FUNDAMENTALS OF ARTIFICIAL INTELLIGENCE

Exercise 6: Inference in First Order Logic – Solutions Winter semester 2023/2024

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Problem 6.1: The man in the painting

(The following puzzle appears in [1] Exercise 9.10.) A man stands in front of a painting and says the following:

> Brothers and sisters have I none, but that man's father is my father's son.

What is the relationship between the man in the painting and the speaker? Use the predicates

$$\begin{aligned} Male(x) &: x \text{ is male.} \\ Father(x, y) &: x \text{ is the father of } y. \\ Son(x, y) &: x \text{ is a son of } y. \\ Parent(x, y) &: x \text{ is a parent of } y. \\ Child(x, y) &: x \text{ is a child of } y. \\ Sibling(x, y) &: x \text{ is a sibling of } y \end{aligned}$$

and the knowledge

- A sibling is another child of one's parents.

$$\forall x, y \quad Sibling(x, y) \Leftrightarrow x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y)$$

- Parent and child are inverse relations.

$$\forall p, c \quad Parent(p, c) \Leftrightarrow Child(c, p)$$

to solve the riddle with first-order logic.

Problem 6.1.1: Define the rule which relates *son*, *child*, and *male*. Define also the rule which relates *father*, *parent*, and *male*.

Solution:

- Every son is a male child, and every male child is a son.

$$\forall s, p \quad Son(s, p) \Leftrightarrow Child(s, p) \land Male(s)$$

- Every father is a male parent, and every male parent is a father.

$$\forall f, c \quad Father(f, c) \Leftrightarrow Parent(f, c) \land Male(f)$$

Problem 6.1.2: Using the constants Me for the speaker and $That$ for the person depicted in the painting, formalize the sentences regarding the sexes of the people in the puzzle.

Solution: From the problem definition, we know that the constant Me is male. Therefore, we add the following sentence to our knowledge base.

$$Male(Me)$$

From the problem definition, we also know that the person in the painting is male.

$$\mathit{Male}(\mathit{That})$$

Problem 6.1.3: Formalize the sentences “Brothers and sisters have I none” and “That man’s father is my father’s son” in first-order logic.

Solution:

- Brothers and sisters have I none.

$$\forall x \quad \neg \mathit{Sibling}(x, \mathit{Me}) \land \neg \mathit{Sibling}(\mathit{Me}, x)$$

- That man’s father is my father’s son.

$$\exists f_1, f_2 \quad \mathit{Father}(f_1, \mathit{That}) \land \mathit{Father}(f_2, \mathit{Me}) \land \mathit{Son}(f_1, f_2)$$

Problem 6.1.4: Solve this puzzle informally and decide what is the relation between the man in the painting and the speaker.

Solution: We analyze the sentence

> . . . but that man’s father is my father’s son.

The phrase “my father’s son” could be either “Me” or “My Sibling”. However, from previous constraint

> Brothers and sisters have I none, . . .

we know that it could not be “My Sibling” because it does not exist. Therefore, the phrase “my father’s son” is “Me”. Using this equality, we know from the first quote above that “That man’s father” is “Me”. Therefore, “That man” is the son of “Me”.

Problem 6.1.5: Using the resolution technique for first-order logic, prove your answer. You can use the two diagrams on the next page to structure your proof.

$\neg Son(That, Me)$

$\neg Parent(Me, That)$

$Son(F_1, F_2)$

$Parent(Me, That)$

Solution: We want to prove $\alpha := Son(That, Me)$. To do so, we first need to transform the different rules we are given into Conjunctive Normal Forms (CNF); this will be done in the following, and the different clauses will then be written explicitly at the end of each transformation:

