Tutorial Robotics IN2067

Exercise Sheet 06

P01

Problem 1

For the RP manipulator shown in Figure 1, we assume the following parameters:

$$l_1 = 0.2, m_1 = 1.$$

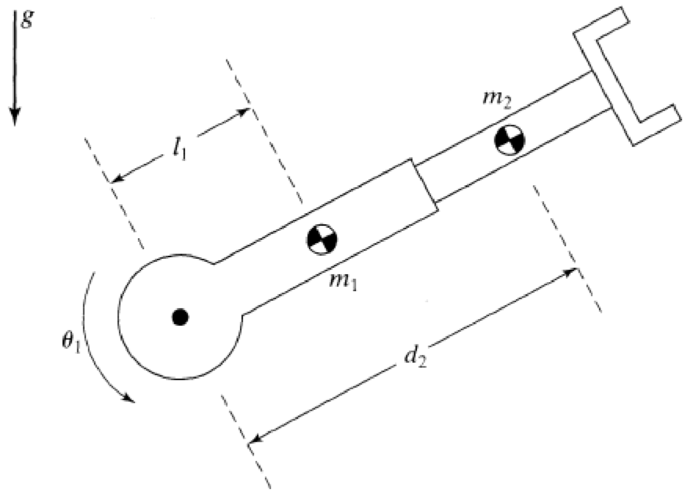

Figure 1: *RP Robot (Problem 2)*

a) Determine the matrices M, V, G of the state space form of the dynamic equations using Lagrange's method, assuming that the inertia tensors are

$$^{C_1}I_1 = \begin{pmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & 0.1 \end{pmatrix}, \quad ^{C_2}I_2 = \begin{pmatrix} I_{xx2} & 0 & 0 \\ 0 & 0.07 & 0 \\ 0 & 0 & I_{zz2} \end{pmatrix}.$$

b) The system is operated through a model-driven PD controller. Determine the form of the matrices α and the vectors β and τ', treating the factors k_{vi} and k_{pi} as variables.

c) Determine values of k_{vi} and k_{pi} such that closed-loop frequencies are 20 rad/s and 25 rad/s for both joints and such that the system is critically damped.

d) Draw a block diagram of the controller.

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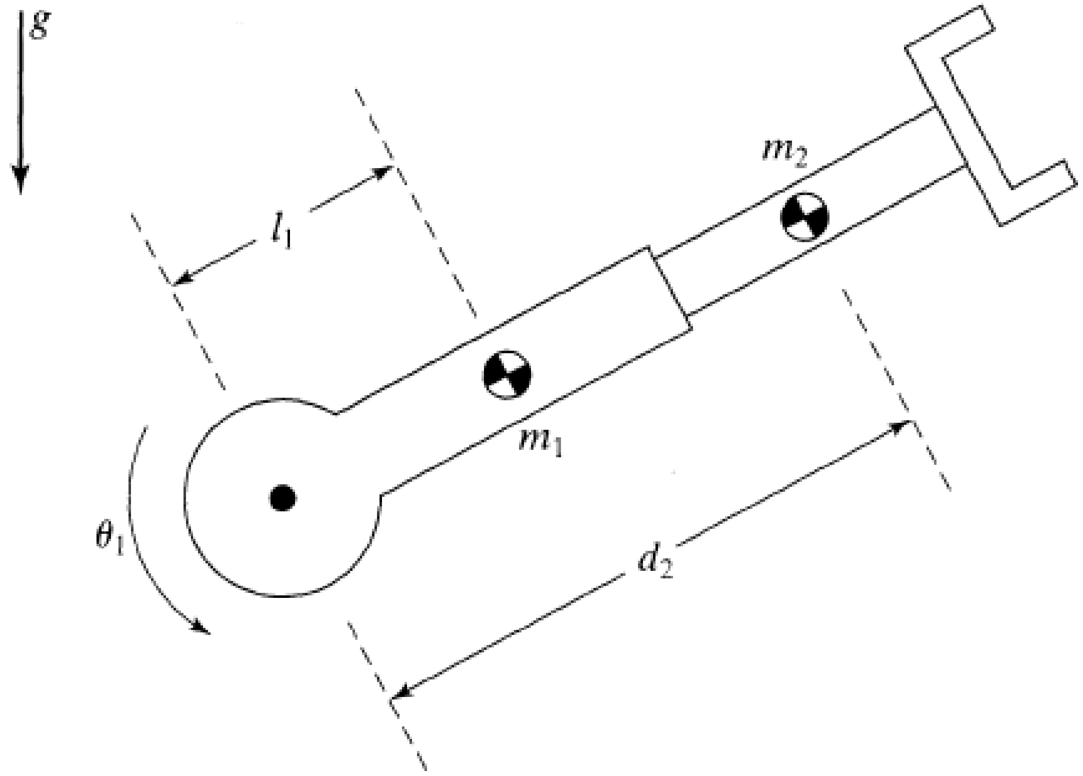

