Tutorial Robotics IN2067

Exercise Sheet 05

P01

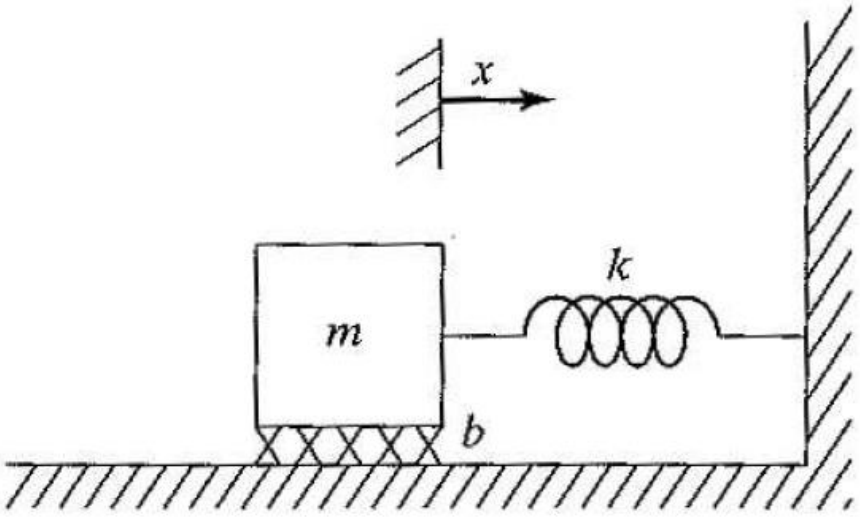


Figure 1: *Simple mass-spring-system.*

Problem 1

Consider a simple mass-spring system (Figure 1) with one object of mass $m = 1$, attached to a spring with stiffness $k = 5$ and affected by friction with a friction constant $b = 4$. The system has a resonant frequency of $\omega_{\text{res}} = 6.0$. Determine k_v and k_p such that the system is critically damped.

P01

- Determine forces acting on the body:
  - Inertia force: $-m\ddot{x}$
    - Always in the opposite direction of acceleration
  - Damping force: $-b\dot{x}$
    - Always in the opposite direction of velocity
  - Elastic force: $-k\Delta x$
    - This is an approximation when Δx is small of the true law which is $-k(\Delta x)^3$
    - In this course, unless otherwise mentioned, we will use $F_{elastic} = -k\Delta x$

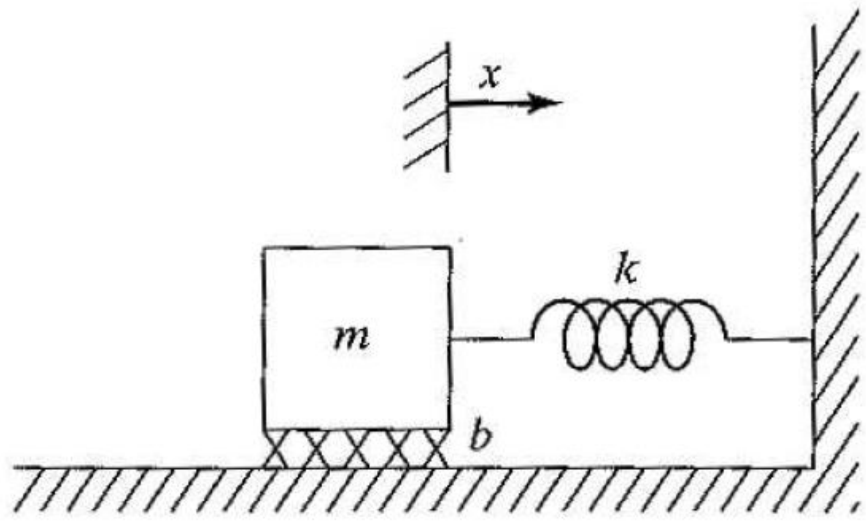


Figure 1: *Simple mass-spring-system.*

- Assumption for this problem: the origin of the coordinate system as shown in the figure corresponds to the resting position of the spring
  - Which means: $x_0 = 0$ and $\Delta x = x - x_0 = x$

P01

- We introduce a controlling force f to keep or bring (in a fast manner!) the rigid body in its resting position

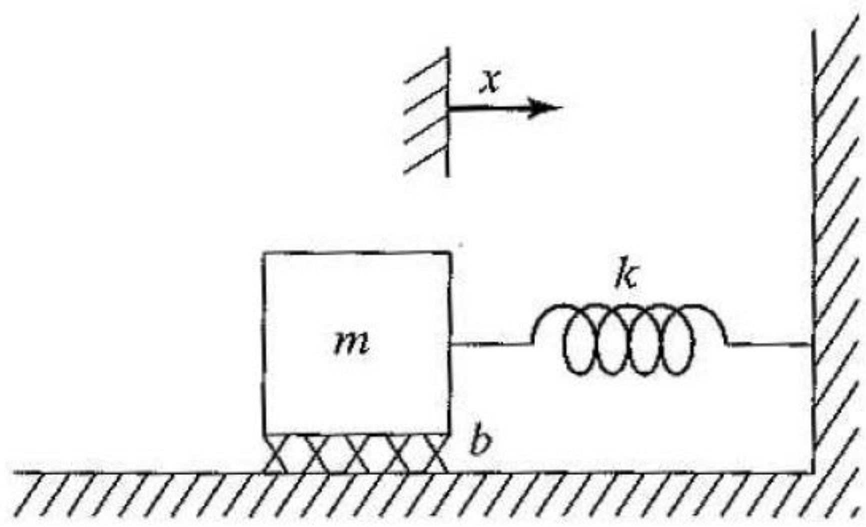
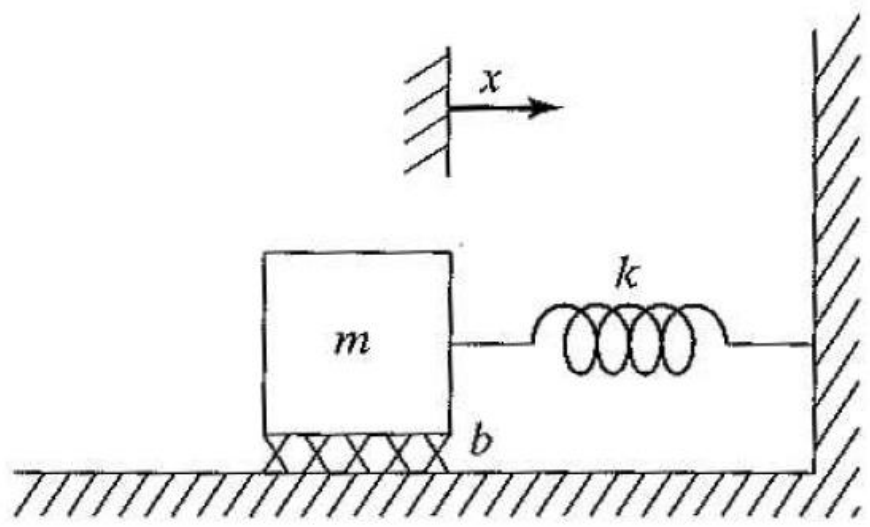


Figure 1: *Simple mass-spring-system.*

P01

- We introduce a controlling force f to keep or bring (in a fast manner!) the rigid body in its resting position
- From Netwon's second law:
  $f \ - m\ddot{x} - b\dot{x} - k\Delta x = 0$



