Fundamentals of Artificial Intelligence
Exercise 8: Bayesian Networks

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Recap Bayesian Networks

Bayesian network

A Bayesian network is a directed acyclic graph, where

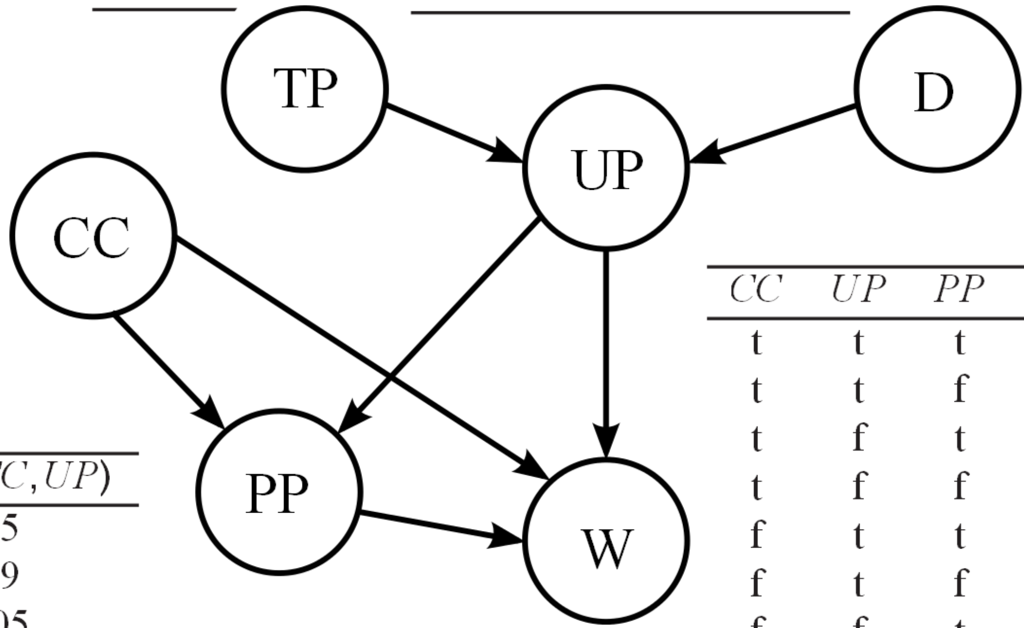
- each node corresponds to a random variable,
- arrows between nodes start at parents,
- each node N_i has a conditional probability distribution $P(X_i|Parents(X_i))$,
- $P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i|parents(X_i))$.

Recap Bayesian Networks - Determine (Conditional) Independence

- Independence: $P(X, Y) = P(X)P(Y)$ or $P(X|Y) = P(X)$
- Conditional independence: $P(X|Y, E) = P(X|E)$

Problem 8.1

Consider the Bayesian network about a race:



$P(tp)$
0.35

$P(d)$
0.25

$P(cc)$
0.3

D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

Figure: A Bayesian Network with Boolean random variables: $D = \mathit{DemotivatedPilot}$, $TP = \mathit{TalentedPilot}$, $CC = \mathit{CompetitiveCar}$, $UP = \mathit{UnderPerformance}$, $PP = \mathit{PolePosition}$, $W = \mathit{Wins}$

Problem 8.1a

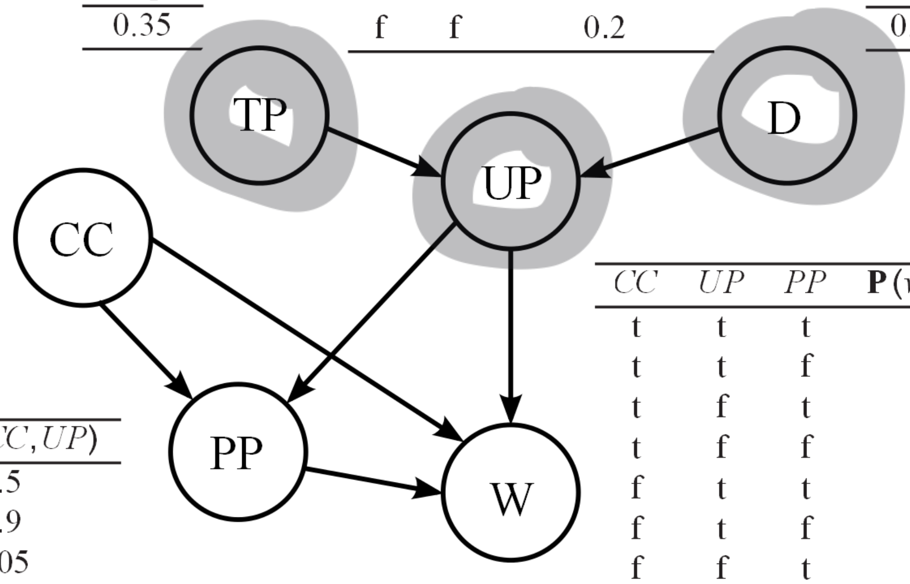
a. Which of these statements are true?
   i. TP, UP and D are independent.

