FUNDAMENTALS OF ARTIFICIAL INTELLIGENCE
Exercise 5: First Order Logic – Solutions Winter semester 2023/2024
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Problem 5.1: Syntax of first-order logic

Recall the formal syntax of first-order logic, in particular the definition of terms and sentences.

Problem 5.1.1: *(Taken from [1] Exercise 2.1)* Let $\mathcal{F} = \{d, f, g\}$ be a set of symbols, where d is a constant, f a function symbol with two arguments, and g a function symbol with three arguments. Which of the following expressions are terms over $\mathcal{F}$? If an expression is not a term, give a reason why. (You may assume that x, y, z are variables.)

1. $g(d, d)$
2. $f(x, g(y, z), d)$
3. $g(x, f(y, z), d)$
4. $g(x, k(y, z), d)$

Solution:

1. This **is not** a term, because g needs three arguments.
2. This **is not** a term, because f needs two arguments.
3. This **is** a term, because
   - g is a function accepting three arguments;
   - x is a variable, hence a term;
   - $f(y, z)$ is a term, because
     - f is a function accepting two arguments;
     - y is a variable, hence a term;
     - z is a variable, hence a term;
   - d is a constant, hence a term.
4. This **is not** a term because there is no function symbol k in $\mathcal{F}$.

Problem 5.1.2: *(Taken from [1] Exercise 2.2)*

Let m be a constant, h a function symbol with one argument, and S and B two predicate symbols, each with two arguments. Which of the following expressions are sentences in first-order logic? If an expression is not a sentence, give a reason why. (You may assume that x, y, z are variables.)

1. $S(m, x)$
2. $B(m, h(m))$
3. $B(B(m, x), y)$
4. $B(x, y) \Rightarrow [\exists z\, S(z, y)]$
5. $S(x, y) \Rightarrow S(y, h(h(x)))$

Solution:

1. This **is** a sentence, because

- S is a predicate accepting two arguments;
- m is a constant, hence a term;
- x is a variable, hence a term.

2. This **is** a sentence, because
   - B is a predicate accepting two arguments;
   - m is a constant, hence a term;
   - $h(m)$ is a term, because
     - h is a function accepting one argument;
     - m is a constant, hence a term.
3. This **is not** a sentence, because the first argument of B, that is $B(m, x)$, is not a *term*, but a *sentence*, because
   - B is a predicate accepting two arguments;
   - m is a constant, hence a term;
   - x is a variable, hence a term.
4. This **is** a sentence, because
   - $B(x, y)$ is a sentence, because
     - B is a predicate accepting two arguments;
     - x, y are variables, hence terms.
   - $\exists z\, S(z, y)$ is a sentence, because
     - z is a variable;
     - $S(z, y)$ is a sentence, because
       * S is a predicate accepting two arguments;
       * z, y are variables, hence terms.
5. This **is** a sentence, because
   - $S(x, y)$ is a sentence, because
     - S is a predicate accepting two arguments;
     - x, y are variables, hence terms.
   - $S(y, h(h(x)))$ is a sentence, because
     - S is a predicate accepting two arguments;
     - y is a variable, hence a term;
     - $h(h(x))$ is a term, because
       * h is a function accepting one argument;
       * $h(x)$ is a term, because
         · h is a function accepting one argument;
         · x is a variable, hence a term.

Problem 5.2: Universal and existential quantifiers

Let us abbreviate the predicate "x is taking the Bus" by $B(x)$, and "x has a Ticket" by $T(x)$. Suppose that we only consider the universe of discourse where there are only three people: Alice, Bob, and Charlie.

Universal quantifiers

Problem 5.2.1: For each of our protagonists, Table 1 lists whether they have a ticket, and whether they take the bus. Suppose that we want to formalize the sentence "All people who take the bus have a ticket." Given the characteristics in Table 1, do you think the sentence evaluates to true? Or false?

x	$B(x)$	$T(x)$
Alice	*True*	*True*
Bob	*False*	*True*
Charlie	*False*	*False*

Table 1: Truth assignments for our universe of discourse

Solution: The sentence evaluates to true according to these conditions, since this sentence applies only to people who take the bus. In this case, only Alice takes the bus. Since she does have a ticket, the sentence is true.

