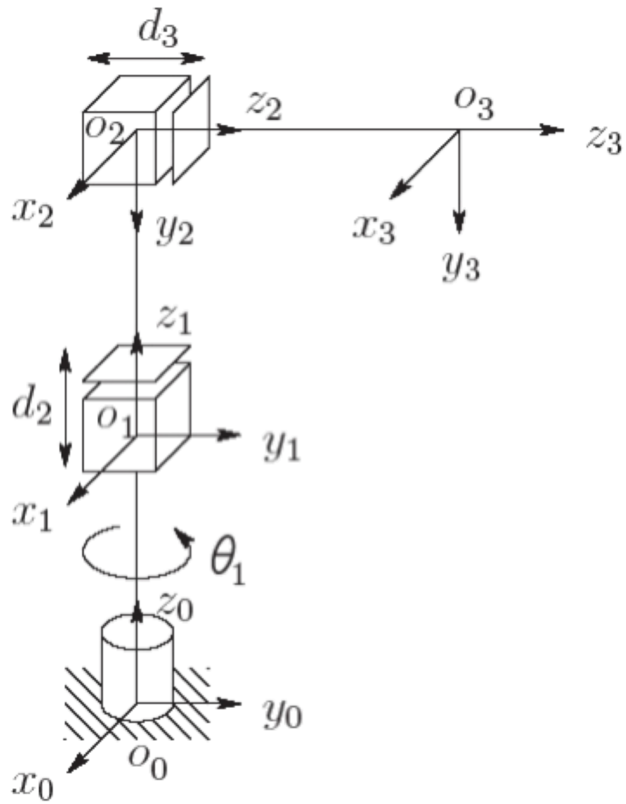
Robot Dynamics

Example: three link cylindrical robot

- Up to this point, we have developed a systematic method to determine the forward and inverse kinematics and the Jacobian for any arbitrary serial manipulator
  - Forward kinematics: mapping from joint variables to position and orientation of the end effector
  - Inverse kinematics: finding joint variables that satisfy a given position and orientation of the end effector
  - Jacobian: mapping from the joint velocities to the end effector linear and angular velocities
- Example: three link cylindrical robot



General system overview

- Ex: position control…

Dynamics Overview

- We want to come up with equations of motion for any *n*DOF system
  - In general, this will consist of *n* coupled second order ODEs
- These systems may be:
  - Linear or nonlinear
  - Conservative or nonconservative
- We want to develop an expression of the form: $\dot{q} = f(q,t)$
- Once we have this, we can use it to choose an appropriate controller that will put our dynamical system in a desired state (configuration)

Euler-Lagrange Equations

- We can derive the equations of motion for any *n*DOF system by using energy methods
  - All we need to know are the conservative (kinetic and potential) and non-conservative (dissipative) terms
- This is a shortcut to describing the motion of each particle in a rigid body along with the constraints that form rigid motions
- For this, we need to first use virtual displacements subject to holonomic constraints, then use the principle of virtual work, then finally use D'Alembert's Principle to derive the Euler-Lagrange equations of motion
- But first, an example…

Ex: 1DOF system

- To illustrate, we derive the equations of motion for a 1DOF system
  - Consider a particle of mass *m*
  - Using Newton's second law:

$$m\ddot{y} = f - mg$$

  - Now define the kinetic and potential energies:

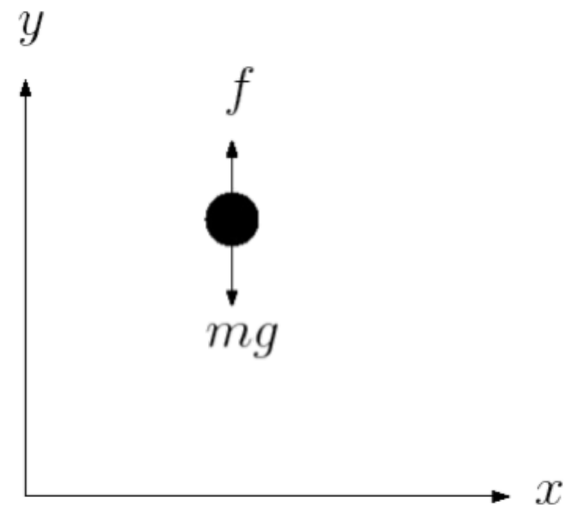
$$K = \frac{1}{2}m\dot{y}^2 \qquad P = mgy$$