Figure 1: *RP Robot (Problem 2)*

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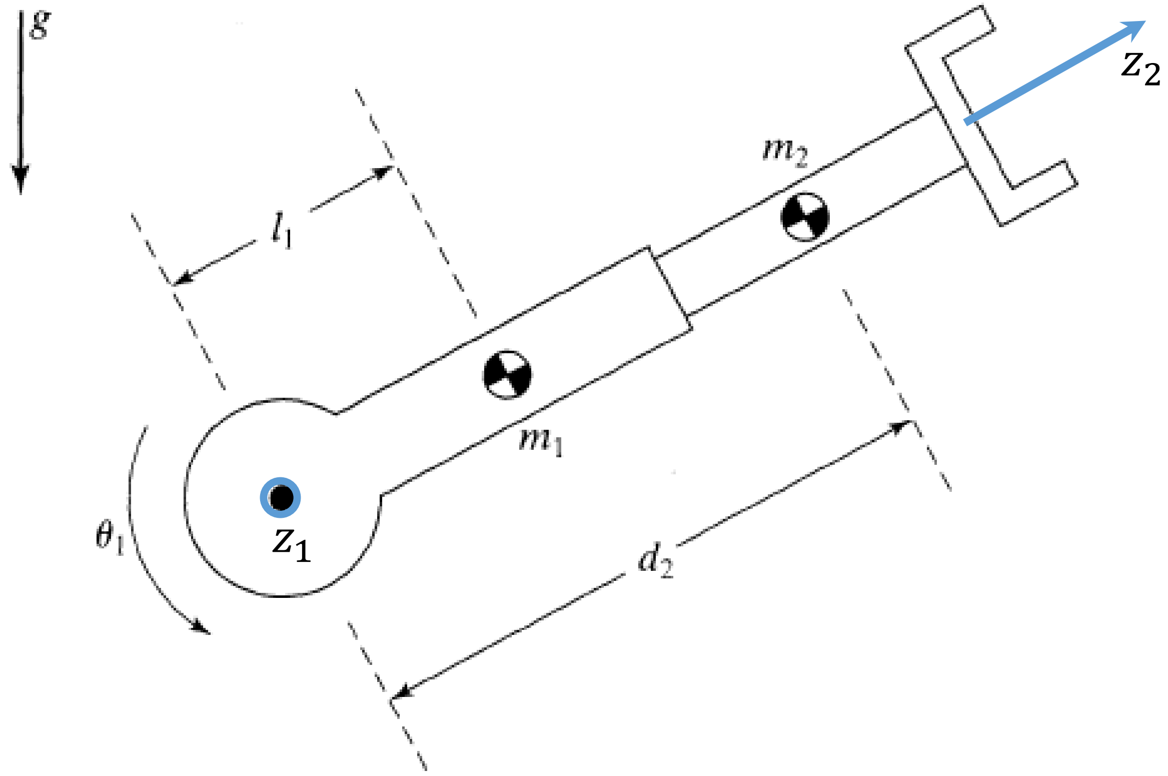