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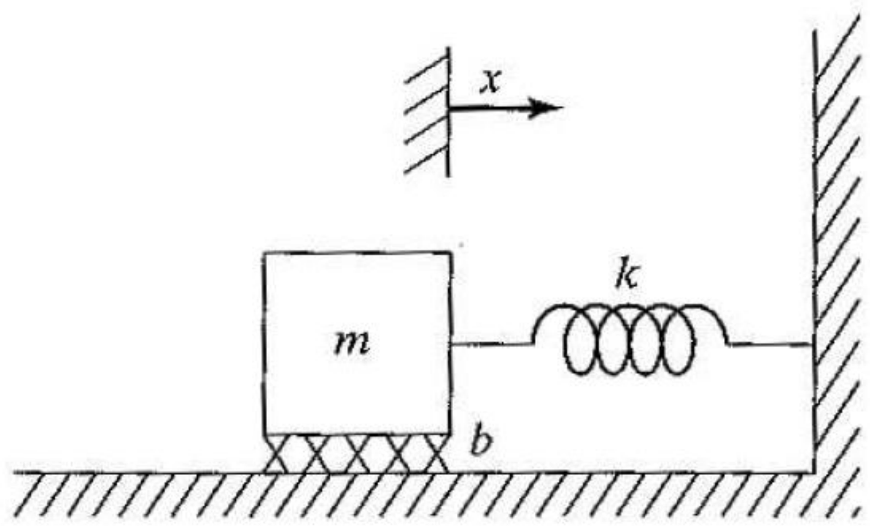

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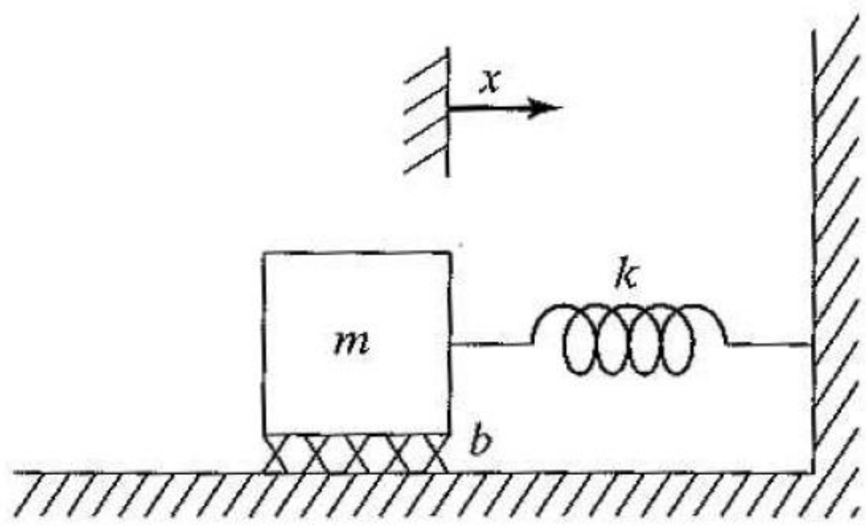
  $m\ddot{x} + b\dot{x} + kx = f$



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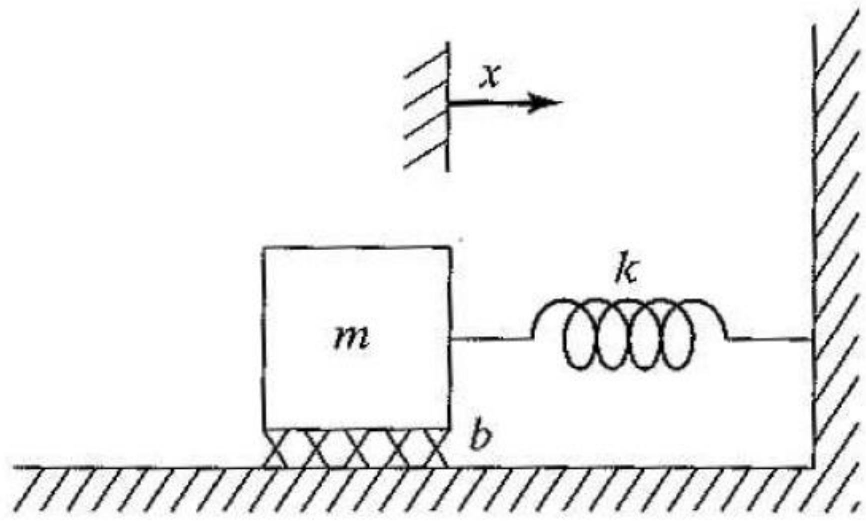

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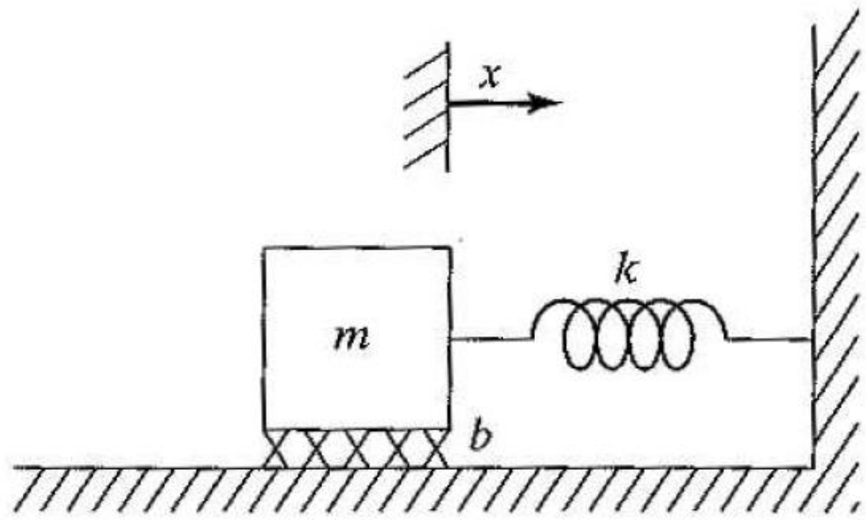
$$m\ddot{x} + (b + k_v)\dot{x} + \left(k + k_p\right)x = 0$$



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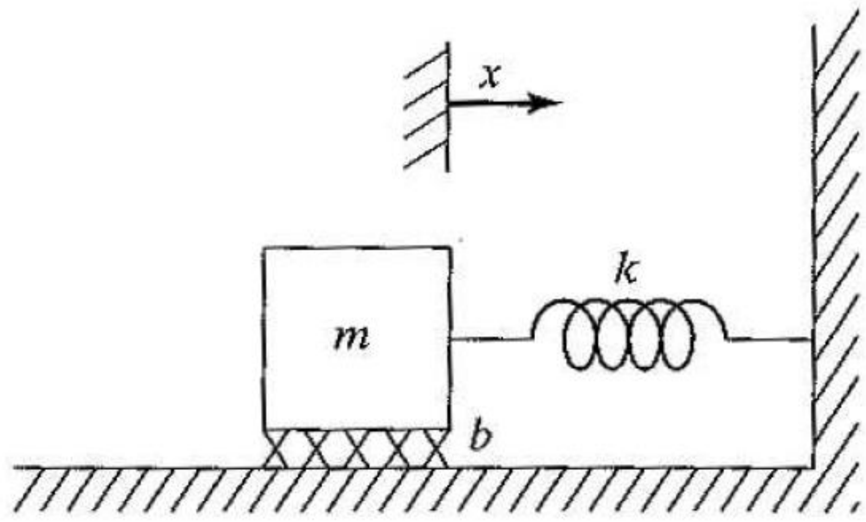
  $m\ddot{x} + (b + k_v)\dot{x} + \left(k + k_p\right)x = 0$
  $\ddot{x} + \frac{b+k_v}{m}\dot{x} + \frac{k+k_p}{m}x = 0$



