Robot Control

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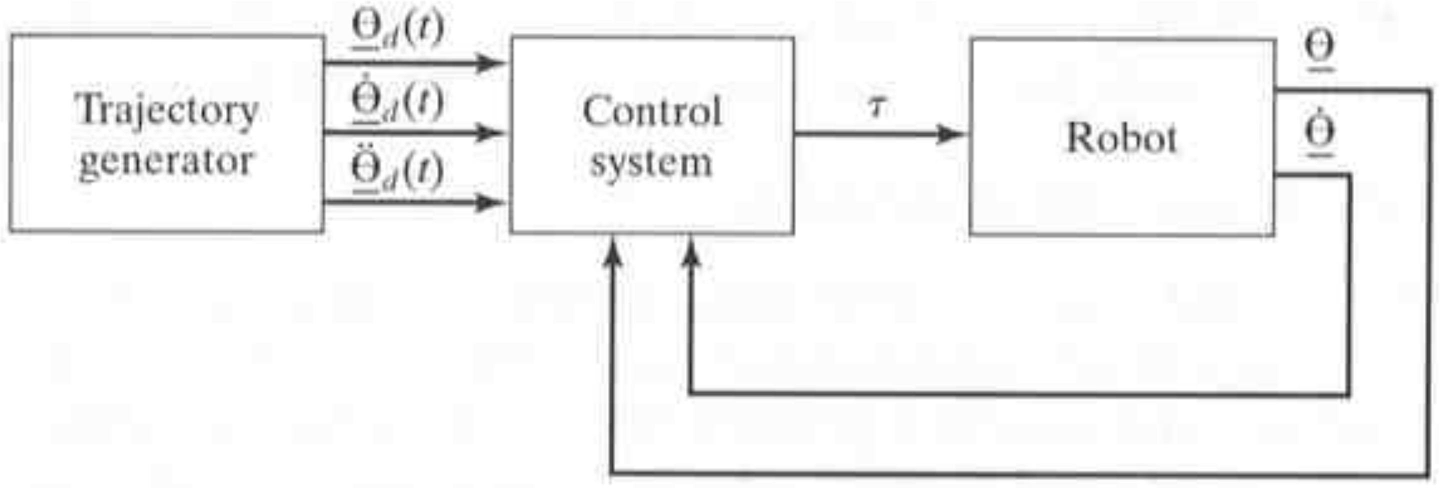
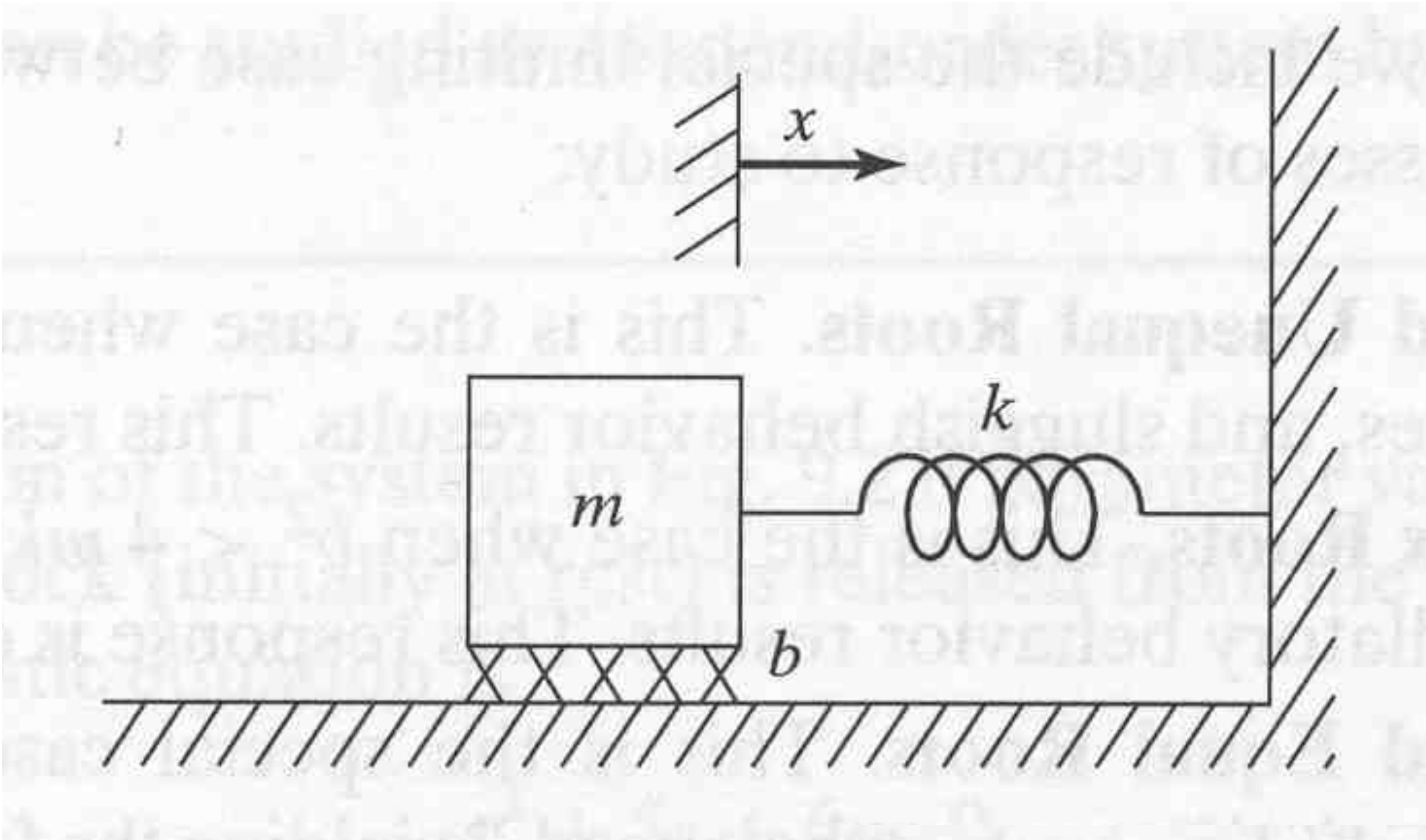


$$\tau = M(\Theta_d)\ddot{\Theta}_d + V(\Theta_d, \dot{\Theta}_d) + G(\Theta_d)$$

Mass –Springer Damper System



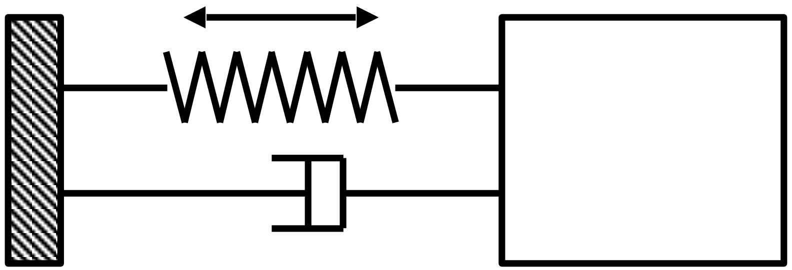
$$m\ddot{x} + b\dot{x} + kx = 0$$

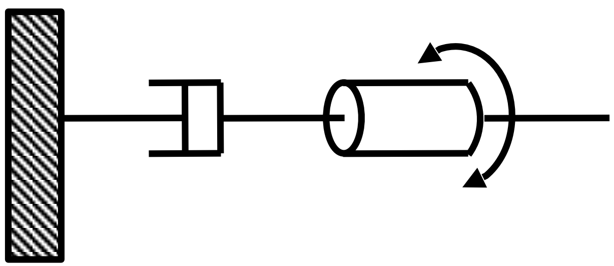
$$ms^2 + bs + k = 0$$

Laplace Transformation

F(s)	f (t) , t > 0
$Y(s) = \int_0^{\infty} \exp(-st)y(t)dt$	definition of a Laplace transform y(t)
Y(s)	inversion formula $y(t) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \exp(st)Y(s)\,ds$
$sY(s) - y(0)$	first derivative $y'(t)$
$s^2 Y(s) - s\,y(0) - y'(0)$	second derivative $y''(t)$
$s^n Y(s) - s^{n-1}[y(0)]$ $- s^{n-2}[y'(0)] - \dots - s[y^{(n-2)}(0)]$ $-[y^{(n-1)}(0)]$	nth derivative $y^{(n)}(t)$
$\frac{1}{s}F(s)$	integration $\int_0^t Y(\tau)d\tau$

Modeling Transient Response





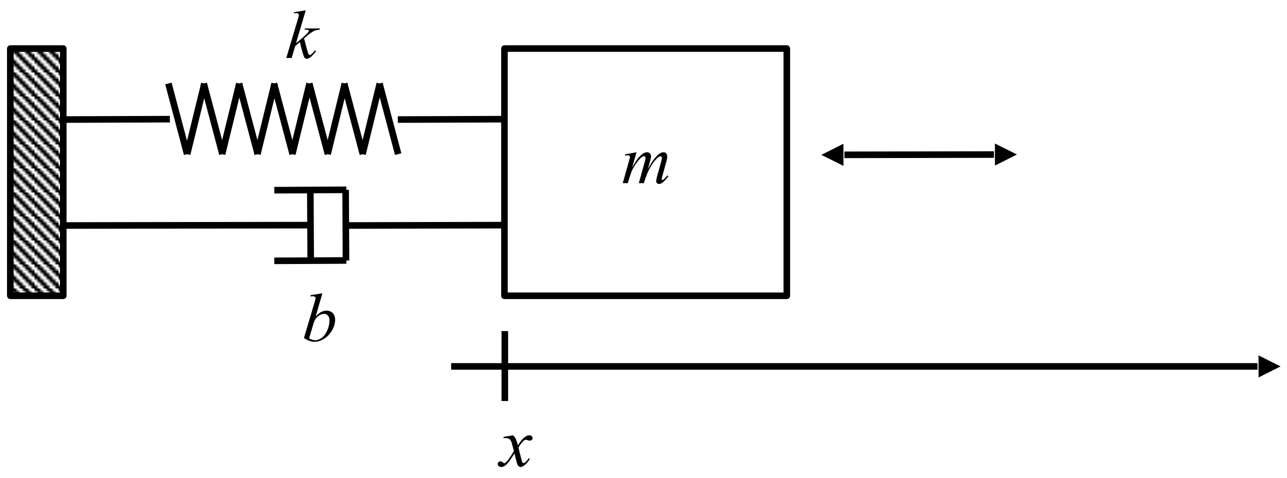
So far our analysis has been purely kinematic:

- The transient response has been ignored
- The inertia, damping, and elasticity of the plant have been ignored

But – no more!