  - Rewrite the above differential equation
    - Left side:

$$m\ddot{y} = \frac{d}{dt}\left(m\dot{y}\right) = \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\dot{y}^2\right) = \frac{d}{dt}\frac{\partial K}{\partial \dot{y}}$$

    - Right side:

$$mg = \frac{\partial}{\partial y}\left(mgy\right) = \frac{\partial P}{\partial y}$$



$$\frac{\partial \frac{1}{2}m\dot{y}^2}{\partial \dot{y}} = m\ddot{y} =$$

Ex: 1DOF system

- Thus we can rewrite the initial equation:

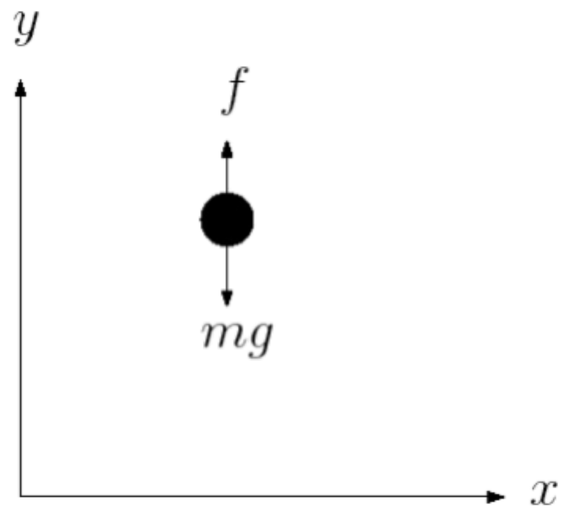
$$\frac{d}{dt}\frac{\partial K}{\partial \dot{y}} = f - \frac{\partial P}{\partial y}$$

- Now we make the following definition:

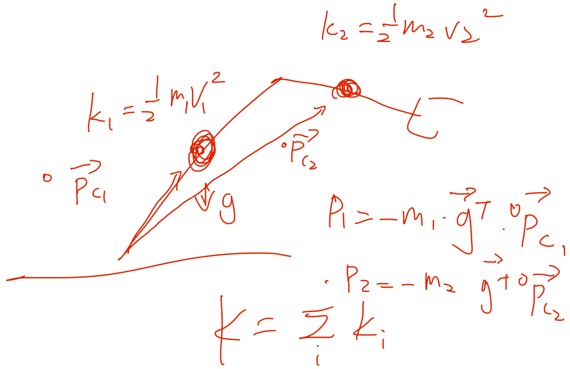
$$L = K - P$$

- *L* is called the *Lagrangian*
  - We can rewrite our equation of motion again:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = f$$



- Thus, to define the equation of motion for this system, all we need is a description of the potential and kinetic energies

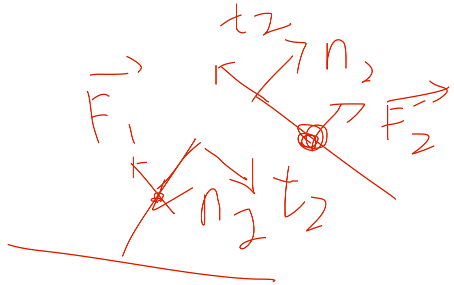


Euler-Lagrange Equations

- If we represent the variables of the system as generalized coordinates, then we can write the equations of motion for an *n*DOF system as:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau_i$$

- We will come back to this, but it is important to recognize the form of the above equation:
  - The left side contains the conservative terms
  - The right side contains the non-conservative terms
- This formulation leads to a set of *n* coupled 2nd order differential equations



