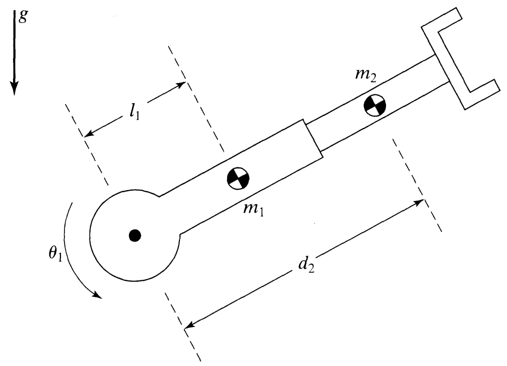


Figure 1: *RP Robot (Problem 2)*

Problem 1

For the RP manipulator shown in Figure 1, we assume the following parameters:

$$l_1 = 0.2, m_1 = 1.$$

a) Determine the matrices M, V, G of the state space form of the dynamic equations using Lagrange's method, assuming that the inertia tensors are

$$^{C_1}I_1 = \begin{pmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & 0.1 \end{pmatrix}, \quad ^{C_2}I_2 = \begin{pmatrix} I_{xx2} & 0 & 0 \\ 0 & 0.07 & 0 \\ 0 & 0 & I_{zz2} \end{pmatrix}.$$

b) The system is operated through a model-driven PD controller. Determine the form of the matrices α and the vectors β and τ', treating the factors k_{vi} and k_{pi} as variables.

c) Determine values of k_{vi} and k_{pi} such that closed-loop frequencies are 20 rad/s and 25 rad/s for both joints and such that the system is critically damped.

d) Draw a block diagram of the controller.

1 a)

$$k_i = \frac{1}{2} m_i V_{c_i}^T V_{c_i} + \frac{1}{2} {}^i w_i^T \, {}^{c_i} I_i \, {}^i w_i$$

$$u_i = -m_i \, {}^0 g^T \, {}^0 P_{c_i} + u_{ref_i}$$

$$\frac{d}{dt} \cdot \frac{\partial k}{\partial \dot{\theta}} - \frac{\partial k}{\partial \theta} + \frac{\partial u}{\partial \theta} = \tau$$

$$u_1 = m_1 \cdot g \cdot L_1 \cdot \sin\theta_1 + m_1 \cdot g \cdot L_1$$

$$u_2 = m_2 \cdot g \cdot d_2 \cdot \sin\theta_1 + m_2 \cdot g \cdot d_{2max}$$

$$k_1 = \frac{1}{2} m_1 v_{c_1}^2 + \frac{1}{2} {}^{c_1} I_1 \cdot {}^1 w_1^2$$

$$= \frac{1}{2} m_1 \cdot (L_1 \cdot {}^1 w_1)^2 + \frac{1}{2} I_{zz_1} \cdot {}^1 w_1^2$$

$$= \frac{1}{2} \cdot 1 \cdot (0.2 \cdot \dot{\theta}_1)^2 + \frac{1}{2} \cdot 0.1 \cdot \dot{\theta}_1^2$$

$$= 0.02 \dot{\theta}_1^2 + 0.05 \dot{\theta}_1^2$$

$$= 0.07 \dot{\theta}_1^2$$

要用矢量

$$k_2 = \frac{1}{2} m_2 \cdot V_{c_2}^2 + \frac{1}{2} I_{zz2} \cdot {}^1 w_1^2$$

$$= \frac{1}{2} m_2 \cdot \left(\sqrt{(\dot{d}_2)^2 + (d_2 \cdot \dot{\theta}_2)^2} \right)^2 + \frac{1}{2} I_{zz2} \cdot \dot{\theta}_1^2$$

$$= \frac{1}{2} m_2 \cdot \left(\dot{d}_2^2 + d_2^2 \cdot \dot{\theta}_1^2 \right) + \frac{1}{2} I_{zz2} \cdot \dot{\theta}_1^2$$

