FUNDAMENTALS OF ARTIFICIAL INTELLIGENCE
Exercise 3: CSP – Solutions — Winter Semester 2023/24
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1 Presented Problems

Problem 3.1: Turning n-ary constraints into binary constraints

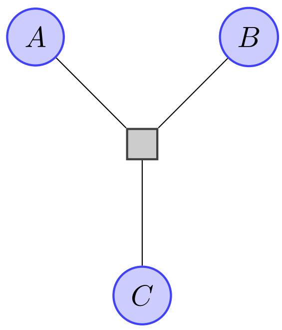
(from *Russell & Norvig 3ed.* q. 7.6) Suppose that we have $CSP = (X, D, E^1)$ with

$$\begin{aligned} X &= \{A, B, C\}, \\ D &= \{\text{dom}(A), \text{dom}(B), \text{dom}(C)\}, \\ E &= \{\langle (A, B, C),\ A + B = C\rangle\}, \end{aligned}$$

where dom(A), dom(B), and dom(C) denote the domain of variable A, B, and C, respectively, and each domain can be $\{0, 1, \ldots, 9\}$ for example.

Problem 3.1.1: Draw the constraint hypergraph for the CSP. In this case, a hypergraph is a graph with two types of nodes. The first type of node represents the *variables*, depicted by ○, and the second type of node represents the constraint, depicted by □. Based on the number of variables involved, what is the type of the constraint?

Solution:



There is only one constraint in set E, and there are three variables involved in this constraint. Therefore, this constraint is a higher-order constraint (3-ary constraint).

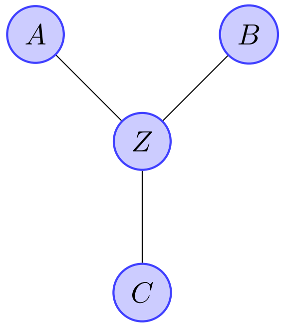
Problem 3.1.2: We can eliminate the higher-order constraint in E by replacing the constraint node □ with a new variable node Z. (We denote this new CSP as CSP'.) What is the domain for variable Z? (Hint: The domain for variable Z can be ordered pairs of other values.) What is the new constraint set E' after introducing the new variable Z?

Solution: The domain for variable Z is a 3-tuple:

$$\text{dom}(Z) = \{(z_1, z_2, z_3) \mid z_1 \in \text{dom}(A) \wedge z_2 \in \text{dom}(B) \wedge z_3 \in \text{dom}(C)\}.$$

Instead of the higher-order constraint E, we now have the constraint set E' containing an unary constraint on node Z, which is $\langle (Z), z_1 + z_2 = z_3\rangle$, and binary constraints between adjacent nodes. The new CSP

[1]the symbol E is taken from German word *Einschränkung*.



can be formulated as $CSP' = (X', D', E')$ with

$$
\begin{aligned}
X' &= \{A, B, C, Z\}, \\
D' &= D \cup \text{dom}(Z), \\
E' &= \{\langle (A, Z),\ \mathit{fst}(Z) = A\rangle,\ \langle (B, Z),\ \mathit{snd}(Z) = B\rangle,\ \langle (C, Z),\ \mathit{thrd}(Z) = C\rangle, \\
&\quad \langle (Z),\ z_1 + z_2 = z_3\rangle\},
\end{aligned}
$$

where *fst, snd,* and *thrd* are operators to get the first, second, and third element of the tuple in Z.

Problem 3.1.3: Modify CSP' such that it only contains binary constraints and formally express the new $CSP'' = (X'', D'', E'')$.

Solution: We can eliminate the unary constraint of a variable by altering its domain such that all values in the domain satisfy the constraint. This is called node-consistency.

Thus, we reduce the domain of variable Z to be

$$\text{dom}(Z) = \{(z_1, z_2, z_3) \mid z_1 + z_2 = z_3 \wedge z_1 \in \text{dom}(A) \wedge z_2 \in \text{dom}(B) \wedge z_3 \in \text{dom}(C)\}.$$

As a result, we obtain a CSP with only binary constraints: $CSP'' = (X'', D'', E'')$ with

$$
\begin{aligned}
X'' &= \{A, B, C, Z\}, \\
D'' &= D \cup \text{dom}(Z), \\
E'' &= \{\langle (A, Z),\ \mathit{fst}(Z) = A\rangle,\ \langle (B, Z),\ \mathit{snd}(Z) = B\rangle,\ \langle (C, Z),\ \mathit{thrd}(Z) = C\rangle\}.
\end{aligned}
$$

Problem 3.1.4: Taking inspiration from previous solutions, how can you generally turn a n-ary constraint into binary constraints?

Solution: Suppose that we have a constraint of order n. We can represent this constraint of order n with a relation $R(x_1, x_2, \ldots, x_n)$ of order n.

1. We replace the n-ary constraint by a new variable Z. The domain of Z is a n-tuple and must be restricted such that it satisfies the relation R (e.g., see the domain of variable Z in Problem 3.1.3).
2. Then, we introduce new binary constraints to match the values of variable Z with the values of the neighboring variables $x_1, x_2, \ldots, x_n$ (i.e., $\mathit{fst}(Z) = x_1$, $\mathit{snd}(Z) = x_2$, and so on).

This procedure is known as hidden transformation[2]. Research also gives guidance for when transforming CSP is beneficial[3].

[2] `https://doi.org/10.1016/S0004-3702(02)00210-2`
[3] `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.110.7551&rep=rep1&type=pdf`

Problem 3.2: Arc consistency and backtracking search for binary constraints

Consider the constraint satisfaction problem in Fig. 1. According to the picture, we have $CSP = (X, D, E)$ with

$$\begin{aligned}
X &= v_1, v_2, v_3, v_4, v_5, \\
D &= \{\text{dom}(v_1), \text{dom}(v_2), \text{dom}(v_3), \text{dom}(v_4), \text{dom}(v_5)\}, \\
E &= \{\langle (v_1, v_2),\ v_2 = v_1 + 1\rangle, \\
&\quad \langle (v_1, v_3),\ v_1 \neq v_3\rangle \\
&\quad \langle (v_2, v_3),\ v_2 \neq v_3\rangle, \\
&\quad \langle (v_3, v_4),\ v_3 \neq v_4\rangle, \\
&\quad \langle (v_3, v_5),\ v_3 \neq v_5\rangle, \\
&\quad \langle (v_4, v_5),\ v_4 \neq v_5\rangle, \\
&\quad \langle (v_1, v_5),\ v_1 \neq v_5\rangle\},
\end{aligned}$$

where $\text{dom}(v_1)$, $\text{dom}(v_2)$, $\text{dom}(v_3)$, $\text{dom}(v_4)$ and $\text{dom}(v_5)$ denote the domain of variable v_1, v_2, v_3, v_4 and v_5, respectively, and each domain is initially $\{2, 3, 4\}$. Note that all constraints in the graph are binary constraints.

