Fundamentals of Artificial Intelligence – Hidden Markov Models

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Organization

1. Time and Uncertainty
2. Hidden Markov Models (HMMs)
3. Inference in Hidden Markov Models
   - Filtering
   - Prediction
   - Smoothing
   - Most Likely Explanation
4. Approximate Inference in Hidden Markov Models

The content is covered in the AI book by the section "Probabilistic Reasoning Over Time" and Sec. 5 of "Natural Language for Communication".

Learning Outcomes

- You know and understand the definition of *stochastic process*, *Markov process*, *Markov property*, and *stationary process*.
- You can compute the probability distribution of a *stationary Markov chain*.
- You can convert *higher-order Markov chains* to a standard Markov chain.
- You can create a *hidden Markov model*.
- You can compute the joint probability distribution of a hidden Markov model.
- You can perform *filtering*, *prediction*, *smoothing*, and find *the most likely explanation* of a hidden Markov model.
- You can perform *particle filtering* for hidden Markov models.

Overview of Probabilistic Methods

This lecture focuses on dynamic environments without actions.

	Static environment	**Dynamic environment**
Without actions	Bayesian networks (lecture 9)	Hidden Markov models (lecture 10)
With actions	Decision networks (lecture 11)	Markov decision processes (lecture 12)

Motivation

The world changes: We need to track and predict it.

Diabetes management vs vehicle diagnosis

- Vehicle diagnosis: We assume that whatever is broken remains broken during the diagnosis.
- Diabetes management: Blood sugar levels change over time, affecting the diagnosis.

Other examples where the dynamics of the system is essential:

- Locating robots,
- tracking the economic activity of a nation,
- language processing,
- smart grid control,
- etc.

Time-Varying Random Variables

- **Basic idea**: Copy state and evidence variables for each time step.
- **Assumption**: The set of variables does not change over time.
- X_t = set of unobservable state variables at time t
  e.g., $BloodSugar_t$, and $StomachContents_t$.
- E_t = set of observable evidence variables at time t
  e.g., $MeasuredBloodSugar_t$, $PulseRate_t$, and $FoodEaten_t$.
- This assumes **discrete time**; step size depends on problem.
- **Notation**: $X_{a:b} = X_a, X_{a+1}, \ldots, X_{b-1}, X_b$.

Conversion to Scalar Random Variables

For finite discrete state spaces, we can assume scalar random variables without loss of generality.

Example

$$\mathrm{X} = [\mathit{FanOfFCB}, \mathit{LivesIn}],$$

where
$\mathit{FanOfFCB} \in \{\mathit{true}, \mathit{false}\}$, $\mathit{LivesIn} \in \{\mathit{Munich}, \mathit{somewhereElseInGermany}\}$.

We introduce the new scalar random variable $\hat{X} \in \{x_1, x_2, x_3, x_4\}$, where

$$x_1 \hat{=} [\mathit{true}, \mathit{Munich}],$$
$$x_2 \hat{=} [\mathit{true}, \mathit{somewhereElseInGermany}],$$
$$x_3 \hat{=} [\mathit{false}, \mathit{Munich}],$$
$$x_4 \hat{=} [\mathit{false}, \mathit{somewhereElseInGermany}].$$

From now on we will only consider scalar random variables.

Definitions

Stochastic Process

A sequence of random variables X_1, X_2, X_3, etc. is a **stochastic process**.

Markov Process

A Markov process is a stochastic process that has the Markov property.

Markov Property

A discrete time stochastic process has the Markov property if

$$P(X_n = x_i | X_{n-1} = x_j, X_{n-2} = x_k, \ldots, X_0 = x_l) = P(X_n = x_i | X_{n-1} = x_j).$$

Stationary Process

A stationary process is a stochastic process whose joint probability distribution does not change when shifted in time. A Markov process is stationary if

$$\forall t: \quad P(X_n = x_i | X_{n-1} = x_j) = P(X_{n+t} = x_i | X_{n-1+t} = x_j).$$

Stationary Markov Chain

A stationary Markov chain is a discrete stationary process with the Markov property. The probability distribution is obtained by the law of total probability:

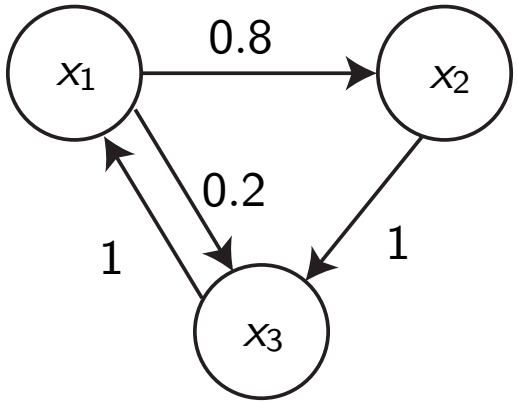
$$P(X_n = x_i) = \sum_{j=1}^{N} P(X_n = x_i | X_{n-1} = x_j) P(X_{n-1} = x_j)$$

Convenient matrix notation:

$$\mathbf{p}_n = \mathbf{T}\mathbf{p}_{n-1},$$

where $(p_i)_n = P(X_n = x_i)$

$$T_{i,j} = P(X_n = x_i | X_{n-1} = x_j).$$



Example:

$$\mathbf{p}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{T} = \begin{bmatrix} 0 & 0 & 1 \\ 0.8 & 0 & 0 \\ 0.2 & 1 & 0 \end{bmatrix}.$$

Stationary Markov Chain: Computing Probabilities

The probabilities can be obtained iteratively using

$$p_n = T p_{n-1},$$

or from

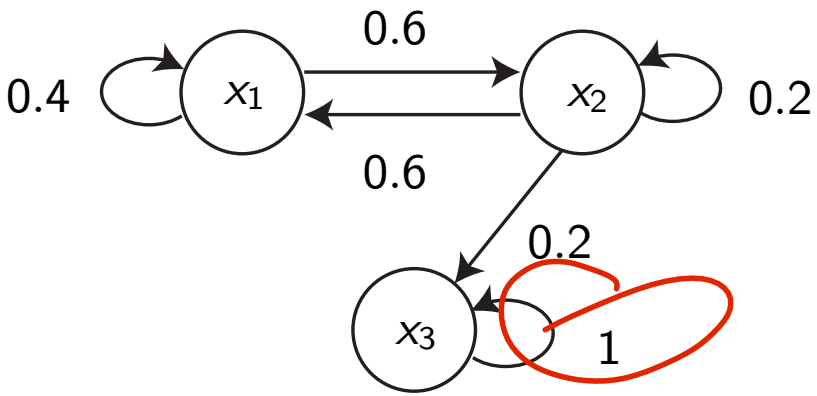
$$p_n = T \dots (T(T p_0)) = T^n p_0. \tag{1}$$

Thus, the probabilities for the previous example become

$$p_1 = T p_0 = \begin{bmatrix} 0 \\ 0.8 \\ 0.2 \end{bmatrix}, \quad p_2 = T p_1 = \begin{bmatrix} 0.2 \\ 0 \\ 0.8 \end{bmatrix}.$$

Tweedback Questions

Given is the following Markov chain:



- What is the one-step probability for $p_0 = [1, 0, 0]^T$?
  - A $p_1 = [0.4, 0.6, 0]^T$
  - B $p_1 = [0.4, 0.4, 0.2]^T$
- What is the probability for $n \to \infty$?
  - A $\lim_{n\to\infty} p_n = [0.4, 0.4, 0.2]^T$
  - B $\lim_{n\to\infty} p_n = [0, 0, 1]^T$

Conversion of Higher-Order Markov Chains

Higher-order Markov chains

A Markov chain of m^{th} order is defined as

$$P(X_n = x_i | X_{n-1} = x_j, X_{n-2} = x_k, \ldots, X_1 = x_o)$$
$$= P(X_n = x_i | X_{n-1} = x_j, X_{n-2} = x_k, \ldots, X_{n-m} = x_l) \text{ for } n > m$$

We can always rewrite a higher-order Markov chain to a normal one by introducing new states corresponding to the sequence of the m previous states:

$$\hat{x}_1 \hat{=} (\overbrace{x_1, \ldots, x_1, x_1}^{m \text{ entries}})$$
$$\hat{x}_2 \hat{=} (x_1, \ldots, x_1, x_2)$$
$$\ldots$$
$$\hat{x}_d \hat{=} (x_1, \ldots, x_1, x_{d_1})$$
$$\hat{x}_{d+1} \hat{=} (x_1, \ldots, x_2, x_1)$$

Conversion of Higher-Order Markov Chains: Example

Before:

$$
\begin{aligned}
&P(X_n = x_1 | X_{n-1} = x_1, X_{n-2} = x_1) = a \\
&P(X_n = x_1 | X_{n-1} = x_1, X_{n-2} = x_2) = b \\
&P(X_n = x_1 | X_{n-1} = x_2, X_{n-2} = x_1) = c \\
&P(X_n = x_1 | X_{n-1} = x_2, X_{n-2} = x_2) = d \\
&P(X_n = x_2 | X_{n-1} = x_1, X_{n-2} = x_1) = e \\
&P(X_n = x_2 | X_{n-1} = x_1, X_{n-2} = x_2) = f \\
&\dots
\end{aligned}
$$

After:

$$
\begin{aligned}
&P(X_n = \hat{x}_1 | X_{n-1} = \hat{x}_1) = a \\
&P(X_n = \hat{x}_1 | X_{n-1} = \hat{x}_2) = b \\
&P(X_n = \hat{x}_2 | X_{n-1} = \hat{x}_3) = c \\
&P(X_n = \hat{x}_2 | X_{n-1} = \hat{x}_4) = d \\
&P(X_n = \hat{x}_3 | X_{n-1} = \hat{x}_1) = e \\
&P(X_n = \hat{x}_3 | X_{n-1} = \hat{x}_2) = f \\
&\dots
\end{aligned}
$$

New states for $\mathcal{D}_x = \{x_1, x_2\}$:

$$
\begin{aligned}
\hat{x}_1 &\hat{=} (x_1, x_1) \\
\hat{x}_2 &\hat{=} (x_1, x_2) \\
\hat{x}_3 &\hat{=} (x_2, x_1) \\
\hat{x}_4 &\hat{=} (x_2, x_2)
\end{aligned}
$$

Sensor Model

In many examples, the state of a system cannot be directly measured (see lecture Cyber-Physical Systems) and has to be inferred from sensor values.

Examples:

- Identify persons from images
- Identify drowsiness of human drivers
- Speech recognition
- Stock market analysis

We assume that the random sensor values E_t only depend on the current state:

$$P(E_t|X_{0:t}, E_{0:t-1}) = P(E_t|X_t).$$

If this is not the case, one simply has to add more states to the system.

Hidden Markov Model

Combining a Markov chain with the previous sensor model results in a **hidden Markov model** (HMM).

After introducing the probabilities

$$(p_i)_n = P(X_n = x_i)$$
$$(\hat{p}_i)_n = P(E_n = e_i)$$

and the matrices

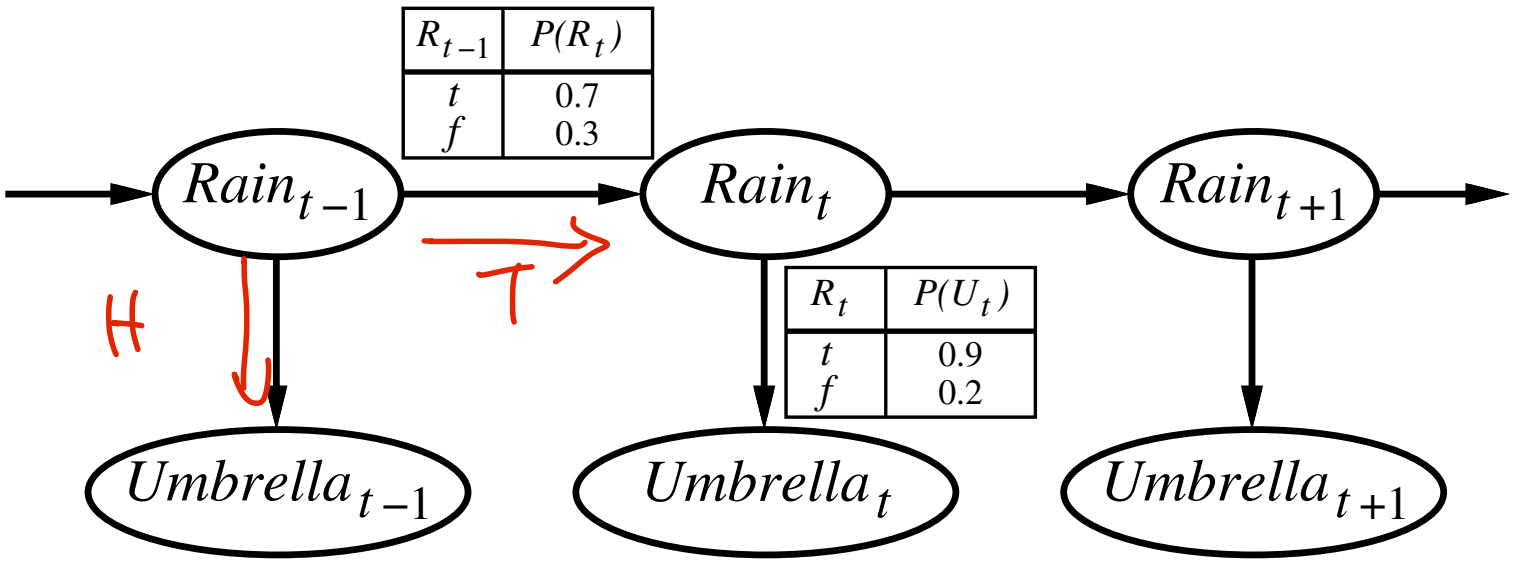
$$T_{i,j} = P(X_n = x_i | X_{n-1} = x_j),$$
$$H_{i,j} = P(E_n = e_i | X_n = x_j).$$

We can compute the probabilities as

$$\mathbf{p}_n = \mathbf{T}\mathbf{p}_{n-1},$$
$$\hat{\mathbf{p}}_n = \mathbf{H}\mathbf{p}_n.$$

Umbrella Example

- You are the security guard stationed at a secret underground installation.
- You want to know if it is rainy today.
- Your only measurement is to check whether the director coming in has an umbrella or not.
- The state is $X_t = Rain_t$ and the measurement is $E_t = Umbrella_t$.



Joint Probability Distribution (1)

Given the initial probability distribution at time 0, $\mathrm{P}(X_0)$, we can compute the joint probability distribution of state and measurement using the chain rule and the Markov property:

$$\mathrm{P}(X_{0:t}, E_{1:t}) = \left(\prod_{i=1}^{t} \mathrm{P}(E_i|X_i)\mathrm{P}(X_i|X_{i-1}) \right) \mathrm{P}(X_0).$$

Explanation of formula by example:

$$\begin{aligned} & P(E_1|X_1)P(X_1|X_0)P(X_0) \\ =& P(E_1|X_1, X_0)P(X_1|X_0)P(X_0) \\ =& P(E_1|X_1, X_0)P(X_1, X_0) \\ =& P(E_1, X_1, X_0). \end{aligned}$$

Joint Probability Distribution (2)

Reminder:

$$\mathrm{P}(X_{0:t}, E_{1:t}) = \left(\prod_{i=1}^{t} \mathrm{P}(E_i|X_i)\mathrm{P}(X_i|X_{i-1}) \right) \mathrm{P}(X_0).$$

Given are the initial probability distribution $P(Rain_0 = true) = 0.2$, $P(Rain_0 = false) = 0.8$ and the conditional probabilities from slide 16.

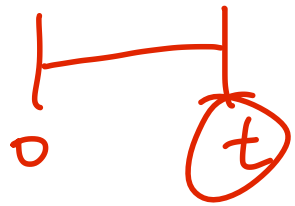
First iteration ($t = 1$):

$$P(r_0, r_1, u_1) = P(u_1|r_1)P(r_1|r_0)P(r_0) = 0.9 \cdot 0.7 \cdot 0.2 = 0.126,$$
$$P(r_0, r_1, \neg u_1) = P(\neg u_1|r_1)P(r_1|r_0)P(r_0) = 0.1 \cdot 0.7 \cdot 0.2 = 0.014, \ldots$$

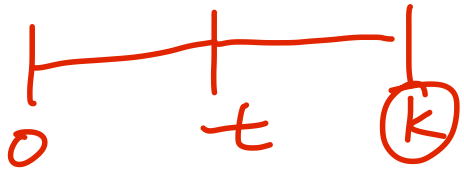
Second iteration ($t = 2$):

$$\begin{aligned} P(r_0, r_1, r_2, u_1, u_2) &= P(u_2|r_2)P(r_2|r_1)P(u_1|r_1)P(r_1|r_0)P(r_0) \\ &= 0.9 \cdot 0.7 \cdot 0.9 \cdot 0.7 \cdot 0.2 = 0.07938, \\ P(r_0, r_1, r_2, \neg u_1, u_2) &= P(u_2|r_2)P(r_2|r_1)P(\neg u_1|r_1)P(r_1|r_0)P(r_0) \\ &= 0.9 \cdot 0.7 \cdot 0.1 \cdot 0.7 \cdot 0.2 = 0.0088, \ldots \end{aligned}$$

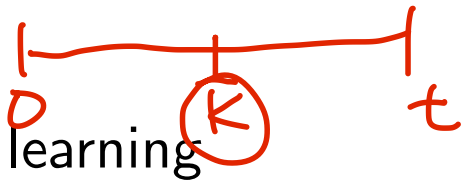
Inference Tasks in Hidden Markov Models



- **Filtering**: $P(X_t|e_{1:t})$
  belief state – input to the decision process of a rational agent
- **Prediction**: $P(X_{t+k}|e_{1:t})$ for $k > 0$
  evaluation of possible action sequences;
  like filtering without the evidence

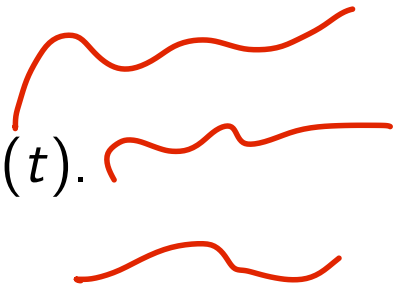


- **Smoothing**: $P(X_k|e_{1:t})$ for $0 \leq k < t$
  better estimate of past states, essential for learning



- **Most likely explanation**: $\arg\max_{x_{1:t}} P(x_{1:t}|e_{1:t})$
  speech recognition, decoding with a noisy channel

We will see that all algorithms have a time complexity of $\mathcal{O}(t)$.



