

# FUNDAMENTALS OF ARTIFICIAL INTELLIGENCE

## Exercise 7: Probability theory and Bayesian networks – Solutions

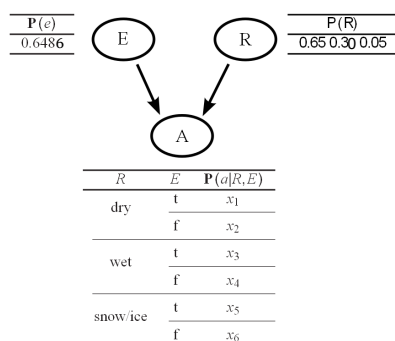
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### Problem 7.1:

- a. The missing probability can be computed knowing that the sum of all the elements of a full joint probability distribution of a set of random variables is 1.  
 $P(R = wet, e, \neg a) = 1 - (0.0607 + 0.0449 + 0.0084 + 0.3605 + 0.0240 + 0.0851 + 0.0654 + 0.0152 + 0.1435 + 0.0400 + 0.0022) = 0.1501$   
 (Note that  $A = true$  is abbreviated with  $a$ , and  $E = true$  with  $e$ ).
- b. The prior probability distribution of  $R$  is obtained by summing over all the possible combinations of the other random variables. This means that we simply have to sum over the columns of the table:  
 $P(R) = [P(R = dry), P(R = wet), P(R = snow/ice)]$  where  
 $P(R = dry) = 0.0607 + 0.3605 + 0.0851 + 0.1435 = 0.65$ ,  
 $P(R = wet) = 0.0449 + 0.1501 + 0.0654 + 0.0400 = 0.30$ ,  
 $P(R = snow/ice) = 1 - (0.65 + 0.30) = 0.05$ .  
 The same is done for  $E$ :  
 $P(E) = [P(e), P(\neg e)]$  where  
 $P(e) = 0.0607 + 0.0449 + 0.0084 + 0.3605 + 0.1501 + 0.0240 = 0.6486$ ,  
 $P(\neg e) = 1 - P(e) = 0.3514$ .
- c. The solution is obtained using the definition of conditional probability:  
 $P(\neg e|a, R = wet) = \frac{P(R=wet, \neg e, a)}{P(a, R=wet)} = \frac{P(R=wet, \neg e, a)}{P(e, a, R=wet) + P(\neg e, a, R=wet)} = \frac{0.0654}{0.0449 + 0.0654} \approx 0.5929$ .
- d. Each random variable is a node of the Bayesian network (BN). The random variables  $R$  and  $E$  can be considered as independent. Additionally, they directly influence  $A$  and thus they are represented as its parents in the BN. Usually we let the causes precede effects for the placement of the nodes. As usual, we write in the conditional probability tables only the values for  $E = true$  and  $A = true$ .

By applying the chain rule, it can be written the following:  $P(A, R, E) = P(A|R, E) \cdot P(R|E) \cdot P(E)$ . Since  $R$  and  $E$  are independent,  $P(R|E) = P(R)$ . Therefore,  $P(A|R, E) = \frac{P(A, R, E)}{P(R) \cdot P(E)}$ . (Note that snow/ice is abbreviated with  $s/i$ .)



