Tutorial Robotics IN2067

Exercise Sheet 01

Notation

Scalar: $a, b, c, \ldots \in \mathbb{R}$

Vector: $\vec{x}, \vec{y}, \vec{z}, \ldots \in \mathbb{R}^n$

Matrix: $\tilde{A}, \tilde{B}, \tilde{C}, \ldots \in \mathbb{R}^{m \times n}$

Usually:

t, τ is the time

$\vec{p}$ is a 3-dimensional position / point in space

$\vec{t}$ is a 3-dimensional translation vector

$\tilde{R}$ is a 3×3-dimensional rotation matrix

$\tilde{T}$ is a 4×4-dimensional coordinate transformation matrix

$\{CF\}$ is a coordinate frame called CF

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- **✖** The axis is pointing into the plane

In an $m \times n$ matrix $\tilde{A}$, the top-left value is $\tilde{A}_{1,1}$ or $a_{1,1}$ (indices start at 1, not 0)

P01

Problem 1

In classic geometry, a rotation is characterized as a length-preserving and orientation-preserving linear transformation. Starting from this characterization, it can also be shown that rotations preserve angles between vectors, which is a fact that you should use to solve the following problem.

The angle φ between two vectors $\vec{x}, \vec{y}$ in a standard euclidean space (with standard scalar product) can be defined through the relation:

$$\cos\varphi = \frac{\vec{x}^{\mathrm{T}}\vec{y}}{|\vec{x}| \cdot |\vec{y}|}$$

Using this definition, show that any rotation matrix R is orthonormal, or $R \cdot R^T = I_3$.

P01

$\tilde{R}$ is a length-preserving linear transformation:
$\forall \vec{x} \in \mathbb{R}^3 \colon \|\vec{x}\| = \left\|\tilde{R}\vec{x}\right\|$

$\tilde{R}$ is an orientation-preserving linear transformation:
The angle between two vectors $\vec{x}, \vec{y} \in \mathbb{R}^3$ and their rotated counterparts is the same:

$$\alpha(\vec{x}, \vec{y}) = \alpha\left(\tilde{R}\vec{x}, \tilde{R}\vec{y}\right) \Rightarrow \frac{\vec{x}^T\vec{y}}{\|\vec{x}\|\|\vec{y}\|} = \frac{\left(\tilde{R}\vec{x}\right)^T\left(\tilde{R}\vec{y}\right)}{\left\|\tilde{R}\vec{x}\right\|\left\|\tilde{R}\vec{y}\right\|} \Rightarrow$$

$$\frac{\vec{x}^T\vec{y}}{\|\vec{x}\|\|\vec{y}\|} = \frac{\left(\tilde{R}\vec{x}\right)^T\left(\tilde{R}\vec{y}\right)}{\|\vec{x}\|\|\vec{y}\|} \Rightarrow \vec{x}^T\vec{y} = \left(\tilde{R}\vec{x}\right)^T\left(\tilde{R}\vec{y}\right) \Rightarrow \vec{x}^T\vec{y} = \vec{x}^T\tilde{R}^T\tilde{R}\vec{y}$$

P01

$\vec{x}^T\vec{y} = \vec{x}^T\tilde{R}^T\tilde{R}\vec{y}, \forall\vec{x},\vec{y} \in \mathbb{R}^3$

Let $\vec{x} \in \left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ and $\vec{y} \in \left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

E.g. for $\vec{x} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\vec{y} = \begin{pmatrix}0\\0\\1\end{pmatrix}: 0 = \begin{pmatrix}0\\1\\0\end{pmatrix}^T\begin{pmatrix}0\\0\\1\end{pmatrix} = \vec{x}^T\vec{y} = \vec{x}^T\tilde{R}^T\tilde{R}\vec{y} =$

$= \begin{pmatrix}0\\1\\0\end{pmatrix}^T\tilde{R}^T\tilde{R}\begin{pmatrix}0\\0\\1\end{pmatrix} = \widetilde{M}_{2,3}$, where $\tilde{R}^T\tilde{R} = \widetilde{M}$.

element in Row 2, 3

P01

It is now easy to see that when $\vec{x} \neq \vec{y}$, the respective entry in $\widetilde{M}$ is 0 and when $\vec{x} = \vec{y}$, the respective entry in $\widetilde{M}$ is 1, making $\widetilde{M}$ equal to I_3

$\Rightarrow \tilde{R}^T \tilde{R} = I_3$

$\Rightarrow \tilde{R}\tilde{R}^T \tilde{R} = \tilde{R}$

$\Rightarrow \tilde{R}\tilde{R}^T \tilde{R}\tilde{R}^{-1} = \tilde{R}\tilde{R}^{-1} = I_3$

$\Rightarrow \tilde{R}\tilde{R}^T = I_3$ q.e.d.

P01

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$\Rightarrow \tilde{R} \tilde{R}^T = I_3$ q.e.d.

Also, because of $\tilde{R}^T \tilde{R} = I_3$ we know that $\tilde{R}^{-1} = \tilde{R}^T$

P02

Problem 2

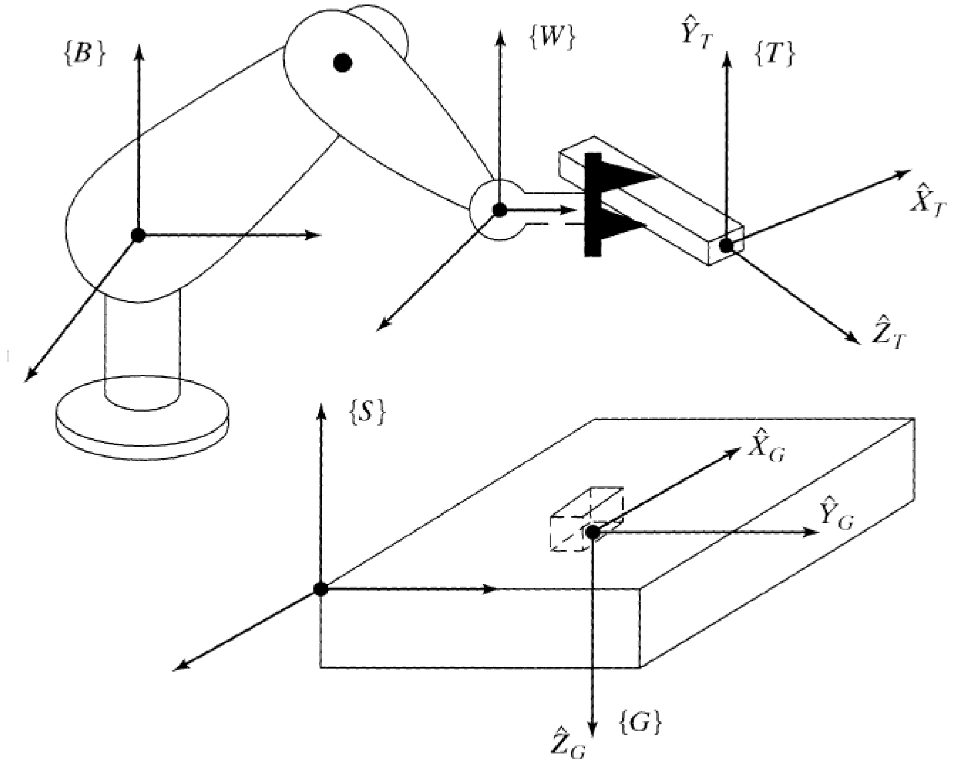


Figure 1: *Coordinate systems (Problem 2)*

Consider the situation shown in Figure 1. There are 5 coordinate frames

$$\{B\}, \{W\}, \{T\}, \{S\}, \{G\}.$$

We are interested in determining the transformation ${}^{W}_{T}\mathrm{T}$. This can be motivated as follows: Imagine that the robot has previously picked up the tool, and that the position of the tool in the gripper is not known very well, but we would like to know it exactly. However, the robot might be equipped with a force sensor and it could try to "feel" around to fit the tool to the goal. Assuming that the transformations ${}^{B}_{S}\mathrm{T}, {}^{S}_{G}\mathrm{T}, {}^{B}_{W}\mathrm{T}$ are known, and that the coordinate frames $\{G\}$ (goal) and $\{T\}$ (tool) are "calibrated", i.e., coincident, how can we determine the unknown transformation ${}^{W}_{T}\mathrm{T}$?

P2

$$ {}^{B}_{A}T = \begin{pmatrix} {}^{B}_{A}R & {}^{B}_{A}t \\ 0\ 0\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{4\times 4} \ ; \quad \begin{pmatrix} {}^{A}_{B}R & {}^{A}_{B}t \\ 0\ 0\ 0 & 1 \end{pmatrix} = {}^{A}_{B}T = \left({}^{B}_{A}T\right)^{-1} = \begin{pmatrix} {}^{B}_{A}R^{T} & -{}^{B}_{A}R^{T}\,{}^{B}_{A}t \\ 0 & 1 \end{pmatrix} $$

$$ {}^{A}_{B}T = {}^{B}_{A}T^{-1} $$

P2 $\;{}^{B}_{A}T = \begin{pmatrix} {}^{B}_{A}R & {}^{B}_{A}t \\ 0\,0\,0 & 1 \end{pmatrix} \in \mathbb{R}^{4\times4}$; $\begin{pmatrix} {}^{A}_{B}R & {}^{A}_{B}t \\ 0\,0\,0 & 1 \end{pmatrix} = {}^{A}_{B}T = \left({}^{B}_{A}T\right)^{-1} = \begin{pmatrix} {}^{B}_{A}R^{T} & -{}^{B}_{A}R^{T}\,{}^{B}_{A}t \\ 0 & 1 \end{pmatrix}$

$${}^{B}p = {}^{B}_{A}T\,{}^{A}p = {}^{B}_{A}R\,{}^{A}p + {}^{B}_{A}t$$

$$\Downarrow$$

$${}^{B}p - {}^{B}_{A}t = {}^{B}_{A}R\,{}^{A}p$$

$$\Updownarrow$$

$$\left({}^{B}_{A}R\right)^{T}\left({}^{B}p - {}^{B}_{A}t\right) = {}^{A}p \iff$$

$$\left.\begin{array}{l} \iff {}^{A}p = {}^{B}_{A}R^{T}\,{}^{B}p - {}^{B}_{A}R^{T}\,{}^{B}_{A}t \\ {}^{A}p = {}^{A}_{B}R\,{}^{B}p + {}^{A}_{B}t \end{array}\right\} \Rightarrow \begin{array}{l} {}^{A}_{B}R = {}^{B}_{A}R^{T} \\ {}^{A}_{B}t = -{}^{B}_{A}R^{T}\,{}^{B}_{A}t \end{array}$$

(P2) $\;{}^{B}_{A}T = \begin{pmatrix} {}^{B}_{A}R & {}^{B}_{A}t \\ 000 & 1 \end{pmatrix} \in \mathbb{R}^{4\times4}$; $\begin{pmatrix} {}^{A}_{B}R & {}^{A}_{B}t \\ 000 & 1 \end{pmatrix} = \boxed{{}^{A}_{B}T = \left({}^{B}_{A}T\right)^{-1}} = \begin{pmatrix} {}^{B}_{A}R^{T} & -{}^{B}_{A}R^{T}\,{}^{B}_{A}t \\ 0 & 1 \end{pmatrix}$

$${}^{A}_{B}T = {}^{B}_{A}T^{-1}$$

$$\underline{{}^{B}p = {}^{B}_{A}T\,{}^{A}p = {}^{B}_{A}R\,{}^{A}p + {}^{B}_{A}t}$$

$$\Downarrow$$

$${}^{B}p - {}^{B}_{A}t = {}^{B}_{A}R\,{}^{A}p$$

$$\Updownarrow$$

$$\left({}^{B}_{A}R\right)^{T}\left({}^{B}p - {}^{B}_{A}t\right) = {}^{A}p \Leftrightarrow$$

$$\boxed{\begin{aligned} \Leftrightarrow {}^{A}p &= {}^{B}_{A}R^{T}\,{}^{B}p - {}^{B}_{A}R^{T}\,{}^{B}_{A}t \\ {}^{A}p &= {}^{A}_{B}R\,{}^{B}p + {}^{A}_{B}t \end{aligned}} \Rightarrow \begin{aligned} {}^{A}_{B}R &= {}^{B}_{A}R^{T} \\ {}^{A}_{B}t &= -{}^{B}_{A}R^{T}\,{}^{B}_{A}t \end{aligned}$$

Let $\{A\}, \{B\}$ be coordinate frames.

$\forall \{C\}$: ${}^{B}_{A}T = {}^{B}_{C}T\;{}^{C}_{A}T$

${}^{B}_{S}T,\; {}^{S}_{G}T,\; {}^{B}_{W}T$; $\{G\} = \{T\}$

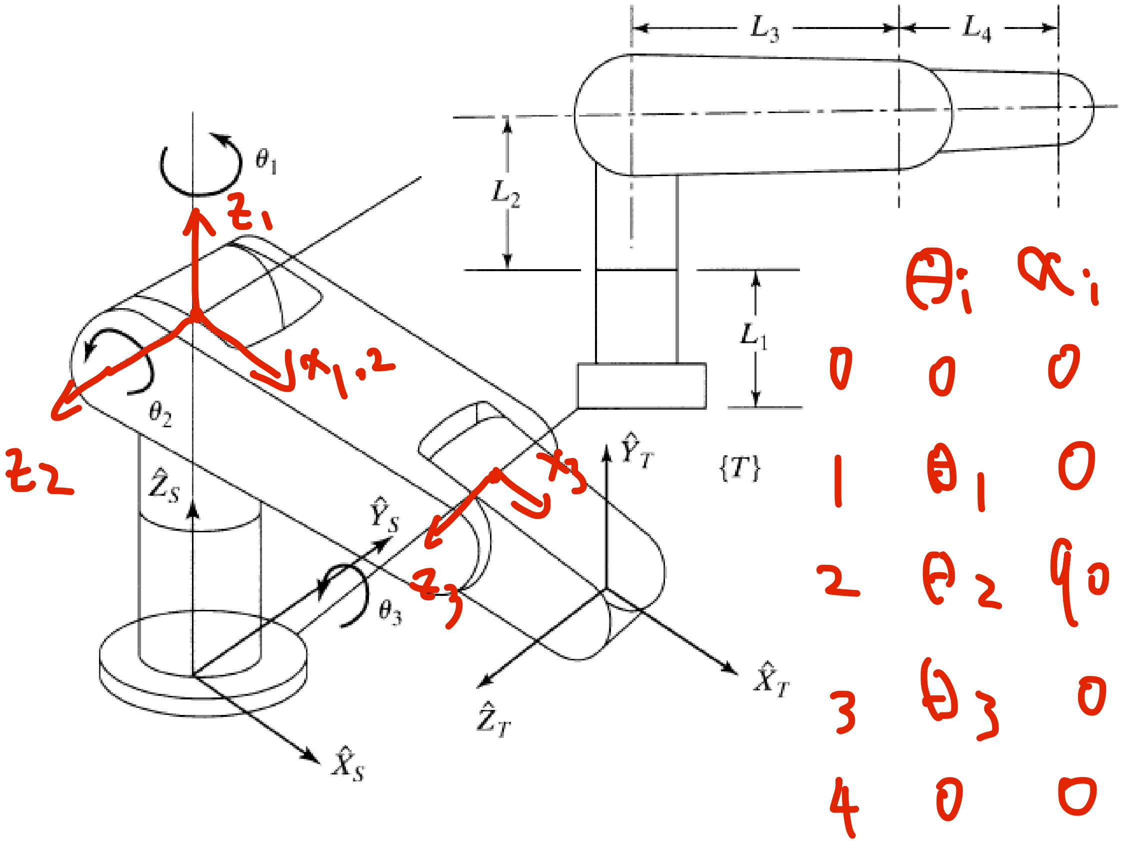
${}^{W}_{T}T$

$${}^{W}_{T}T = {}^{W}_{G}T = \left.\begin{matrix} {}^{W}_{X}T\;{}^{X}_{G}T \\ X = S \end{matrix}\right\} \Rightarrow {}^{W}_{T}T = {}^{W}_{S}T\;{}^{S}_{G}T = \left.\begin{matrix} {}^{W}_{Y}T\;{}^{Y}_{S}T\;{}^{S}_{G}T \\ Y = B \end{matrix}\right\} \Rightarrow {}^{W}_{T}T = {}^{W}_{B}T\;{}^{B}_{S}T\;{}^{S}_{G}T =$$

$$= {}^{B}_{W}T^{-1}\;{}^{B}_{S}T\;{}^{S}_{G}T$$

Problem 3

P03



	θ_i	α_i	a_i	d_i
0	0	0	0	0
1	θ_1	0	0	l_1+l_2
2	θ_2	90	0	0
3	θ_3	0	l_3	0
4	0	0	l_4	0

Figure 2: *Nonplanar 3R Robot (Problem 3)*

Figure 2 shows a robot arm with three rotational axes, thus three degrees of freedom. As shown in the diagram, all robot joints are in zero position. Determine the positions of the robot arm coordinate systems according to the Denavit-Hartenberg convention and derive the corresponding coordinate transformations.

P03

- DH-convention simplifies general 3D transformations to a sequence of:
  a) Rotation along current x-Axis (x_{i-1}) to make z_{i-1} and z_i parallel/aligned
  b) Translation along current x-Axis (x_{i-1})
  c) Translation along new z-Axis (z_i; obtained from z_{i-1} by the rotation in a)
  d) Rotation along new z-Axis (z_i)

- ${}^{i}_{i-1}T(\alpha, a, d, \theta) = rot(z_i, \theta) \cdot trans(z_i, d) \cdot trans(x_{i-1}, a) \cdot rot(x_{i-1}, \alpha)$
  used to obtain a point p's coordinates as seen from CF $\{i\}$ when the point is represented in $\{i-1\}$:
  ${}^{i}p = {}^{i}_{i-1}T \cdot {}^{i-1}p$

- ${}^{i-1}_{i}T(\alpha, a, d, \theta) = rot(x_{i-1}, \alpha) \cdot trans(x_{i-1}, a) \cdot trans(z_i, d) \cdot rot(z_i, \theta)$
  used to obtain a point p's coordinates as seen from CF $\{i-1\}$ when the point is represented in $\{i\}$:
  ${}^{i-1}p = {}^{i-1}_{i}T \cdot {}^{i}p$

P03

$$
{}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- ${}^{i}_{i-1}T$ will be useful later for propagating forces and velocities joint to joint from the base until the robot's gripper.
- $\boxed{{}^{i-1}_{i}T}$ is useful for determining the forward kinematics of the robot

$$
{}^{i-1}_{i}T(\alpha, a, d, \theta) = rot(x,\alpha)\cdot trans(x,a)\cdot trans(z,d)\cdot rot(z,\theta) =
$$

$$
= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha & -s\alpha & 0 \\ 0 & s\alpha & c\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \boxed{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \boxed{d} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\theta & -s\theta & 0 & 0 \\ s\theta & c\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} c\theta & -s\theta & 0 & a \\ s\theta c\alpha & c\theta c\alpha & -s\alpha & -d\, s\alpha \\ s\theta s\alpha & c\theta s\alpha & c\alpha & d\, c\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

P03

Procedure to determine coordinate frames:

- Step 1: Determine z-Axis for each joint
  - If the joint is prismatic: z-Axis is along the direction of movement
  - If the joint is rotational: z-Axis is along the direction of rotation

- 如果z i-1和z i是一对斜线：确定垂直于两条斜线的直线（距离最短的直线）。这条线与zi-1的交点是坐标框架i - 1的原点，x i-1在这条线上，朝向zi。

- 如果zi-1和zi只相交于一点：该点就是坐标系i - 1的原点。x i-1是z i-1与z i之间的交积。选择x i-1的方向，使得DH参数的α i-1参数>0 .