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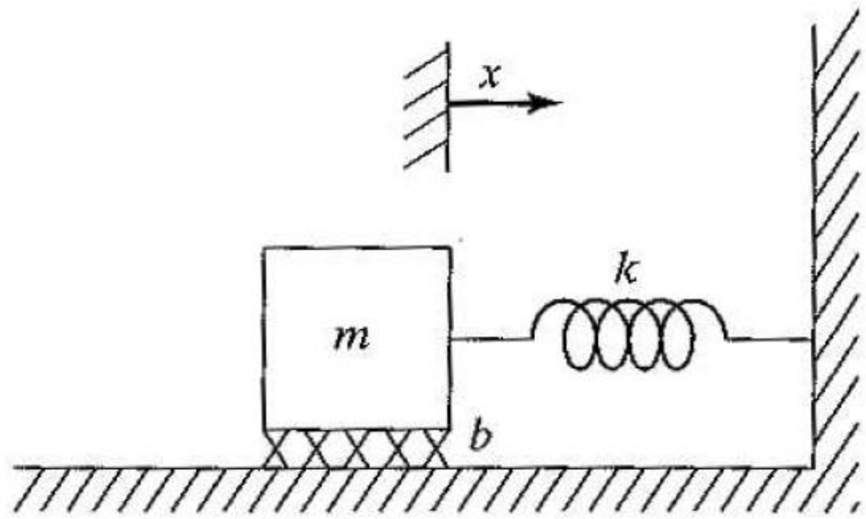


Figure 1: *Simple mass-spring-system.*

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

P01

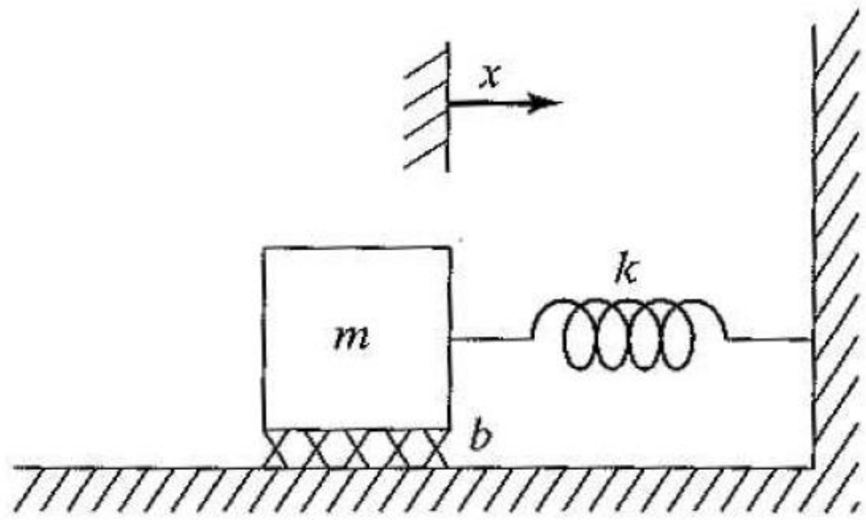


Figure 1: *Simple mass-spring-system.*

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\ddot{x} + \frac{(b+k_v)}{m}\dot{x} + \left(\frac{k+k_p}{m}\right)x = 0$$

P01

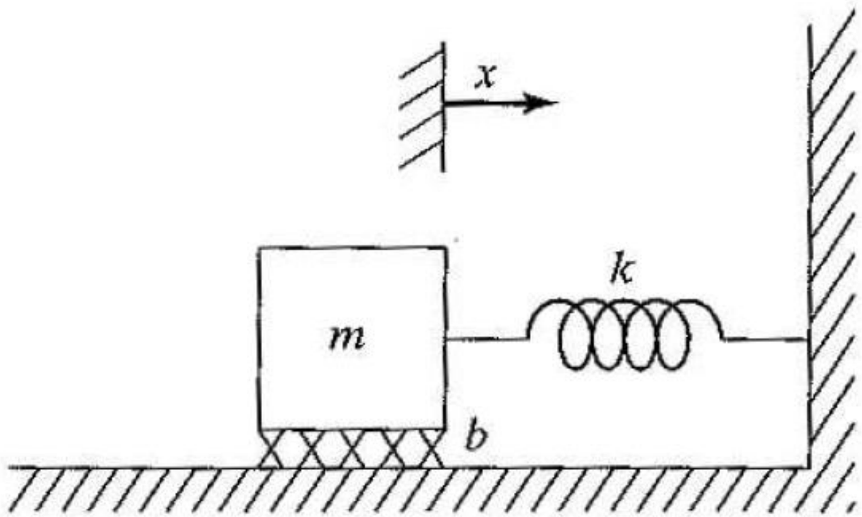


Figure 1: *Simple mass-spring-system.*

$$\left.\begin{aligned} & s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \\ & \ddot{x} + \frac{(b+k_v)}{m}\dot{x} + \left(\frac{k+k_p}{m}\right)x = 0 \end{aligned}\right\} \Rightarrow \omega_n^2 = \frac{k+k_p}{m}$$

P01

$$\omega_{res} \geq 2\omega_n = 2\sqrt{k+k_p}$$

$$\left.\begin{aligned} 3 &\geq \sqrt{5+k_p} \\ k_p + k \geq 0 &\Rightarrow k_p \geq -k \end{aligned}\right\} \Rightarrow \underline{\underline{4 \geq k_p \geq -5}}$$

