Computer Vision II: Multiple View Geometry (IN2228)

Chapter 08 3D-3D Geometry

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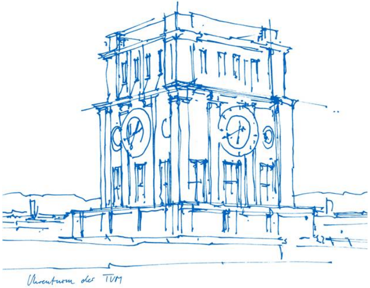


Explanation for Linear Systems of PnP

➢ Recap on System Generation

✓ DLT (direct, one-step)

$$\mathbf{t}_1^T\mathbf{P} - \mathbf{t}_3^T\mathbf{P}u_1 = 0,$$

$$\mathbf{t}_2^T\mathbf{P} - \mathbf{t}_3^T\mathbf{P}v_1 = 0.$$

Constraint of one correspondence

$$\Rightarrow \begin{pmatrix} \mathbf{P}_1^T & 0 & -u_1\mathbf{P}_1^T \\ 0 & \mathbf{P}_1^T & -v_1\mathbf{P}_1^T \\ \vdots & \vdots & \vdots \\ \mathbf{P}_N^T & 0 & -u_N\mathbf{P}_N^T \\ 0 & \mathbf{P}_N^T & -v_N\mathbf{P}_N^T \end{pmatrix} \boxed{\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{pmatrix}} = 0$$

Parameters of transformation

✓ EPnP (indirect, two-step)

$$\begin{cases} \sum_{j=1}^{4}\left(\alpha_{ij} f_x x_j^c + \alpha_{ij}\left(c_x - u_i\right) z_j^c\right) = 0 \\ \sum_{j=1}^{4}\left(\alpha_{ij} f_y y_j^c + \alpha_{ij}\left(c_y - v_i\right) z_j^c\right) = 0 \end{cases} \Rightarrow \mathbf{M}\mathbf{x} = 0$$

$2n \times 12$

Coordinates of control points

$$\boxed{\begin{matrix} \begin{bmatrix} x_j^c \\ y_j^c \\ z_j^c \end{bmatrix} \\ \cdots \end{matrix}} \quad \mathbf{c}_j, \quad j = 1, \cdots, 4$$

Explanation for Linear Systems of PnP

➢ Use Redundant Points to Improve Accuracy

- ✓ If we have prior knowledge that all the correspondences are inliers, we can use all the correspondences to generate an **over-determined** linear system.
- ✓ The result is the least-squared solution.
- ✓ It is helpful for noise compensation.