Figure 1: *RP Robot (Problem 2)*

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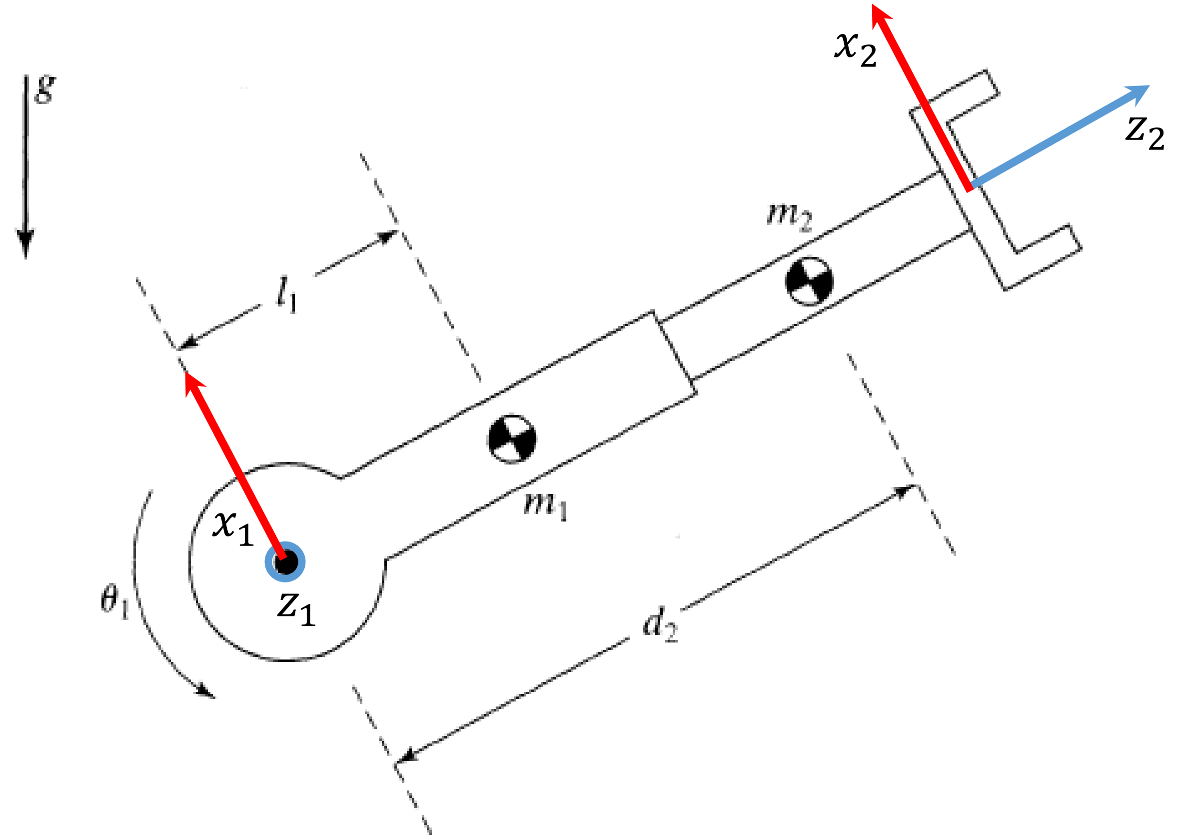