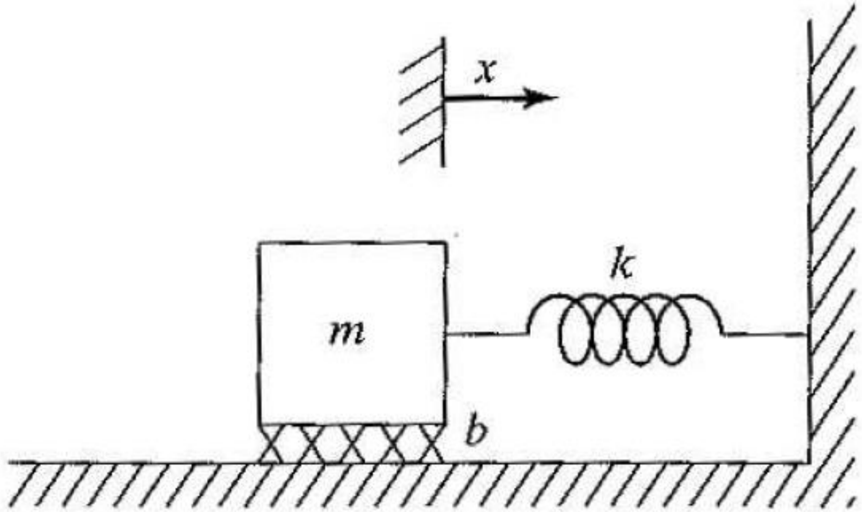


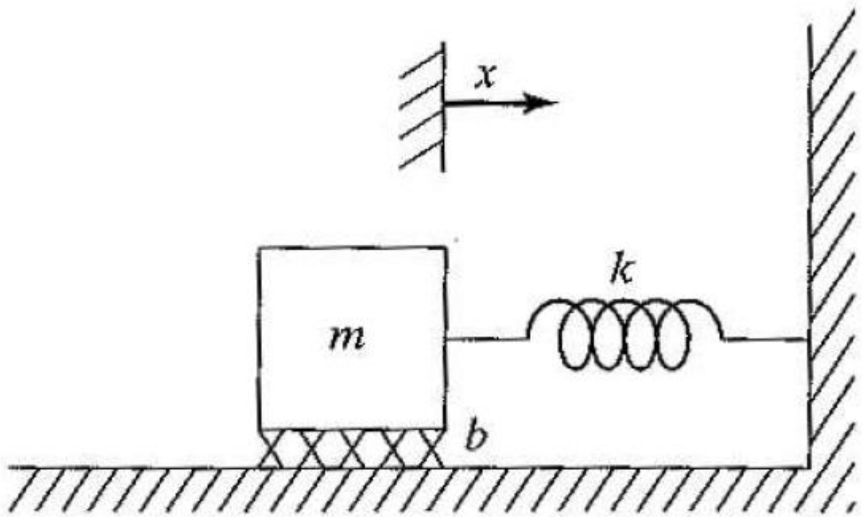
Figure 1: *Simple mass-spring-system.*

Note that we can always assume that k' and b' are positive values. Negative values would lead to an unstable system.

$$\left.\begin{aligned} & s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \\ & \ddot{x} + \frac{(b+k_v)}{m}\dot{x} + \left(\frac{k+k_p}{m}\right)x = 0 \end{aligned}\right\} \Rightarrow \omega_n^2 = \frac{k+k_p}{m}$$

P01

$$\Delta = (b + k_v)^2 - 4m(k + k_p) = 0$$ for critically damped



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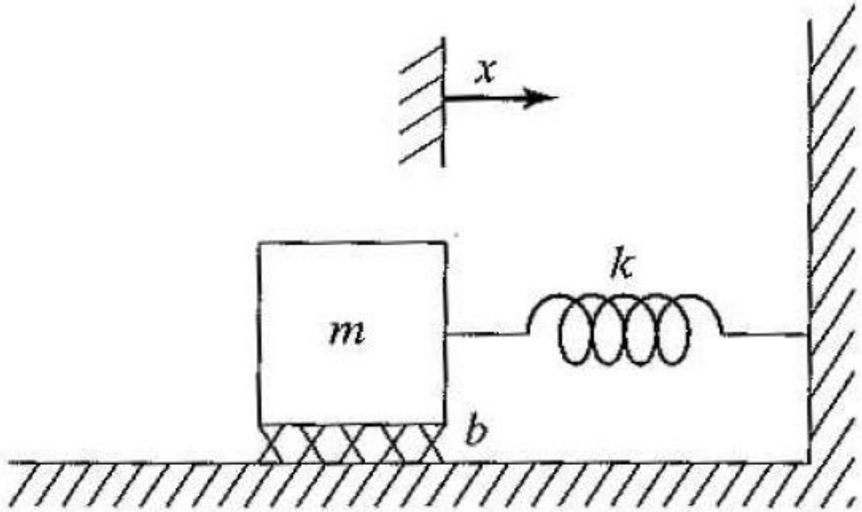
P01

$$\Delta = (b + k_v)^2 - 4m(k + k_p) = 0 \quad \text{for critically damped}$$

$$k_v = -b \pm \sqrt{4m(k + k_p)} = -b \pm 2\sqrt{k + k_p}$$



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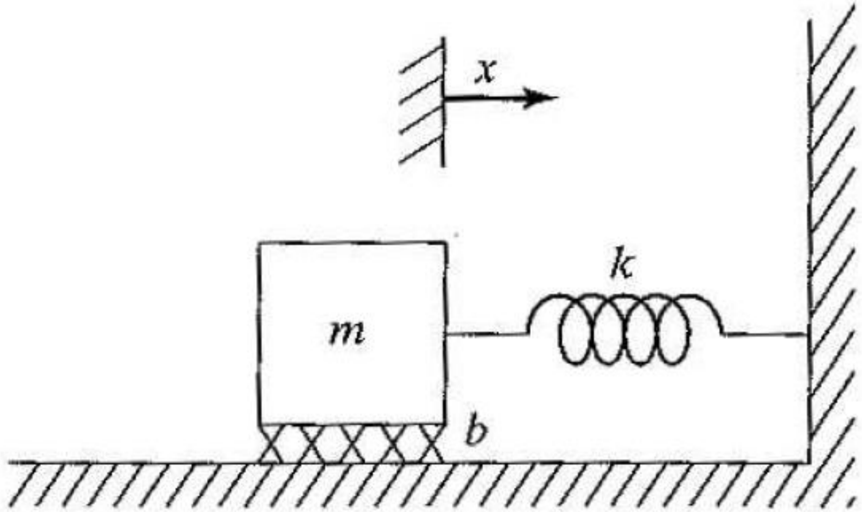


Figure 1: *Simple mass-spring-system.*

$$\left.\begin{aligned} &\to k_v = -b \pm 2\sqrt{k + k_p} \\ &k_v + b \geq 0 \Rightarrow k_v \geq -b \end{aligned}\right\} \Rightarrow \underline{\underline{k_v = -b + 2\sqrt{k + k_p}}}$$

Note that we can always assume that k' and b' are positive values. Negative values would lead to an unstable system.

P01

$$4 \geq k_p \geq -5$$

$$k_v = -b + 2\sqrt{k + k_p}$$

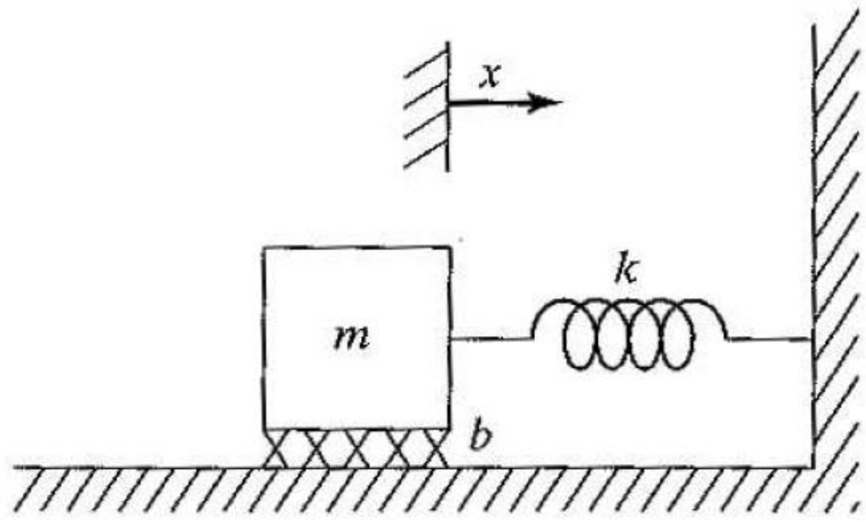


Figure 1: *Simple mass-spring-system.*

- For critically damping the system the above relations must hold.
- We always want a “stiff” system ⇒ $k_p + k$ and $k_v + b$ should be as large as possible

