Multiple View Geometry: Exercise 1

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Exercise: May 03, 2023

Math Background

1. Show for each of the following sets (1) whether they are linearly independent, (2) whether they span $\mathbb{R}^3$ and (3) whether they form a basis of $\mathbb{R}^3$:

   (a) $B_1 = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$

   (b) $B_2 = \left\{ \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\}$

   (c) $B_3 = \left\{ \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}3\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$

2. Which of the following sets forms a group (with matrix-multiplication)? Prove or disprove!

   (a) $G_1 := \left\{A \in \mathbb{R}^{n\times n} \middle| \det(A) \neq 0 \wedge A^\top = A\right\}$

   (b) $G_2 := \{A \in \mathbb{R}^{n\times n} | \det(A) = -1\}$

   (c) $G_3 := \{A \in \mathbb{R}^{n\times n} | \det(A) > 0\}$

3. Prove or disprove: There exist vectors $\mathbf{v}_1, ..., \mathbf{v}_5 \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$, which are pairwise orthogonal, i.e.

$$\forall i, j = 1, ..., 5 : \quad i \neq j \implies \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$$

4. Which groups have you seen in the lecture? Write down the names and the correct inclusions! (e.g.: group A $\subset$ group B)

5. Let A be a symmetric matrix, and λ_a, λ_b eigenvalues with eigenvectors v_a and v_b. Prove: if v_a and v_b are not orthogonal, it follows: $\lambda_a = \lambda_b$.

   Hint: What can you say about $\langle Av_a, v_b \rangle$?

6. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with the orthonormal basis of eigenvectors $v_1, \ldots, v_n$ and eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. Find all vectors x, that minimize the following term:

$$\min_{||x||=1} x^\top Ax$$

1. Show for each of the following sets (1) whether they are linearly independent, (2) whether they span $\mathbb{R}^3$ and (3) whether they form a basis of $\mathbb{R}^3$:

   (a) $B_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ det $B_1 \neq 0$

   (b) $B_2 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

   (c) $B_3 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

a linear independent

span R^3

basis

b) linear independent

span R^3 need 3 vector

~~basis~~ not basis

c) not independent

span R^3

not basis $\Leftarrow$ no more than 3 independent basis vector

2. Which of the following sets forms a group (with matrix-multiplication)? Prove or disprove!

(a) $G_1 := \{A \in \mathbb{R}^{n\times n} | \det(A) \neq 0 \wedge A^\top = A\}$

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(c) $G_3 := \{A \in \mathbb{R}^{n\times n} | \det(A) > 0\}$

(a) not closed under a group

$(AB)^T = B^T A^T = BA \neq AB$

(b) consist no neutral element

(c)

3. Prove or disprove: There exist vectors $\mathbf{v}_1, ..., \mathbf{v}_5 \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$, which are pairwise orthogonal, i.e.

$$\forall i, j = 1, ..., 5 : \quad i \neq j \implies \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$$

in 3 demenison, exist maximal 3 orthogonal vector which can genen te frame

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GL, Linear group

SO, special ortogonal

SE, special ecualidan

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Hint: What can you say about $\langle Av_a, v_b \rangle$?

$$Z = v_a^T A v_b = Z^T = v_b^T A^T v_a = v_b^T A v_a$$

$$\Downarrow \qquad\qquad (A v_b)^T v_a \qquad\qquad \Downarrow$$

$$v_a^T \lambda_b v_b = \qquad \lambda_b v_b^T v_a = v_b^T \lambda_a v_a$$

$$(\lambda_b - \lambda_a) \underline{(v_b^T v_a)} = 0$$

6. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with the orthonormal basis of eigenvectors $v_1, \ldots, v_n$ and eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. Find all vectors x, that minimize the following term:

$$z = \min_{||x||=1} x^\top A x$$

How many solutions exist? How can the term be maximized?

Hint: Use the expression $x = \sum_{i=1}^{n} \alpha_i v_i$ with coefficients $\alpha_i \in \mathbb{R}$ and compute appropriate coefficients!

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j v_i^\top A v_j$$

$$\frac{\partial Z}{\partial V} = \sum_{i=1}^{n} \sum_{j=1}^{n} 2\alpha_i \alpha_j \underbrace{A v_j}_{\lambda_j v_j} = 0$$

$$= 0$$

7. Let $A \in \mathbb{R}^{m\times n}$. Prove that $\text{kernel}(A) = \text{kernel}(A^\top A)$.

Hint: Consider a) $x \in \text{kernel}(A) \Rightarrow x \in \text{kernel}(A^\top A)$
and b) $x \in \text{kernel}(A^\top A) \Rightarrow x \in \text{kernel}(A)$.

$x \in \text{kenel}(A) \Rightarrow Ax = 0 \Rightarrow A^T A x = 0$, $x \in \text{kernel}(A^T A)$

propeity

8. Singular Value Decomposition (SVD)

Let $A = USV^\top$ be the SVD of A.

(a) Write down possible dimensions for A, U, S and V.

(b) What are the similarities and differences between the SVD and the eigenvalue decomposition?

(c) What do you know about the relationship between U, S, V and the eigenvalues and eigenvectors of $A^\top A$ and $AA^\top$?

(d) What is the interpretation of the entries in S and what do the entries of S tell us about A?

a) $A \in R^{m\times n}$ $V \in R^{m\times m}$ $S \in R^{m\times r}$ $V \in R^{n\times r}$

b) A must be squard matrix

$v^T A^T A v = \lambda^2 v^T v$

$\cancel{V^T} V S^T \cancel{U^T U} S V^T V = \lambda^2 v^T v$

How many solutions exist? How can the term be maximized?

Hint: Use the expression $x = \sum_{i=1}^{n} \alpha_i v_i$ with coefficients $\alpha_i \in \mathbb{R}$ and compute appropriate coefficients!

7. Let $A \in \mathbb{R}^{m \times n}$. Prove that kernel$(A) =$ kernel$(A^\top A)$.

   Hint: Consider a) $x \in$ kernel$(A) \Rightarrow x \in$ kernel$(A^\top A)$
   and b) $x \in$ kernel$(A^\top A) \Rightarrow x \in$ kernel(A).

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