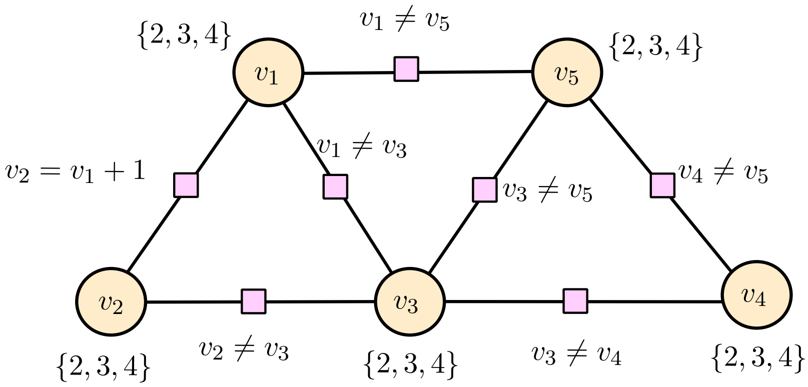


Figure 1: Constraint graph for Problem 3.2

Problem 3.2.1: Sort the variables once by their domain size (i.e. number of remaining values) and once by their degree (i.e. number of constraints on other unassigned variables).

Solution: Note that the minimum-remaining-values (MRV) heuristic always chooses the variable that has the smaller domain, while degree heuristics chooses the variable with the highest degree, i.e., involved in the highest number of constraints.

Initially all variables have the same domain size, i.e., 3. Using Fig. 1, we can count the degree of each variable and then sort the variables by their degree (cf. Table 1).

Variable	Degree
v_3	4
v_1, v_5	3
v_2, v_4	2

Table 1: Variables sorted by their degree.

Problem 3.2.2: Perform backtracking search by hand to solve the CSP problem: Determine which variable to expand next by applying the minimum-remaining-values (MRV) heuristic; if there is a tie, use degree heuristics; if there is a tie again, choose the variable with the lower index. Use least-constraining-value heuristics to decide which value to assign; if there is a tie, set the value to 3; if this is not possible choose the lowest value. After each assignment, perform forward checking as inference. Backtrack if you find an inconsistency.

Solution: For each recursion depth, the assigned variable and the remaining values for each variable (after performing forward checking) is shown in Table 2. If the number of remaining values is zero for at least one variable, we need to backtrack. In the following, we describe the search steps (where the bold number denotes the depth of recursion).

		current domains					degree					
step	assignment	v_1	v_2	v_3	v_4	v_5	v_1	v_2	v_3	v_4	v_5	backtrack
0	$\emptyset$	$\{2,3,4\}$	$\{2,3,4\}$	$\{2,3,4\}$	$\{2,3,4\}$	$\{2,3,4\}$	3	2	4	2	3	
1	$v_3 = 3$	$\{2,4\}$	$\{2,4\}$	/	$\{2,4\}$	$\{2,4\}$	2	1	/	1	2	
2	$v_1 = 2$	/	$\emptyset$	/	$\{2,4\}$	$\{4\}$	/	0	/	1	1	backtrack to 2
2	$v_1 = 4$	/	$\emptyset$	/	$\{2,4\}$	$\{2\}$	/	0	/	1	1	backtrack to 1
1	$v_3 = 2$	$\{3,4\}$	$\{3,4\}$	/	$\{3,4\}$	$\{3,4\}$	2	1	/	1	2	
2	$v_1 = 3$	/	$\{4\}$	/	$\{3,4\}$	$\{4\}$	/	0	/	1	1	
3	$v_5 = 4$	/	$\{4\}$	/	$\{3\}$	/	/	0	/	0	/	
4	$v_2 = 4$	/	/	/	$\{3\}$	/	/	/	/	0	/	
5	$v_4 = 3$	/	/	/	/	/	/	/	/	/	/	

Table 2: Backtracking search with forward checking as inference. For each variable, the number of remaining values (domain size) and the number of constraints (degree) is listed. The rows correspond to the steps of the backtracking algorithm, we assign a value to one variable and perform forward checking (i.e. updating the domain of connected variables).

1 We apply MRV heuristics to choose the variable to expand first. Since all variables have the same number of remaining values, MRV holds a tie. Thus, we use degree heuristics and choose the variable with the highest degree (cf. Table 1), i.e., v_3. To choose a value, we use least-constraining-value heuristic. Every value of v_3 will leave the same number of values for the neighboring variables v_1, v_2, v_4 and v_5. Since we have a tie, we choose $v_3 = 3$ (see exercise description). Then, we perform forward checking: The reduced domain of v_1, v_2, v_4 and v_5 are all equal to $\{2,4\}$. All remaining variables were connected to the assigned variable v_3 by a constraint, thus we reduce their degree by one.

2 All variables have the same number of remaining values, but v_1 and v_5 have a higher degree. We choose variable v_1, since it has the lower index. Applying least-constraining-value heuristic shows that both values of v_1 leave the same number of values for v_2 and v_5. Since we cannot assign the value of 3, we assign the lowest value $v_1 = 2$. For forward checking, we update the variable v_2 (and v_5) and we see that its domain size is reduced to zero, and we backtrack immediately to the assignment of v_1 restoring the domains.

2 The only other possible value for v_1 is 4. Therefore, we assign $v_1 = 4$. During forward checking, we update the variable v_2 (and v_5) and we see that its domain size is again reduced to zero, so we backtrack and restore the domains. Since we already tried all possible values for v_1, we backtrack to the assignment of v_3.

1 We assign $v_3 = 2$ and reduce the domain of all other variables to $\{3,4\}$.

2 Analogously to the previous try, we have a tie between v_1 and v_5 and choose variable v_1. Using least-constraining-value heuristic, we choose $v_1 = 3$ (Note: Assignment $v_1 = 4$ would leave no options for v_2). With forward checking we reduce the domains of v_2 and v_5.