1. A sibling is another child of one's parents ($\Leftarrow$).

$$\begin{aligned}
& \forall x, y \quad Sibling(x, y) \Leftarrow (x \neq y) \wedge \exists p \quad Parent(p, x) \wedge Parent(p, y) \\
& \quad \text{(Removing implication)} \\
\equiv\ & \forall x, y \quad Sibling(x, y) \vee \neg((x \neq y) \wedge \exists p \quad Parent(p, x) \wedge Parent(p, y)) \\
& \quad \text{(Pushing } \neg \text{ inwards)} \\
\equiv\ & \forall x, y \quad Sibling(x, y) \vee (x = y) \vee \forall p \quad \neg Parent(p, x) \vee \neg Parent(p, y) \\
& \quad \text{(Dropping universal quantifier)} \\
\equiv\ & Sibling(x, y) \vee (x = y) \vee \neg Parent(p, x) \vee \neg Parent(p, y)
\end{aligned}$$

$$\boxed{Sibling(x, y) \vee (x = y) \vee \neg Parent(p, x) \vee \neg Parent(p, y)} \tag{1}$$

2. A sibling is another child of one's parents ($\Rightarrow$).

$$\begin{aligned}
& \forall x, y \quad Sibling(x, y) \Rightarrow ((x \neq y) \wedge \exists p \quad Parent(p, x) \wedge Parent(p, y)) \\
& \quad \text{(Removing implication)} \\
\equiv\ & \forall x, y \quad \neg Sibling(x, y) \vee ((x \neq y) \wedge \exists p \quad Parent(p, x) \wedge Parent(p, y)) \\
& \quad \text{(Skolemization)} \\
\equiv\ & \forall x, y \quad \neg Sibling(x, y) \vee ((x \neq y) \wedge Parent(F(x, y), x) \wedge Parent(F(x, y), y)) \\
& \quad \text{(Dropping universal quantifier)} \\
\equiv\ & \neg Sibling(x, y) \vee ((x \neq y) \wedge Parent(F(x, y), x) \wedge Parent(F(x, y), y)) \\
& \quad \text{(Distributivity of } \vee \text{ over } \wedge \text{)} \\
\equiv\ & (\neg Sibling(x, y) \vee (x \neq y)) \wedge (\neg Sibling(x, y) \vee Parent(F(x, y), x)) \wedge \ldots \\
& \quad \ldots \wedge (\neg Sibling(x, y) \vee Parent(F(x, y), y))
\end{aligned}$$

$$\boxed{\neg Sibling(x, y) \vee (x \neq y)} \tag{2}$$

$$\boxed{\neg Sibling(x, y) \vee Parent(F(x, y), x)} \tag{3}$$

$$\boxed{\neg Sibling(x, y) \vee Parent(F(x, y), y)} \tag{4}$$

3. Parent and child are inverse relations ($\Rightarrow$).

$$\begin{aligned}
& \forall p, c \quad Parent(p, c) \Rightarrow Child(c, p) \\
& \quad \text{(Removing implication)} \\
\equiv\ & \forall p, c \quad \neg Parent(p, c) \vee Child(c, p) \\
& \quad \text{(Dropping universal quantifier)} \\
\equiv\ & \neg Parent(p, c) \vee Child(c, p)
\end{aligned}$$

$$\boxed{\neg Parent(p, c) \vee Child(c, p)} \tag{5}$$

4. Parent and child are inverse relations ($\Leftarrow$).

$$\begin{aligned} & \forall p, c \quad Child(c,p) \;\Rightarrow\; Parent(p,c) \\ & \quad \text{(Removing implication)} \\ \equiv\; & \forall p, c \quad \neg Child(c,p) \;\vee\; Parent(p,c) \\ & \quad \text{(Dropping universal quantifier)} \\ \equiv\; & \neg Child(c,p) \vee Parent(p,c) \end{aligned}$$

$$\boxed{\neg Child(c,p) \vee Parent(p,c)} \tag{6}$$

5. Every son is a male child ($\Rightarrow$).

$$\begin{aligned} & \forall s, p \quad Son(s,p) \;\Rightarrow\; Child(s,p) \wedge Male(s) \\ & \quad \text{(Removing implication)} \\ \equiv\; & \forall s, p \quad \neg Son(s,p) \;\vee\; (Child(s,p) \wedge Male(s)) \\ & \quad \text{(Dropping universal quantifier)} \\ \equiv\; & \neg Son(s,p) \;\vee\; (Child(s,p) \wedge Male(s)) \\ & \quad \text{(Distributivity of } \vee \text{ over } \wedge\text{)} \\ & (\neg Son(s,p) \;\vee\; Child(s,p)) \wedge (\neg Son(s,p) \;\vee\; Male(s)) \end{aligned}$$

