Solution 1

Let $\vec{p}, \vec{q}$ be two unit vectors as specified in the problem description. The angles between those vectors as well as the vector's lengths remain unchanged after arbitrary rotations, and we have (with an arbitrary rotation matrix R):

$$\cos(\angle(\vec{p},\vec{q})) = \vec{p}^{\mathrm{T}}\vec{q} = (R\vec{p})^{\mathrm{T}}(R\vec{q}) = \cos(\angle(\vec{R}p,\vec{R}q)) \quad \Leftrightarrow$$
$$\vec{p}^{\mathrm{T}}\vec{q} = \vec{p}^{\mathrm{T}}R^{\mathrm{T}}R\vec{q}$$

This holds for all unit vectors $\vec{p}, \vec{q}$ and all rotation matrices R. In particular, we can choose the canonical unit vectors $\{(1,0,0)^{\mathrm{T}}, (0,1,0)^{\mathrm{T}}, (0,0,1)^{\mathrm{T}}\}$ for $\vec{p}, \vec{q}$, which yields $R^{\mathrm{T}}R = \mathrm{I}$, from which we can furthermore derive

$$R^{\mathrm{T}}R = \mathrm{I} \quad \Leftrightarrow \quad R^{-1} = R^{\mathrm{T}},$$

as well as

$$R^{\mathrm{T}}RR^{\mathrm{T}} = R^{\mathrm{T}} \quad \Leftrightarrow \quad RR^{\mathrm{T}} = \mathrm{I}. \quad \square$$

Recapitulation of Coordinate Transformations

When dealing with kinematics, we are primarily interested in the position and orientation of parts of the robot. Thus, we need some way to describe those quantities. It is clear that positions can be specified using a regular 3D vector $D \in \mathbb{R}^3$. For representing a rotation, several possibilities have been discussed in the lecture. We will, unless indicated otherwise, use the most straightforward representation, which is a rotation matrix R.

A very convenient way to specify rotation R and translation D simultaneously through specification of only one matrix is made possible through use of so-called homogeneous coordinates. This means that instead of using vectors $(x, y, z)^T$ from $\mathbb{R}^3$ to specify positions, we use vectors from $\mathbb{R}^4$ with a fourth component that is always set to 1:

$$\vec{p} = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Using this convention, it is possible to specify a complete coordinate transformation (rotation and translation) through one 4×4 matrix as follows:

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} & d_1 \\ r_{21} & r_{22} & r_{23} & d_2 \\ r_{31} & r_{32} & r_{33} & d_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Here, r_{ij} are the 9 components of the rotation matrix R, and d_i are the 3 translation parameters. Multiplying p with above matrix, we see that the result is:

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} & d_1 \\ r_{21} & r_{22} & r_{23} & d_2 \\ r_{31} & r_{32} & r_{33} & d_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} R \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \\ 1 \end{pmatrix}$$

Thus, we have found a way to specify rotation and translation that is at the same time computationally convenient and elegant. With this specification of coordinate transformations, we can also easily compute the result two or more consecutive transformations as follows:

$$^{A}_{C}\mathrm{T} = {}^{A}_{B}\mathrm{T} \cdot {}^{B}_{C}\mathrm{T}$$

Note that transformations $^{A}_{B}\mathrm{T}$ are to be interpreted as transformations from coordinates that are relative to frame B to frame A, or equivalently:

$$^{A}_{B}\mathrm{T}\,{}^{B}p = {}^{A}p$$

Which means that $^{A}_{B}\mathrm{T}$ will compute how a vector that is expressed in coordinate system B "looks like" if observed from coordinate system A.

Solution 2

After calibration, $\{T\}$ and $\{G\}$ are coincident. Thus, we know that the following equation holds:

$$^{B}_{W}\mathrm{T}\,{}^{W}_{T}\mathrm{T} = {}^{B}_{S}\mathrm{T}\,{}^{S}_{G}\mathrm{T}$$

In this equation, all transformations except for $^{W}_{T}\mathrm{T}$ are known. All those transformations are basically 4×4 full-rank matrices, so we can rearrange above equation to compute the transformation we are interested in:

$$^{W}_{T}\mathrm{T} = \left({}^{B}_{W}\mathrm{T}\right)^{-1} {}^{B}_{S}\mathrm{T}\,{}^{S}_{G}\mathrm{T}$$

Note that this exercise could of course also be solved using any coordinate transformation representation, but it is probably easier to solve when thinking of transformations in terms of 4×4 homogeneous matrices, as explained in the recapitulation above.

