Fundamentals of Artificial Intelligence
Exercise 8: Bayesian Networks

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Recap Bayesian Networks

Bayesian network

A Bayesian network is a directed acyclic graph, where

- each node corresponds to a random variable,
- arrows between nodes start at parents,
- each node N_i has a conditional probability distribution $P(X_i | Parents(X_i))$,
- $P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$.

Recap Bayesian Networks - Determine (Conditional) Independence

- Independence: $P(X, Y) = P(X)P(Y)$ or $P(X|Y) = P(X)$
- Conditional independence: $P(X|Y, E) = P(X|E)$

Problem 8.1

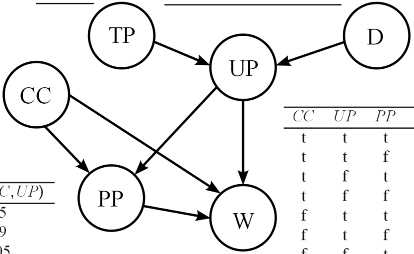
Consider the Bayesian network about a race:

$P(tp)$
0.35

D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(d)$
0.25

$P(cc)$
0.3



CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

Figure: A Bayesian Network with Boolean random variables: $D = DemotivatedPilot$, $TP = TalentedPilot$, $CC = CompetitiveCar$, $UP = UnderPerformance$, $PP = PolePosition$, $W = Wins$

Problem 8.1a

a. Which of these statements are true?

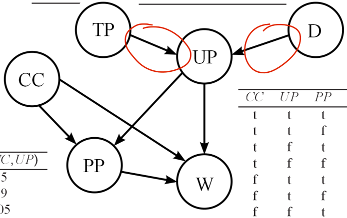
   i. TP, UP and D are independent.

$P(tp)$
0.35

D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(d)$
0.25

$P(cc)$
0.3



connected

CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

Problem 8.1a

a. Which of these statements are true?

   ii. TP and D are independent. ✓

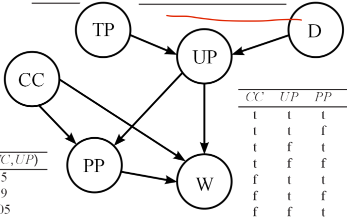
no connect

D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(tp)$
0.35

$P(d)$
0.25

$P(cc)$
0.3

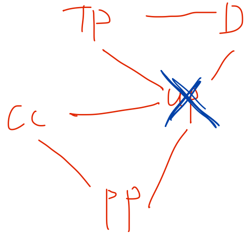


CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

Problem 8.1a

a. Which of these statements are true?
   iii. PP and D are conditionally independent given UP.



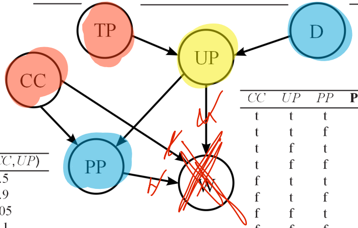
blocked

$P(tp)$
0.35

D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(d)$
0.25

$P(cc)$
0.3



CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

Problem 8.1a

a. Which of these statements are true?
   iv. TP and D are conditionally independent given CC. ✓

TP D

CC

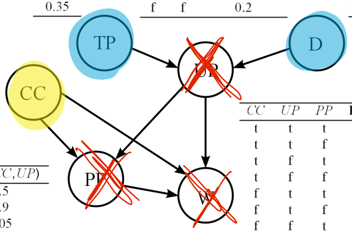
no connect

D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(tp)$
0.35

$P(d)$
0.25

$P(cc)$
0.3



CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

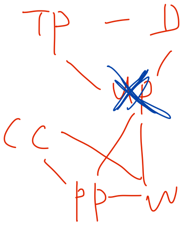
CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

Problem 8.1a

Given

a. Which of these statements are true?

v. $\mathbf{P}(D|TP, UP) = \mathbf{P}(D|TP, CC, UP, PP, W)$.