Writing the Equations of Motion

The prototypical second order system is the mass spring damper.
Let's analyze this system:

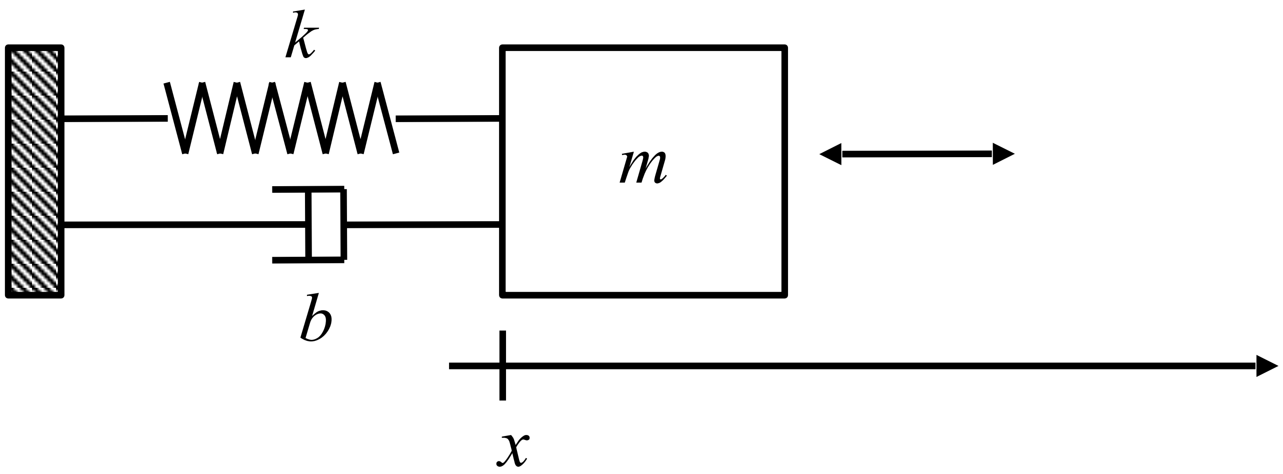


Force exerted by the spring: $f = -kx$

Force exerted by the damper: $f = -b\dot{x}$

Force exerted by the inertia of the mass: $f = -m\ddot{x}$

Writing the Equations of Motion



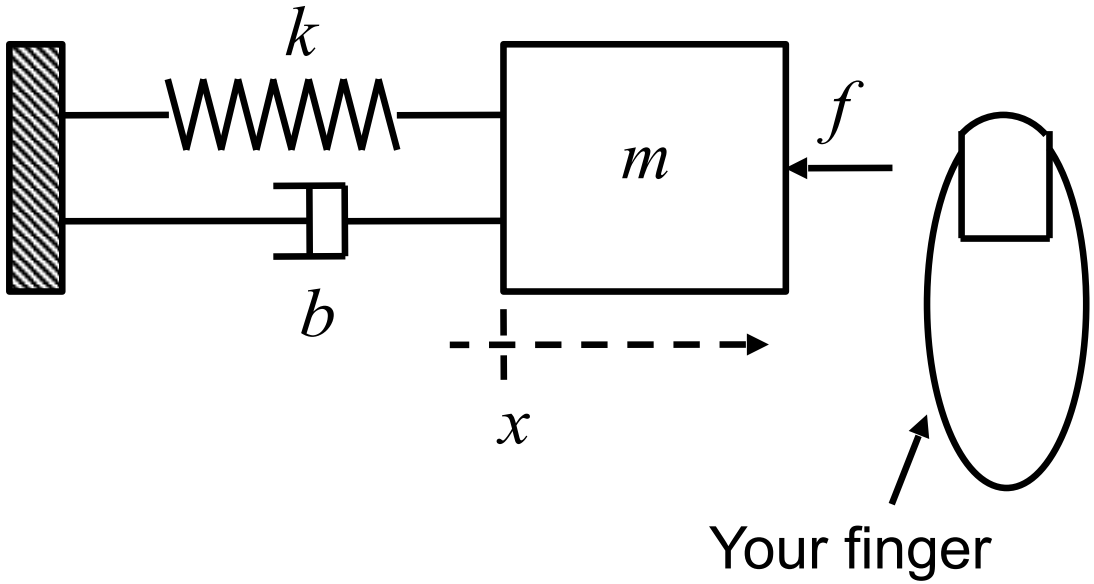
Consider the motion of the mass

- there are no other forces acting on the mass
- therefore, the equation of motion is the sum of the forces:

$$0 = m\ddot{x} + b\dot{x} + kx$$

Why is this a linear system?

Writing the Equations of Motion



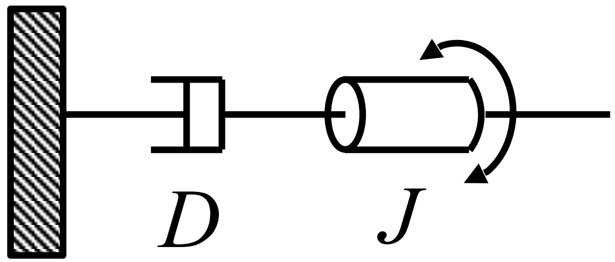
Suppose that your finger "plucks" the mass (i.e. applies a dirac delta transient force):

$$f = m\ddot{x} + b\dot{x} + kx$$

Writing the Equations of Motion: Example 1

Write the equation of motion for a torsional damped inertia:

- For example, a rotating shaft with friction



Torque due to inertia: $\tau = -J\ddot{\theta}$

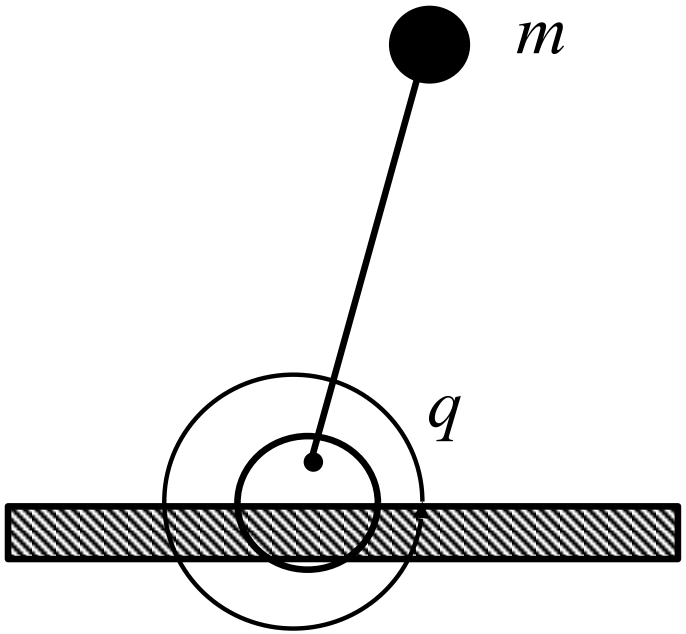
Torque due to damping: $\tau = -D\dot{\theta}$

$$\tau = J\ddot{\theta} + D\dot{\theta}$$

Writing the Equations of Motion: Example 2

Write the equations of motion for the following:

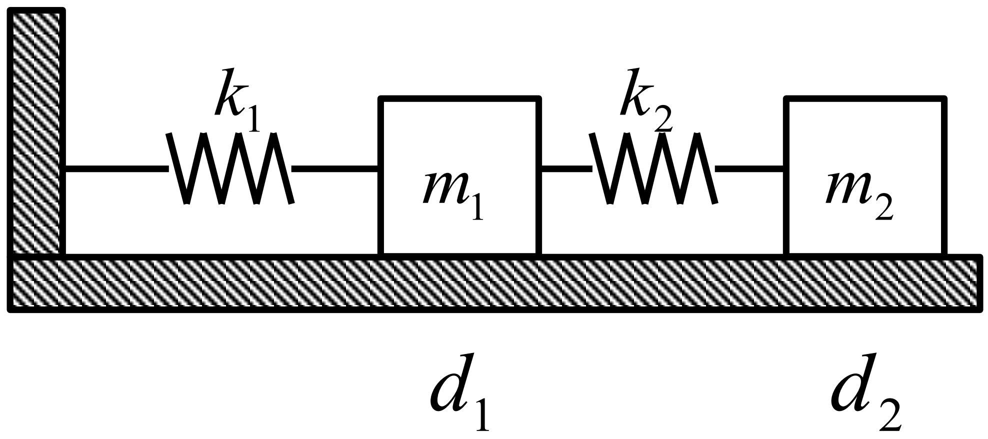
- assume a frictionless joint



$$0 = -ml^2\ddot{q} - mgl\cos(q)$$

Writing the Equations of Motion: Example 3

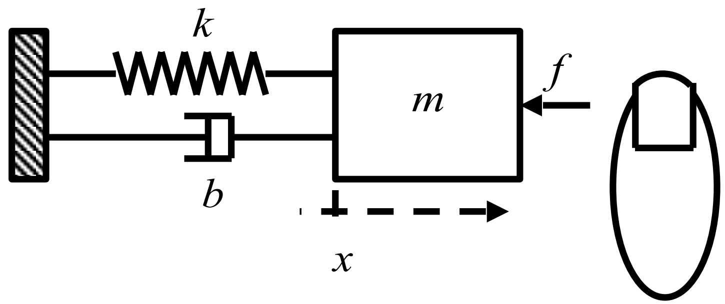
Write the equations of motion for the following:



$$0 = -m_1\ddot{x}_1 - d_1\dot{x}_1 - k_1x_1 + k_2\left(x_2 - x_1\right)$$

$$0 = -m_2\ddot{x}_2 - d_2\dot{x}_2 + k_2\left(x_1 - x_2\right)$$

The Laplace Transform



How does the mass respond to the transient force?

- You have to solve this differential equation:

$$f = m\ddot{x} + b\dot{x} + kx$$

We will analyze it using the Laplace transform:

$$L[f(t)] = F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt$$

And the inverse Laplace transform:

$$L^{-1}[F(s)] = f(t)u(t)$$

The Laplace Transform: Properties

Linearity:

$$L[kf(t)] = kF(s)$$

$$L[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$$

Frequency shift: $L[e^{-at} f(t)] = F(s+a)$

Time shift: $L[f(t-T)] = e^{-sT} F(s)$

Scaling: $L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

The Laplace Transform: Properties

Differentiation: $$L\left[\frac{df}{dt}\right] = sF(s) - f(0^-)$$

$$L\left[\frac{d^2 f}{dt^2}\right] = s^2 F(s) - sf(0^-) - f'(0^-)$$

Integration: $$L\left[\int_{0^-}^{t} f(\tau)\,d\tau\right] = \frac{1}{s}F(s)$$

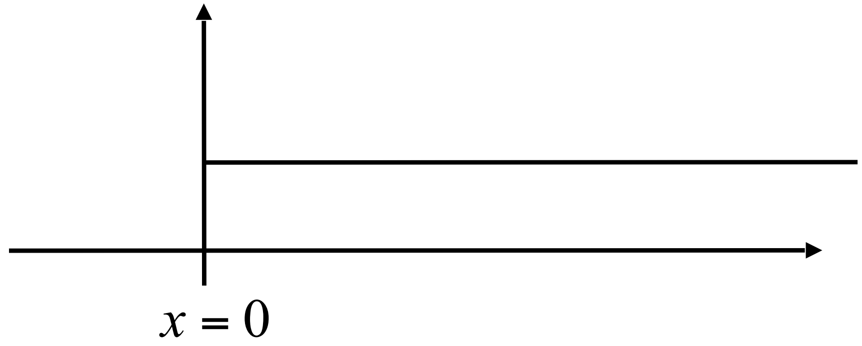
Final value theorem: $$f(\infty) = \lim_{s \to 0} sF(s)$$