? 如果我们事先知道所有的对应关系都是异常值，我们可以使用所有的对应关系来生成一个超定点的线性系统。

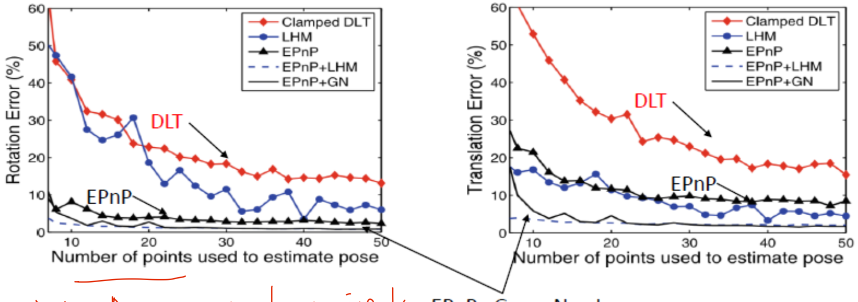
? 其结果是最小平方解。

? 它对噪声补偿很有帮助。

Explanation for Linear Systems of PnP

➢ Experimental Illustration of Redundant Case

✓ The more inlier points we use, the higher the algorithm accuracy is



points ↑ error ↓ noise ↓

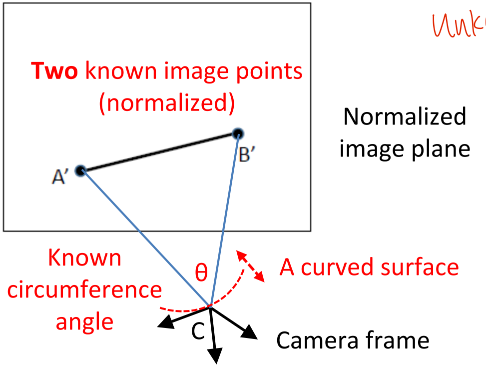
Explanation for 2-Point Configuration

- ➢ Recap on Our Analysis Method

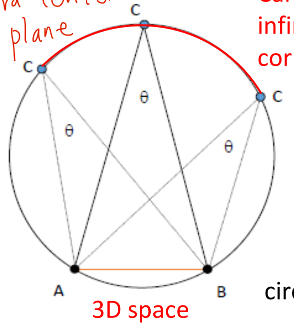
[?] 根据归一化的图像点计算圆周率的角度。

[?] 找到满足圆周角约束的最佳摄像机中心。

- ✓ Compute circumference angle based on the normalized image points.
- ✓ Find the optimal camera center satisfying the constraint of circumference angle.



Unknown: camera center
image plane



Camera positions? An infinite number of positions correspond to the angle θ

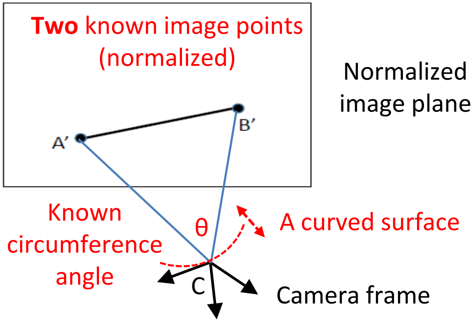
Theory of circumference angle

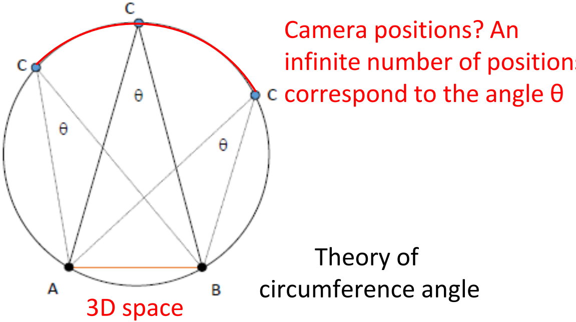
Explanation for 2-Point Configuration

➢ Recap on Our Analysis Method

✓ Can we enforce the constraint of distance (focal length)?

- No. We do **not know image plane**. We can treat image plane and camera center as a whole part.
- The angle is computed based on image points, but we should consider the relationship between 3D point and camera center (see right figure).





Today's Outline

- Overview of 3D-3D Geometry
- Non-iterative Method: SVD-based Method
- Iterative Method: Iterative closest point (ICP)

Overview of 3D-3D Geometry

从本质上讲，以下两种类型的表述是等同的。

? 第一类：第一和第二坐标系中都有N个点
例如：在EPnP中，四个控制点是静态的。我们的目标是确定它们在世界框架和摄像机框架中的坐标。

? 第二种类型：单一坐标系中的N+N点
例子：点组在单一坐标系中移动。

➢ Problem formulation

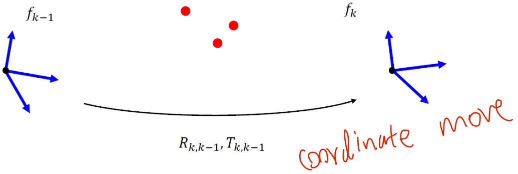
In essence, the following two types of formulations are equivalent.

✓ First type: *N* points in both first and second coordinate systems

Example: in **EPnP**, four control point are static. We aim to determine their coordinate in both world frame and camera frame.