Problem 5.2.2: Now, consider the formula

$$\forall x \quad B(x) \wedge T(x).$$

By using the extended interpretation for universal quantifiers (in other words, the transformation to propositional logic), decide whether this formula is true or false in our universe of discourse. Do you think this formula is a faithful formalization of "All people who take the bus have a ticket"? If not, what would be a better formula?

Solution: This formula evaluates to false. To see this, we apply the extended interpretation of universal quantifiers, which yields the sentence

$$[B(\text{Alice}) \wedge T(\text{Alice})] \quad \wedge \quad [B(\text{Bob}) \wedge T(\text{Bob})] \quad \wedge \quad [B(\text{Charlie}) \wedge T(\text{Charlie})].$$

Since B(Bob) and B(Charlie) are false, the whole extended interpretation is also false. Therefore, this formalization is not a faithful representation of our sentence. The correct formalization would be

$$\forall x \quad B(x) \Rightarrow T(x).$$

Existential quantifiers

Problem 5.2.3: Now consider the changed truth assignments in Table 2. Suppose that we want to formalize the sentence "Some people who take the bus have a ticket"[1]. Given the characteristics in Table 2, do you think the sentence evaluates to true? Or false?

x	$B(x)$	$T(x)$
Alice	*False*	*True*
Bob	*False*	*True*
Charlie	*False*	*False*

Table 2: Changed truth assignments: now Alice does not take the bus

Solution: The sentence evaluates to false if applied to the specific case of Alice, Bob, and Charlie, since no one is taking the bus, no matter whether they have a ticket or not.

[1]This has to be understood as "There is **at least** one person that takes the bus, who has a ticket".

Problem 5.2.4: Now, consider the formula

$$\exists x \quad B(x) \Rightarrow T(x).$$

By using the extended interpretation for existential quantifiers, decide whether this formula is true or false in our universe of discourse. Do you think this formula is a faithful formalization of "Some people who take the bus have a ticket"? If not, what would be a better formula?

Solution: This formula evaluates to true. To see this, we apply the extended interpretation of existential quantifiers, which yields the sentence

$$[B(\text{Alice}) \Rightarrow T(\text{Alice})] \quad \vee \quad [B(\text{Bob}) \Rightarrow T(\text{Bob})] \quad \vee \quad [B(\text{Charlie}) \Rightarrow T(\text{Charlie})].$$

Since $B(\text{Alice})$ is false, $B(\text{Alice}) \Rightarrow T(\text{Alice})$ is true. Thus, the whole extended interpretation is true. Therefore, this formalization is not a faithful representation of our sentence. The correct formalization would be

$$\exists x \quad B(x) \wedge T(x).$$

Problem 5.3: Formalization to First Order Logic

Problem 5.3.1: *(Taken from [1] Exercise 2.1)*

Formalize the next sentences using the following predicates and constant:

$$\begin{aligned} A(x,y) &: x \text{ admires } y \\ P(x) &: x \text{ is a professor} \\ d &: \text{Dan} \end{aligned}$$

1. Dan admires every professor.
2. Some professor admires Dan.
3. Dan admires himself.

Solution:

1. Dan admires every professor:
$$\forall x \quad P(x) \Rightarrow A(d,x)$$
2. Some professor admires Dan:
$$\exists x \quad P(x) \;\wedge\; A(x,d)$$
3. Dan admires himself:
$$A(d,d)$$

Problem 5.3.2: *(Taken from [1] Exercise 2.1)*

Formalize the next sentences using the following predicates:

$$\begin{aligned} A(x,y) &: x \text{ attended } y \\ S(x) &: x \text{ is a student} \\ L(x) &: x \text{ is a lecture} \end{aligned}$$

1. No student attended every lecture.
2. No lecture was attended by every student.
3. No lecture was attended by any student.

