrew is wrong when believing his sister?

$$\underbrace{P(\neg t, s_1, s_2, \dots, s_n)}_{a} = P(s_1, \dots, s_n \mid \neg t)\, P(\neg t) = \left(\tfrac{1}{2}\right)^n \tfrac{3}{10} \leq \left(\tfrac{1}{2}\right)^n = \underbrace{P(s_1, \dots, s_n \mid \neg t)}_{b}$$