Solution 3

The frames of the coordinate systems are shown in Figure 1. The DH parameter set for this robot looks as follows:

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	$0°$	$l_1 + l_2$	θ_1
2	0	$90°$	0	θ_2
3	l_3	$0°$	0	θ_3
(4)	l_4	$0°$	0	$0°$

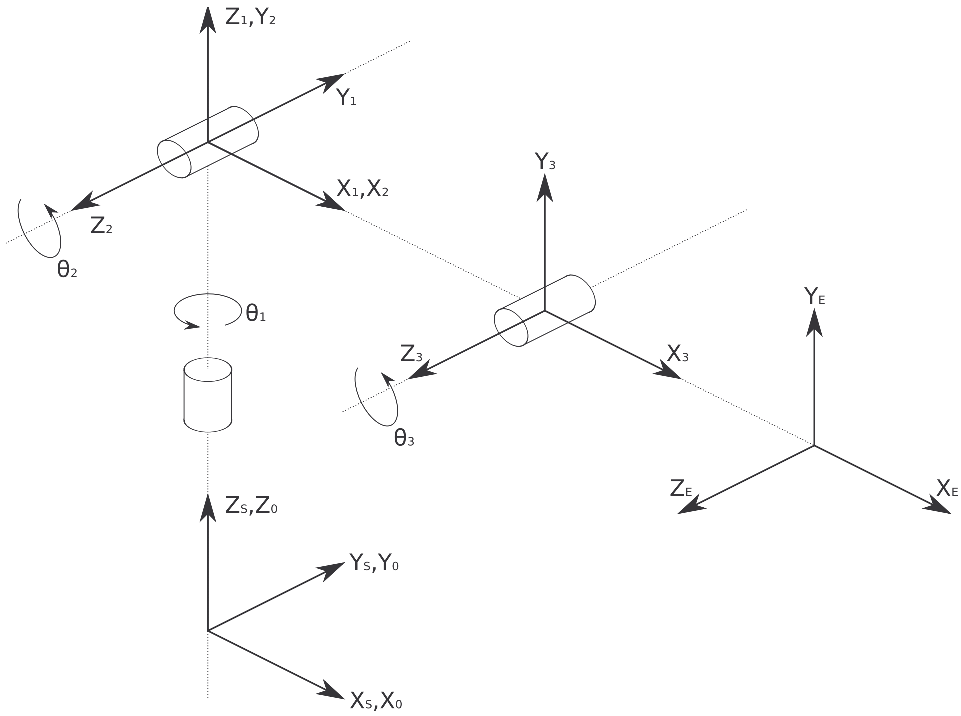

Figure 1: *Choice of coordinate systems according to DH convention (problem 3)*

When asked for the DH parameters of a robot, we would actually only need to specify the first three parameter sets, or the first three lines of above table. These transformations describe the position and orientation of the coordinate systems that are located at the beginning of each arm. Thus, to describe the tool position, a fourth transformation is needed, which is specified in the fourth line of above table. The fourth transformation in this case corresponds to a simple fixed translation along the x_3 axis.

After determining the parameters, we can formulate the translation matrices. After substituting the values above into the formula for translation matrices, we obtain:

$$ {}^{0}_{1}\mathrm{T} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & l_2 + l_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{1}_{2}\mathrm{T} = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{2}_{3}\mathrm{T} = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & l_3 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$
{}^{3}_{4}\mathrm{T} = \begin{pmatrix} 1 & 0 & 0 & l_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

For further computations, we can make use of the following trigonometric identities:

$$
\begin{aligned}
\sin(\theta_1 \pm \theta_2) &= \sin\theta_1 \cos\theta_2 \pm \cos\theta_1 \sin\theta_2 \\
\cos(\theta_1 \pm \theta_2) &= \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2
\end{aligned}
$$

Furthermore it is useful to recall that transformation matrices exhibit a certain structure that allows to simplify the following computations:

$$
T := \begin{pmatrix} R & t \\ \mathbf{0} & 1 \end{pmatrix}, T' := \begin{pmatrix} R' & t' \\ \mathbf{0} & 1 \end{pmatrix}, T \cdot T' = \begin{pmatrix} R \cdot R' & R \cdot t' + t \\ \mathbf{0} & 1 \end{pmatrix}
$$

The matrices R, R' are rotation matrices of size 3×3, and t, t' are 3 element column vectors, and $\mathbf{0}$ is to be interpreted as row vector $(0, 0, 0)$. We see that coordinate transformation matrices can be subdivided into blocks, and that the product of transformation matrices can be broken down into somewhat simpler products of those blocks.