$P(tp)$
0.35

D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(d)$
0.25

$P(cc)$
0.3



CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

Problem 8.1a

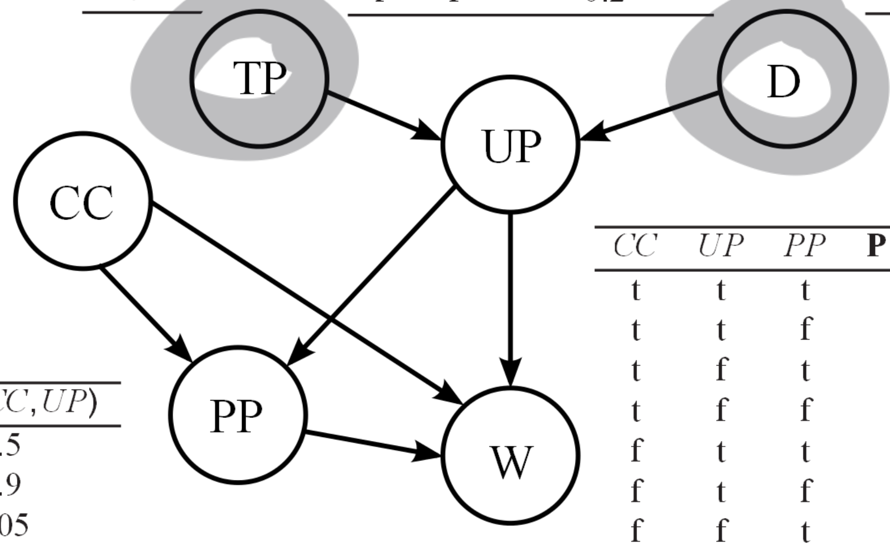
a. Which of these statements are true?
   ii. TP and D are independent.

$P(tp)$
0.35

D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(d)$
0.25

$P(cc)$
0.3



CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

Problem 8.1a

a. Which of these statements are true?

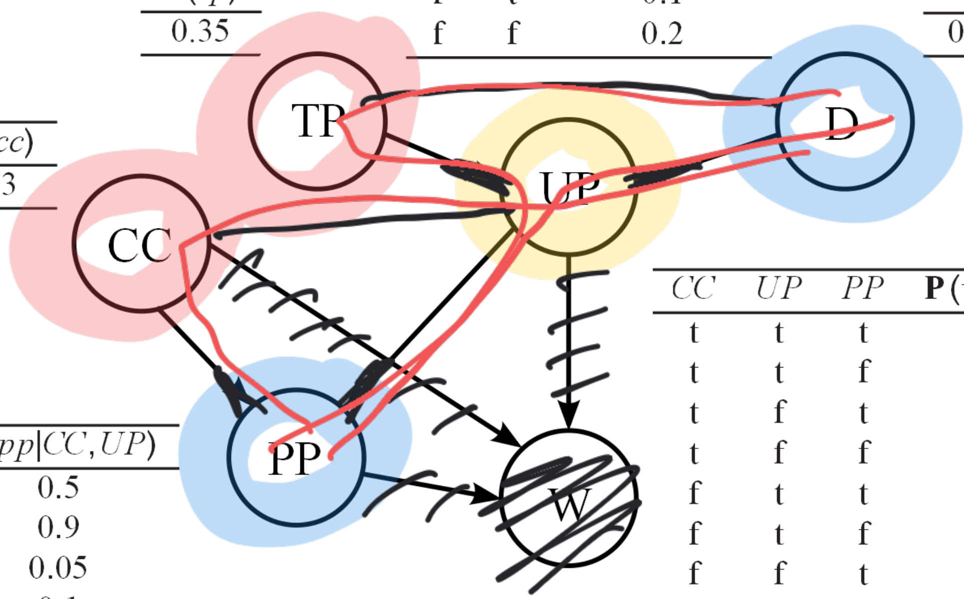
  iii. PP and D are conditionally independent given UP.

D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(tp)$
0.35

$P(d)$
0.25

$P(cc)$
0.3



CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

Problem 8.1a

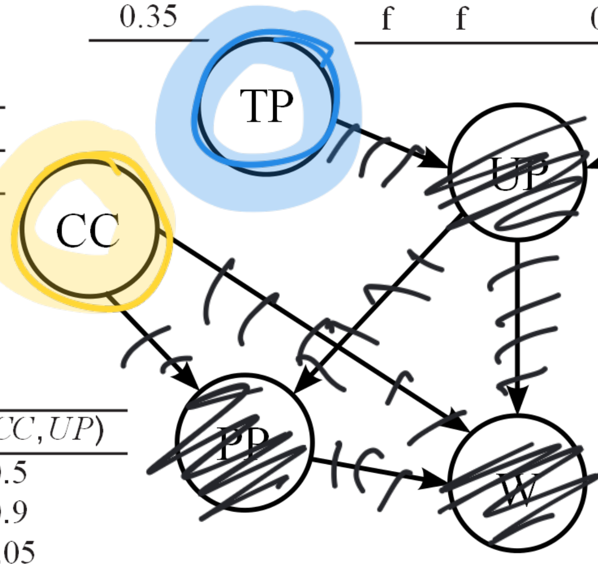
a. Which of these statements are true?
   iv. TP and D are conditionally independent given CC.

$P(tp)$
0.35

D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(d)$
0.25

$P(cc)$
0.3



CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

Problem 8.1a

a. Which of these statements are true?

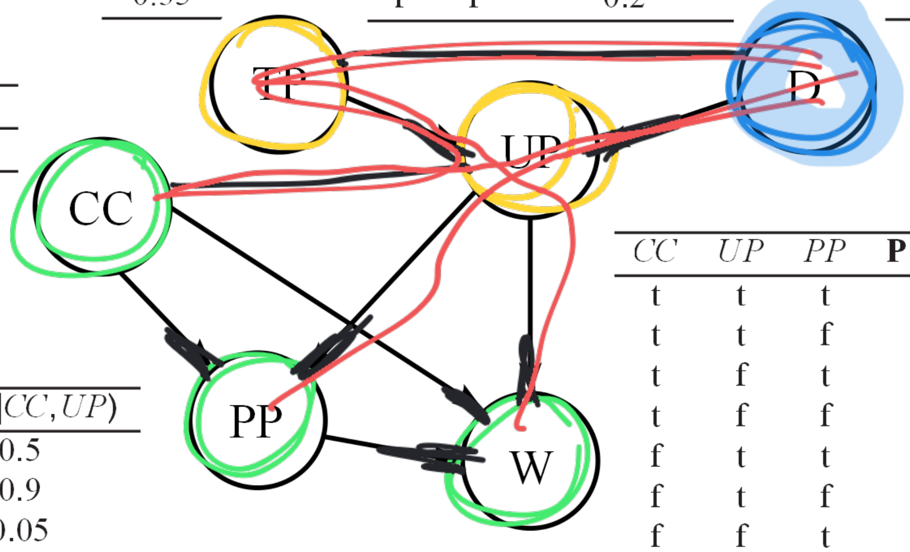
   v. $\mathbf{P}(D|TP, UP) = \mathbf{P}(D|TP, CC, UP, PP, W)$.