$$K = \left(0.07 + \frac{1}{2} m_2 \cdot d_2^2 + \frac{1}{2} \overset{I_{yy2}}{\cancel{I_{zz2}}}\right) \cdot \dot{\theta}_1^2 + \frac{1}{2} m_2 \cdot \dot{d}_2^2$$

$$u = m_1 g L_1 \cdot \sin\theta_1 + m_2 \cdot g \cdot d_2 \cdot \sin\theta_1 + m_1 g l_1 + m_2 g \cdot d_{2max}$$

$$\frac{\partial k}{\partial \dot{\theta}} = \begin{pmatrix} (0.14 + m_2 \cdot d_2^2 + I_{\cancel{zz}2}^{yy}) \cdot \dot{\theta}_1 \\ m_2 \cdot \dot{d}_2 \end{pmatrix}$$

$$\frac{\partial k}{\partial \theta} = \begin{pmatrix} 0 \\ m_2 \cdot \dot{\theta}_1^2 \cdot d_2 \end{pmatrix}$$

$$\frac{\partial u}{\partial \theta} = \begin{pmatrix} m_1 \cdot g \cdot l_1 \cdot \cos\theta_1 + m_2 \cdot g \cdot d_2 \cdot \cos\theta_1 \\ m_2 \cdot g \cdot \sin\theta_1 \end{pmatrix}$$

chain rule

$$\tau = \begin{pmatrix} (0.14 + m_2 \cdot d_2^2 + I_{zz2}) \cdot \ddot{\theta}_1 + \underline{2 m_2 \cdot \dot{\theta}_1 d_2 \cdot \dot{d}_2} + (m_1 \cdot l_1 + m_2 \cdot d_2) \cdot g \cdot \cos\theta_1 \\ m_2 \cdot \ddot{d}_2 - m_2 \cdot \dot{\theta}_1^2 \cdot d_2 + m_2 \cdot g \cdot \sin\theta_1 \end{pmatrix}$$

$$M(\theta) = \begin{pmatrix} 0.14 + m_2 \cdot d_2^2 + I_{yy2} & 0 \\ 0 & m_2 \end{pmatrix}$$

$$V(\dot{\theta}, \theta) = \begin{pmatrix} 2 m_2 \cdot \dot{\theta}_1 d_2 \cdot \dot{d}_2 \\ -m_2 \cdot \dot{\theta}_1^2 \cdot d_2 \end{pmatrix}$$

$$G(\theta) = \begin{pmatrix} (m_1 l_1 + m_2 \cdot d_2) \cdot g \cdot \cos\theta_1 \\ m_2 \cdot g \cdot \sin\theta_1 \end{pmatrix}$$

b) $\tau = \alpha \tau' + \beta$ 10-12

$$\begin{cases} \alpha = M(\Theta) \\ \beta = V(\Theta, \dot{\Theta}) + G(\Theta) + F(\Theta, \dot{\Theta}) \end{cases}$$

$$M(\Theta) = \begin{pmatrix} 0.14 + m_2 \cdot d_2^2 + Zyy_2 & 0 \\ 0 & m_2 \end{pmatrix}$$

$$\tau' = \ddot{\Theta}_d + k_v \dot{E} + k_p E$$

$$V(\dot{\Theta}, \Theta) = \begin{pmatrix} 2 m_2 \cdot \dot{\theta}_1 d_2 \cdot \dot{d}_2 \\ -m_2 \cdot \dot{\theta}_1^2 \cdot d_2 \end{pmatrix}$$

$$E = \Theta_d - \Theta$$

$$G(\Theta) = \begin{pmatrix} (m_1 l_1 + m_2 \cdot d_2) \cdot g \cdot \cos\theta_1 \\ m_2 \cdot g \cdot \sin\theta_1 \end{pmatrix}$$

---

c) $\ddot{\Theta} = \tau' = \ddot{\Theta}_d + k_v \dot{E} + k_p E \iff \ddot{E} + k_v \dot{E} + k_p E = 0$