Force and Torque relation

Thus we have

$$\mathcal{F} \cdot \delta\mathcal{X} = \tau \cdot \delta\Theta, \tag{5.91}$$

where $\mathcal{F}$ is a 6×1 Cartesian force-moment vector acting at the end-effector, $\delta\mathcal{X}$ is a 6×1 infinitesimal Cartesian displacement of the end-effector, τ is a 6×1 vector of torques at the joints, and $\delta\Theta$ is a 6×1 vector of infinitesimal joint displacements. Expression (5.91) can also be written

$$\mathcal{F}^T \delta\mathcal{X} = \tau^T \delta\Theta. \tag{5.92}$$

The definition of the Jacobian is

$$\delta\mathcal{X} = J\delta\Theta, \tag{5.93}$$

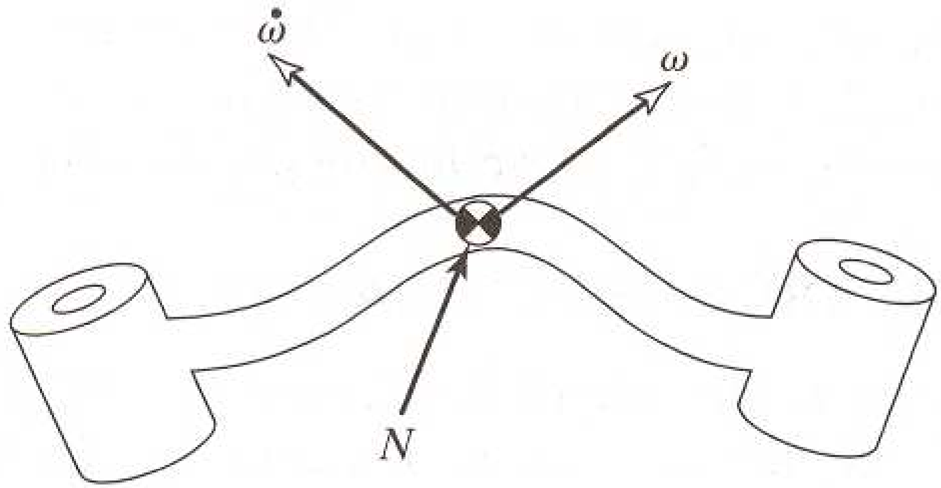
and so we may write

$$\mathcal{F}^T J\delta\Theta = \tau^T \delta\Theta. \tag{5.94}$$

which must hold for all $\delta\Theta$. and so we have

$$\mathcal{F}^T J = \tau^T. \tag{5.95}$$

Motion of the center of mass



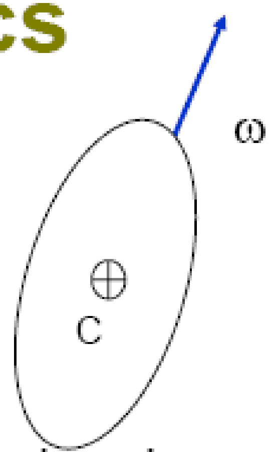
Newton-Euler Quations

$\vec{P} = m\vec{v}$

$F \to 0 \longrightarrow K = \frac{1}{2}mv^2$

$F = ma$

Rigid Body Dynamics



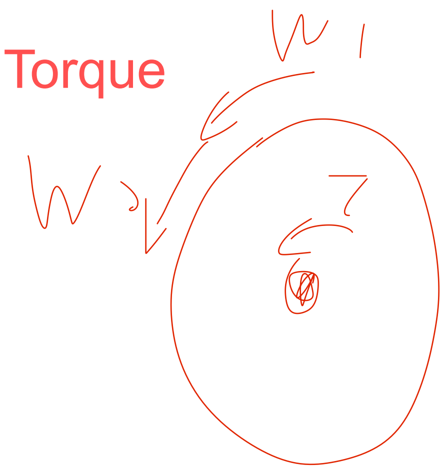
Newton's law: $\sum F = m\mathbf{a}_c$

Euler's Equation: $\sum \mathbf{M}_c = \dot{\mathbf{H}}_c$

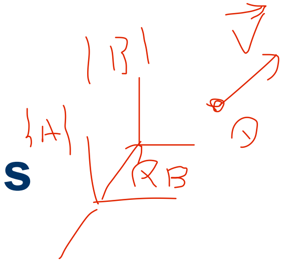
The rate of change of the angular momentum is given by

$$\tau = \dot{\mathbf{H}}_c = \boldsymbol{\omega} \times \mathbf{I}_c \boldsymbol{\omega} + \mathbf{I}_c \dot{\boldsymbol{\omega}}$$