The Laplace Transform: Examples

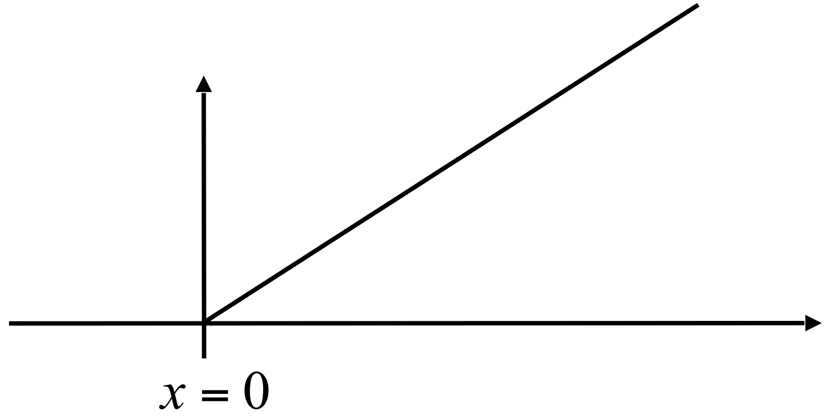
$$L[f(t)] = F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt$$

Step function: $u(t)$

$$L[u(t)] = \int_{0^-}^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_{0^-}^{\infty} = \frac{1}{s}$$



Ramp function: $tu(t)$



$$L[tu(t)] = \int_{0^-}^{\infty} t e^{-st} dt$$

$$= -\frac{t}{s} e^{-st} + \frac{1}{s} \int_{0^-}^{\infty} e^{-st} dt$$

$$= -\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} = \frac{1}{s^2}$$

The Laplace Transform: Examples

Given: $f(t) = Ae^{-at}u(t)$, find $F(s)$

$$L[u(t)] = \frac{1}{s}$$

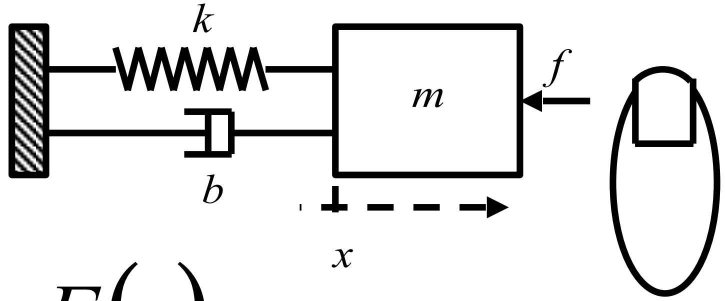
$$L\left[e^{-at}u(t)\right] = \frac{1}{s+a}$$

$$L\left[e^{-at}f(t)\right] = F(s+a)$$

$$L\left[Ae^{-at}u(t)\right] = \frac{A}{s+a}$$

$$L[kf(t)] = kF(s)$$

SMD Laplace Transform



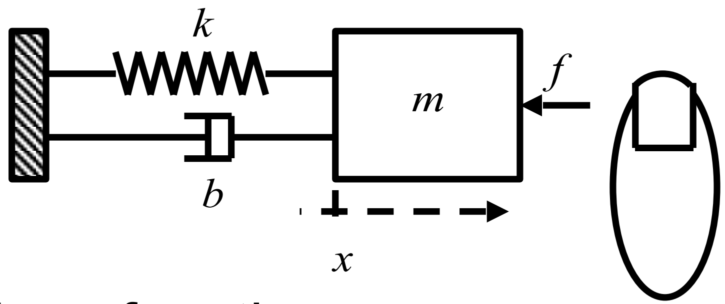
Given: $f(t) = m\ddot{x} + b\dot{x} + kx$, find $F(s)$

$$L\left[\frac{d^2x}{dt^2}\right] = s^2x$$

$$L\left[\frac{dx}{dt}\right] = sx$$

$$F(s) = s^2mX(s) + sbX(s) + kX(s)$$

SMD Transform



The Laplace transform of the SMD equation of motion:

$$F(s) = s^2 mX(s) + sbX(s) + kX(s)$$

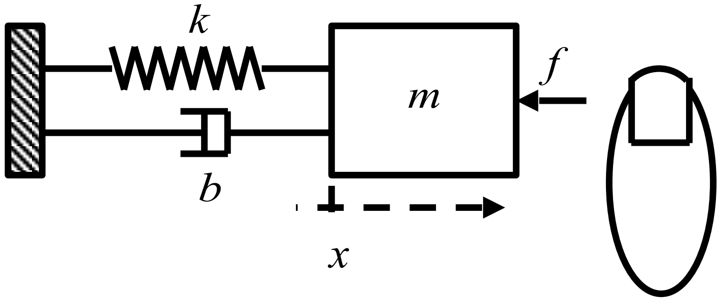
SMD transfer function:

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

This is called the “characteristic equation” of the transfer function.

Multiplying by this expression converts forces to displacements in the frequency domain

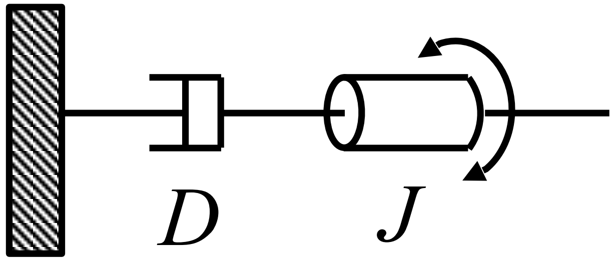
SMD Transform



$$F(s) \longrightarrow \boxed{\frac{1}{ms^2+bs+k}} \longrightarrow X(s)$$

System Transform: Example

Write the transform for the following system:



$$\tau = J\ddot{\theta} + D\dot{\theta}$$

$$T(s) = J\Theta(s)s^2 + D\Theta(s)s$$

$$\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + Ds}$$

Time out for partial fraction expansion!

Suppose you have the following: $$\frac{1}{(s+a)(s+b)(s+c)}$$

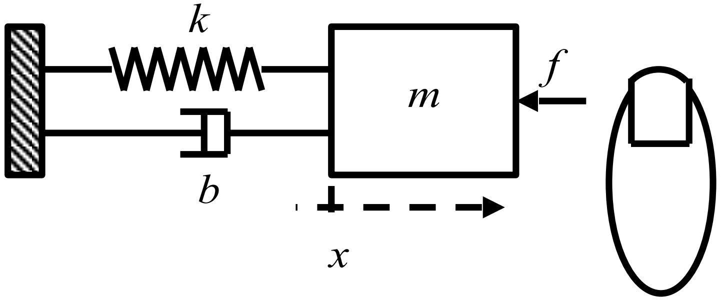
You can always decompose the denominator as follows:

$$\frac{1}{(s+a)(s+b)(s+c)} = \frac{K_1}{(s+a)} + \frac{K_2}{(s+b)} + \frac{K_3}{(s+c)}$$

for some values of K_1, K_2, K_3

You have to solve for these constants (you don't need to know how for this course).

SMD Transient Response: Example



$$G(s) = \frac{1}{s^2 + 5s + 6}$$

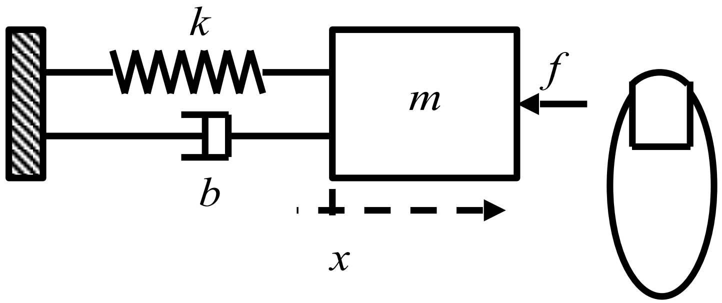
$$G(s) = \frac{1}{(s+2)(s+3)} = \frac{K_1}{(s+2)} + \frac{K_2}{(s+3)}$$

$$g(t) = K_1 e^{-2t} + K_2 e^{-3t}$$

Note that the transient response decomposes into two exponentials

- The e^{-2t} one dominates…

SMD Transient Response: Example



If you reduce damping term, then the mass will oscillate:

$$G(s) = \frac{1}{s^2 + 2s + 6}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note that in this case, the characteristic equation has complex roots:

$$-1 \pm \frac{\sqrt{4-24}}{2} = -1 \pm \frac{i\sqrt{20}}{2}$$

SMD Transient Response: Example

The roots have an imaginary component: $$-1 \pm \frac{\sqrt{4-24}}{2} = -1 \pm \frac{i\sqrt{20}}{2}$$

$$G(s) = \frac{K_1}{\left(s+1+\frac{i\sqrt{20}}{2}\right)} + \frac{K_2}{\left(s+1-\frac{i\sqrt{20}}{2}\right)}$$

Remember Euler's identity: $$e^{j\sigma} + e^{-j\sigma} = 2\cos(\sigma)$$

SMD Transient Response: Example (corrected)

$$e^{j\sigma} + e^{-j\sigma} = 2\cos(\sigma)$$

Frequency domain: $$G(s) = \frac{K_1}{\left(s + 1 + \frac{i\sqrt{20}}{2}\right)} + \frac{K_2}{\left(s + 1 - \frac{i\sqrt{20}}{2}\right)}$$

Time domain: $$g(t) = K_1 e^{\left(-1 + \frac{j\sqrt{20}}{2}\right)t} + K_2 e^{\left(-1 - \frac{j\sqrt{20}}{2}\right)t}$$

$$= e^{-t}\left(K_1 e^{\frac{j\sqrt{20}}{2}t} + K_2 e^{\frac{-j\sqrt{20}}{2}t}\right)$$

$$= e^{-t}\left(K_1 \cos\left(-\frac{\sqrt{20}}{2}t\right) + K_1 \sin\left(-\frac{\sqrt{20}}{2}t\right) + K_2 \cos\left(\frac{\sqrt{20}}{2}t\right) + K_2 \sin\left(\frac{\sqrt{20}}{2}t\right)\right)$$

SMD Transient Response: Example (corrected)

$$g(t) = e^{-t}\left(K_1 e^{\frac{j\sqrt{20}}{2}t} + K_2 e^{\frac{-j\sqrt{20}}{2}t} \right)$$

$$= e^{-t}\left(K_1 \cos\left(-\frac{\sqrt{20}}{2}t \right) + K_1 \sin\left(-\frac{\sqrt{20}}{2}t \right) + K_2 \cos\left(\frac{\sqrt{20}}{2}t \right) + K_2 \sin\left(\frac{\sqrt{20}}{2}t \right) \right)$$

$$= e^{-t}\left((K_1 + K_2)\cos\left(\frac{\sqrt{20}}{2}t \right) + (K_2 - K_1)\sin\left(\frac{\sqrt{20}}{2}t \right) \right)$$

$$= K_3 e^{-t}\left(K_4 \cos\left(\frac{\sqrt{20}}{2}t \right) + K_5 \sin\left(\frac{\sqrt{20}}{2}t \right) \right)$$

$$= K_5 e^{-t} \cos\left(\frac{\sqrt{20}}{2}t - \phi \right)$$

The bottom line: imaginary roots in the characteristic equation indicate that the system will oscillate.