D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(tp)$
0.35

$P(d)$
0.25

$P(cc)$
0.3

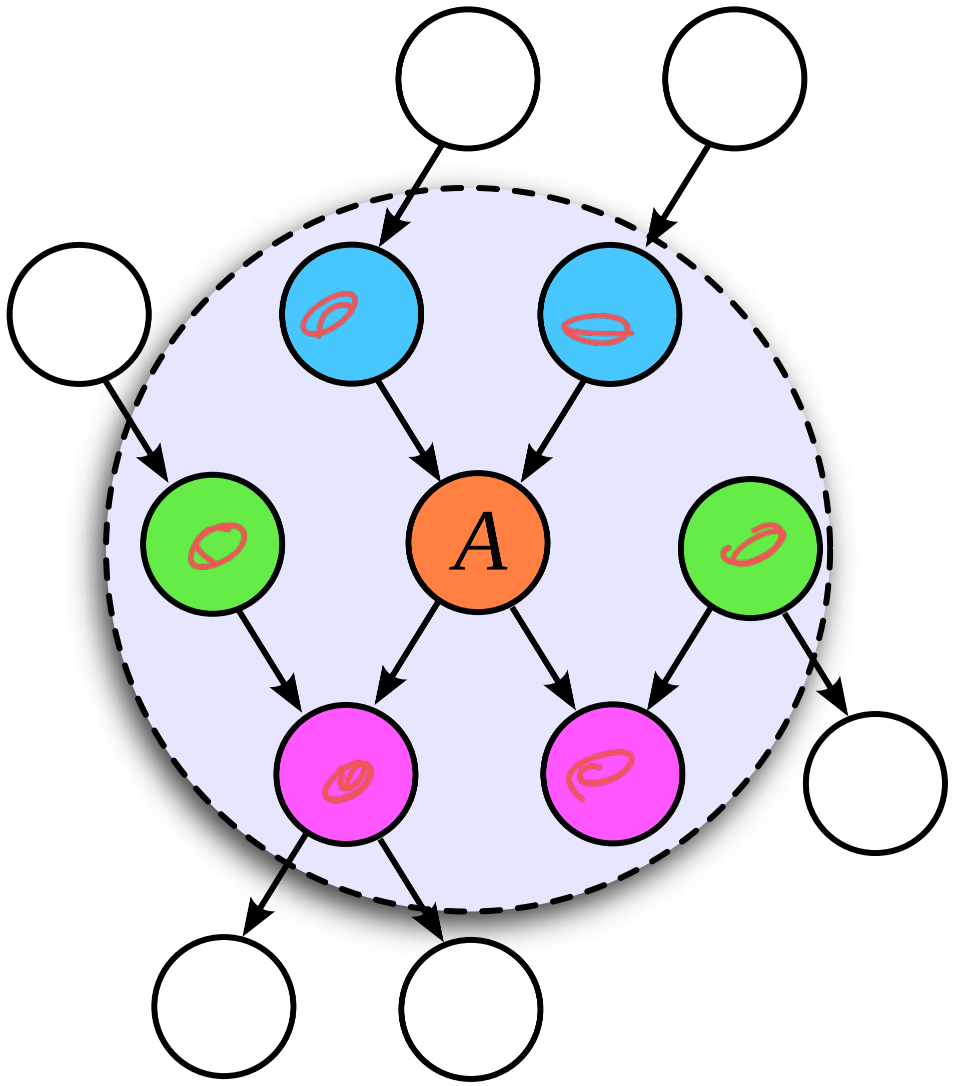


CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

Recap Bayesian Networks - Markov Blanket

- A node in the Bayesian network is conditionally independent of all other nodes given its parents, children, and children's parents



Problem 8.1a

a. Which of these statements are true?

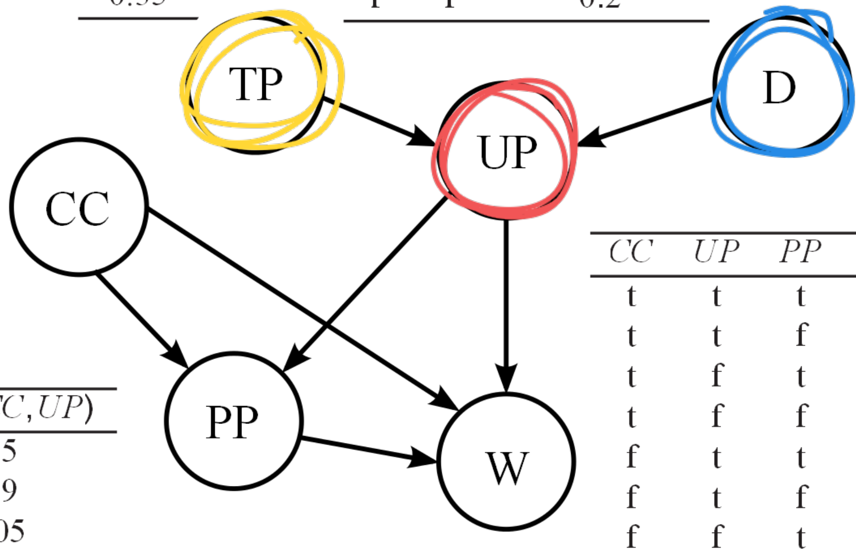
   v. $\mathbf{P}(D|TP, UP) = \mathbf{P}(D|TP, CC, UP, PP, W)$.

