Fundamentals of Artificial Intelligence
Exercise 6: Inference in First Order Logic

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Problem 6.1: The man in the painting

A man stands in front of a painting and says the following:

> *Brothers and sisters have I none, but that man's father is my father's son.*

What is the relationship between the man in the painting and the speaker?

Problem 6.1: The man in the painting

To solve the riddle with first-order logic, use the predicates

$$\begin{aligned} Male(x) &: x \text{ is male.} \\ Father(x, y) &: x \text{ is the father of } y. \\ Son(x, y) &: x \text{ is a son of } y. \\ Parent(x, y) &: x \text{ is a parent of } y. \\ Child(x, y) &: x \text{ is a child of } y. \\ Sibling(x, y) &: x \text{ is a sibling of } y \end{aligned}$$

and the knowledge

- A sibling is another child of one's parents.

$$\forall x, y \quad Sibling(x, y) \Leftrightarrow x \neq y \wedge \exists p \quad Parent(p, x) \wedge Parent(p, y)$$

- Parent and child are inverse relations.

$$\forall p, c \quad Parent(p, c) \Leftrightarrow Child(c, p)$$

Problem 6.1: The man in the painting

Problem 6.1.1: Define the rule which relates *son*, *child*, and *male*. Define also the rule which relates *father*, *parent*, and *male*.

- Every son is a male child, and every male child is a son:

$\forall s, p \quad Son(s,p) \Leftrightarrow Male(s) \wedge Child(s,p)$

- Every father is a male parent, and every male parent is a father:

$\forall f, c \quad Father(f,c) \Leftrightarrow Male(f) \wedge Parent(f,c)$

Problem 6.1: The man in the painting

Problem 6.1.2: Using the constants *Me* for the speaker and *That* for the person depicted in the painting, formalize the sentences regarding the sexes of the people in the puzzle.

$Male(Me)$

$Male(That)$

Problem 6.1: The man in the painting

Problem 6.1.3: Formalize the sentences "Brothers and sisters have I none" and "That man's father is my father's son" in first-order logic.

- Brothers and sisters have I none: I don't have sibling.

$\forall x\ \neg Sibling(x, Me)$

- That man's father is my father's son:

$\underbrace{\text{That man's father}}_{f_1}$ is $\underbrace{\text{my father's}}_{f_2}$ son:

$\exists f_1, f_2 \quad Father(f_1, That) \wedge Father(f_2, Me) \wedge Son(f_1, f_2)$

Problem 6.1: The man in the painting

Problem 6.1.4: Solve this puzzle informally and decide what is the relation between the man in the painting and the speaker.

add $KB \models \alpha$ $\alpha = Son(That, Me)$

Problem 6.1: The man in the painting

Problem 6.1.5: Using the resolution technique for first-order logic, prove your answer.

Reminder: Conversion to CNF

The following steps need to be performed to convert a first-order logic formula into Conjunctive Normal Form:

- **Eliminate implications** $A \Rightarrow B \equiv \neg A \vee B$
- **Move $\neg$ inwards** $\neg \forall x \; x \equiv \exists x \neg x$
- **Standardize variables**
- **Skolemization**
- **Drop universal quantifiers**
- **Distribute $\vee$ over $\wedge$**

Problem 6.1: The man in the painting

"A sibling is another child of one's parents."

$$\forall x, y \quad Sibling(x, y) \Leftrightarrow x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y)$$

Problem 6.1: The man in the painting

"A sibling is another child of one's parents."

$$
\begin{aligned}
&\forall x, y \quad Sibling(x, y) \Leftrightarrow x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y) \\
&\equiv \forall x, y \quad [Sibling(x, y) \Rightarrow x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y)] \\
&\quad \land [Sibling(x, y) \Leftarrow x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y)]
\end{aligned}
$$

Problem 6.1: The man in the painting

"A sibling is another child of one's parents.", direction $\Rightarrow$

$$\forall x, y \quad Sibling(x, y) \;\Rightarrow\; x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y)$$

Problem 6.1: The man in the painting

"A sibling is another child of one's parents.", direction $\Rightarrow$

$$\forall x, y \quad Sibling(x, y) \Rightarrow x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y)$$
$$\equiv \forall x, y \quad \neg Sibling(x, y) \lor [x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y)]$$