Torque



Velocities and accelerations

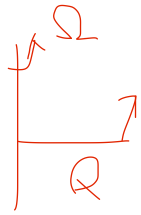


${}^{A}\vec{V}_Q = {}^{A}_{B}\hat{R}\,{}^{B}\vec{V}_Q + {}^{A}\Omega_B\,{}^{A}_{B}R\,{}^{B}Q$

Rotation   W angle V

Linear acceleration

We start by restating (5.12), an important result from Chapter 5, which describe the velocity of a vector ${}^{B}Q$ as seen from frame $\{A\}$ when the origins are coincident



$V = \Omega Q$

$$ {}^{A}V_Q = {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}Q. \quad (6.5 $$

The left-hand side of this equation describes how ${}^{A}Q$ is changing in time. So, becaus origins are coincident, we could rewrite (6.5) as

$$ \frac{d}{dt}({}^{A}_{B}R\,{}^{B}Q) = {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}Q. \quad (6.6 $$

This form of the equation will be useful when deriving the corresponding acceleratio equation.

By differentiating (6.5), we can derive expressions for the acceleration of ${}^{B}($ as viewed from $\{A\}$ when the origins of $\{A\}$ and $\{B\}$ coincide:

$$ {}^{A}\dot{V}_Q = \frac{d}{dt}({}^{A}_{B}R\,{}^{B}V_Q) + {}^{A}\dot{\Omega}_B \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}\Omega_B \times \frac{d}{dt}({}^{A}_{B}R\,{}^{B}Q). \quad (6.7 $$

Now we apply (6.6) twice—once to the first term, and once to the last term. Th right-hand side of equation (6.7) becomes

$$ \begin{aligned} {}^{A}_{B}R\,{}^{B}\dot{V}_Q &+ {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\dot{\Omega}_B \times {}^{A}_{B}R\,{}^{B}Q \\ &+ {}^{A}\Omega_B \times ({}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}Q). \end{aligned} \quad (6.8 $$

Combining two terms, we get

$$ {}^{A}_{B}R\,{}^{B}\dot{V}_Q + 2\,{}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\dot{\Omega}_B \times {}^{A}_{B}R\,{}^{B}Q + {}^{A}\Omega_B \times ({}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}Q). \quad (6.9 $$

Finally, to generalize to the case in which the origins are not coincident, we add one term which gives the linear acceleration of the origin of $\{B\}$, resulting in the final general formula:

$$
\begin{aligned}
&{}^{A}\dot{V}_{BORG} + {}^{A}_{B}R\,{}^{B}\dot{V}_{Q} + 2\,{}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\dot{\Omega}_{B} \times {}^{A}_{B}R\,{}^{B}Q \\
&\quad + {}^{A}\Omega_{B} \times ({}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q).
\end{aligned}
\tag{6.10}
$$

A particular case that is worth pointing out is when ${}^{B}Q$ is constant, or

$$
{}^{B}V_{Q} = {}^{B}\dot{V}_{Q} = 0. \tag{6.11}
$$

In this case, (6.10) simplifies to

$$
{}^{A}\dot{V}_{Q} = {}^{A}\dot{V}_{BORG} + {}^{A}\Omega_{B} \times ({}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q) + {}^{A}\dot{\Omega}_{B} \times {}^{A}_{B}R\,{}^{B}Q. \tag{6.12}
$$

We will use this result in calculating the linear acceleration of the links of a manipulator with rotational joints. When a prismatic joint is present, the more general form of (6.10) will be used.

