FUNDAMENTALS OF ARTIFICIAL INTELLIGENCE

Exercise 7: Probability theory and Bayesian networks – Problems Winter Semester 2023/24

Jonathan Külz

1 Presented Problems

Problem 7.1:

Consider the following probabilistic inference problem with random variables for driver expertise, road conditions and accidents. The joint probability distribution is provided in the table below (values are guessed). The first variable $R \in \{dry, wet, snow/ice\}$ is a discrete random variable and represents the considered road conditions. The rest of the variables are Boolean: E is associated to the event that the driver is experienced or not and A to the event that an accident happens or not. The joint probability distribution is given by the table below.

E	A	$\mathbf{P}(R = dry, E, A)$	$\mathbf{P}(R = wet, E, A)$	$\mathbf{P}(R = snow/ice, E, A)$
t	t	0.0607	0.0449	0.0084
t	f	0.3605	?	0.0240
f	t	0.0851	0.0654	0.0152
f	f	0.1435	0.0400	0.0022

a. Calculate the value of the missing probability in the table.
b. What is the prior probability distribution of the random variables R and E?
c. What is the probability that the driver is not experienced given that there is an accident and the road is wet?
d. Construct a corresponding Bayesian network with the conditional probability tables (calculate the required table entries).

Problem 7.2:

(Taken from **russel** Ex. 13.21) Suppose you are a witness to a nighttime hit-and-run accident involving a taxi in Athens. All taxis in Athens are blue or green. You swear, under oath, that the taxi was blue. Extensive testing shows that, under the dim lighting conditions, discrimination between blue and green is 75% reliable.

a. Is it possible to calculate the most likely color for the taxi?
b. What if you know that 9 out of 10 Athenian taxis are green?

Problem 7.3:

(Zero-knowledge proof) For Christmas, the siblings Josie and Andrew are each given a box of sweets from their grandmother. They look the same from the outside, but they know that one contains more chocolate while the other has more nuts. Both would like to have the box with more chocolate. Josie claims she can hear the difference when shaking the box with perfect accuracy, but Andrew doesn't believe her.

a. How can Josie convince Andrew she can indeed hear the difference without opening the present and without giving away which box has more chocolate?
b. Based on experience, Andrew believes his sister lies 3 out of 10 times. How should he update his (conditional) belief about her statement after Josie has successfully mastered his test[1] n times?
c. Given n successful tests, what's the (unconditional) probability that Andrew is wrong when believing his sister?

[1] Solution for 7.3 a) Andrew gives Josie one of the boxes and allows her to shake it. Then he leaves the room and, without her seeing it, either exchanges the box or comes back with the same one. He then asks Josie if the box makes the same noise as before.

Problem 7.1:

Consider the following probabilistic inference problem with random variables for driver expertise, road conditions and accidents. The joint probability distribution is provided in the table below (values are guessed). The first variable $R \in \{dry, wet, snow/ice\}$ is a discrete random variable and represents the considered road conditions. The rest of the variables are Boolean: E is associated to the event that the driver is experienced or not and A to the event that an accident happens or not. The joint probability distribution is given by the table below.

E	A	$\mathbf{P}(R = dry, E, A)$	$\mathbf{P}(R = wet, E, A)$	$\mathbf{P}(R = snow/ice, E, A)$
t	t	0.0607	0.0449	0.0084
t	f	0.3605	?	0.0240
f	t	0.0851	0.0654	0.0152
f	f	0.1435	0.0400	0.0022

a. Calculate the value of the missing probability in the table.

b. What is the prior probability distribution of the random variables R and E?

c. What is the probability that the driver is not experienced given that there is an accident and the road is wet?

d. Construct a corresponding Bayesian network with the conditional probability tables (calculate the required table entries).