- 如果z i-1和z i平行：我们选择i - 1的原点，使DH参数的d i参数为0 . x i-1是沿着从z i-1到z i的线。

- 如果z i-1和z i重合：i - 1的原点就是i的原点，x i-1是任意的，但为了简单起见，选择x i-1 = x i 。

P03

Procedure to determine coordinate frames:

- Step 2: Determine origin and x-Axis for each joint/coordinate frame
  Look at the relation between z_{i-1} and z_i:
  - If z_{i-1} and z_i are a pair of skew lines: determine the line perpendicular on both skew lines (line with shortest distance). The intersection of this line and z_{i-1} is the origin of the coordinate frame $\{i-1\}$ and x_{i-1} is on this line towards z_i.
  - If z_{i-1} and z_i intersect in only one point: the point is the origin of the coordinate frame $\{i-1\}$. x_{i-1} is the cross-product between z_{i-1} and z_i. Choose the direction of x_{i-1} such that the α_{i-1} parameter of the DH parameters is > 0.
  - If z_{i-1} and z_i are parallel: we choose the origin of $\{i-1\}$ such that the d_i parameter of the DH parameters is 0. x_{i-1} is along the line from z_{i-1} to z_i.
  - If z_{i-1} and z_i coincide: the origin of $\{i-1\}$ is the origin of $\{i\}$ and x_{i-1} is arbitrary, but for simplicity, choose $x_{i-1} = x_i$.

- 如果z_{i}-1和z_{i}是一对倾斜线：每个参数都可以确定（对未来数据没有依赖性）。

- 如果z_i-1和z_i只相交于一个点：每个参数都可以确定（对未来数据没有依赖性）。

- 如果z_i-1和z_i平行：{ i - 1}的原点取决于{ i }的原点转到下一个坐标框架{ i }，然后返回{ i - 1}。

- 如果z_i-1和z_i重合：{ i - 1}的原点和x轴依赖于{ i }。转到下一个坐标框架{ i }，然后返回到{ i - 1}。

P03

Procedure to determine coordinate frames:

- Step 2: Determine origin and x-Axis for each joint/coordinate frame
  Look at the relation between z_{i-1} and z_i:
  - If z_{i-1} and z_i are a pair of skew lines: every parameter can be determined (*there is no dependency on future data*).
  - If z_{i-1} and z_i intersect in only one point: every parameter can be determined (*there is no dependency on future data*).
  - If z_{i-1} and z_i are parallel: the origin of $\{i-1\}$ depends on the origin of $\{i\}$
    Go to the next coordinate frame $\{i\}$ and then return to $\{i-1\}$.
  - If z_{i-1} and z_i coincide: the origin and x-axis of $\{i-1\}$ depend on $\{i\}$
    Go to the next coordinate frame $\{i\}$ and then return to $\{i-1\}$.

P03

Procedure to determine coordinate frames:

- Step 3: Determine the y-Axis for each joint as $y_{i-1} = z_{i-1} \times x_{i-1}$.

P03

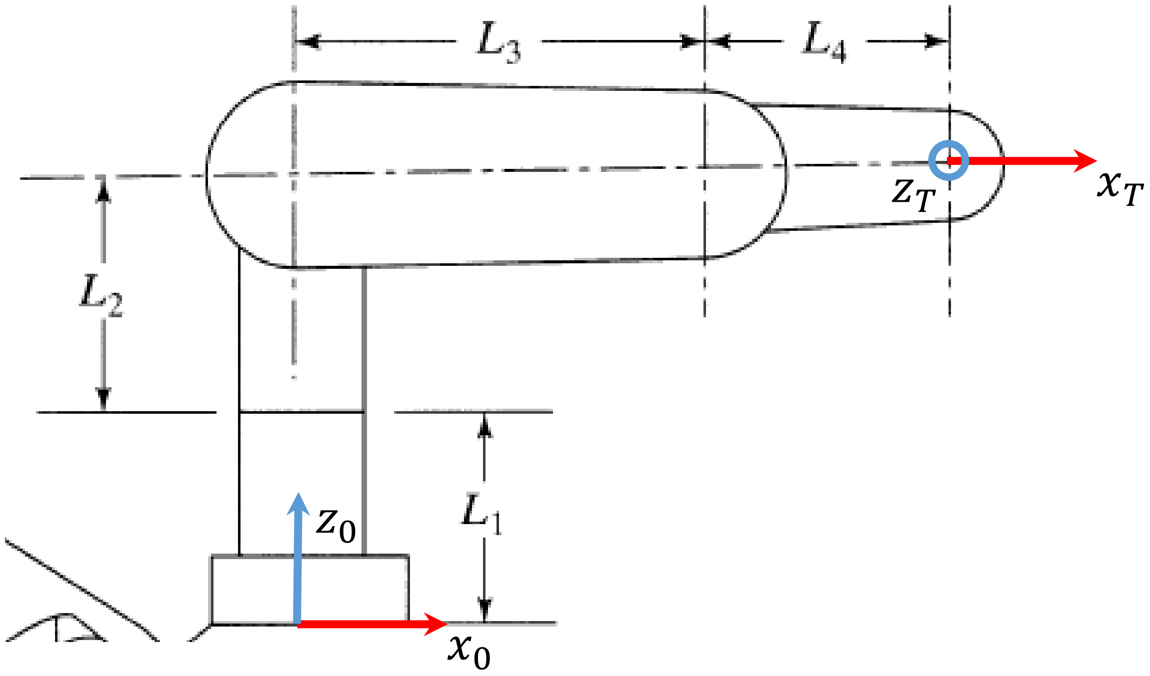
Procedure to determine coordinate frames:

- Step 3: Determine the y-Axis for each joint as $y_{i-1} = z_{i-1} \times x_{i-1}$.

- Step 4: Fill out the DH parameters table

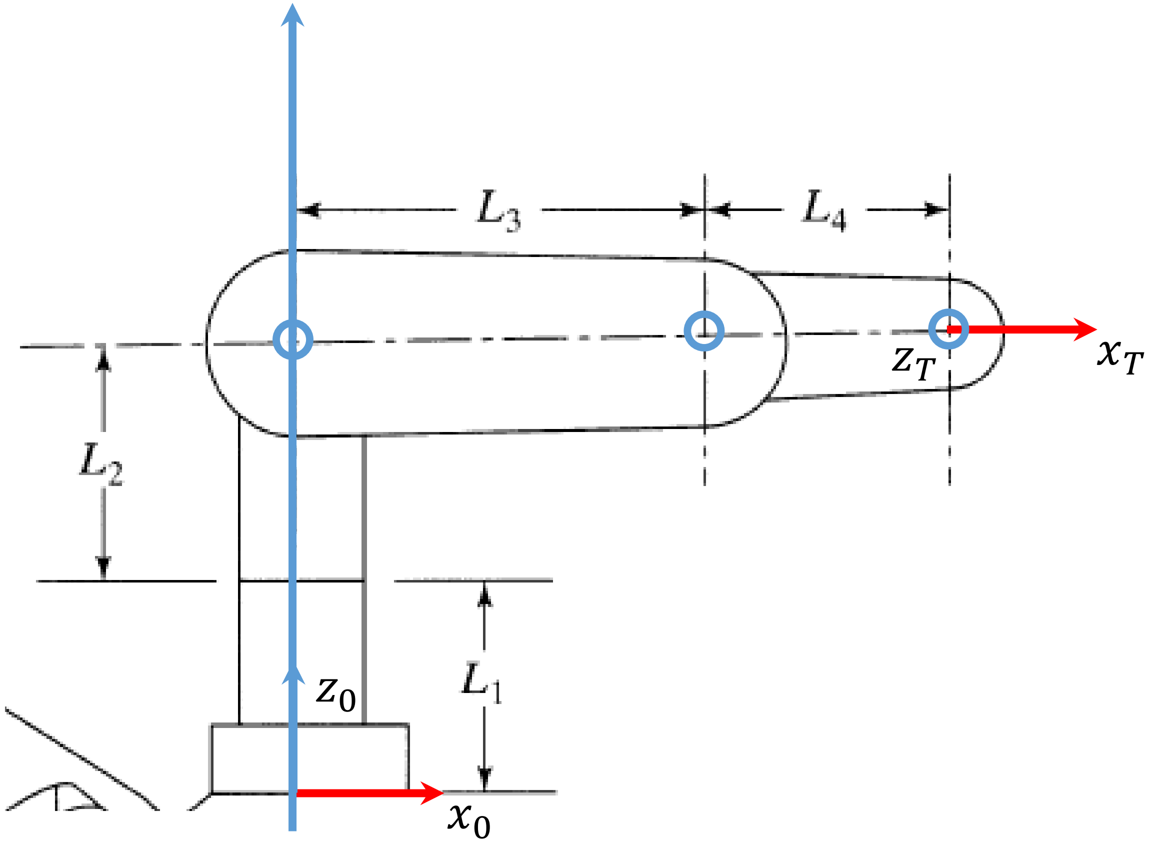
P03

- Step 0: Draw $\{0\}$ and $\{T\}$



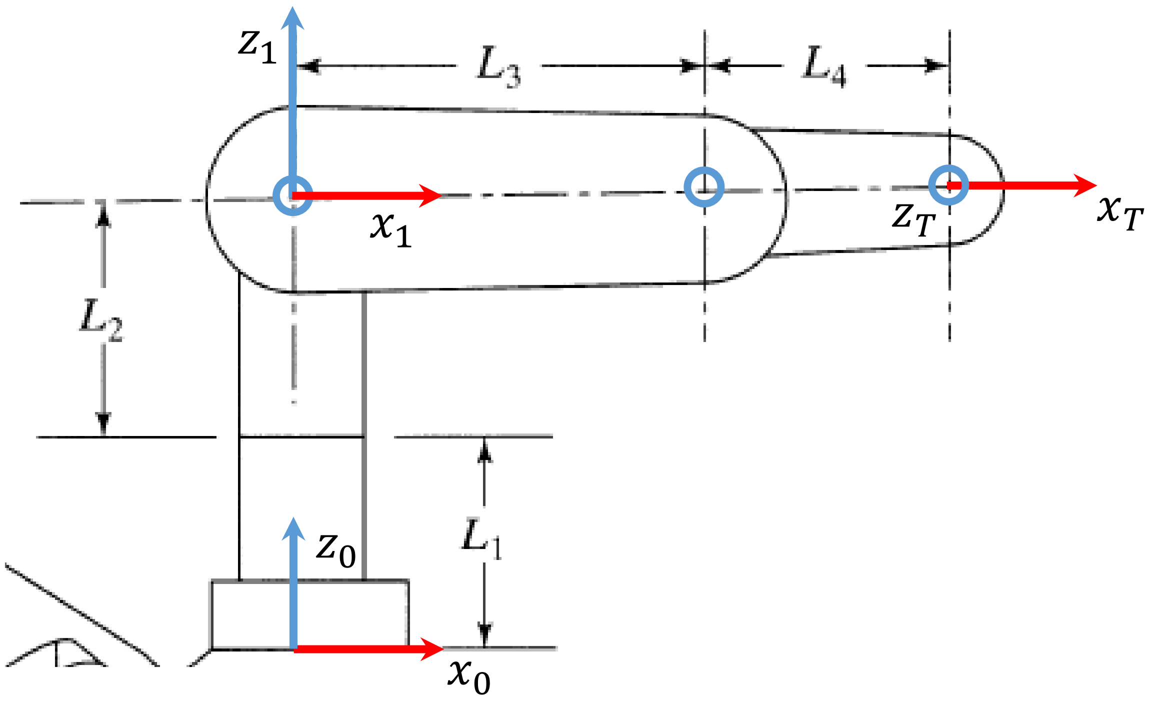
P03

- Step 0: Draw $\{0\}$ and $\{T\}$
- Step 1: Determine z-axes



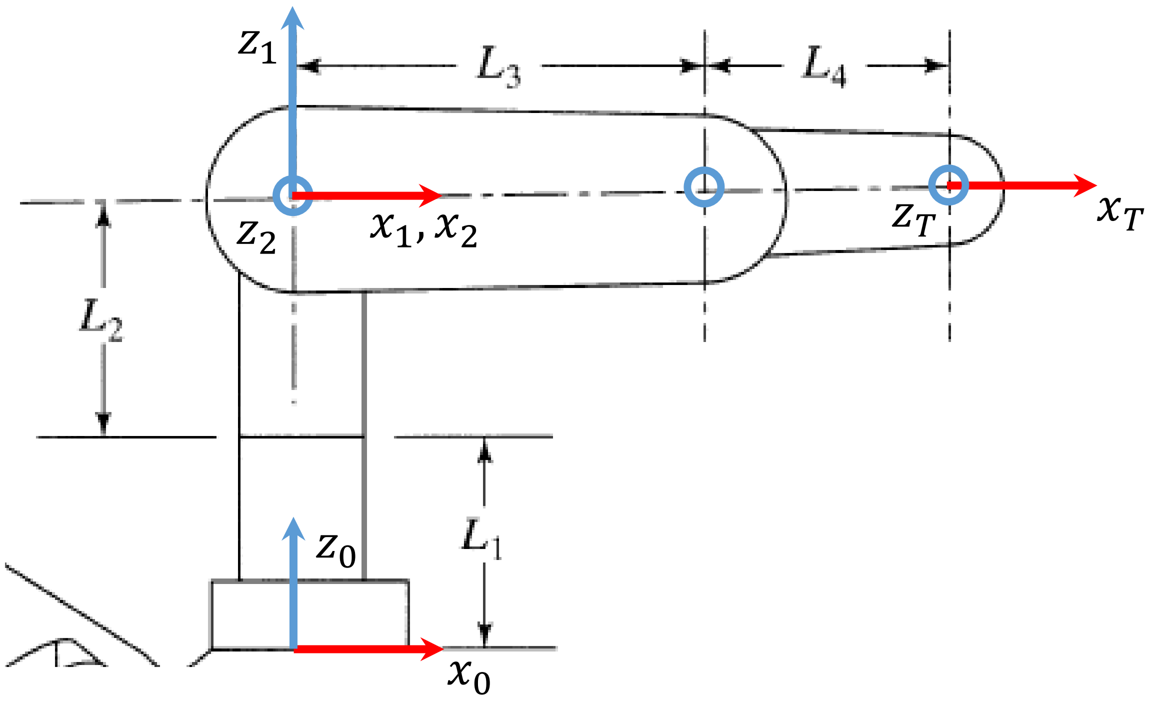
P03

- Step 0: Draw $\{0\}$ and $\{T\}$
- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes



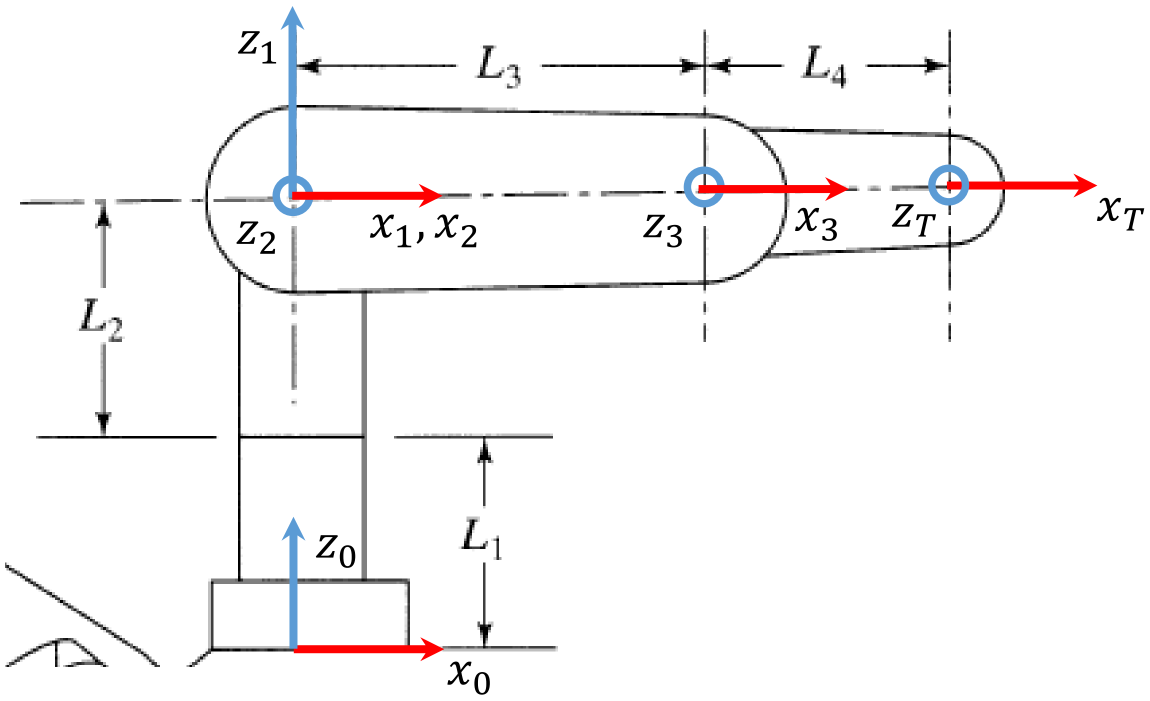
P03

- Step 0: Draw $\{0\}$ and $\{T\}$
- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes



P03

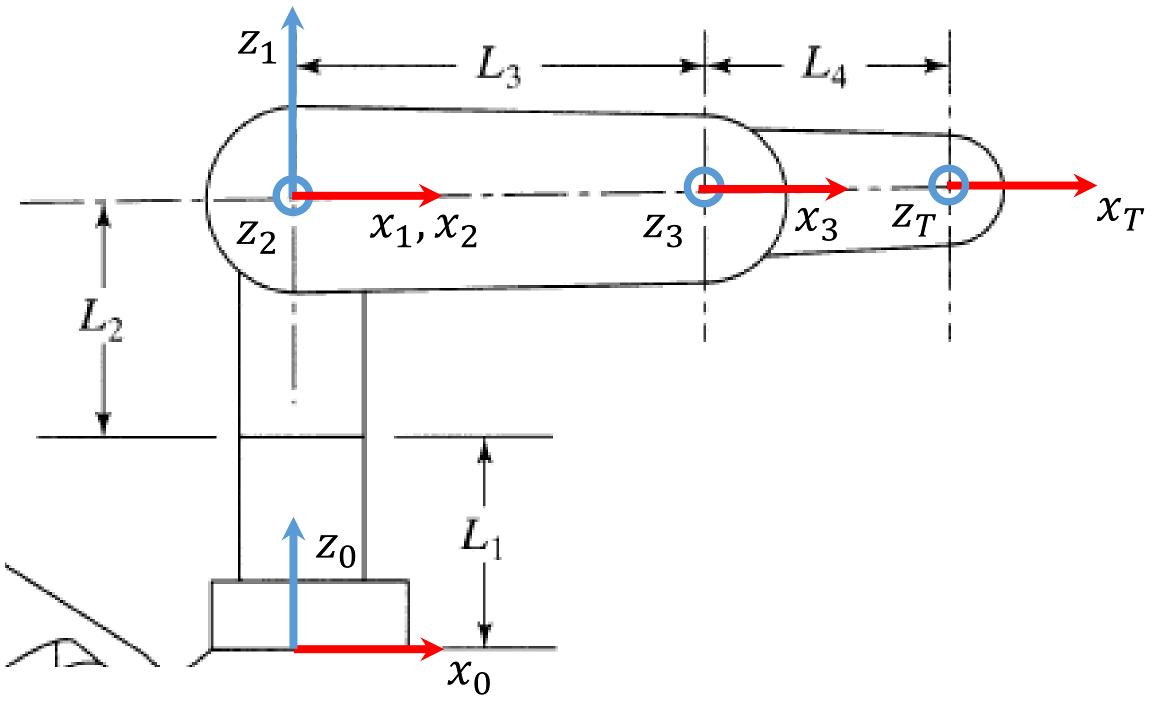
- Step 0: Draw $\{0\}$ and $\{T\}$
- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes



P03

- Step 0: Draw $\{0\}$ and $\{T\}$
- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes
- Step 3: Fill out DH table

CF	α	a	d	Θ
1	$0°$	0	$L_1 + L_2$	θ_1
2	$90°$	0	0	θ_2
3	$0°$	L_3	0	θ_3
$T = 4$	$0°$	L_4	0	$0°$



P03

Coordinate Transformations $\Rightarrow {}^0_1T, {}^1_2T, {}^2_3T, {}^3_4T$

CF	$\boldsymbol{\alpha}$	a	d	Θ
1	$0°$	0	$L_1 + L_2$	θ_1
2	$90°$	0	0	θ_2
3	$0°$	L_3	0	θ_3
$T = 4$	$0°$	L_4	0	$0°$

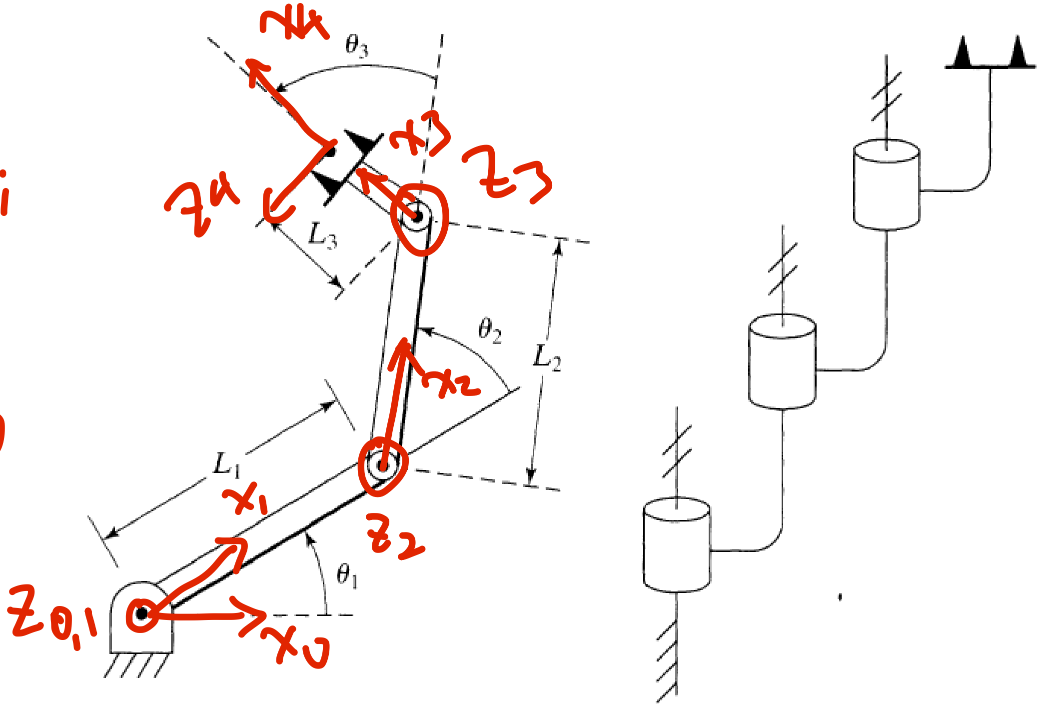
$${}^0_1T = \begin{pmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & L_1+L_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \quad {}^1_2T = \begin{pmatrix} C_2 & -S_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^2_3T = \begin{pmatrix} C_3 & -S_3 & 0 & L_3 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^3_4T = \begin{pmatrix} 1 & 0 & 0 & L_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem 4

P04

j	θ_i	α_i	a_i	d_i
0	0	0	0	0
1	θ_1	0	0	0
2	θ_2	0	l_1	0
3	θ_3	0	l_2	0
4	0	0	l_3	0



(a) (b)

Figure 3: *Planar 3R Robot (Problem 4)*

Consider the planar 3R robot shown in Figure 3. All axes of rotation are parallel, as indicated by the hash marks.

a) Identify the DH parameters of this system.

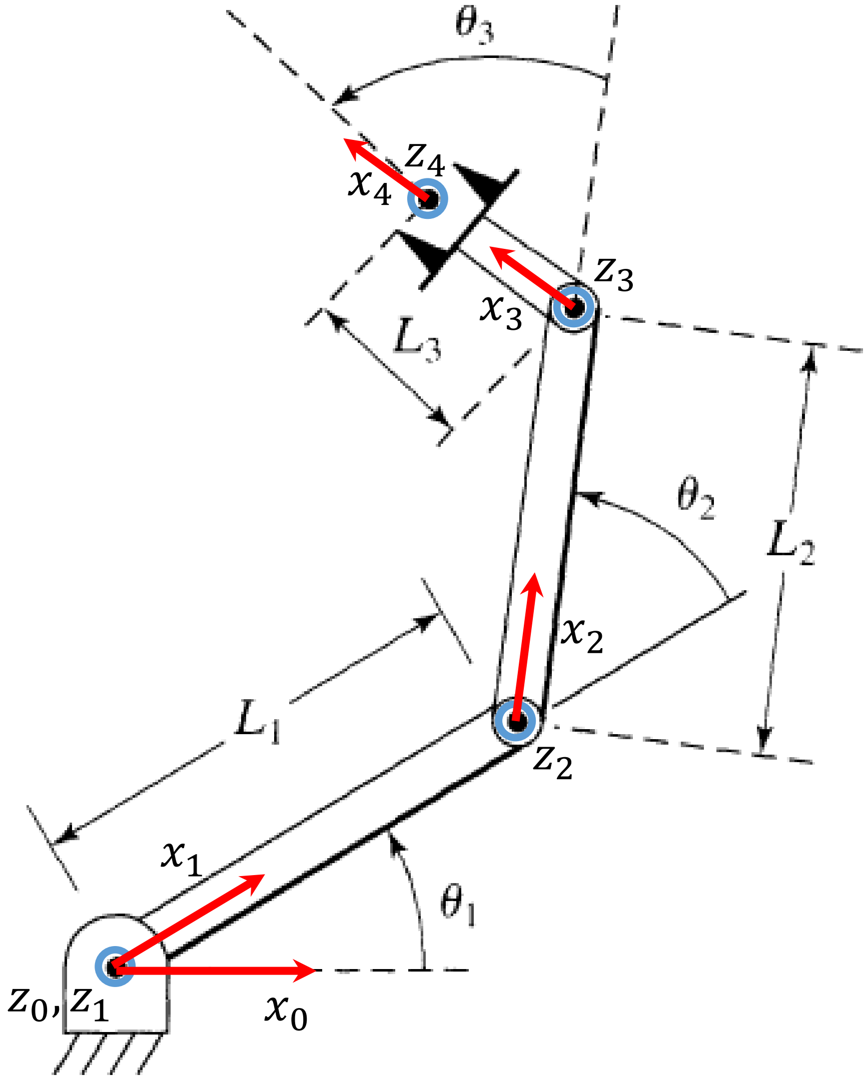
b) Determine its forward kinematics.

c) Given a position (cartesian coordinates of gripper center and gripper angle), can you think of a simple way to find out if such a position is reachable for the robot?

P04

- DH parameters

CF	α	a	d	Θ
1	$0°$	0	0	θ_1
2	$0°$	L_1	0	θ_2
3	$0°$	L_2	0	θ_3
4	$0°$	L_3	0	$0°$



P04

- DH parameters ⇒ Forward Kinematics: ${}^{0}_{4}T = {}^{0}_{1}T \cdot {}^{1}_{2}T \cdot {}^{2}_{3}T \cdot {}^{3}_{4}T$

CF	α	a	d	Θ
1	$0°$	0	0	θ_1
2	$0°$	L_1	0	θ_2
3	$0°$	L_2	0	θ_3
4	$0°$	L_3	0	$0°$

$$
{}^{0}_{4}T = \begin{pmatrix} c_{123} & -s_{123} & 0 & L_1 c_1 + L_2 c_{12} + L_3 c_{123} \\ s_{123} & c_{123} & 0 & L_1 s_1 + L_2 s_{12} + L_3 s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

- $c_{12} = \cos(\theta_1 + \theta_2)$
- $s_{123} = \sin(\theta_1 + \theta_2 + \theta_3)$

P04

$${}^0P_3 + L_3 \begin{pmatrix} c\theta_G \\ s\theta_G \\ 0 \end{pmatrix} = {}^0P_G$$

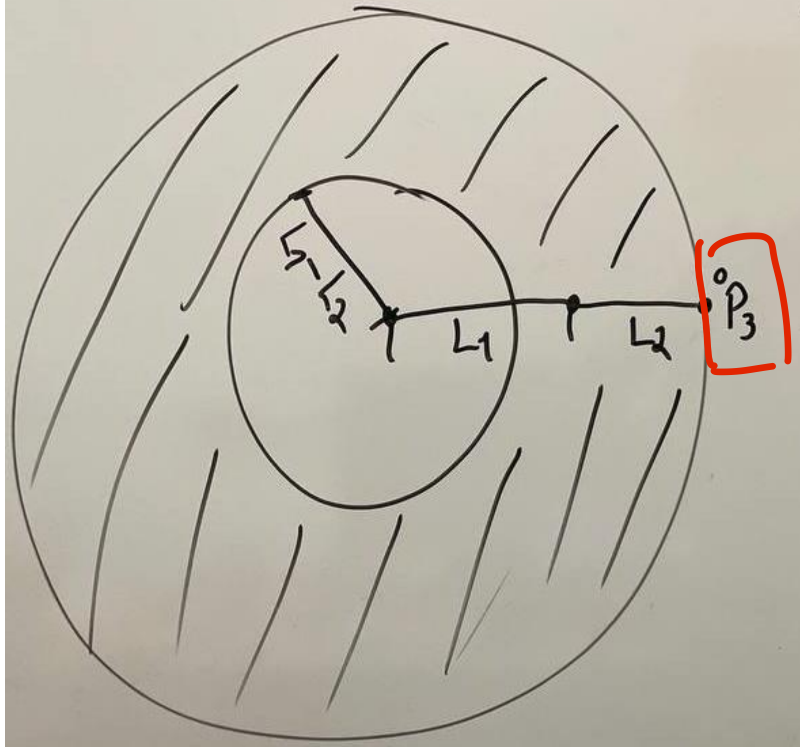
- Reachability condition:

c) ${}^0P_G$ = gripper point

θ_G = gripper orientation

inequality conditions involving ${}^0P_G$ and θ_G

$${}^0P_3 = {}^0P_G - L_3 \begin{pmatrix} c\theta_G \\ s\theta_G \\ 0 \end{pmatrix} \longleftarrow$$ ${}^0P_3$ is determined by J_1 and J_2



${}^0P_3$ can take any value in the strafed area: $\max(0, L_1 - L_2) \leq \|{}^0P_3\| \leq L_1 + L_2$

$$\Rightarrow \max(0, L_1 - L_2) \leq \left\| {}^0P_G - L_3 \begin{pmatrix} c\theta_G \\ s\theta_G \\ 0 \end{pmatrix} \right\| \leq L_1 + L_2$$

Problem 5

P05

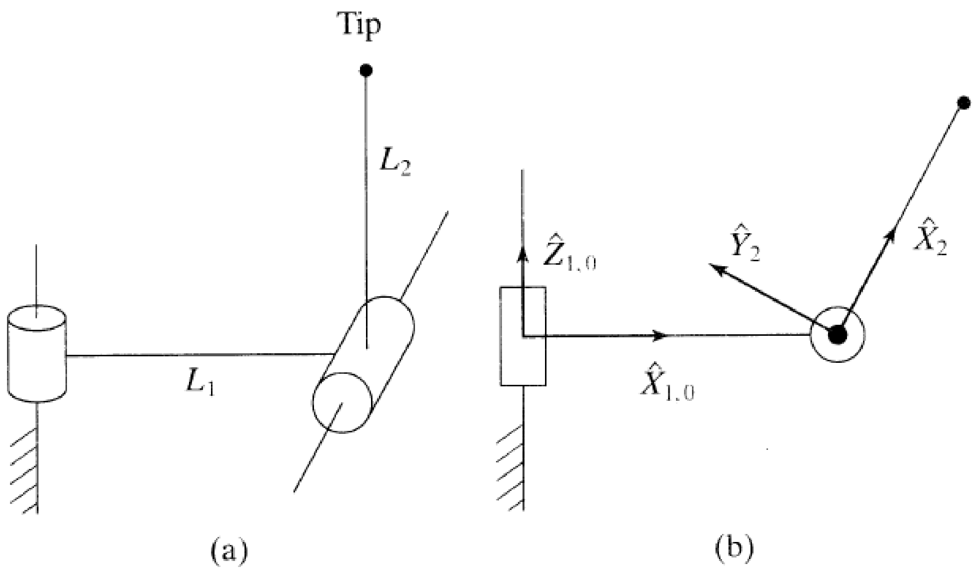


Figure 4: *2R Robot, problem 5*

Consider the 2R robot shown in Figure 4. The link transformation matrices have been determined, and the coordinate transformation for the second link is:

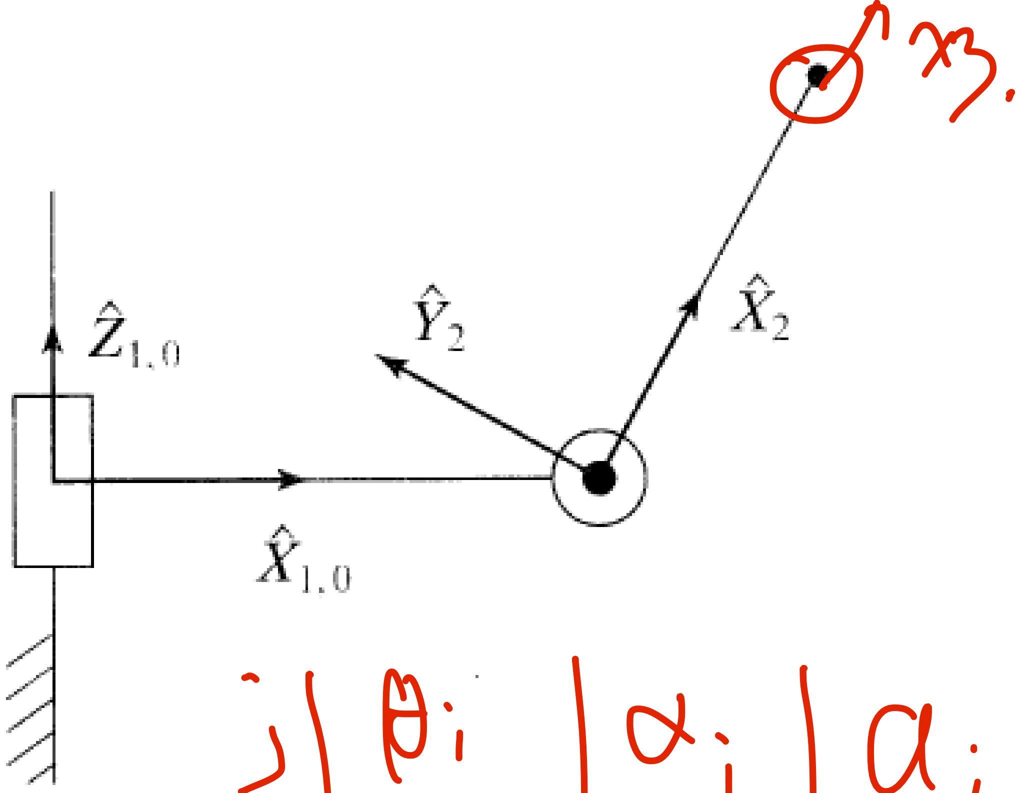
$$ {}^0_2T = \begin{pmatrix} c_1c_2 & -c_1s_2 & s_1 & l_1c_1 \\ s_1c_2 & -s_1s_2 & -c_1 & l_1s_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

The link-frame assignments are shown in Figure 4. Note that the frames $\{0\}$ and $\{1\}$ are coincident when $\Theta_1 = 0$. The length of the second link is l_2. Find an expression for the vector ${}^0P_{\text{tip}}$, which locates the tip of the arm relative to the $\{0\}$ frame.