Overall, we obtain the following result (using the trigonometric identities indicated above):

$$
{}^{0}_{1}\mathrm{T}{}^{1}_{2}\mathrm{T}{}^{2}_{3}\mathrm{T} = \begin{pmatrix} \cos\theta_1 \, \cos(\theta_3 + \theta_2) & -\cos\theta_1 \, \sin(\theta_3 + \theta_2) & \sin\theta_1 & l_3 \, \cos\theta_1 \, \cos\theta_2 \\ \sin\theta_1 \, \cos(\theta_3 + \theta_2) & -\sin\theta_1 \, \sin(\theta_3 + \theta_2) & -\cos\theta_1 & l_3 \, \sin\theta_1 \, \cos\theta_2 \\ \sin(\theta_3 + \theta_2) & \cos(\theta_3 + \theta_2) & 0 & l_3 \, \sin\theta_2 + l_2 + l_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

One final remark, for the sake of completeness: While this would be enough to fully answer the question posed on the exercise sheet (and it would be enough to answer a similar question posed in an exam), it is not sufficient for determining the location and orientation of the tool. To calculate this, we need to calculate the product ${}^{0}_{1}\mathrm{T} \cdots {}^{3}_{4}\mathrm{T}$, which is:

$$
\begin{pmatrix} \cos\theta_1 \, \cos(\theta_3 + \theta_2) & -\cos\theta_1 \, \sin(\theta_3 + \theta_2) & \sin\theta_1 & \cos\theta_1 \, (l_4 \, \cos(\theta_3 + \theta_2) + l_3 \, \cos\theta_2) \\ \sin\theta_1 \, \cos(\theta_3 + \theta_2) & -\sin\theta_1 \, \sin(\theta_3 + \theta_2) & -\cos\theta_1 & \sin\theta_1 \, (l_4 \, \cos(\theta_3 + \theta_2) + l_3 \, \cos\theta_2) \\ \sin(\theta_3 + \theta_2) & \cos(\theta_3 + \theta_2) & 0 & l_4 \, \sin(\theta_3 + \theta_2) + l_3 \, \sin\theta_2 + l_2 + l_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Solution 4

The z_i-axes are chosen such that they lie along the joint axes, and thus all of the z_i-axes are parallel in this exercise. The common perpendiculars (which determine the x_i-axes) can be chosen freely in the sense that the z-coordinate of the origin can be chosen arbitrarily along the joint axis. It makes most sense, however, to have the origin at the joint center, since the origin usually lies there. This has the advantage that, if all origins are placed like this, all of the d_i parameters are 0, which in turn simplifies the calculation. Positions and orientations of the coordinate systems are shown in Figure 2.

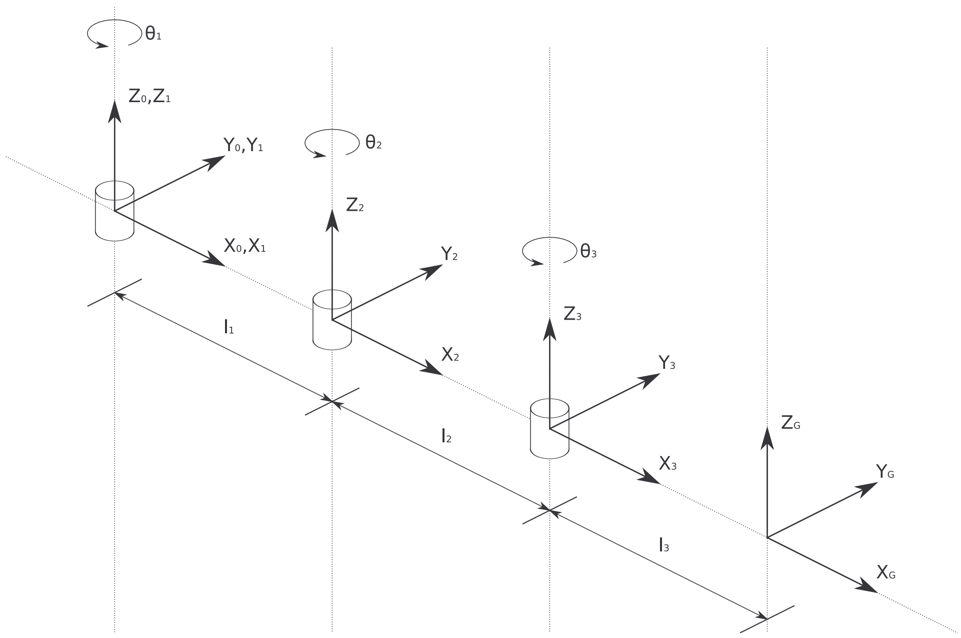

Figure 2: *Robot from problem 4*

a)