a. $\sum p(w) = 1 \Rightarrow 0.1501$

b. $p(R)$ / $p(E)$

$p(R) = \sum_{E,A} P(R, E, A) \Rightarrow p(R = dry) = 0.6498$

$p(R = wet) = 0.3004$

$p(R = snow/ice) = 0.0498$

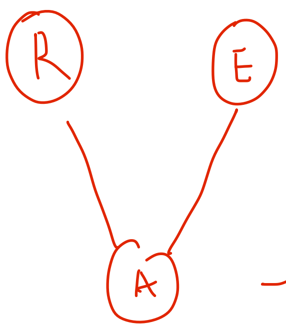
$p(E) = \sum_{R,A} P(E, R, A) \Rightarrow p(E = true) = 0.6486$

$p(E = false) = 0.3514$

c. $p(\neg e \mid a, r = wet) = \dfrac{p(\neg e, a, r = wet)}{\sum_e P(a, r = wet, e)} = \dfrac{0.0654}{0.0654 + 0.0449} = 0.5929$

d.

R	P(R)
dry	0.6498
wet	0.3004
snow/ice	0.0498



E	P(E)
t	0.6486
f	0.3514

R	E	P(a | R,E)
dry	t	0.1441
	f	0.3723
wet	t	0.2303
	f	0.6205
snow/ice	t	0.2593
	f	0.8736

Problem 7.2:

(Taken from **russel** Ex. 13.21) Suppose you are a witness to a nighttime hit-and-run accident involving a taxi in Athens. All taxis in Athens are blue or green. You swear, under oath, that the taxi was blue. Extensive testing shows that, under the dim lighting conditions, discrimination between blue and green is 75% reliable.

a. Is it possible to calculate the most likely color for the taxi? no , no piror knowledge

b. What if you know that 9 out of 10 Athenian taxis are green?

$p(\text{taxi}=\text{green}) = 0.9 \quad p(\text{taxi}=\text{blue}) = 0.1$

$p(\text{ob}=\text{green} \mid \text{taxi}=\text{green}) = 0.75$

$p(\text{ob}=\text{blue} \mid \text{taxi}=\text{green}) = 0.25$

$p(\text{ob}=\text{green} \mid \text{taxi}=\text{blue}) = 0.25$

$p(\text{ob}=\text{blue} \mid \text{taxi}=\text{blue}) = 0.75$

$$p(\text{taxi}=\text{blue} \mid \text{ob}=\text{blue}) = \frac{p(\text{ob}=\text{blue} \mid \text{taxi}=\text{blue}) \cdot p(\text{taxi}=\text{blue})}{p(\text{ob}=\text{blue})}.$$

$$\approx 0.75 \cdot 0.1 = 0.075$$

$$p(\text{taxi}=\text{green} \mid \text{ob}=\text{blue}) = \frac{p(\text{ob}=\text{blue} \mid \text{taxi}=\text{green})\, p(\text{taxi}=\text{green})}{p(\text{ob}=\text{blue})}$$

$$= 0.25 \cdot 0.9 = 0.225$$

$$\langle \text{blue}, \text{green} \rangle = \langle \frac{1}{4}, \frac{3}{4} \rangle$$

Problem 7.3:

(Zero-knowledge proof) For Christmas, the siblings Josie and Andrew are each given a box of sweets from their grandmother. They look the same from the outside, but they know that one contains more chocolate while the other has more nuts. Both would like to have the box with more chocolate. Josie claims she can hear the difference when shaking the box with perfect accuracy, but Andrew doesn't believe her.

a. How can Josie convince Andrew she can indeed hear the difference without opening the present and without giving away which box has more chocolate?

b. Based on experience, Andrew believes his sister lies 3 out of 10 times. How should he update his (conditional) belief about her statement after Josie has successfully mastered his test[1] n times?

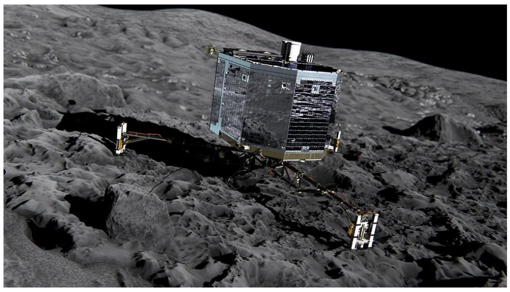
c. Given n successful tests, what's the (unconditional) probability that Andrew is wrong when believing his sister?

a. Repeat in process → Unlikely tell the difference

2 Additional Problems

Problem 7.4:

Suppose that we send a space robot on a comet to collect and analyze samples from the terrain. Due to the uncertain environment we know that the probability that the robot finds a correct pose to extract useful samples is 45%. Fortunately, the robot has a measurement system that senses a correct pose for taking a sample with 80% of reliability, performing one measurement. This sensing system has the same reliability for the detection of an unacceptable pose (the probability values are guessed).