Filtering (1)

Aim

Devise a **recursive** state estimation algorithm:

$$\mathrm{P}(X_{t+1}|e_{1:t+1}) = f(e_{t+1}, \mathrm{P}(X_t|e_{1:t}))$$

Such a recursive algorithm can be obtained as follows:

$$\begin{aligned}
&\mathrm{P}(X_{t+1}|e_{1:t+1}) \\
&= \mathrm{P}(X_{t+1}|e_{1:t}, e_{t+1}) && \text{(dividing the evidence)} \\
&= \alpha \mathrm{P}(e_{t+1}|X_{t+1}, e_{1:t})\mathrm{P}(X_{t+1}|e_{1:t}) && \text{(using Bayes' rule)} \\
&= \alpha \mathrm{P}(e_{t+1}|X_{t+1})\mathrm{P}(X_{t+1}|e_{1:t}) && \text{(Markov assumption on sensors)}
\end{aligned}$$

The probability $\mathrm{P}(X_{t+1}|e_{1:t})$ represents a one-step prediction of the next state as discussed on the next slide.

$$p(X_t \mid e_{1:t}) = p(X_t \mid e_{1:t-1}, e_t)$$

$$= \alpha \cdot p(e_t \mid X_t, \cancel{e_{1:t-1}}) \cdot p(X_t \mid e_{1:t-1})$$

$$= \alpha \cdot p(e_t \mid X_t) \cdot p(X_t \mid e_{1:t-1})$$

sensor model

$$\rightarrow p(X_t \mid e_{1:t-1}) = \sum_{X_{t-1}} p(X_t, X_{t-1} \mid e_{1:t-1})$$

$$= \sum_{X_{t-1}} p(X_t \mid X_{t-1}, \cancel{e_{1:t-1}}) \cdot p(X_{t-1} \mid e_{1:t-1})$$

$$= \sum_{X_{t-1}} p(X_t \mid X_{t-1}) \cdot p(X_{t-1} \mid e_{1:t-1})$$

Transition model

$$p(X_t \mid e_{1:t}) = \alpha \cdot p(e_t \mid X_t) \sum_{X_{t-1}} p(X_t \mid X_{t-1}) \cdot p(X_{t-1} \mid e_{1:t-1})$$

Filtering (2)

Reminder:

$$\mathrm{P}(X_{t+1}|e_{1:t+1}) = \alpha \mathrm{P}(e_{t+1}|X_{t+1})\mathrm{P}(X_{t+1}|e_{1:t})$$

Prediction by summing out X_t:

$$\mathrm{P}(X_{t+1}|e_{1:t+1}) = \alpha \mathrm{P}(e_{t+1}|X_{t+1}) \underbrace{\sum_{x_t} \mathrm{P}(X_{t+1}|x_t, e_{1:t}) P(x_t|e_{1:t})}_{\mathrm{P}(X_{t+1}|e_{1:t})}$$

$$= \alpha \mathrm{P}(e_{t+1}|X_{t+1}) \sum_{x_t} \mathrm{P}(X_{t+1}|x_t) P(x_t|e_{1:t}). \quad \text{(Markov assumption)} \tag{2}$$

- $\mathrm{P}(e_{t+1}|X_{t+1}) = \mathrm{P}(e_t|X_t)$ is directly obtained from the sensor model.
- $\mathrm{P}(X_{t+1}|x_t)$ comes from the transition model.
- $P(x_t|e_{1:t})$ comes from the current state distribution.
- Algorithm is time and space **constant** (independent of t).

Filtering: Matrix Notation

Reminder:

$$\mathrm{P}(X_{t+1}|e_{1:t+1}) = \alpha \mathrm{P}(e_{t+1}|X_{t+1}) \sum_{x_t} \mathrm{P}(X_{t+1}|x_t) P(x_t|e_{1:t}).$$

To bring the filtering algorithm in matrix notation, we introduce

$$(f_i)_{1:t} = P(X_t = x_i|e_{1:t}),$$

$$(O_{ij})_t = \begin{cases} P(e_t|X_t = x_i), \text{ if } j = i \\ 0, \text{ otherwise.} \end{cases}$$

This makes it possible to write the filtering algorithm as

$$\mathrm{f}_{1:t+1} = \alpha \mathrm{O}_{t+1} \mathrm{T} \mathrm{f}_{1:t},$$

where T was the transition matrix. Note that the transition matrix is defined as its transpose in the AI book.

Filtering: Umbrella Example (1)

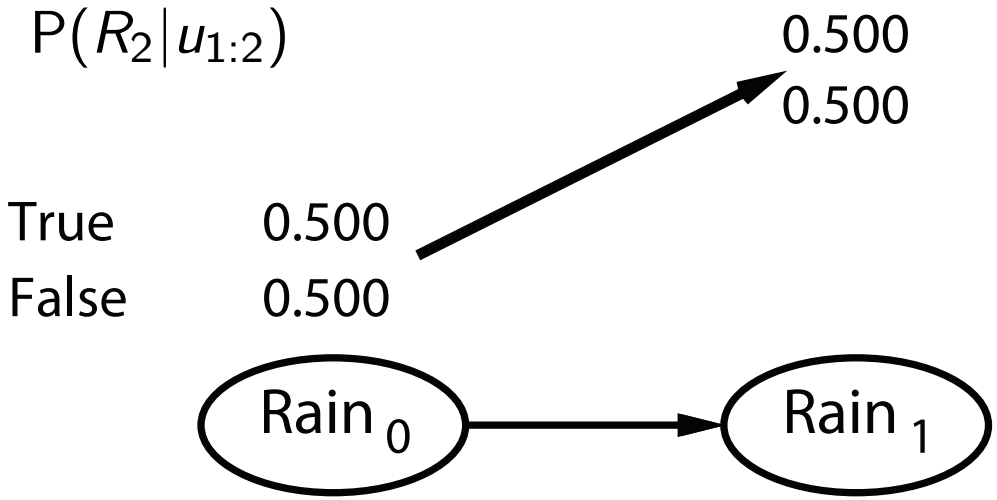
Query: $\mathrm{P}(R_2|u_{1:2})$

True	0.500
False	0.500

Day 0: No observations; Only the security guard's belief, which is $\mathrm{P}(R_0) = \langle 0.5, 0.5 \rangle$.

Filtering: Umbrella Example (2)

Query: $\mathrm{P}(R_2|u_{1:2})$

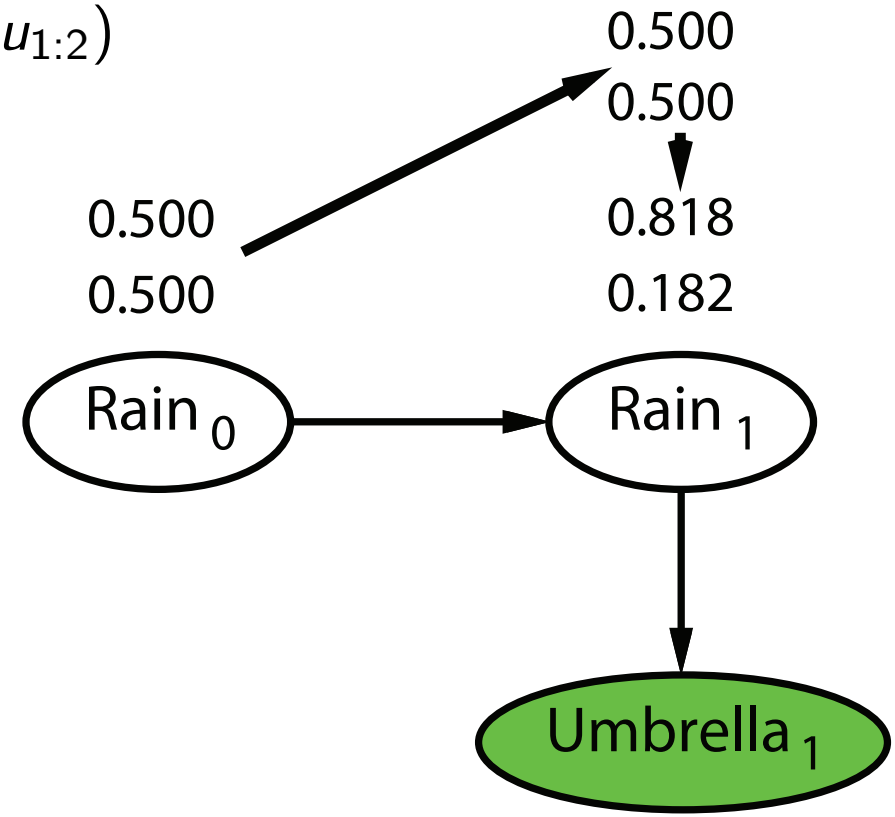


Day 1: The prediction from $t = 0$ to $t = 1$ is

$$\mathrm{P}(R_1) = \sum_{r_0} \mathrm{P}(R_1|r_0)P(r_0) = \langle 0.7, 0.3\rangle \times 0.5 + \langle 0.3, 0.7\rangle \times 0.5 = \langle 0.5, 0.5\rangle.$$

Filtering: Umbrella Example (3)

Query: $\mathrm{P}(R_2|u_{1:2})$

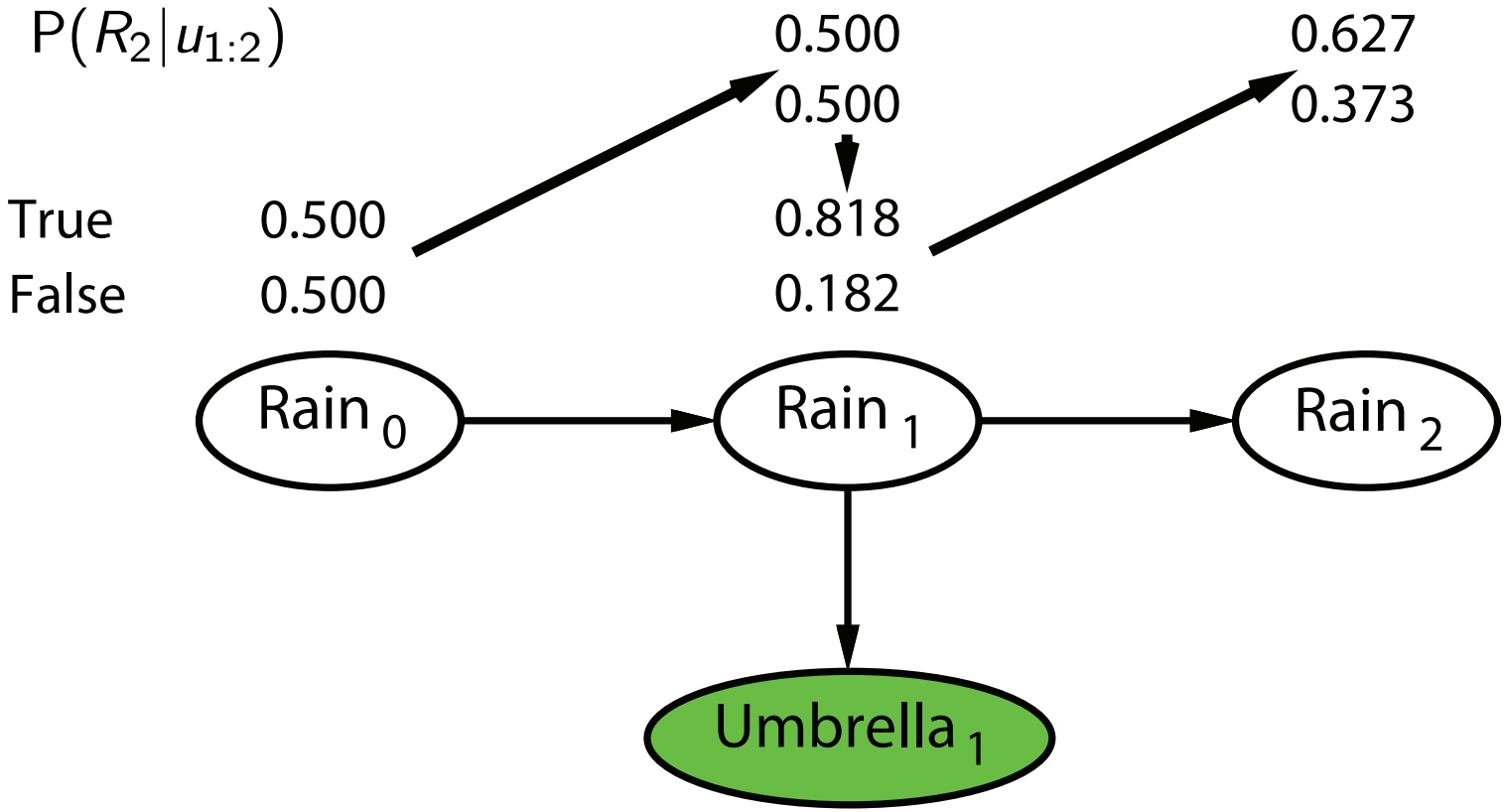


Day 1: The umbrella appears, we incorporate the measurement

$$\mathrm{P}(R_1|u_1) = \alpha \mathrm{P}(u_1|R_1)\mathrm{P}(R_1) = \alpha(\langle 0.9, 0.2\rangle \times \langle 0.5, 0.5\rangle) = \alpha\langle 0.45, 0.1\rangle$$
$$\approx \langle 0.818, 0.182\rangle.$$

Filtering: Umbrella Example (4)

Query: $\mathrm{P}(R_2|u_{1:2})$

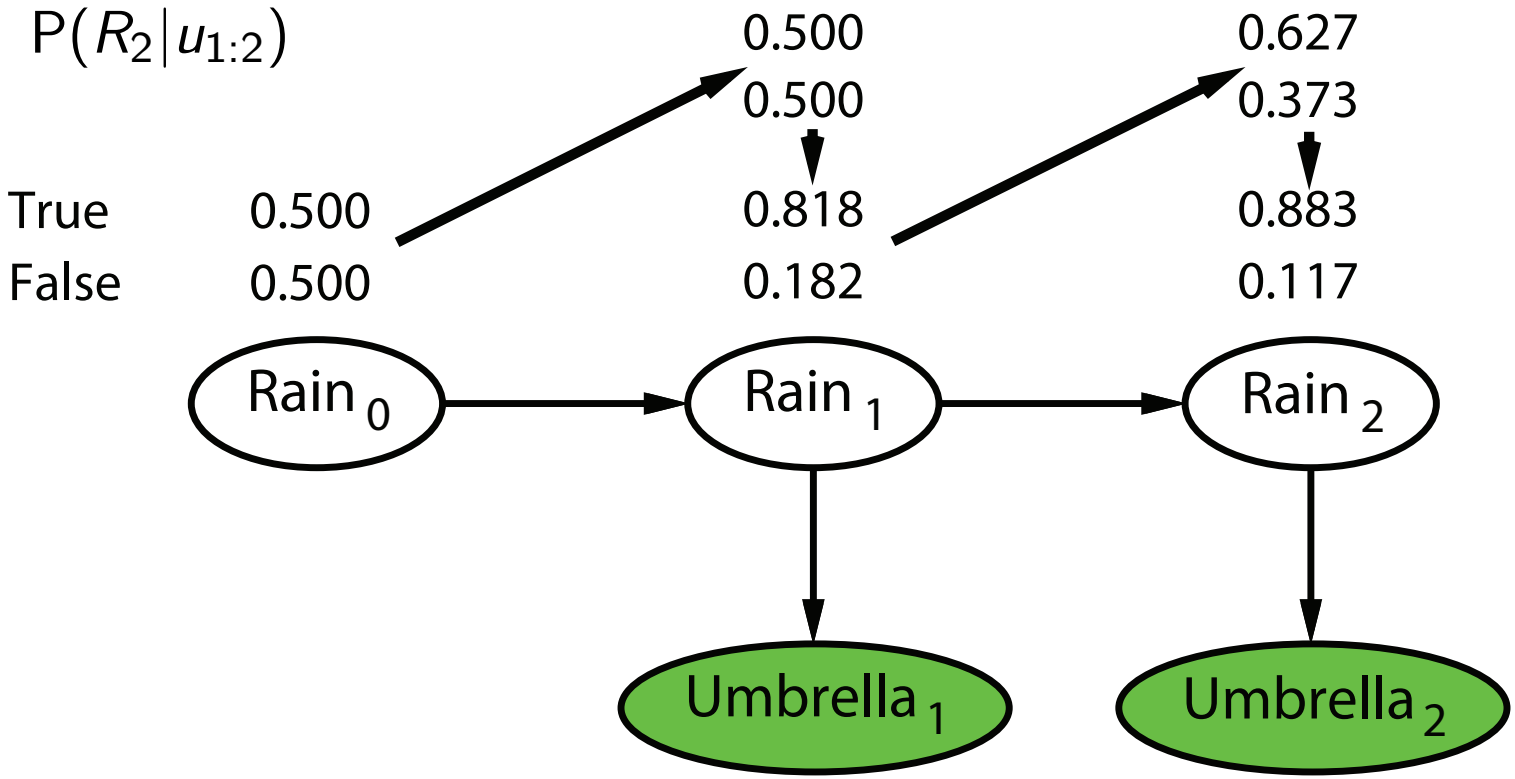


Day 2: The prediction from $t = 1$ to $t = 2$ is

$$\mathrm{P}(R_2|u_1) = \sum_{r_1} \mathrm{P}(R_2|r_1)P(r_1|u_1) = \langle 0.7, 0.3\rangle \times 0.818 + \langle 0.3, 0.7\rangle \times 0.182$$
$$\approx \langle 0.627, 0.373\rangle.$$

Filtering: Umbrella Example (5)

Query: $\mathrm{P}(R_2|u_{1:2})$



Day 2: The umbrella appears, we incorporate the measurement

$$\mathrm{P}(R_2|u_1, u_2) = \alpha \mathrm{P}(u_2|R_2)\mathrm{P}(R_2|u_1) = \alpha(\langle 0.9, 0.2\rangle \times \langle 0.627, 0.373\rangle)$$
$$= \alpha\langle 0.565, 0.075\rangle \approx \langle 0.883, 0.117\rangle.$$

Filtering: Umbrella Example in Matrix Notation

Observation matrices:

$$\mathsf{O}_1 = \mathsf{O}_2 = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

Transition matrix:

$$\mathsf{T} = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}.$$

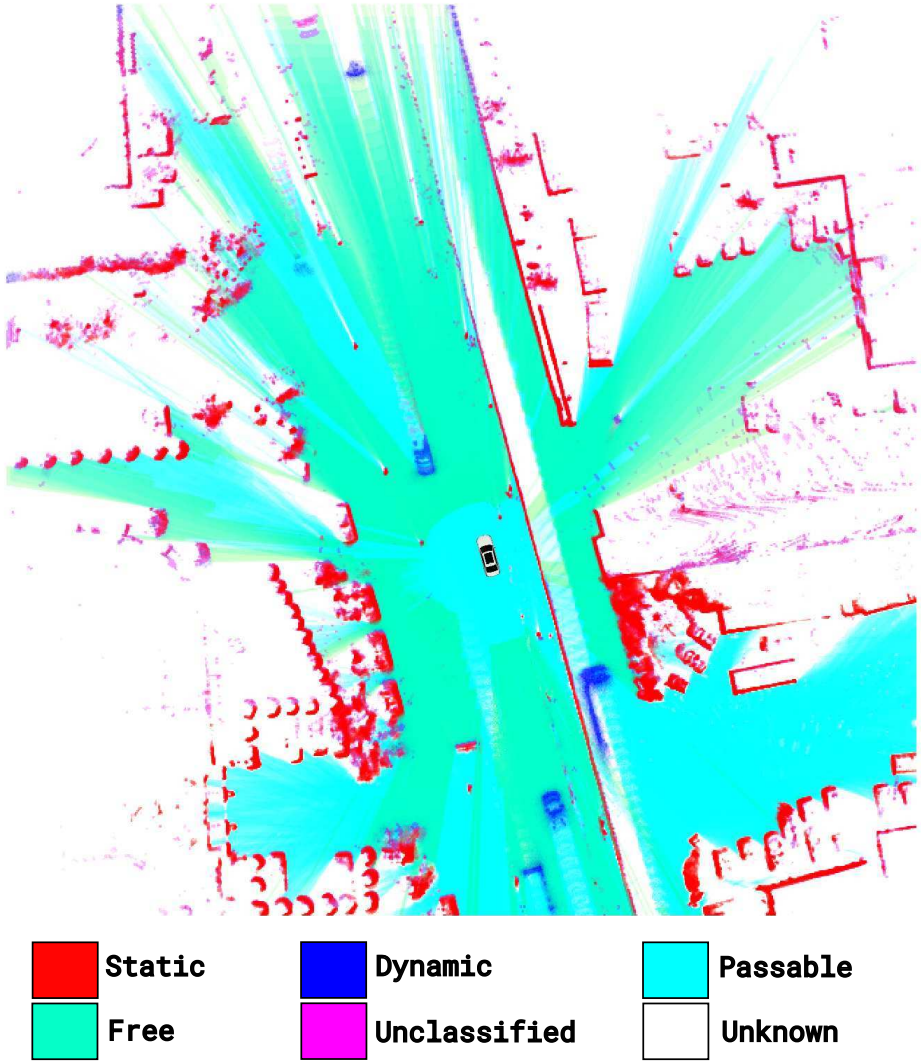
$$\mathsf{f}_{1:t+1} = \alpha \mathsf{O}_{t+1} \mathsf{T} \mathsf{f}_{1:t}$$

$$\mathsf{f}_{1:1} = \alpha \begin{bmatrix} 0.9 & 0 \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \alpha \begin{bmatrix} 0.45 \\ 0.1 \end{bmatrix} \approx \begin{bmatrix} 0.818 \\ 0.182 \end{bmatrix}$$

$$\mathsf{f}_{1:2} = \alpha \begin{bmatrix} 0.9 & 0 \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.818 \\ 0.182 \end{bmatrix} = \alpha \begin{bmatrix} 0.5645 \\ 0.0746 \end{bmatrix} \approx \begin{bmatrix} 0.883 \\ 0.117 \end{bmatrix}$$

Filtering Example: Dynamic Grid Generation[1]

- Problem: Estimating the occupancy probability of a grid.
- The actual occupancy state is hidden, we only have LiDAR sensor measurements as observations.