D	TP	$\mathbf{P}(up\|D,TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(tp)$
0.35

$P(d)$
0.25

$P(cc)$
0.3



CC	UP	$\mathbf{P}(pp\|CC,UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w\|CC,UP,PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

Problem 8.1b

b. Write the formula for computing the joint probability distribution in terms of the conditional probabilities exploiting the conditional independences in the considered network.

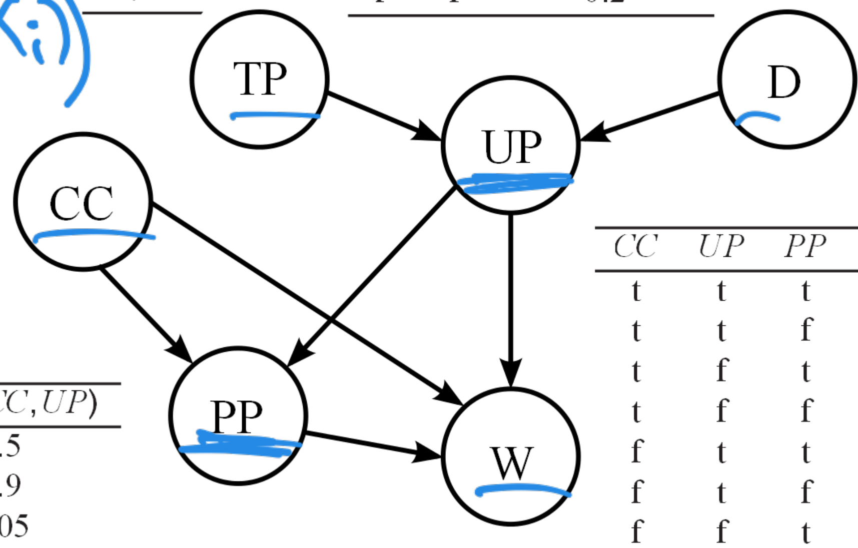
$$P(x_1, \dots x_n) = \prod P(x_i \mid parents(X_i))$$

D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(tp)$
0.35

$P(d)$
0.25

$P(cc)$
0.3



CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

$$P(D, TP, CC, UP, PP, W) = P(D)\,P(TP)\,P(CC)\,P(UP \mid TP, D)\,P(PP \mid CC, UP)\,P(W \mid PP, CC, UP)$$

Problem 8.1c

c. Calculate $P(\neg d, tp, cc, \neg up, pp, w)$ and $P(\neg d, tp, cc, \neg up, \neg pp, w)$.

D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

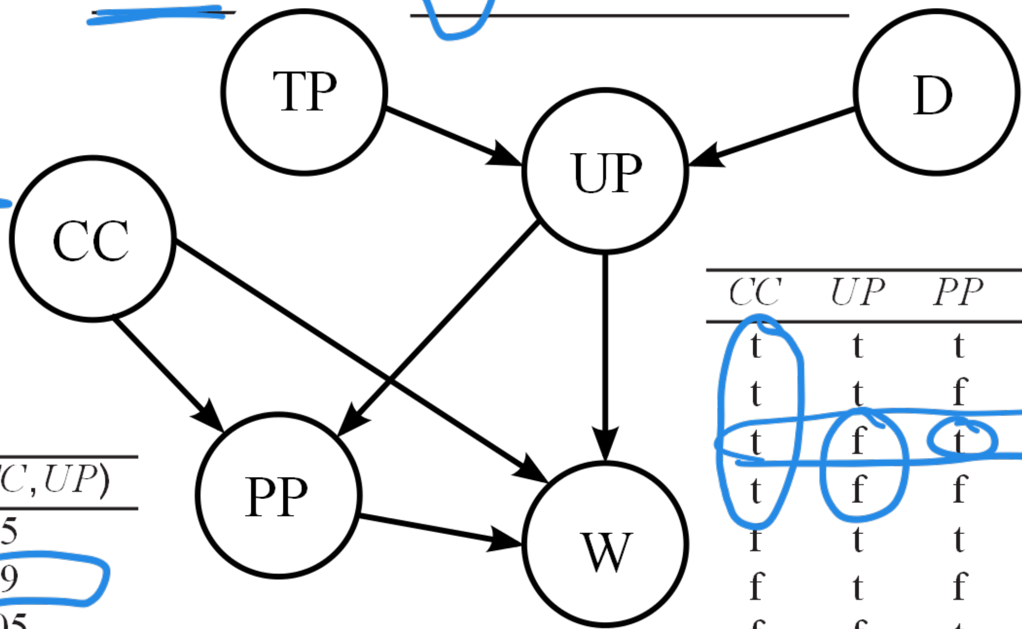
$1-0.1$

$P(tp)$
0.35

$P(d)$
0.25

$1-0.25$

$P(cc)$
0.3



$P(D, TP, UP, CC, PP, W)$
$= P(D)\,P(TP)\,P(CC)$
$\cdot P(UP \mid TP, D)$
$\cdot P(PP \mid CC, UP)$
$P(W \mid PP, CC, UP)$

CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

$P(\neg d, tp, cc, \neg up, pp, w) = \underset{0.75}{P(\neg d)} \cdot \underset{0.35}{P(tp)}\ \underset{0.3}{P(cc)}\ \underset{0.9}{P(\neg up \mid tp, \neg d)}\ \underset{0.9}{P(pp \mid cc, \neg up)}\ \underset{0.9}{P(w \mid pp, cc, \neg up)} = 0.0574$

$P(\neg d, tp, cc, \neg up, \neg pp, w) = \ldots = 0.0035$

Problem 8.1d

d. Calculate the probability that the pilot wins given that he is talented, motivated and starts from the pole position.

w tp ¬d

pp

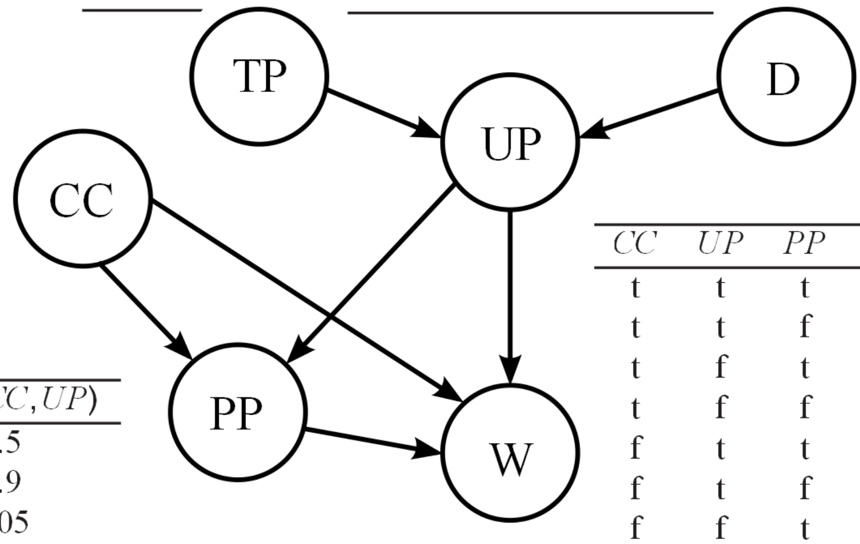
$P(w \mid tp, \neg d, pp)$

D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(tp)$
0.35

$P(d)$
0.25

$P(cc)$
0.3



CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

Problem 8.1d

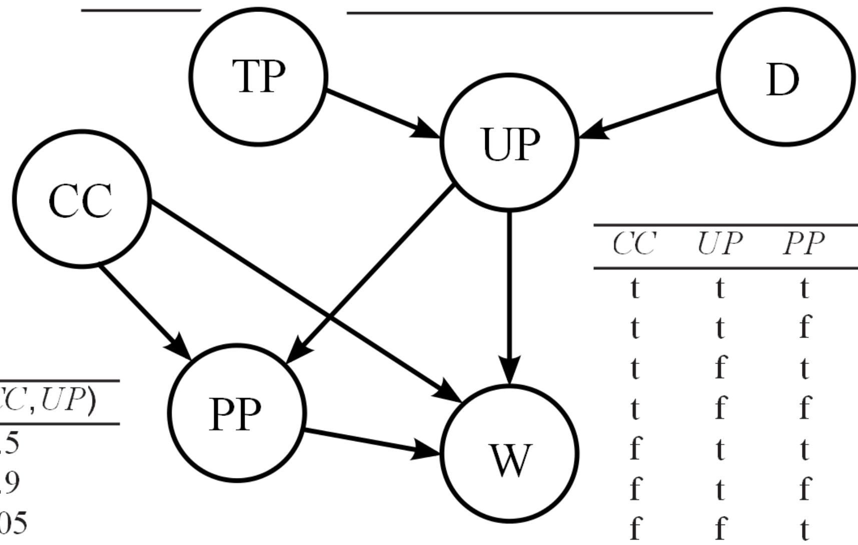
- $P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$

D	TP	$\mathbf{P}(up\|D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(tp)$
0.35

$P(d)$
0.25

$P(cc)$
0.3



CC	UP	$\mathbf{P}(pp\|CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w\|CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

$$P(w | tp, \neg d, pp) = \frac{P(w, tp, \neg d, pp)}{P(tp, \neg d, pp)}$$

$$= P(w, tp, \neg d, pp) + P(\neg w, tp, \neg d, pp)$$

$$:= \frac{1}{\alpha}$$

$$P(w, tp, \neg d, pp) = \sum_{UP} \sum_{CC} P(w, tp, \neg d, pp, UP, CC)$$

$$= \sum_{UP} \sum_{CC} P(tp) P(\neg d) P(CC) P(UP | tp, \neg d) P(pp | CC, UP) \cdot P(w | pp, UP, CC)$$

$$= P(tp) P(\neg d) \sum_{UP} P(UP | tp, \neg d) \sum_{CC} P(CC) P(pp | CC, UP) \cdot P(w | pp, UP, CC)$$

Problem 8.1d

$$= P(tp)\,P(\neg d)$$

$UP = up$ $\quad$ $CC = cc$

$$\Big\{ P(up \mid tp, \neg d)\,\big[P(cc)\,P(pp \mid cc, up)\,P(w \mid pp, up, cc)$$

$CC = \neg cc$

$$+ P(\neg cc)\,P(pp \mid \neg cc, up)\,P(w \mid pp, up, \neg cc) \big]$$

$UP = \neg up$ $\quad$ $CC = cc$

$$+ P(\neg up \mid tp, \neg d)\,\big[P(cc)\,P(pp \mid cc, \neg up)\,P(w \mid pp, \neg up, cc)$$