$$\boxed{\neg Son(s,p) \;\vee\; Child(s,p)} \tag{7}$$

$$\boxed{\neg Son(s,p) \;\vee\; Male(s)} \tag{8}$$

6. Every son is a male child ($\Leftarrow$).

$$\begin{aligned} & \forall s, p \quad Son(s,p) \;\Leftarrow\; Child(s,p) \wedge Male(s) \\ & \quad \text{(Removing implication)} \\ \equiv\; & \forall s, p \quad Son(s,p) \;\vee\; \neg(Child(s,p) \wedge Male(s)) \\ & \quad \text{(Dropping universal quantifier)} \\ \equiv\; & Son(s,p) \;\vee\; \neg(Child(s,p) \wedge Male(s)) \\ & \quad \text{(de Morgan's rule)} \\ \equiv\; & Son(s,p) \;\vee\; \neg Child(s,p) \vee \neg Male(s) \end{aligned}$$

$$\boxed{Son(s,p) \;\vee\; \neg Child(s,p) \vee \neg Male(s)} \tag{9}$$

7. Every father is a male parent ($\Rightarrow$).

$$\begin{aligned} & \forall p, c \quad Father(p,c) \;\Rightarrow\; Parent(p,c) \wedge Male(p) \\ & \quad \text{(Removing implication)} \\ \equiv\; & \forall p, c \quad \neg Father(p,c) \;\vee\; (Parent(p,c) \wedge Male(p)) \\ & \quad \text{(Dropping universal quantifier)} \\ \equiv\; & \neg Father(p,c) \;\vee\; (Parent(p,c) \wedge Male(p)) \\ & \quad \text{(Distributivity of } \vee \text{ over } \wedge\text{)} \\ \equiv\; & (\neg Father(p,c) \;\vee\; Parent(p,c)) \;\wedge\; (\neg Father(p,c) \;\vee\; Male(p)) \end{aligned}$$

$$\boxed{\neg Father(p,c) \;\vee\; Parent(p,c)} \tag{10}$$

$$\boxed{\neg Father(p,c) \;\vee\; Male(p)} \tag{11}$$

8. Every father is a male parent ($\Leftarrow$).

$$\begin{aligned}
& \forall p, c \quad \mathit{Father}(p,c) \;\Leftarrow\; \mathit{Parent}(p,c) \wedge \mathit{Male}(p) \\
& \quad \text{(Removing implication)} \\
\equiv\; & \forall p, c \quad \mathit{Father}(p,c) \;\vee\; \neg(\mathit{Parent}(p,c) \wedge \mathit{Male}(p)) \\
& \quad \text{(de Morgan's rule)} \\
\equiv\; & \forall p, c \quad \mathit{Father}(p,c) \;\vee\; \neg \mathit{Parent}(p,c) \vee \neg \mathit{Male}(p) \\
& \quad \text{(Dropping universal quantifier)} \\
\equiv\; & \mathit{Father}(p,c) \;\vee\; \neg \mathit{Parent}(p,c) \vee \neg \mathit{Male}(p)
\end{aligned}$$

$$\boxed{\mathit{Father}(p,c) \;\vee\; \neg \mathit{Parent}(p,c) \vee \neg \mathit{Male}(p)} \tag{12}$$

9. Brothers and sisters have I none.

$$\begin{aligned}
& \forall x \quad \neg \mathit{Sibling}(x, \mathit{Me}) \wedge \neg \mathit{Sibling}(\mathit{Me}, x) \\
& \quad \text{(Dropping quantifier)} \\
\equiv\; & \neg \mathit{Sibling}(x, \mathit{Me}) \wedge \neg \mathit{Sibling}(\mathit{Me}, x)
\end{aligned}$$