(P5) DH-Table:

	α	a	d	θ
1	$0°$	0	0	θ_1
2	$90°$	L_1	0	θ_2
3	$0°$	L_2	0	$0°$

${}^0P_{tip}$: fwd kinematics ; position only



i	θ_i	α_i	a_i	d_i
0	0	0	0	0
1	θ_1	0	0	0
2	θ_2	90	L_1	0
3	0	0	L_2	0

(P5) DH-Table:

	α	a	d	θ
1	$0°$	0	0	θ_1
2	$90°$	L_1	0	θ_2
3	$0°$	L_2	0	$0°$

Rows 1–2 $\rightarrow {}^{0}_{2}T$

Row 3 $\rightarrow$ $${}^{2}_{3}T = \begin{pmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

${}^{0}P_{tip}$: forward kinematics ; position only

P5 DH-Table:

	α	a	d	θ
1	$0°$	0	0	θ_1
2	$90°$	L_1	0	θ_2
3	$0°$	L_2	0	$0°$

Rows 1–2 $\Rightarrow {}^0_2T$

Row 3 $\rightarrow$

$$ {}^2_3T = \begin{pmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

${}^0P_{tip}$: fwd kinematics ; position only

$$ {}^0_3T = {}^0_2T \, {}^2_3T = \begin{pmatrix} c_1c_2 & -c_1s_2 & s_1 & l_1c_1 + L_2c_1c_2 \\ s_1c_2 & -s_1s_2 & -c_1 & l_1s_1 + L_2s_1c_2 \\ s_2 & c_2 & 0 & 0 + L_2s_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

position only

$$ {}^0P_{tip} = \begin{pmatrix} l_1c_1 + L_2c_1c_2 \\ l_1s_1 + L_2s_1c_2 \\ L_2s_2 \end{pmatrix} $$

P06

Problem 6

$\theta_4 = 90$?

	θ_i	α_i	a_i	d_i
0	0	0	0	0
4	θ_4	0	0	0
5	θ_5	ϕ	0	0
6	θ_6	180°	0	0

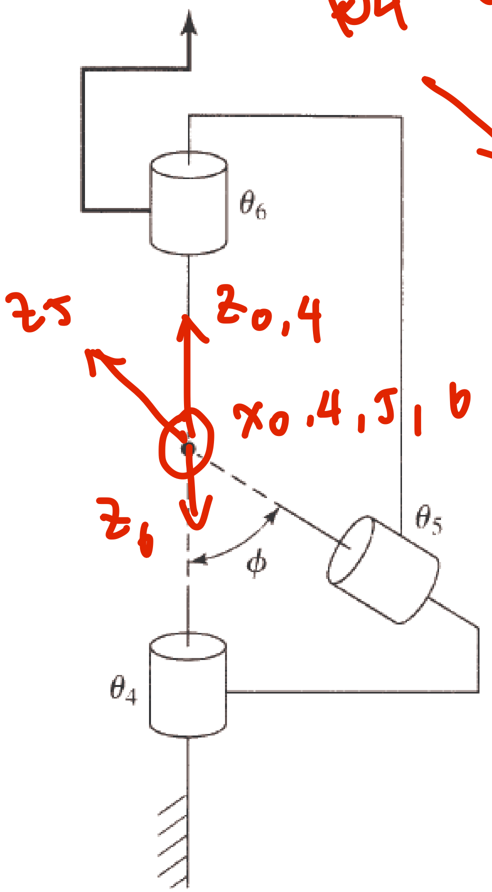
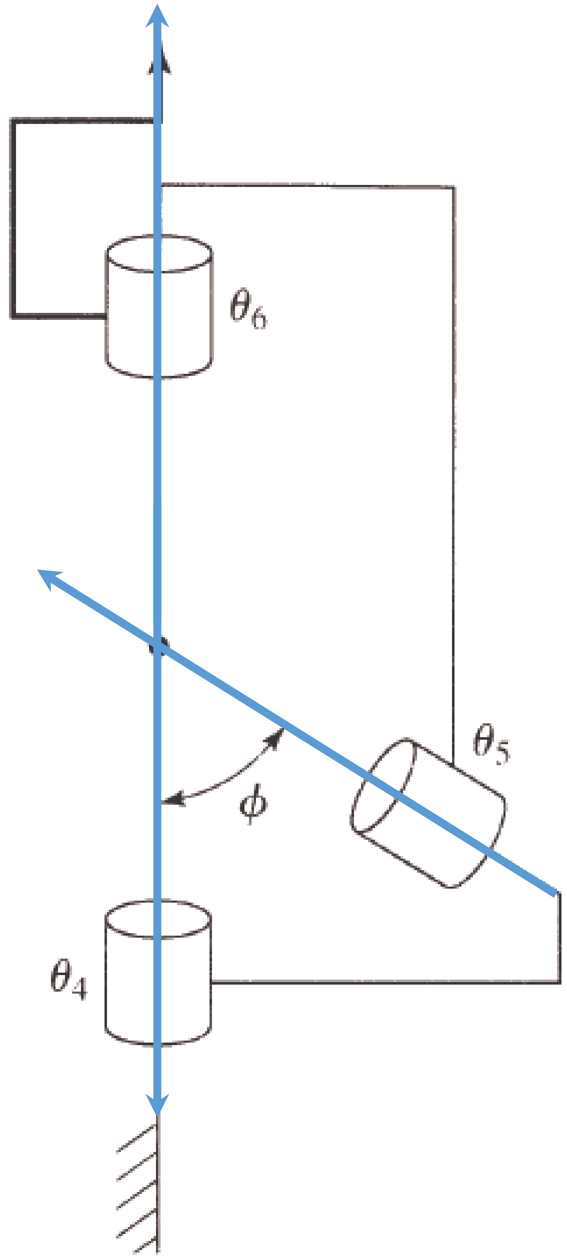


Figure 5: *3R nonorthogonal-axis robot (problem 6)*

Figure 5 shows a schematic of a wrist that has three intersecting axes that are not orthogonal. Assign link frames to the wrist (as if it were a regular 3R manipulator) and give the link parameters.

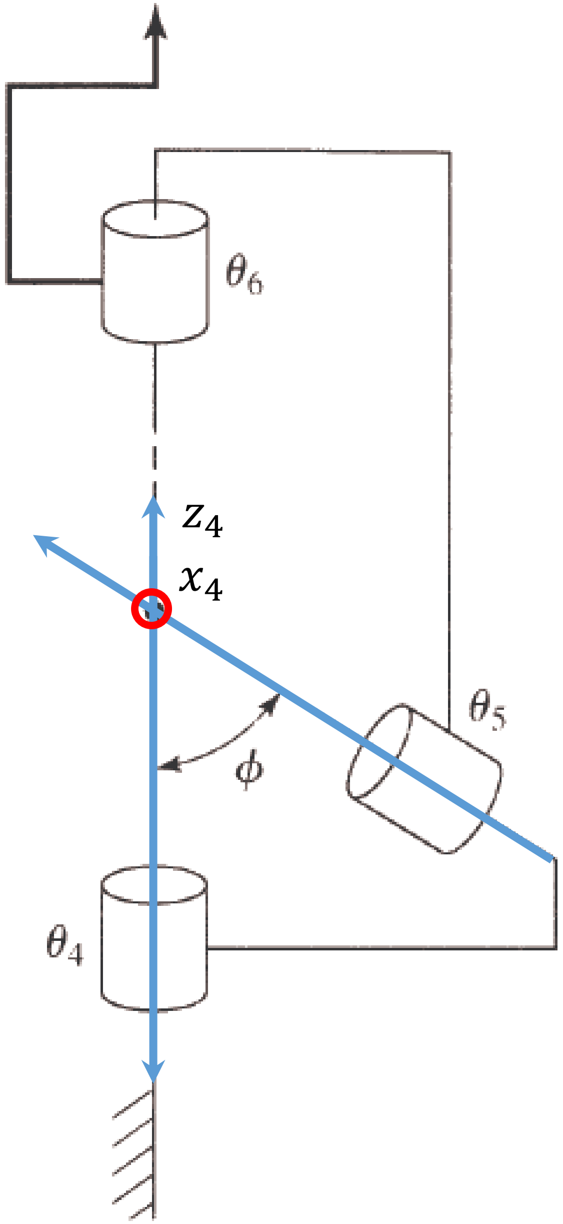
P06

- Step 1: Determine z-axes



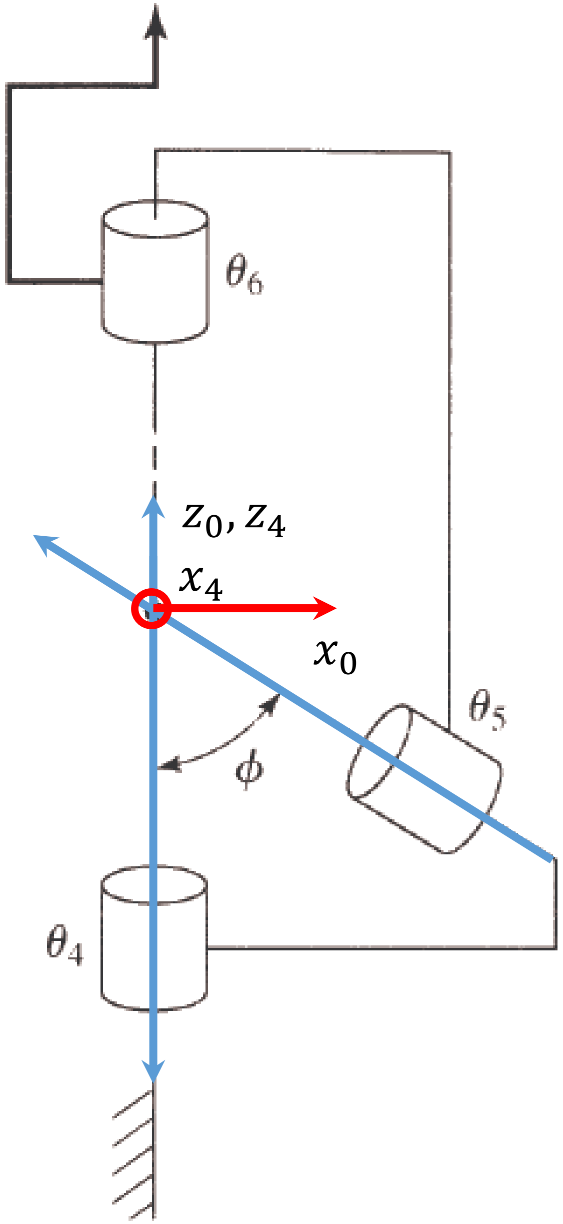
P06

- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes



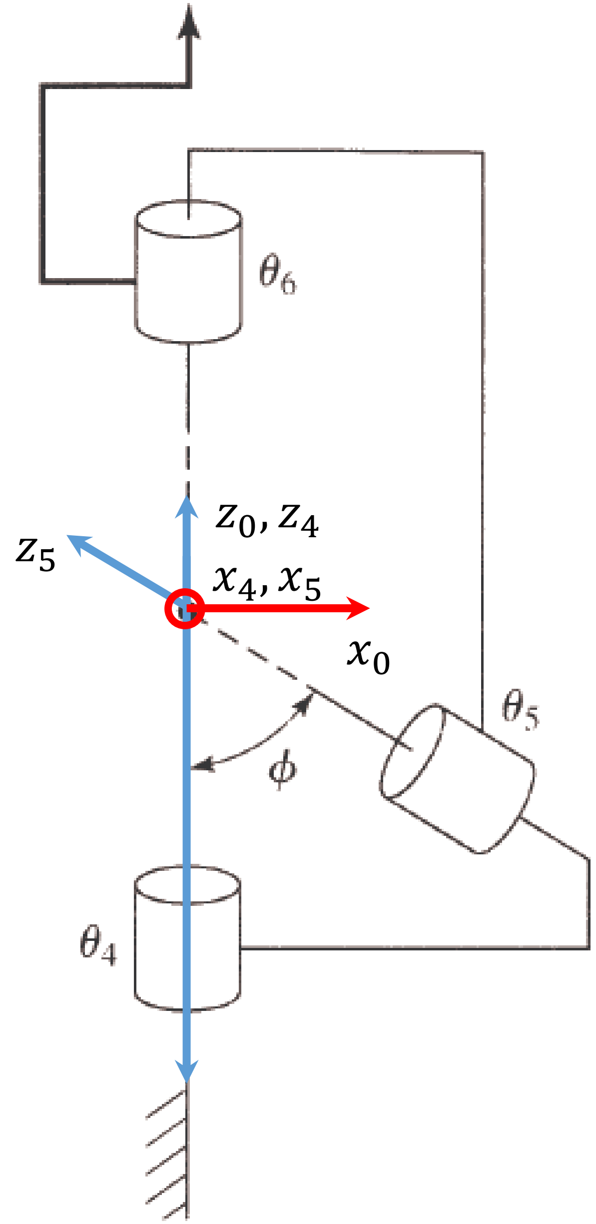
P06

- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes



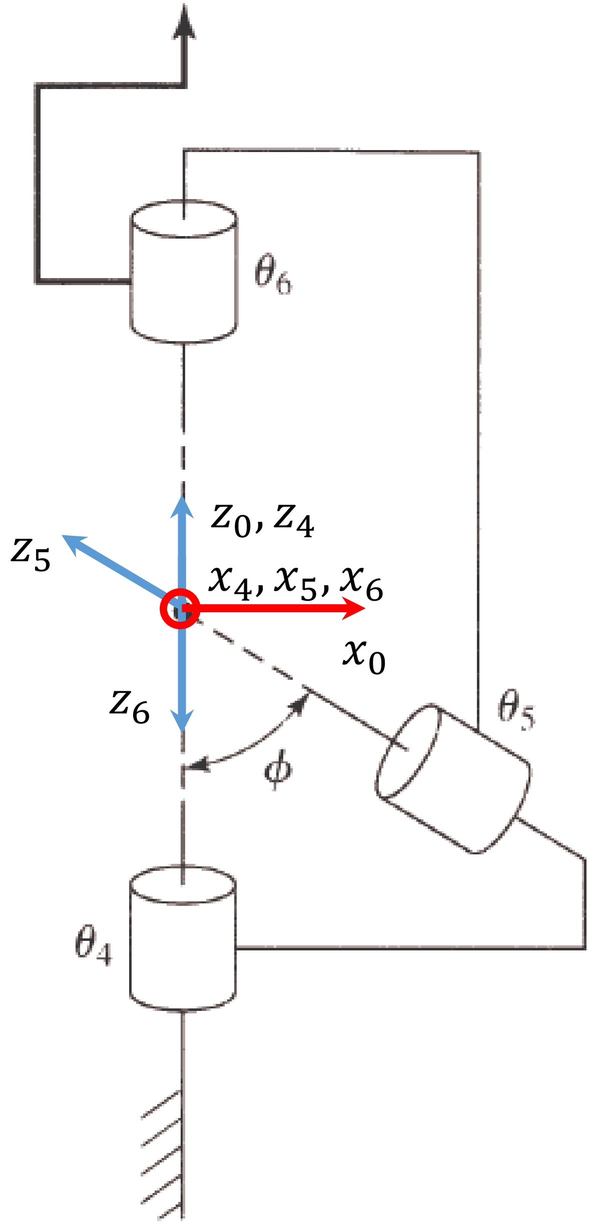
P06

- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes



P06

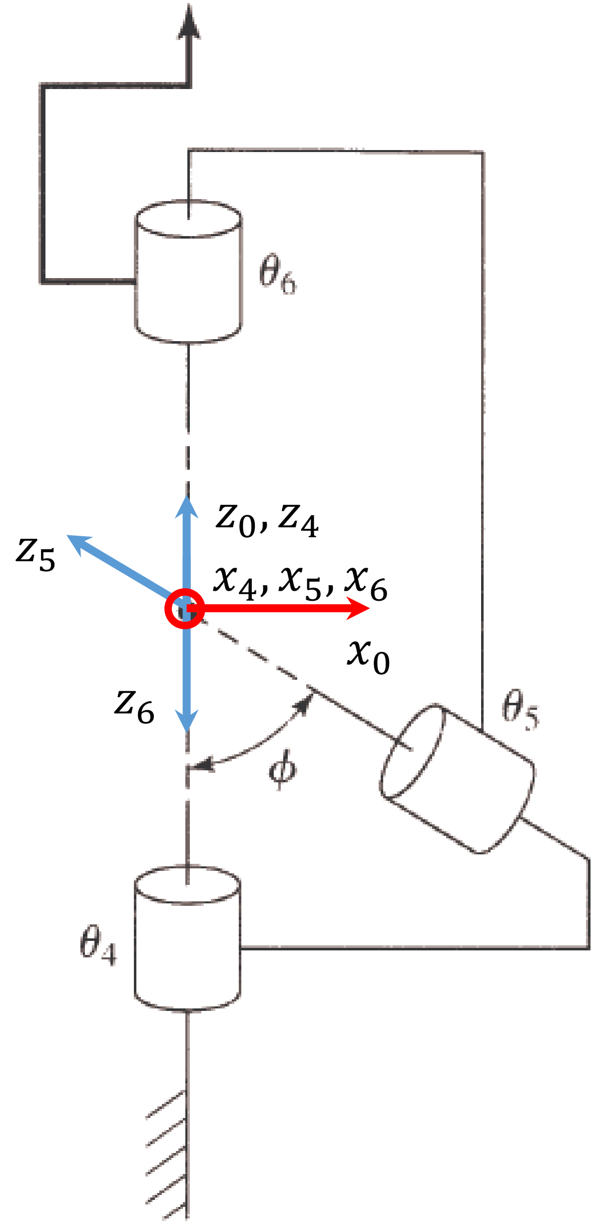
- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes



P06

- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes
- Step 3: Fill DH table

CF	α	a	d	Θ
4	$0°$	0	0	$\theta_4 = -90°$
5	ϕ	0	0	θ_5
6	$180° - \phi$	0	0	θ_6



P07

Problem 7

The formula for determining the transformation between consecutive frames within the Denavit-Hartenberg-Convention is given as:

$$
{}^{i-1}_{i}\mathrm{T} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_{i-1} \\ \sin\theta_i \cos\alpha_{i-1} & \cos\theta_i \cos\alpha_{i-1} & -\sin\alpha_{i-1} & -\sin\alpha_{i-1} d_i \\ \sin\theta_i \sin\alpha_{i-1} & \cos\theta_i \sin\alpha_{i-1} & \cos\alpha_{i-1} & \cos\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

Is is possible to express arbitrary rigid body transformations this way?