The joint variables are denoted by $\theta_1, \theta_2, \theta_3$, roboter constants are l_1, l_2, l_3.

b)

The DH parameters are:

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0°	0	θ_1
2	L_1	0°	0	θ_2
3	L_2	0°	0	θ_3
(4)	L_3	0°	0	0°

Which means that the transformations look like this:

$$
{}^0_1\mathrm{T} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^1_2\mathrm{T} = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^2_3\mathrm{T} = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^3_G\mathrm{T} = \begin{pmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

The overall rotation can easily be determined by exploiting the fact that all rotations occur about the same axis. This means that the orientation of the gripper can be described using only one angle, and that angle will be $\theta_1 + \theta_2 + \theta_3$.

Furthermore, it is also quite easy to determine the origin position of the goal frame using geometric reasoning. The orientations of the coordinate systems $\{1\}, \{2\}, \{3\}$ are determined by the angles θ_1, $\theta_1 + \theta_2$, $\theta_1 + \theta_2 + \theta_3$, respectively. The transformations occuring are shifts in X_1, X_2, X_3 directions, which are easily computed as

$$l_1 \begin{pmatrix} \cos(\theta_1) \\ \sin(\theta_1) \end{pmatrix}, \quad l_2 \begin{pmatrix} \cos(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) \end{pmatrix}, \quad l_3 \begin{pmatrix} \cos(\theta_1 + \theta_2 + \theta_3) \\ \sin(\theta_1 + \theta_2 + \theta_3) \end{pmatrix}.$$

All these translations add up, so we can conclude that the overall transformation must be:

$$\begin{pmatrix} \cos(\theta_3 + \theta_2 + \theta_1) & -\sin(\theta_3 + \theta_2 + \theta_1) & 0 & l_3 \cos(\theta_3 + \theta_2 + \theta_1) + l_2 \cos(\theta_2 + \theta_1) + l_1 \cos\theta_1 \\ \sin(\theta_3 + \theta_2 + \theta_1) & \cos(\theta_3 + \theta_2 + \theta_1) & 0 & l_3 \sin(\theta_3 + \theta_2 + \theta_1) + l_2 \sin(\theta_2 + \theta_1) + l_1 \sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

c)

The tool position for above robot can be completely specified through X, Y coordinates together with an angle describing the desired gripper orientation. Figure 3 shows the situation in a top-down view. You can see the arm maximally stretched, in a configuration where all angles θ_i are 0, and another configuration with nonzero angles. Furthermore, gripper and third link are shown for two unreachable positions in red.

With the following reasoning, it is easy to establish a reachability test for this robot. We shall denote the X, Y coordinates of the desired gripper position with ${}^0P_G \in \mathbb{R}^2$, and the desired gripper orientation with θ_G, measured against the orientation of the base system $\{0\}$.

- The complete position description determines position and orientation of the gripper. This in turn determines the X, Y-position of the third joint. We shall denote this position with ${}^0P_3 \in \mathbb{R}^2$, and we note that it can be computed from ${}^0P_G$ and θ_G by shifting in negative X_G direction by l_3:

$${}^0P_3 = {}^0P_G - l_3 \begin{pmatrix} \cos(\theta_G) \\ \sin(\theta_G) \end{pmatrix}$$

- The configuration θ_1, θ_2 of the first two joints also determines the X, Y location of the third joint. In Figure 3, for the fully stretched configuration, where $\theta_1 = 0$, the set of reachable positions (letting θ_2 range from $0°$ to $360°$) is the line of the blue circle. You can imagine that, as θ_1 ranges from $0°$ to $360°$, the circle rotates about the origin, making up the grey ring-shaped area describing all reachable positions for system $\{3\}$.