$$\begin{aligned}
 x_1 &= P(a|R = dry, e) = \frac{P(R=dry, e, a)}{P(R=dry, e)} = \frac{P(R=dry, e, a)}{P(R=dry) \cdot P(e)} = \frac{0.0607}{0.65 \cdot 0.6486} = 0.1440 \\
 x_2 &= P(a|R = dry, \neg e) = \frac{P(R=dry, \neg e, a)}{P(R=dry, \neg e)} = \frac{P(R=dry, \neg e, a)}{P(R=dry) \cdot P(\neg e)} = \frac{0.0851}{0.65 \cdot 0.3514} = 0.3726 \\
 x_3 &= P(a|R = wet, e) = \frac{P(R=wet, e, a)}{P(R=wet, e)} = \frac{P(R=wet, e, a)}{P(R=wet) \cdot P(e)} = \frac{0.0449}{0.30 \cdot 0.6486} = 0.2309 \\
 x_4 &= P(a|R = wet, \neg e) = \frac{P(R=wet, \neg e, a)}{P(R=wet, \neg e)} = \frac{P(R=wet, \neg e, a)}{P(R=wet) \cdot P(\neg e)} = \frac{0.0654}{0.30 \cdot 0.3514} = 0.6204 \\
 x_5 &= P(a|R = s/i, e) = \frac{P(R=s/i, e, a)}{P(R=s/i, e)} = \frac{P(R=s/i, e, a)}{P(R=s/i) \cdot P(e)} = \frac{0.0084}{0.05 \cdot 0.6486} = 0.2590 \\
 x_6 &= P(a|R = s/i, \neg e) = \frac{P(R=s/i, \neg e, a)}{P(R=s/i, \neg e)} = \frac{P(R=s/i, \neg e, a)}{P(R=s/i) \cdot P(\neg e)} = \frac{0.0152}{0.05 \cdot 0.3514} = 0.8651
 \end{aligned}$$

### Problem 7.2:

- a. The answer is no. To calculate this posterior probability we should have a useful prior distribution of the random variables for the color of the taxis to use Bayes' rule.
- b. To answer this question we have to compute the probability that the taxi of the accident has a particular color given our observation (posterior). We first define two Boolean random variables:  $B$  which is true if the taxi

is blue (abbreviated with  $b$ ) and false if the taxi is green ( $\neg b$ ) and  $OB$  which is true if the observed color is blue (abbreviated with  $ob$ ) and false if it is green ( $\neg ob$ ). Remember that all the taxis are green or blue and that we swear that the taxi was blue. Actually, our reliability is only 75% for discriminating blue and green taxis. We can write this as:

$$P(ob|b) = 0.75 \text{ and } P(\neg ob|b) = 0.25,$$

$$P(\neg ob|\neg b) = 0.75 \text{ and } P(ob|\neg b) = 0.25.$$

We also know the prior distribution of  $B$ :

$$P(b) = \frac{1}{10} \text{ and } P(\neg b) = \frac{9}{10}.$$

In order to compute the posterior probability  $P(b|ob)$  we use Bayes' rule with normalization.

$$P(b|ob) = \alpha P(ob|b) P(b)$$

$$P(\neg b|ob) = \alpha P(ob|\neg b) P(\neg b)$$

$$\alpha [P(ob|b) P(b) + P(ob|\neg b) P(\neg b)] = 1$$

$$\alpha = 1 / (\frac{3}{4} \frac{1}{10} + \frac{1}{4} \frac{9}{10}) = \frac{10}{3}$$

Result:

$$P(b|ob) = \frac{10}{3} \frac{3}{4} \frac{1}{10} = \frac{1}{4}.$$

This means that the probability that the taxi was blue given our observation is 25%. It is more likely that the taxi was green!

### Problem 7.3:

- a. Andrew gives Josie one of the boxes and allows her to shake it. Then he leaves the room and, without her seeing it, either exchanges the box or comes back with the same one. He then asks Josie if the box makes the same noise as before. By repeating this process over and over again, it becomes increasingly unlikely that Josie is guessing the truth answer correctly. ~~Still~~, Andrew does not know which of the boxes contains more chocolate.
- b. We start by introducing two Boolean variables:  $T$  indicates whether Josie is telling the truth ( $T = \text{true}$  abbreviated with  $t$ ) or not ( $\neg t$ ) and  $S$  indicates whether she successfully determines the correct answer to Andrews question ( $S = \text{true}$ , abbreviated with  $s$ ) or fails his test ( $\neg s$ ). We know that:

$$P(t) = 0.70,$$

$$P(s|t) = 1,$$

$$P(s|\neg t) = 0.50 \text{ if Josie guesses randomly.}$$

The probability that Josie is truthfull, given  $n$  succesfull trials  $(s_1, \dots, s_n)$  can be determined using Bayes' rule with normalization:

$$P(t | s_1, \dots, s_n) = \alpha P(s_1, \dots, s_n | t) P(t) = 0.7\alpha, \text{ where}$$

$$\alpha = \frac{1}{P(s_1, \dots, s_n | t) P(t) + P(s_1, \dots, s_n | \neg t) P(\neg t)} = \frac{1}{P(t) + P(s | \neg t)^n P(\neg t)}$$

$$P(t | s_1, \dots, s_n) = \frac{\frac{7}{10}}{\frac{7}{10} + \frac{3}{10} \left(\frac{1}{2}\right)^n} = \frac{7}{7 + 3 \frac{1}{2^n}}$$

$$P(\neg t | s_1, \dots, s_n) = 1 - P(t | s_1, \dots, s_n) = \frac{3}{3 + 7 \cdot 2^n}$$

- c. We are looking for the probability that Josie was lying and still gives the right answer  $n$  times  $P(s_1, \dots, s_n, \neg t)$ . This probability relates to the conditional probability that she gets the answer right given she was lying via  $P(s_1, \dots, s_n, \neg t) = P(s_1, \dots, s_n | \neg t) P(\neg t)$ , which we can evaluate to  $P(s_1, \dots, s_n, \neg t) = (1/2)^n 3/10$ .<sup>1</sup>

<sup>1</sup>While semantically similar, there's a difference in asking for the probability that the test will lead to Andrew wrongfully believing his sister and asking for the probability he believes her while assuming that Josie is lying. The first case accounts for the possibility that Josie is telling the truth, so consequentially  $P(s_1, \dots, s_n, \neg t) \leq P(s_1, \dots, s_n | \neg t)$ .

### Problem 7.4:

- a. We first define two Boolean random variables: CP indicates whether the robot is in a correct pose ( $CP = true$  abbreviated with  $cp$ ) or not ( $\neg cp$ ) and PM that indicates whether a measurement for correct pose is positive ( $PM = true$  abbreviated with  $pm$ ) or not ( $\neg pm$ ). We can write what we know about the reliability of each measurement as:

$$P(pm|cp) = 0.80,$$

$$P(\neg pm|\neg cp) = 0.80.$$

We also know the prior probability of a correct pose of the robot:

$$P(cp) = 0.45 \text{ and thus } P(\neg cp) = 0.55.$$

The posterior probability that the robot is in a good pose given that we have a positive measurement can be computed using Bayes' rule with normalization as

$$P(cp|pm) = \alpha P(pm|cp) P(cp),$$

$$P(\neg cp|pm) = \alpha P(\neg pm|\neg cp) P(\neg cp).$$

Considering that  $P(pm|cp) = 1 - P(\neg pm|\neg cp) = 0.2$ , we can finally answer the question:

$$\alpha [P(pm|cp) P(cp) + P(\neg pm|\neg cp) P(\neg cp)] = 1$$

$$\alpha = 1/(0.8 \cdot 0.45 + 0.2 \cdot 0.55) \approx 2.1277$$

Result:

$$P(cp|pm) \approx 2.1277 \cdot 0.8 \cdot 0.45 \approx 0.7660.$$

- b. We introduce an additional random variable for the result of the second measurement  $PM2$  (abbreviated with  $pm2$  when true and  $\neg pm2$  when false). What we need is the probability that the robot is in a good pose given two positive measurements. This can be written as:

$$P(cp|pm, pm2) = \alpha P(pm, pm2|cp) P(cp).$$

The two measurements are not independent in general because they both depend from the pose of the robot.

We can assume that given a fixed pose of the robot, the two measurements are independent (conditional independence). This simplifies the previous equation and lets us write:

$$P(cp|pm, pm2) = \alpha P(pm|cp) P(pm2|cp) P(cp),$$

$$P(\neg cp|pm, pm2) = \alpha P(\neg pm|\neg cp) P(\neg pm2|\neg cp) P(\neg cp).$$

$$\alpha [P(pm|cp) P(pm2|cp) P(cp) + P(\neg pm|\neg cp) P(\neg pm2|\neg cp) P(\neg cp)] = 1$$

$$\alpha = 1/(0.8 \cdot 0.8 \cdot 0.45 + 0.2 \cdot 0.2 \cdot 0.55) \approx 3.2258$$

Result:

$$P(cp|pm, pm2) \approx 3.2258 \cdot 0.8 \cdot 0.8 \cdot 0.45 \approx 0.9290.$$

The answer is yes, the robot can now collect a sample.