P07

- Minimal number of parameters to represent an arbitrary rigid body transformation is 6
- DH-convention uses only 4 parameters for
  - x-rotation
  - x-translation
  - z-translation
  - z-rotation
- There is no way of representing a rotation around the y-axis
- y- and z-position are coupled (not independent of orientation)

P07

- The element ${}^{i-1}_{i}T_{1,3}$ is always 0 in the DH-transformation matrix, already suggesting that it is not possible to represent an arbitrary rotation $R = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{pmatrix}$

$$
{}^{i-1}_{i}\mathrm{T} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_{i-1} \\ \sin\theta_i\cos\alpha_{i-1} & \cos\theta_i\cos\alpha_{i-1} & -\sin\alpha_{i-1} & -\sin\alpha_{i-1}d_i \\ \sin\theta_i\sin\alpha_{i-1} & \cos\theta_i\sin\alpha_{i-1} & \cos\alpha_{i-1} & \cos\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

P08

Problem 8

i	θ_i	α_i	a_i	d_i
0	0	0	0	0
1	θ_1	0	0	l_1
2	θ_2	$-\alpha_2$	l_2	l_3
3	0			
4				
5				
6	0			

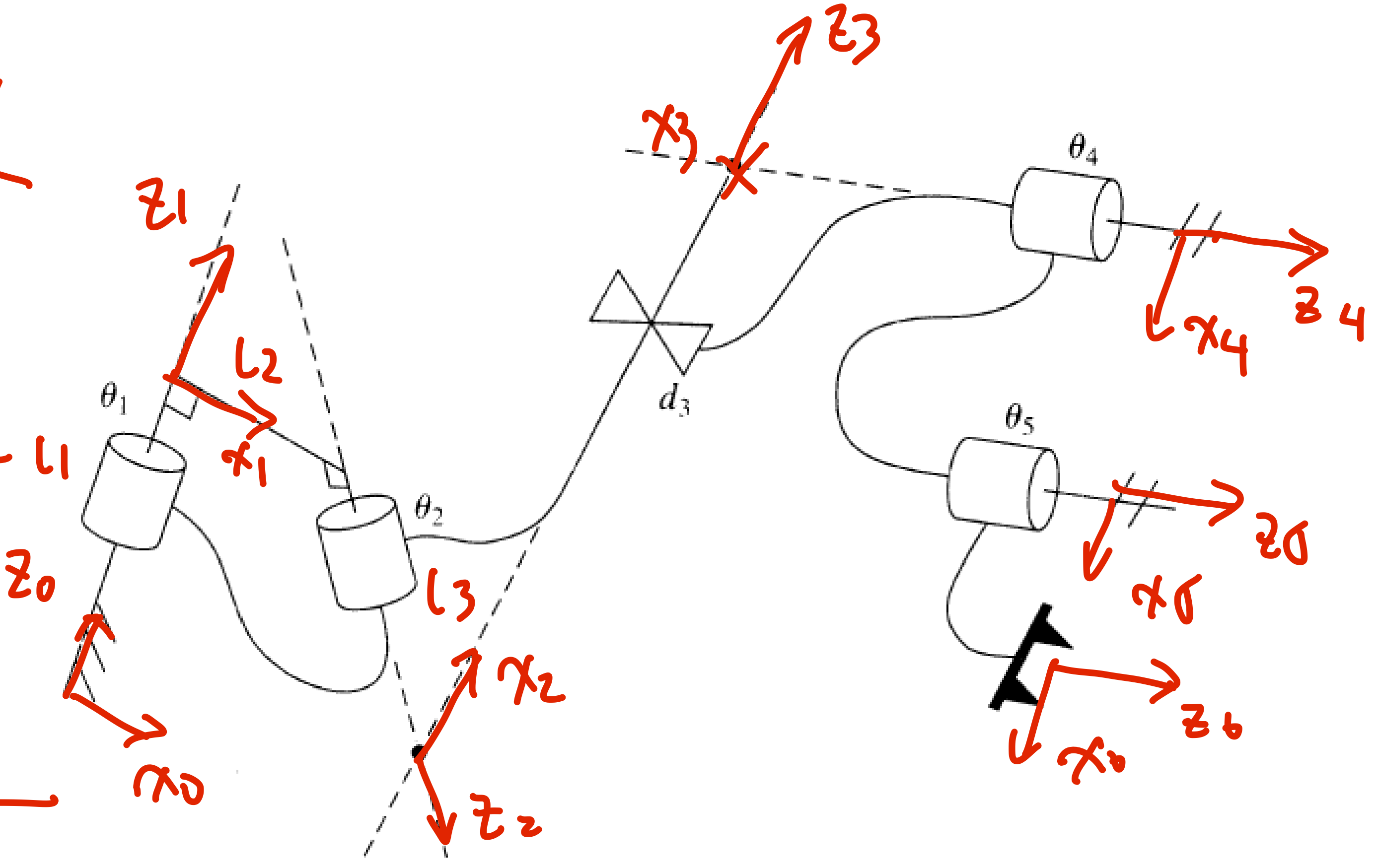
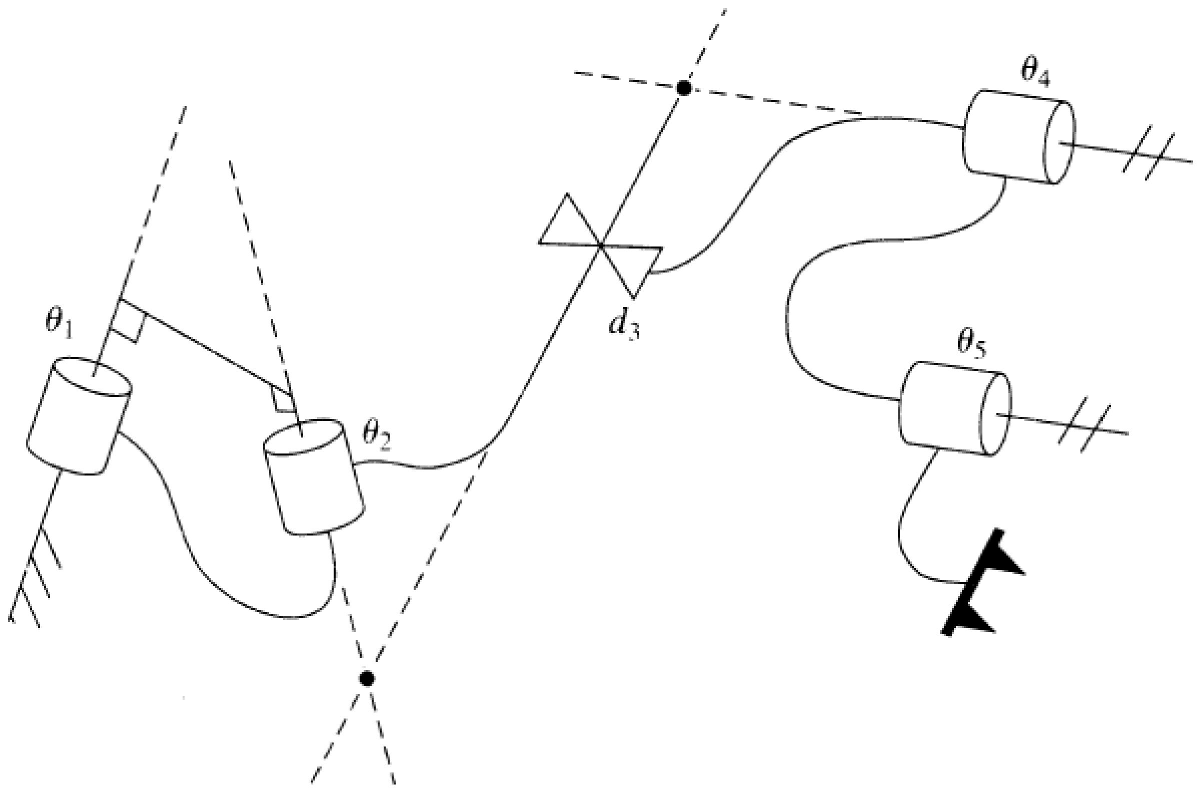


Figure 6: *Schematic of a 2RP2R manipulator (problem 8).*

Show the attachment of link frames for the manipulator shown in Figure 6.

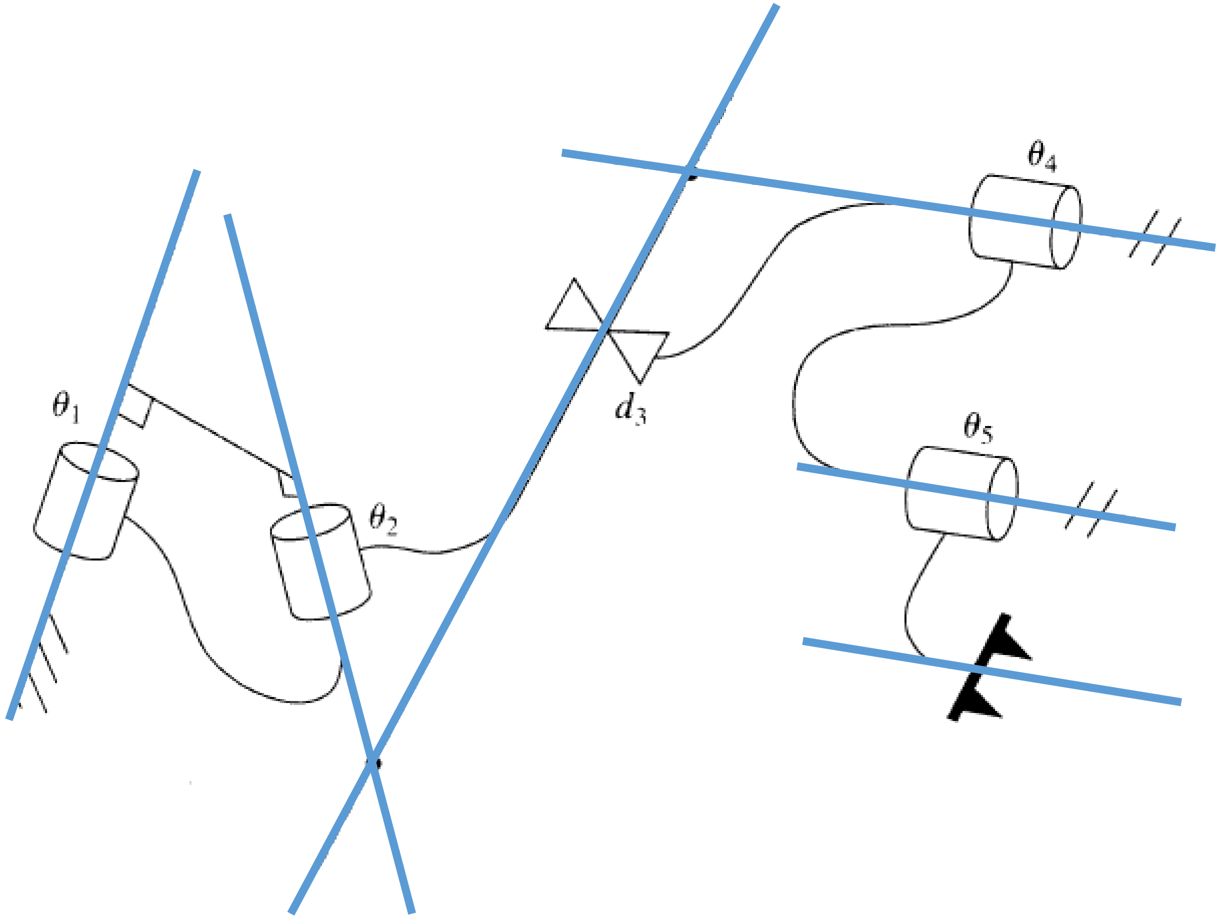
P08

- Step 1: Determine z-axes



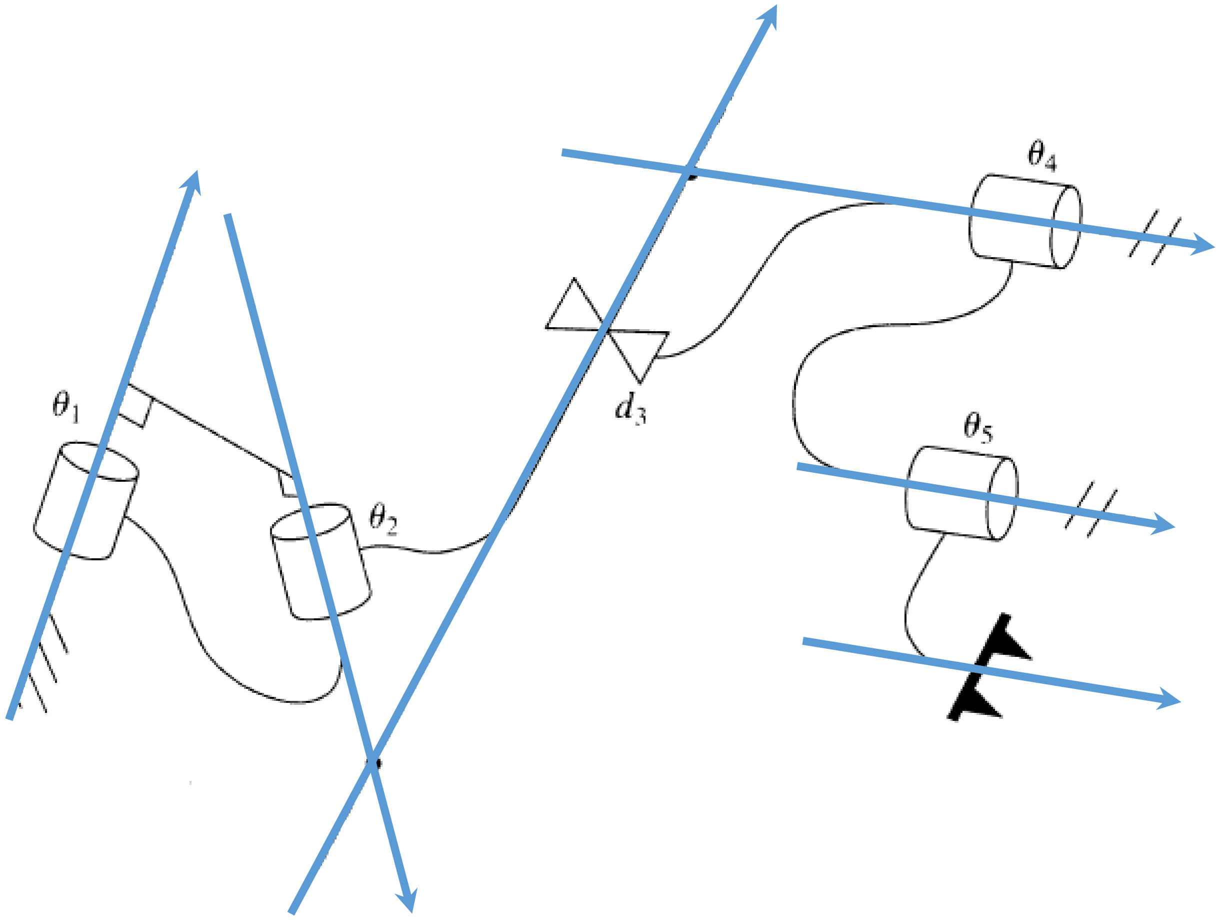
P08

- Step 1: Determine z-axes
  - we don't know the positive direction



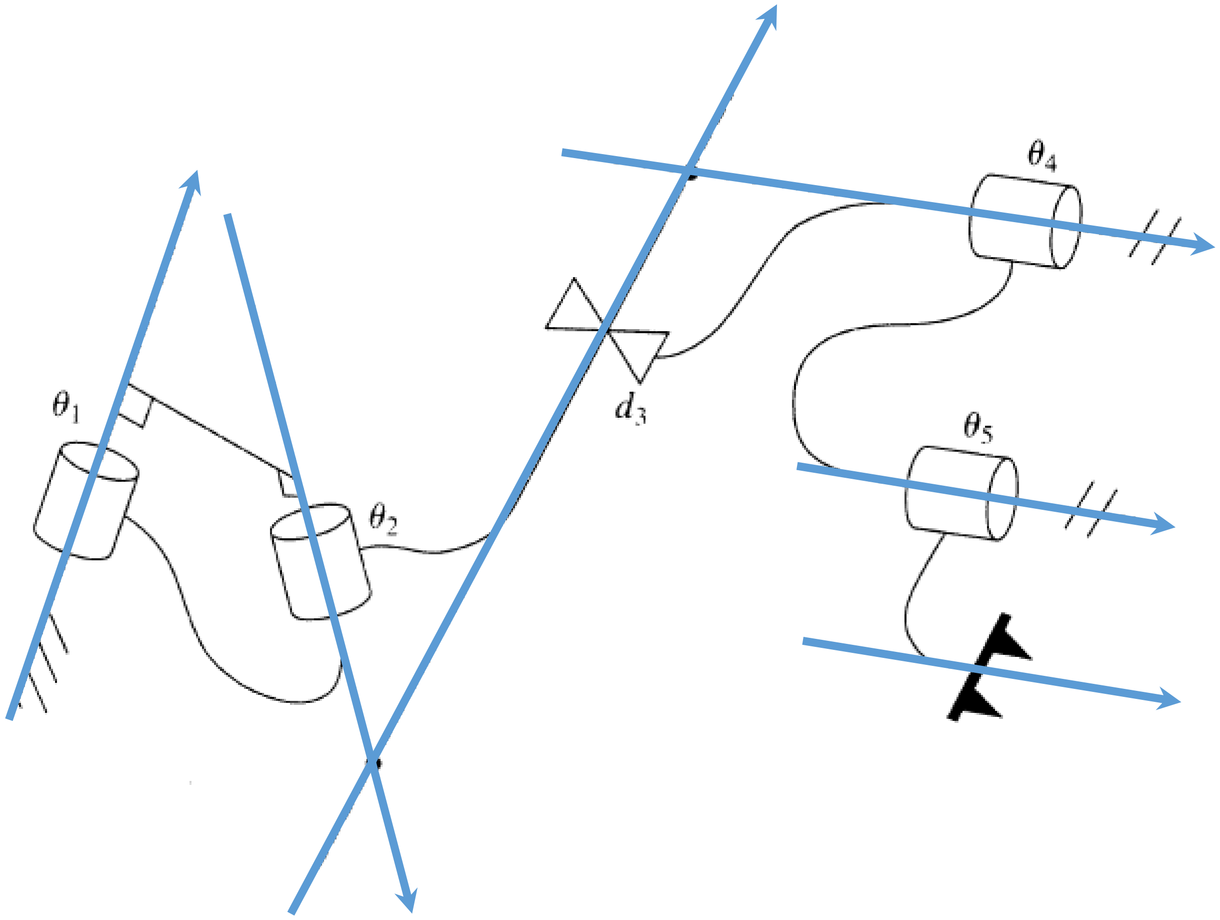
P08

- Step 1: Determine z-axes
  - we don't know the positive direction
  - set direction ☺