- Finally, we see that a position is reachable exactly when ${}^0P_3$, as induced by ${}^0P_G$ and θ_G, is within the grey area. Note that for the two unreachable positions, the positions of the third joints are outside that area. We can also specify this mathematically:

$$l_1 - l_2 \leq \left| {}^0P_G - l_3 \begin{pmatrix} \cos(\theta_G) \\ \sin(\theta_G) \end{pmatrix} \right| \leq l_1 + l_2$$

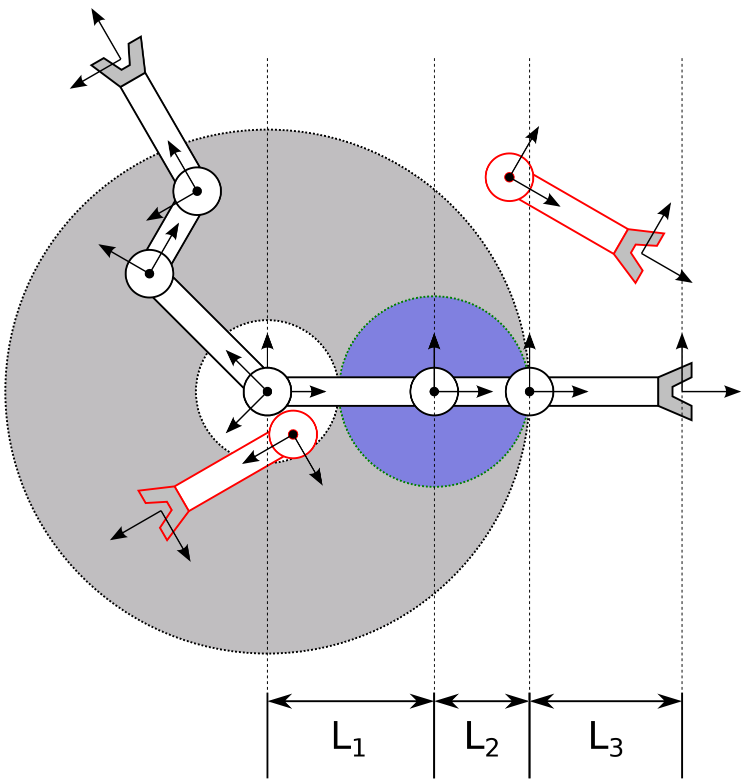

Figure 3: *Check of reachability for the 3R manipulator.*

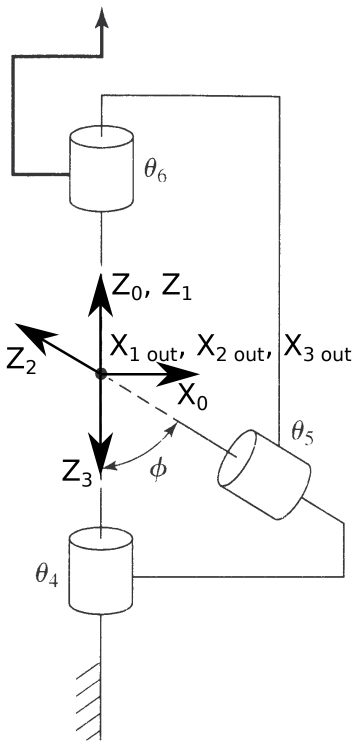

Figure 4: *Solution to Problem 6*

Solution 5

The tip position relative to system $\{2\}$ is specified in the drawing as:

$$^{2}P_{\text{tip}} = \begin{pmatrix} l_2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

This means that the tip position, as seen from the coordinate system $\{2\}$, is just shifted in x direction about an offset of l_2. This means that we can compute the sought tip position $^{0}P_{\text{tip}}$ simply as:

$$^{0}P_{\text{tip}} = {}^{0}_{2}\mathrm{T}\,{}^{2}P_{\text{tip}} = \begin{pmatrix} \mathrm{c}_1\mathrm{c}_2 & -\mathrm{c}_1\mathrm{s}_2 & \mathrm{s}_1 & l_1\mathrm{c}_1 \\ \mathrm{s}_1\mathrm{c}_2 & -\mathrm{s}_1\mathrm{s}_2 & -\mathrm{c}_1 & l_1\mathrm{s}_1 \\ \mathrm{s}_2 & \mathrm{c}_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} {}^{2}P_{\text{tip}} = \begin{pmatrix} l_2\mathrm{c}_1\mathrm{c}_2 + l_1\mathrm{c}_1 \\ l_2\mathrm{s}_1\mathrm{c}_2 + l_1\mathrm{s}_1 \\ l_2\mathrm{s}_2 \\ 1 \end{pmatrix}$$

Solution 6

You can see the result of the link frame attachment procedure in Figure 4. Note that the coordinate systems $\{0\}$ and $\{3\}$ are not unique. The coordinate systems have been chosen to make things as simple as possible, which explains why they all have the same origin in the intersection point of the joint axes. Also, note that there are some ambiguities in the original problem description: First of all, rotation directions are not indicated (rotation directions influence the directions of the z axes), and also the direction for x axes can be chosen freely whenever joint axes intersect, so different solutions are possible.