$$\boxed{\neg \mathit{Sibling}(x, \mathit{Me})} \tag{13}$$

$$\boxed{\neg \mathit{Sibling}(\mathit{Me}, x)} \tag{14}$$

10. That man's father is my father's son.

$$\begin{aligned}
& \exists f_1, f_2 \quad \mathit{Father}(f_1, \mathit{That}) \wedge \mathit{Father}(f_2, \mathit{Me}) \wedge \mathit{Son}(f_1, f_2) \\
& \quad \text{(Skolemization)} \\
\equiv\; & \mathit{Father}(F_1, \mathit{That}) \wedge \mathit{Father}(F_2, \mathit{Me}) \wedge \mathit{Son}(F_1, F_2)
\end{aligned}$$

$$\boxed{\mathit{Father}(F_1, \mathit{That})} \tag{15}$$

$$\boxed{\mathit{Father}(F_2, \mathit{Me})} \tag{16}$$

$$\boxed{\mathit{Son}(F_1, F_2)} \tag{17}$$

11. Sex of the person in the painting.

$$\boxed{\mathit{Male}(\mathit{That})} \tag{18}$$

12. Sex of the person standing in front of the painting.

$$\boxed{\mathit{Male}(\mathit{Me})} \tag{19}$$

13. Negation of the goal.

$$\boxed{\neg \mathit{Son}(\mathit{That}, \mathit{Me})} \tag{20}$$

Formal proof using the Resolution Algorithm: We start with the negation of the goal:

$$
\begin{array}{ll}
 & \neg Son(That, Me) \\
 & \quad \text{(Rule 9 with } \{s/That, p/Me\}\text{)} \\
\rightarrow & \neg Child(That, Me) \vee \neg Male(That) \\
 & \quad \text{(Rule 18)} \\
\rightarrow & \neg Child(That, Me) \\
 & \quad \text{(Rule 5, with } \{c/That, p/Me\}\text{)} \\
\rightarrow & \neg Parent(Me, That)
\end{array}
$$

Graphically, this can be represented as follows:

$$
\begin{tikzcd}
\boxed{\mathbf{Son}(\mathbf{s}, \mathbf{p}) \vee \neg Child(s, p) \vee \neg Male(s)} \arrow[dr] & \boxed{\neg Son(That, Me)} \arrow[d] \\
\boxed{\mathbf{Male}(\mathbf{That})} \arrow[dr] & \boxed{\neg Child(That, Me) \vee \neg Male(That)} \arrow[d] \\
\boxed{\neg Parent(p, c) \vee \mathbf{Child}(\mathbf{c}, \mathbf{p})} \arrow[dr] & \boxed{\neg Child(That, Me)} \arrow[d] \\
 & \boxed{\neg Parent(Me, That)}
\end{tikzcd}
$$

Figure 1: Resolution proof for $\neg Parent(Me, That)$

We add the conclusion to our resolution clauses:

$$\boxed{\neg Parent(Me, That)} \tag{21}$$

We, then start again with the clause 17, which we shall combine in the very end with 21:

$$
\begin{array}{ll}
 & Son(F_1, F_2) \\
 & \quad \text{(Rule 7 with } \{s/F_1, p/F_2\}\text{)} \\
\rightarrow & Child(F_1, F_2) \\
 & \quad \text{(Rule 6 with } \{c/F_1, p/F_2\}\text{)} \\
\rightarrow & Parent(F_2, F_1) \\
 & \quad \text{(Rule 1 with } \{x/F_1, p/F_2\}\text{)} \\
\rightarrow & Sibling(F_1, y) \vee (F_1 = y) \vee \neg Parent(F_2, y) \\
 & \quad \text{(Rule 10 with } \{p/F_2, c/y\}\text{)} \\
\rightarrow & Sibling(F_1, y) \vee (F_1 = y) \vee \neg Father(F_2, y) \\
 & \quad \text{(Rule 16 with } \{y/Me\}\text{)} \\
\rightarrow & Sibling(F_1, Me) \vee (F_1 = Me) \\
 & \quad \text{(Rule 13 with } \{x/F_1\}\text{)} \\
\rightarrow & (F_1 = Me) \\
 & \quad \text{(Demodulation rule with rule 15)} \\
\rightarrow & Father(Me, That) \\
 & \quad \text{(Rule 10 with } \{p/Me, c/That\}\text{)} \\
\rightarrow & Parent(Me, That) \\
 & \quad \text{(The last clause added 21)} \\
\rightarrow & \emptyset
\end{array}
$$