P01

$$4 \geq k_p \geq -5 \qquad k_v = -b + 2\sqrt{k + k_p}$$

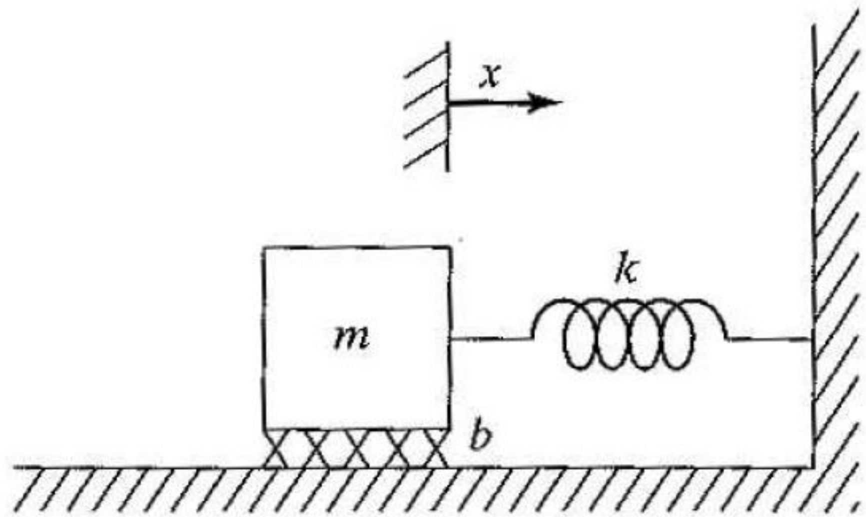


Figure 1: *Simple mass-spring-system.*

- For critically damping the system the above relations must hold.
- We always want a “stiff” system ⇒ $k_p + k$ and $k_v + b$ should be as large as possible
- Solution: $k_p = 4$ and $k_v = 2$

P02

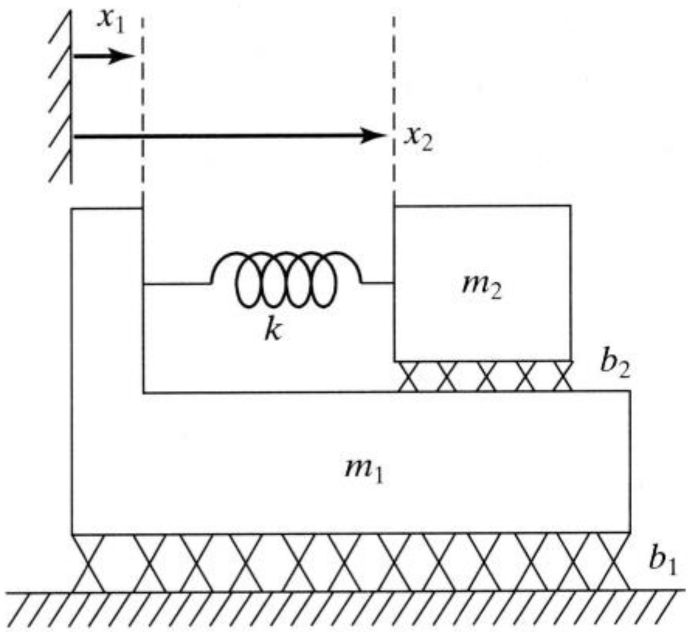


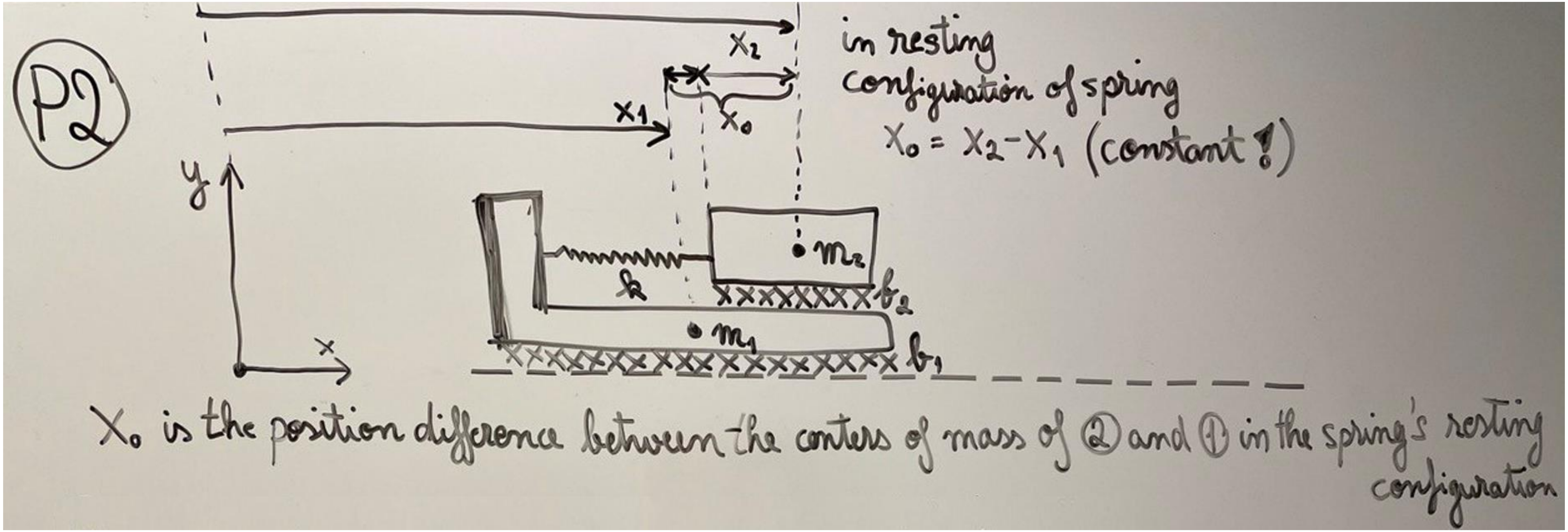
Figure 2: *Complex mass-spring-system (Problem 2)*

Problem 2

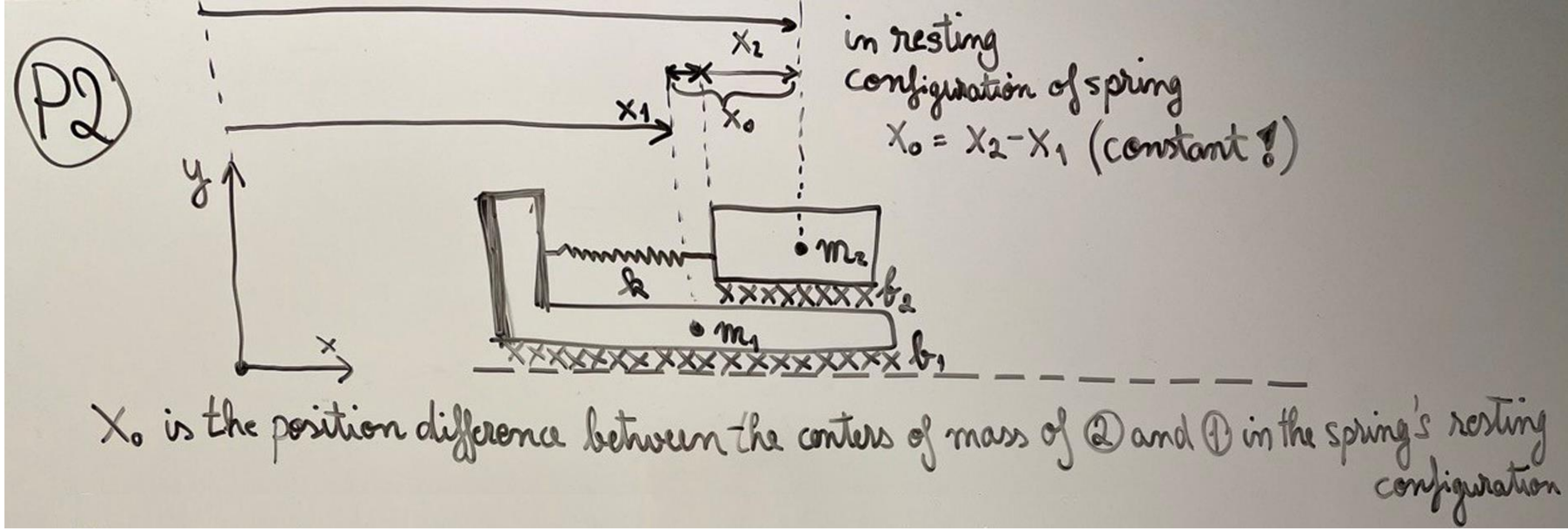
Derive a PD controlling scheme for the system shown in Figure 2 that allows following of trajectories for both objects and critically damps the error. The steps you should perform are the following:

- Determine forces that apply to both objects, derive equations of motion.
- Apply the control law partitioning principle. Explicitely show model-based portion and servo portion of the control law.
- Formulate the error equation.

P02

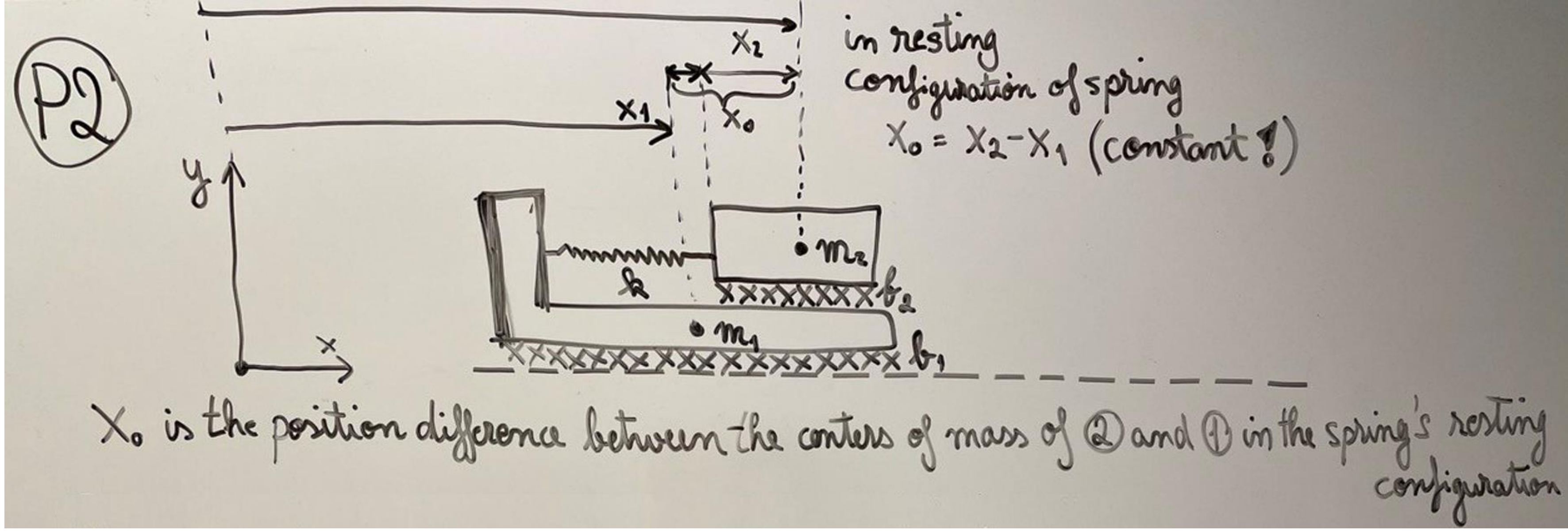

P2
x_2
x_1
x_0
in resting configuration of spring
$x_0 = x_2 - x_1$ (constant!)
y
x
k
m_2
b_2
m_1
b_1
x_0 is the position difference between the centers of mass of ② and ① in the spring's resting configuration

P02



- Step 1: Determine forces acting on the rigid bodies

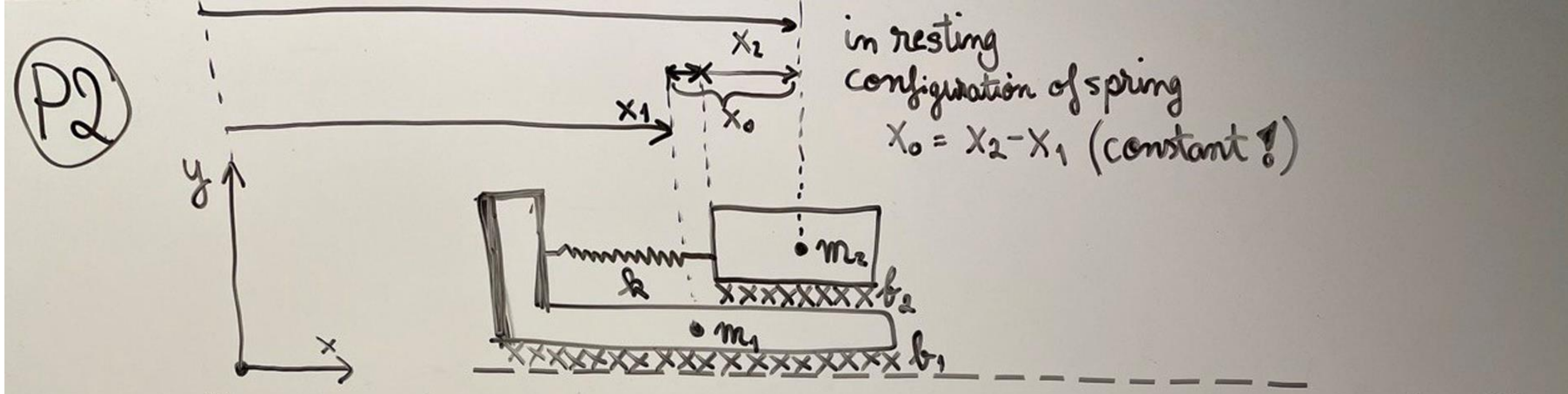
P02



- Step 1: Determine forces acting on the rigid bodies

②: – in x-direction:
- spring force: $-k(x_2-(x_1+x_0))$
- damping force: $-b_2(\dot{x}_2-\dot{x}_1)$
- inertia force: $-m_2\ddot{x}_2$