Figure 1: *RP Robot (Problem 2)*

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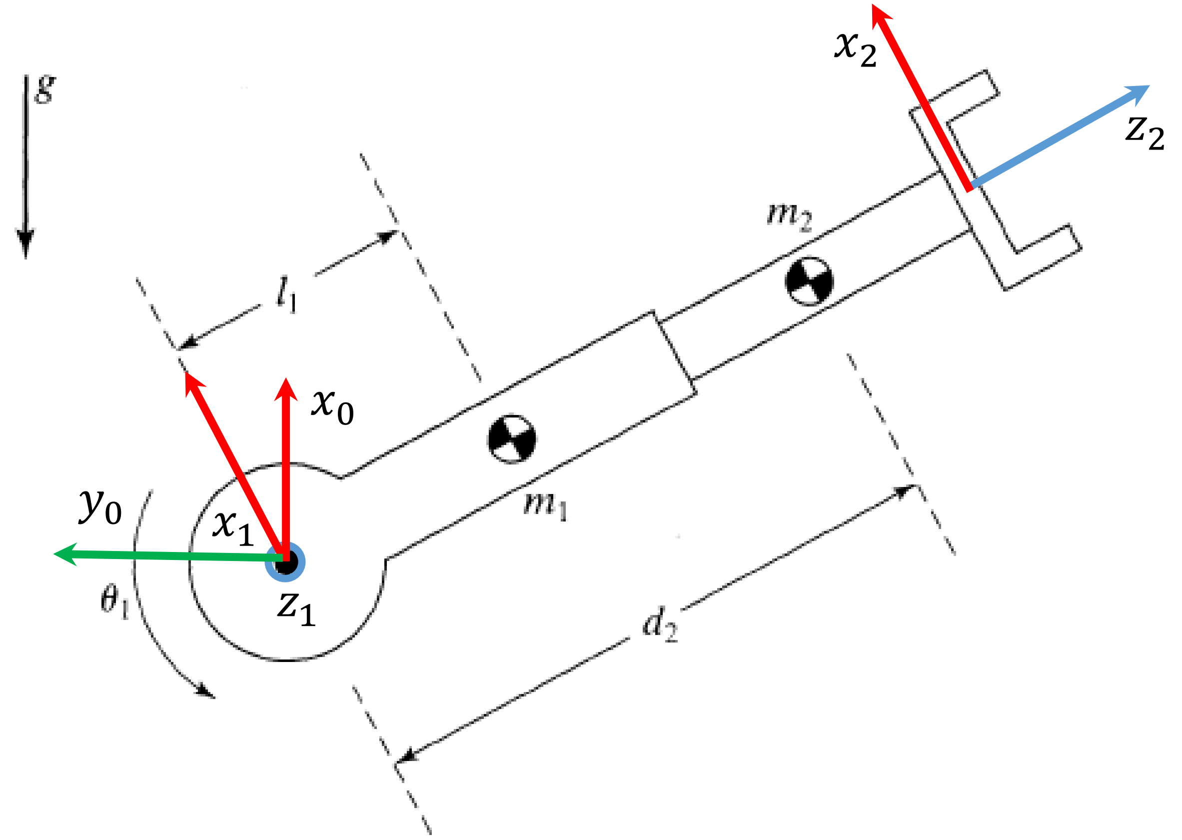

Figure 1: *RP Robot (Problem 2)*

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DH- Table:

	α	a	d	θ
1	0°	0	0	θ_1
2	90°	0	d_2	0°

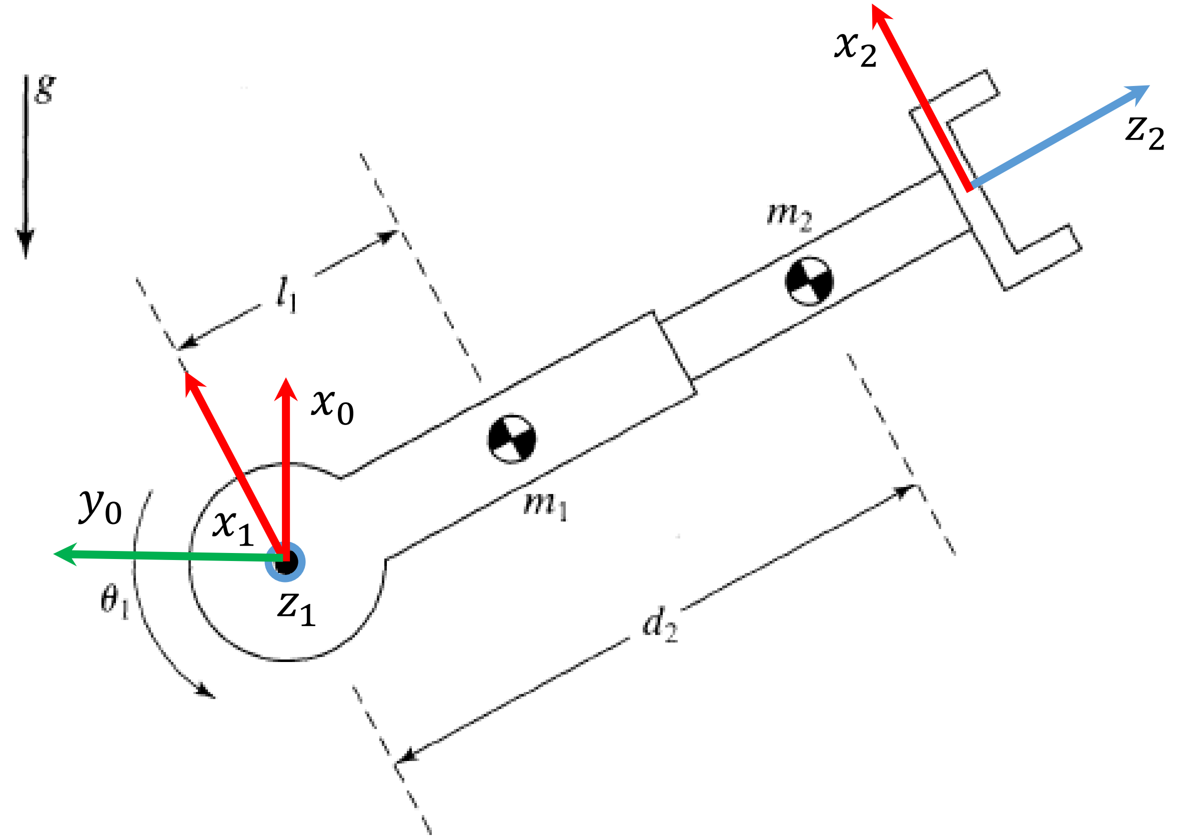

Figure 1: *RP Robot (Problem 2)*

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DH-Table:

	α	a	d	θ
1	$0°$	0	0	θ_1
2	$90°$	0	d_2	$0°$

$${}^{1}_{2}R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow {}^{2}_{1}R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