3 Notice that the domain size of v_2 is equal to one and the variable itself is not involved in any constraints. Since we did not backtrack, this actually means that a valid assignment for v_2 has already been found. But according to MRV heuristics and degree heuristics we select $v_5 = 4$ and reduce the domain of v_4.

4 The variable v_2 and v_4 have the same number of remaining values and the same degree. Since it is a tie, we choose the variable with the lower index, i.e., v_2. Thus, we assign $v_2 = 4$.

5 In the final step we choose $v_4 = 3$.

Problem 3.2.3: Perform backtracking search again, but with a different inference: Determine which variable to expand next by applying the minimum-remaining-values (MRV) heuristic; if there is a tie, use degree heuristics; if there is a tie again, choose the variable with the lower index. Use least-constraining-value heuristics to decide which value to assign; if there is a tie, set the value to 3; if this is not possible choose the lowest value. After each assignment, perform the arc consistency algorithm. Backtrack if you find an inconsistency.

Solution: For each recursion depth, the assigned variable and the remaining values for each variable is shown in Table 3 (after performing arc consistency algorithm). If the number of remaining values is zero for at least one variable, we need to backtrack. In the following, we describe the search steps (where the bold number denotes the depth of recursion).

		current domains					degree					
step	assignment	v_1	v_2	v_3	v_4	v_5	v_1	v_2	v_3	v_4	v_5	backtrack
0	$\emptyset$	$\{2,3,4\}$	$\{2,3,4\}$	$\{2,3,4\}$	$\{2,3,4\}$	$\{2,3,4\}$	3	2	4	2	3	
1	$v_3 = 3$	$\{2,4\}$	$\emptyset$	/	$\{2,4\}$	$\{2,4\}$	2	1	/	1	2	backtrack to 1
1	$v_3 = 2$	$\{3\}$	$\{4\}$	/	$\{3\}$	$\{4\}$	2	1	/	1	2	
2	$v_1 = 3$	/	$\{4\}$	/	$\{3\}$	$\{4\}$	/	0	/	1	1	
3	$v_4 = 3$	/	$\{4\}$	/	/	$\{4\}$	/	0	/	/	0	
4	$v_2 = 4$	/	/	/	/	$\{4\}$	/	/	/	/	0	
5	$v_5 = 4$	/	/	/	/	/	/	/	/	/	/	

Table 3: Backtracking search with arc consistency algorithm as inference. For each variable, the number of remaining values (domain size) and the number of constraints (degree) is listed. The rows correspond to the steps of the backtracking algorithm, we assign a value to one variable and perform arc consistency algorithm (i.e. updating the domain of all variables).

1 Analogously to the first step of the previous subproblem, we choose $v_3 = 3$. We apply the arc consistency algorithm, initializing the queue with the arcs (v_1, v_3), (v_2, v_3), (v_4, v_3) and (v_5, v_3):

- We pop the first element from the queue, i.e. (v_1, v_3), and remove the inconsistent values. Since we remove the value 3 from the domain of v_1, we add the arcs to all neighbours of v_1 (except v_3) to the queue, i.e., (v_2, v_1) and (v_5, v_1).
- In the next step, we check arc consistency for (v_2, v_3) and thus remove the value 3 from the domain of v_2. Again we add the arcs to all neighbours of v_2 (except v_3) to the queue, i.e., (v_1, v_2).

- In the same way the inconsistent value 3 is removed from the domains of v_4 and v_5 and the arcs to all neighbours of v_4 and v_5 (except v_3), respectively, are added to the queue. The queue after these steps is: (v_2, v_1), (v_5, v_1), (v_1, v_2), (v_5, v_4), (v_1, v_5), (v_4, v_5).
- In the following step, we make v_2 arc consistent with v_1. Therefore, we remove the values 2 and 4 from the domain of v_2. The arc consistency algorithm returns failure, since the size of the domain of v_2 is zero. We immediately backtrack to the assignment of v_3 restoring the domains.

1 According to the exercise description, we assign to v_3 the lowest value: we choose $v_3 = 2$. We apply again the arc consistency algorithm, initializing the queue with the arcs (v_1, v_3), (v_2, v_3), (v_4, v_3) and (v_5, v_3):

- Similar to the previous assignment, the value 2 is removed from the domains of v_1, v_2, v_4 and v_5, and the arcs to all neighbours of v_1, v_2, v_4 and v_5 (except v_3), respectively, are added to the queue. The queue after these steps is: (v_2, v_1), (v_5, v_1), (v_1, v_2), (v_5, v_4), (v_1, v_5), (v_4, v_5).
- After that, we remove the inconsistent value 3 from v_2 in order to make v_2 arc consistent with v_1. We add the arcs to all neighbours of v_2 (except v_1), i.e., (v_3, v_2).
- (v_5, v_1) is already arc consistent. Thus, we do not remove any values from the domain of v_5 and do not add any new arcs to the queue.
- Next, we make v_1 arc consistent with v_2 by removing the value 4 from the domain of v_1. Then we add the arcs to all neighbours of v_1 (except v_2) to the queue, i.e., (v_3, v_1) and (v_5, v_1). The queue after these steps is: (v_5, v_4), (v_1, v_5), (v_4, v_5), (v_3, v_2), (v_3, v_1), (v_5, v_1).
- Since (v_5, v_4), (v_1, v_5), (v_4, v_5), (v_3, v_2) and (v_3, v_1) are already arc consistent, these arcs are popped from the queue and no new arcs are added to the queue.
- In order to make v_5 arc consistent with v_1, the value 3 is removed from the domain of v_5. We add the arcs to all neighbours of v_5 (except v_1) to the queue, i.e., (v_3, v_5), (v_4, v_5) to the queue.
- (v_3, v_5) is already arc consistent. Thus, this arc is popped from the queue without any changes in the domain of v_3.
- Now we remove the inconsistent value 4 from the domain of v_4 in order to make v_4 arc consistent with v_5. We add the arcs to all neighbours of v_4 (except v_5) to the queue, i.e., (v_3, v_4).
- Since (v_3, v_4) is already arc consistent, this arc is popped from the queue and the algorithm terminates, as the queue is empty.