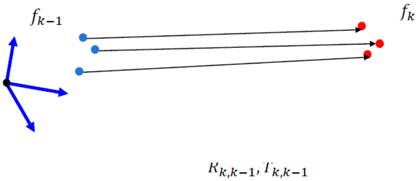
✓ Second type: *N*+*N* points in a single coordinate system

Example: Point set moves in a single coordinate system.



First type

3D points move



Second type

Overview of 3D-3D Geometry

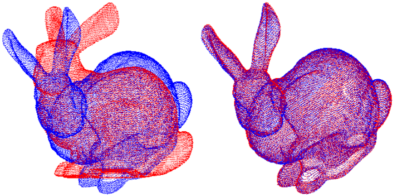
➢ Two Sub-problems

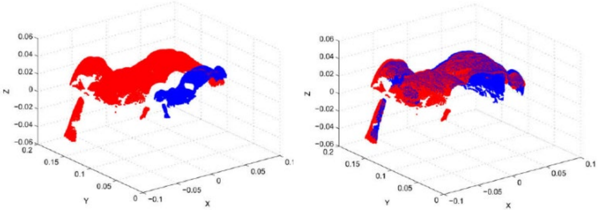
✓ 3D-3D Correspondence Establishment

✓ Transformation Estimation

- Case of SE(3)
- Case of Sim(3)

$$\mathrm{SE}(3) \quad \mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \quad \Rightarrow \quad \mathrm{Sim}(3) \quad \mathbf{T}_S = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$





Overview of 3D-3D Geometry

从三维到三维的特征对应中进行运动估计（也称为点云注册问题）

? 输入：两个点集$f f k k$- 1和$f k k$在三维中。它们是通过三角测量或立体视觉获得的。它们也可以是虚拟点（例如，EPnP中的控制点）。
? 最小情况下的解决方案涉及三个3D-3D点的对应关系。
? 解下列方程组，即未知的R和T：

➢ Intuitive Illustration

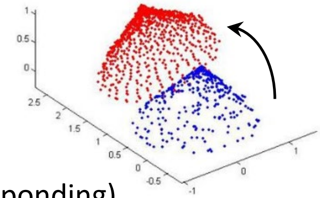
Motion estimation from 3D-to-3D feature correspondences (also known as point cloud registration problem)

- ✓ Input: Two point sets f_{k-1} and f_k in 3D. They are obtained by triangulation or stereo vision. They can also be virtual points (e.g., control points in EPnP).
- ✓ The minimal-case solution involves **three** 3D-3D point correspondences.
- ✓ Solving the following system of equations w.r.t. unknown R and T:

$$\begin{bmatrix} X^i_{k-1} \\ Y^i_{k-1} \\ Z^i_{k-1} \end{bmatrix} = \left[\boxed{\begin{matrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{matrix}} \; \boxed{\begin{matrix} t_1 \\ t_2 \\ t_3 \end{matrix}}\right] \cdot \begin{bmatrix} X^i_k \\ Y^i_k \\ Z^i_k \\ 1 \end{bmatrix}$$

where i is the feature ID.

Overview of 3D-3D Geometry



➢ Formal Definition

✓ Input: two point sets (we do not know which two points are corresponding)

$$X = \{x_1, ..., x_{N_x}\}$$

$$P = \{p_1, ..., p_{N_p}\}$$

Number of points are unnecessarily the same

✓ Goal: Find the optimal translation t and rotation R minimizing the sum of the squared error

$$E(R,t) = \frac{1}{N_p} \sum_{i=1}^{N_p} ||x_i - R p_i - t||^2$$

Unknown

Point to transform

where x_i and p_i are **unknown-but-sought** corresponding points.