Problem 6.1: The man in the painting

"A sibling is another child of one's parents.", direction $\Rightarrow$

$$\forall x, y \quad Sibling(x, y) \ \Rightarrow \ x \neq y \wedge \exists p \quad Parent(p, x) \wedge Parent(p, y)$$
$$\equiv \forall x, y \quad \neg Sibling(x, y) \vee [x \neq y \wedge \exists p \quad Parent(p, x) \wedge Parent(p, y)]$$
$$\equiv \forall x, y \quad \neg Sibling(x, y) \vee [x \neq y \wedge Parent(F(x, y), x) \wedge Parent(F(x, y), y)]$$

Problem 6.1: The man in the painting

"A sibling is another child of one's parents.", direction $\Rightarrow$

$$
\begin{aligned}
&\forall x, y \quad Sibling(x, y) \;\Rightarrow\; x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y) \\
&\equiv \forall x, y \quad \neg Sibling(x, y) \lor [x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y)] \\
&\equiv \forall x, y \quad \neg Sibling(x, y) \lor [x \neq y \land Parent(F(x, y), x) \land Parent(F(x, y), y)] \\
&\equiv \forall x, y \quad (\neg Sibling(x, y) \lor (x \neq y)) \land (\neg Sibling(x, y) \lor Parent(F(x, y), x)) \\
&\qquad\quad \land (\neg Sibling(x, y) \lor Parent(F(x, y), y))
\end{aligned}
$$

Problem 6.1: The man in the painting

"A sibling is another child of one's parents.", direction $\Leftarrow$

$$\forall x, y \quad Sibling(x, y) \;\Leftarrow\; x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y)$$
$$\equiv \forall x, y \quad x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y) \;\Rightarrow\; Sibling(x, y)$$

Problem 6.1: The man in the painting

"A sibling is another child of one's parents.", direction $\Leftarrow$

$$
\begin{aligned}
&\forall x, y \quad Sibling(x, y) \;\Leftarrow\; x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y) \\
&\equiv \forall x, y \quad x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y) \;\Rightarrow\; Sibling(x, y) \\
&\equiv \forall x, y \quad \neg(x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y)) \;\lor\; Sibling(x, y)
\end{aligned}
$$

Problem 6.1: The man in the painting

"A sibling is another child of one's parents.", direction $\Leftarrow$

$$
\begin{aligned}
&\forall x, y \quad Sibling(x, y) \;\Leftarrow\; x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y) \\
&\equiv \forall x, y \quad x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y) \;\Rightarrow\; Sibling(x, y) \\
&\equiv \forall x, y \quad \neg(x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y)) \;\lor\; Sibling(x, y) \\
&\equiv \forall x, y \quad (x = y \lor \forall p \quad \neg Parent(p, x) \lor \neg Parent(p, y)) \;\lor\; Sibling(x, y)
\end{aligned}
$$

Problem 6.1: The man in the painting

"A sibling is another child of one's parents.", direction $\Leftarrow$

$$
\begin{aligned}
&\forall x, y \quad Sibling(x, y) \;\Leftarrow\; x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y) \\
&\equiv \forall x, y \quad x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y) \;\Rightarrow\; Sibling(x, y) \\
&\equiv \forall x, y \quad \neg(x \neq y \land \exists p \quad Parent(p, x) \land Parent(p, y)) \;\lor\; Sibling(x, y) \\
&\equiv \forall x, y \quad (x = y \lor \forall p \quad \neg Parent(p, x) \lor \neg Parent(p, y)) \;\lor\; Sibling(x, y) \\
&\equiv \forall x, y, p \quad x = y \lor \neg Parent(p, x) \lor \neg Parent(p, y) \;\lor\; Sibling(x, y)
\end{aligned}
$$

Problem 6.1: The man in the painting

"That man's father is my father's son."

$$\underline{\exists f_1, f_2} \quad Father(f_1, That) \land Father(f_2, Me) \land Son(f_1, f_2)$$

Problem 6.1: The man in the painting

“That man’s father is my father’s son.”

no $\forall$

$$\begin{aligned} \exists f_1, f_2 \quad & Father(f_1, That) \wedge Father(f_2, Me) \wedge Son(f_1, f_2) \\ \equiv \quad & Father(F_1, That) \wedge Father(F_2, Me) \wedge Son(F_1, F_2) \end{aligned}$$