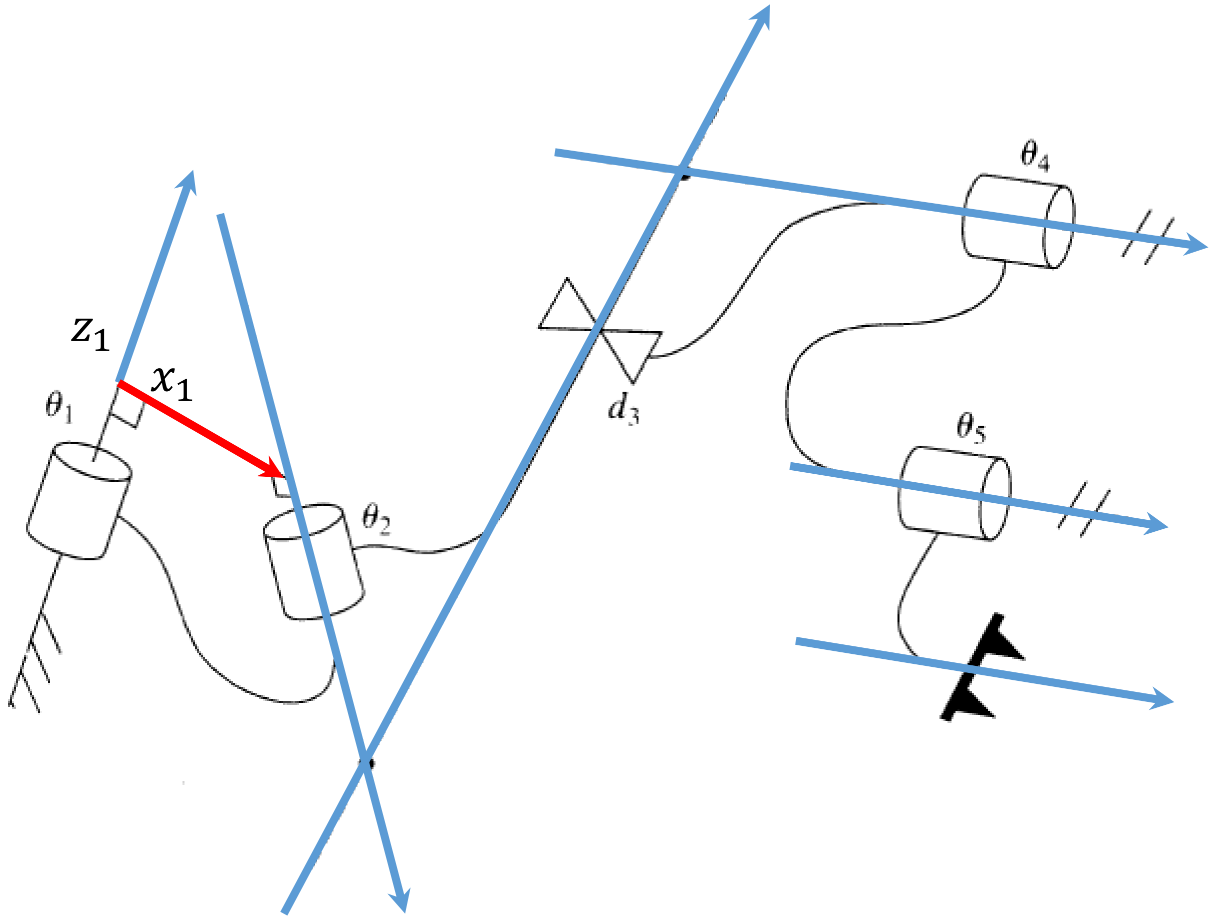
P08

- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes



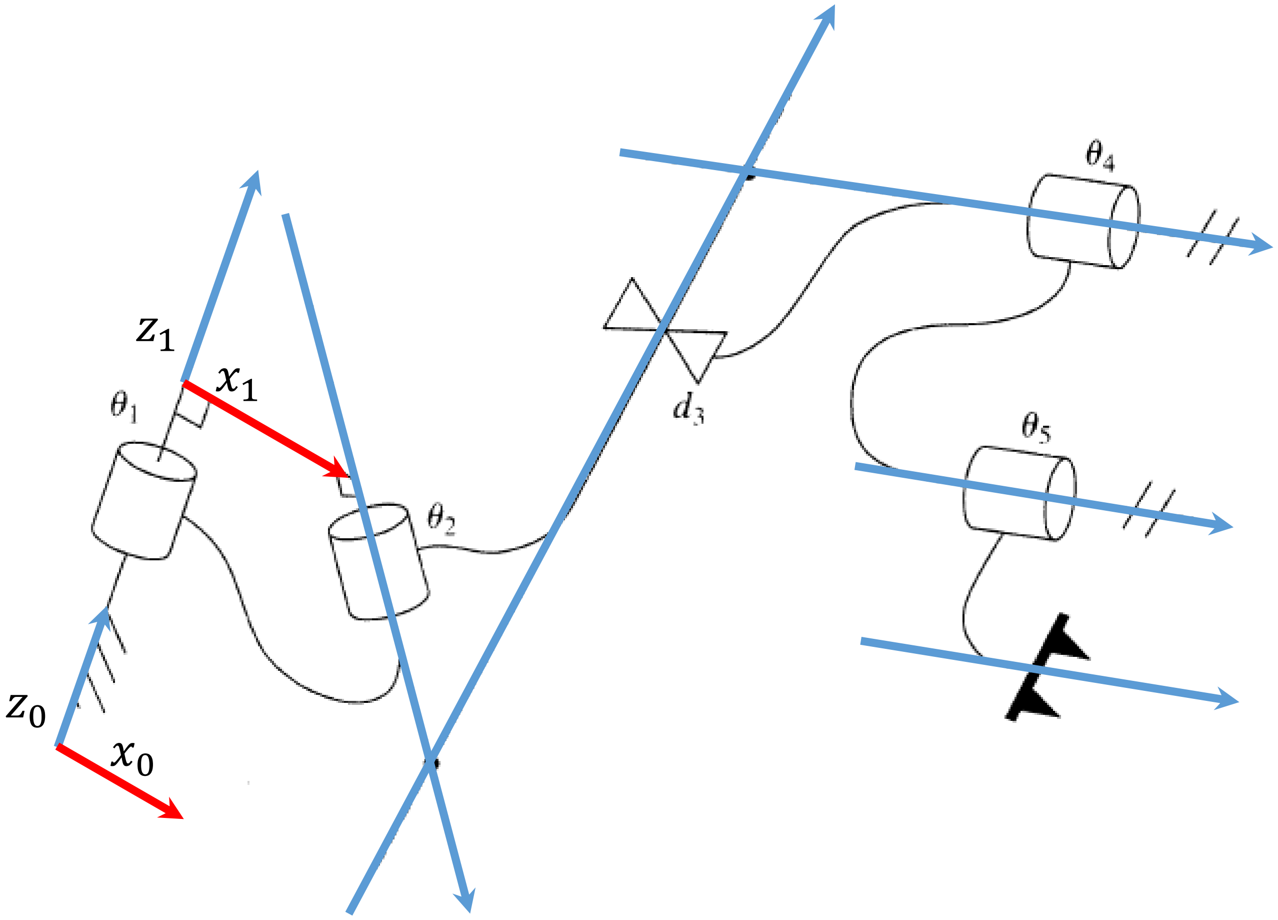
P08

- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes



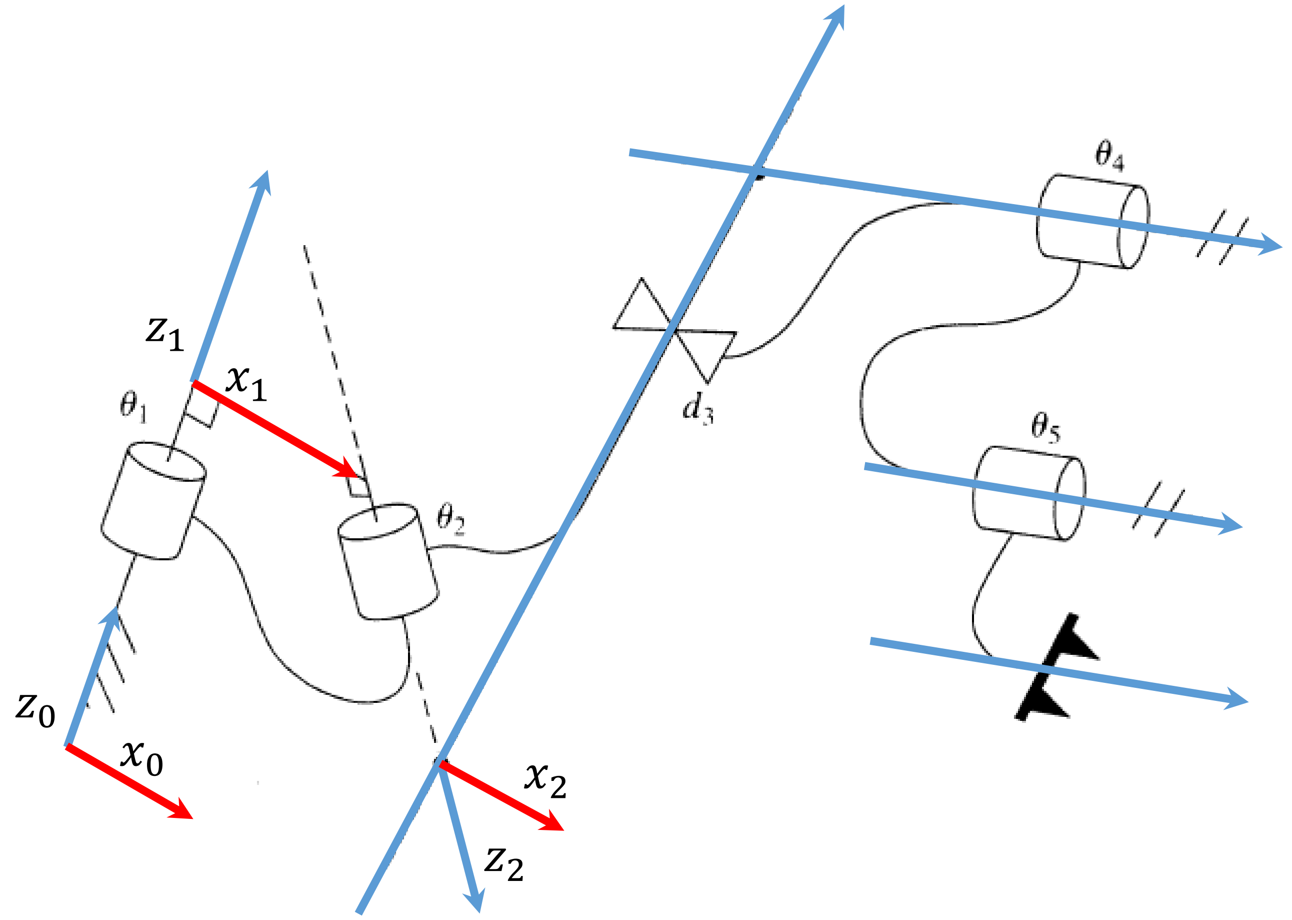
P08

- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes



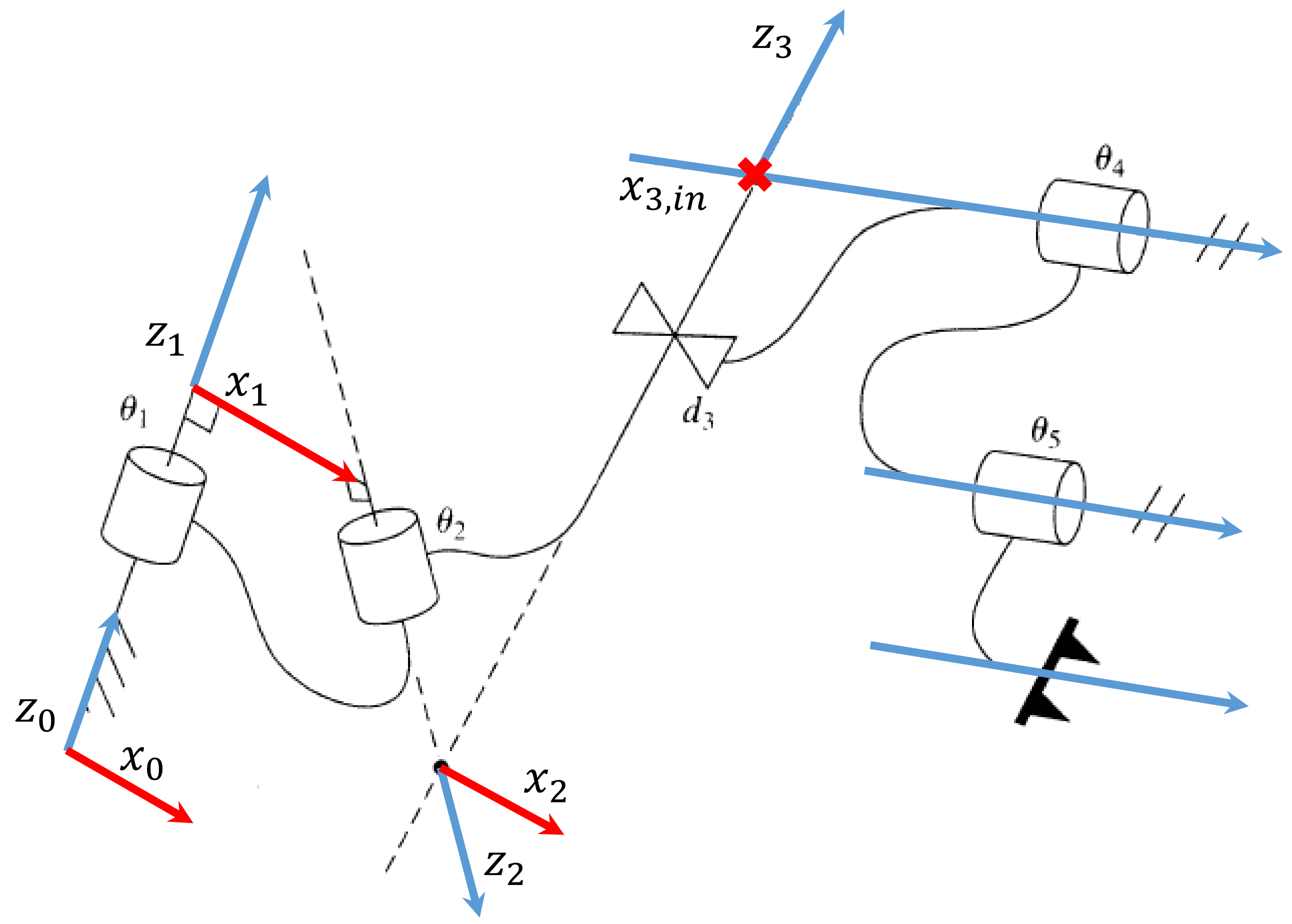
P08

- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes



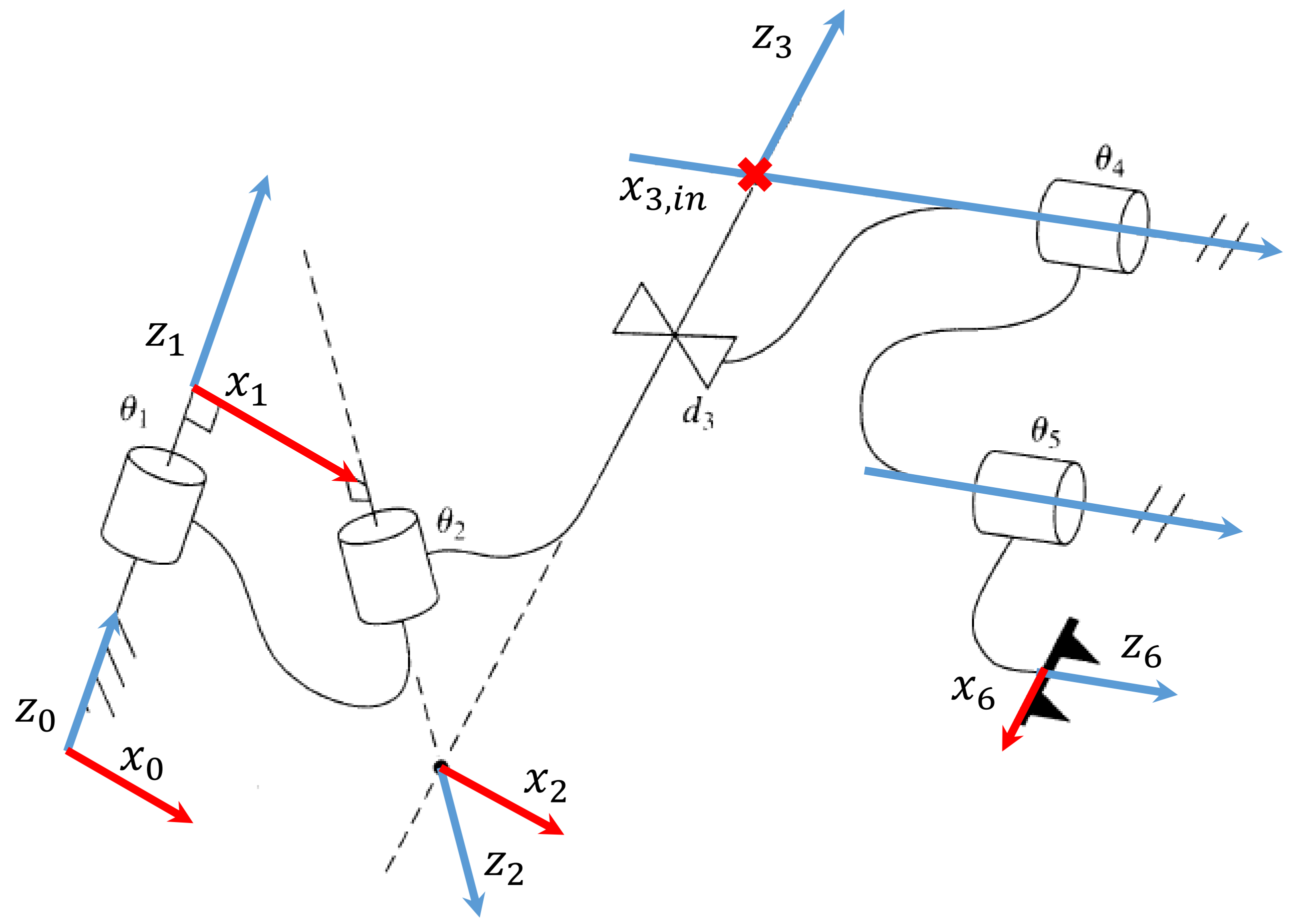
P08

- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes



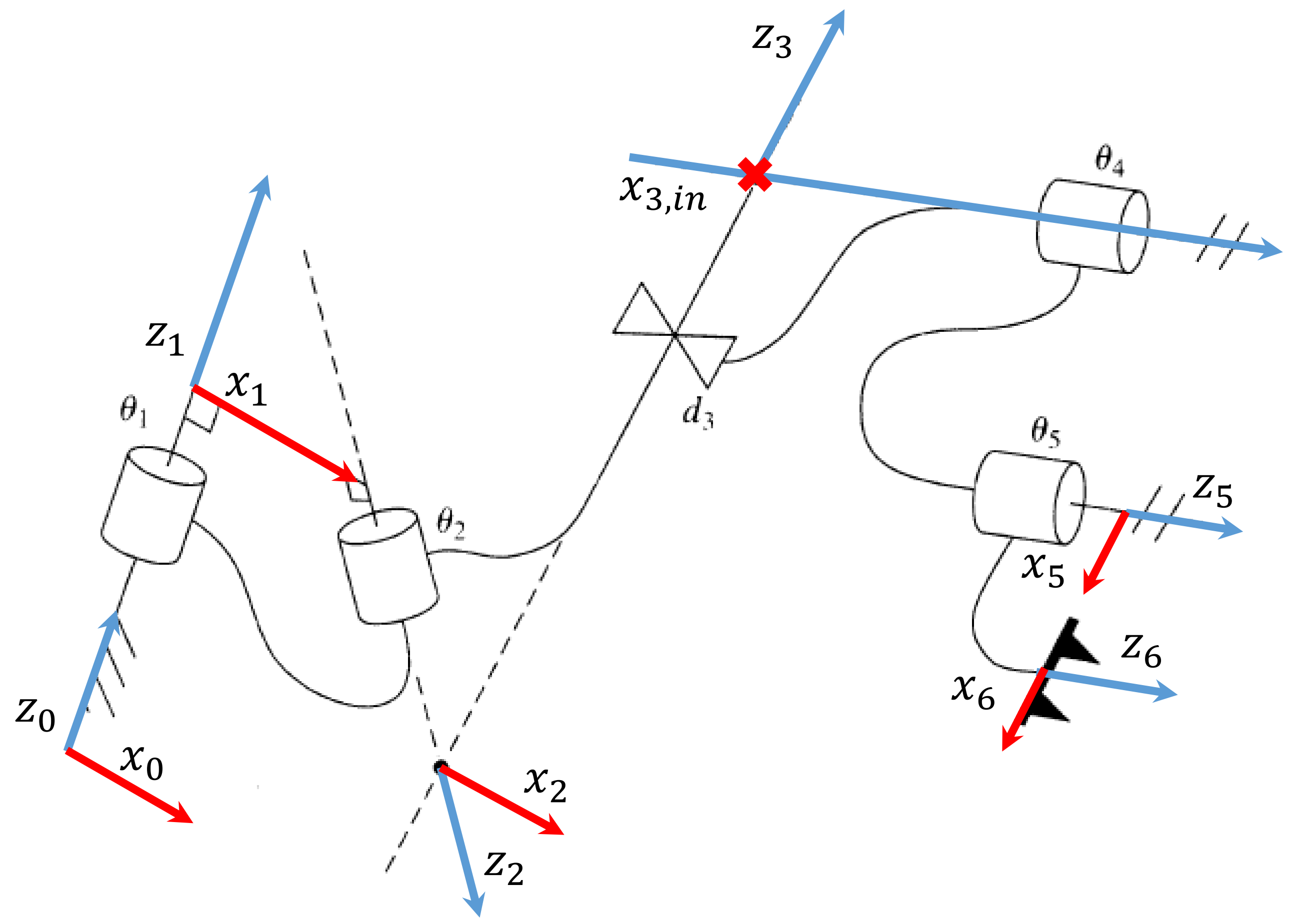