The DH parameter set, based on the indicated choice of coordinate frames, looks like this:

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	$0°$	0	$\theta_1 = -90°$
2	0	ϕ	0	$\theta_2 = 0°$
3	0	$180° - \phi$	0	$\theta_3 = 0°$

Solution 7

There is a simple way to see immediately that it is not possible to describe arbitrary coordinate system transformations using DH-style transformations: Look at the entry in row 1 and column 3 of the transformation matrix. You'll see that this entry is always 0. This means that any rotation that would result in a nonzero entry there cannot be realized using this transformation formula.

Analyzing the situation further, one can see that this observation corresponds to the fact that no rotation around the y axis is accounted for within the DH convention. There are only two rotations, one about the x axis and one about the z axis, and three rotations would be needed to allow arbitrary rotations.

Please be aware that this means that one of the rules of the DH convention, as stated in Craig's book, is **not** correct. Since we have just seen that the DH convention is not powerful enough to describe general coordinate transformations, it is obviously not possible to freely choose position and orientation of frame $\{0\}$, as opposed to what is claimed in Craig's book in the elaboration about first and last links in the chain (the respective paragraph starts with "Frame $\{0\}$ is arbitrary, ...").

Solution 8

The solution is shown in Figure 5. Note that the solution is not unique: First of all, rotation directions are not indicated (rotation directions influence the directions of the z axes), and also the direction for x axes can be chosen freely whenever joint axes intersect. The coordinate systems $\{4\}$ and $\{5\}$ are chosen such that d_5 and d_6 are 0 - this is also the reason why a gripper coordinate system has been introduced.