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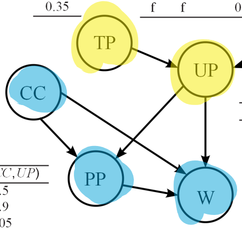
$P(tp)$
0.35

D	TP	$\mathbf{P}(up\|D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(d)$
0.25

$P(cc)$
0.3

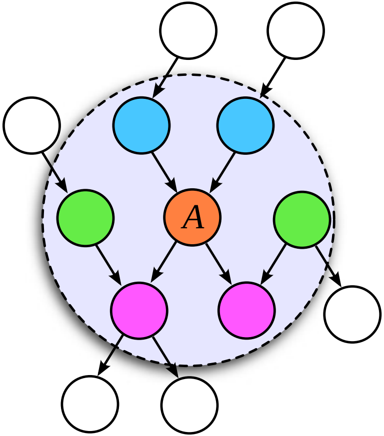


CC	UP	$\mathbf{P}(pp\|CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w\|CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

Recap Bayesian Networks - Markov Blanket

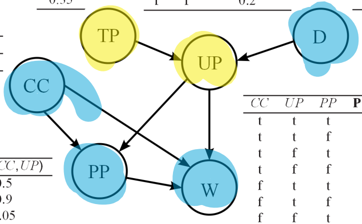
- A node in the Bayesian network is conditionally independent of all other nodes given its parents, children, and children's parents



Problem 8.1a

a. Which of these statements are true?

  v. $\mathbf{P}(D|TP, UP) = \mathbf{P}(D|TP, CC, UP, PP, W)$.



$P(tp)$
0.35

D	TP	$\mathbf{P}(up|D,TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(d)$
0.25

$P(cc)$
0.3

CC	UP	PP	$\mathbf{P}(w|CC,UP,PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

CC	UP	$\mathbf{P}(pp|CC,UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

Problem 8.1b

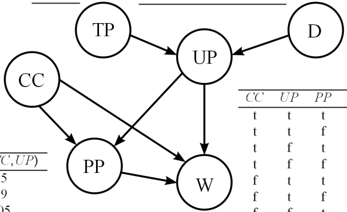
b. Write the formula for computing the joint probability distribution in terms of the conditional probabilities exploiting the conditional independences in the considered network.

D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(tp)$
0.35

$P(d)$
0.25

$P(cc)$
0.3



$$P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i \mid parents(X_i)).$$

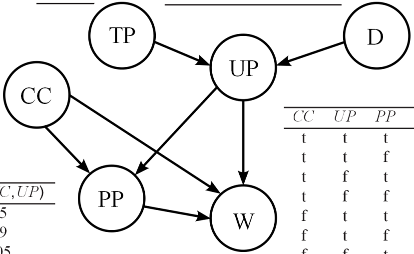
CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

$$P(D, TP, UP, CC, PP, W) = P(D) \cdot P(TP) \cdot P(UP \mid TP, D) \cdot P(CC) \cdot P(PP \mid CC, UP) \cdot P(W \mid PP, CC, UP)$$

Problem 8.1c

c. Calculate $P(\neg d, tp, cc, \neg up, pp, w)$ and $P(\neg d, tp, cc, \neg up, \neg pp, w)$.



D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(tp)$
0.35

$P(d)$
0.25

$P(cc)$
0.3

CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

$P(\neg d, tp, cc, \neg up, pp, w) =$

$P(\neg d, tp, cc, \neg up, \neg pp, w) =$

$$p(D, TP, UP, CC, PP, W) = p(D)\cdot p(TP)\cdot p(UP \mid TP, D)\cdot p(CC)\cdot p(PP \mid CC, UP)\cdot p(W \mid PP, CC, UP)$$

$$= p(\neg d)\, p(tp)\cdot p(\neg up \mid tp, \neg d)\cdot p(cc)\cdot p(pp \mid cc, \neg up)\cdot p(w \mid pp, cc, \neg up)$$

$$= (1-0.25)\cdot 0.35\cdot (1-0.1)\cdot 0.3\cdot 0.9\cdot 0.9$$

$$= 0.0574$$

Problem 8.1d

d. Calculate the probability that the pilot wins given that he is talented, motivated and starts from the pole position.