The resulting CSP has only one possible value per variable and, since we did not backtrack, it is arc consistent. This means that we have actually already found a solution for the CSP. In the next steps we assign the found values to the variables in the order defined by the exercise description.

2 Since v_1 and v_5 have both the highest degree, we have a tie and assign $v_1 = 3$.

3 Similar to the previous step, we have a tie between v_4 and v_5. Thus, we assign $v_4 = 3$.

4 Now we have a tie between v_2 and v_5 and assign $v_2 = 4$.

5 Finally, we assign $v_5 = 4$.

Problem 3.2.4: Consider the CSP in Fig. 1 at its initial state. Is the CSP arc consistent? Is this a convenient initial condition if we plan to apply backtracking search? Describe the domains after the arc consistency algorithm has been applied to the CSP as a preprocessing step.

Solution: In order to find out whether the CSP is arc consistent or not, we could apply the arc consistency algorithm by initializing the queue with all arcs in the CSP. If one or more domains are pruned, then the CSP was initially not arc consistent. However, this is actually not strictly necessary in this case, since the CSP has only few variables with small domains and relatively simple constraints. In general, variables X and Y with equal domains $2, 3, 4$ involved in a $X \neq Y$ constraint are always arc consistent with each other: independent of which value from the domain $2, 3, 4$ is selected for X, there will always be a feasible value among $2, 3, 4$ for Y. Therefore, in our CSP all variables involved in an inequality constraint are already arc consistent which each other.

Let us now consider the constraint $v_2 = v_1 + 1$ between variable v_1 and v_2, both having domain $2, 3, 4$. If we assigned $v_1 = 4$, there would not be any value left for v_2, since $4 + 1 = 5$ and 5 is not in the domain of v_2. This means that the arc (v_1, v_2) is not arc consistent. A similar reasoning holds for the arc (v_2, v_1). Therefore, the CSP is not arc consistent and thus is not in a convenient initial situation for applying backtracking search, since those will probably lead to a higher number of iteration and/or backtracking steps. In order to make both arcs (v_1, v_2) and (v_2, v_1) arc consistent, we need to prune the domains of v_1 and v_2 to $\{2, 3\}$ and $\{3, 4\}$ respectively. The resulting CSP is shown in Fig. 2.

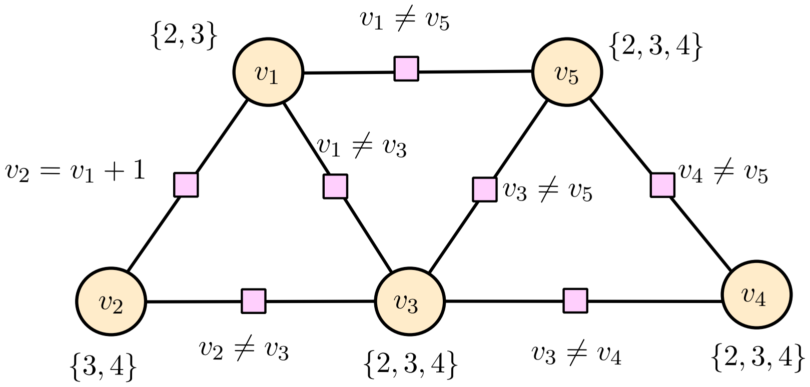


Figure 2: Constraint graph after applying the arc consistency algorithm for preprocessing

Problem 3.2.5: Perform backtracking search after the preprocessing step of the previous task: Determine which variable to expand next by applying the minimum-remaining-values (MRV) heuristic; if there is a tie, use degree heuristics; if there is a tie again, choose one randomly. Use least-constraining-value heuristics to decide which value to assign; if there is a tie, choose one randomly. After each assignment, perform arc consistency algorithm. Backtrack if you find an inconsistency. Assume the data structure of the queue is a set, i.e., if we add an element to the queue which is already in the queue, the element will not be added a second time (each element is unique).

Solution: For each recursion depth, the assigned variable and the remaining values for each variable is shown in Table 4 (after performing arc consistency algorithm). The search starts after we used the arc consistency algorithm for preprocessing, i.e., we made the CSP arc consistent. In the following, we describe the search steps (where the bold number denotes the depth of recursion).

1 The variables with minimum remaining values are v_1 and v_2, but v_1 has a higher degree. Since the least-constraining-value heuristic holds a tie, we randomly assign $v_1 = 2$. We apply the arc consistency algorithm, initializing the queue with the arcs (v_2, v_1), (v_3, v_1) and (v_5, v_1):

- We pop the first element from the queue, i.e. (v_2, v_1), and remove the inconsistent value 4 from the domain of v_2. Then we add the arcs to all neighbours of v_2 (except v_1) to the queue, i.e., (v_3, v_2).
- In the following step, v_3 is checked for arc consistency with v_1. We have to remove the value 2 from the domain of v_3 to make it arc consistent with v_1. Again, we add the arcs to all neighbours of v_3 (except v_1) to the queue, i.e., (v_2, v_3), (v_4, v_3), and (v_5, v_3).

		current domains					degree					
step	assignment	v_1	v_2	v_3	v_4	v_5	v_1	v_2	v_3	v_4	v_5	backtrack
0	$\emptyset$	$\{2,3\}$	$\{3,4\}$	$\{2,3,4\}$	$\{2,3,4\}$	$\{2,3,4\}$	3	2	4	2	3	
1	$v_1 = 2$	/	$\{3\}$	$\{4\}$	$\{2\}$	$\{3\}$	/	1	3	2	2	
2	$v_3 = 4$	/	$\{3\}$	/	$\{2\}$	$\{3\}$	/	0	/	1	1	
3	$v_4 = 2$	/	$\{3\}$	/	/	$\{3\}$	/	0	/	/	0	
4	$v_5 = 3$	/	$\{3\}$	/	/	/	/	0	/	/	/	
5	$v_2 = 3$	/	/	/	/	/	/	/	/	/	/	

Table 4: For each variable, the number of remaining values (domain size) and the number of constraints (degree) is listed. The rows correspond to the steps of the backtracking algorithm, we assign a value to one variable and perform arc consistency algorithm (i.e. updating the domain of connected variables).