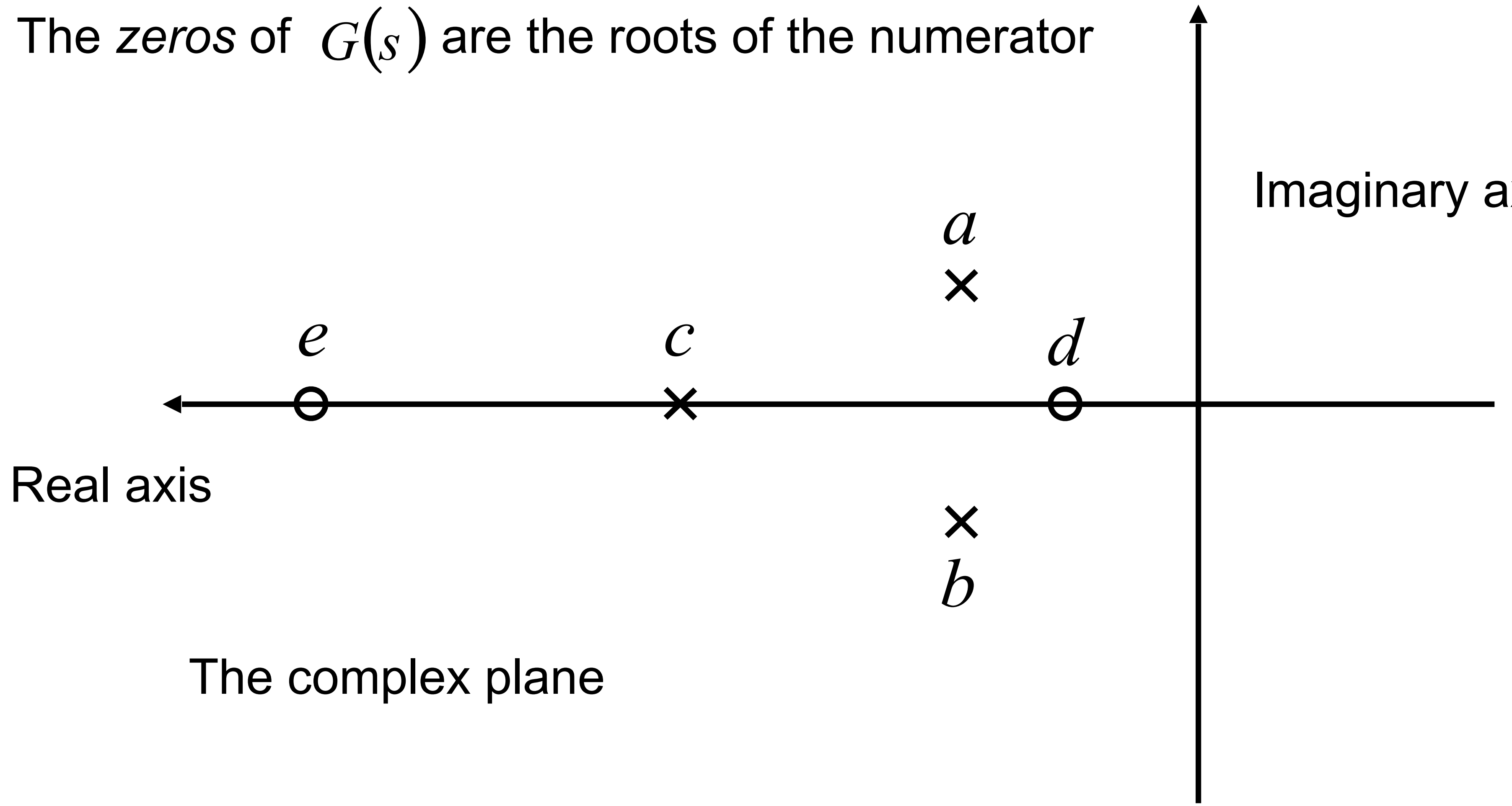
Pole-Zero Plot

Consider the following transfer fn:

$$G(s) = \frac{(s-d)(s-e)}{(s-a)(s-b)(s-c)}$$

How do we characterize the transient response?

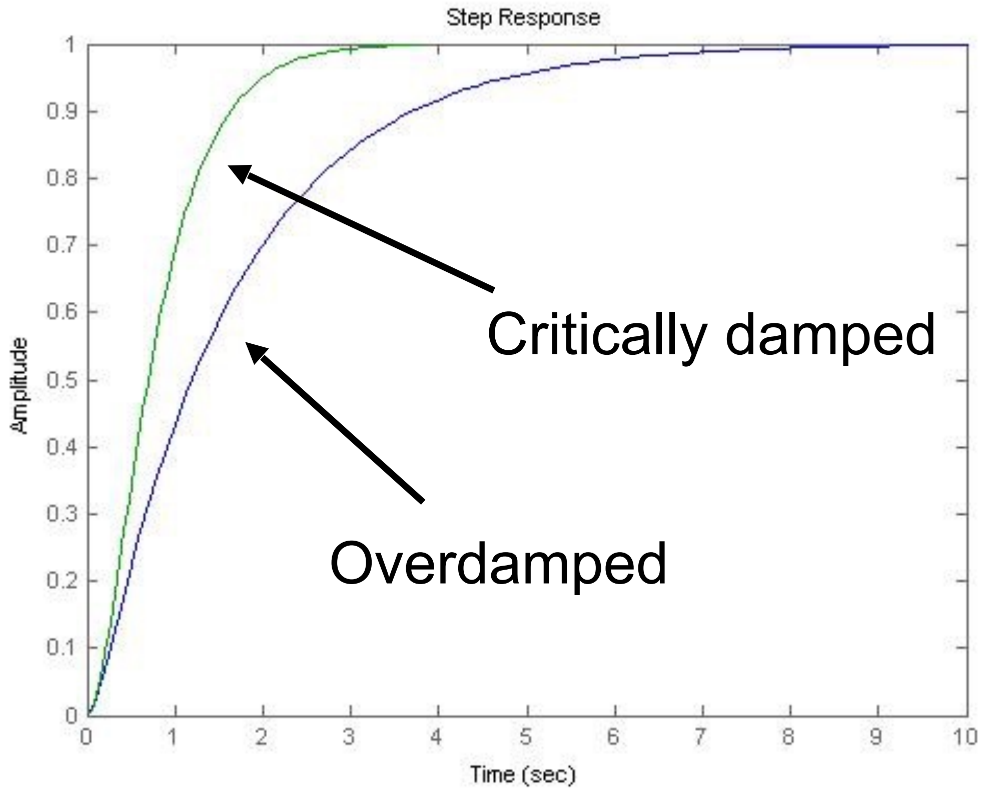
- The *poles* of $G(s)$ are the roots of the denominator
- The *zeros* of $G(s)$ are the roots of the numerator



Transient Response

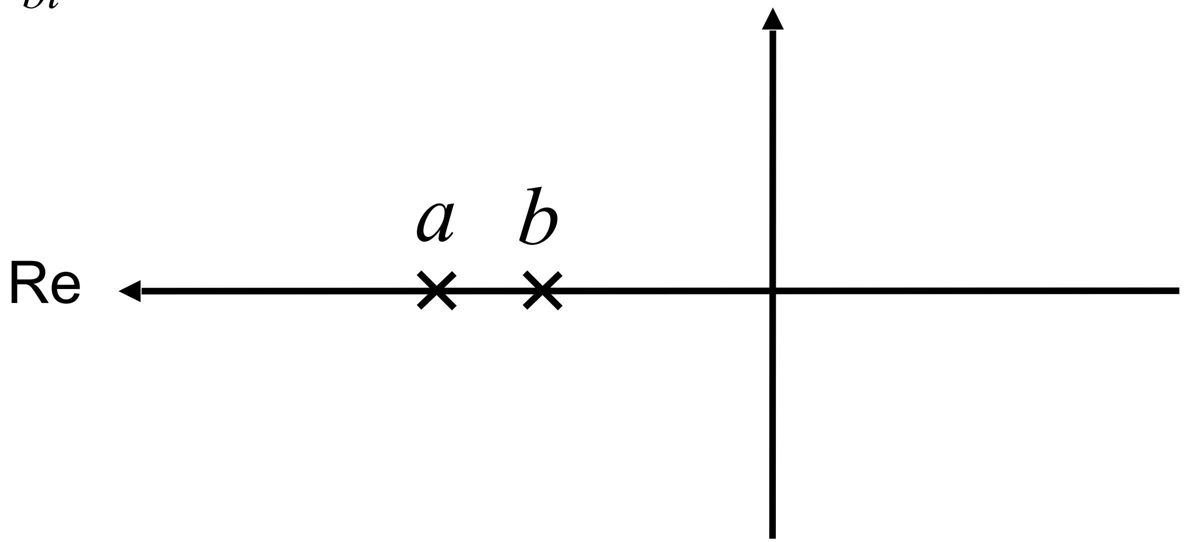
Different negative real poles correspond to non-oscillatory exponential decay