[1] Ye, Egon, et al. "Offline Dynamic Grid Generation for Automotive Environment Perception Using Temporal Inference Methods." IEEE Robotics and Automation Letters 6.3 (2021): 5501-5508.

Prediction

- The task of **prediction** can be seen simply as filtering without the addition of new evidence.
- The filtering process already incorporates a one-step prediction.

It is trivial to see that

$$\mathrm{P}(X_{t+k+1}|e_{1:t}) = \sum_{x_{t+k}} \mathrm{P}(X_{t+k+1}|x_{t+k}) P(x_{t+k}|e_{1:t}).$$

Comments:

- As $k \to \infty$, $P(x_{t+k}|e_{1:t})$ tends to the **stationary distribution** of the included Markov chain, where $\mathrm{p}_{t+k} = \mathrm{T}^k \mathrm{p}_t$ (see slide 10).
- It is obvious that we cannot accurately predict the state when the time horizon is relatively long.

$$P(X_{t+k} \mid e_{1:t}) = \sum_{X_{t+k-1}} P(X_{t+k}, X_{t+k-1} \mid e_{1:t})$$

$$= \sum_{X_{t+k-1}} P(X_{t+k} \mid X_{t+k-1}, \cancel{e_{1:t}}) \cdot P(X_{t+k-1} \mid e_{1:t})$$

$$= \sum_{X_{t+k-1}} \underbrace{P(X_{t+k} \mid X_{t+k-1})}_{\text{Transition model}} P(X_{t+k-1} \mid e_{1:t})$$

Prediction: Umbrella Example (1)

Reminder: We use the probabilities

$$(p_i)_n = P(X_n = x_i)$$
$$(\hat{p}_i)_n = P(E_n = e_i)$$

and the matrices

$$T_{i,j} = P(X_n = x_i | X_{n-1} = x_j),$$
$$H_{i,j} = P(E_n = e_i | X_n = x_j).$$

Umbrella example

Given $x_1 \hat{=} r, x_2 \hat{=} \neg r, e_1 = u, e_2 = \neg u$, the initial probability distribution, and the conditional probabilities from slide 16, we have:

$$\mathrm{p}_0 = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}, \quad \mathrm{T} = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}, \quad \mathrm{H} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}.$$

Prediction: Umbrella Example (2)

Using (1) on slide 10, one obtains

$$\begin{aligned} p_n &= T^n p_0, \\ \hat{p}_n &= H p_n. \end{aligned}$$

Umbrella example

$$p_0 = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}, \quad T^{100} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \quad H = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix},$$

$$p_{100} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \quad \hat{p}_{100} = \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix}.$$

Example: Safety Assessment of Autonomous Cars

Not relevant for the exam

Trajectory planner:

- Planned trajectory

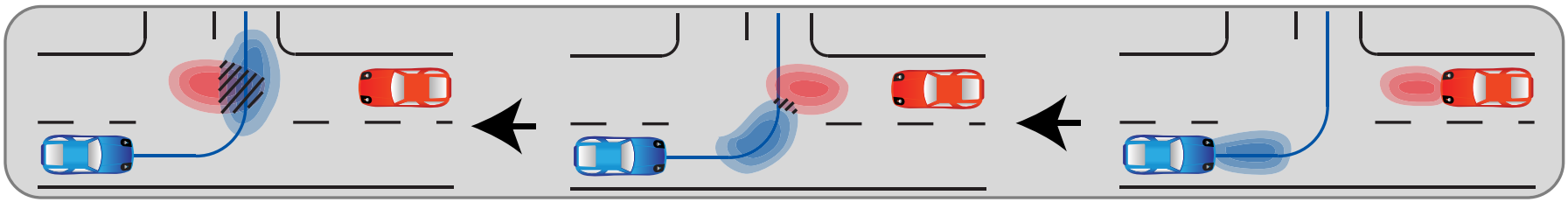
Environment sensors:

- Road geometry
- Static obstacles
- Dynamic obstacles

Cycle time ≈ 0.5 sec

Prediction horizon $>$ Cycle time
$\rightarrow$ Prediction has to be faster than real time.

Safety Verification: Predict situation for each cycle $\rightarrow$ Crash probabilities

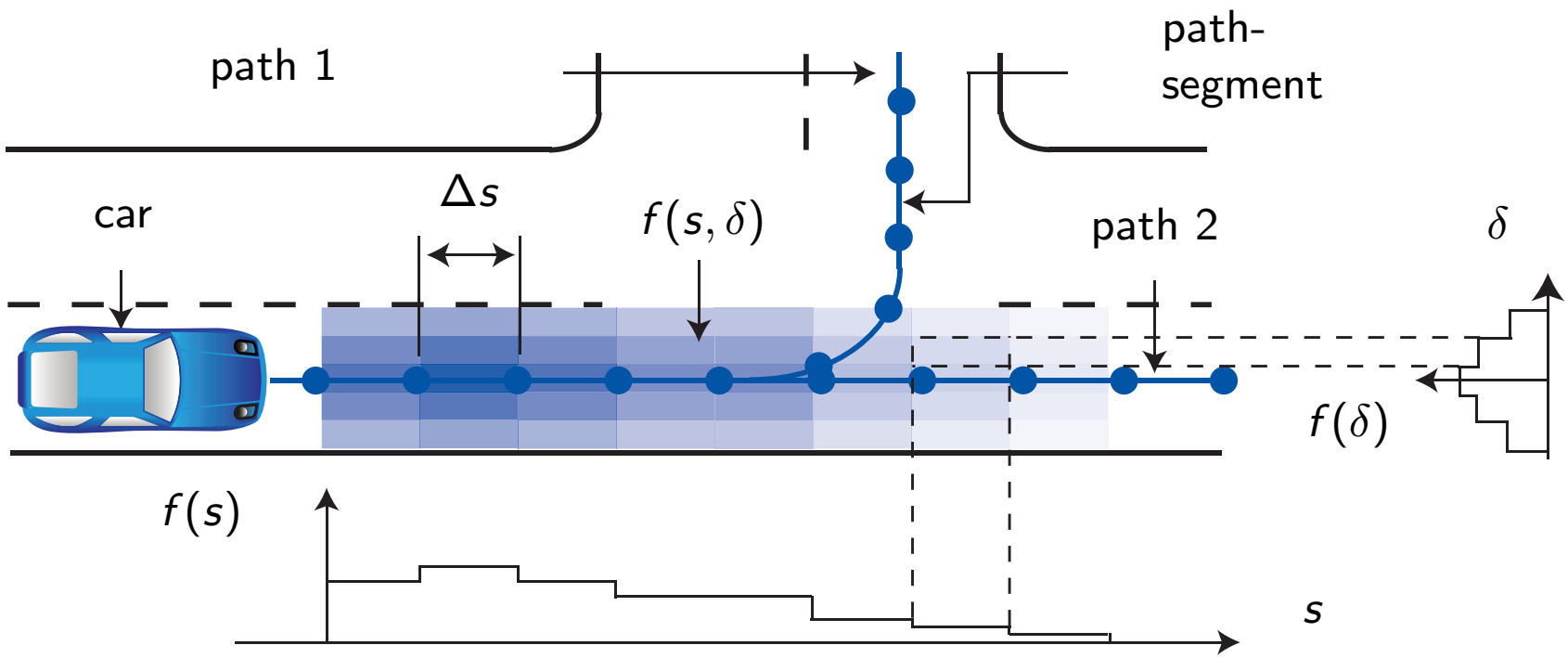


Modeling of Other Traffic Participants

Not relevant for the exam

Structured environments

- Vehicles follow preferred paths such as traffic lanes.
- Longitudinal distribution $f(s)$ from hidden Markov model, lateral distribution $f(\delta)$ is static.

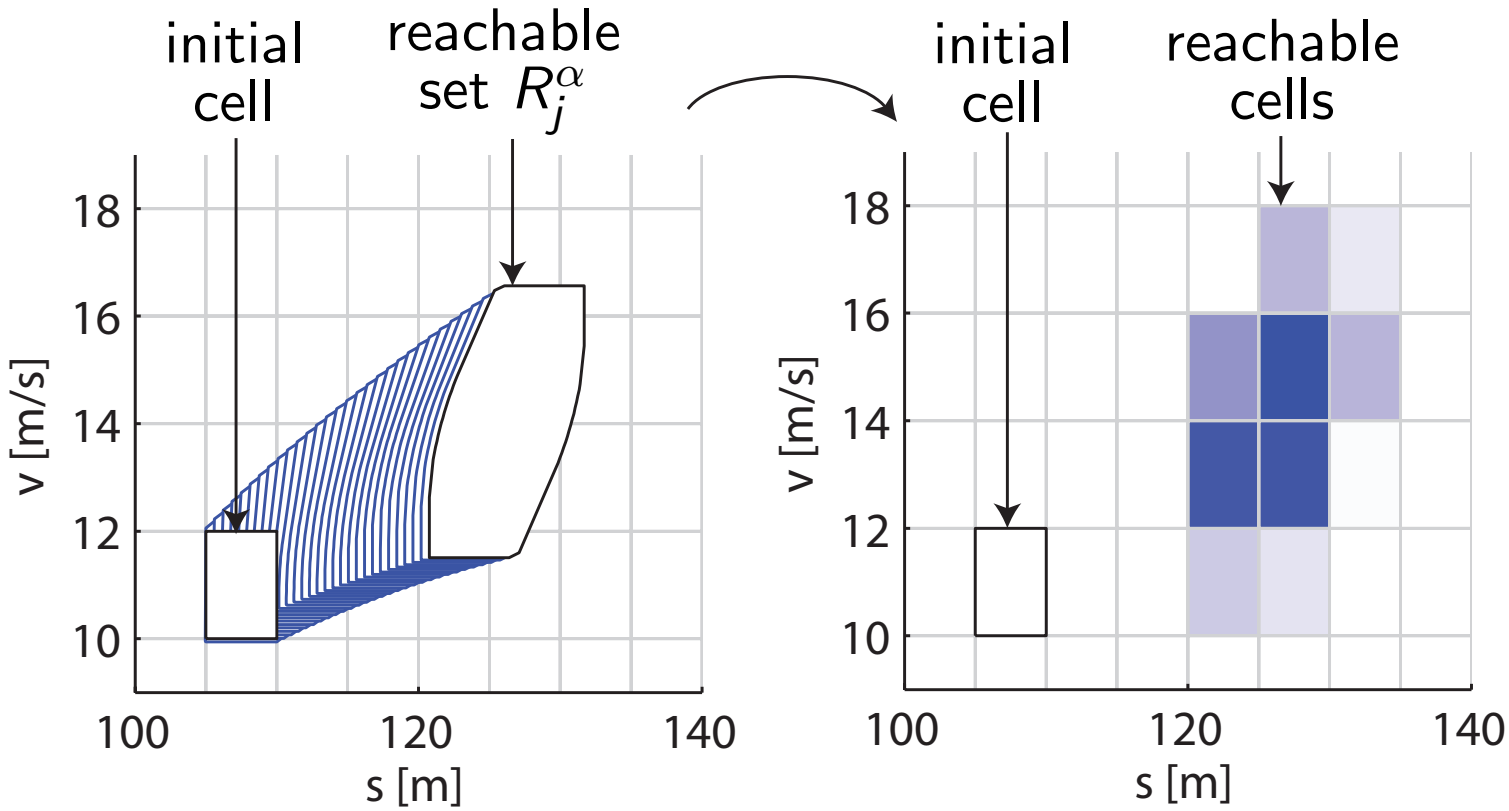


Abstraction to Hidden Markov Models via Reachability Analysis

Not relevant for the exam

Prediction: $p_n = T\, p_{n-1}$.

Computation of the transition matrix T via reachability analysis.