### Problem 7.5:

- a. This is true for any nontrivial solution  $P(a|b) \neq 0 \wedge P(b|a) \neq 0$ .

$$P(a, b, c) = P(a|b, c) P(b|c) P(c) = P(b|a, c) P(a|c) P(c).$$

$$\text{If } P(a|b, c) = P(b|a, c):$$

$$\cancel{P(a|b, c)} P(b|c) \cancel{P(c)} = \cancel{P(b|a, c)} P(a|c) \cancel{P(c)}$$

$$P(b|c) = P(a|c). \quad \square$$

- b. This is not always true.

Let us consider e.g. the case that:  $P(a) = \frac{1}{K_a}$ ,  $P(b) = \frac{1}{K_b}$ ,  $P(c) = \frac{1}{K_c}$ , with  $K_i > 0$  for  $i = a, b, c$ .

$$\text{If } P(a, b, c) > \frac{1}{K_a K_b K_c},$$

$$P(a, b, c) = P(a|b, c) P(b|c) P(c) > \frac{1}{K_a K_b K_c}, \text{ thus}$$

$$P(b|c) > \frac{1}{K_a K_b K_c} \frac{1}{P(c) P(a|b, c)}.$$

Now imposing  $P(a|b, c) = P(a)$  we get:

$$P(b|c) > \frac{1}{K_a K_b K_c} \frac{1}{P(c) P(a)},$$

$$P(b|c) > \frac{1}{K_b}, \text{ and thus } P(b|c) \neq P(b). \quad \square$$

- c. This is not always true.

Let us consider three Boolean random variables A, B and C and assume independence between A and B.

For compactness we use the notation  $a$  for  $A = true$  and  $\neg a$  for  $A = false$ , the same for the other random

variables.

Consider the case that the random variable  $C$  assumes the following values:

- True if  $\langle a, b \rangle$  or  $\langle \neg a, \neg b \rangle$ ;
- False if  $\langle \neg a, b \rangle$  or  $\langle a, \neg b \rangle$ .

$$P(c|a, b) = 1, P(\neg c|a, b) = 0;$$

$$P(c|\neg a, \neg b) = 1, P(\neg c|\neg a, \neg b) = 0;$$

$$P(c|\neg a, b) = 0, P(\neg c|\neg a, b) = 1;$$

$$P(c|a, \neg b) = 0, P(\neg c|a, \neg b) = 1.$$

We can now calculate the prior probability of  $c$  as:

$$P(c) = P(c|a, b)P(a, b) + P(c|\neg a, \neg b)P(\neg a, \neg b) + P(c|\neg a, b)P(\neg a, b) + P(c|a, \neg b)P(a, \neg b).$$

Exploiting the independence between  $A$  and  $B$  we can write:

$$P(c) = P(a)P(b) + P(\neg a)P(\neg b).$$

Let us introduce numbers:

$$P(a) = \frac{1}{2},$$

$$P(b) = \frac{1}{2},$$

$$P(c) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

In this case we obtain:

$$P(a, c) = P(c|a, b)P(a)P(b) + P(c|a, \neg b)P(a)P(\neg b) = \frac{1}{4},$$

$$P(b, c) = P(c|a, b)P(a)P(b) + P(c|\neg a, b)P(\neg a)P(b) = \frac{1}{4},$$

$$P(a, b, c) = P(c|a, b)P(a)P(b) = \frac{1}{4}.$$

It is now possible to show that in this case the statement of this exercise is not true:

$$P(a|b, c) = \frac{P(a, b, c)}{P(b, c)} = 1 \neq P(a|c) = \frac{P(a, c)}{P(c)} = \frac{1}{2}. \quad \square$$