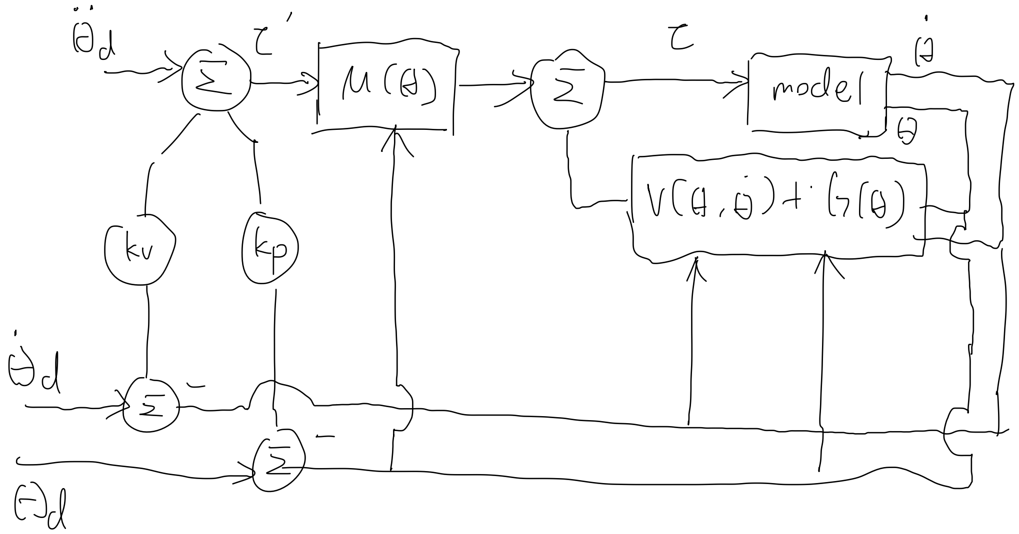
$$\begin{cases} \ddot{e}_1 + k_{v_1} \dot{e}_1 + k_{p_1} e_1 = 0 \\ \ddot{e}_2 + k_{v_2} \dot{e}_2 + k_{p_2} e_2 = 0 \end{cases} \Rightarrow s^2 + 2\xi w_n \cdot s + w_n^2 = 0$$

$w_n = \sqrt{k_{p_i}}$ critically damped $\iff \Delta = 0$

$k_{v_i}^2 - 4k_{p_i} = 0 \iff k_{v_i} = 2\sqrt{k_{p_i}}$

$$\begin{matrix} w_{n_1} = 20 \\ w_{n_2} = 25 \end{matrix} \Rightarrow \begin{matrix} k_{p_1} = 400 & k_{v_1} = 40 \\ k_{p_2} = 625 & k_{v_2} = 50 \end{matrix}$$

d)



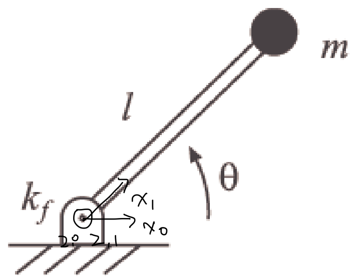


Figure 2: *Simple Robot with mass at distal end of link.*

Problem 2

$I = 0$

Consider the robot shown in Figure 2. The robot has only one joint and one link with lenght l, and at the distal end of the link there is a point mass m. The mass of the link is neglected, thus, the center of mass is also located at the end of the link. The joint is affected by friction with a friction constant k_f. The inertia tensor associated with the link is denoted by I_m. You do not need to consider gravity.

a) Determine the equations of motion for this system. The computation of the inertia tensor can be performed easily if the following formula for an accumulation of point-shaped masses is used:

$$I = \sum_i m_i \begin{pmatrix} y_i^2 + z_i^2 & -x_i y_i & -x_i z_i \\ -y_i x_i & x_i^2 + z_i^2 & -y_i z_i \\ -z_i x_i & -z_i y_i & x_i^2 + y_i^2 \end{pmatrix}$$

b) Assume that a desired position Θ_d has been specified. Design a closed-loop controller that uses only $\Theta(t), \dot{\Theta}(t)$ and receives Θ_d as input.

c) Draw a block diagram of the controller.