gravity vector $= \begin{pmatrix} -g \\ 0 \\ 0 \end{pmatrix}$

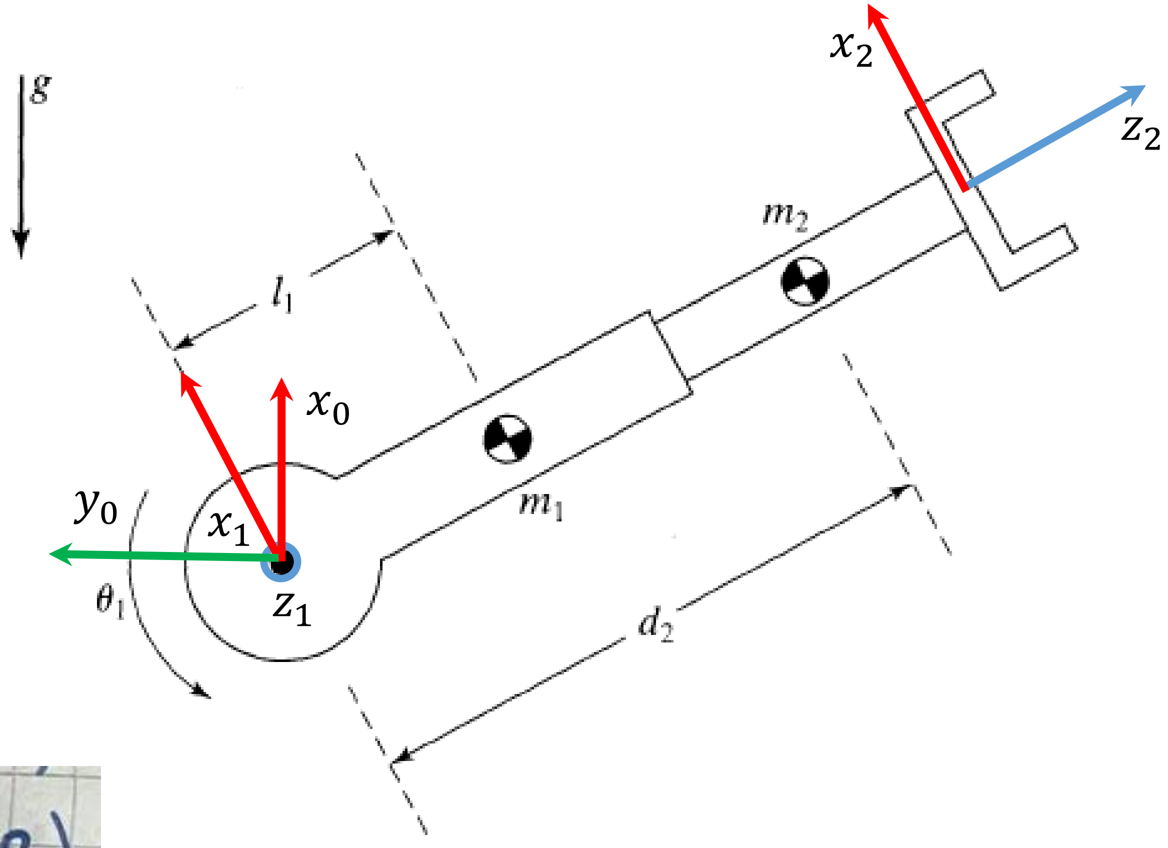

Figure 1: *RP Robot (Problem 2)*

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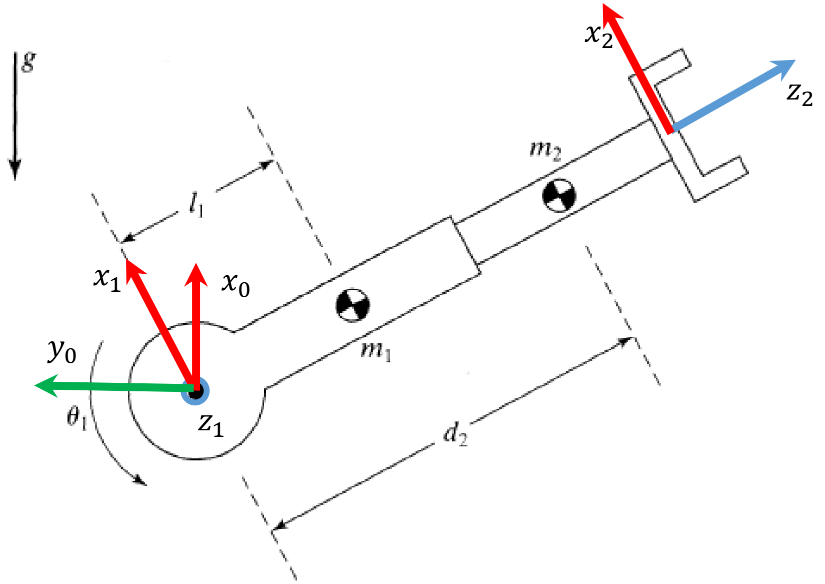
- Lagrange Method



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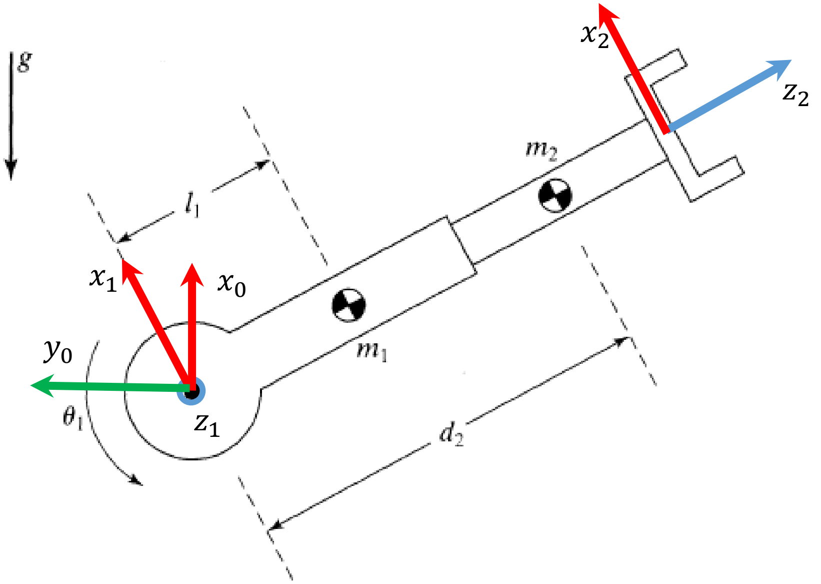
- Lagrange Method

$$ {}^1\omega_1 = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} \qquad {}^2\omega_2 = {}^2_1R\,{}^1\omega_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{pmatrix} $$