- Overdamped



$$G(s) = \frac{1}{(s+a)(s+b)}$$

$$g(t) = K_1 e^{-at} + K_2 e^{-bt}$$



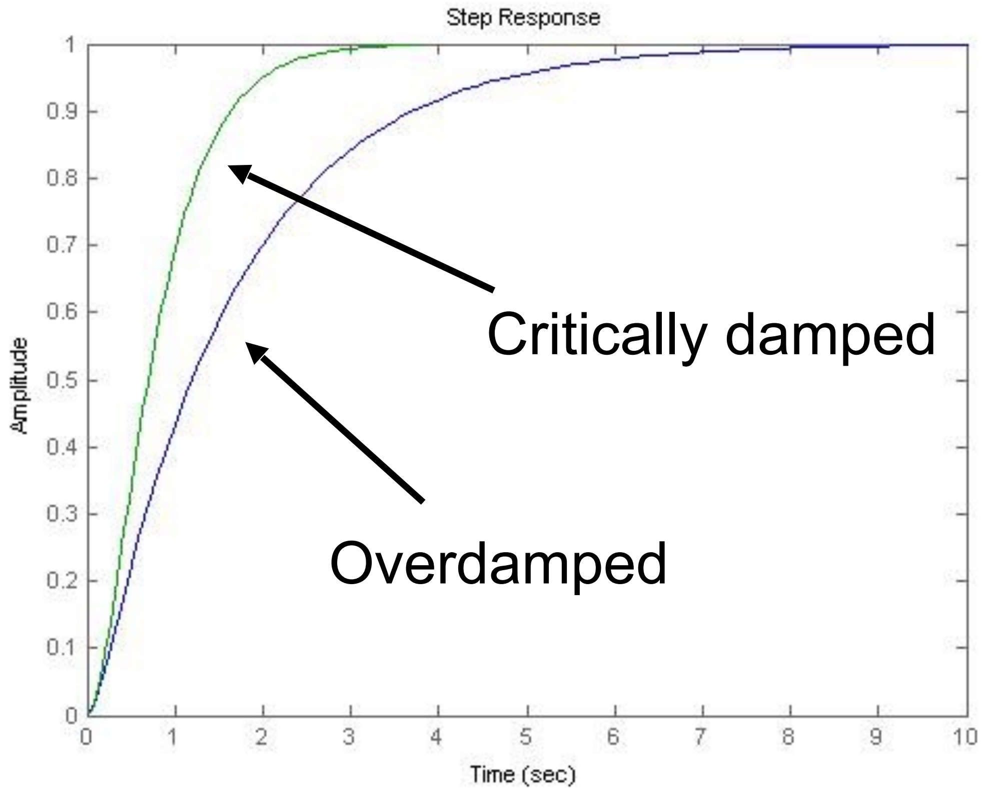
Transient Response

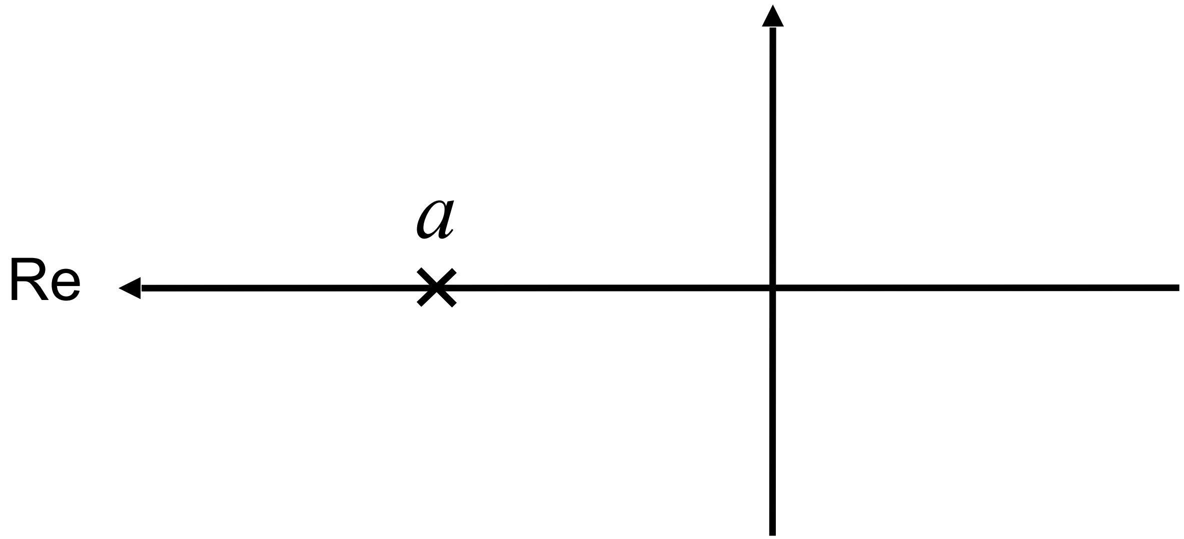
Repeated negative real poles correspond to the “fastest” non-oscillatory exponential decay possible in a second order system

- Critically damped

$$G(s) = \frac{1}{(s+a)(s+a)}$$

$$g(t) = 2K_1 e^{-at}$$

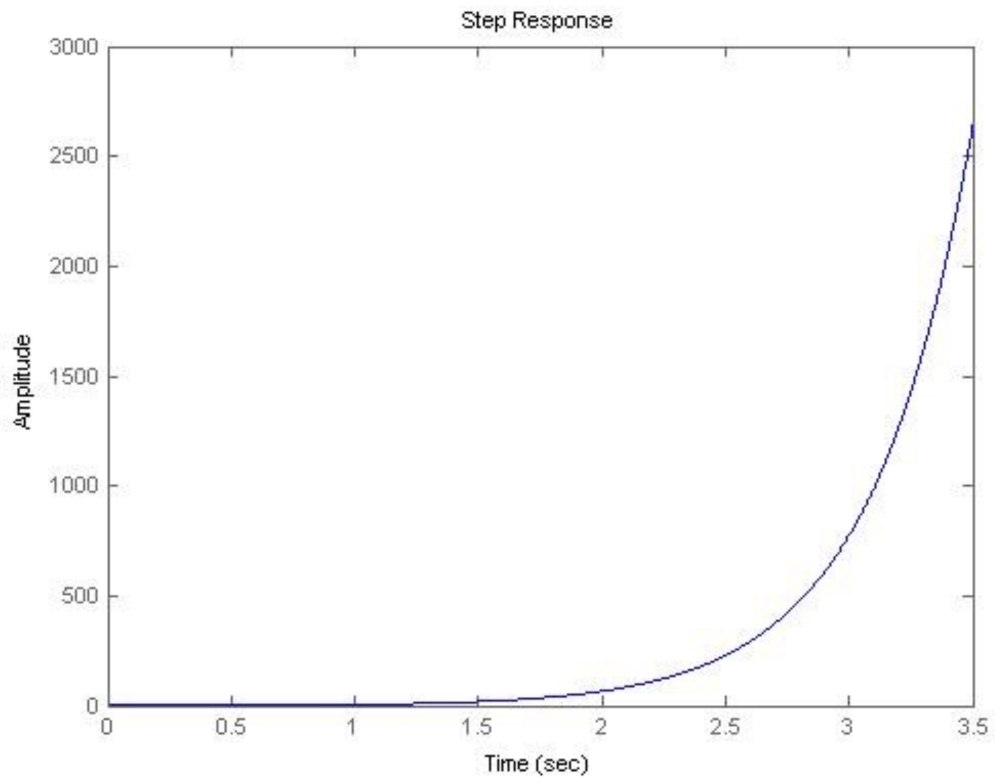


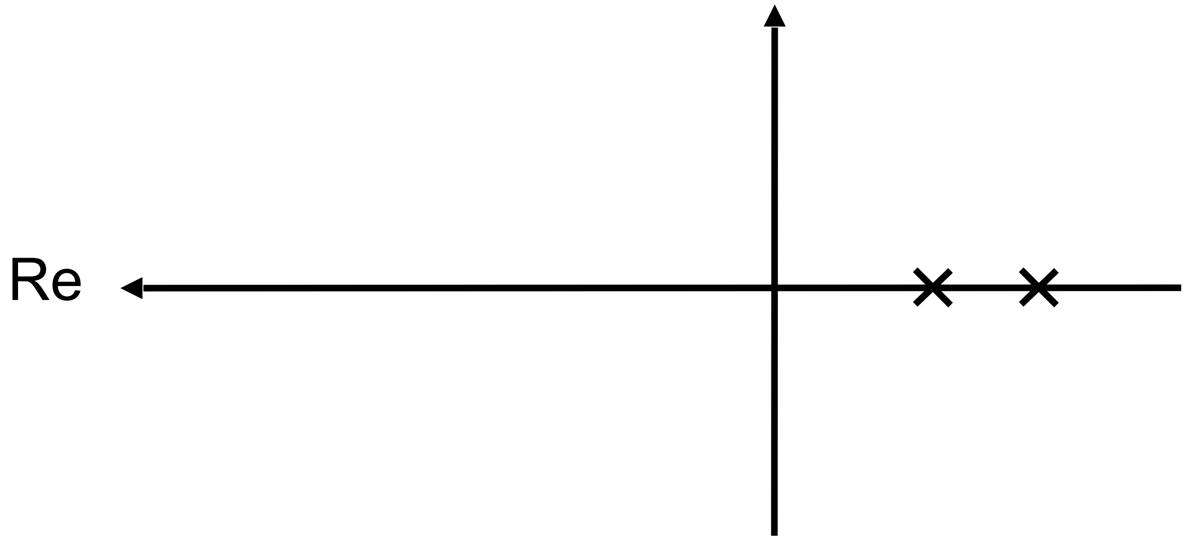


Transient Response

Positive real poles correspond to non-oscillatory exponential increase

- Not BIBO stable

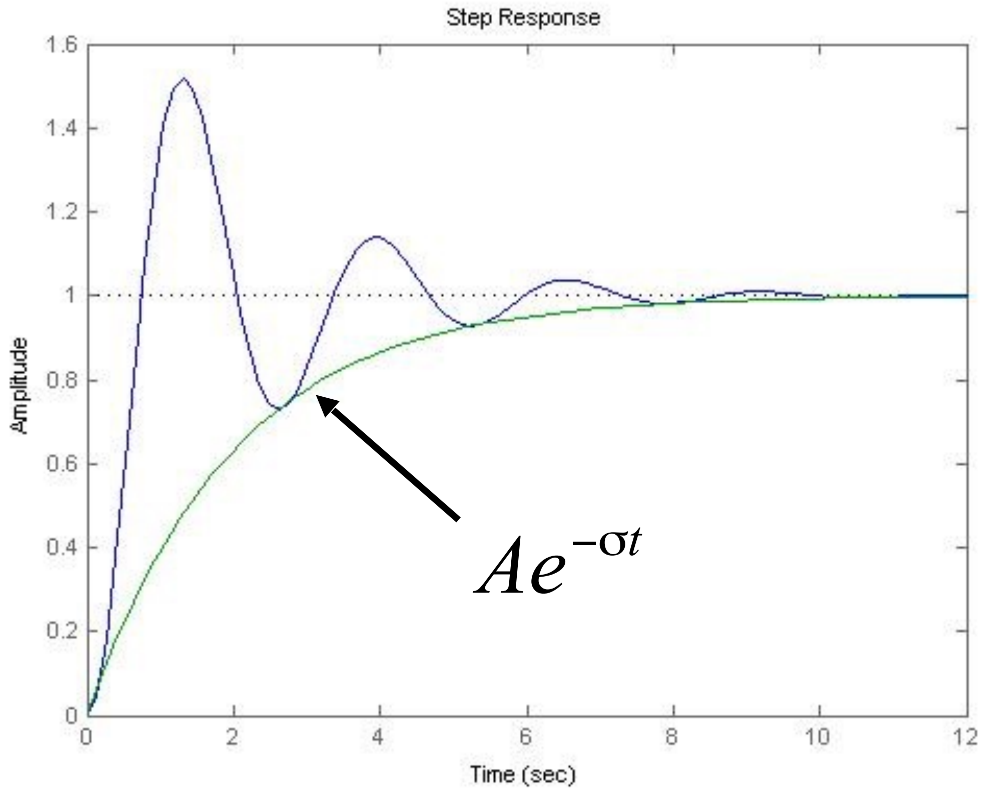




Transient Response

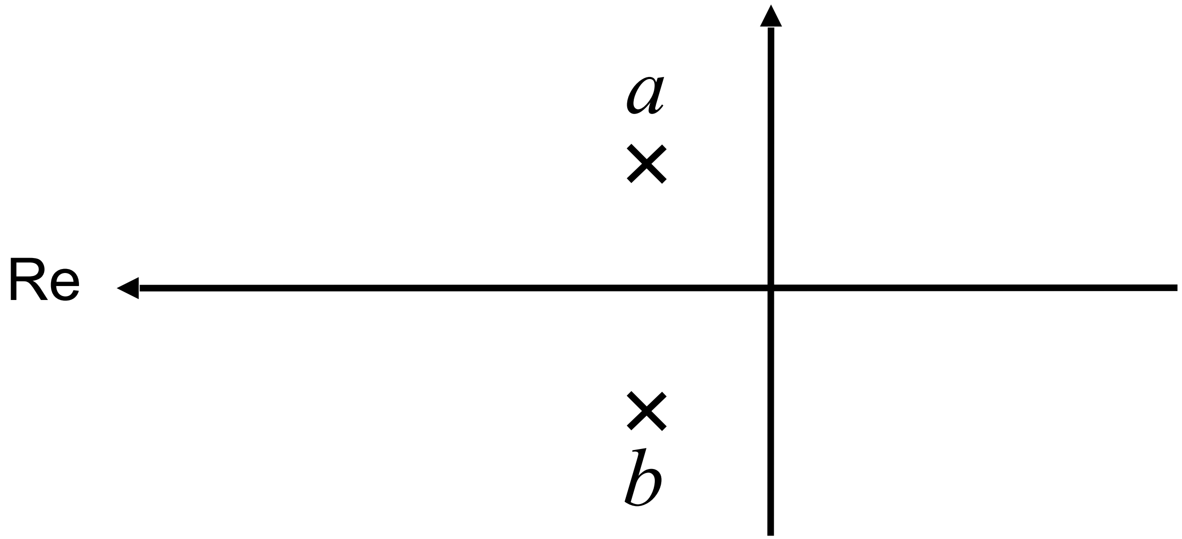
If there is an imaginary component to the root, then the system oscillates