$CC = \neg cc$

$$+ P(\neg cc)\,P(pp \mid \neg cc, \neg up)\,P(w \mid pp, \neg up, \neg cc) \big] \Big\}$$

$$= 0.35 \cdot 0.75 \,\{ 0.1\,[0.3 \cdot 0.5 \cdot 0.4 + 0.7\ 0.05\ 0.05]$$

$$+ 0.9\,[0.3\ 0.9\ 0.9 + 0.7\ 0.1\ 0.15]\} = 0.0615$$

D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$1 - 0.1$

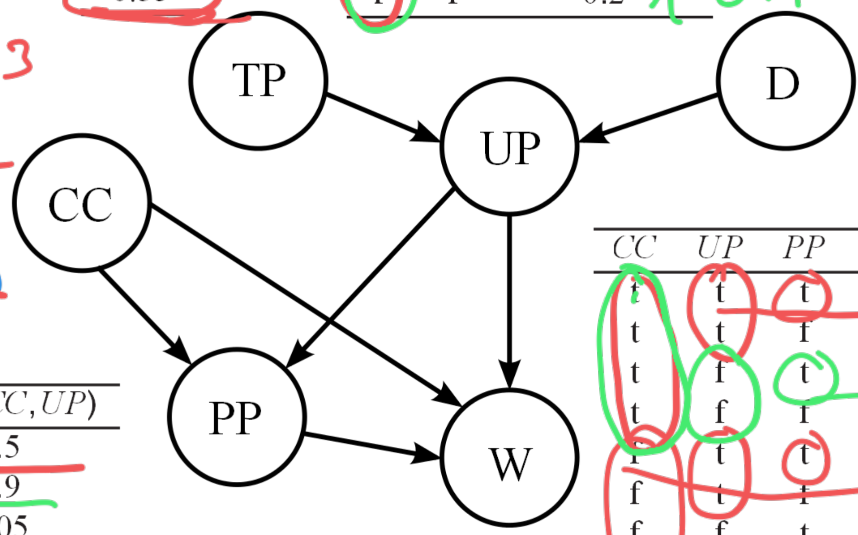
$P(tp)$
0.35

$1 - 0.25$

$P(d)$
0.25

$1 - 0.3$

$P(cc)$
0.3



CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

Problem 8.1d

$P(\neg w, +p, \neg d, pp) = \ldots = 0.0237$

$\alpha = \frac{1}{P(w, +p, \neg d, pp) + P(\neg w, +p, \neg d, pp)} = \frac{1}{0.0615 + 0.0237} \approx 11.7371$

$P(w \mid +p, \neg d, pp) = \alpha \cdot P(w, +p, \neg d, pp) = 11.7371 \cdot 0.0615$

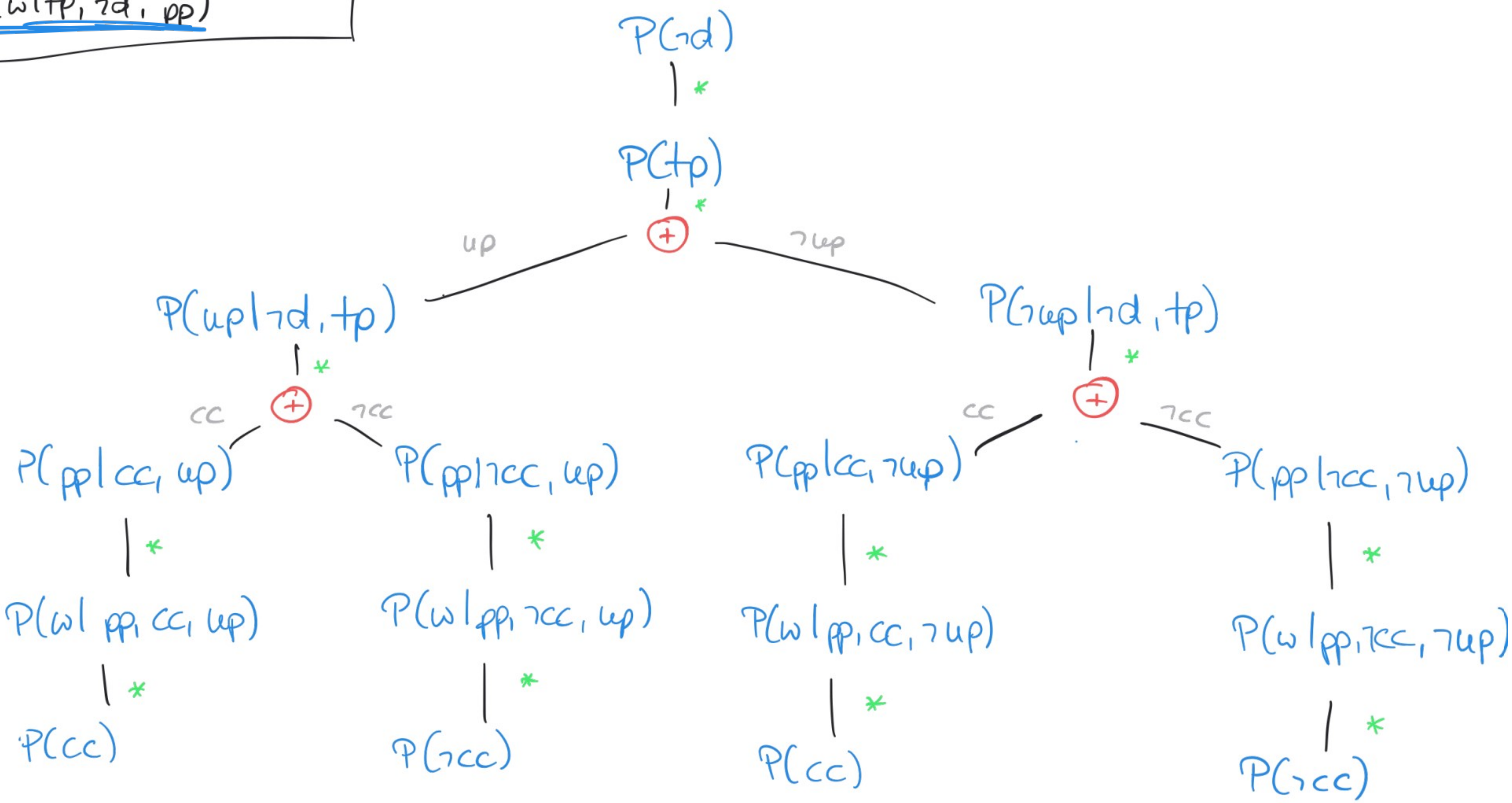
$\approx \underline{\underline{0.7218}}$

