Multiple View Geometry: Exercise 5

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`Wednesdays 16:00-18:15 at Hörsaal 2, "Interims I" (5620.01.102), and on RBG Live`

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Note: For more readability we will call $T_\times$ from the lecture $\hat{T}$ in this exercise sheet.

1. In this task, we are considering the non-zero essential matrix $E = \hat{T}R$ with $T \in \mathbb{R}^3$ and $R \in \mathrm{SO}(3)$. Let $R_Z\left(\pm\frac{\pi}{2}\right)$ be the rotation by $\pm\frac{\pi}{2}$ around the z-axis.

   Extra Information: The non-zero essential matrix has the singular value decomposition $E = U\Sigma V^T$ with $\Sigma = \mathrm{diag}\{\sigma, \sigma, 0\}$ for some $\sigma > 0$ and $U, V \in \mathrm{SO}(3)$. There exist exactly two options for $(\hat{T}, R)$:

$$\left(\hat{T}_1, R_1\right) = \left(UR_Z\left(+\frac{\pi}{2}\right)\Sigma\, U^\top, \quad UR_Z\left(+\frac{\pi}{2}\right)^\top V^\top\right) \tag{1}$$

$$\left(\hat{T}_2, R_2\right) = \left(UR_Z\left(-\frac{\pi}{2}\right)\Sigma\, U^\top, \quad UR_Z\left(-\frac{\pi}{2}\right)^\top V^\top\right) \tag{2}$$

   Show that by using (1) and (2), the following properties hold:

   (a) $\hat{T}_1, \hat{T}_2 \in so(3)$ (i.e. $\hat{T}_1, \hat{T}_2$ are skew-symmetric matrices)

   (b) $R_1, R_2 \in \mathrm{SO}(3)$ (i.e. R_1, R_2 are rotation matrices)

2. Consider the matrices $E = \hat{T}R$ and $H = R + Tu^\top$ with $R \in \mathbb{R}^{3\times 3}$ and $T, u \in \mathbb{R}^3$.
   Show that the following holds:

   (a) $E = \hat{T}H$

   (b) $H^\top E + E^\top H = 0$

3. Let $F \in \mathbb{R}^{3\times 3}$ be the fundamental matrix for the cameras C_1 and C_2. Show that the following holds for the epipoles e_1 and e_2:

$$Fe_1 = 0 \qquad \text{and} \qquad e_2^\top F = 0$$

   Hint: Use the visualizations and information from the lecture (Chapter 6, Slides: 15/16) to determine e_1 and e_2.

1. In this task, we are considering the non-zero essential matrix $E = \hat{T}R$ with $T \in \mathbb{R}^3$ and $R \in \mathrm{SO}(3)$. Let $R_Z\left(\pm\frac{\pi}{2}\right)$ be the rotation by $\pm\frac{\pi}{2}$ around the z-axis.

Extra Information: The non-zero essential matrix has the singular value decomposition $E = U\Sigma V^T$ with $\Sigma = \operatorname{diag}\{\sigma, \sigma, 0\}$ for some $\sigma > 0$ and $U, V \in \mathrm{SO}(3)$. There exist exactly two options for $(\hat{T}, R)$:

$$\left(\hat{T}_1, R_1\right) = \left(U R_Z\left(+\frac{\pi}{2}\right)\Sigma U^\top,\quad U R_Z\left(+\frac{\pi}{2}\right)^\top V^\top\right) \tag{1}$$

$$\left(\hat{T}_2, R_2\right) = \left(U R_Z\left(-\frac{\pi}{2}\right)\Sigma U^\top,\quad U R_Z\left(-\frac{\pi}{2}\right)^\top V^\top\right) \tag{2}$$

Show that by using $\boxed{1}$ and $\boxed{2}$, the following properties hold:

(a) $\hat{T}_1, \hat{T}_2 \in so(3)$ (i.e. $\hat{T}_1, \hat{T}_2$ are skew-symmetric matrices)