- Underdamped



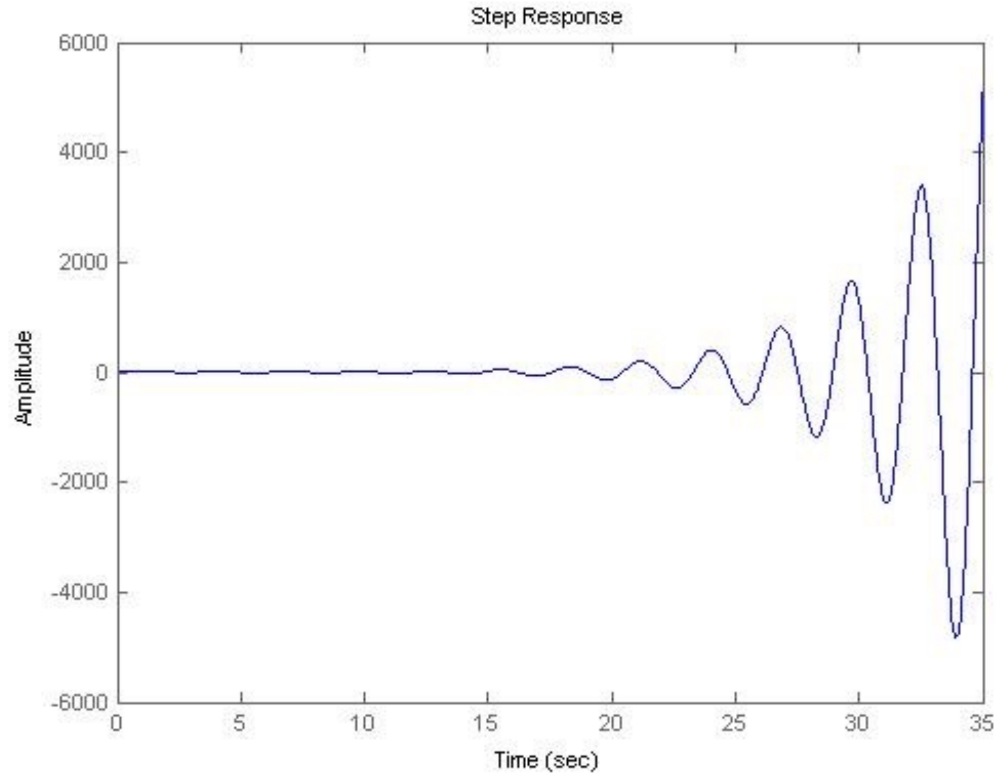
$$G(s) = \frac{1}{(s+\sigma+j\omega)(s+\sigma-j\omega)}$$

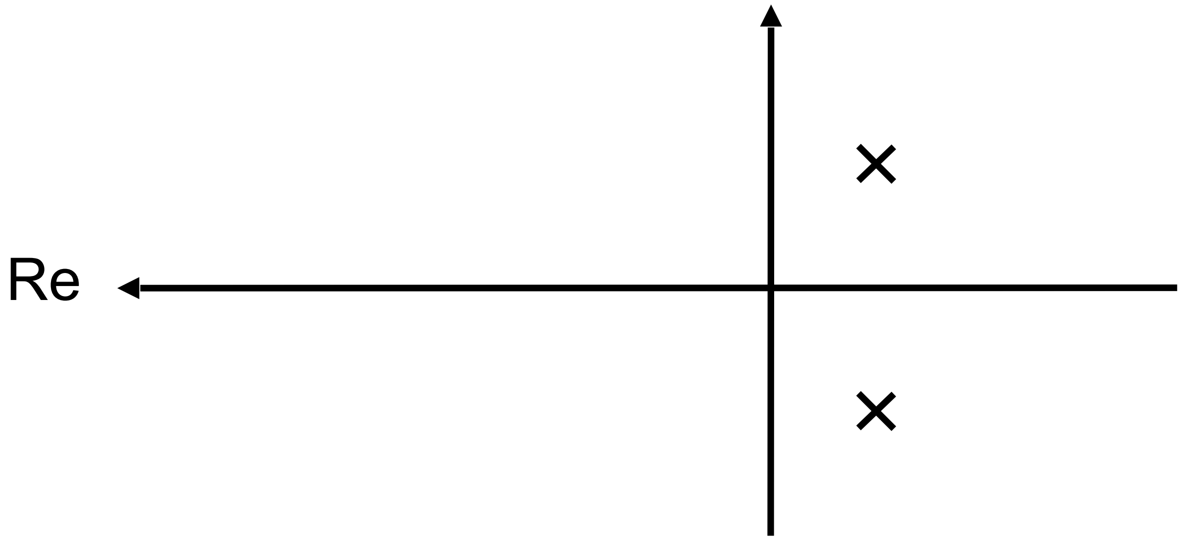
$$g(t) = Ae^{-\sigma t}\cos(\omega t - \phi)$$



Transient Response

If the real component of a complex root is positive, then the system is not BIBO stable

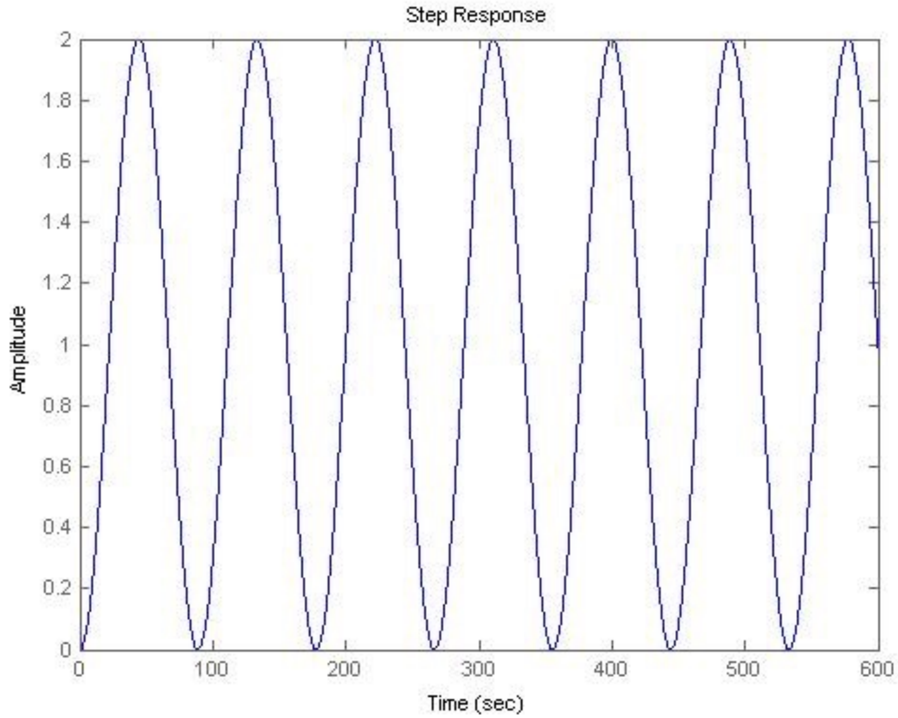


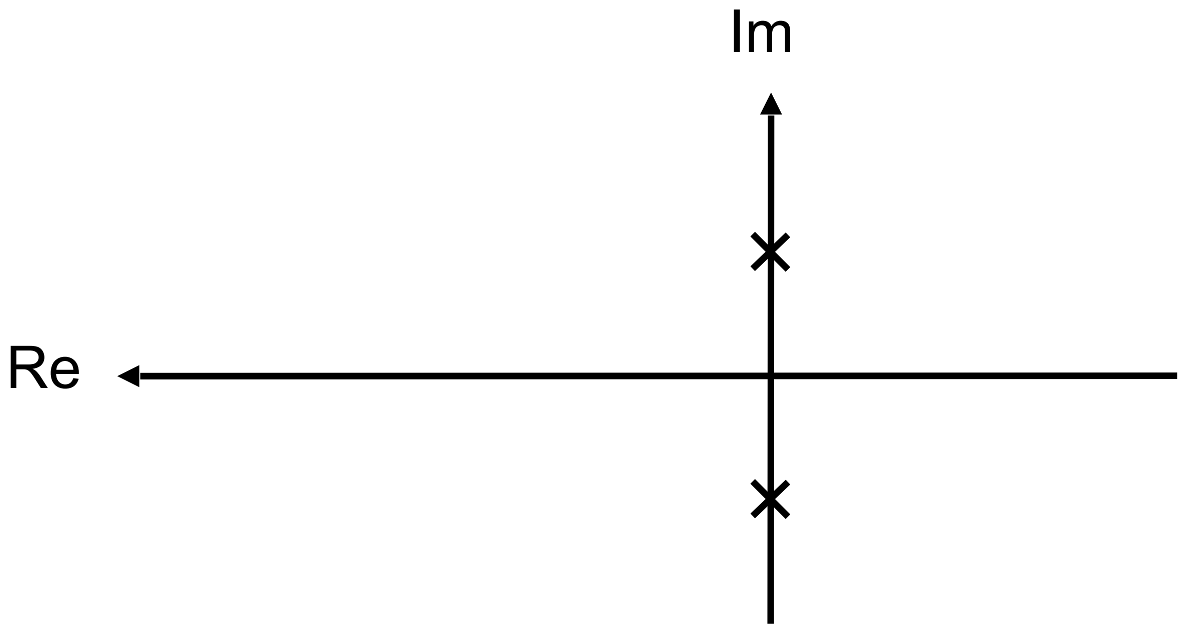


Transient Response

Purely imaginary roots cause the system to oscillate forever

- undamped





Second Order Transient Response

Second order systems can be characterized in terms of the following:

- Natural frequency: the frequency of oscillation w/o damping
- Damping ratio: exponential decay frequency / natural frequency

Second Order Transient Response

Consider the transfer fn:

$$G(s) = \frac{b}{s^2 + as + b}$$

With a zero damping term, this becomes:

$$G(s) = \frac{b}{s^2 + b}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The natural frequency is the frequency of
"pure" oscillation:

$$\frac{\sqrt{4b}}{2} = \sqrt{b} = \omega_n$$

Second Order Transient Response

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Damping ratio: exponential decay frequency / natural frequency

The damping ratio is: $\zeta = \dfrac{\frac{a}{2}}{\omega_n}$ $\quad a = 2\zeta\omega_n$

Therefore, the transfer fn can be re-written as:

$$G(s) = \frac{b}{s^2 + as + b} \longrightarrow G(s) = \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

Second Order Transient Response

The damping ratio characterizes whether the second order system is underdamped, overdamped, or critically damped:

$\zeta < 1 \longleftarrow$ underdamped

$\zeta = 1 \longleftarrow$ critically damped

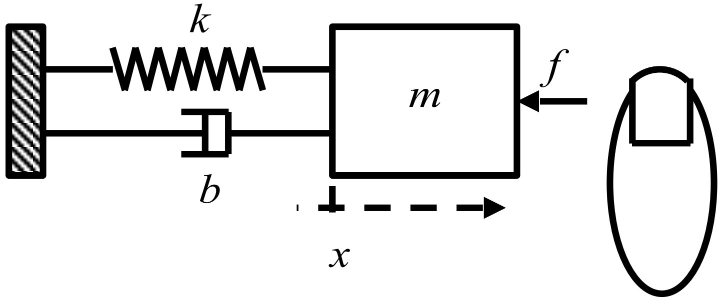
$\zeta > 1 \longleftarrow$ overdamped

If $\zeta = 1$, then $s^2 + 2\zeta\omega_n s + \omega_n^2 = (s + \omega_n)(s + \omega_n)$

Second Order Transient Response: Example (corrected)

Is the following SMD system over/under/critically damped?

$$m = 2 \quad b = 4 \quad k = 8$$



$$\frac{X(s)}{F(s)} = \frac{\frac{1}{2}}{s^2 + 2s + 4} = \frac{\frac{1}{2}}{(s+2)(s+2)}$$

Therefore, it's critically damped.