- Next, we look at the arc (v_5, v_1). We remove the inconsistent value 2 from the domain of v_5 and add the arcs to all neighbours of v_5 (except v_1) to the queue. Thus, the queue after this step is: $(v_3, v_2), (v_2, v_3), (v_4, v_3), (v_5, v_3), (v_3, v_5), (v_4, v_5)$.
- The arc (v_3, v_2) is not consistent, since there is no value in the domain of v_2 which fulfills the constraint $v_2 \neq v_3$ for the value 3 in the domain of v_3. We remove the inconsistent value 3 from the domain of v_3 and add the arcs to all neighbours of v_3 (except v_2) to the queue, i.e., (v_1, v_3), (v_4, v_3) and (v_5, v_3). Since the data structure of the queue is a set, and (v_4, v_3) and (v_5, v_3) are already in the queue, these arcs are not added to the queue a second time. Thus, the queue after this step is: $(v_2, v_3), (v_4, v_3), (v_5, v_3), (v_3, v_5), (v_4, v_5), (v_1, v_3)$.
- (v_2, v_3) is already arc consistent. Therefore, we do not remove any values from the domain of v_2 and do not add any new arcs to the queue.
- In the next step, we remove the inconsistent value 4 from the domain of v_4 to make v_4 arc consistent with v_3. We add (v_5, v_4) to the queue.
- In order to make v_5 arc consistent with v_3, we have to remove the value 4 from the domain of v_5. Then we add the arcs to all neighbours of v_5 (except v_3) to the queue, i.e., (v_1, v_5) and (v_4, v_5). However, as (v_4, v_5) is already in the queue, only (v_1, v_5) is added.
- Since (v_3, v_5) is already arc consistent, this arc is popped from the queue and no new arcs are added to the queue. The queue after these steps is: $(v_4, v_5), (v_1, v_3), (v_5, v_4), (v_1, v_5)$
- Next, we remove the value 3 from the domain of v_4 to make it arc consistent with v_5. We add the arcs to all neighbours of v_4 (except v_5) to the queue, i.e., (v_3, v_4).
- Since the remaining arcs (v_1, v_3), (v_5, v_4), (v_1, v_5) and (v_3, v_4) are already arc consistent, these arcs are popped from the queue and no new arcs are added to the queue. As the queue is empty, the algorithm terminates.

After applying the arc consistency algorithm the resulting CSP has only one possible value per variable and, since we did not backtrack, it is arc consistent. We have actually already found a solution for the CSP. Note that this would also hold for a different assignment of v_1. In the next steps we assign the found values to the variables in the order defined by the exercise description.

2 Since v_3 has the highest degree, we assign $v_3 = 4$.

3 In the next step, v_4 and v_5 have both the highest degree, we have a tie and assign randomly $v_4 = 2$.

4 Similar to the previous step, we have a tie between v_2 and v_5. Thus, we assign randomly $v_5 = 3$.

5 Finally, we assign $v_2 = 3$.

Problem 3.2.6: For each of the previous performances of backtracking search with a different inference (forward checking, arc consistency, arc consistency after preprocessing), compare the number of iterations and the number of times you needed to backtrack.

Solution:

Forward Checking. The backtracking search required eight iterations and two backtracking steps. (Note: Assigning $v_3 = 2$ or $v_3 = 4$ in the first step would have solved the CSP without backtracking once.) There are no solutions for the assignment $v_3 = 3$, but forward checking was not able to identify this inconsistency immediately and we noticed it only after ruling out all options for v_1.

Arc Consistency Algorithm. A total of six iterations and one backtracking step was required. Note that the arc consistency algorithm was able to immediately identify that the assignment $v_3 = 3$ leads to a subproblem with no solution. Furthermore, by looking at the state of all variable domains after assigning $v_3 = 2$ and performing the inference we could tell that a solution exists. That was not the case for forward checking. This fact underlines a fundamental difference between the two inferences: forward checking establishes arc consistency between the neighbours of the assigned variable and the assigned variable itself (i.e. makes all arcs arc consistent in both directions for a radius of length one from the assigned variable); the arc consistency algorithm makes the whole CSP arc consistent.

Arc Consistency Algorithm after preprocessing. The backtracking search finished within five iterations without backtracking. While applying the arc consistency algorithm as a preprocessing step does not exclude the possibility of backtracking, it surely speeds up the whole search. Note that arc consistency algorithm is not always the best inference, since it is more resource consuming. However, applying the arc consistency algorithm as a preprocessing step is always a good choice. For a practical example, take a look at the additional problems, where we solve this CSP with backtracking search applying forward checking after preprocessing with the arc consistency algorithm.

2 Additional Problems

Problem 3.3: Arc consistency and backtracking search for binary constraints

Consider again the constraint satisfaction problem from Problem 3.2 described in Fig. 1.

Problem 3.3.1: Perform backtracking search by hand with forward checking as inference after preprocessing with the arc consistency algorithm: Apply the arc consistency algorithm to the CSP initializing the queue with all arcs, then start backtracking search. Determine which variable to expand next by applying the minimum-remaining-values (MRV) heuristic; if there is a tie, use degree heuristics; if there is a tie again, choose one randomly. Use least-constraining-value heuristics to decide which value to assign; if there is a tie, choose randomly. After each assignment, perform forward checking as inference. Backtrack if you find an inconsistency.

Solution: For each recursion depth, the assigned variable and the remaining values for each variable are shown (after performing forward checking) in Table 5. In the following, we describe the search steps (where the bold number denotes the depth of recursion).

1 MRV heuristics holds a tie between variables v_1 and v_2. Since v_1 has a higher degree and least-constraining-value heuristic holds a tie, we assign randomly $v_1 = 2$. Using forward checking

		current domains					degree					
step	assignment	v_1	v_2	v_3	v_4	v_5	v_1	v_2	v_3	v_4	v_5	backtrack
0	$\emptyset$	$\{2,3\}$	$\{3,4\}$	$\{2,3,4\}$	$\{2,3,4\}$	$\{2,3,4\}$	3	2	4	2	3	
1	$v_1 = 2$	/	$\{3\}$	$\{3,4\}$	$\{2,3,4\}$	$\{3,4\}$	/	1	3	2	2	
2	$v_2 = 3$	/	/	$\{4\}$	$\{2,3,4\}$	$\{3,4\}$	/	/	2	2	2	
3	$v_3 = 4$	/	/	/	$\{2,3\}$	$\{3\}$	/	/	/	1	1	
4	$v_5 = 3$	/	/	/	$\{2\}$	/	/	/	/	0	/	
5	$v_4 = 2$	/	/	/	/	/	/	/	/	/	/	

Table 5: Backtracking search with forward checking as inference after preprocessing with the arc consistency algorithm.

prunes the domains of v_2, v_3 and v_5.