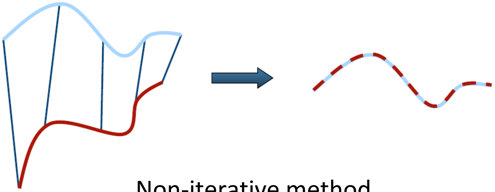
Overview of 3D-3D Geometry

➤ Two Configurations

[?] 如果正确的对应关系是已知的，正确的旋转和平移可以用封闭的形式计算出来（非迭代法）。

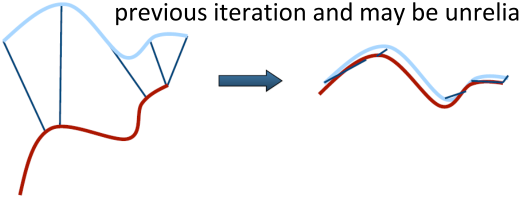
[?] 如果不知道正确的对应关系，通常不可能在一个步骤中确定最佳旋转和平移。我们必须进行迭代。

- ✓ If the correct correspondences are known, the correct rotation and translation can be calculated in closed form (non-iterative method).
- ✓ If the correct correspondences are not known, it is generally impossible to determine the optimal rotation and translation in one step. We have to perform **iterations**.



Non-iterative method
(closed-form method)

Correspondence is determined by the previous iteration and may be unreliable



Iterative method

Overview of 3D-3D Geometry

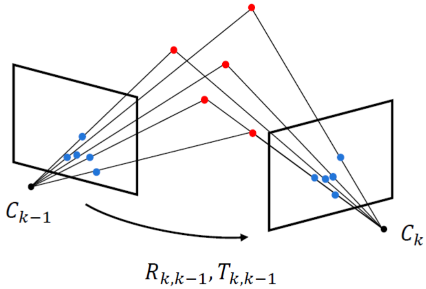
➢ Comparison with 2D-2D Geometry

Motion estimation from 2D-to-2D feature correspondences

✓ Both feature correspondences f_{k-1} and f_k are in image coordinates (2D)

✓ The minimal case solution involves 5 feature correspondences

✓ Popular algorithms:
- 8-point algorithm
- 5-point algorithm



Overview of 3D-3D Geometry

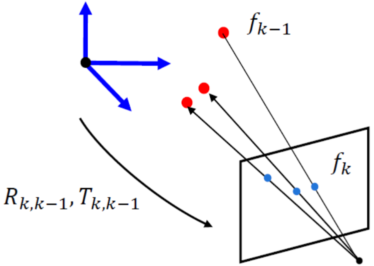
➢ Comparison with 3D-2D Geometry

Motion estimation from 3D-to-2D feature correspondences, i.e., Perspective-n-Points (PnP) problem)

✓ Feature f_{k-1} is in 3D and feature f_k in 2D

✓ Popular algorithms:

- DLT algorithm: at least 6 point correspondences
- P3P algorithm: minimal case with 3 point correspondences
- EPNP algorithm: at least 6 point correspondences



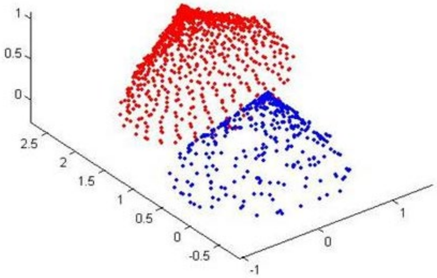
Non-iterative Method

- SE(3)

This case is mainly introduced today

- Sim(3)

✓ Horn's method [1]
✓ Umeyama's method [2]



[1] Berthold K. P. Horn, "Closed-form solution of absolute orientation using unit quaternions," in Journal of the Optical Society of America A, vol. 4, no. 2, pp. 629-642, 1987.
[2] Umeyama S. Least-squares estimation of transformation parameters between two point patterns. IEEE Trans Pattern Anal Mach Intell. 1991;13:376-380. doi:10.1109/34.88573.

Non-iterative Method

➢ Preprocessing Step

✓ Computing center of mass

$$\mu_x = \frac{1}{N_x}\sum_{i=1}^{N_x} x_i \quad \text{and} \quad \mu_p = \frac{1}{N_p}\sum_{i=1}^{N_p} p_i$$

Here, we can simply assume that $N_x = N_p$

✓ Point set normalization

We subtract the corresponding **center of mass** from each point in the two point sets

$$X' = \{x_i - \mu_x\} = \{x'_i\}$$
$$P' = \{p_i - \mu_p\} = \{p'_i\}$$

We use the normalized point sets to calculate the transformation.