Problem 6.1: The man in the painting

Goal: $\alpha = Son(That, Me)$

add $\alpha = \neg Son(That, Me)$

Problem 6.1: The man in the painting

$\boxed{Sibling(x, y) \lor (x = y) \lor \neg Parent(p, x) \lor \neg Parent(p, y)}$

$\boxed{\neg Sibling(x, y) \lor (x \neq y)}$

$\boxed{\neg Sibling(x, y) \lor Parent(F(x, y), x)}$

$\boxed{\neg Sibling(x, y) \lor Parent(F(x, y), y)}$

$\boxed{\neg Parent(p, c) \lor Child(c, p)}$

$\boxed{\neg Child(c, p) \lor Parent(p, c)}$

$\boxed{\neg Son(s, p) \lor Child(s, p)}$ $\boxed{\neg Son(s, p) \lor Male(s)}$

$\boxed{Son(s, p) \lor \neg Child(s, p) \lor \neg Male(s)}$

$\boxed{\neg Father(p, c) \lor Parent(p, c)}$

$\boxed{\neg Father(p, c) \lor Male(p)}$

$\boxed{Father(p, c) \lor \neg Parent(p, c) \lor \neg Male(p)}$

$\boxed{\neg Sibling(Me, x)}$ $\boxed{\neg Sibling(x, Me)}$

$\boxed{Father(F_1, That)}$ $\boxed{Father(F_2, Me)}$ $\boxed{Son(F_1, F_2)}$

$\boxed{Male(That)}$ $\boxed{Male(Me)}$ $\boxed{\neg Son(That, Me)}$

Reminder: Resolution Inference Rule

The resolution rule of propositional logic can be lifted to first-order logic:

Resolution rule for first-order logic

$$\frac{l_1 \vee \ldots \vee l_k, \quad m_1 \vee \ldots \vee m_n}{\texttt{Subst}(\theta, l_1 \vee \ldots \vee l_{i-1} \vee l_{i+1} \vee \ldots \vee l_k \vee m_1 \vee \ldots \vee m_{j-1} \vee m_{j+1} \vee \ldots \vee m_n)},$$

where $\texttt{Unify}(l_i, \neg m_j) = \theta$.

Example: We can resolve the two clauses

$$[Animal(F(x)) \vee Loves(G(x), x)] \text{ and } [\neg Loves(u, v) \vee \neg Kills(u, v)]$$

by eliminating the complementary literals $Loves(G(x), x)$ and $\neg Loves(u, v)$, with unifier $\theta = \{u/G(x), v/x\}$, to produce the **resolvent** clause

$$[Animal(F(x)) \vee \neg Kills(G(x), x)].$$

Problem 6.1: The man in the painting

$Sibling(x, y) \lor (x = y) \lor \neg Parent(p, x) \lor \neg Parent(p, y)$

$\neg Sibling(x, y) \lor (x \neq y)$

$\neg Sibling(x, y) \lor Parent(F(x, y), x)$

$\neg Sibling(x, y) \lor Parent(F(x, y), y)$

$\neg Parent(p, c) \lor Child(c, p)$

$\neg Child(c, p) \lor Parent(p, c)$

$\neg Son(s, p) \lor Child(s, p)$ $\quad$ $\neg Son(s, p) \lor Male(s)$

$Son(s, p) \lor \neg Child(s, p) \lor \neg Male(s)$

$\neg Father(p, c) \lor Parent(p, c)$

$\neg Father(p, c) \lor Male(p)$

$Father(p, c) \lor \neg Parent(p, c) \lor \neg Male(p)$

$\neg Sibling(Me, x)$ $\quad$ $\neg Sibling(x, Me)$

$Father(F_1, That)$ $\quad$ $Father(F_2, Me)$ $\quad$ $Son(F_1, F_2)$

$Male(That)$ $\quad$ $Male(Me)$ $\quad$ $\neg Son(That, Me)$

$\neg Son(\cancel{That, Me})$ $\quad$ $Son(\cancel{s,p}) \lor \neg Child(s,p) \lor \neg Male(s)$

$Male(\cancel{That})$ $\quad$ $\neg Child(That, Me) \lor \cancel{\neg Male(That)}$