2 The variable with minimum remaining values is v_2. Since there is just one value left, we assign $v_2 = 3$. Forward checking leaves only one value for v_3.

3 The only possible value for v_3 is 4. Therefore, we assign $v_3 = 4$ and prune the domain of the neighboring variables v_4 and v_5.

4 There is only one value left for v_5. We assign $v_5 = 3$ and forward checking prunes the domain of v_4.

5 The only assignment left is $v_4 = 2$.

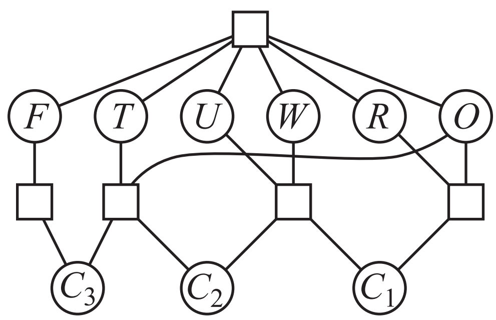
The search ended without backtracking once. Furthermore, choosing the variable and the value to assign was straightforward. Choosing a different value for v_1 in the first step does not change these aspects.

Problem 3.4: Solving a CSP by hand performing backtracking search with minimum-remaining-values (MRV) and degree heuristics, least-constraining-value heuristics, and forward checking

(from *Russell & Norvig 3ed.* q. 7.5) Suppose that we have the cryptarithmetic problem as shown in Fig. 3.

$$\begin{array}{ccccc} & & T & W & O \\ + & & T & W & O \\ \hline & F & O & U & R \end{array}$$

(a)



(b)

Figure 3: (a) A cryptarithmetic problem. Each letter stands for a distinct digit; the aim is to find a substitution of digits for letters such that the resulting sum is arithmetically correct. (b) The constraint hypergraph for the cryptarithmetic problem, showing the `Alldiff` constraint (square box at the top) as well as the column addition constraints (four square boxes in the middle). The variables $C1$, $C2$, and $C3$ represent the carry digits for the three columns.

We model the cryptarithmetic problem as $CSP = (X, D, C)$ with

$$\begin{array}{rcl} X & = & \{F, T, U, W, R, O, C_1, C_2, C_3\} \\ D & = & \{\text{numbers}, \ldots, \text{numbers}, \text{binary}, \text{binary}, \text{binary}\} \\ C & = & \{\langle (O, R, C_1),\ O + O = R + 10 \cdot C_1 \rangle, \\ & & \langle (U, W, C_1, C_2),\ C_1 + W + W = U + 10 \cdot C_2 \rangle, \\ & & \langle (O, T, C_2, C_3),\ C_2 + T + T = O + 10 \cdot C_3 \rangle, \\ & & \langle (C_3, F),\ C_3 = F \rangle, \\ & & \langle (F, T, U, W, R, O),\ \texttt{Alldiff}(F, T, U, W, R, O) \rangle\}, \end{array}$$

where numbers $= \{0, 1, 2, \ldots, 9\}$ and binary $= \{0, 1\}$.

Problem 3.4.1: Replace all boxes which correspond to higher-order constraints by binary constraints. Use the approach of Problem 3.1 and introduce variables such as X_1, X_2, and X_3, etc.

Solution: For constraint $\langle (O, R, C_1),\ O + O = R + 10 \cdot C_1 \rangle$, we introduce variable X_1 with a domain of

$$X_1 \in \{(o, r, c_1) \mid o \in \text{numbers} \ \wedge\ r \in \text{numbers} \ \wedge\ c_1 \in \text{binary} \ \wedge\ o + o = r + 10 \cdot c_1\}.$$

Similarly, we also define X_2 and X_3 as

$$\begin{array}{rcl} X_2 & \in & \{(u, w, c_1, c_2) \mid u, w \in \text{numbers} \ \wedge\ c_1, c_2 \in \text{binary} \ \wedge\ c_1 + w + w = u + 10 \cdot c_2\}, \\ X_3 & \in & \{(o, t, c_2, c_3) \mid o, t \in \text{numbers} \ \wedge\ c_2, c_3 \in \text{binary} \ \wedge\ c_2 + t + t = o + 10 \cdot c_3\}. \end{array}$$

We also need a variable X_4 for the `Alldiff` constraint:

$$X_4 \in \{(f, t, u, w, r, o) \mid f, t, u, w, r, o \in \text{numbers} \ \wedge\ \texttt{Alldiff}(f, t, u, w, r, o)\}.$$

Thus, we get the constraint hypergraph as shown in Fig. 4, and the new constraint set C' is

$$\begin{array}{rcl} C' & = & \{\langle (O, X_1),\ \textit{fst}(X_1) = O \rangle, \\ & & \langle (R, X_1),\ \textit{snd}(X_1) = R \rangle, \\ & & \langle (C_1, X_1),\ \textit{thrd}(X_1) = C \rangle, \\ & & \langle (U, X_2),\ \textit{fst}(X_2) = U \rangle, \\ & & \langle (W, X_2),\ \textit{snd}(X_2) = W \rangle, \\ & & \langle (C_1, X_2),\ \textit{thrd}(X_2) = C_1 \rangle, \\ & & \langle (C_2, X_2),\ \textit{frth}(X_2) = C_2 \rangle, \\ & & \langle (O, X_3),\ \textit{fst}(X_3) = O \rangle, \\ & & \langle (T, X_3),\ \textit{snd}(X_3) = T \rangle, \\ & & \langle (C_2, X_3),\ \textit{thrd}(X_3) = C_2 \rangle, \\ & & \langle (C_3, X_3),\ \textit{frth}(X_3) = C_3 \rangle, \\ & & \langle (C_3, F),\ C_3 = F \rangle, \\ & & \langle (F, X_4),\ \textit{fst}(X_4) = F \rangle, \\ & & \langle (T, X_4),\ \textit{snd}(X_4) = T \rangle, \\ & & \langle (U, X_4),\ \textit{thrd}(X_4) = U \rangle, \\ & & \langle (W, X_4),\ \textit{frth}(X_4) = W \rangle, \\ & & \langle (R, X_4),\ \textit{ffth}(X_4) = R \rangle, \\ & & \langle (O, X_4),\ \textit{sxth}(X_4) = O \rangle\}. \end{array}$$