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

If $b = 2$ instead:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

then $\frac{X(s)}{F(s)} = \frac{\frac{1}{2}}{s^2 + s + 4}$

and the roots are: $\frac{-1 \pm \sqrt{-15}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{15}}{2}$

F(s)	f (t) , t > 0
$Y(s) = \int_0^{\infty} \exp(-st)y(t)dt$	definition of a Laplace transform y(t)
Y(s)	inversion formula $y(t) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \exp(st)Y(s)\,ds$
$sY(s) - y(0)$	first derivative $y'(t)$
$s^2Y(s) - sy(0) - y'(0)$	second derivative $y''(t)$
$s^nY(s) - s^{n-1}[y(0)] - s^{n-2}[y'(0)] - \cdots - s[y^{(n-2)}(0)] - [y^{(n-1)}(0)]$	nth derivative $y^{(n)}(t)$
$\frac{1}{s}F(s)$	integration $\int_0^t Y(\tau)d\tau$

The characteristic equation is

$$s^2 + 5s + 6 = 0, \tag{9.7}$$

which has the roots $s_1 = -2$ and $s_2 = -3$. Hence, the response has the form

$$x(t) = c_1 e^{-2t} + c_2 e^{-3t}. \tag{9.8}$$

We now use the given initial conditions, $x(0) = -1$ and $\dot{x}(0) = 0$, to compute c_1 and c_2. To satisfy these conditions at $t = 0$, we must have

$$c_1 + c_2 = -1$$

and

$$-2c_1 - 3c_2 = 0, \tag{9.9}$$

which are satisfied by $c_1 = -3$ and $c_2 = 2$. So, the motion of the system for $t \geq 0$ is given by

$$x(t) = -3e^{-2t} + 2e^{-3t}. \tag{9.10}$$

Complex roots

For the case where the characteristic equation has complex roots of the form

$$s_1 = \lambda + \mu i,$$
$$s_2 = \lambda - \mu i, \tag{9.11}$$

it is still the case that the solution has the form

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}. \tag{9.12}$$

However, equation (9.12) is difficult to use directly, because it involves imaginary numbers explicitly. It can be shown (see Exercise 9.1) that **Euler's formula**,

$$e^{ix} = \cos x + i \sin x, \tag{9.13}$$

allows the solution (9.12) to be manipulated into the form

$$x(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t). \tag{9.14}$$

As before, the coefficients c_1 and c_2 are constants that can be computed for any given set of initial conditions (i.e., initial position and velocity of the block). If we write the constants c_1 and c_2 in the form

$$c_1 = r \cos \delta,$$
$$c_2 = r \sin \delta, \tag{9.15}$$

$$x(t) = re^{\lambda t}\cos(\mu t - \delta), \tag{9.16}$$

where

$$r = \sqrt{c_1^2 + c_2^2},$$
$$\delta = \mathrm{Atan2}(c_2, c_1). \tag{9.17}$$

In this form, it is easier to see that the resulting motion is an oscillation whose amplitude is exponentially decreasing toward zero.

Another common way of describing oscillatory second-order systems is in terms of **damping ratio** and **natural frequency**. These terms are defined by the parameterization of the characteristic equation given by

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0, \tag{9.18}$$

where ζ is the damping ratio (a dimensionless number between 0 and 1) and ω_n is the natural frequency.[2] Relationships between the pole locations and these parameters are

$$\lambda = -\zeta\omega_n$$

and

$$\mu = \omega_n\sqrt{1-\zeta^2}. \tag{9.19}$$

In this terminology, μ, the imaginary part of the poles, is sometimes called the **damped natural frequency**. For a damped spring–mass system such as the one in Fig. 9.2, the damping ratio and natural frequency are, respectively,

$$\zeta = \frac{b}{2\sqrt{km}},$$
$$\omega_n = \sqrt{k/m}. \tag{9.20}$$

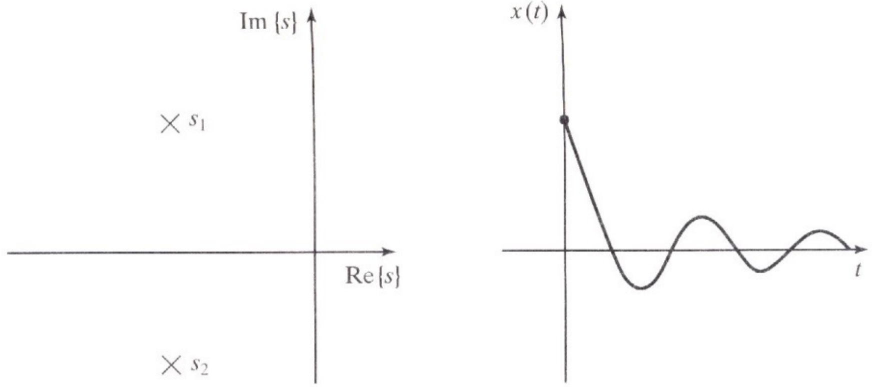


FIGURE 9.4: Root location and response to initial conditions for an underdamped system.

which has the roots $s_i = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$. Hence, the response has the form

$$x(t) = e^{-\frac{t}{2}} \left(c_1 \cos \frac{\sqrt{3}}{2} t + c_2 \sin \frac{\sqrt{3}}{2} t \right). \tag{9.22}$$

We now use the given initial conditions, $x(0) = -1$ and $\dot{x}(0) = 0$, to compute c_1 and c_2. To satisfy these conditions at $t = 0$, we must have

$$c_1 = -1$$

and

$$-\frac{1}{2} c_1 - \frac{\sqrt{3}}{2} c_2 = 0, \tag{9.23}$$

which are satisfied by $c_1 = -1$ and $c_2 = \frac{\sqrt{3}}{3}$. So, the motion of the system for $t \geq 0$ is given by

$$x(t) = e^{-\frac{t}{2}} \left(-\cos \frac{\sqrt{3}}{2} t - \frac{\sqrt{3}}{3} \sin \frac{\sqrt{3}}{2} t \right). \tag{9.24}$$

This result can also be put in the form of (9.16), as

$$x(t) = \frac{2\sqrt{3}}{3} e^{-\frac{t}{2}} \cos \left(\frac{\sqrt{3}}{2} t + 120^\circ \right). \tag{9.25}$$

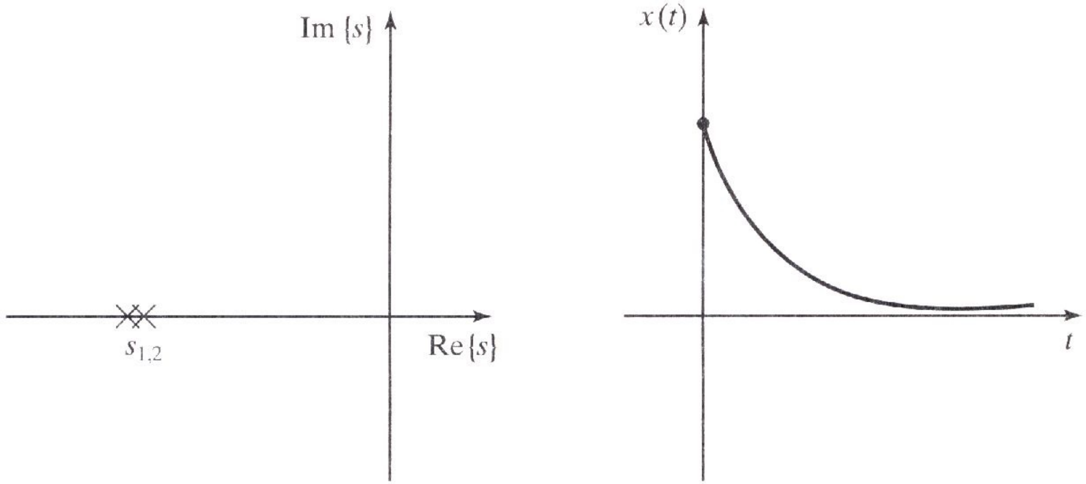


FIGURE 9.5: Root location and response to initial conditions for a critically damped system.

where, in this case, $s_1 = s_2 = -\frac{b}{2m}$, so (9.26) can be written

$$x(t) = (c_1 + c_2 t)e^{-\frac{b}{2m}t}. \tag{9.27}$$

In case it is not clear, a quick application of **l'Hôpital's rule** [2] shows that, for any c_1, c_2, and a,

$$\lim_{t\to\infty}(c_1 + c_2 t)e^{-at} = 0. \tag{9.28}$$

Figure 9.5 shows an example of pole locations and the corresponding time response to a nonzero initial condition. When the poles of a second-order system are real and equal, the system exhibits critically damped motion, the fastest possible nonoscillatory response.

$$m\ddot{x} + b\dot{x} + kx = f. \tag{9.34}$$

Let's also assume that we have sensors capable of detecting the block's position and velocity. We now propose a **control law** which computes the force that should be applied by the actuator as a function of the sensed feedback:

$$f = -k_p x - k_v \dot{x}. \tag{9.35}$$

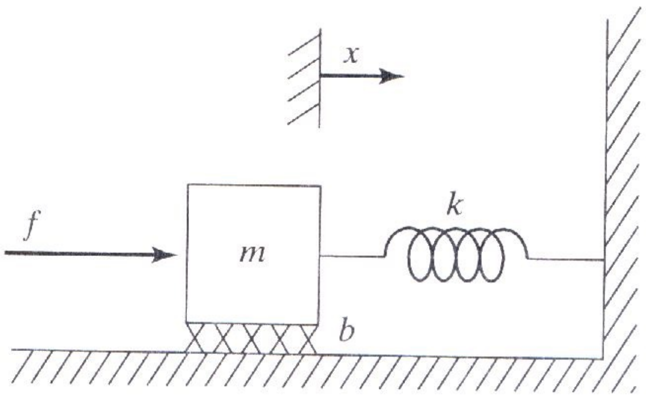


FIGURE 9.6: A damped spring–mass system with an actuator.