P02



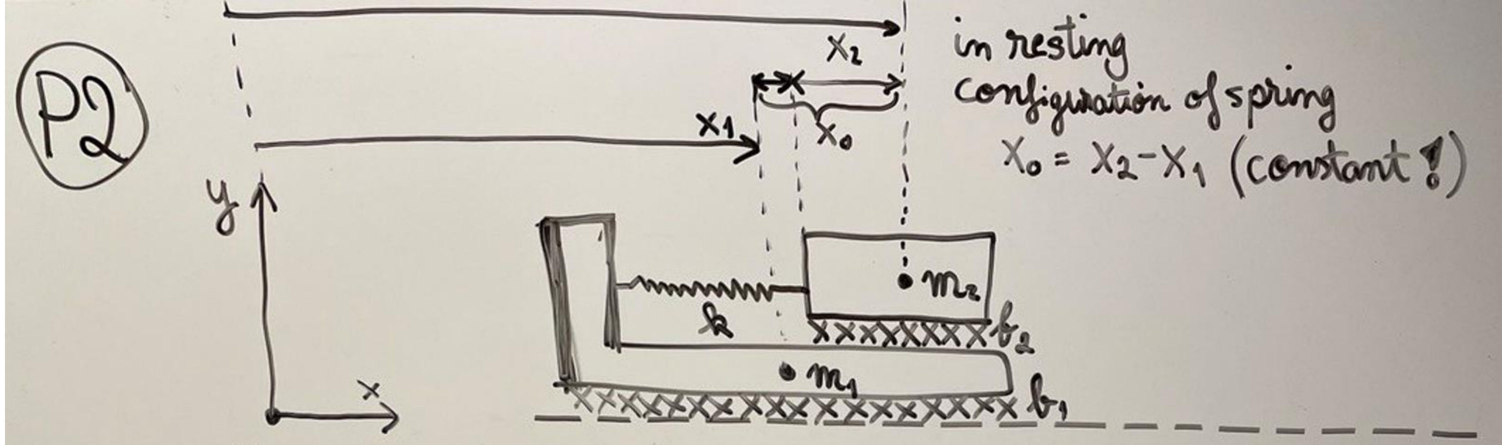
x_0 is the position difference between the centers of mass of ② and ① in the spring's resting configuration

- Step 1: Determine forces acting on the rigid bodies

② : – in x-direction:
- spring force: $-k(x_2-(x_1+x_0))$
- damping force: $-b_2(\dot{x}_2-\dot{x}_1)$
- inertia force: $-m_2\ddot{x}_2$

① : – in x-direction:
- damping force: $-b_1\dot{x}_1$
- inertia force: $-m_1\ddot{x}_1$
- damping force from ②: $b_2(\dot{x}_2-\dot{x}_1)$
- spring force from ②: $k(x_2-(x_1+x_0))$

P02



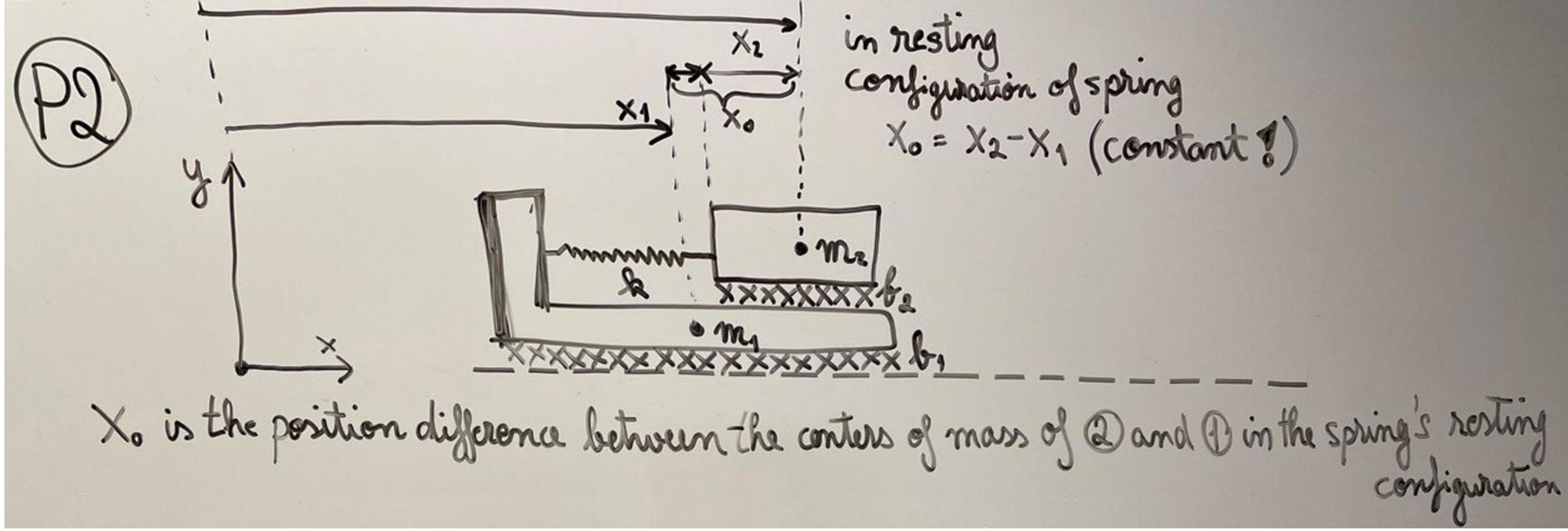
x_0 is the position difference between the centers of mass of ② and ① in the spring's resting configuration

- Step 2: Determine equations of motion (incl. controlling force)

$$\begin{cases} m_1 \ddot{x}_1 + b_1 \dot{x}_1 - b_2(\dot{x}_2 - \dot{x}_1) - k(x_2 - (x_1 + x_0)) = f_1 \\ m_2 \ddot{x}_2 + b_2(\dot{x}_2 - \dot{x}_1) + k(x_2 - (x_1 + x_0)) = f_2 \end{cases}$$

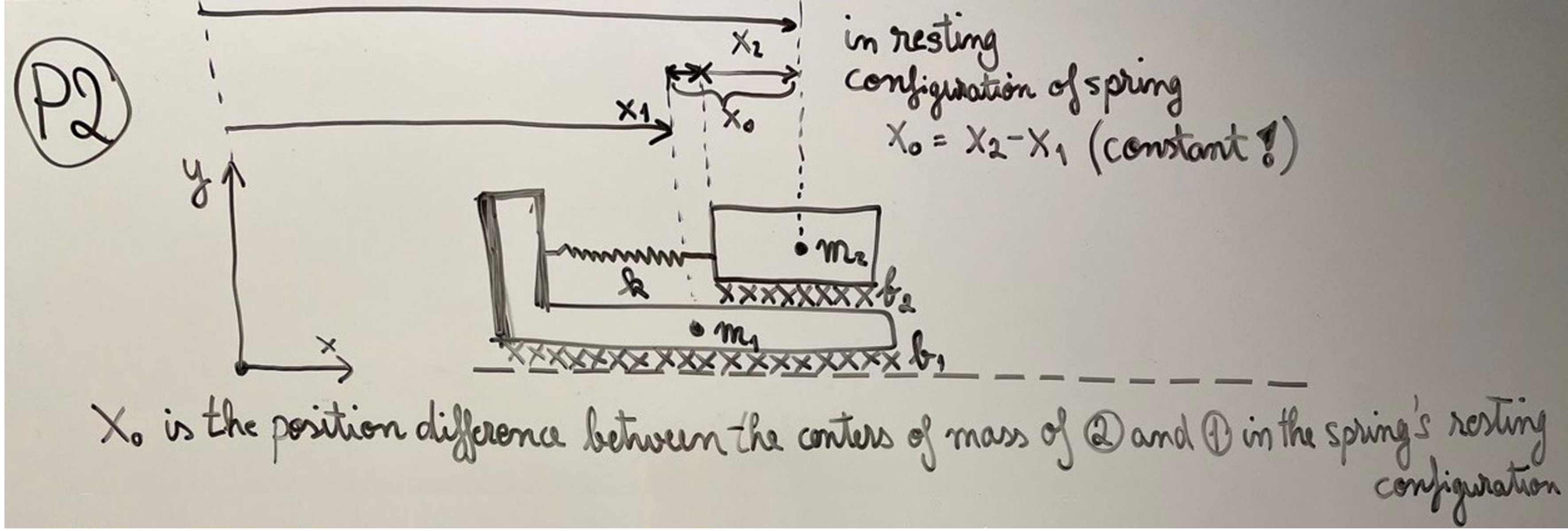
$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} b_1 + b_2 & -b_2 \\ -b_2 & b_2 \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} k x_0 \\ -k x_0 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