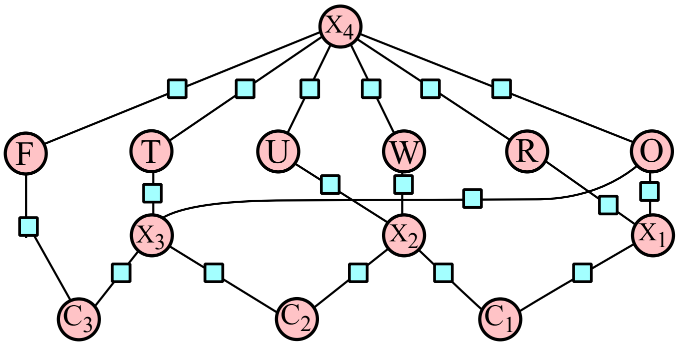


Figure 4: The constraint hypergraph for the cryptarithmetic problem after turning all constraints into binary constraints.

Problem 3.4.2: Sort the variables once by their domain size (i.e. number of remaining values) and once by their degree (i.e. number of constraints on other unassigned variables).

Solution:

Domain size Table 6 sorts the variables by their domain size. Whereas the domain size of the variables of the original CSP is easy to determine (see D in the given CSP formulation), we discuss the size of X_1, X_2, X_3, and X_4 in the following.

Variable	Number of remaining values
C_1, C_2, C_3	2
F, T, U, W, R, O	10
X_1	10
X_2, X_3	20
X_4	$P(10, 6) = 151.200$

Table 6: Variables sorted by their number of remaining values.

Combinatorial analysis for X_1. If we assume $c_1 = 0$, we must find all value pairs of (o, r) which satisfy $o + o = r$. Since $r \in \{0, \ldots, 9\}$ and r must be even, $(o, r) \in \{(0, 0), (1, 2), (2, 4), (3, 6), (4, 8)\}$. Thus, we have 5 possible values for the tuple (o, r).

If we assume $c_1 = 1$, we have the same 5 possible values for r as before, since $o + o = r + 10$ can also only be satisfied if r is even. Together with $o = \frac{r}{2} + 5$, it follows that $5 \leq o \leq 9$ and thus $o \in \{5, 6, 7, 8, 9\}$. In total, we have 10 possible values for variable $X_1 = (o, r, c_1)$.

Combinatorial analysis for X_2 and X_3. Variables X_2 and X_3 have similar structure of their domain. Therefore, we analyze only X_2, and the result applies to X_3 analogously.

If we assume $c_1 = 0$, the constraint becomes $2w = u + 10 \cdot c_2$, which is equivalent to the one of X_1. Thus, we have 10 possible values for (u, w, c_2).

If we assume that $(c_1, c_2) = (1, 0)$, the constraint becomes $1 + 2w = u$. With $u \leq 9$, it follows that $2w \leq 8$. Again, 5 value pairs satisfy this inequality. If we assume that $(c_1, c_2) = (1, 1)$, the constraint becomes $1 + 2w = u + 10$ and $2w - 9 = u$. Since the minimum value for u is 0, it follows that $w \in \{5, \ldots 9\}$. Thus, we have 5 possible values for this case.

In total, X_2 has 20 possible values, as well as variable X_3.

Combinatorial analysis for X_4. Since we can assign 10 values to 6 variables which have to be different, the total number of possibilities is $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = \frac{10!}{4!} = \frac{10!}{(10-6)!} = P(10, 6)$, where $P(n, k)$ denotes the k-permutations of n.

Degree In Fig. 4, we can count the degree of each variable. Table 7 sorts the variables by their degree.

Variable	Degree
X_4	6
X_2, X_3	4
X_1	3
O	3
F, T, U, W, R	2
C_1, C_2, C_3	2

Table 7: Variables sorted by the their degree.

Problem 3.4.3: Perform backtracking search to solve the cryptarithmetic problem: Determine which variable to expand next by applying the minimum-remaining-values (MRV) heuristic; if there is a tie, use degree heuristics; if there is a tie again, choose one randomly. Use least-constraining-value heuristics to decide which value to assign. After each assignment, perform forward checking. Backtrack if you find an inconsistency.

Solution: The term backtracking search is used for a depth-first search that chooses values for one variable at a time and backtracks when a variable has no values left to assign (see lecture 5, slide 15). Whenever a variable is assigned, the forward checking process establishes arc consistency for this variable with its neighbors (i.e. reduces their domain so that all their values satisfy the binary constraints; see lecture 5, slide 26).

Table 8 summarizes the search: For each recursion depth, the assigned variable and the remaining values for each variable is shown (after performing forward checking). In the following, we describe the search steps (where the bold number denotes the depth of recursion).

1 We use MRV heuristics to choose the variable to expand first. The variables C_1, C_2, and C_3 have the minimum number of remaining values (cf. Table 6), and also the same degree (cf. Table 7). Therefore, we randomly choose C_3 to expand first. To choose a value, we use least-constraining-value heuristic. Every value of C_3 will leave the same number of values for variables F and X_3. Thus, we just choose $C_3 = 1$. Then, we perform forward checking: The reduced domain of F is $\{1\}$, and for X_3 it is

$$X_3 \in \{(o, t, c_2, 1) \mid o, t, c_2 \in \text{numbers} \ \wedge \ c_2 + t + t = o + 10\}.$$

2 Using MRV heuristic, we choose variable F. We assign it with value $F = 1$. For forward checking, we update the variable X_4 such that the first value of the tuple is always equal to 1.