Graphically, this can be represented as follows:

$\neg \mathbf{Son}(\mathbf{s}, \mathbf{p}) \ \lor \ \mathit{Child}(s, p)$	$\mathit{Son}(F_1, F_2)$
$\neg \mathbf{Child}(\mathbf{c}, \mathbf{p}) \lor \mathit{Parent}(p, c)$	$\mathit{Child}(F_1, F_2)$
$\mathit{Sibling}(x, y) \ \lor \ (x = y) \lor \neg \mathbf{Parent}(\mathbf{p}, \mathbf{x}) \lor \neg \mathit{Parent}(p, y)$	$\mathit{Parent}(F_2, F_1)$
$\neg \mathit{Father}(p, c) \ \lor \ \mathbf{Parent}(\mathbf{p}, \mathbf{c})$	$\mathit{Sibling}(F_1, y) \lor (F_1 = y) \lor \neg \mathit{Parent}(F_2, y)$
$\mathbf{Father}(\mathbf{F_2}, \mathbf{Me})$	$\mathit{Sibling}(F_1, y) \lor (F_1 = y) \lor \neg \mathit{Father}(F_2, y)$
$\neg \mathbf{Sibling}(\mathbf{x}, \mathbf{Me})$	$\mathit{Sibling}(F_1, \mathit{Me}) \lor (F_1 = \mathit{Me})$
$\mathbf{Father}(\mathbf{F_1}, \mathbf{That})$	$F_1 = \mathit{Me}$
$\neg \mathbf{Father}(\mathbf{p}, \mathbf{c}) \ \lor \ \mathit{Parent}(p, c)$	$\mathit{Father}(\mathit{Me}, \mathit{That})$
	$\mathit{Parent}(\mathit{Me}, \mathit{That})$

Figure 2: Resolution proof for $\mathit{Parent}(\mathit{Me}, \mathit{That})$

At the end of the day, we end up with the empty clause, proving that our knowledge base does indeed entail the original claim $\mathit{Son}(\mathit{That}, \mathit{Me})$.

Problem 6.2: Backward chaining

(The following exercise is taken from [1] Exercise 9.9.) Suppose you are given the following axioms:

1. $0 \leq 3$
2. $7 \leq 9$
3. $\forall x \quad x \leq x$
4. $\forall x \quad x \leq x + 0$
5. $\forall x \quad x + 0 \leq x$
6. $\forall x, y \quad x + y \leq y + x$
7. $\forall w, x, y, z \quad w \leq y \land x \leq z \ \Rightarrow \ w + x \leq y + z$
8. $\forall x, y, z \quad x \leq y \land y \leq z \ \Rightarrow \ x \leq z.$

Give a backward-chaining proof of the sentence $7 \leq 3 + 9$. (Be sure, of course, to use only the axioms given here, not anything else you may know about arithmetic.) Show only the steps that lead to success, not the irrelevant steps.

Solution: For the sake of simplicity, we shall not go through a "complete" backwards chaining algorithm, in the sense that we will only apply one rule at a time to go from assumption to assumption; in contrast, if we were to use the actual algorithm we would need to apply every available rule at every step.

Before we actually apply the algorithm, let us first consider a summary of the proof. In the following derivation, the step numbers on the side indicate the steps in the algorithm. Moreover, we add the subscript i to the variables in the i-th rule to make it easier to see which rule introduced a variable.