P08

- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes



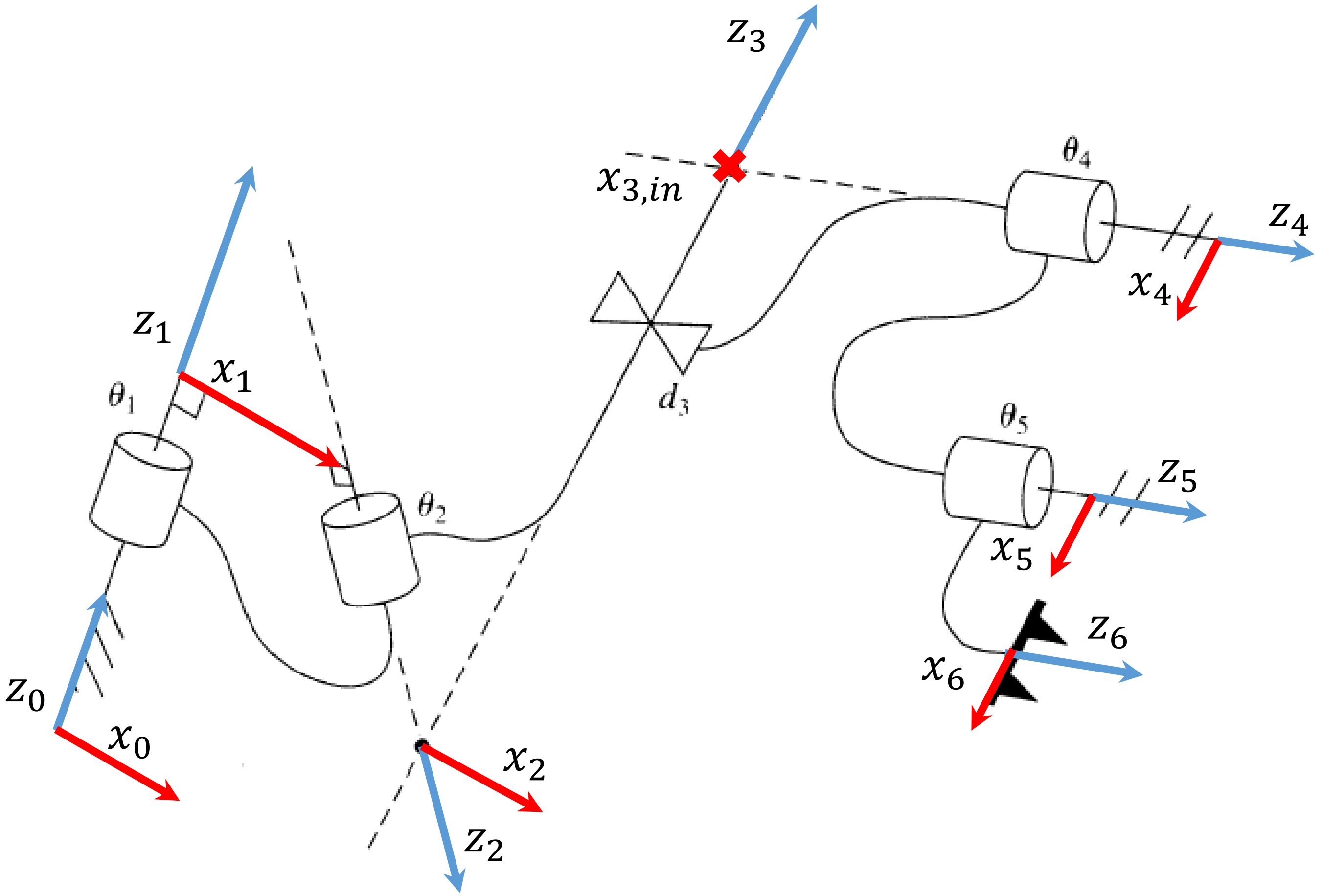
P08

- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes



P08

- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes



P09

Problem 9

	θ_i	α_i	a_i	d_i
0	0	0	0	0
1	θ_1	90	0	0
2	0	90	l_1	d_2
3	θ_3			
4				

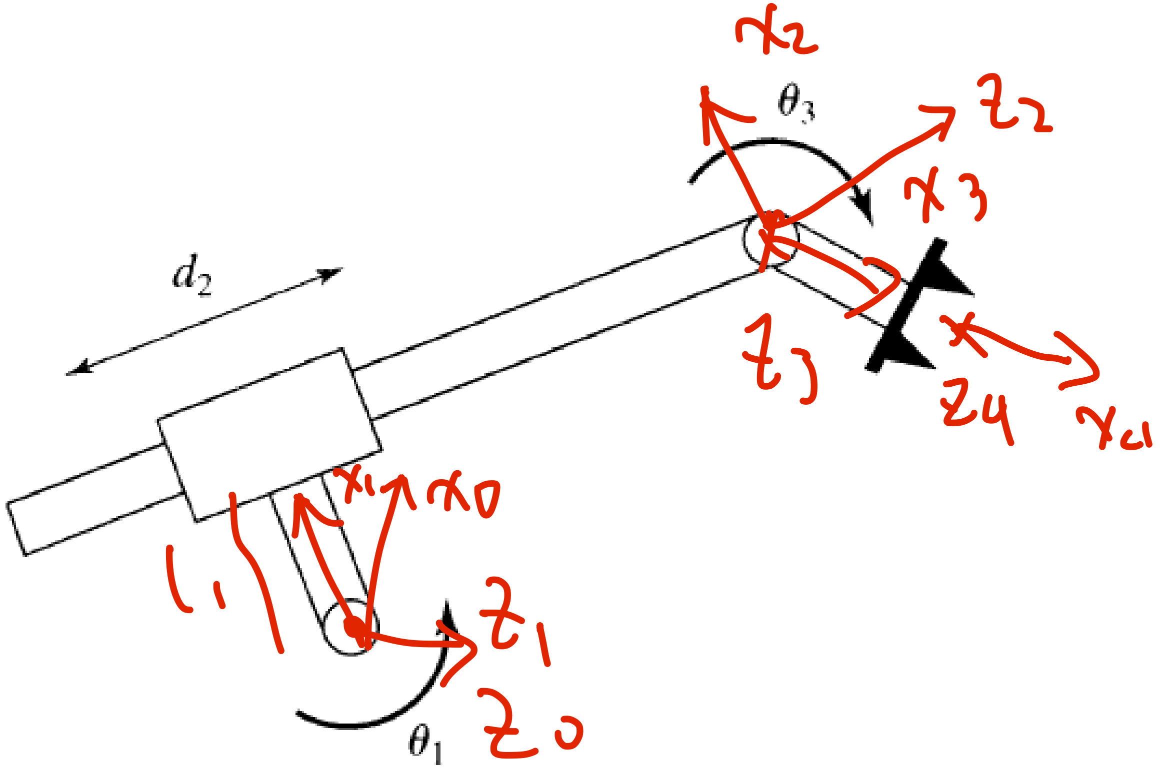
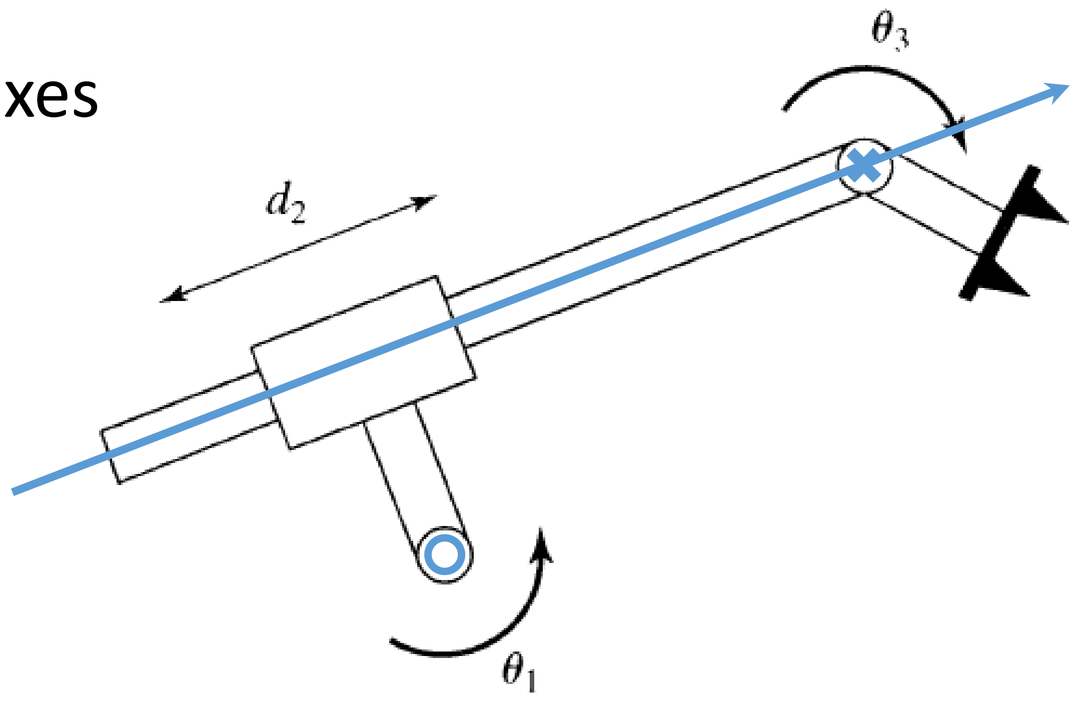


Figure 7: *RPR planar robot (problem 9)*

Assign link frames to the robot shown in Figure 7, and give approximate linkage parameters.