w tp $\neg d$

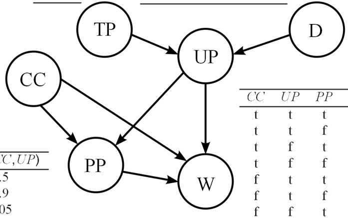
pp

$P(tp)$
0.35

D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(d)$
0.25

$P(cc)$
0.3



CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

$$P(w \mid tp, \neg d, pp) = \frac{P(w, tp, \neg d, pp)}{P(tp, \neg d, pp)} = \frac{P(w, tp, \neg d, pp)}{\underbrace{P(w, tp, \neg d, pp)}_{0.0615} + \underbrace{P(\neg w, tp, \neg d, pp)}_{0.0237}}$$

$\alpha = 11.737$

$$P(w, tp, \neg d, pp) = \sum_{cc} \sum_{up} P(w, tp, \neg d, pp, cc, up)$$

$$= \sum_{cc} \sum_{up} P(tp)\, P(\neg d) \cdot P(cc) \cdot P(up \mid tp, \neg d) \cdot P(pp \mid cc, up) \cdot P(w \mid pp, up, cc)$$

$$= P(tp)\, P(\neg d) \sum_{up} P(up \mid tp, \neg d) \sum_{cc} P(cc) \cdot P(pp \mid cc, up) \cdot P(w \mid pp, up, cc)$$

$$= 0.35 \cdot 0.75 \cdot \Big[\underset{0.1}{P(up \mid tp, \neg d)} \cdot \Big[\underset{0.3}{P(cc)} \cdot \underset{0.5}{P(pp \mid cc, up)} \cdot \underset{0.4}{P(w \mid pp, up, cc)} + \underset{0.7}{P(\neg cc)}\, \underset{0.05}{P(pp \mid \neg cc, up)} \cdot \underset{0.05}{P(w \mid pp, up, \neg cc)} \Big] +$$

6.175×10^{-3}

0.22815

$$\underset{0.9}{P(\neg up \mid tp, \neg d)} \cdot \Big[\underset{0.3}{P(cc)} \cdot \underset{0.9}{P(pp \mid cc, \neg up)} \cdot \underset{0.9}{P(w \mid pp, \neg up, cc)} + \underset{0.7}{P(\neg cc)} \cdot \underset{0.1}{P(pp \mid \neg cc, \neg up)} \cdot \underset{0.15}{P(w \mid pp, \neg up, \neg cc)} \Big] \Big]$$

$= 0.0615$

$$P(w \mid tp, \neg d, pp) = \alpha \cdot P(w, tp, \neg d, pp) = 11.7371 \cdot 0.0615 \approx \ldots$$

Problem 8.1d

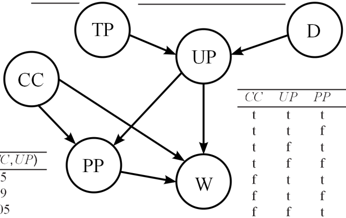
- $P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$

D	TP	$\mathbf{P}(up\|D,TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(tp)$
0.35

$P(d)$
0.25

$P(cc)$
0.3



CC	UP	PP	$\mathbf{P}(w\|CC,UP,PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

CC	UP	$\mathbf{P}(pp\|CC,UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

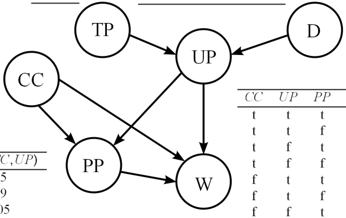
Problem 8.1d

$P(tp)$
0.35

D	TP	$\mathbf{P}(up \mid D, TP)$
t	t	0.5
t	f	0.8
f	t	0.1
f	f	0.2

$P(d)$
0.25

$P(cc)$
0.3



CC	UP	PP	$\mathbf{P}(w \mid CC, UP, PP)$
t	t	t	0.4
t	t	f	0.15
t	f	t	0.9
t	f	f	0.5
f	t	t	0.05
f	t	f	0.01
f	f	t	0.15
f	f	f	0.1