Non-iterative Method

- Transformation Recovery

✓ Singular Value Decomposition

We compute matrix W by

$$W = \sum_{i=1}^{N_p} x_i' p_i'^T$$

We conduct the singular value decomposition (SVD) of W by:

$$W = U \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} V^T$$

where $\sigma_1 \geq \sigma_2 \geq \sigma_3$ are the singular values of W

Non-iterative Method

- Transformation Recovery

✓ Computation of rotation and translation

The optimal solution of transformation is unique and is given by:

$$
\begin{aligned}
R &= UV^T \\
t &= \mu_x - R\mu_p
\end{aligned}
$$

The conclusion is very precise, but how can we obtain this result? [1]

[1] "Least-Squares Fitting of Two 3-D Point Sets", K. S. Arun, T. S. Huang, and S. D. Blostein

Non-iterative Method

➢ Derivation Behind Conclusion

$$R = UV^T$$
$$\boxed{t = \mu_x - R\mu_p}$$

Previous conclusion

Due to limited, only some key steps are provided.

Center of mass

$$E(R,t) = \sum_{i=1}^{n} ||y_i - Rx_i - t||^2$$
$$= \sum_{i=1}^{n} ||y_i - Rx_i - t - y_o + y_o - Rx_o + Rx_o||^2$$

...

$$= \sum_{i=1}^{n} ||y_i - y_o - R(x_i - x_o)||^2 + n||y_o - Rx_o - t||^2$$

This part is only w.r.t R

Independent from specific points.
We can force this part to be 0. After obtaining R, we can obtain t

Non-iterative Method

➢ Derivation Behind Conclusion

Due to limited, only some key steps are provided.

$$R^* = \arg\min_R \sum_{i=1}^{n} \|y_i - y_o - R(x_i - x_o)\|^2$$

$$= \arg\min_R \sum_{i=1}^{n} \|y'_i - Rx'_i\|^2$$ Normalized points

$$= \arg\min_R \sum_{i=1}^{n} \left(y_i'^T y'_i + x_i'^T \boxed{R^T R} x'_i - 2y_i'^T R x'_i\right)$$ Expansion

$$= \arg\min_R \sum_{i=1}^{n} \left(-2y_i'^T R x'_i\right)$$ Neglect the part independent from R

$$= \arg\max_R \sum_{i=1}^{n} \left(y_i'^T R x'_i\right)$$ Reformulate a minimization problem as a maximization problem

...

$$= \arg\max_R trace\left(R \sum_{i=1}^{n} \boxed{x'_i y_i'^T}\right)$$

$$W = \sum_{i=1}^{N_p} \boxed{x'_i p_i'^T}$$

$$W = U \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} V^T$$

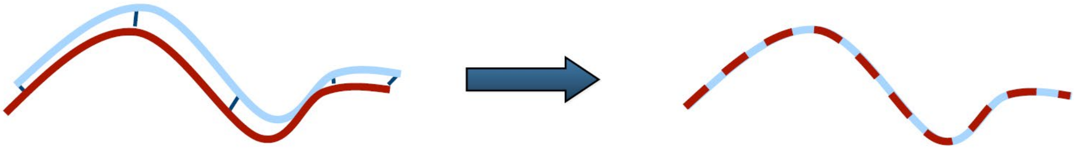
$$\boxed{R = UV^T}$$

Previous conclusion

Iterative closest point (ICP)

➢ Overview

- ✓ Idea: Iteratively align two point sets
- ✓ Iterative Closest Points (ICP) algorithm [1]
- ✓ Converges if corresponding points are "close enough"