$$ {}^{A}\Omega_{B} + {}^{A}_{B}\hat{R}\, {}^{B}\Omega_{C} = {}^{A}\Omega_{C} $$

Angular acceleration

Consider the case in which $\{B\}$ is rotating relative to $\{A\}$ with ${}^{A}\Omega_{B}$ and $\{C\}$ is rotating relative to $\{B\}$ with ${}^{B}\Omega_{C}$. To calculate ${}^{A}\Omega_{C}$, we sum the vectors in frame $\{A\}$:

$$ {}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\, {}^{B}\Omega_{C}. \tag{6.13} $$

By differentiating, we obtain

$$ {}^{A}\dot{\Omega}_{C} = {}^{A}\dot{\Omega}_{B} + \frac{d}{dt}({}^{A}_{B}R\, {}^{B}\Omega_{C}). \tag{6.14} $$

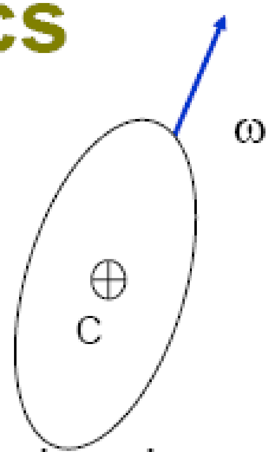
Now, applying (6.6) to the last term of (6.14), we get

$$ {}^{A}\dot{\Omega}_{C} = {}^{A}\dot{\Omega}_{B} + {}^{A}_{B}R\, {}^{B}\dot{\Omega}_{C} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\, {}^{B}\Omega_{C}. \tag{6.15} $$

We will use this result to calculate the angular acceleration of the links of a manipulator.

Newton-Euler Quations

Rigid Body Dynamics



Newton's law: $\sum F = m\mathbf{a}_c$

Euler's Equation: $\sum \mathbf{M}_c = \dot{\mathbf{H}}_c$

The rate of change of the angular momentum is given by

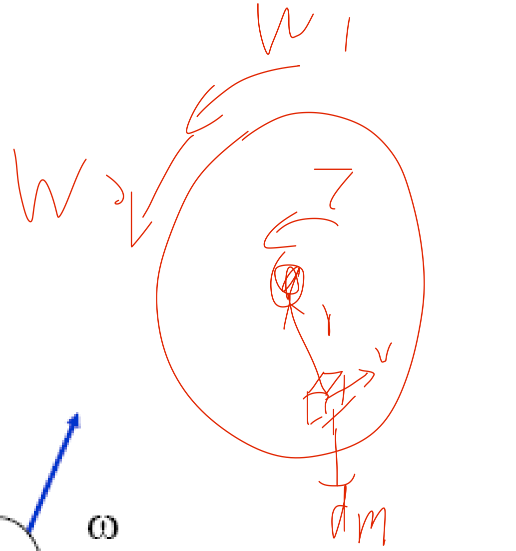
$$\dot{\mathbf{H}}_c = \boldsymbol{\omega} \times \mathbf{I}_c \boldsymbol{\omega} + \mathbf{I}_c \dot{\boldsymbol{\omega}}$$

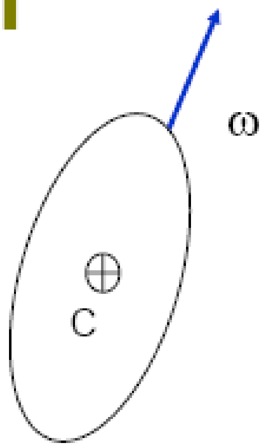
change direction

change magr

Angular Momentum

Angular Momentum





$$\vec{V} = \vec{w} \times \vec{r}$$

Definition: Angular momentum is the total moment of the momentum of a rigid body's constituting particles:

$$\mathbf{H}_c = \int_{\mathcal{B}} \mathbf{r} \times \mathbf{v} \, dm = \mathbf{I}_c \boldsymbol{\omega}$$