$\neg Parent(p,c) \lor Child(\cancel{c,p})$ $\quad$ $\cancel{\neg Child(That, Me)}$

$\underline{\neg Parent(Me, That)}$

Problem 6.1: The man in the painting

$\neg\mathbf{Son(s,p)} \lor Child(s,p)$	$Son(F_1, F_2)$
$\neg\mathbf{Child(c,p)} \lor Parent(p,c)$	$Child(F_1, F_2)$
$Sibling(x,y) \lor (x = y) \lor \neg\mathbf{Parent(p,x)} \lor \neg Parent(p,y)$	$Parent(F_2, F_1)$
$\neg Father(p,c) \lor \mathbf{Parent(p,c)}$	$Sibling(F_1, y) \lor (F_1 = y) \lor \neg Parent(F_2, y)$
$\mathbf{Father(F_2, Me)}$	$Sibling(F_1, y) \lor (F_1 = y) \lor \neg Father(F_2, y)$
$\neg\mathbf{Sibling(x, Me)}$	$Sibling(F_1, Me) \lor (F_1 = Me)$
$\mathbf{Father(F_1, That)}$	$F_1 = Me$
$\neg\mathbf{Father(p,c)} \lor Parent(p,c)$	$Father(Me, That)$
$\neg Parent(Me, That)$	$Parent(Me, That)$

$\emptyset$

Problem 6.2: Backward chaining

Suppose you are given the following axioms:

1. $0 \leq 3$
2. $7 \leq 9$
3. $\forall x \quad x \leq x$
4. $\forall x \quad x \leq x + 0$
5. $\forall x \quad x + 0 \leq x$
6. $\forall x, y \quad x + y \leq y + x$
7. $\forall w, x, y, z \quad w \leq y \land x \leq z \ \Rightarrow \ w + x \leq y + z$
8. $\forall x, y, z \quad x \leq y \land y \leq z \ \Rightarrow \ x \leq z$.

Give a backward-chaining proof of the sentence $7 \leq 3 + 9$. (Be sure, of course, to use only the axioms given here, not anything else you may know about arithmetic.) Show only the steps that lead to success, not the irrelevant steps.

Reminder: Backward-Chaining Algorithm

function FOL-BC-Ask (KB,$goals$,θ) **returns** a set of substitutions

inputs: KB, a knowledge base
$goals$, a list of conjuncts forming a query (θ already applied)
θ, the current substitution, initially the empty substitution $\emptyset$

local variables: $answers$, a set of substitutions, initially empty

if $goals$ is empty **then return** $\{\theta\}$
$q' \leftarrow \texttt{Subst}(\theta, \texttt{First}(goals))$
for each sentence r **in** KB where $\texttt{Standardize-Apart}(r) = (p_1 \wedge \ldots \wedge p_n \Rightarrow q)$
and $\theta' \leftarrow \texttt{Unify}(q, q')$ succeeds
    $new\_goals \leftarrow [p_1, \ldots, p_n | \texttt{Rest}(goals)]$
    $answers \leftarrow \texttt{FOL-BC-Ask}(KB, new\_goals, \texttt{Compose}(\theta', \theta)) \cup answers$
return $answers$

Step 1: $(7 \leq 3+9) \Leftarrow (7 \leq y \land y \leq 3+9)$

goals : $\{7 \leq 3+9\}$

$q' \leftarrow \textsc{Subst}(\emptyset,\ 7 \leq 3+9\ )$

Using rule 8: $\boxed{\forall x_8, y_8, z_8 \quad x_8 \leq y_8 \land y_8 \leq z_8 \;\Rightarrow\; x_8 \leq z_8}$

$\theta' \leftarrow \{x_8/7,\ z_8/3+9\}$

new goals $\leftarrow \{x_8 \leq y_8,\ y_8 \leq z_8\}$

Step 2: $(7 \leq y \wedge y \leq 3+9) \Leftarrow (True \wedge 7+0 \leq 3+9)$
$\Leftarrow (7+0 \leq 3+9)$

goals : $\{\underline{x_8 \leq y_8}, y_8 \leq z_8\}$

$q' \leftarrow \text{SUBST}(\{x_8/7,\ z_8/3+9\},$ $x_8 \leq y_8$ $)$

$7 \leq y_8$

Using rule 4: $\boxed{\forall x_4 \quad x_4 \leq x_4 + 0}$

$\theta' \leftarrow$ $\{x_4/7, \underline{y_8/7+0}\}$

new goals $\leftarrow$ $\emptyset$

Step 3: $(7+0 \leq 3+9) \Leftarrow (7+0 \leq y \wedge y \leq 3+9)$

goals : $\{y_8 \leq z_8\}$

$q' \leftarrow \text{SUBST}(\{x_8/7,\ z_8/3{+}9,\ x_4/7,\ y_8/7{+}0\},\ \underset{7+0\ \leq\ 3+9}{y_8 \leq z_8}\qquad)$