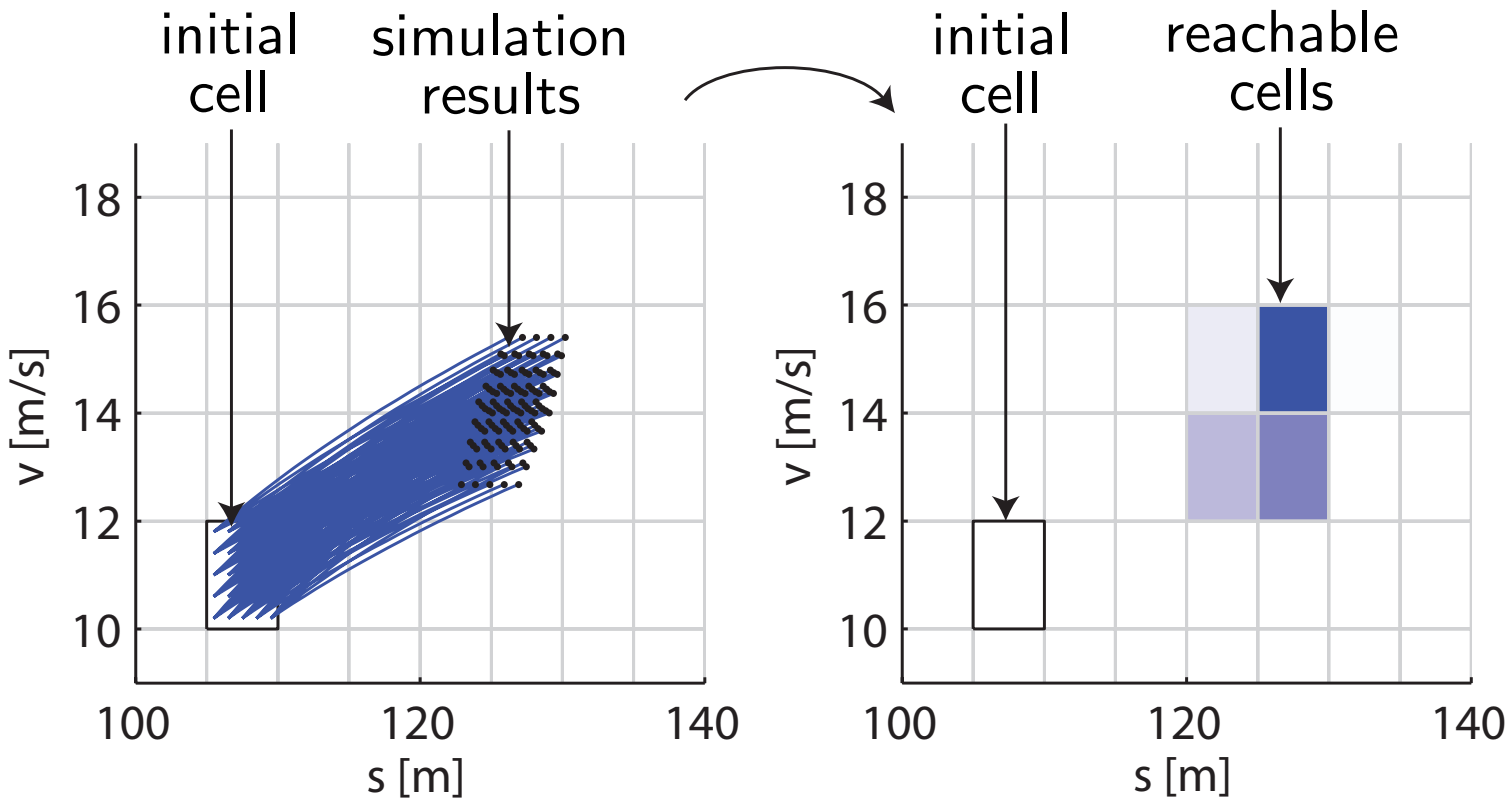


Abstraction to Hidden Markov Models via Simulation Techniques

Not relevant for the exam

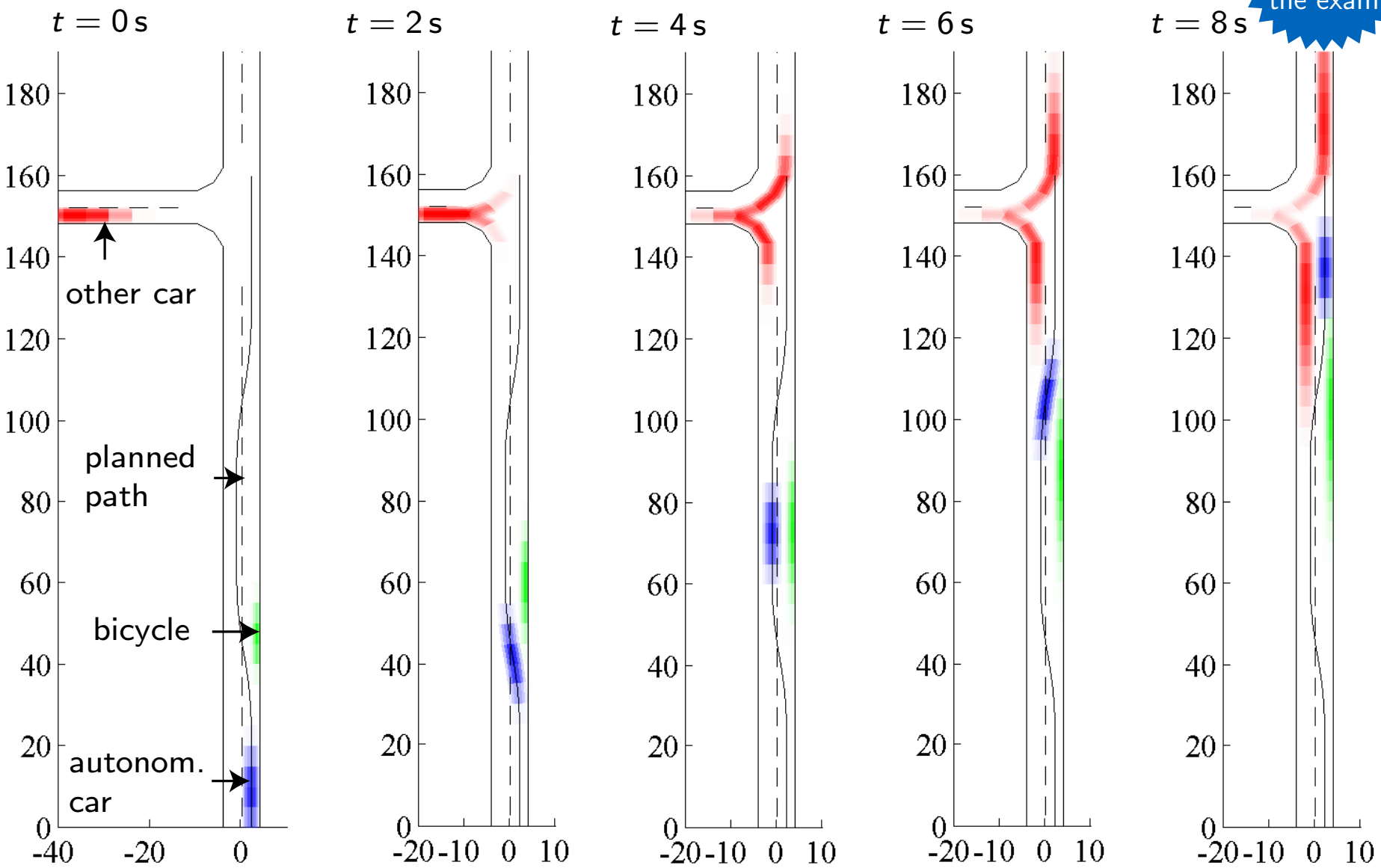
Prediction: $p_n = T\, p_{n-1}$.

Computation of the transition matrix T via simulation techniques.



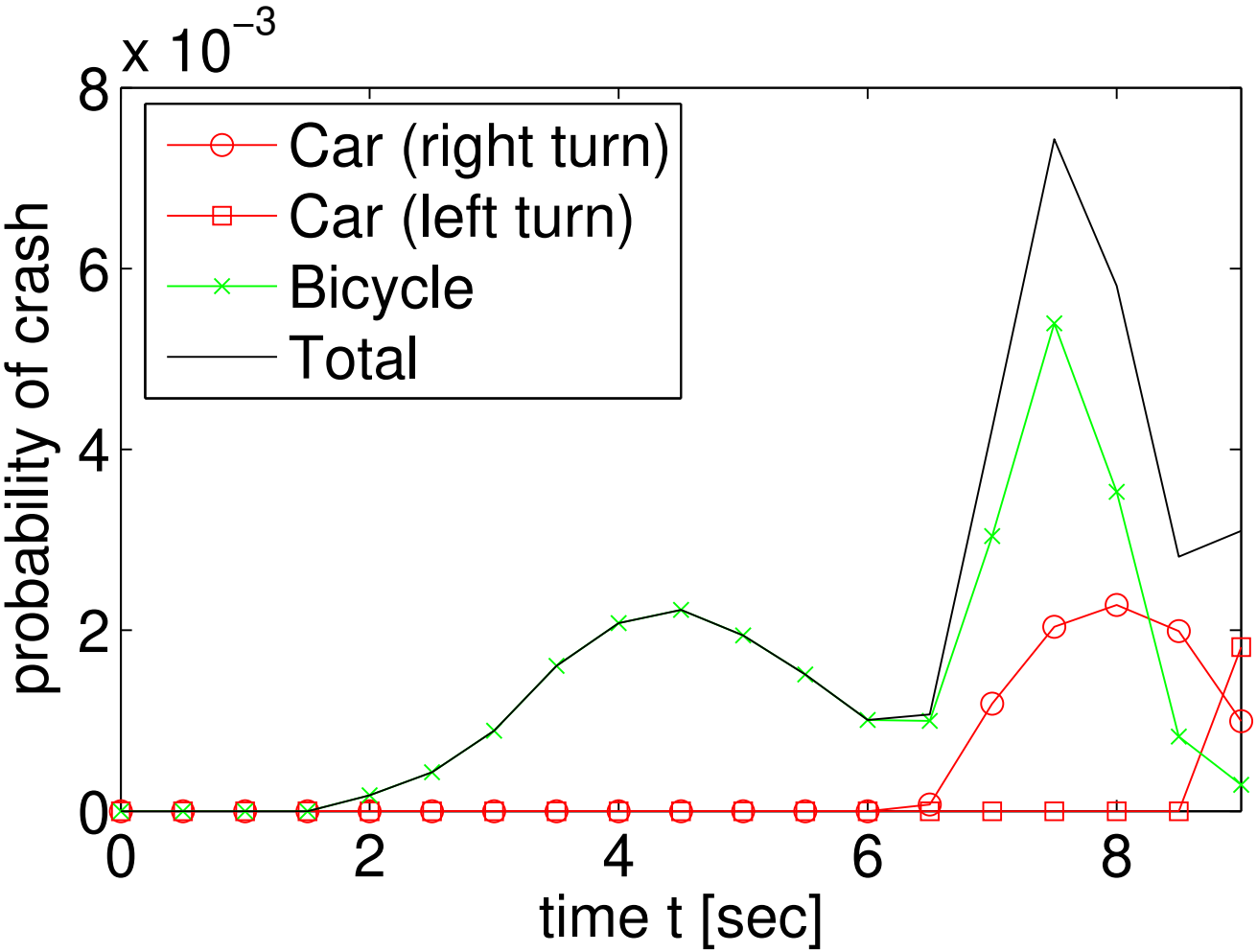
Overtaking Scenario

Not relevant for the exam



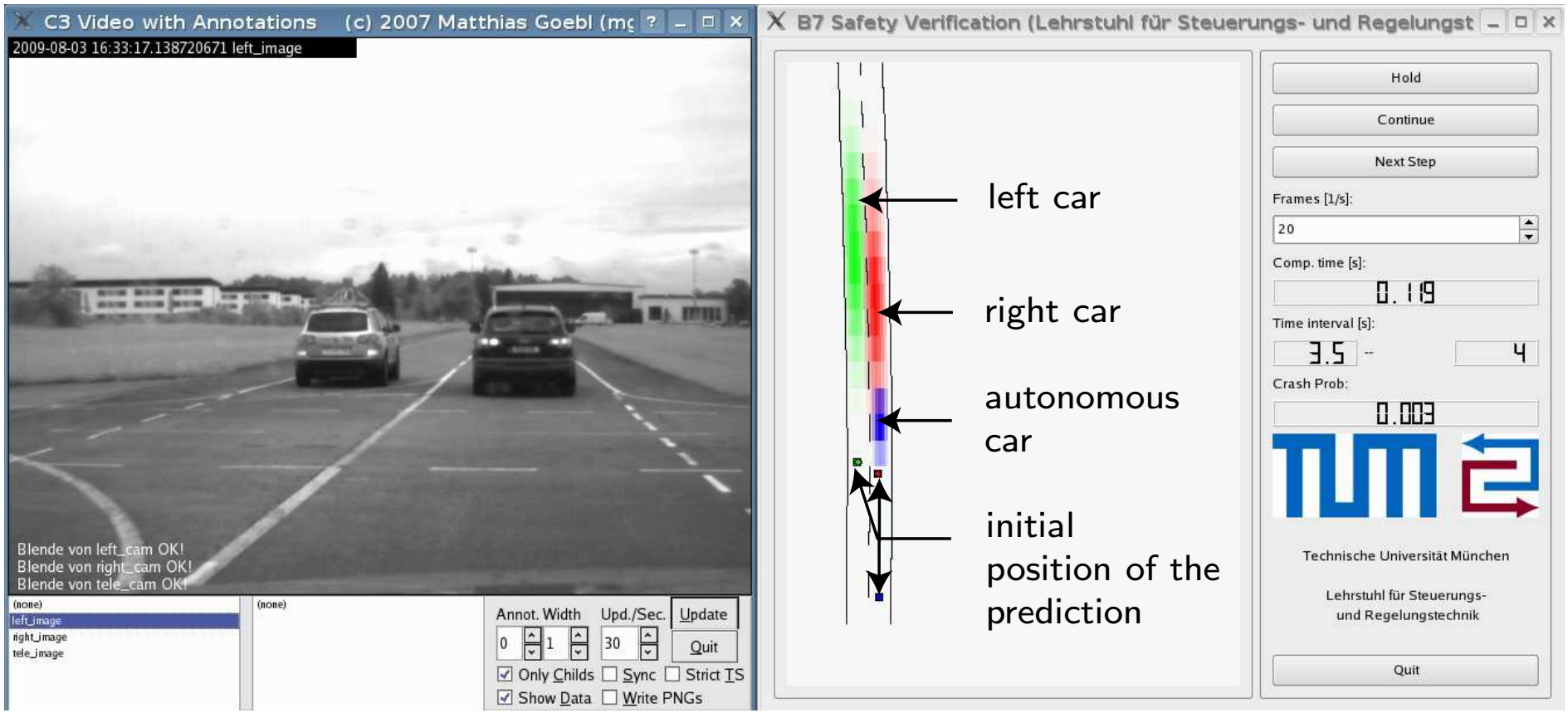
Crash Probability

Not relevant for the exam



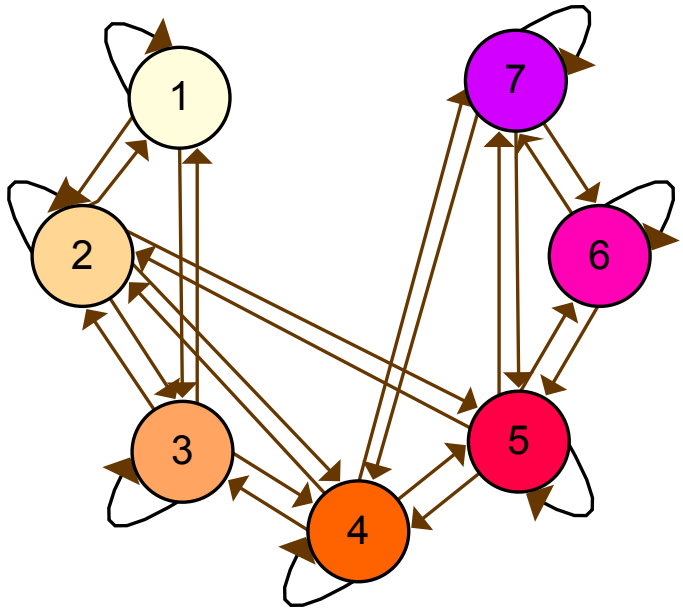
Test Drive

Not relevant for the exam



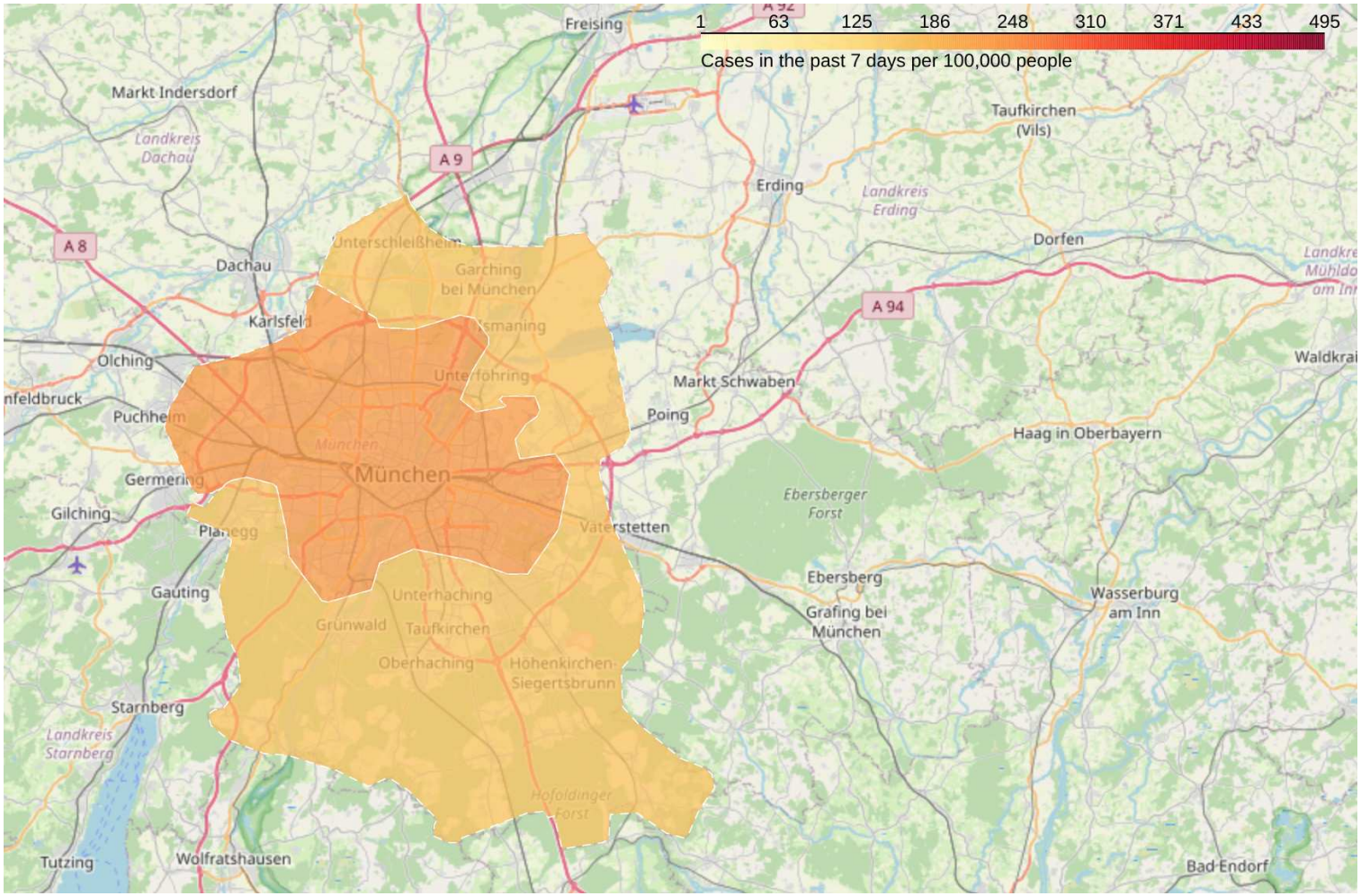
Prediction Example: Spatio-Temporal Prediction of COVID-19[2]

- In order to control COVID-19 and future pandemics, we need an understanding of the disease's evolution as well as the effectiveness of intervention methods.
- Our observations are the reported cases and deaths in several countries and regions.
- An HMM with seven (hidden) severity states is used to predict the severity level of different regions over a fixed time period.



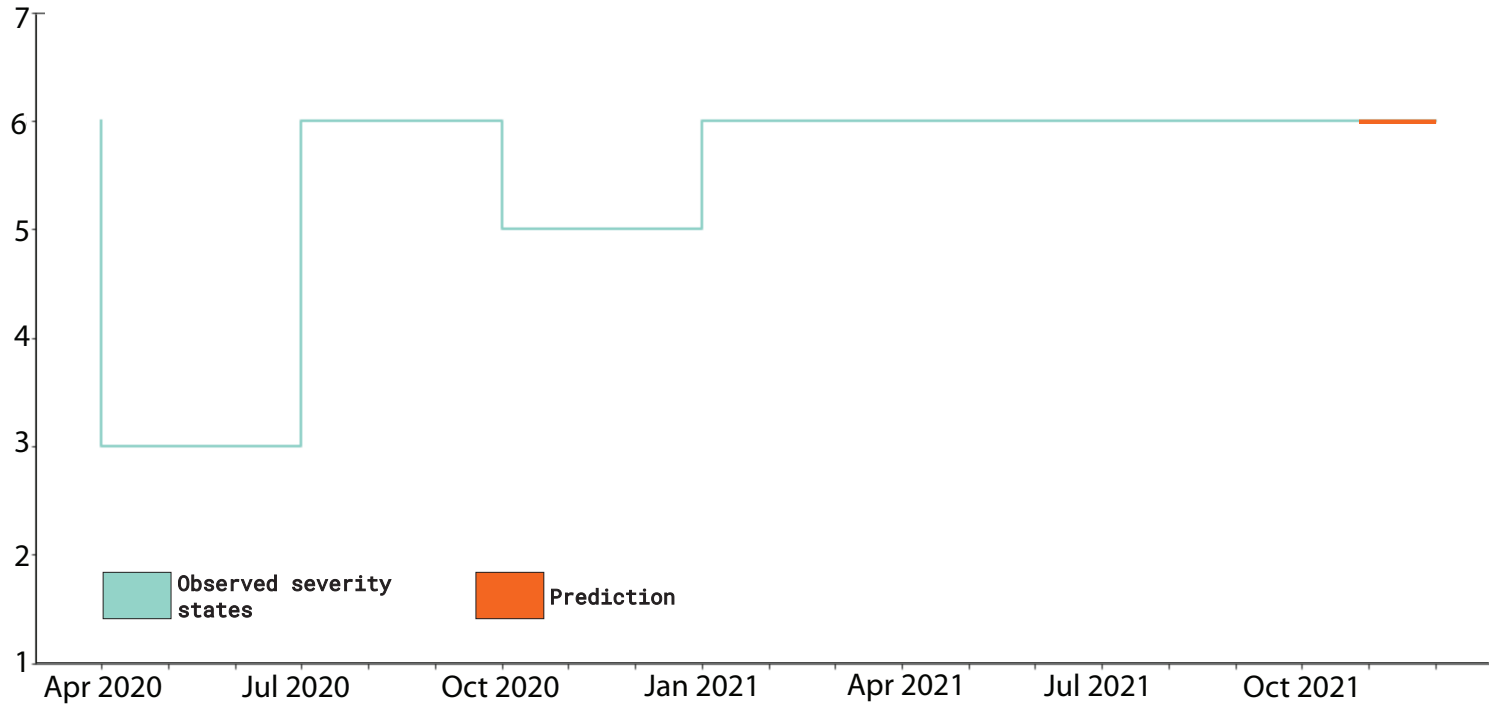
[2] S. Zhou et al., "Application of Hidden Markov Models to Analyze, Group and Visualize Spatio-Temporal COVID-19 Data," in IEEE Access, vol. 9, pp. 134384-134401, 2021

Prediction Example: COVID-19 Cases per 100.000 in the last 7 Days in the Munich Region, 09.12.2020

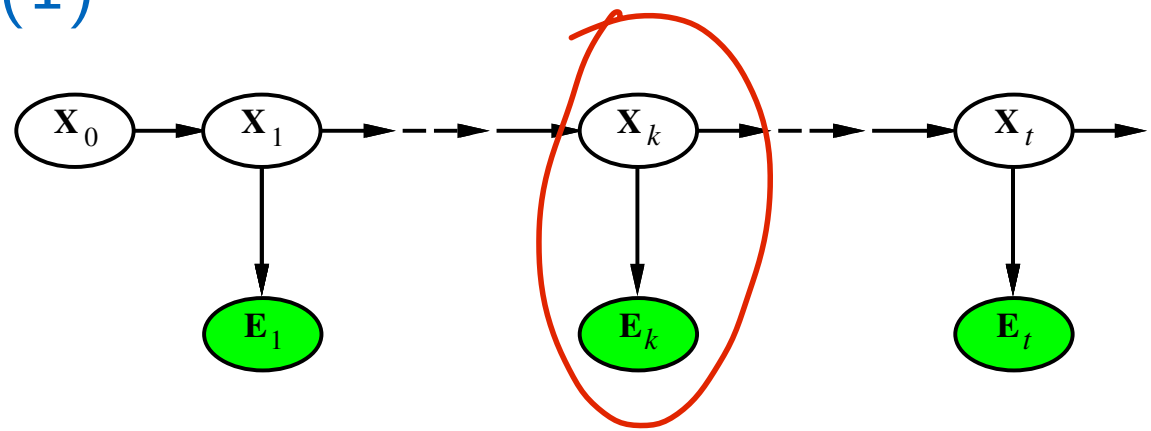


Prediction Example: COVID-19 Severity Prediction in Munich

In the HMM exercise of 2021/22, we predicted the severity states in November and December 2021. The result, state 6, was actually correct.