And I_c is the intertia matrix given by $\mathbf{I}_c = \int_{\mathcal{B}} \mathbf{S}(\mathbf{r})^T \mathbf{S}(\mathbf{r}) \, dm$

$$\mathbf{I}_c = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

$$I_{xx} = \int_{\mathcal{B}} \left(y^2 + z^2\right) dm \qquad I_{xy} = \int_{\mathcal{B}} xy \, dm$$

$$I_{yy} = \int_{\mathcal{B}} \left(x^2 + z^2\right) dm \qquad I_{xz} = \int_{\mathcal{B}} xz \, dm$$

$$I_{zz} = \int_{\mathcal{B}} \left(x^2 + y^2\right) dm \qquad I_{yz} = \int_{\mathcal{B}} yz \, dm$$

$$\vec{r} \times \vec{v} = \vec{r} \times \vec{w} \times \vec{r}$$

Inertial Matrix

$$I_{xx} = \int_0^h \int_0^l \int_0^\omega (y^2 + z^2)\rho\, dx\, dy\, dz$$

$$= \int_0^h \int_0^l (y^2 + z^2)\omega\rho\, dy\, dz$$

$$= \int_0^h \left(\frac{l^3}{3} + z^2 l\right) \omega\rho\, dz$$

$$= \left(\frac{hl^3\omega}{3} + \frac{h^3 l\omega}{3}\right)\rho$$

$$= \frac{m}{3}(l^2 + h^2),$$

$$I_{xx} = \int (y^2 + z^2)\, dm$$

ρ m

$$m = \rho V$$

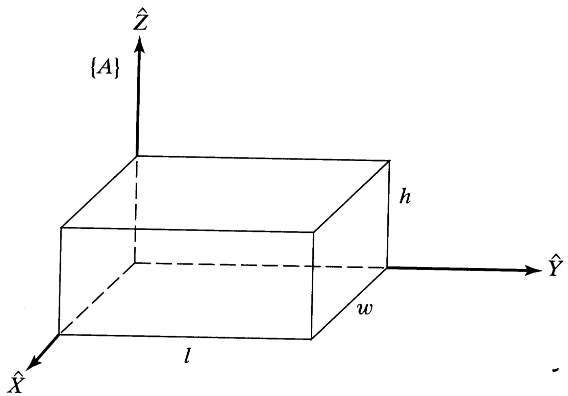
$$dm = \rho\, dv = \rho\, dx\, dy\, dz$$

where m is the total mass of the body. Permuting the terms, we can get I_{yy} and I_{zz} by inspection:

$$I_{yy} = \frac{m}{3}(\omega^2 + h^2)$$

and

$$I_{zz} = \frac{m}{3}(l^2 + \omega^2).$$



$$
\begin{aligned}
I_{xy} &= \int_0^h \int_0^l \int_0^\omega xy\rho \, dx \, dy \, dz \\
&= \int_0^h \int_0^l \frac{\omega^2}{2} y\rho \, dy \, dz \\
&= \int_0^h \frac{\omega^2 l^2}{4} \rho \, dz \\
&= \frac{m}{4}\omega l.
\end{aligned}
$$

Permuting the terms, we get

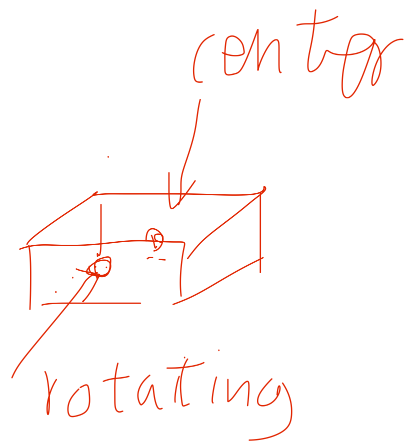
$$I_{xz} = \frac{m}{4} h\omega$$

and

$$I_{yz} = \frac{m}{4} hl.$$

Hence, the inertia tensor for this object is

$$
{}^{A}I = \begin{bmatrix} \frac{m}{3}(l^2 + h^2) & -\frac{m}{4}\omega l & -\frac{m}{4}h\omega \\ -\frac{m}{4}\omega l & \frac{m}{3}(\omega^2 + h^2) & -\frac{m}{4}hl \\ -\frac{m}{4}h\omega & -\frac{m}{4}hl & \frac{m}{3}(l^2 + \omega^2) \end{bmatrix}.
$$



Find the inertia tensor for the same solid body described for Example 6.1 when it is described in a coordinate system with origin at the body's center of mass.