$$
\begin{array}{ll}
& 7 \leq 3+9 \\
& \left.\begin{array}{ll} & \text{(Rule 8 with } \{x_8/7, z_8/3+9\}\text{)} \\ \Leftarrow & 7 \leq y_8 \quad \wedge \quad y_8 \leq 3+9 \end{array}\right\} \text{Step 1} \\
& \left.\begin{array}{ll} & \text{(Rule 4 with } \{x_4/7, y_8/7+0\}\text{)} \\ \Leftarrow & \mathit{True} \quad \wedge \quad 7+0 \leq 3+9 \\ & \text{(True rule)} \\ \Leftarrow & 7+0 \leq 3+9 \end{array}\right\} \text{Step 2} \\
& \left.\begin{array}{ll} & \text{(Rule 8 with } \{x_8/7+0, z_8/3+9\}\text{)} \\ \Leftarrow & 7+0 \leq y_8 \quad \wedge \quad y_8 \leq 3+9 \end{array}\right\} \text{Step 3} \\
& \left.\begin{array}{ll} & \text{(Rule 6 with } \{y_8/0+7, x_6/7, y_6/0\}\text{)} \\ \Leftarrow & \mathit{True} \quad \wedge \quad 0+7 \leq 3+9 \\ & \text{(True rule)} \\ \Leftarrow & 0+7 \leq 3+9 \end{array}\right\} \text{Step 4} \\
& \left.\begin{array}{ll} & \text{(Rule 7 with } \{w_7/0, y_7/3, x_7/7, z_7/9\}\text{)} \\ \Leftarrow & 0 \leq 3 \quad \wedge \quad \leq 7 \leq 9 \end{array}\right\} \text{Step 5} \\
& \left.\begin{array}{ll} & \text{(Rule 1 and Rule 2)} \\ \Leftarrow & \mathit{True} \quad \wedge \quad \mathit{True} \\ & \text{(True rule)} \\ \Leftarrow & \mathit{True} \end{array}\right\} \text{Steps 6 \& 7}
\end{array}
$$

We now consider how the algorithm shown in the lecture is applied in each step. For the first 3 steps, we describe in detail which operations one needs to make. We start with the assumption we would like to prove, that is to say $7 \leq 3+9$:

Step 1: Since our goal is to prove $7 \leq 3+9$, we set the variable `goals` as follows:

`goals :=` $\{7 \leq 3+9\}$

Since this is the start of the algorithm, there are no additional substitutions to be made here:

$q' \leftarrow \textsc{Subst}(\emptyset, 7 \leq 3+9)$

We now decide to use Rule 8, that is to say

$$\boxed{\forall x_8, y_8, z_8 \quad x_8 \leq y_8 \wedge y_8 \leq z_8 \;\Rightarrow\; x_8 \leq z_8}$$

To apply this on our `goals`, we need to unify the variables from $7 \leq 3+9$ and $x_8 \leq z_8$, which can be done by substituting $x_8/7$ and $z_8/(3+9)$. We store the substitutions we have to make:

$\theta' \leftarrow \{x_8/7,\ z_8/3+9\}$

Now, to use Rule 8, we have to assume that $x_8 \leq y_8$ and $y_8 \leq z_8$ hold, therefore we store them as our new goals for the next step:

`new_goals` $\leftarrow \{x_8 \leq y_8,\ y_8 \leq z_8\}$

Step 2: From the previous step, we get the following goals:

`goals :=` $\{x_8 \leq y_8,\ y_8 \leq z_8\}$

as well as the substitutions given by θ' from the previous step; evaluating these substitutions on the first goal $x_8 \leq y_8$ is done by

$q' \leftarrow \textsc{Subst}(\{x_8/7,\ z_8/3+9\}, x_8 \leq y_8)$

and yields the formula $7 \leq y_8$. To continue, we wish to apply Rule 4

$$\boxed{\forall x_4, \quad x_4 \leq x_4 + 0}$$

by identifying $x_4 \leq x_4 + 0$ with our $q' = 7 \leq y_8$. This can be done by choosing $x_4/7$ and $y_8/7+0$:

$\theta' \leftarrow \{x_4/7, y_8/7+0\}$

Since Rule 4 does not have any conditions to be true, it does not add new goals to be proven. Therefore, our new goals consist just of the initial goals we have not treated in this step:

`new_goals` $\leftarrow \{y_8 \leq z_8\}$

Step 3: From the previous step, we get the following goals:

`goals :=` $\{y_8 \leq z_8\}$

As well as the following substitutions, applied on the first (and only) goal $y_8 \leq z_8$:

$q' \leftarrow \textsc{Subst}(\{x_8/7,\ z_8/3+9,\ x_4/7,\ y_8/7+0\}, y_8 \leq z_8)$

This yields $q' = 7 + 0 \leq 3 + 9$ We now wish to apply Rule 8 again, i.e.

$$\boxed{\forall x_8', y_8', z_8' \quad x_8' \leq y_8' \land y_8' \leq z_8' \;\Rightarrow\; x_8' \leq z_8'}$$

for which we need to identify $x_8' \leq z_8'$ with $7 + 0 \leq 3 + 9$; this can be done by using the substitutions

$\theta' \leftarrow \{x_8'/7+0,\ z_8'/3+9\}$

To use Rule 8, we need to prove that $x_8' \leq y_8'$ and $y_8' \leq z_8'$ holds with our substitutions, therefore we add those two sentences to our goals:

`new_goals` $\leftarrow \{x_8' \leq y_8',\ y_8' \leq z_8'\}$

Step 4: `goals :=` $\{x_8' \leq y_8',\ y_8' \leq z_8'\}$
$q' \leftarrow \textsc{Subst}(\{x_8/7,\ z_8/3+9,\ x_4/7,\ y_8/7+0,\ x_8'/7+0,\ z_8'/3+9\}, x_8' \leq y_8') = [7+0 \leq y_8']$
`Using Rule 6:` $\boxed{\forall x_6, y_6 \quad x_6 + y_6 \leq 0 + 7}$
$\theta' \leftarrow \{y_8'/0+7,\ x_6/7,\ y_6/0\}$
`new_goals` $\leftarrow \{y_8' \leq z_8'\}$

Step 5: `goals :=` $\{y_8' \leq z_8'\}$
$q' \leftarrow \textsc{Subst}(\{x_8/7,\ z_8/3+9,\ x_4/7,\ y_8/7+0,\ x_8'/7+0,\ z_8'/3+9, y_8'/0+7, \ldots$
$\ldots,\ x_6/7,\ y_6/0\}, y_8' \leq z_8') = [0+7 \leq 3+9]$
`Using Rule 7:` $\boxed{\forall w_7, x_7, y_7, z_7 \quad w_7 \leq y_7 \land x_7 \leq z_7 \;\Rightarrow\; w_7 + x_7 \leq y_7 + z_7}$
$\theta' \leftarrow \{w_7/0,\ y_7/3,\ x_7/7,\ z_7/9\}$
`new_goals` $\leftarrow \{w_7 \leq y_7,\ x_7 \leq z_7\}$

Step 6: `goals :=` $\{w_7 \leq y_7,\, x_7 \leq z_7\}$

$q' \leftarrow \textsc{Subst}(\{x_8/7,\, z_8/3+9,\, x_4/7,\, y_8/7+0,\, x_8'/7+0,\, z_8'/3+9, y_8'/0+7,\, x_6/7,\, y_6/0,\, w_7/0,\, y_7/3, ...$
$..., x_7/7,\, z_7/9\}, w_7 \leq y_7) = [0 \leq 3]$

`Using Rule 1:` $\boxed{0 \leq 3}$

$\theta' \leftarrow \emptyset$

`new_goals` $\leftarrow \{x_7 \leq z_7\}$

Step 7: `goals :=` $\{x_7 \leq z_7\}$

$q' \leftarrow \textsc{Subst}(\{x_8/7,\, z_8/3+9,\, x_4/7,\, y_8/7+0,\, x_8'/7+0,\, z_8'/3+9, y_8'/0+7,\, x_6/7,\, y_6/0,\, w_7/0,\, y_7/3, ...$
$..., x_7/7,\, z_7/9\}, w_7 \leq y_7) = [7 \leq 9]$

`Using Rule 2:` $\boxed{7 \leq 9}$

$\theta' \leftarrow \emptyset$

`new_goals` $\leftarrow \emptyset$

Since at the end of the algorithm we end up with an empty goal set, we have proven that the knowledge base does indeed entail our original assumption $7 \leq 3 + 9$.

References

[1] S. Russell and P. Norvig, *Artificial Intelligence: A Modern Approach*. Prentice Hall, 2010.