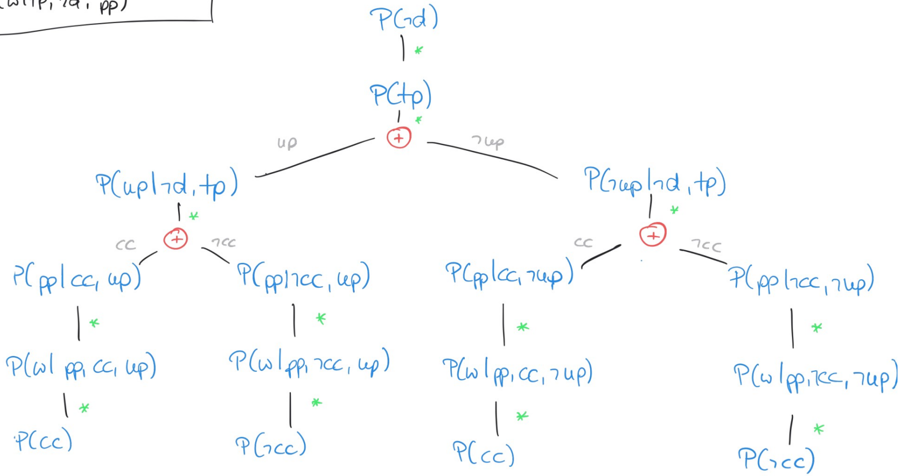
CC	UP	$\mathbf{P}(pp \mid CC, UP)$
t	t	0.5
t	f	0.9
f	t	0.05
f	f	0.1

Problem 8.1d

Problem 8.1d

Enumeration Tree for

$P(w \mid tp, \neg d, pp)$



Problem 8.2a

Consider a driving situation that contains the Boolean random variables $H = Hurry$, $CD = CarefulDriver$, $DF = DriveFast$, $A = Accident$ and $GF = GetFined$.

a. Draw the Bayesian network corresponding to:

$$\boxed{\mathbf{P}(H, CD, DF, A, GF) = \mathbf{P}(GF|DF, A)\,\mathbf{P}(A|DF)\,\mathbf{P}(DF|CD, H)\,\mathbf{P}(CD)\,\mathbf{P}(H)\,.}$$

$P(h)$
0.5

$P(cd)$
0.6

DF	$\mathbf{P}(a\|DF)$
t	0.7
f	0.25

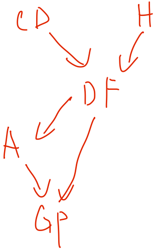
CD	H	$\mathbf{P}(df\|CD, H)$
t	t	0.15
t	f	0.01
f	t	0.99
f	f	0.1

DF	A	$\mathbf{P}(gf\|DF, A)$
t	t	0.99
t	f	0.4
f	t	0.5
f	f	0.05

Problem 8.2a

a. Draw the Bayesian network corresponding to:

$$\boxed{\mathbf{P}(H, CD, DF, A, GF) = \mathbf{P}(GF|DF, A)\,\mathbf{P}(A|DF)\,\mathbf{P}(DF|CD, H)\,\mathbf{P}(CD)\,\mathbf{P}(H)\,.}$$



Problem 8.2b

$$\boxed{\mathbf{P}(H, CD, DF, A, GF) = \mathbf{P}(GF|DF, A)\,\mathbf{P}(A|DF)\,\mathbf{P}(DF|CD, H)\,\mathbf{P}(CD)\,\mathbf{P}(H)\,.}$$

b. Calculate $\mathbf{P}(CD|\neg a, gf)$ using enumeration.

$$p(CD|\neg a, gf) = \frac{p(CD, \neg a, gf)}{p(CD, \neg a, gf) + p(\neg CD, \neg a, gf)}$$

$$= \alpha \sum_{DF} \sum_{H} p(CD, \neg a, gf, DF, H)$$

$$= \alpha\, p(CD) \sum_{DF} p(gf|DF, \neg a) \cdot p(\neg a|DF) \sum_{H} p(H) \cdot p(DF|CD, H)$$