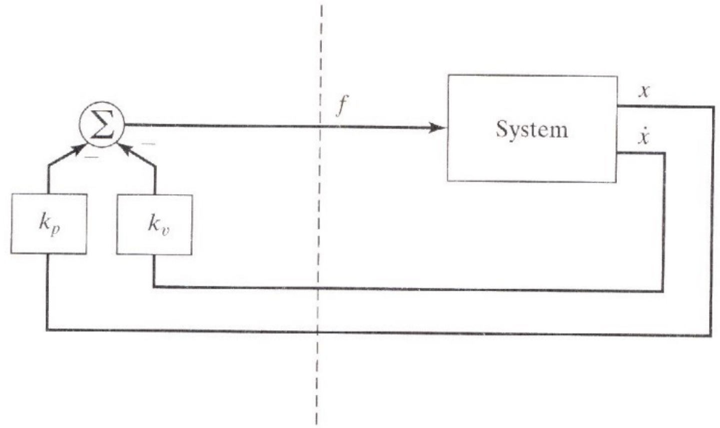


FIGURE 9.7: A closed-loop control system. The control computer (to the left of the dashed line) reads sensor input and writes actuator output commands.

of disturbance forces applied to the block. In a later section, we will construct a **trajectory-following** control system, which can cause the block to follow a desired position trajectory.

By equating the open-loop dynamics of (9.34) with the control law of (9.35), we can derive the closed-loop dynamics as

$$m\ddot{x} + b\dot{x} + kx = -k_p x - k_v \dot{x}, \tag{9.36}$$

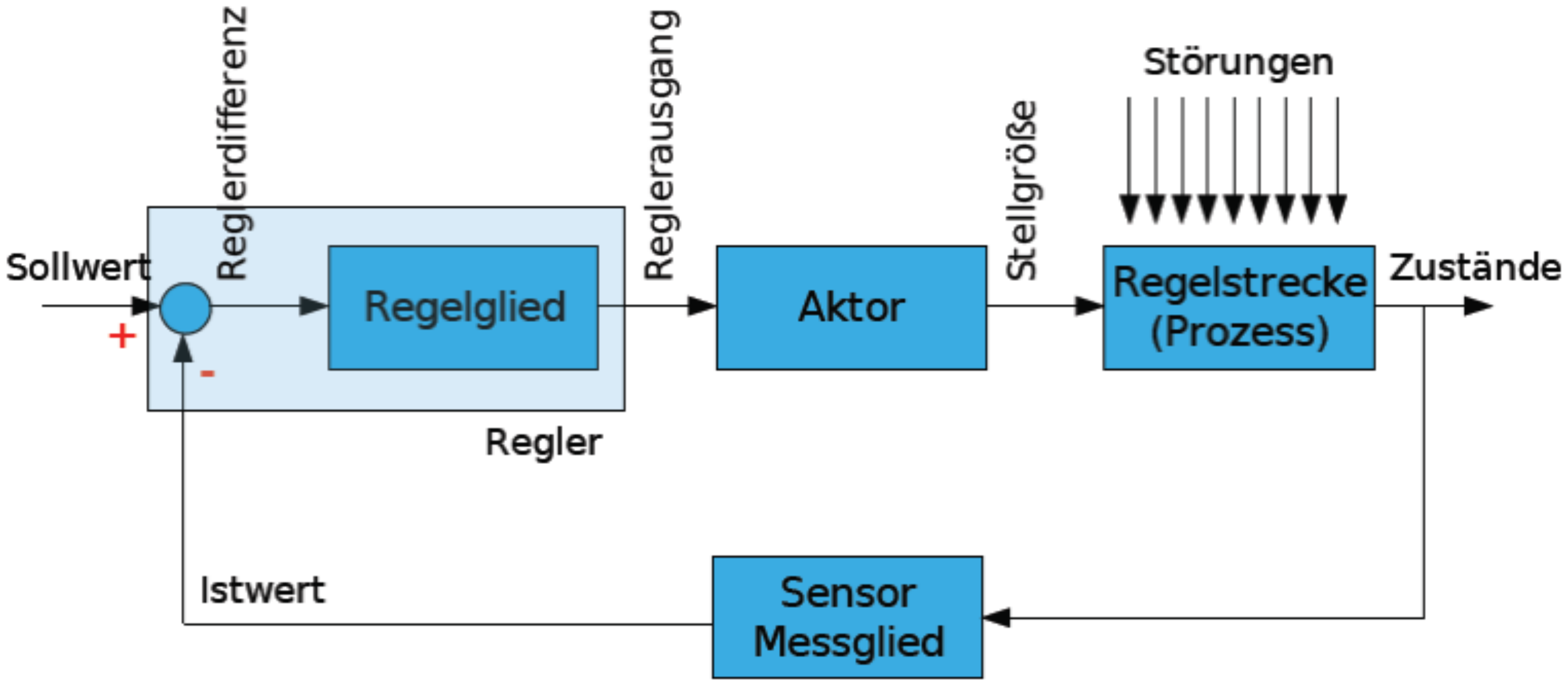
or

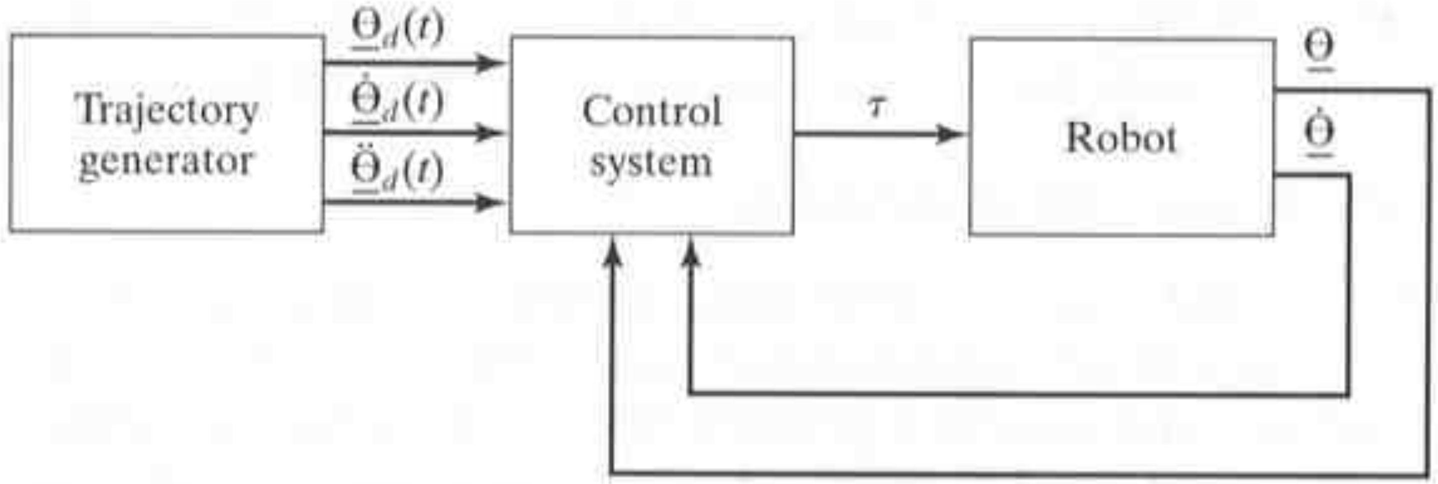
$$m\ddot{x} + (b + k_v)\dot{x} + (k + k_p)x = 0, \tag{9.37}$$

or

$$m\ddot{x} + b'\dot{x} + k'x = 0, \tag{9.38}$$

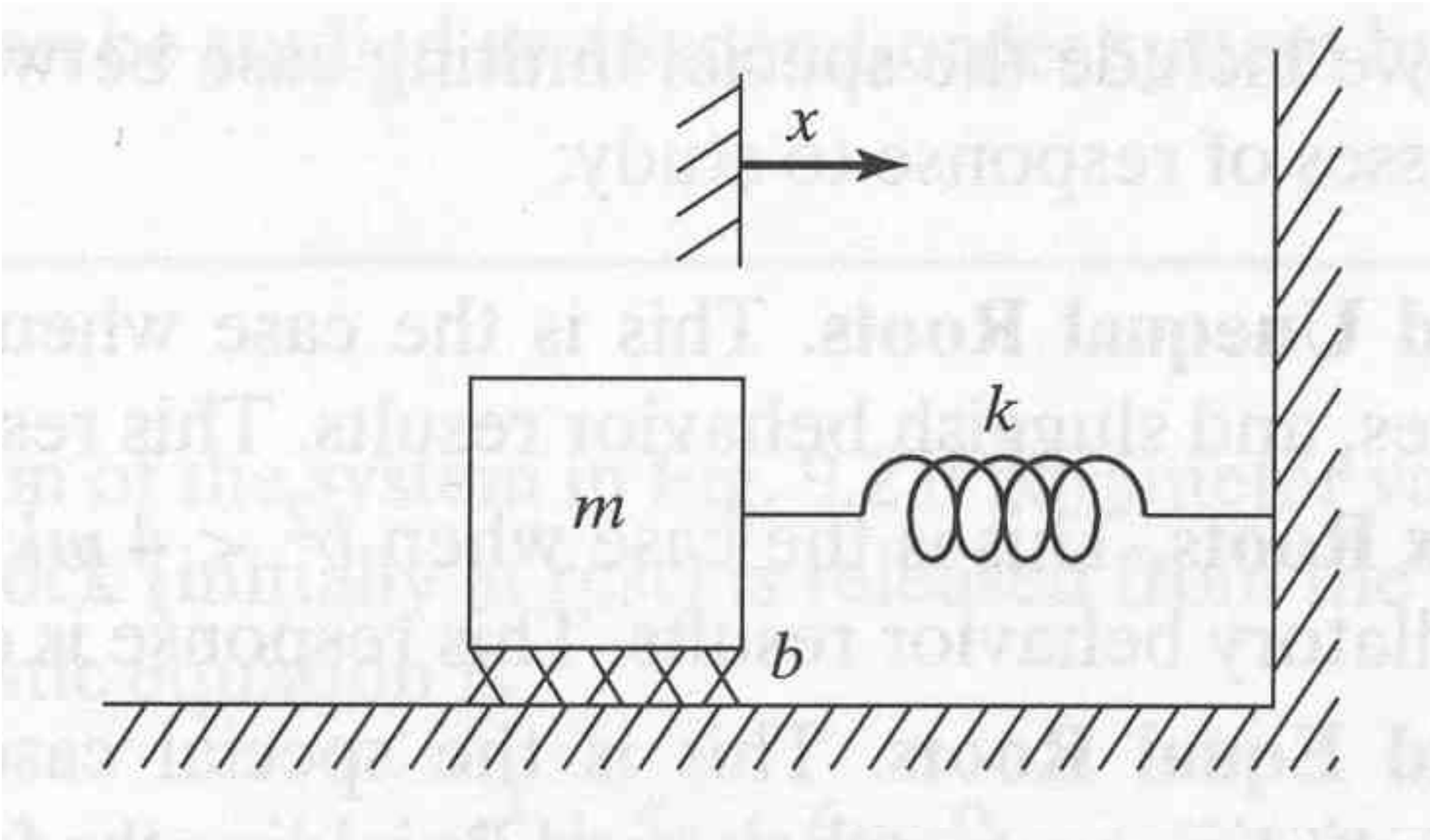
Control Loop





$$\tau = M(\Theta_d)\ddot{\Theta}_d + V(\Theta_d, \dot{\Theta}_d) + G(\Theta_d)$$

Masse-Feder System



$$m\ddot{x} + b\dot{x} + kx = 0$$

$$ms^2 + bs + k = 0$$