DH

	α	a	d	θ
1	0	0	0	θ_1

$${}^{i-1}_{\ i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1_0T = \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^0\omega_0 = {}^0\dot\omega_0 = {}^0V_0 = {}^0\dot V_0 = {}^2f_2 = {}^2n_2 = 0$$

$${}^1P_{c_1} = \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix} \quad {}^{c_1}I_1 = 0$$

$${}^1\omega_1 = \cancel{{}^1_0R\,{}^0\omega_0} + \dot\theta_1 \cdot {}^1\hat Z_1 = \begin{pmatrix} 0 \\ 0 \\ \dot\theta_1 \end{pmatrix}$$

$${}^1\dot\omega_1 = {}^1_0R\,\cancel{{}^0\dot\omega_0} + {}^1_0R\,\cancel{{}^0\omega_0 \times \dot\theta_1\,{}^1\hat Z_1} + \ddot\theta_1 \cdot {}^1\hat Z_1 = \begin{pmatrix} 0 \\ 0 \\ \ddot\theta_1 \end{pmatrix}$$

$${}^1\dot V_1 = {}^1_0R(\cancel{{}^0\dot\omega_0} \times {}^0P_1 + \cancel{{}^0\omega_0 \times ({}^0\omega_0 \times {}^0P_1)} + {}^0\dot V_0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$${}^1\dot V_{c_1} = {}^1\dot\omega_1 \times {}^1P_{c_1} + {}^1\omega_1 \times ({}^1\omega_1 \times {}^1P_{c_1}) + {}^1\dot V_1 = \begin{pmatrix} 0 \\ 0 \\ \ddot\theta_1 \end{pmatrix} \times \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot\theta_1 \end{pmatrix} \times \left(\begin{pmatrix} 0 \\ 0 \\ \dot\theta_1 \end{pmatrix} \times \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 0 \\ l \cdot \ddot\theta_1 \\ 0 \end{pmatrix} + \begin{pmatrix} -l \cdot \dot\theta_1^2 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -l \cdot \dot\theta_1^2 \\ l \cdot \ddot\theta_1 \\ 0 \end{pmatrix}$$

$${}^1F_1 = m_1 \cdot {}^1\dot V_{c_1} = \begin{pmatrix} -ml \cdot \dot\theta_1^2 \\ ml \cdot \ddot\theta_1 \\ 0 \end{pmatrix}$$

$${}^1N_1 = \cancel{{}^{c_1}I_1}\,{}^1\dot\omega_1 + {}^1\omega_1 \times \cancel{{}^{c_1}I_1}\,{}^1\omega_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$${}^1f_1 = {}^1_2R\,\cancel{{}^2f_2} + {}^1F_1 = \begin{pmatrix} -m \cdot l \cdot \dot\theta_1^2 \\ ml \cdot \ddot\theta_1 \\ 0 \end{pmatrix}$$

$${}^1n_1 = \cancel{{}^1N_1} + \cancel{{}^1_2R}\,{}^2n_2 + {}^1P_{c_1} \times {}^1F_1 + {}^1P_2 \times \cancel{{}^1_2R\,{}^2f_2} = \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} -ml \cdot \dot\theta_1^2 \\ m \cdot l \cdot \ddot\theta_1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ ml^2 \cdot \ddot\theta_1 \end{pmatrix}$$

$$\tau_1 = ml^2\ddot\theta_1 \quad \Rightarrow M(\theta) = ml^2$$

$$\tau = \alpha\tau' + \beta$$

$$\alpha = M(\theta) = ml^2$$

$$\tau' = \ddot\theta_1$$

$$\beta = V(\cancel{\theta, \dot\theta}) + \cancel{G(\theta)} + F(\theta, \dot\theta) = k_f \cdot \dot\theta$$

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b) $\ddot\theta = \tau' = \ddot\theta_d + k_v\dot E + k_p E$ $\qquad E = \theta_d - \theta$

$\Rightarrow$ only θ_d, $\dot\theta_d$ and $\ddot\theta_d = 0$

$$\Rightarrow -\ddot\theta \underbrace{- k_v \cdot \dot\theta + k_p e}_{= \tau'} = 0$$