Solution:

1. No student attended every lecture:

$$\neg\exists x\forall y \quad S(x) \ \wedge \ (L(y) \Rightarrow A(x,y)),$$

   or equivalently "Every student has a lecture they do not attend"

$$\forall x\exists y \quad S(x) \Rightarrow (L(y) \ \wedge \ \neg A(x,y))$$

2. No lecture was attended by every student:

$$\neg\exists x\forall y \quad L(x) \ \wedge \ (S(y) \Rightarrow A(y,x)),$$

   or equivalently "Every lecture must have some student missing the lecture"

$$\forall x\exists y \quad L(x) \Rightarrow (S(y) \ \wedge \ \neg A(y,x))$$

3. No lecture was attended by any student:

$$\neg\exists x\exists y \quad L(x) \ \wedge \ S(y) \ \wedge \ A(y,x),$$

   or equivalently "Every lecture is missed by every student"

$$\forall x\forall y \quad L(x) \Rightarrow (S(y) \Rightarrow \neg A(y,x)),$$

   or equivalently

$$\forall x\forall y \quad (L(x) \ \wedge \ S(y)) \Rightarrow \neg A(y,x)$$

Problem 5.3.3: *(Taken from [2])*

Let the constant h stand for Holmes (Sherlock Holmes) and m for Moriarty (Professor Moriarty). Let us abbreviate "x can trap y" with the predicate $T(x,y)$. Give symbolic rendition of the following statements:

1. Holmes can trap everyone who can trap Moriarty.
2. Holmes can trap everyone whom Moriarty can trap.
3. Holmes can trap everyone who can be trapped by Moriarty.
4. If someone can trap Moriarty, then Holmes can.
5. If everyone can trap Moriarty, then Holmes can.
6. Anyone who can trap Holmes can trap Moriarty.
7. No one can trap Holmes unless that person can trap Moriarty.
8. Everyone can trap someone who cannot trap Moriarty.
9. Everyone who can trap Holmes can trap everyone whom Holmes can trap.

Solution:

1. Holmes can trap everyone who can trap Moriarty:

$$\forall x \quad T(x, m) \Rightarrow T(h, x)$$

2. Holmes can trap everyone whom Moriarty can trap:

$$\forall x \quad T(m, x) \Rightarrow T(h, x)$$

3. Holmes can trap everyone who can be trapped by Moriarty:

$$\forall x \quad T(m, x) \Rightarrow T(h, x)$$

4. If someone can trap Moriarty, then Holmes can:

$$(\exists x \quad T(x, m)) \Rightarrow T(h, m)$$

5. If everyone can trap Moriarty, then Holmes can:

$$(\forall x \quad T(x, m)) \Rightarrow T(h, m)$$

6. Anyone who can trap Holmes can trap Moriarty:

$$\forall x \quad T(x, h) \Rightarrow T(x, m)$$

7. No one can trap Holmes unless that person can trap Moriarty:

$$\forall x \quad \neg T(x, m) \Rightarrow \neg T(x, h),$$

   or equivalently

$$\forall x \quad T(x, h) \Rightarrow T(x, m)$$

   or equivalently "No one can trap Holmes but not Moriarty"

$$\neg \exists x \quad \neg T(x, m) \wedge T(x, h)$$

8. Everyone can trap someone who cannot trap Moriarty:

$$\forall x \, \exists y \quad \neg T(y, m) \wedge T(x, y)$$

9. Everyone who can trap Holmes can trap everyone whom Holmes can trap:

$$\forall x \quad T(x, h) \Rightarrow (\forall y \quad T(h, y) \Rightarrow T(x, y)),$$

   or equivalently

$$\forall x, y \quad T(x, h) \Rightarrow (T(h, y) \Rightarrow T(x, y)),$$

   or equivalently

$$\forall x, y \quad T(x, h) \wedge T(h, y) \Rightarrow T(x, y)$$

Observe that statements 2 and 3, as well as statements 6 and 7 are equivalent.

References

[1] M. Huth and M. D. Ryan, *Logic in computer science - modelling and reasoning about systems (2. ed.)* Cambridge University Press, 2004.

[2] D. Gries and F. B. Schneider, *A Logical Approach to Discrete Math*. Springer-Verlag New York, Inc., 1993.