[1] P. J. Besl and N. D. McKay, "A method for registration of 3-D shapes," in IEEE Transactions on Pattern Analysis and Machine Intelligence, vol. 14, no. 2, pp. 239-256, Feb. 1992

Iterative closest point (ICP)

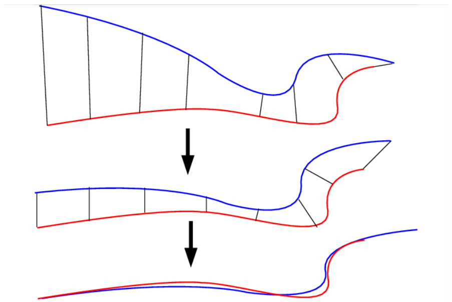
➢ Intuitive Illustration

主要问题是要确定正确的数据关联。我们将一对距离最小的点作为 "时间上 "的三维-三维对应关系。

鉴于关联点，可以使用SVD有效地计算转换。

✓ The major problem is to determine the correct data associations. We treat a pair of points with the smallest distance as a “temporal” 3D-3D correspondence.

✓ Given the associated points, the transformation can be computed efficiently using SVD.



A set of points is chosen along each line.
One point set (blue) is iteratively transformed to minimize the distance between each pair of points.

Iterative closest point (ICP)

➢ Detailed Procedures

✓ Determine corresponding points based on the smallest distance
✓ Compute rotation R, translation t via SVD
✓ Apply R and t to the points of the set to be registered
✓ Compute the error E(R,t)
✓ If error decreased and error > threshold
- Repeat these steps
- Stop and output final alignment, otherwise

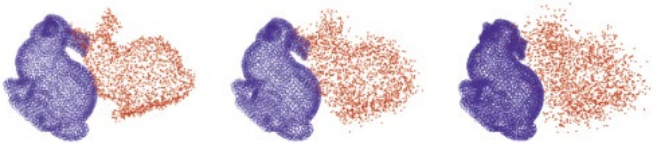
Iterative closest point (ICP)

➢ Variants

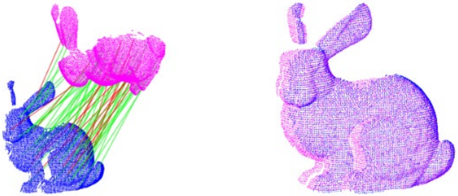
very noisy

✓ Several improvements have been proposed at different stages:

- Weighting the correspondences (mainly for high accuracy)
- Rejecting outlier point pairs (mainly for high robustness)



Some inlier correspondences are noisy. They should be assigned relatively small weights.



Outliers must be removed to correctly align point sets

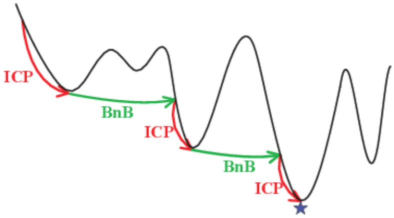
not asked in exam

Iterative closest point (ICP)

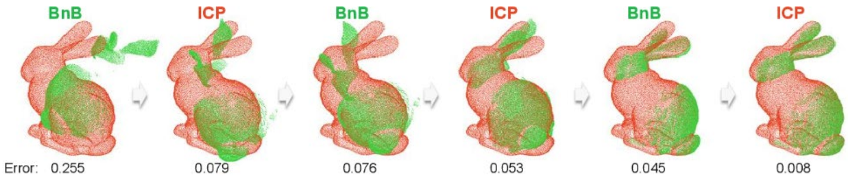
➢ Variants

✓ Several improvements have been proposed at different stages:

- Jump out of local minima based on global search method, i.e., branch-and-bound (BnB) (mainly for stability).
- Combine ICP and BnB to improve the efficiency of pure BnB.



Error evolution



Transformation of green point set (red point set remain unchanged)

Summary

- Overview of 3D-3D Geometry
- Non-iterative Method: SVD-based Method
- Iterative Method: Iterative closest point (ICP)

Thank you for your listening!

If you have any questions, please come to me :-)