Artistic impression of a comet lander: the Rosetta's Philae lander. Source: ESA.

a. Consider that the robot is in some pose and takes a measurement with positive result. What is the probability that the robot is in a correct pose for collecting a sample?

b. Consider that the robot should take a sample if the probability that it is useful is at least 90%. Can the robot collect a sample if it takes another measurement and the result is again positive? Calculate the corresponding posterior probability.

Problem 7.5:

(Taken from **russel** Ex. 13.3) For each of the following statements, either prove it is true or give a counterexample.

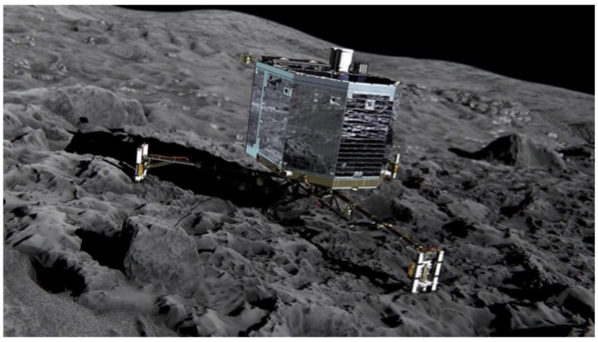
a. If $P(a|b,c) = P(b|a,c)$, then $P(a|c) = P(b|c)$.

b. If $P(a|b,c) = P(a)$, then $P(b|c) = P(b)$.

c. If $P(a|b) = P(a)$, then $P(a|b,c) = P(a|c)$.

Problem 7.4:

Suppose that we send a space robot on a comet to collect and analyze samples from the terrain. Due to the uncertain environment we know that the probability that the robot finds a correct pose to extract useful samples is 45%. Fortunately, the robot has a measurement system that senses a correct pose for taking a sample with 80% of reliability, performing one measurement. This sensing system has the same reliability for the detection of an unacceptable pose (the probability values are guessed).



Artistic impression of a comet lander: the Rosetta's Philae lander. Source: ESA.

a. Consider that the robot is in some pose and takes a measurement with positive result. What is the probability that the robot is in a correct pose for collecting a sample?

b. Consider that the robot should take a sample if the probability that it is useful is at least 90%. Can the robot collect a sample if it takes another measurement and the result is again positive? Calculate the corresponding posterior probability.

$P(cp) = 0.45$

$p(pm \mid cp) = 0.8 \quad p(\neg pm \mid \neg cp) = 0.8$

a. $p(cp \mid pm) = \dfrac{p(pm \mid cp) \cdot p(cp)}{p(pm)} \simeq 0.8 \cdot 0.45 = 0.36$

$p(\neg cp \mid pm) = \dfrac{p(pm \mid \neg cp) \cdot p(\neg cp)}{p(pm)} \simeq 0.2 \cdot 0.55 = 0.11$

$\langle cp, \neg cp \rangle = \langle 0.77, 0.23 \rangle$

0.288

b. $p(cp \mid pm \cdot pm2) = \alpha\, \underbrace{P(pm \mid cp)}_{0.8} \cdot \underbrace{p(pm2 \mid cp)}_{0.8} \cdot \underbrace{P(cp)}_{0.45}$

$p(\neg cp \mid pm \cdot pm2) = \alpha\, \underbrace{p(pm \mid \neg cp)}_{0.2} \cdot \underbrace{p(pm2 \mid \neg cp)}_{0.2} \cdot \underbrace{p(\neg cp)}_{0.55}$

0.022

$\langle cp, \neg cp \rangle = \langle 0.93, 0.07 \rangle$ ✓