Smoothing (1)



- **Smoothing** is the process of computing the distribution over past states given evidence up to the present: $\mathrm{P}(X_k|e_{1:t})$ for $0 \leq k < t$.
- In anticipation of creating another recursive algorithm (as for filtering), we divide the evidence $e_{1:t}$ into $e_{1:k}$ and $e_{k+1:t}$:

$$\begin{aligned}
&\mathrm{P}(X_k|e_{1:t}) = \mathrm{P}(X_k|e_{1:k}, e_{k+1:t}) \\
&= \alpha' \mathrm{P}(X_k, e_{1:k}, e_{k+1:t}) \qquad \text{(normalization)} \\
&= \alpha' \mathrm{P}(e_{k+1:t}|X_k, e_{1:k}) \mathrm{P}(X_k|e_{1:k}) P(e_{1:k}) \qquad \text{(using chain rule)} \\
&= \alpha \mathrm{P}(X_k|e_{1:k}) \mathrm{P}(e_{k+1:t}|X_k, e_{1:k}) \qquad \text{(remove } P(e_{1:k})\text{)} \\
&= \alpha \mathrm{P}(X_k|e_{1:k}) \mathrm{P}(e_{k+1:t}|X_k) \qquad \text{(using conditional independence)} \\
&= \alpha \mathrm{f}_{1:k} \times \mathrm{b}_{k+1:t} \qquad \text{(f: forward; b: backward)} \qquad (3)
\end{aligned}$$

$$p(x_k \mid e_{1:t}) = p(x_k \mid e_{1:k}, e_{k+1:t})$$

$$= \alpha p(x_k, e_{1:k}, e_{k+1:t})$$

$$= \alpha p(e_{k+1:t} \mid x_k, e_{1:k}) \cdot p(x_k \mid e_{1:k}) \cdot p(e_{1:k})$$

$$= \alpha \underbrace{p(e_{k+1:t} \mid x_k)}_{b_{k+1:t}} \cdot \underbrace{p(x_k \mid e_{1:k})}_{f_{1:t}}$$

$$p(e_{k+1:t} \mid x_k) = \sum_{x_{k+1}} p(e_{k+1:t}, x_{k+1} \mid x_k)$$

$$= \sum_{x_{k+1}} p(e_{k+1:t} \mid x_{k+1}, x_k) \, p(x_{k+1} \mid x_k)$$

$$= \sum_{x_{k+1}} p(e_{k+1}, e_{k+2:t} \mid x_{k+1}) \cdot p(x_{k+1} \mid x_k)$$

$$= \sum_{x_{k+1}} p(e_{k+1} \mid x_{k+1}) \, p(e_{k+2:t} \mid x_{k+1}) \, p(x_{k+1} \mid x_k)$$

sensor model

Transition model

$$p(e_{k+1:t} \mid x_k) = \sum_{x_{k+1}} p(e_{k+1} \mid x_{k+1}) \cdot p(e_{k+2:t} \mid x_{k+1}) \, p(x_{k+1} \mid x_k)$$

$$b_{k+1:t} = T^{\top} O_{k+1} \cdot b_{k+2:t}$$

Smoothing (2)

Reminder: $$\mathrm{P}(X_k|e_{1:t}) = \alpha \underbrace{\mathrm{P}(X_k|e_{1:k})}_{\mathrm{f}_{1:k}} \underbrace{\mathrm{P}(e_{k+1:t}|X_k)}_{\mathrm{b}_{k+1:t}}$$

- The forward factor $\mathrm{f}_{1:k}$ is computed as for the filtering on slide 21.
- The backward factor $\mathrm{b}_{k+1:t}$ is obtained by the following recursion:

$$\begin{aligned}
&\mathrm{P}(e_{k+1:t}|X_k) \\
&= \sum_{x_{k+1}} \mathrm{P}(e_{k+1:t}, x_{k+1}|X_k) && \text{(rule for total probability)} \\
&= \sum_{x_{k+1}} \mathrm{P}(e_{k+1:t}|X_k, x_{k+1})\mathrm{P}(x_{k+1}|X_k) && \left(P(a|b,c)P(b|c) = \frac{P(a,b,c)}{P(b,c)}\frac{P(b,c)}{P(c)} = \frac{P(a,b,c)}{P(c)} = P(a,b|c)\right) \\
&= \sum_{x_{k+1}} P(e_{k+1:t}|x_{k+1})\mathrm{P}(x_{k+1}|X_k) && \text{(by conditional independence)} \\
&= \sum_{x_{k+1}} P(e_{k+1}, e_{k+2:t}|x_{k+1})\mathrm{P}(x_{k+1}|X_k) && \text{(by evidence splitting)} \\
&= \sum_{x_{k+1}} P(e_{k+1}|x_{k+1})P(e_{k+2:t}|x_{k+1})\mathrm{P}(x_{k+1}|X_k) && \text{(by cond. ind.)}
\end{aligned}$$

sensor model, future, transition model

Smoothing (3)

Reminder: $$\mathrm{P}(e_{k+1:t}|X_k) = \sum_{x_{k+1}} P(e_{k+1}|x_{k+1})P(e_{k+2:t}|x_{k+1})\mathrm{P}(x_{k+1}|X_k) \quad (4)$$

- $P(e_{k+1}|x_{k+1})$ is the sensor model, see slide 14.
- $\mathrm{P}(x_{k+1}|X_k)$ is the transition probability, see slide 9.
- $P(e_{k+2:t}|x_{k+1})$ is obtained by recursive execution of the above formula backwards in time.
- The backwards recursion is denoted by

$$\mathrm{b}_{k+1:t} = \texttt{Backward}(\mathrm{b}_{k+2:t}, e_{k+1}),$$

  which implements (4).
- The forward recursion is denoted by

$$\mathrm{f}_{1:t+1} = \alpha\,\texttt{Forward}(\mathrm{f}_{1:t}, e_{t+1}),$$

  which implements (2) on slide 21.
- The backward phase is initialized by $\mathrm{b}_{t+1:t} = \mathrm{P}(e_{t+1:t}|X_t) = P(\quad|X_t) = 1$, where 1 is a vector of ones (probability of observing future evidence is 0).

Smoothing: Matrix Notation

Reminder: $$\mathrm{P}(e_{k+1:t}|X_k) = \sum_{x_{k+1}} P(e_{k+1}|x_{k+1})P(e_{k+2:t}|x_{k+1})\mathrm{P}(x_{k+1}|X_k)$$

To bring the filtering algorithm in matrix notation, we introduce

$$(b_i)_{k+1:t} = P(e_{k+1:t}|X_k = x_i)$$

and use again

$$(O_{ij})_t = \begin{cases} P(e_t|X_t = x_i), \text{ if } j = i \\ 0, \text{ otherwise.} \end{cases}$$

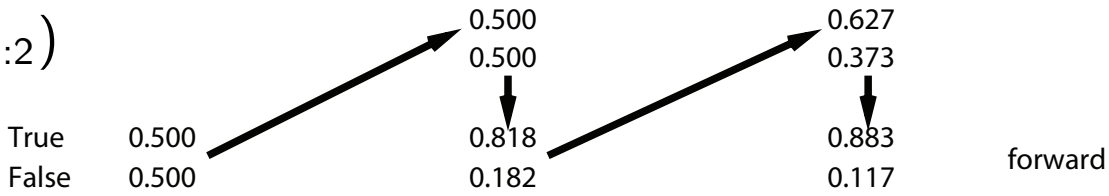
This makes it possible to write (4) as

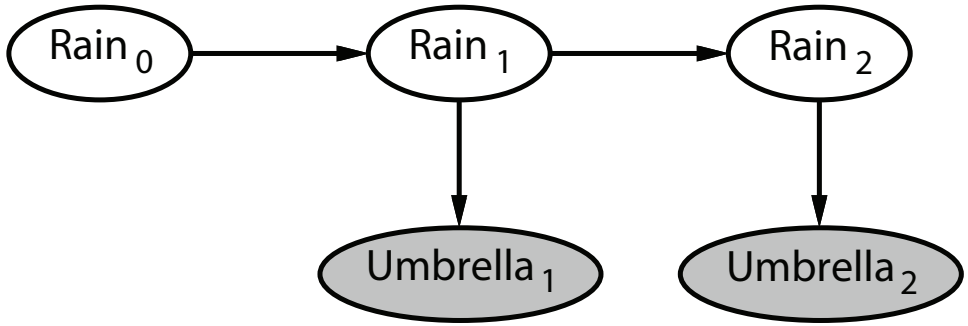
$$\mathrm{b}_{k+1:t} = \mathrm{T}^T \mathrm{O}_{k+1} \mathrm{b}_{k+2:t},$$

where T was the transition matrix. Note that the transition matrix is defined as its transpose in the AI book.

Smoothing: Umbrella Example (1)

Query: $\mathrm{P}(R_1|u_{1:2})$

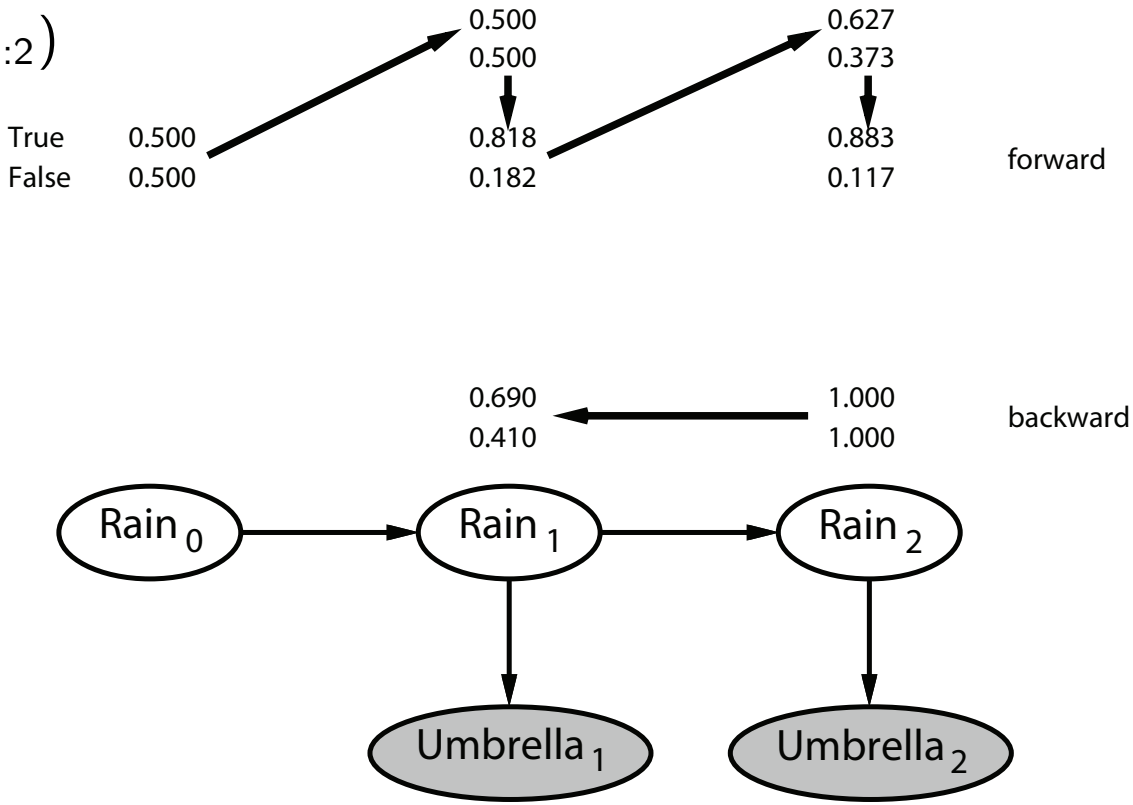




From (3) on slide 43, we have $\mathrm{P}(R_1|u_{1:2}) = \alpha\mathrm{P}(R_1|u_1)\mathrm{P}(u_2|R_1)$. $\mathrm{P}(R_1|u_1)$ we already know from forward filtering to be $\langle 0.818, 0.182\rangle$.

Smoothing: Umbrella Example (2)

Query: $\mathrm{P}(R_1|u_{1:2})$

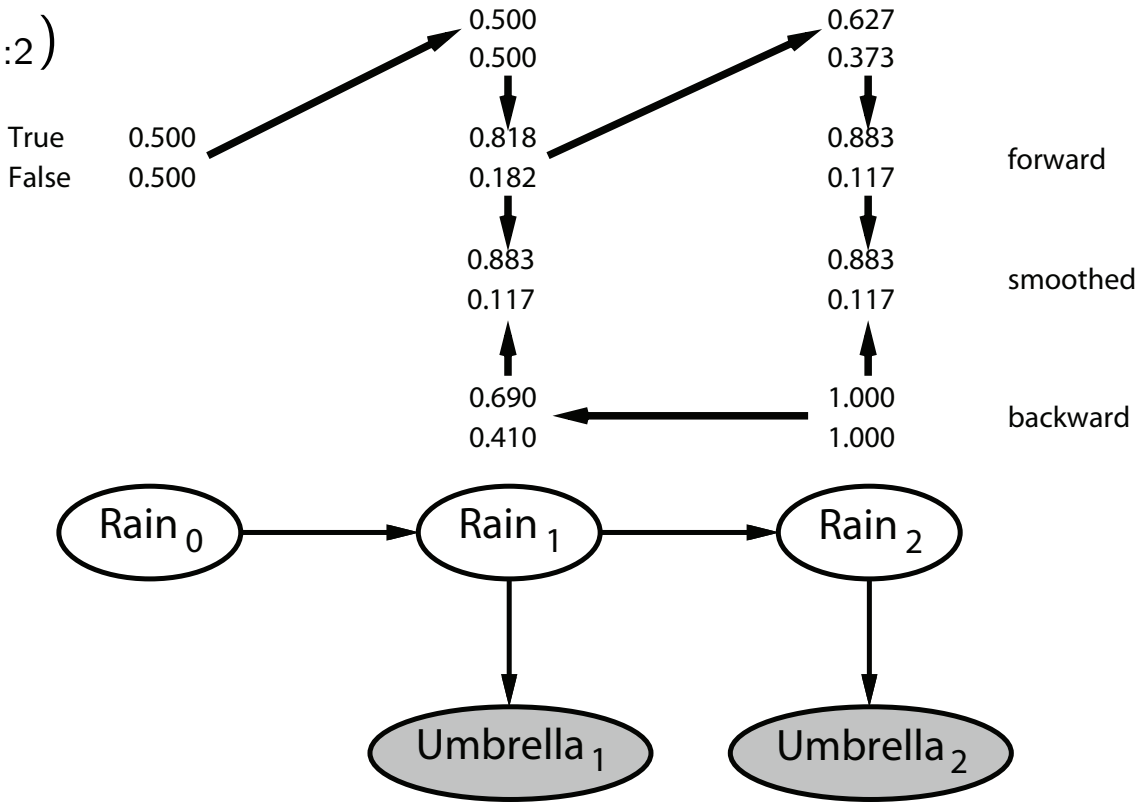


The second term can be computed by the backward recursion in eq. (4):

$$\mathrm{P}(u_2|R_1) = \sum_{r_2} P(u_2|r_2)P(\quad|r_2)\mathrm{P}(r_2|R_1)$$
$$= (0.9 \times 1 \times \langle 0.7, 0.3\rangle) + (0.2 \times 1 \times \langle 0.3, 0.7\rangle) = \langle 0.69, 0.41\rangle.$$

Smoothing: Umbrella Example (3)

Query: $\mathrm{P}(R_1|u_{1:2})$



Inserting the previous results into $\mathrm{P}(R_1|u_{1:2}) = \alpha \mathrm{P}(R_1|u_1)\mathrm{P}(u_2|R_1)$ yields:

$$\mathrm{P}(R_1|u_{1:2}) = \alpha\langle 0.818, 0.182\rangle \times \langle 0.69, 0.41\rangle \approx \langle 0.883, 0.117\rangle.$$

The smoothed estimate for rain is higher than for the filtered one, since the umbrella on day 2 makes it more likely that day 1 was rainy.