Problem 8.1d

Enumeration Tree for

$P(w \mid tp, \neg d, pp)$

$P(w, tp, \neg d, pp)$ $\neg w$



Problem 8.2a

Consider a driving situation that contains the Boolean random variables $H = Hurry$, $CD = CarefulDriver$, $DF = DriveFast$, $A = Accident$ and $GF = GetFined$.

a. Draw the Bayesian network corresponding to:

$$\boxed{\mathbf{P}(H, CD, DF, A, GF) = \mathbf{P}(GF|DF, A)\,\mathbf{P}(A|DF)\,\mathbf{P}(DF|CD, H)\,\mathbf{P}(CD)\,\mathbf{P}(H).}$$

$P(h)$
0.5

$P(cd)$
0.6

DF	$\mathbf{P}(a\|DF)$
t	0.7
f	0.25

CD	H	$\mathbf{P}(df\|CD, H)$
t	t	0.15
t	f	0.01
f	t	0.99
f	f	0.1

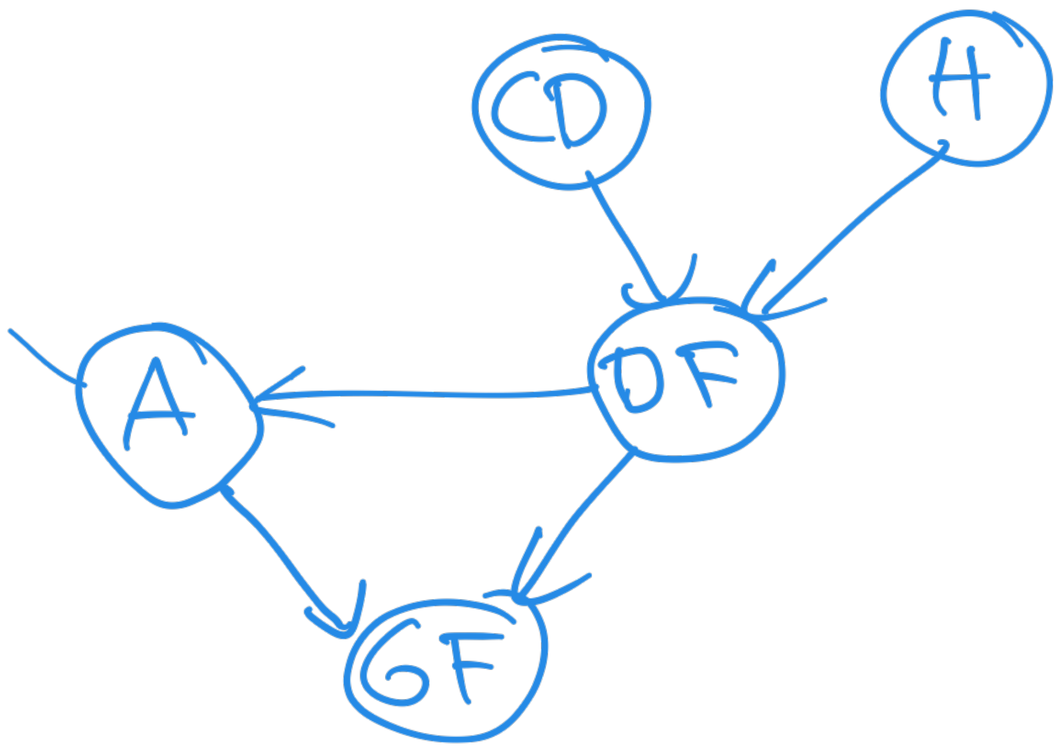
DF	A	$\mathbf{P}(gf\|DF, A)$
t	t	0.99
t	f	0.4
f	t	0.5
f	f	0.05

Problem 8.2a

a. Draw the Bayesian network corresponding to:

$$\boxed{\mathbf{P}(H, CD, DF, A, GF) = \mathbf{P}(GF|DF, A)\,\mathbf{P}(A|DF)\,\mathbf{P}(DF|CD, H)\,\mathbf{P}(CD)\,\mathbf{P}(H)\,.}$$

$P(x_1, \ldots, x_n) = \prod P(x_i \,|\, parents(X_i))$



Problem 8.2b

$$\boxed{\mathbf{P}(H, CD, DF, A, GF) = \mathbf{P}(GF|DF, A)\,\mathbf{P}(A|DF)\,\mathbf{P}(DF|CD, H)\,\mathbf{P}(CD)\,\mathbf{P}(H)\,.}$$

b. Calculate $\mathbf{P}(CD|\underline{\neg a, gf})$ using enumeration.