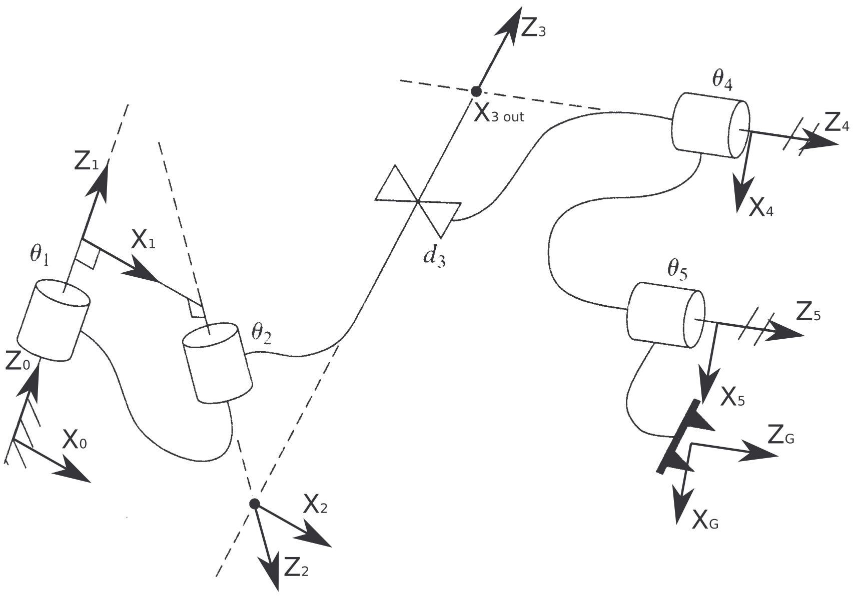

Figure 5: *Solution to Problem 8*

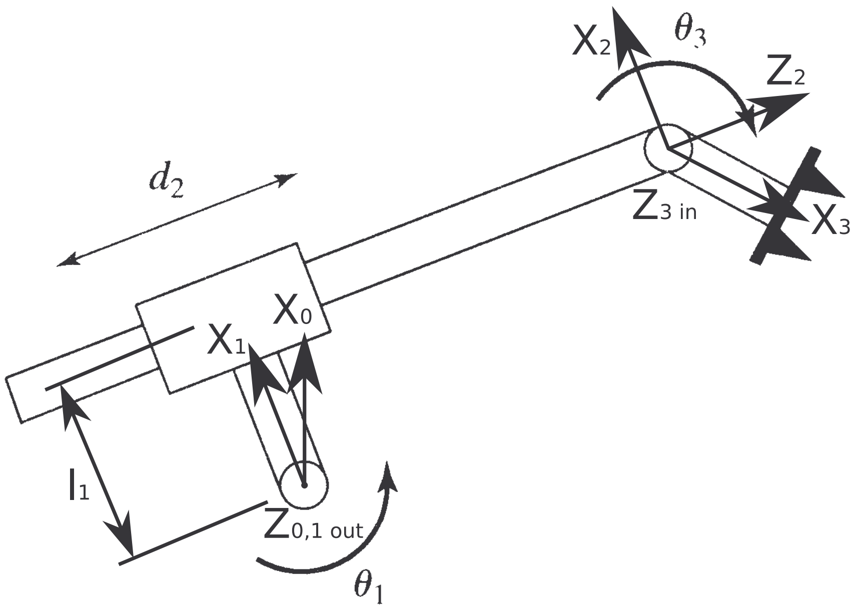

Figure 6: *Solution to Problem 9*

Solution 9

The solution is shown in Figure 6. Again, the coordinate frame choices are not unique. The DH parameters are, with estimates for parameters of the current configuration:

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	$0°$	0	$\theta_1 \approx 15°$
2	l_1	$90°$	d_2	$0°$
3	0	$90°$	0	$\theta_3 \approx 135°$

Note that the base coordinate system has been chosen arbitrarily such that the X_0-axis points "upwards." Furthermore, the third coordinate system has also been chosen arbitrarily in such a way that the X_3-axis is pointing along the final link of the robot. Be aware that, even though this choice evidently makes sense, it is **not** a result of the DH convention, and by no means enforced by it.

Solution 10

In this exercise we are dealing with a 3R robot. A fourth rotational axis is shown in the diagram, but this joint is actually treated as end effector. The coordinate systems can be derived as usual according to the DH conventions. They are shown in Figure 7.

The DH parameters of the system look like this:

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	$0°$	0	θ_1
2	1	$0°$	0	θ_2
3	0	$45°$	$\sqrt{2}$	θ_3
4	$\sqrt{2}$	$0°$	0	0

Now we are supposed to compute θ_3 such that the end effector position with 3D coordinates $(\bullet, \bullet, 1.707)^{\mathrm{T}}$ can be reached. If we look closely at the system, we notice that the z coordinate of the end effector can only be influenced through θ_3. Any rotation about the first and second joint axes will *not* change the Z-coordinate of the end effector origin at all. This explains why we are asked to compute only θ_3 in order to reach a certain z coordinate.

This can be motivated as follows: When computing the inverse kinematics for any robot, we should try to make our lives easier by finding a small sub-task that is easy to solve. This can, in many cases, be substantially easier than solving the whole problem of inverse kinematics at once. In our case, it turns out that fixing θ_3 first is really easy to do. The rest of the inverse kinematics procedure (which is not performed here, due to lack of time) is then reduced to a simple inverse kinematics computation for a 2R robot as discussed in the lecture.

Concentrating on the problem of adjusting θ_3 such that a specific Z-coordinate is attained for the goal frame 4, we can focus on the transformation between the second

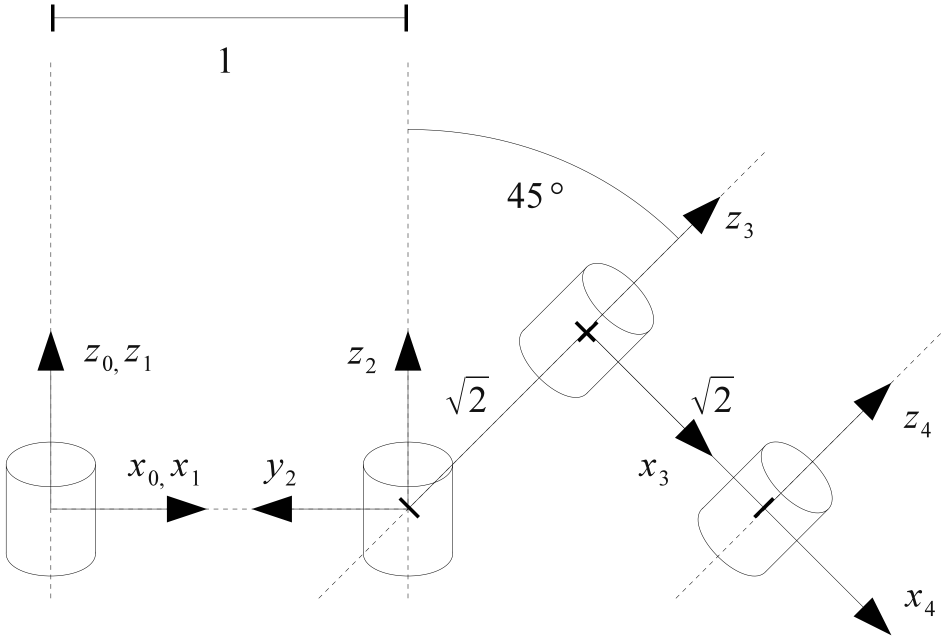

Figure 7: *Robot from problem* 10*, under configuration* $\theta_1 = 0°, \theta_2 = 90°, \theta_3 = -90°$.

and the fourth system, according to the argumentation above. Computing ${}^2_4\mathrm{T}\,{}^4O_4 = {}^2_4\mathrm{T}(0,0,0,1)^T$ will give the position of the origin of system 4 in the coordinate frame of system 2.