Smoothing: Umbrella Example in Matrix Notation

Observation matrices:

$$\mathsf{O}_1 = \mathsf{O}_2 = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

Transition matrix:

$$\mathsf{T} = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}.$$

$$\mathsf{b}_{k+1:t} = \mathsf{T}^T \mathsf{O}_{k+1} \mathsf{b}_{k+2:t}$$

$$\mathsf{b}_{3:2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathsf{b}_{2:2} = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}^T \begin{bmatrix} 0.9 & 0 \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.69 \\ 0.41 \end{bmatrix}$$

Forward-Backward Algorithm

(HMM.ipynb)

function Forward-Backward (ev,*prior*) **returns** a vector of probability distributions

inputs: ev, a vector of evidence values for steps $1, \ldots, t$
prior, the prior distribution on the initial state, $\mathrm{P}(\mathrm{X}_0)$

local variables: fv, a vector of forward messages for steps $0, \ldots, t$
b, backward message representation, initially all ones
sv, a vector of smoothed estimates for steps $1, \ldots, t$

$\mathrm{fv}[0] \leftarrow \mathit{prior}$
for $i = 1$ **to** t **do**
  $\mathrm{fv}[i] \leftarrow \texttt{Forward}(\mathrm{fv}[i-1], \mathrm{ev}[i])$ (see eq. (2) on slide 21)
for $i = t$ **downto** 1 **do**
  $\mathrm{sv}[i] \leftarrow \texttt{Normalize}(\mathrm{fv}[i] \times \mathrm{b})$
  $\mathrm{b} \leftarrow \texttt{Backward}(\mathrm{b}, \mathrm{ev}[i])$ (see eq. (4) on slide 45)
return sv

Unlike filtering, the space complexity is $\mathcal{O}(t)$.

Munich Motion Dataset of Natural Driving (MONA)

- Recordings of five consecutive days.
- Three different perspectives, stationary camera setup.
- 130h of 4k@25fps video material.

Creator: Maximilian Dörrbecker CC BY-SA 2.5

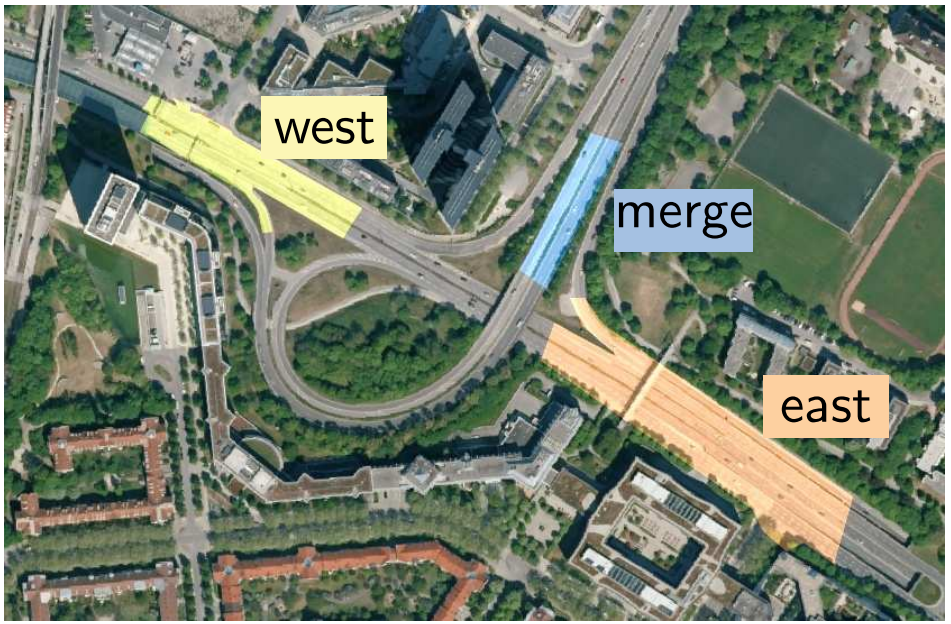
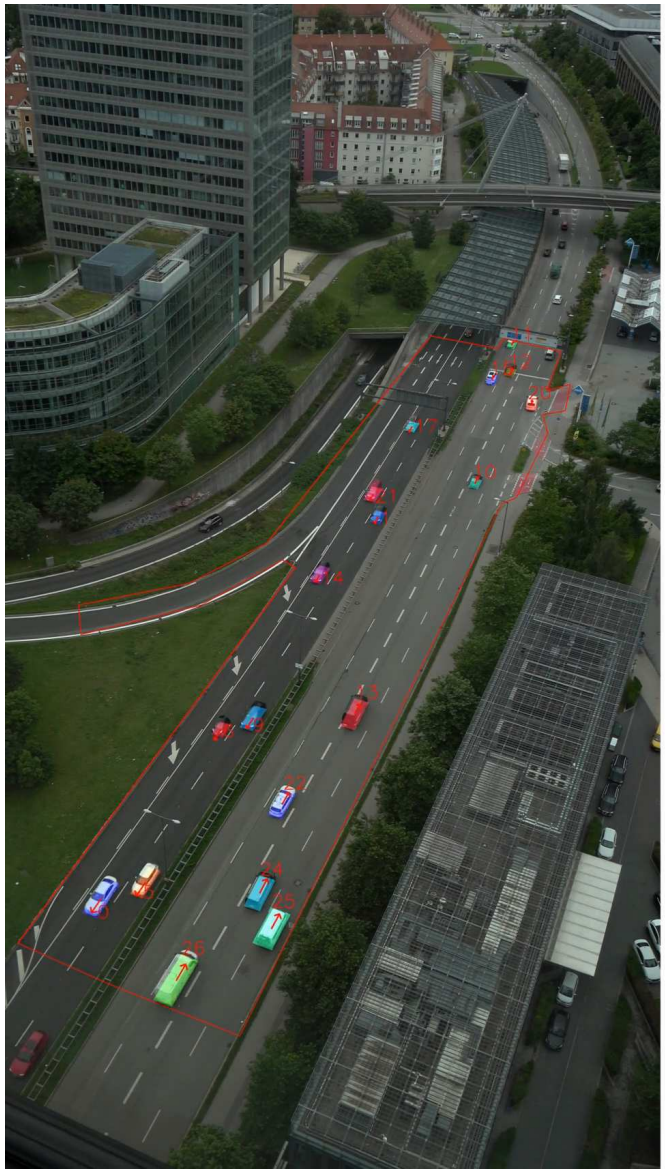
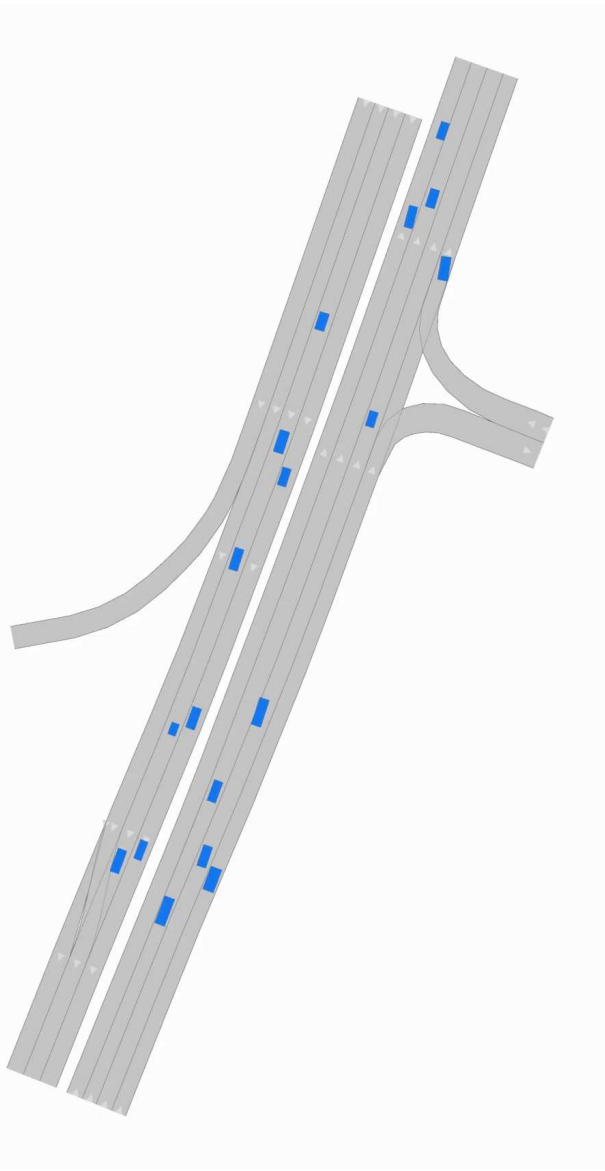


Image source: Bayr. Vermessungsverwaltung (CC BY 3.0 DE).

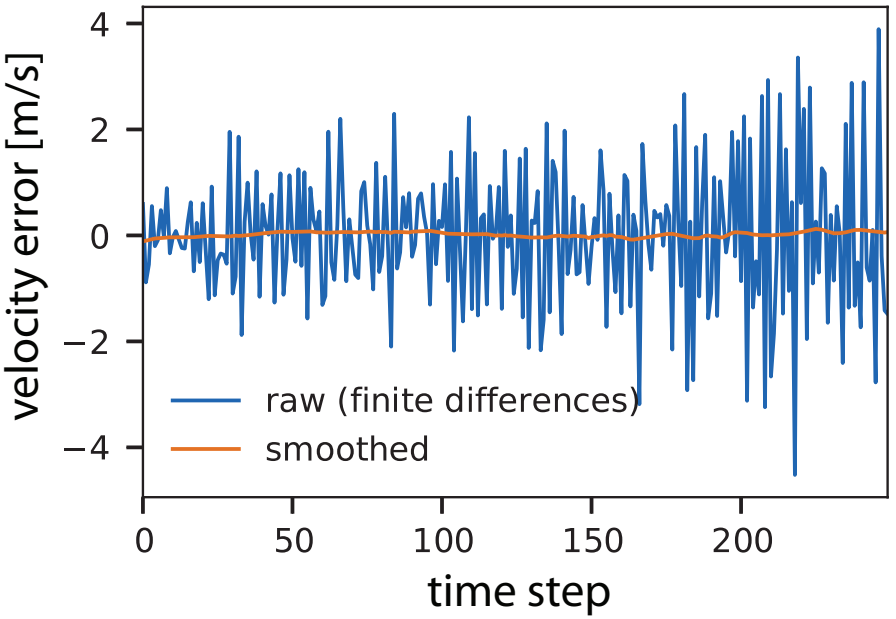
MONA Dataset: West

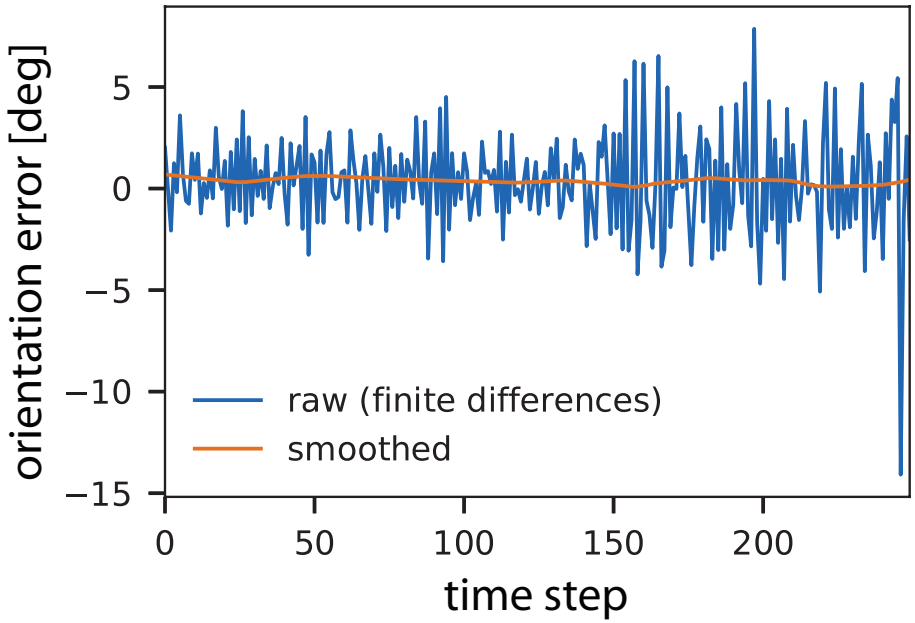




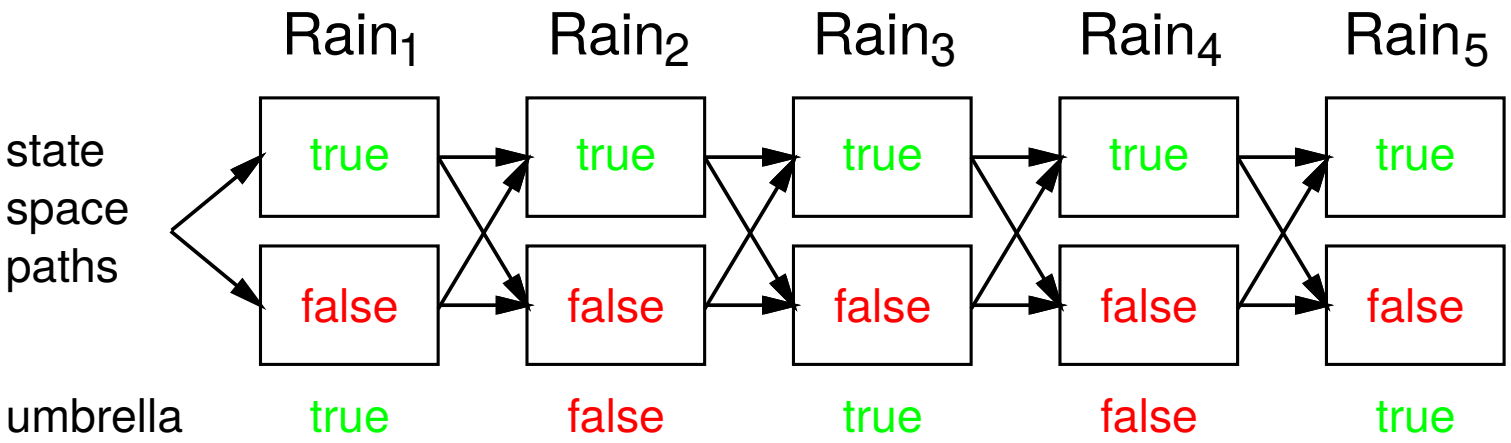
Detection Error Reduction through Smoothing

- Smoothing effectively reduces the error in orientation and velocity of detected vehicles.
- The ground truth was obtained by an autonomous vehicle.
- Rauch-Tung-Striebel smoothing (continuous version of the presented technique) was used due to the continuous nature of the data.



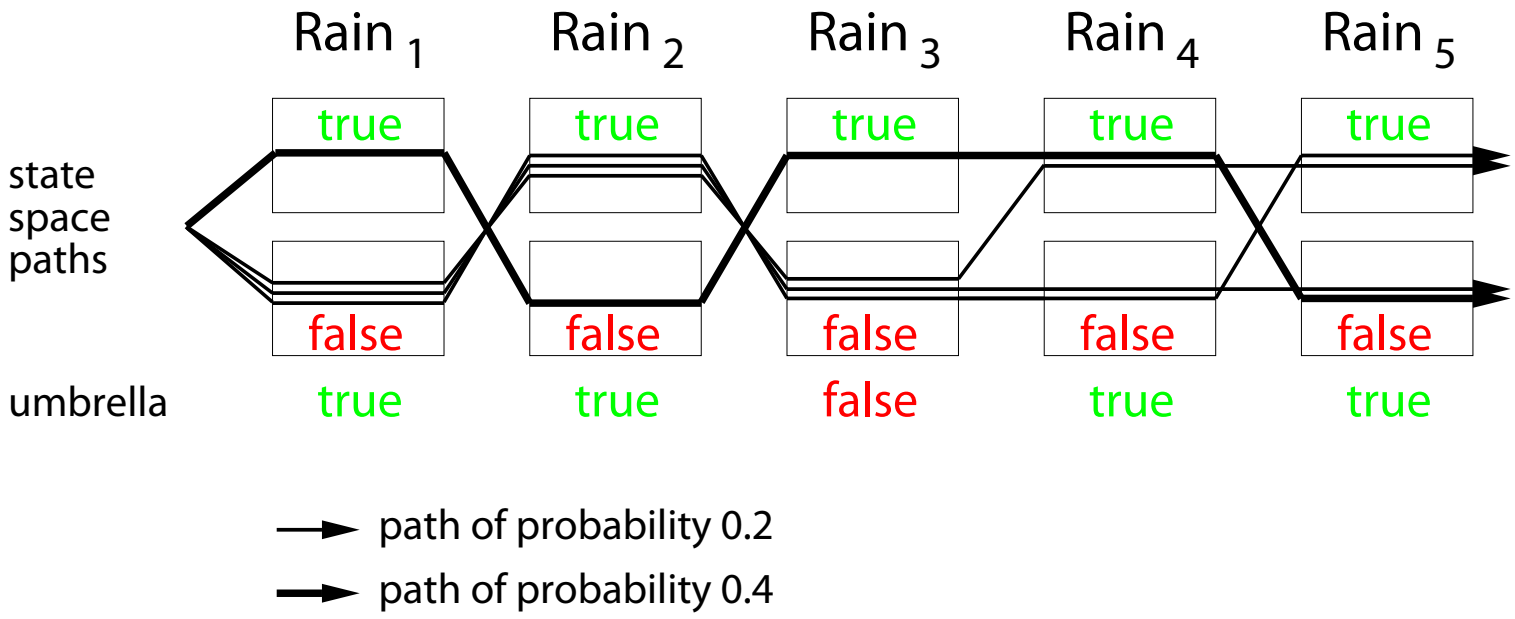


Most Likely Explanation



- Given is the above sequence of umbrellas. What is the weather sequence most likely to explain this?
- In all, there are 2^5 possible weather sequences.
- Is there a way to find the most likely one without enumerating all sequences?
- Try smoothening (linear-time procedure): find the distribution for the weather at each time step; then construct the sequence using at each step the most likely one.
- But: **Most likely sequence $\neq$ sequence of most likely states!**

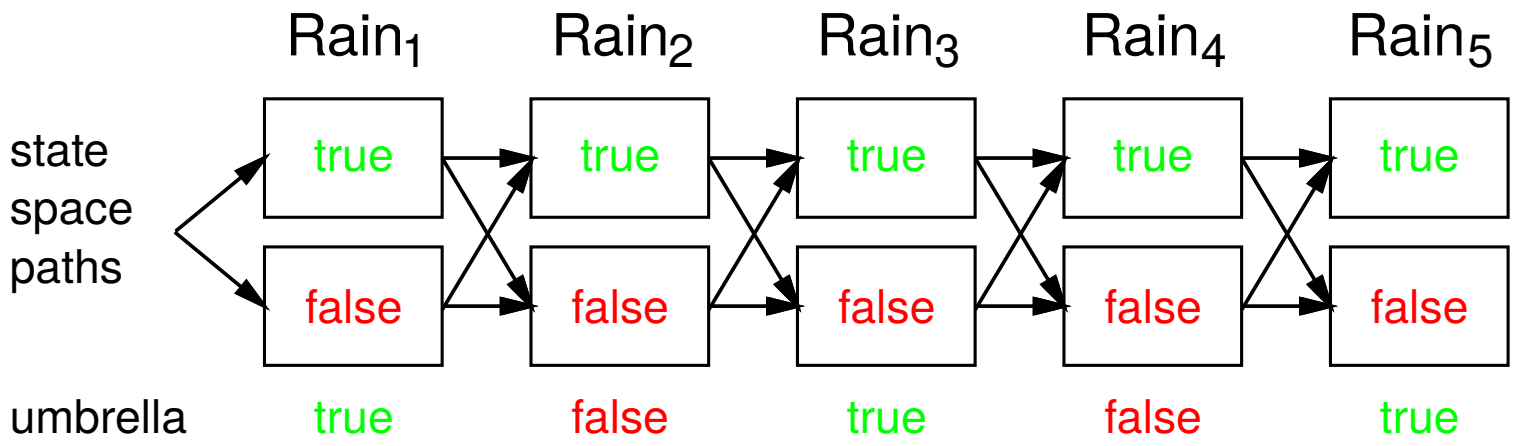
Example: Most Likely Sequence $\neq$ Sequence of Most Likely States



Most likely sequence: true, false, true, true, false

Sequence of most likely states: false, true, false, true , false

Recursive Procedure for the Most Likely Explanation



- Each sequence is a path through a graph whose nodes are the possible states at each time step (see figure above).
- Let us focus on the path that reaches $Rain_5 = true$.
- Because of the Markov property, the most likely path to the state $Rain_5 = true$ consists of
  - the most likely path to **some** state at time 4
  - followed by a transition to $Rain_5 = true$.
- The state at time 4 becomes a part of the most likely path to $Rain_5 = true$.