Using rule 8: $\boxed{\forall x_8', y_8', z_8' \quad x_8' \leq y_8' \wedge y_8' \leq z_8' \ \Rightarrow\ x_8' \leq z_8'}$

$\theta' \leftarrow \{x_8'/7+0,\ z_8'/3+9\}$

new goals $\leftarrow$ $\{x_8' \leq y_8',\ y_8' \leq z_8'\}$

Step 4: $(7+0 \leq y \wedge y \leq 3+9) \Leftarrow (True \wedge 0+7 \leq 3+9)$
$\Leftarrow (0+7 \leq 3+9)$

goals : $\{x'_8 \leq y'_8, y'_8 \leq z'_8\}$

$q' \leftarrow \textsc{Subst}(\{x_8/7,\ z_8/3+9,\ x_4/7,\ y_8/7+0,\ x'_8/7+0,$
$z'_8/3+9\},\ x'_8 \leq y'_8 \qquad)$

$7+0 \leq y'_8$

Using rule 6: $\boxed{\forall x_6, y_6 \quad x_6 + y_6 \leq y_6 + x_6}$

$\theta' \leftarrow \{x_6/7 \,.\, y_6/0 \,,\, 0+7/y'_8\}$

new goals $\leftarrow$

Step 5: $(0 + 7 \leq 3 + 9) \Leftarrow (0 \leq 3 \land 7 \leq 9)$

goals : $\{y'_8 \leq z'_8\}$

$$q' \leftarrow \textsc{Subst}(\{x_8/7,\ z_8/3+9,\ x_4/7,\ y_8/7+0,\ x'_8/7+0,$$
$$z'_8/3+9,\ y'_8/0+7,\ x_6/7,\ y_6/0\},\ y'_8 \leq z'_8\ )$$

$0+7 \leq 3+9$

Using rule 7:

$$\boxed{\forall w_7, x_7, y_7, z_7 \quad w_7 \leq y_7 \land x_7 \leq z_7 \ \Rightarrow\ w_7 + x_7 \leq y_7 + z_7}$$

$\theta' \leftarrow \{w_7/0,\ x_7/7,\ y_7/3,\ z_7/9\}$

new goals $\leftarrow \{w_7 \leq y_7,\ x_7 \leq z_7\}$

Step 6: $(0 \leq 3 \wedge 7 \leq 9) \Leftarrow (\mathit{True} \wedge 7 \leq 9) \Leftarrow (7 \leq 9)$

goals : $\{w_7 \leq y_7, x_7 \leq z_7\}$

$$q' \leftarrow \textsc{Subst}(\{x_8/7,\ z_8/3+9,\ x_4/7,\ y_8/7+0,\ x_8'/7+0,\ z_8'/3+9,\ y_8'/0+7,\ x_6/7,\ y_6/0,\ w_7/0,\ y_7/3,\ x_7/7,\ z_7/9\},\ w_7 \leq y_7 \qquad)$$

$0 \leq 3$

Using rule 1: $\boxed{0 \leq 3}$

$\theta' \leftarrow$

new goals $\leftarrow$

Step 7: $(7 \leq 9) \Leftarrow \mathit{True}$

goals : $\{x_7 \leq z_7\}$

$$q' \leftarrow \textsc{Subst}(\{x_8/7,\ z_8/3+9,\ x_4/7,\ y_8/7+0,\ x_8'/7+0,\ z_8'/3+9,\ y_8'/0+7,\ x_6/7,\ y_6/0,\ w_7/0,\ y_7/3,\ x_7/7,\ z_7/9\},\qquad)$$

Using rule 2: $\boxed{7 \leq 9}$

$\theta' \leftarrow$

new goals $\leftarrow$