P02



- Step 3: Control Law Partitioning
  $M\ddot{x} + B\dot{x} + Kx + x_0 = f$
  Instead of modeling f as $\left(-K_v\dot{x} + K_px\right)$, we define $f = \alpha f' + \beta$
  and we choose f' such that it fits our controlling goal:
  "*[controller to] follow trajectories for both objects and critically damp the error*"

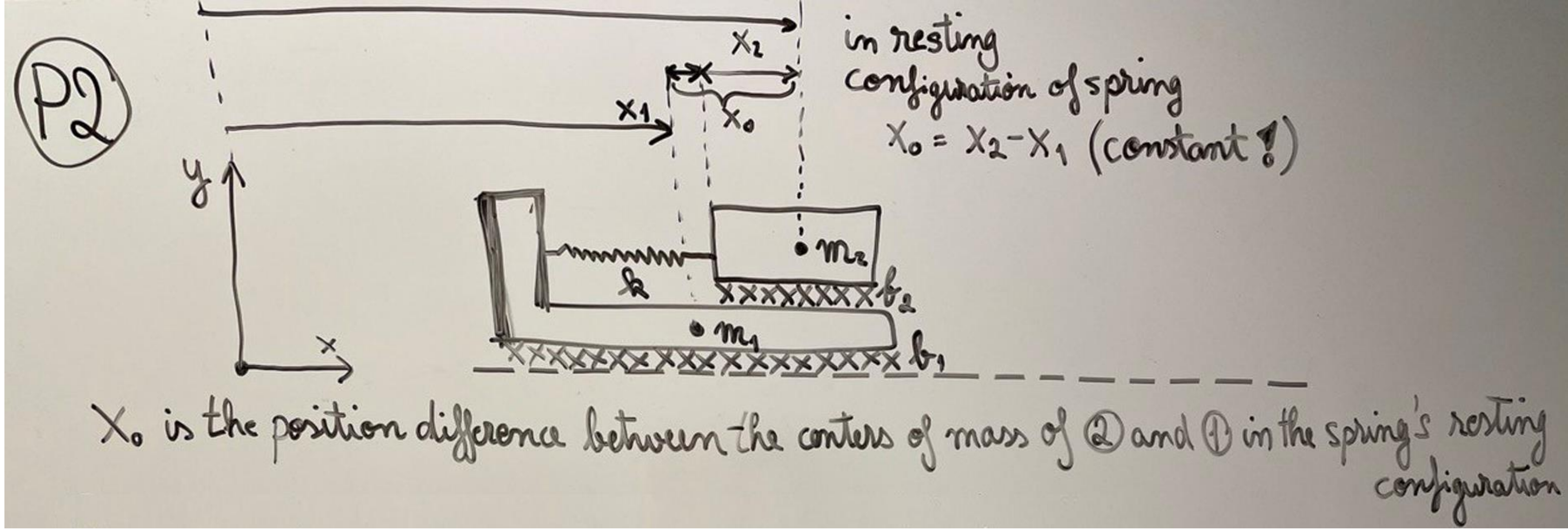
P02



- Step 3: Control Law Partitioning
  $M\ddot{x} + B\dot{x} + Kx + x_0 = f$
  Instead of modeling f as $-\left(K_v\dot{x} + K_p x\right)$, we define $f = \alpha f' + \beta$
  and we choose f' such that it fits our controlling goal:
  "*[controller to] follow trajectories for both objects and critically damp the error*"

  So we choose $f' = \ddot{x}_d + K_v\dot{e} + K_p e = \ddot{x}_d + K_v(\dot{x}_d - \dot{x}) + K_p(x_d - x)$

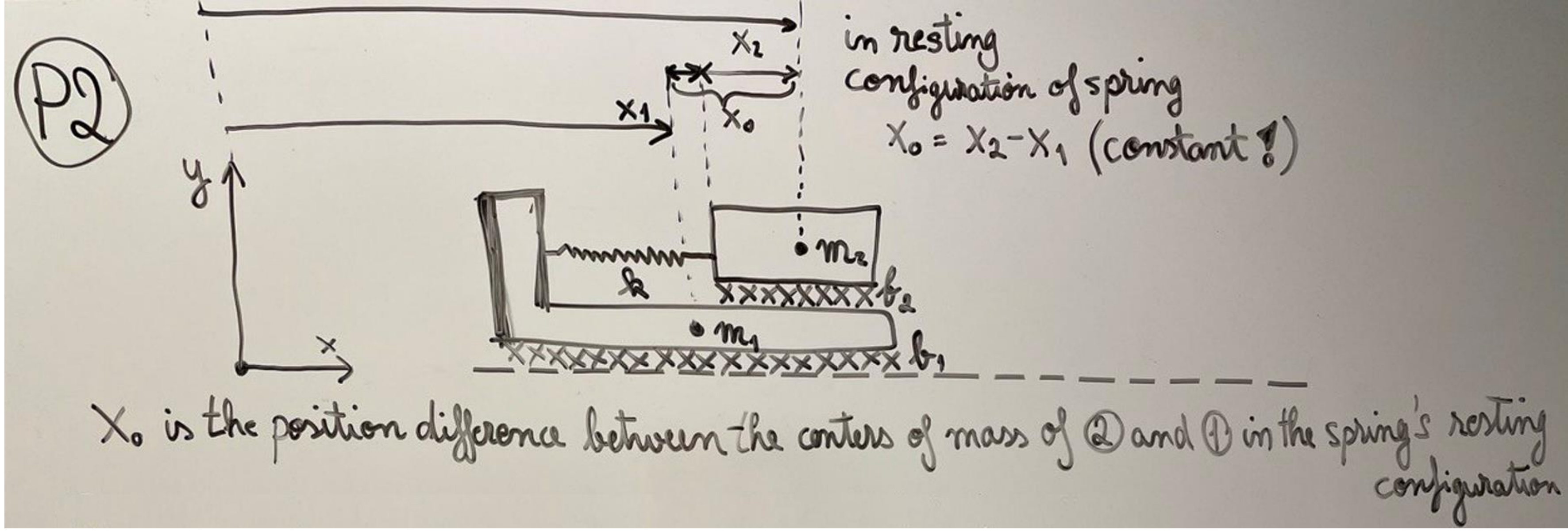
P02



- Step 3: Control Law Partitioning
  $M\ddot{x} + B\dot{x} + Kx + x_0 = \alpha(\ddot{x}_d + K_v\dot{e} + K_p e) + \beta$

  Model part: $\alpha = M, \beta = B\dot{x} + Kx + x_0$
  Servo part: $\ddot{x} = \ddot{x}_d + K_v\dot{e} + K_p e$

P02



- Step 4: Error equation
  $\ddot{x} = \ddot{x}_d + K_v\dot{e} + K_pe \Leftrightarrow$
  $\ddot{x}_d - \ddot{x} + K_v\dot{e} + K_pe = 0 \Leftrightarrow$
  $\ddot{e} + K_v\dot{e} + K_pe = 0$

- Choose $k_{vi} = 2\sqrt{k_{pi}}$ for each $i \in \{1, 2\}$ to critically damp the error