Problem 8.2b

$$\mathbf{P}\left(CD|\neg a, gf\right) = \alpha\mathbf{P}\left(CD\right)\sum\nolimits_{DF}\mathbf{P}\left(gf|\neg a, DF\right)\mathbf{P}\left(\neg a|DF\right)\sum\nolimits_{H}\mathbf{P}\left(DF|CD, H\right)\mathbf{P}\left(H\right)$$

Problem 8.2b

Problem 8.2b

Problem 8.2c

c. Calculate $\mathbf{P}(CD|\neg a, gf)$ using variable elimination.

dim 1 dim 2 dim 3

$$\mathbf{P}(CD|\neg a, gf) = \alpha \underbrace{\mathbf{P}(CD)}_{f_1(CD)} \sum_{DF} \underbrace{\mathbf{P}(gf|\neg a, DF)}_{f_2(DF)} \underbrace{\mathbf{P}(\neg a|DF)}_{f_3(DF)} \sum_{H} \underbrace{\mathbf{P}(DF|CD, H)}_{f_4(DF, CD, H)} \underbrace{\mathbf{P}(H)}_{f_5(H)}$$

$$f_4(DF, CD, H) = \begin{array}{c} \\ df \\ \neg df \end{array} \left\{ \underbrace{\begin{array}{cc} cd & \neg cd \\ \end{array}}_{} \begin{bmatrix} 0.15 & 0.99 \\ 0.85 & 0.01 \end{bmatrix}_{h} \begin{bmatrix} 0.01 & 0.1 \\ 0.99 & 0.9 \end{bmatrix}_{\neg h} \right\}$$

$$f_5(H) = \left\{ [0.5] \; [0.5] \right\}_{h}$$

$$f_4 \times f_5 = \left\{ \begin{bmatrix} 0.075 & 0.495 \\ 0.425 & 0.005 \end{bmatrix} \begin{bmatrix} 0.005 & 0.05 \\ 0.495 & 0.45 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 0.08 & 0.545 \\ 0.92 & 0.455 \end{bmatrix}$$

$$\sum_{DF} f_2(DF) \times f_3(DF) \times f_6 \qquad\qquad f_1(CD) \times f_7$$

$$\begin{array}{c} df \\ \neg df \end{array} \begin{bmatrix} 0.4 \\ 0.05 \end{bmatrix} \times \begin{bmatrix} 0.3 \\ 0.75 \end{bmatrix} \times \begin{bmatrix} 0.08 & 0.545 \\ 0.92 & 0.455 \end{bmatrix}$$

$$= \begin{bmatrix} 0.12 \\ 0.0375 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0441 & 0.0825 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.0441 & 0.0825 \end{bmatrix}$$

$$= 0.05946$$

$$\alpha = \frac{1}{0.05946} \cdot \begin{bmatrix} 0.0265 & 0.0330 \end{bmatrix}$$

$$= [16.806]$$

$$\Rightarrow \begin{bmatrix} 0.4454 & 0.5546 \end{bmatrix}$$

Problem 8.2c

$$\mathbf{f}_4(DF, CD, H) = \left\{ \begin{bmatrix} 0.15 & 0.99 \\ 0.85 & 0.01 \end{bmatrix} \begin{bmatrix} 0.01 & 0.1 \\ 0.99 & 0.9 \end{bmatrix} \right\}, \mathbf{f}_1(CD) = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix},$$

$$\mathbf{f}_2(DF) = \begin{bmatrix} 0.4 \\ 0.05 \end{bmatrix}, \mathbf{f}_3(DF) = \begin{bmatrix} 0.3 \\ 0.75 \end{bmatrix}, \ \mathbf{f}_5(H) = \{ \begin{bmatrix} 0.5 \end{bmatrix} \begin{bmatrix} 0.5 \end{bmatrix} \}.$$

Problem 8.2c

Problem 8.2c

Problem 8.2d

d. Compare the number of operations required to compute the result in **b** and **c** .

enumeration 18 × 6 +

variable Elimination 16 × 6 +

Problem 8.2d

Evaluation Tree for $P(cd \mid \neg a, gf)$