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1 a) T skew-symetric iff $T = -T^T$

$$U R_Z(\tfrac{\pi}{2}) \Sigma U^T \overset{?}{=} -\left(U R_Z(\tfrac{\pi}{2}) \Sigma U^T\right)^T$$

$$R_Z\left(\frac{\pi}{2}\right) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_Z\left(\frac{\pi}{2}\right)\Sigma = \begin{pmatrix} 0 & \sigma & 0 \\ -\sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Sigma R_Z^T\left(-\frac{\pi}{2}\right) = \begin{pmatrix} 0 & -\sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow R_Z(\tfrac{\pi}{2})\Sigma = -\Sigma R_Z^T(-\tfrac{\pi}{2})$$

b) orthogonal $R R^T = 1$

$$R \cdot R^T = U \underline{R_Z(+\tfrac{\pi}{2})^T} \cdot \cancel{V^T \cdot V} \cdot \underline{R_Z(-\tfrac{\pi}{2})} \cdot U^T$$

$$= 1$$

2. Consider the matrices $E = \hat{T}R$ and $H = R + Tu^\top$ with $R \in \mathbb{R}^{3\times3}$ and $T, u \in \mathbb{R}^3$. Show that the following holds:

(a) $E = \hat{T}H$

(b) $H^\top E + E^\top H = 0$

(a)

$$\hat{T}(R + Tu^T) \stackrel{?}{=} E$$

$$\hat{T}R + \underbrace{\hat{T}Tu^T}_{\substack{\shortparallel \\ (T \times T)u^T = 0}} \stackrel{?}{=} E$$

(b) $(R + Tu^T)^T E + E^T(R + Tu^T) \stackrel{?}{=} 0$

$$R^T E + \underline{uT^T E} + E^T R + \underline{E^T T u^T} \stackrel{?}{=} 0$$

$$R^T \hat{T} R + 0 + R^T \hat{T}^T R + 0 = 0$$

$$R^T \hat{T} R - R^T \hat{T} R = 0$$

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3. Let $F \in \mathbb{R}^{3\times 3}$ be the fundamental matrix for the cameras C_1 and C_2. Show that the following holds for the epipoles e_1 and e_2:

$$Fe_1 = 0 \qquad \text{and} \qquad e_2^\top F = 0$$

Hint: Use the visualizations and information from the lecture (Chapter 6, Slides: 15/16) to determine e_1 and e_2.

assume $g_{21} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$ $\quad g_{12} = g_{21}^{-1} = \begin{bmatrix} R^T & -R^T T \\ 0 & 1 \end{bmatrix}$

O_1 in $CS1$: $[0, 0, 0, 1]^T$

O_2 in CS_2 : $g_{21}[0, 0, 0, 1]^T = \begin{bmatrix} T \\ 1 \end{bmatrix}$

e_2 are the pixel coordinates of O_1 projected into image 2

$$\lambda_2 e_2 = K_2 \Pi_0 \begin{bmatrix} T \\ 1 \end{bmatrix} = K_2 T$$

O_2 in CS_2 : $[0, 0, 0, 1]^T$

O_2 in CS_1 : $g_{12}[0, 0, 0, 1]^T = \begin{bmatrix} -R^T T \\ 1 \end{bmatrix}$

e_1 are the pixel coordinates of O_2 projected into image 1

$$\lambda_1 e_1 = K_1 \Pi_0 \begin{bmatrix} -R^T T \\ 1 \end{bmatrix} = -K_1 \cdot R^T T$$

$$Fe_1 = \underbrace{(K_2^{-T} \cancel{\hat{T}} R \cancel{K_1^{-1}})}_{F} \cdot \underbrace{(-\tfrac{1}{\lambda_1} \cancel{K_1} \cancel{R^T} \cancel{T})}_{e_1} = 0$$

$$e_2^T F = \underbrace{(\tfrac{1}{\lambda_2} \cancel{T^T} \cancel{K_2^T})}_{e_2^{-1}} \cdot \underbrace{(\cancel{K_2^{-T}} \cancel{\hat{T}} R K_1^{-1})}_{F} = 0$$