Viterbi Algorithm

(HMM.ipynb)

Previous slide: a recursive relationship between the most likely path to each state x_{t+1} and the most likely path to each state x_t, which we formalize:

$$\max_{x_1 \dots x_t} P(x_1, \dots, x_t, X_{t+1} | e_{1:t+1}) = \max_{x_1 \dots x_t} P(x_1, \dots, x_t, X_{t+1} | e_{1:t}, e_{t+1})$$

$$= \max_{x_1 \dots x_t} \alpha P(x_1, \dots, x_t, X_{t+1}, e_{t+1} | e_{1:t}) \qquad \text{(normalization)}$$

$$= \max_{x_1 \dots x_t} \alpha P(e_{t+1} | x_1, \dots, x_t, X_{t+1}, e_{1:t}) P(x_1, \dots, x_t, X_{t+1} | e_{1:t}) \qquad (P(a,b|c) = \frac{P(a,b,c)}{P(a,c)} \frac{P(a,c)}{P(c)} = P(b|a,c)P(a|c))$$

$$= \alpha P(e_{t+1} | X_{t+1}) \max_{x_1 \dots x_t} P(x_1, \dots, x_t, X_{t+1} | e_{1:t}) \qquad \text{(conditional independence)}$$

$$= \alpha P(e_{t+1} | X_{t+1}) \max_{x_t} \left(P(X_{t+1} | x_t) \max_{x_1 \dots x_{t-1}} P(x_1, \dots, x_{t-1}, x_t | e_{1:t}) \right)$$

- The algorithm is called **Viterbi algorithm** and is similar in structure compared to the filtering procedure in eq. (2) on slide 21.
- Unlike filtering, which uses constant space, the space requirement is $\mathcal{O}(t)$ since one has to keep the pointers for the best sequence to each state.

Viterbi Algorithm: Interpretation of the max-Operator

The Viterbi algorithm requires the evaluation of the max-operator:

$$\max_{x_1 \ldots x_t} \mathrm{P}(x_1, \ldots, x_t, X_{t+1} | e_{1:t+1})$$

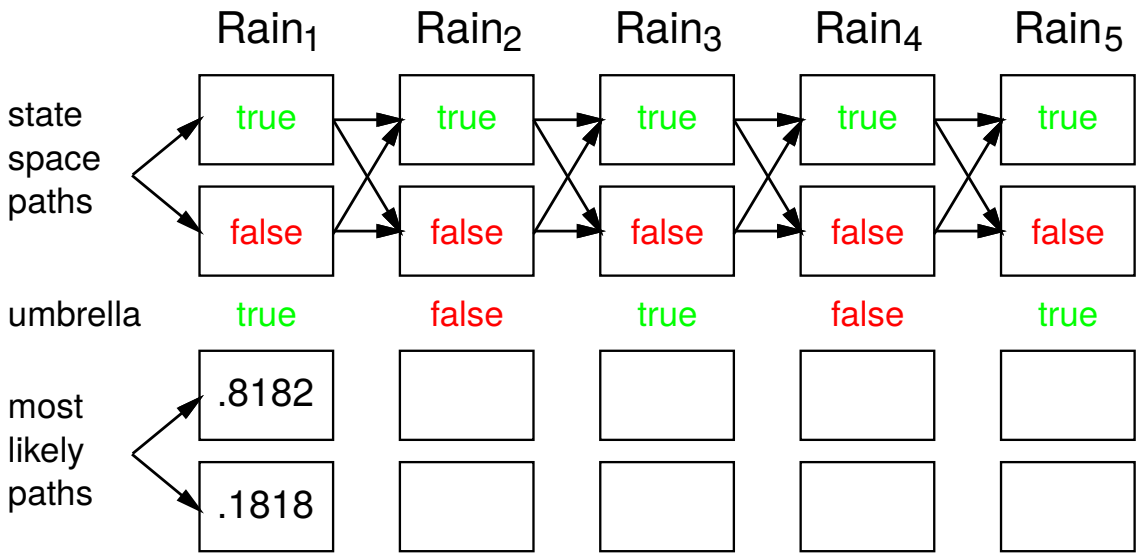
$$= \alpha \mathrm{P}(e_{t+1} | X_{t+1}) \max_{x_t} \left(\mathrm{P}(X_{t+1} | x_t) \max_{x_1 \ldots x_{t-1}} P(x_1, \ldots, x_{t-1}, x_t | e_{1:t}) \right) .$$

The semantics of the max-operator in combination with the P operator is demonstrated by example for $X_t \in \{true, false\}$:

$$\max_{x_t} \mathrm{P}(X_{t+1} | x_t) =$$

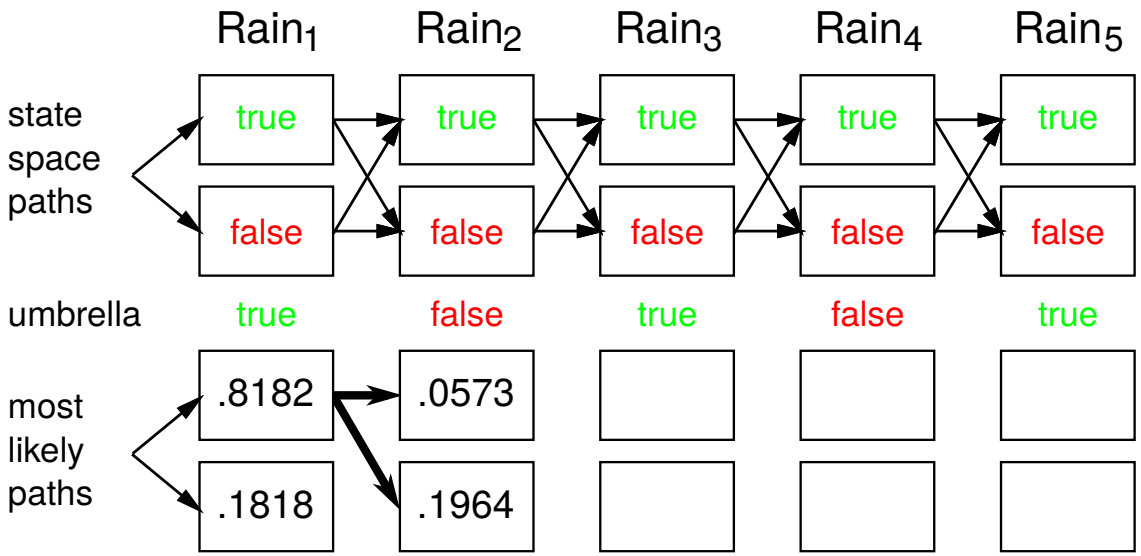
$$\left\langle \max_{x_t} P(X_{t+1} = true | x_t), \max_{x_t} P(X_{t+1} = false | x_t) \right\rangle .$$

Most Likely Explanation: Umbrella Example (1)



Initialization with filtering:
$\mathrm{P}(R_1|u_{1:1}) = \langle 0.8182, 0.1818 \rangle$.

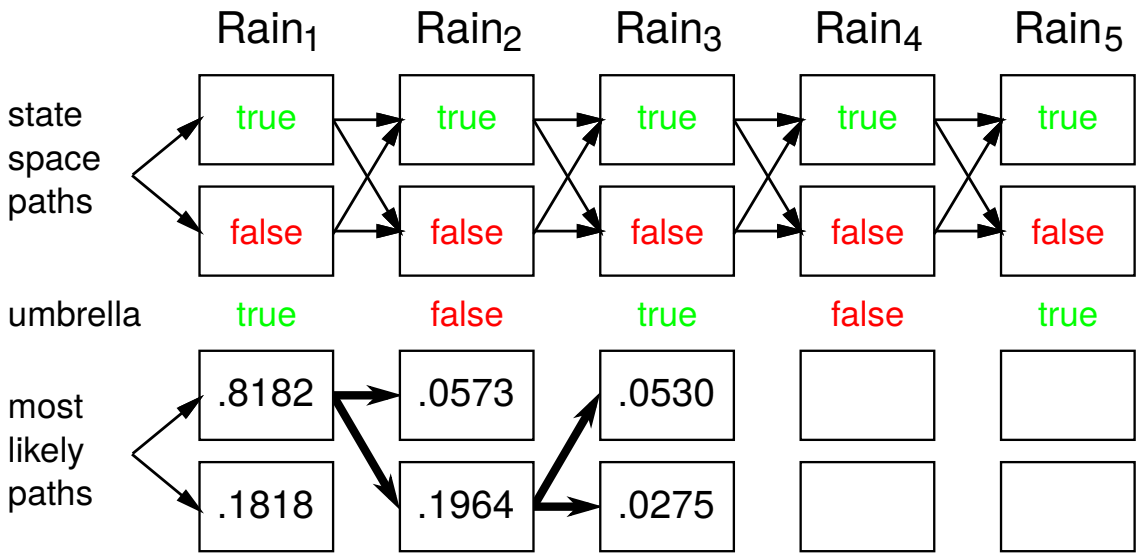
Most Likely Explanation: Umbrella Example (2)



We omit the optional normalization from now on.

- $r_1 = \mathit{true}$: $\mathrm{P}(R_2|r_1)P(r_1|u_{1:1}) = \langle 0.7, 0.3\rangle \times 0.8182 = \langle 0.5727, 0.2455\rangle$
- $r_1 = \mathit{false}$: $\mathrm{P}(R_2|r_1)P(r_1|u_{1:1}) = \langle 0.3, 0.7\rangle \times 0.1818 = \langle 0.0545, 0.1273\rangle$
- $\max_{r_1} \mathrm{P}(r_1, R_2|u_{1:2}) = \mathrm{P}(u_2|R_2) \max_{r_1} (\mathrm{P}(R_2|r_1)P(r_1|u_{1:1})) = \langle 0.1, 0.8\rangle \times \langle 0.5727, 0.2455\rangle = \langle 0.0573, 0.1964\rangle$

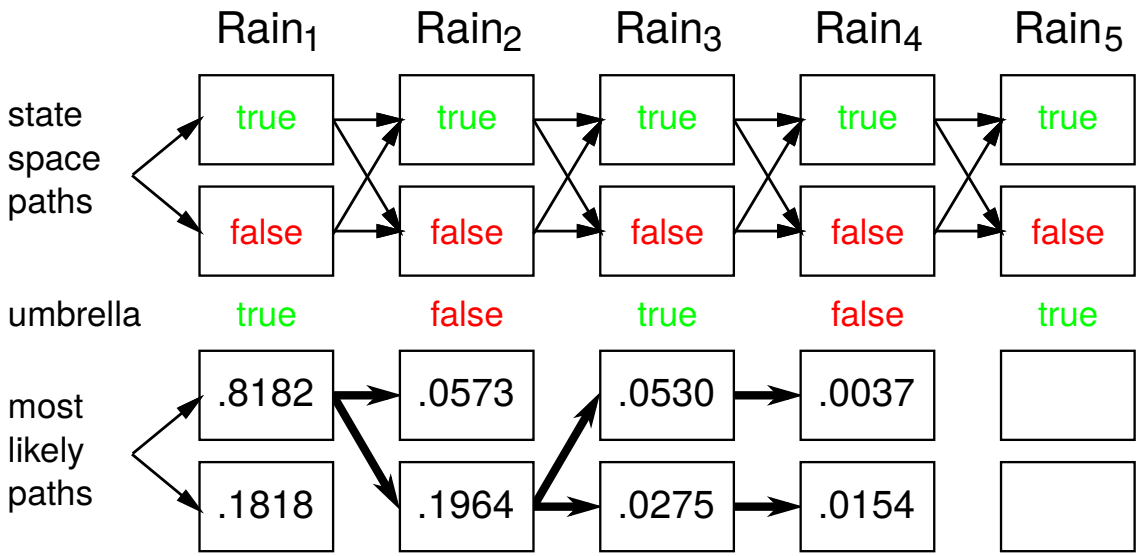
Most Likely Explanation: Umbrella Example (3)



We omit the optional normalization from now on.

- $r_2 = \mathit{true}: \quad \mathrm{P}(R_3|r_2) \max_{r_1} P(r_1, r_2|u_{1:2})$
  $= \langle 0.7, 0.3\rangle \times 0.0573 = \langle 0.0401, 0.0172\rangle$
- $r_2 = \mathit{false}: \quad \mathrm{P}(R_3|r_2) \max_{r_1} P(r_1, r_2|u_{1:2})$
  $= \langle 0.3, 0.7\rangle \times 0.1964 = \langle 0.0589, 0.1375\rangle$
- $\max_{r_1,r_2} \mathrm{P}(r_1, r_2, R_3|u_{1:3}) = \mathrm{P}(u_3|R_3) \max_{r_2} (\mathrm{P}(R_3|r_2) \max_{r_1} P(r_1, r_2|u_{1:2})) =$
  $\langle 0.9, 0.2\rangle \times \langle 0.0589, 0.1375\rangle = \langle 0.0530, 0.0275\rangle$

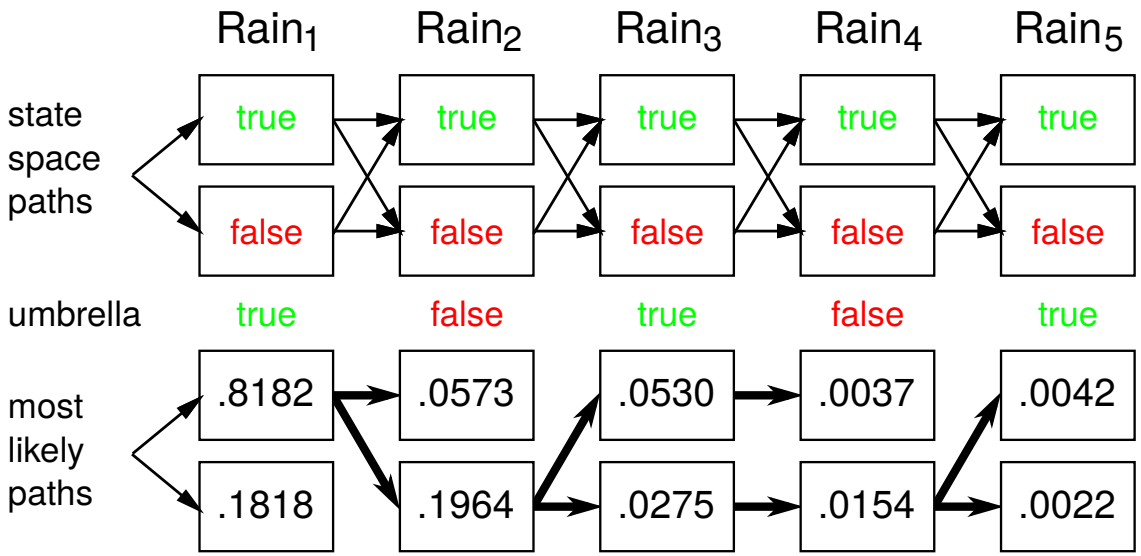
Most Likely Explanation: Umbrella Example (4)



We omit the optional normalization from now on.

- $r_3 = \mathit{true}: \quad \mathrm{P}(R_4|r_3) \max_{r_1,r_2} P(r_1,\ldots,r_3|u_{1:3})$
  $= \langle 0.7, 0.3\rangle \times 0.0530 = \langle 0.0371, 0.0159\rangle$
- $r_3 = \mathit{false}: \quad \mathrm{P}(R_4|r_3) \max_{r_1,r_2} P(r_1,\ldots,r_3|u_{1:3})$
  $= \langle 0.3, 0.7\rangle \times 0.0275 = \langle 0.0082, 0.0192\rangle$
- $\max_{r_1,\ldots,r_3} \mathrm{P}(r_1,\ldots,r_3,R_4|u_{1:4}) =$
  $\mathrm{P}(u_4|R_4) \max_{r_3} (\mathrm{P}(R_4|r_3) \max_{r_1,r_2} P(r_1,\ldots,r_3|u_{1:3})) =$
  $\langle 0.1, 0.8\rangle \times \langle 0.0371, 0.0192\rangle = \langle 0.0037, 0.0154\rangle$ (here: different r_3 values!)

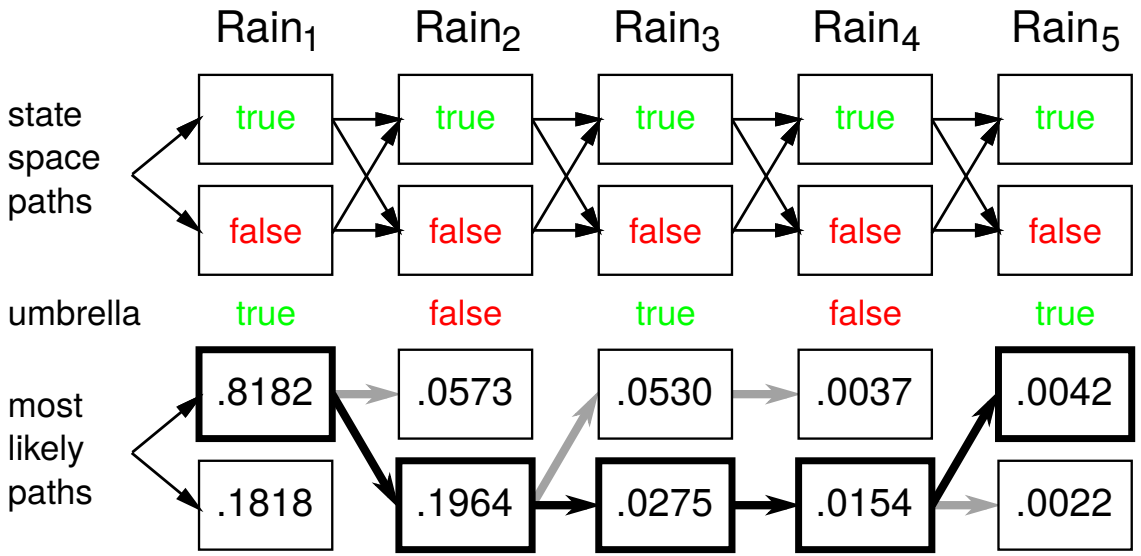
Most Likely Explanation: Umbrella Example (5)



We omit the optional normalization from now on.