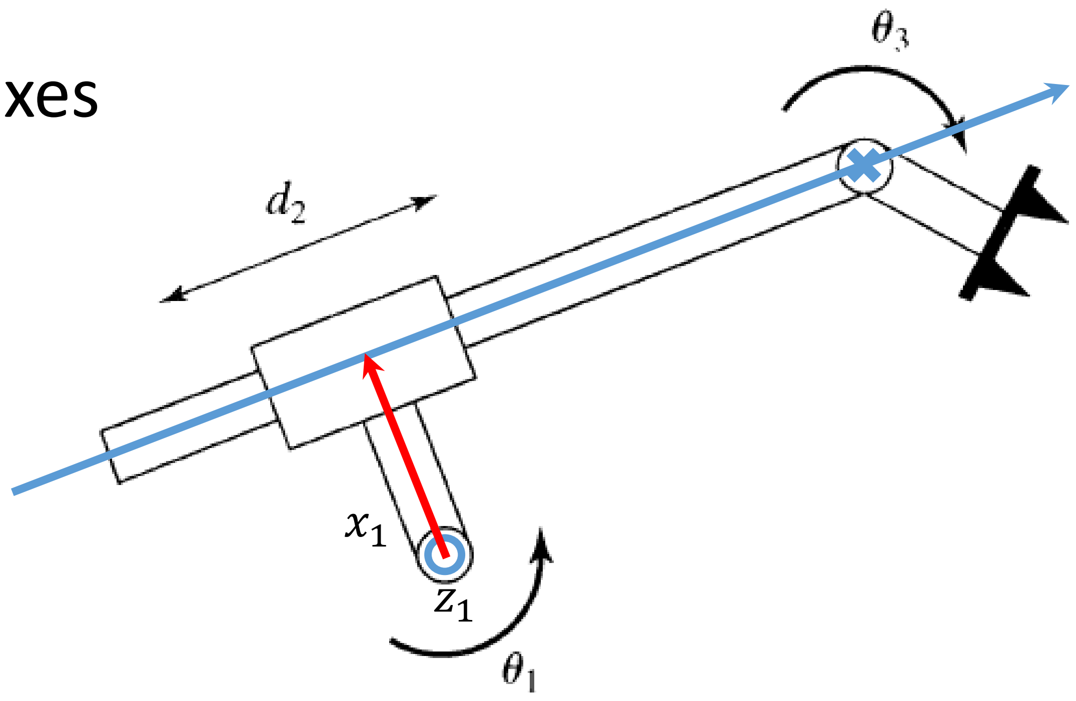
P09

- Step 1: Determine z-axes



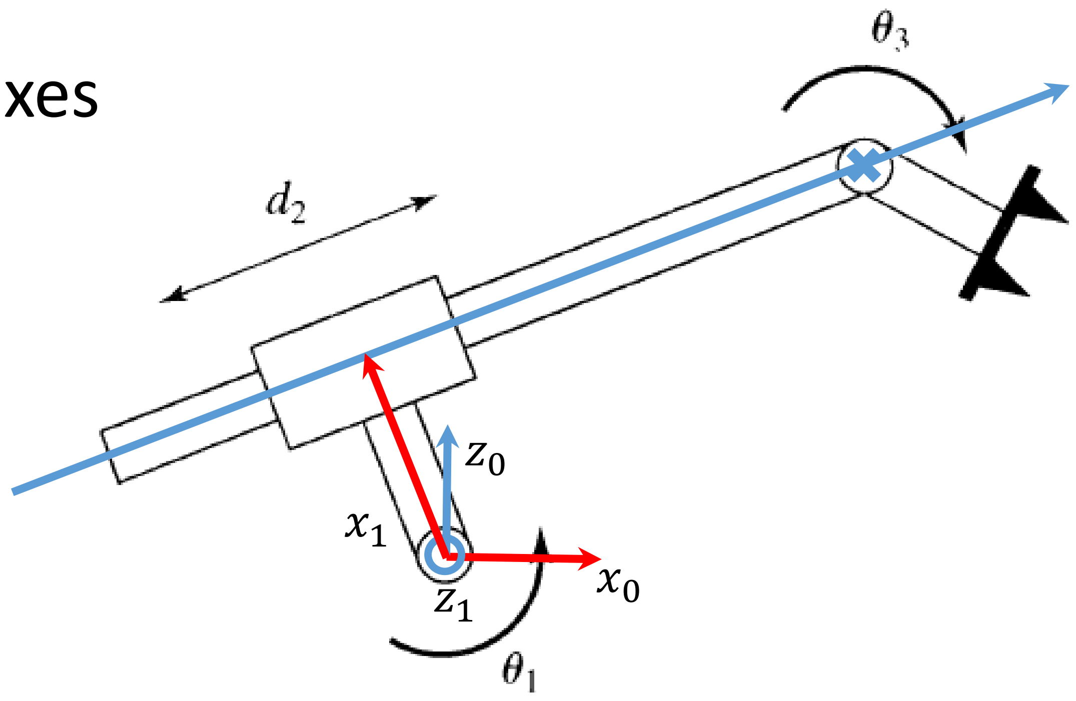
P09

- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes



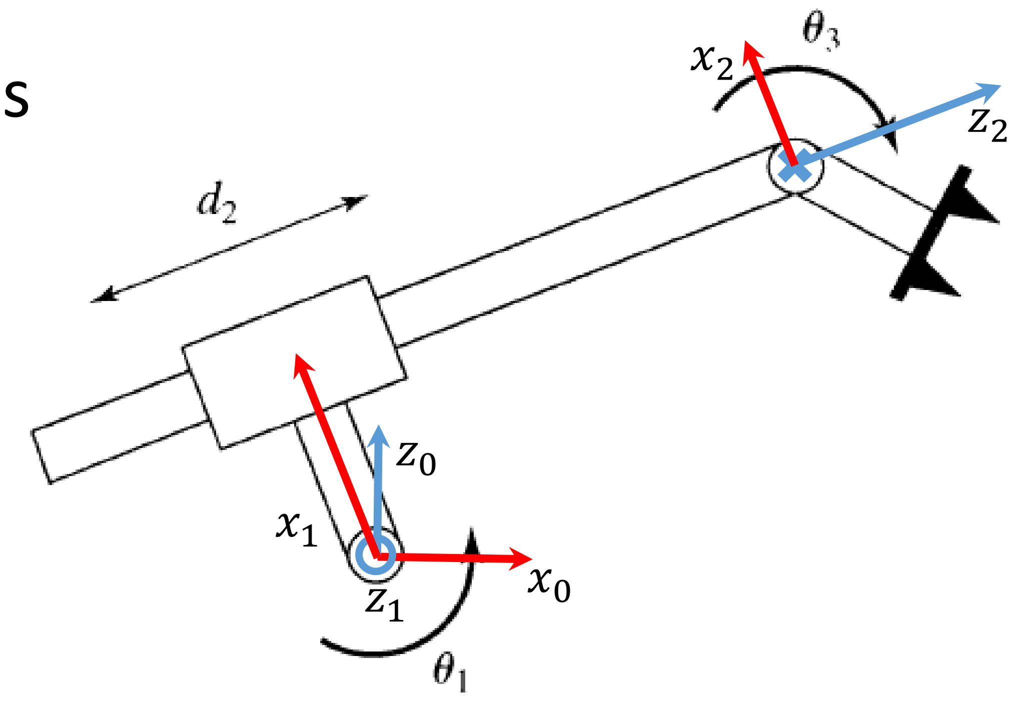
P09

- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes



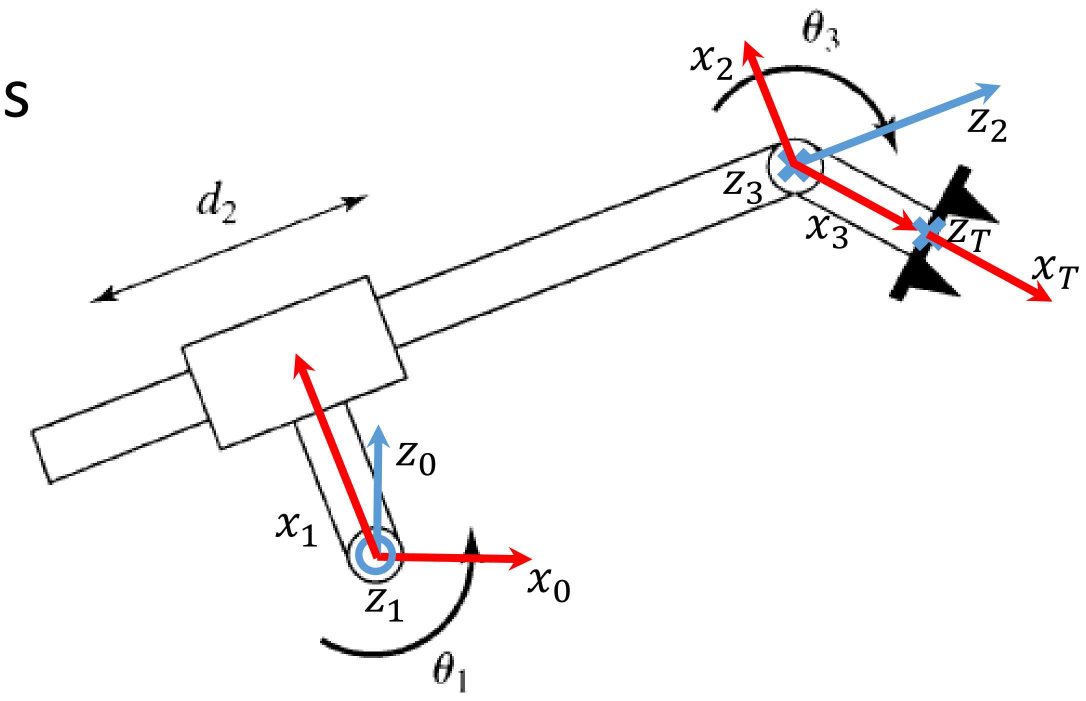
P09

- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes



P09

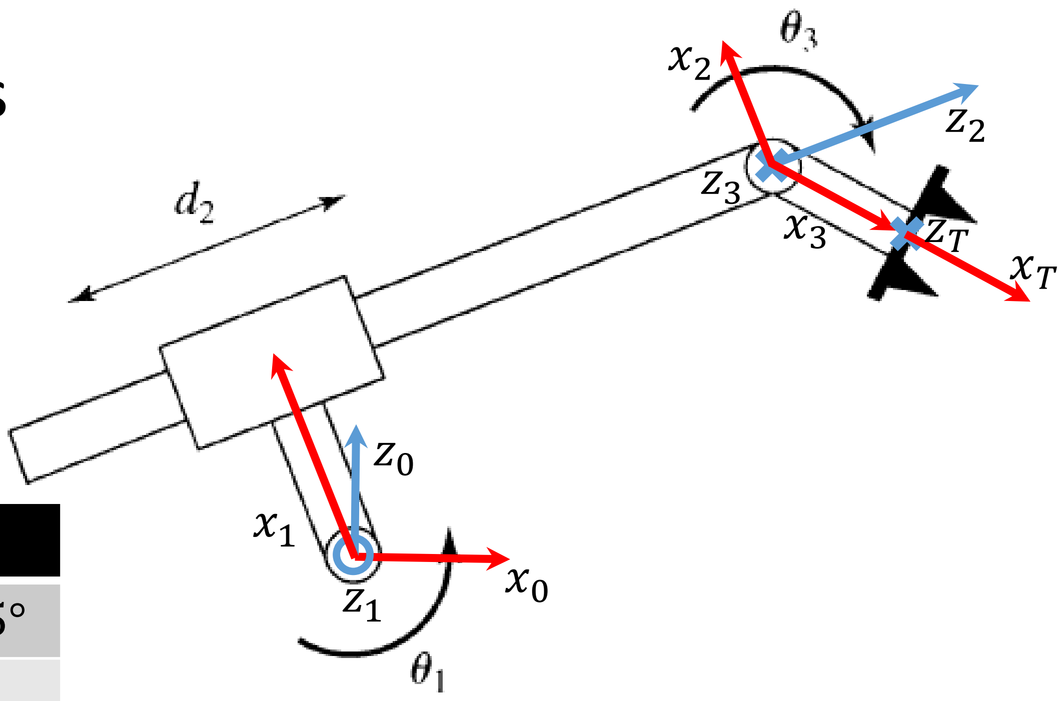
- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes



P09

- Step 1: Determine z-axes
- Step 2: Determine origins and x-axes
- Step 3: Fill out DH-Table

CF	α	a	d	Θ
1	$90°$	0	0	$\theta_1 = 105°$
2	$90°$	L_1	d_2	$0°$
3	$90°$	0	0	$\theta_3 = 135°$
T	$0°$	L_3	0	$0°$



P10

Problem 10

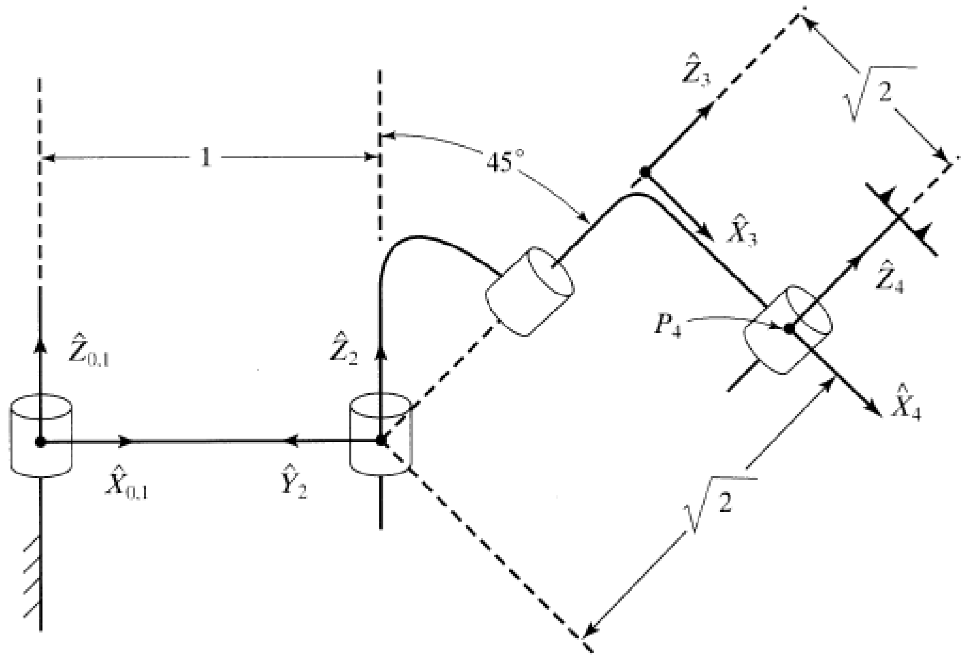


Figure 8: *4R Robot (Problem 10)*

For the 4R manipulator shown in Figure 8, we have the following set of DH parameters:

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0°	0	θ_1
2	1	0°	0	θ_2
3	0	45°	$\sqrt{2}$	θ_3
4	$\sqrt{2}$	0°	0	0

The manipulator is shown for the joint configuration

$$\Theta = [0°, 90°, -90°, 0°].$$

We ignore collisions between arms, so we assume that all joints can move freely within the range $[-180°, 180°]$. Determine values of θ_3 such that positions

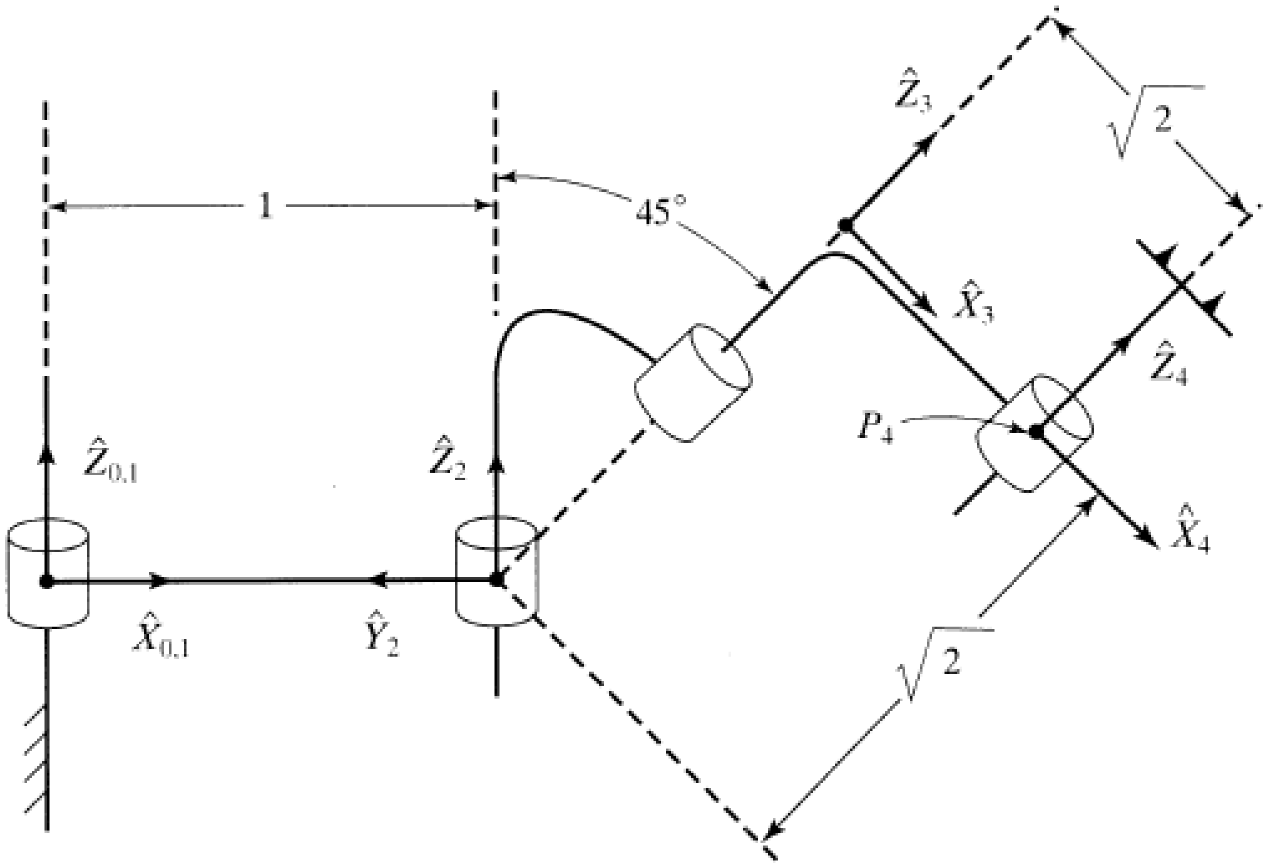
$$^0\mathrm{P}_{4\mathrm{ORG}} = [\bullet, \bullet, 1.707]^T$$

are reachable for the fourth frame.

P10

- Fill DH table
  (not considering joint 4)

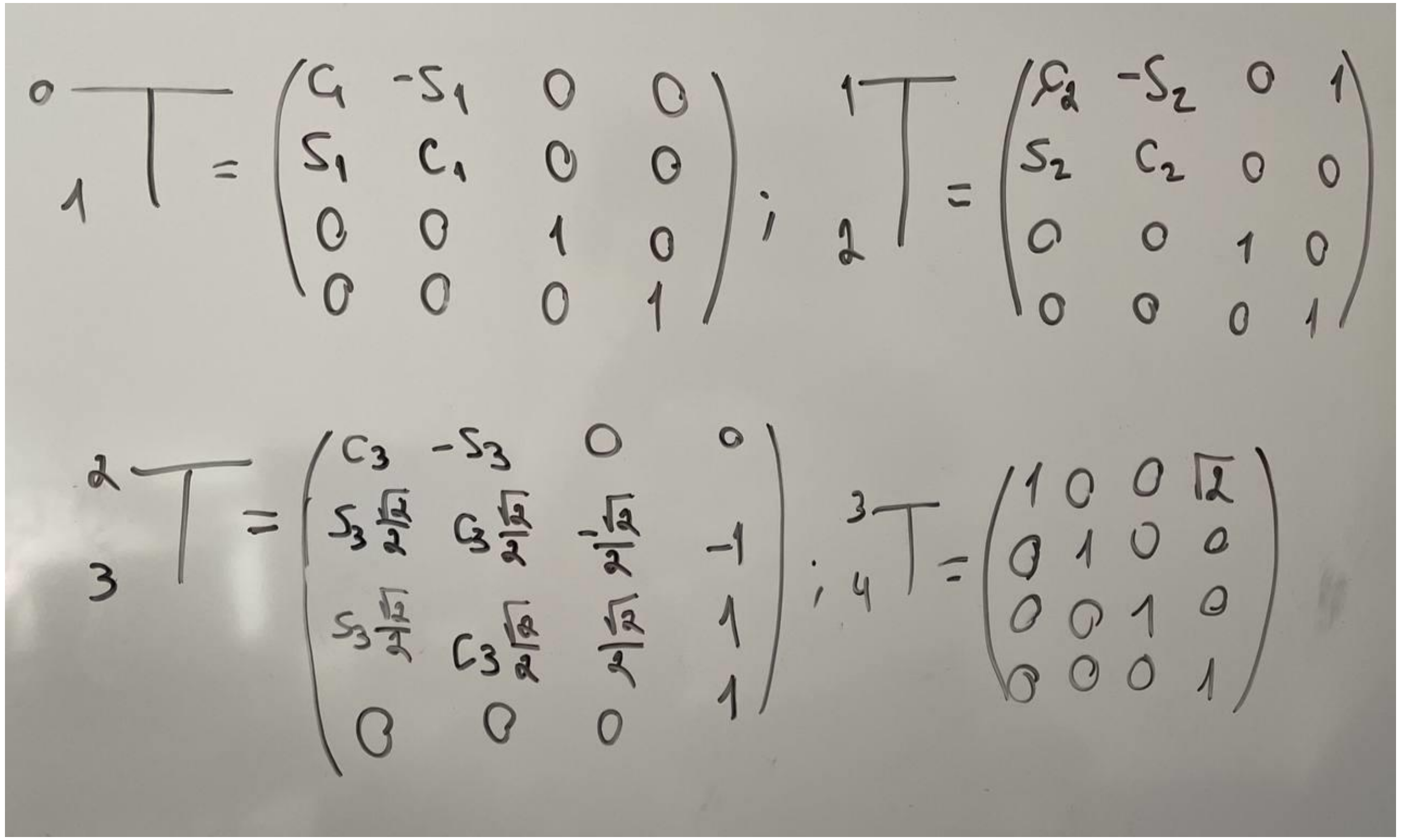
CF	α	a	d	Θ
1	$0°$	0	0	$\theta_1 = 0°$
2	$0°$	1	0	$\theta_2 = 90°$
3	$45°$	0	$\sqrt{2}$	$\theta_3 = -90°$
4	$0°$	$\sqrt{2}$	0	$0°$



P10

- Fill DH table
  (not considering joint 4)

CF	α	a	d	Θ
1	$0°$	0	0	$\theta_1 = 0°$
2	$0°$	1	0	$\theta_2 = 90°$
3	$45°$	0	$\sqrt{2}$	$\theta_3 = -90°$
4	$0°$	$\sqrt{2}$	0	$0°$



P10

- Fill DH table
  (not considering joint 4)

CF	α	a	d	Θ
1	$0°$	0	0	$\theta_1 = 0°$
2	$0°$	1	0	$\theta_2 = 90°$
3	$45°$	0	$\sqrt{2}$	$\theta_3 = -90°$
4	$0°$	$\sqrt{2}$	0	$0°$

$${}^{0}_{1}T = \begin{pmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad {}^{1}_{2}T = \begin{pmatrix} C_2 & -S_2 & 0 & 1 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^{2}_{3}T = \begin{pmatrix} C_3 & -S_3 & 0 & 0 \\ S_3\frac{\sqrt{2}}{2} & C_3\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -1 \\ S_3\frac{\sqrt{2}}{2} & C_3\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad {}^{3}_{4}T = \begin{pmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^{0}_{2}T = \begin{pmatrix} C_{12} & -S_{12} & 0 & C_1 \\ S_{12} & C_{12} & 0 & S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad {}^{2}_{4}T = \begin{pmatrix} C_3 & -S_3 & 0 & C_3\sqrt{2} \\ S_3\frac{\sqrt{2}}{2} & C_3\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & S_3 - 1 \\ S_3\frac{\sqrt{2}}{2} & C_3\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & S_3 + 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

P10

- Fill DH table (not considering joint 4)

CF	α	a	d	Θ
1	0°	0	0	$\theta_1 = 0°$
2	0°	1	0	$\theta_2 = 90°$
3	45°	0	$\sqrt{2}$	$\theta_3 = -90°$
4	0°	$\sqrt{2}$	0	0°

$$
{}^0_4T = \left(\begin{array}{ccc|c} & & & {}^0_4t_x \\ & {}^0_4R & & {}^0_4t_y \\ & & & S_3+1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)
$$

$$
{}^0_2T = \begin{pmatrix} C_{12} & -S_{12} & 0 & C_1 \\ S_{12} & C_{12} & 0 & S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\qquad
{}^2_4T = \begin{pmatrix} C_3 & -S_3 & 0 & C_3\sqrt{2} \\ S_3\frac{\sqrt{2}}{2} & C_3\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & S_3-1 \\ S_3\frac{\sqrt{2}}{2} & C_3\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & S_3+1 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

P10

- Fill DH table
  (not considering joint 4)

CF	α	a	d	Θ
1	$0°$	0	0	$\theta_1 = 0°$
2	$0°$	1	0	$\theta_2 = 90°$
3	$45°$	0	$\sqrt{2}$	$\theta_3 = -90°$
4	$0°$	$\sqrt{2}$	0	$0°$