3 The variables with minimum remaining values are C_1 and C_2, and they have the same degree. We just randomly choose C_2. Since least-constraining-value yields a tie, we just choose to assign $C_2 = 1$. Performing forward checking, we update the variable X_2 and X_3 such that the c_2 component will always be 1.

step	C_1	C_2	C_3	X_1	X_2	X_3	X_4	F	T	U	W	R	O	assignment
0	2	2	2	10	20	20	$P(10, 6)$	10	10	10	10	10	10	$\emptyset$
1	2	2	0	10	20	10	$P(10, 6)$	1	10	10	10	10	10	$C_3 = 1$
2	2	2	0	10	20	10	$P(9, 5)$	0	10	10	10	10	10	$F = 1$
3	2	0	0	10	10	5	$P(9, 5)$	0	10	10	10	10	10	$C_2 = 1$
4	0	0	0	5	5	5	$P(9, 5)$	0	10	10	10	10	10	$C_1 = 1$
5	0	0	0	5	5	0	$P(9, 5)$	0	1	10	10	10	1	$X_3 = (5, 7, 1, 1)$
6	0	0	0	1	5	0	$P(8, 4)$	0	1	10	10	10	0	$O = 5$
7	0	0	0	0	5	0	$P(8, 4)$	0	1	10	10	1	0	$X_1 = (5, 0, 1)$
8	0	0	0	0	5	0	$P(7, 3)$	0	0	10	10	1	0	$T = 7$
9	0	0	0	0	5	0	$P(6, 2)$	0	0	10	10	0	0	$R = 0$
10	0	0	0	0	0	0	$P(6, 2)$	0	0	1	1	0	0	$X_2 = (3, 6, 1, 1)$
11	0	0	0	0	0	0	$P(5, 1)$	0	0	0	1	0	0	$U = 3$
12	0	0	0	0	0	0	1	0	0	0	0	0	0	$W = 6$
13	0	0	0	0	0	0	0	0	0	0	0	0	0	$X_4 = (1, 7, 3, 6, 0, 5)$

Table 8: For each variable, the number of remaining values is listed (rather than the list of the actual values, as in the previous executions of backtracking search). In each step (the rows corresponds to the recursion depth), we assign a value to one variable and perform forward checking (i.e. updating the domain of connected variables).

4 Next, C_1 is chosen by MRV. Again, both possible values are the same in terms of least-constraining-value heuristic. We just choose $C_1 = 1$. Performing forward checking, we constraint the c_1 component of X_1 and X_2 to be 1.

At this point, it might be helpful to list the possible values of X_1, X_2, and X_3:

$$X_1 \in \{(o, r, c_1) \mid (5, 0, 1), (6, 2, 1), (7, 4, 1), (8, 6, 1), (9, 8, 1)\}$$
$$X_2 \in \{(u, w, c_1, c_2) \mid (1, 5, 1, 1), (3, 6, 1, 1), (5, 7, 1, 1), (7, 8, 1, 1), (9, 9, 1, 1)\}$$
$$X_3 \in \{(o, t, c_2, c_3) \mid (1, 5, 1, 1), (3, 6, 1, 1), (5, 7, 1, 1), (7, 8, 1, 1), (9, 9, 1, 1)\}$$

5 The variables X_1, X_2, and X_3 have the same number of remaining values and the same degree of 2. (Recall that the degree is the number of constraints on other unassigned variables.) We randomly choose X_3. Possible values for X_3 are

$$X_3 = \{(\mathbf{1}, \mathbf{5}, \mathbf{1}, \mathbf{1}), (3, 6, 1, 1), (5, 7, 1, 1), (7, 8, 1, 1), (9, 9, 1, 1)\}$$

We try the least-constraining-heuristic; but whichever value we choose, the domains of T and O are both reduced to one. We just assign $X_3 = (1, 5, 1, 1)$. By forward checking, the domain size of T and O is reduced to 1.

6 Between T and O (both have only one remaining value), we choose O because it has a higher degree (which is 2), and assign $O = 1$ (which is the only possible value). Forward checking reduces the domain size of X_1 to 0 (see possible values above) and the domain size of X_4 to 0. **Oh no!** The assigned value is not consistent. We backtrack immediately.

5 We again expand X_3 with possible values of

$$X_3 = \{(1, 5, 1, 1), (\mathbf{3}, \mathbf{6}, \mathbf{1}, \mathbf{1}), (5, 7, 1, 1), (7, 8, 1, 1), (9, 9, 1, 1)\}$$

So let us now try $X_3 = (3, 6, 1, 1)$.

6 Thus, we assign $O = 3$, which reduces the domain size of X_1 to 0. Backtrack again...

5 We again expand X_3 with possible values of

$$X_3 = \{(1,5,1,1),(3,6,1,1),(\mathbf{5},\mathbf{7},\mathbf{1},\mathbf{1}),(7,8,1,1),(9,9,1,1)\}$$

So let us now try $X_3 = (5,7,1,1)$.

6 Thus, we assign $O = 5$, which reduces the domain size of X_1 to 1 and of X_4 to $P(8,4)$.

7 T and X_1 have both the least number of remaining values. Since they have the same degree of 1, we randomly choose X_1. We assign X_1 with the only possible value $X_1 = (5,0,1)$. It leaves R with only 1 possible value.

8 T and R have the same number of remaining values and degree. We randomly choose T, and assign it with $T = 7$. This reduces the number of possible values for variable X_4 to $P(7,3)$.

9 The next smallest domain is R, and we assign with the only possible value $R = 0$. By forward checking, we can reduce the domain size of X_4 to $P(6,2)$.

10 The next smallest domain is X_2, and the possible values are

$$X_2 = \{(\mathbf{1},\mathbf{5},\mathbf{1},\mathbf{1}),(3,6,1,1),(5,7,1,1),(7,8,1,1),(9,9,1,1)\}.$$

Since the least-constraining-value heuristics still does not tell us anything, we just try $X_2 = (1,5,1,1)$. By performing forward checking, the number of remaining values of U and W is reduced to 1.

11 The next variable to assign is either U or W (same MRV and degree). We randomly choose U and assign the only possible value $U = 1$. By forward checking, we see that this reduces the domain size of X_4 to 0. Backtrack!

10 We again expand X_2

$$X_2 = \{(1,5,1,1),(\mathbf{3},\mathbf{6},\mathbf{1},\mathbf{1}),(5,7,1,1),(7,8,1,1),(9,9,1,1)\},$$

and try $X_2 = (3,6,1,1)$. By performing forward checking, the domain of U and W are reduced to $\{3\}$ and $\{6\}$, respectively.

11 Now, only U, W, and X_4 are not yet assigned. Since U and W have a domain size of 1, but the same degree, we randomly choose U and assign $U = 3$. Forward checking reduces the domain size of X_4 to $P(5,1) = 5$.

12 Next variable is $W = 6$. Forward checking reduces the domain size of X_4 to $P(4,0) = 1$.

13 The last one to fill is variable $X_4 = (1,7,3,6,0,5)$. And we have found a solution satisfying all (binary) constraints.