- $r_4 = \mathit{true}: \quad \mathrm{P}(R_5|r_4) \max_{r_1,\ldots,r_3} P(r_1,\ldots,r_4|u_{1:4})$
  $= \langle 0.7, 0.3\rangle \times 0.0037 = \langle 0.0026, 0.0011\rangle$
- $r_4 = \mathit{false}: \quad \mathrm{P}(R_5|r_4) \max_{r_1,\ldots,r_3} P(r_1,\ldots,r_4|u_{1:4})$
  $= \langle 0.3, 0.7\rangle \times 0.0154 = \langle 0.0046, 0.0108\rangle$
- $\max_{r_1,\ldots,r_4} \mathrm{P}(r_1,\ldots,r_4,R_5|u_{1:5}) =$
  $\mathrm{P}(u_5|R_5) \max_{r_4} (\mathrm{P}(R_5|r_4) \max_{r_1,\ldots,r_3} P(r_1,\ldots,r_4|u_{1:4})) =$
  $\langle 0.9, 0.2\rangle \times \langle 0.0046, 0.0108\rangle = \langle 0.0042, 0.0022\rangle$

Most Likely Explanation: Umbrella Example (6)



Obtaining the most likely sequence:

- start with the most likely final state
- backtracking along the path which maximized the probability of the final state

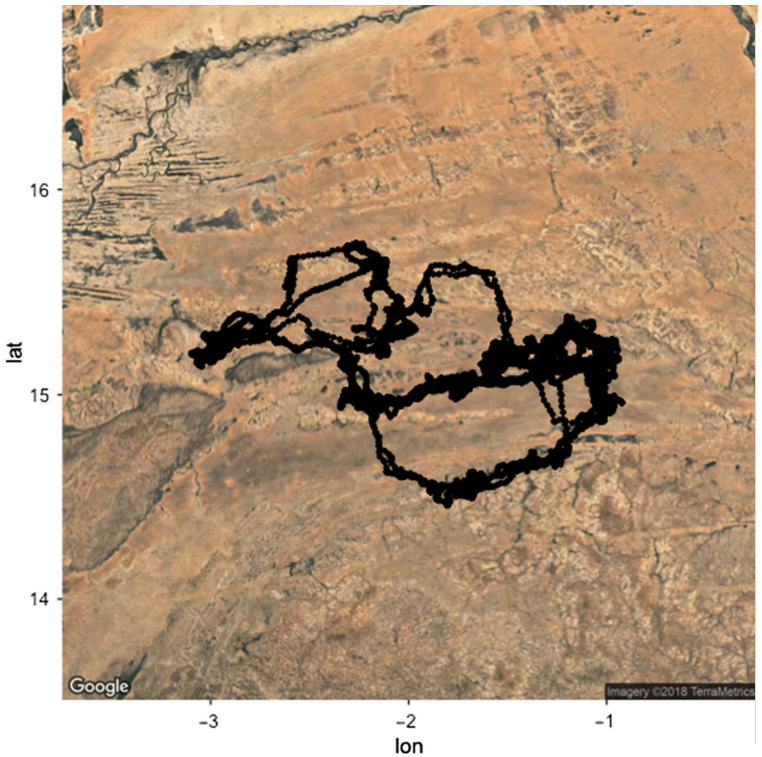
Viterbi Example: Most Likely Sequence of Animal Movement States

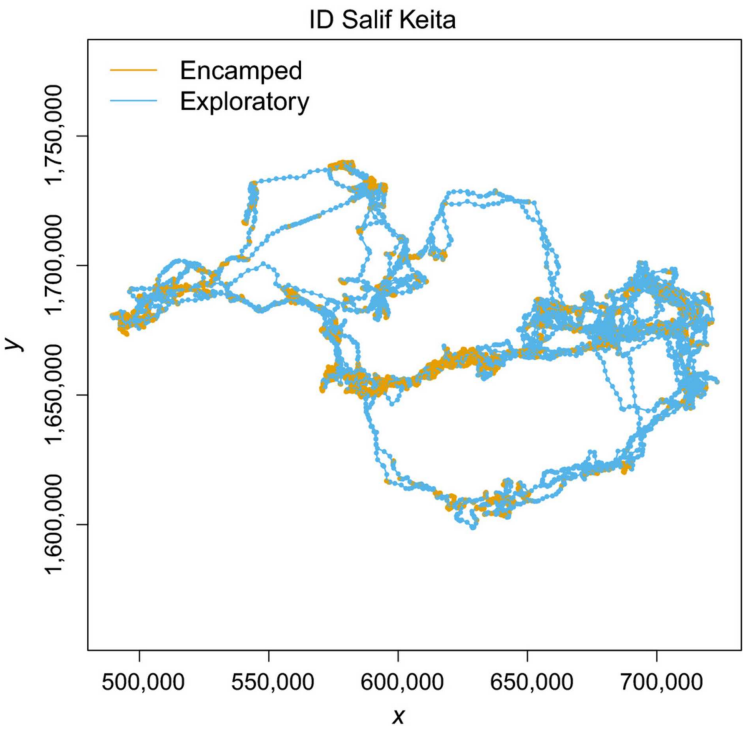
- State sequences are used to detect changes in animal behavior.
- `momentuHMM` [3] is an R package for generalized hidden Markov models of animal movement.
- An African elephant bull's step length and the outside temperature were tracked.

Creator: Rolf Dobberstein

[3] McClintock, B. T., & Michelot, T. (2018). momentuHMM: R package for generalized hidden Markov models of animal movement. Methods in Ecology and Evolution, 9(6), 1518-1530.

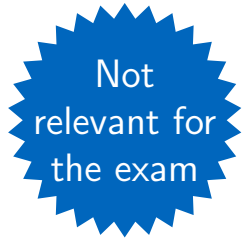
Viterbi Example: Movement States of Elephant Bull





- A state of slow undirected movement "encamped", and a state of faster and more directed movement "exploratory" were identified.
- Step lengths
  - decreased for the "encamped" state as temperature increased, and
  - decreased in the late evening and early morning for both states.

Estimation of Continuous State Variables

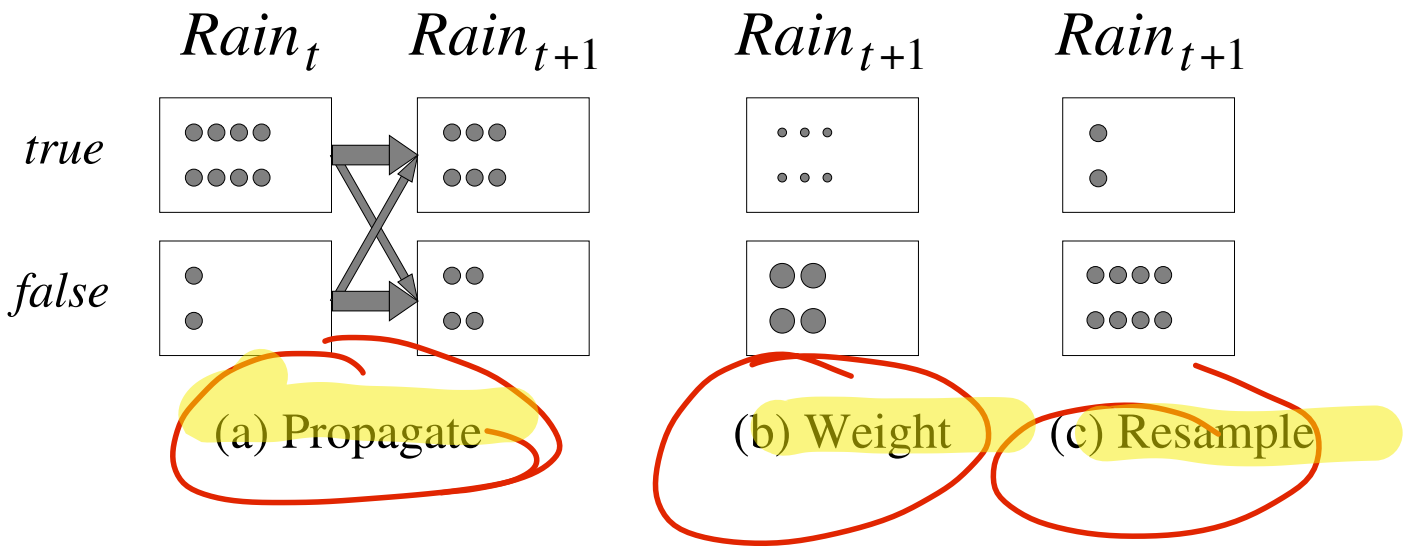


- So far we have only considered estimation of hidden Markov models, which have discrete states.
- Many real-world problems have continuous states, such as position, velocities, forces, temperatures, etc.
- Those problems arise e.g., in
  - robotics,
  - automated driving,
  - smart grids,
  - automated processes, such as in chemical plants,
  - surveillance of automated processes,
  - etc.
- Those aspects are covered in the lecture *Cyber-Physical Systems*.

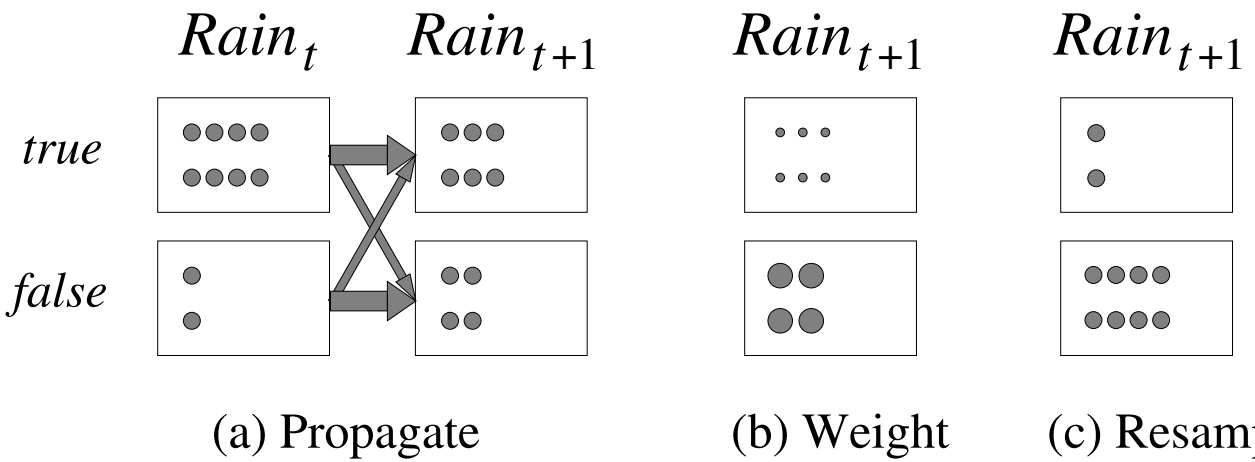
Particle Filtering (1)

(jupyter HMM.ipynb)

- Particle filtering can be interpreted as a Monte Carlo method for hidden Markov models.
- The approach can also be applied to continuous systems.
- Basic idea: ensure that the population of samples ("particles") tracks the high-likelihood regions of the state-space.
- Widely used for tracking nonlinear systems, especially in computer vision.



Particle Filtering (2)



(a) Propagate (b) Weight (c) Resample

a) At time t, 8 samples indicate $Rain = true$ and 2 indicate $Rain = false$. Propagation through the transition model yields 6 samples for $Rain = true$ and 4 for $Rain = false$ at time $t + 1$.

b) $Umbrella = false$ is observed at $t + 1$. Each sample is weighted by its likelihood for the observation, as indicated by the size of the circles.

c) A new set of 10 samples is generated by weighted random selection from the current set, resulting in 2 samples for $Rain = true$ and 8 for $Rain = false$.

Consistency of Particle Filtering

Not relevant for the exam

- Assume consistency at time t: $N(x_t|e_{1:t})/N = P(x_t|e_{1:t})$. (5)
- Propagate forward: populations of x_{t+1} are

$$N(x_{t+1}|e_{1:t}) = \sum_{x_t} P(x_{t+1}|x_t)N(x_t|e_{1:t}). \quad (6)$$

- Weight samples by their likelihood for e_{t+1}:

$$W(x_{t+1}|e_{1:t+1}) = P(e_{t+1}|x_{t+1})N(x_{t+1}|e_{1:t}). \quad (7)$$

- Re-sample to obtain populations proportional to W:

$$\begin{aligned} N(x_{t+1}|e_{1:t+1})/N &= \alpha W(x_{t+1}|e_{1:t+1}) \\ &= \alpha P(e_{t+1}|x_{t+1})N(x_{t+1}|e_{1:t}) && \text{(using (7))} \\ &= \alpha P(e_{t+1}|x_{t+1}) \sum_{x_t} P(x_{t+1}|x_t)N(x_t|e_{1:t}) && \text{(using (6))} \\ &= \alpha' P(e_{t+1}|x_{t+1}) \sum_{x_t} P(x_{t+1}|x_t)P(x_t|e_{1:t}) && \text{(using (5))} \\ &= P(x_{t+1}|e_{1:t+1}) && \text{(using (2))} \end{aligned}$$

Example: Robot Localization

Not relevant for the exam

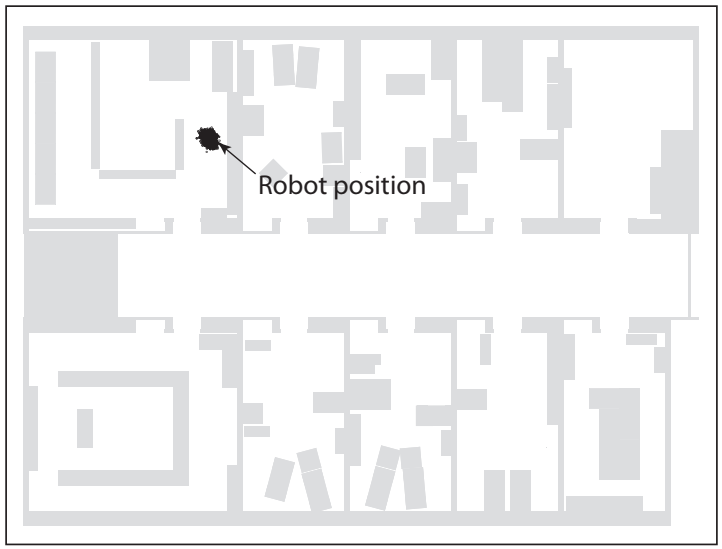
A particular instance of filtering is **localization**.

Localization

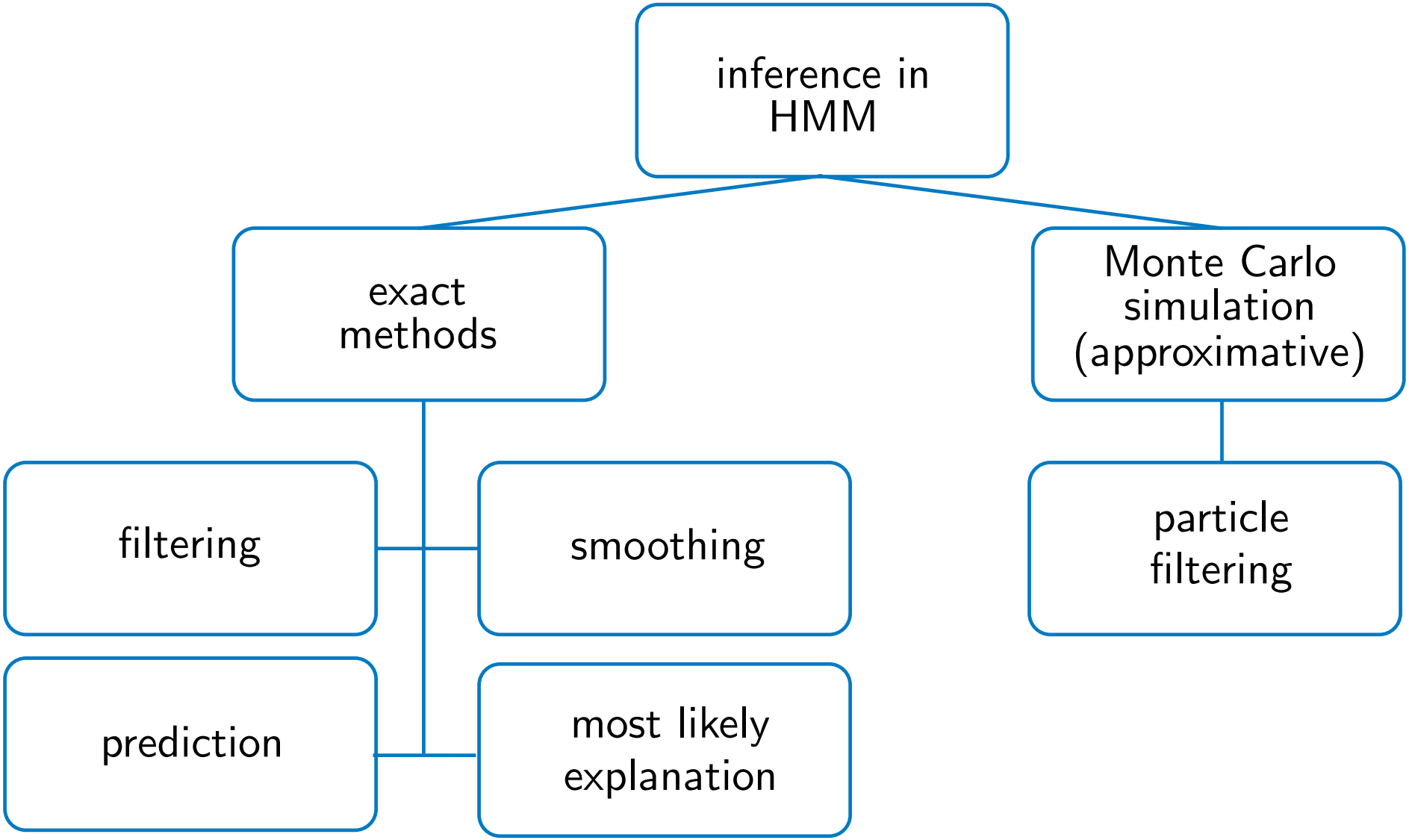
Localization is the problem of finding out where things are – including the robot itself.

To keep things simple, we assume that

- the robot moves slowly on a flat 2D surface,
- the robot knows the exact map of the environment (see figure below).



Overview of HMM Inference Methods



Summary

- Temporal models use state and sensor variables replicated over time.
- Markov assumptions and stationarity assumption, so we only need
  - transition model $\mathrm{P}(X_t|X_{t-1})$,
  - sensor model $\mathrm{P}(E_t|X_t)$.
- Tasks are filtering, prediction, smoothing, most likely sequence; **all done recursively with constant cost per time step**.
- Hidden Markov models have a single discrete state variable; this is no loss of generality.
- Hidden Markov models are used in numerous applications, such as speech recognition.
- Particle filtering is a good approximative filtering algorithm.