Problem 8.2b

$$\mathbf{P}(CD|\neg a, gf) = \alpha \mathbf{P}(CD) \sum_{DF} \mathbf{P}(gf|\neg a, DF)\, \mathbf{P}(\neg a|DF) \sum_{H} \mathbf{P}(DF|CD, H)\, \mathbf{P}(H)$$

Problem 8.2b

Problem 8.2b

Problem 8.2c

c. Calculate $\mathbf{P}(CD|\neg a, gf)$ using variable elimination.

$$\mathbf{P}(CD|\neg a, gf) = \alpha \underbrace{\mathbf{P}(CD)}_{f_1(CD)} \sum_{DF} \underbrace{\mathbf{P}(gf|\neg a, DF)}_{f_2(DF)} \underbrace{\mathbf{P}(\neg a|DF)}_{f_3(DF)} \sum_{H} \underbrace{\mathbf{P}(DF|CD,H)}_{f_4(DF,CD,H)} \underbrace{\mathbf{P}(H)}_{f_5(H)}$$

$f_7(CD)$ $f_6(DF, CD)$

$f_4 = P(DF|CD, H)$

dim 1 dim 2 dim 3

$$DF \left\{ \begin{array}{c} \\ df \\ \neg df \end{array} \begin{array}{cc} cd & \neg cd \\ \left[\begin{array}{c} P(df|cd,h) \\ P(\neg df|cd,h) \end{array}\right. & \left.\begin{array}{c} P(df|\neg cd,h) \\ P(\neg df|\neg cd,h) \end{array}\right] \end{array} \left[\begin{array}{cc} P(df|cd,\neg h) & P(df|\neg cd,\neg h) \\ P(\neg df|cd,\neg h) & P(\neg df|\neg cd,\neg h) \end{array}\right] \right\}$$

CD

h $\neg h$

H

Problem 8.2c

$$\mathbf{f_4}(DF, CD, H) = \left\{ \begin{bmatrix} 0.15 & 0.99 \\ 0.85 & 0.01 \end{bmatrix} \begin{bmatrix} 0.01 & 0.1 \\ 0.99 & 0.9 \end{bmatrix} \right\}, \mathbf{f_1}(CD) = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix},$$

$$\mathbf{f_2}(DF) = \begin{bmatrix} 0.4 \\ 0.05 \end{bmatrix}, \mathbf{f_3}(DF) = \begin{bmatrix} 0.3 \\ 0.75 \end{bmatrix}, \quad \mathbf{f_5}(H) = \{[\ 0.5\ ][\ 0.5\ ]\}.$$

$$f_4(DF, CD, H) \times f_5(H) = \left\{ \underbrace{\begin{bmatrix} 0.15 & 0.99 \\ 0.85 & 0.01 \end{bmatrix}}_{h} \underbrace{\begin{bmatrix} 0.01 & 0.1 \\ 0.99 & 0.9 \end{bmatrix}}_{\neg h} \right\} \{ \underbrace{[0.5]}_{h} \underbrace{[0.5]}_{\neg h} \}$$

$$= \left\{ \begin{bmatrix} 0.075 & 0.495 \\ 0.425 & 0.005 \end{bmatrix} \begin{bmatrix} 0.005 & 0.05 \\ 0.495 & 0.45 \end{bmatrix} \right\}$$

8m

4a

$$\sum_H f_4 \times f_5 = \begin{array}{c|c} \begin{array}{cc} cd & \neg cd \end{array} & \\ \begin{bmatrix} 0.080 & 0.545 \\ 0.92 & 0.455 \end{bmatrix} & \begin{array}{c} df \\ \neg df \end{array} \end{array} = \begin{bmatrix} P(df \mid cd) & P(df \mid \neg cd) \\ P(\neg df \mid cd) & P(\neg df \mid \neg cd) \end{bmatrix}$$

$$:= f_6(DF, CD)$$

Problem 8.2c

$$\sum_{DF} f_2(DF) \times f_3(DF) \times f_6(DF, CD)$$

$$= \sum_{DF} \begin{bmatrix} 0.4 \\ 0.05 \end{bmatrix} \times \begin{bmatrix} 0.3 \\ 0.75 \end{bmatrix} \times \begin{bmatrix} 0.08 & 0.545 \\ 0.92 & 0.455 \end{bmatrix}$$

2m

$$= \sum_{DF} \begin{bmatrix} 0.12 \\ 0.0375 \end{bmatrix} \times \begin{bmatrix} 0.08 & 0.545 \\ 0.92 & 0.455 \end{bmatrix}$$

4m

$$= \sum_{DF} \begin{array}{c} \begin{bmatrix} 0.0096 & 0.0654 \\ 0.0345 & 0.0191 \end{bmatrix} \end{array} \begin{array}{l} df \\ \neg df \end{array}$$

2a

$$= \begin{array}{cc} cd & \neg cd \\ [0.0441 & 0.0825] \end{array} := f_7(CD)$$

Problem 8.2c

$$P(CD \mid \neg a, gf) = \alpha \cdot f_1(CD) \times f_7(CD)$$

2m

$$= \alpha \,[0.6 \;\; 0.4] \times [0.0441 \;\; 0.0825]$$

$$= \alpha \,[\underbrace{\overset{cd}{0.0265}}_{P(cd, \neg a, gf)} \;\; \underbrace{\overset{\neg cd}{0.0330}}_{P(\neg cd, \neg a, gf)}] = [\overbrace{0.4454}^{P(cd \mid \neg a, gf)} \;\; \overbrace{0.5546}^{P(\neg cd \mid \neg a, gf)}]$$

$$= 1$$

$$\alpha = \frac{1}{0.0265 + 0.033} \approx 16.8067$$

Problem 8.2d

d. Compare the number of operations required to compute the result in **b** and **c** .

enumeration: 18 multiplications
6 additions

variable elimination: 16 multiplications
6 additions

Problem 8.2d

Evaluation Tree for
P(cd | ¬a, gf)

P(cd, ¬a, gf) → 9 mult
→ 3 add.

P(¬cd, ¬a, gf) → 9 mult
→ 3 add