We can apply the parallel axis theorem, (6.25), where

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \frac{1}{2} \begin{bmatrix} w \\ l \\ h \end{bmatrix}.$$

Then we find

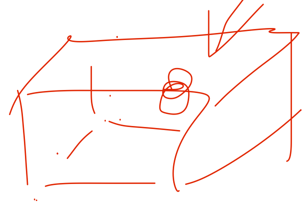
$$\begin{aligned} {}^{C}I_{zz} &= \frac{m}{12}(w^2 + l^2), \\ {}^{C}I_{xy} &= 0. \end{aligned} \tag{6.27}$$

The other elements are found by symmetry. The resulting inertia tensor written in the frame at the center of mass is

$${}^{C}I = \begin{bmatrix} \frac{m}{12}(h^2 + l^2) & 0 & 0 \\ 0 & \frac{m}{12}(w^2 + h^2) & 0 \\ 0 & 0 & \frac{m}{12}(l^2 + w^2) \end{bmatrix}. \tag{6.28}$$

Since the result is diagonal, frame $\{C\}$ must represent the principal axes of this body. ■

center



- Inertia matrix is a positive-definite symmetric matrix
- The inertia matrix is not in general constant and is frame-dependent:

$$\mathbf{I}_c^i = R_i^T \mathbf{I}_c R_i$$

- Any rigid body has a set of principal directions with respect to which the inertia matrix is diagonal

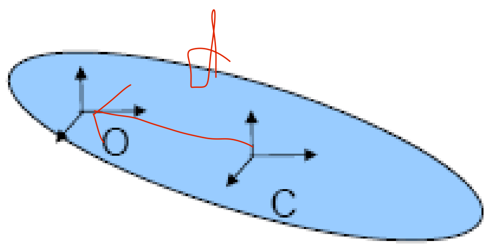
Example

Special cases

- If xy is the pane of symmetry, then $I_{xz}=I_{yz}=0$. Similarly if xz or yz is the plane of symmetry then the corresponding products of inertia are zero.
- If body is axisymmetric (e.g., symmetric about **z**) then the inertia matrix is diagonal and 2 of the moments of inertia equal (e.g., $I_{xx}=I_{yy}$ if **z** is the axis of symmetry)

Parallel-Axis Theorem

The moment of inertia about an arbitrary point O with axes parallel to the original xyz is given by



$$\mathbf{I}_o = \mathbf{I}_C + m\mathbf{S}(\mathbf{d})^T\mathbf{S}(\mathbf{d}) = \begin{bmatrix} I_{xx} + md_x^2 & -\left(I_{xy} + md_x d_y\right) & -\left(I_{xz} + md_x d_z\right) \\ * & I_{yy} + md_y^2 & -\left(I_{yz} + md_y d_z\right) \\ * & * & I_{zz} + md_z^2 \end{bmatrix}$$