The immediate idea to approach this problem might be as follows:

a) Compute ${}^2_4\mathrm{T}$ by carrying out the matrix multiplication ${}^2_3\mathrm{T}\,{}^3_4\mathrm{T}$.

b) Multiply the resulting matrix with $(0,0,0,1)^T$.

c) Look at the third entry of the resulting vector (the Z coordinate), and determine how θ_3 influences its value.

This procedure is definitely correct and would yield a helpful result. However, it is also extremely inefficient! If we closely analyze what we are doing there, we see that a lot of intermediate results are computed that are ignored in the end:

- We compute the full matrix ${}^2_4\mathrm{T}$, even though we are only really interested in it's last column. This is because ${}^2_4\mathrm{T}(0,0,0,1)^T$, computed in the second step, extracts only that column.

- Furthermore, from that column, we are only interested in the third component, which corresponds to the Z coordinate. So why should we compute the X and Y coordinates?

This already suggests that there should be a more efficient way to perform the required computation. What we are ultimately computing using the three-step recipe described above is the following:

$$\begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix} {}^2_3\mathrm{T}{}^3_4\mathrm{T} \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Note that the multiplication of some homogeneous coordinate vector from the left side with the row vector $(0, 0, 1, 0)$ extracts the Z coordinate. As we shall see, we can evaluate this formula in a quite elegant way with minimal effort.

The first step of above computation could then be:

$$\begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix} {}^2_3\mathrm{T}{}^3_4\mathrm{T} \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix} {}^2_3\mathrm{T} \left({}^3_4\mathrm{T} \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right) = \begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix} {}^2_3\mathrm{T} \begin{pmatrix}\sqrt{2}\\0\\0\\1\end{pmatrix}$$

So, instead of multiplying the coordinate transformation matrices ${}^2_3\mathrm{T}$ and ${}^3_4\mathrm{T}$, we directly extract the last column of ${}^3_4\mathrm{T}$. The DH parameter set tells us that the last column of that transformation represents a translation in X direction by $\sqrt{2}$.

Finally, we have to evaluate the vector-matrix-vector product above, to compute the final result. Taking into consideration the structure of both vectors (i.e., their zero entries), we see that we do not need to form the whole transformation matrix ${}^2_3\mathrm{T}$. Instead, we merely need to determine two values of that matrix:

$$\begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix} \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \frac{\sqrt{2}\sin(\theta_3)}{2} & \bullet & \bullet & 1 \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix} \begin{pmatrix}\sqrt{2}\\0\\0\\1\end{pmatrix} = \sin(\theta_3) + 1$$

All entries of the transformation matrix that we do not explicitely need to compute are marked with the $\bullet$ symbol. We have also used the fact that $\sin(\theta_3)\cos(\alpha_2) = \sin(\theta_3)\cos(45°) = \sin(\theta_3)\frac{\sqrt{2}}{2}$.

The full transformation matrices, just for reference, look as follows:

$${}^2_3\mathrm{T} = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & 0 \\ \frac{\sin\theta_3}{\sqrt{2}} & \frac{\cos\theta_3}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -1 \\ \frac{\sin\theta_3}{\sqrt{2}} & \frac{\cos\theta_3}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^3_4\mathrm{T} = \begin{pmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It is easily seen that explicitely computing ${}^2_3\mathrm{T}{}^3_4\mathrm{T}$, or even explicitely writing down ${}^2_3\mathrm{T}$, causes a lot of unnecessary work.

Finally, we see that we have to solve $\sin\theta_3 + 1 = 1.707$, or equivalently $\sin\theta_3 = 0.707$. The solution to this, as given by a calculator, is a value of $\theta_3 = 45°$. Knowing that the sin function is symmetric about the value $90°$, we conclude that there is another solution at $90° + 45° = 135°$. So, all solutions are characterized by $\theta_3 \in \{45°, 135°\}$. Those two solutions are mirrored on the y_2, z_2-plane.