$I_{xy} + md_x \cdot d_y$

$\vec{F_2} = \vec{f_1} - \vec{f_2}$

$F_i = m_i a_i$

torque

$F = \begin{pmatrix} \vec{F} \\ \vec{N} \end{pmatrix}$

Inward Calculations

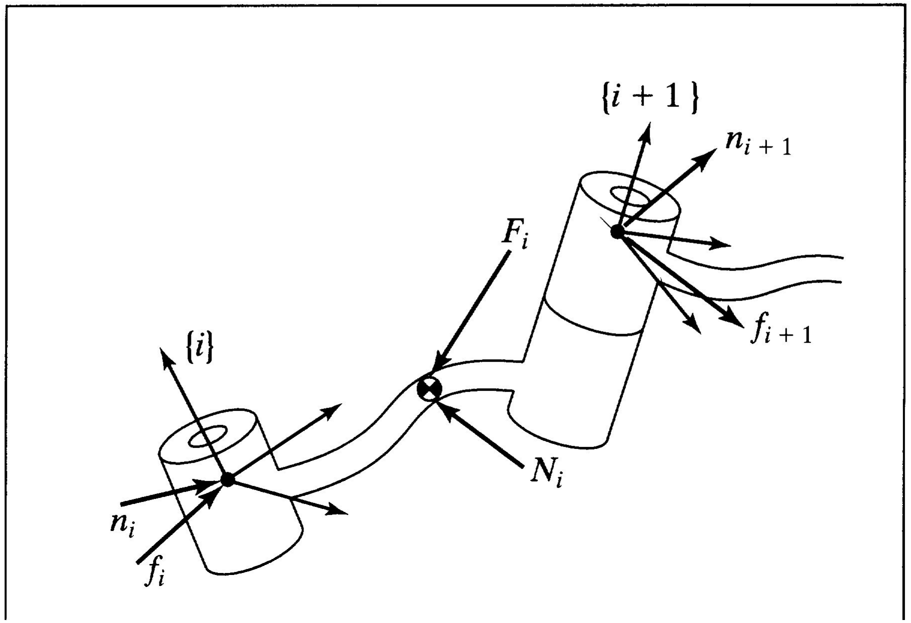
${}^iF_i = {}^if_i - {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}$

f_{i+1}, n_{i+1} | lase $= 0$

$N_2 = \dot{H}_2$ $\vec{n_1} = \vec{n_2} + N_2 +$

$$ {}^iF_i = {}^if_i - {}^{i}_{i+1}R^{i+1}f_{i+1}. $$

$$ {}^iN_i = {}^in_i - {}^in_{i+1} + (-{}^iP_{C_i}) \times {}^if_i - ({}^iP_{i+1} - {}^iP_{C_i}) \times {}^if_{i+1}. $$

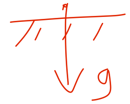


Iterative Closed Loop Form (Newton-Euler)

Outward iterations: $i : 0 \rightarrow 5$

cheat paper

$$
\begin{aligned}
{}^{i+1}\omega_{i+1} &= {}^{i+1}_{i}R\, {}^{i}\omega_i + \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}, \\
{}^{i+1}\dot{\omega}_{i+1} &= {}^{i+1}_{i}R\, {}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\, {}^{i}\omega_i \times \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}, \\
{}^{i+1}\dot{v}_{i+1} &= {}^{i+1}_{i}R({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_i), \\
{}^{i+1}\dot{v}_{C_{i+1}} &= {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} \\
&\quad + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}) + {}^{i+1}\dot{v}_{i+1}, \\
{}^{i+1}F_{i+1} &= m_{i+1}\, {}^{i+1}\dot{v}_{C_{i+1}}, \\
{}^{i+1}N_{i+1} &= {}^{C_{i+1}}I_{i+1}\, {}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\, {}^{i+1}\omega_{i+1}.
\end{aligned}
$$



$^0\vec{V}_0 = 0$ $\quad \vec{W}_0 = 0$

$^0\dot{\vec{V}}_0 = -g$ $\quad \dot{\vec{W}}_0 = 0$

Inward iterations: $i : 6 \rightarrow 1$

$$
\begin{aligned}
{}^{i}f_i &= {}^{i}_{i+1}R\, {}^{i+1}f_{i+1} + {}^{i}F_i, \\
{}^{i}n_i &= {}^{i}N_i + {}^{i}_{i+1}R\, {}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i \\
&\quad + {}^{i}P_{i+1} \times {}^{i}_{i+1}R\, {}^{i+1}f_{i+1}, \\
\tau_i &= {}^{i}n_i^T\, {}^{i}\hat{Z}_i.
\end{aligned}
$$

x和y的力矩会有机器人本体承受

只对旋转节有效