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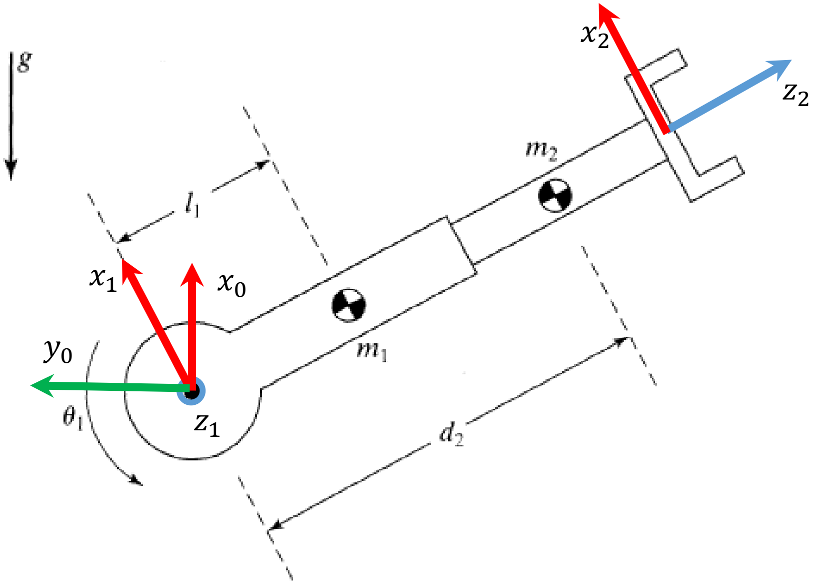

- Lagrange Method

$$ {}^1\omega_1 = \begin{pmatrix} 0 \\ 0 \\ \dot\theta_1 \end{pmatrix} \qquad {}^2\omega_2 = {}^2_1R\,{}^1\omega_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \dot\theta_1 \\ 0 \end{pmatrix} $$

$$ {}^0P_{c_1} = \begin{pmatrix} l_1 s_1 \\ -l_1 c_1 \\ 0 \end{pmatrix} \Rightarrow {}^0v_{c_1} = \frac{d}{dt}\,{}^0P_{c_1} = \begin{pmatrix} l_1 c_1 \dot\theta_1 \\ +l_1 s_1 \dot\theta_1 \\ 0 \end{pmatrix} $$

$$ {}^0P_{c_2} = \begin{pmatrix} d_2 s_1 \\ -d_2 c_1 \\ 0 \end{pmatrix} \Rightarrow {}^0v_{c_2} = \frac{d}{dt}\,{}^0P_{c_2} = \begin{pmatrix} \dot d_2 s_1 + \dot\theta_1 d_2 c_1 \\ -\dot d_2 c_1 + d_2 \dot\theta_1 s_1 \\ 0 \end{pmatrix} $$

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- Lagrange Method

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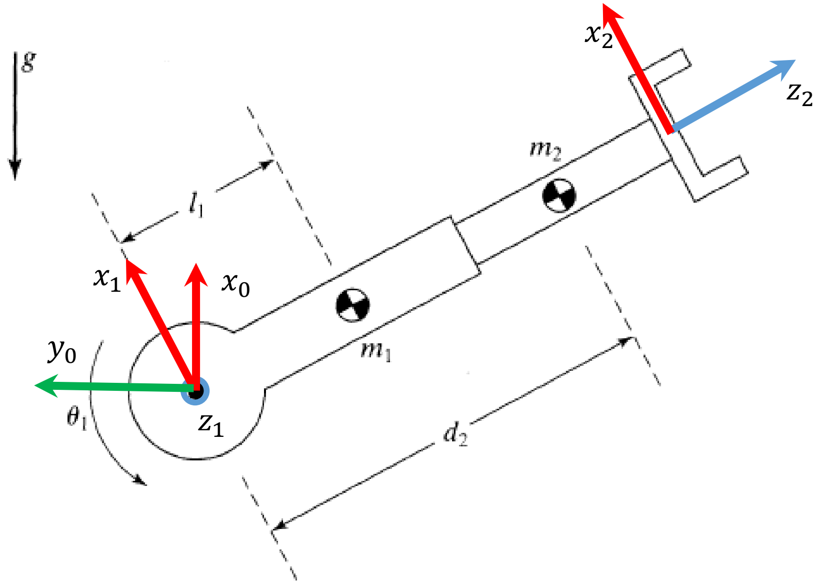
$$

$$
{}^{c_1}I_1 = \begin{pmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{pmatrix} \qquad {}^{c_2}I_2 = \begin{pmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{pmatrix}
$$



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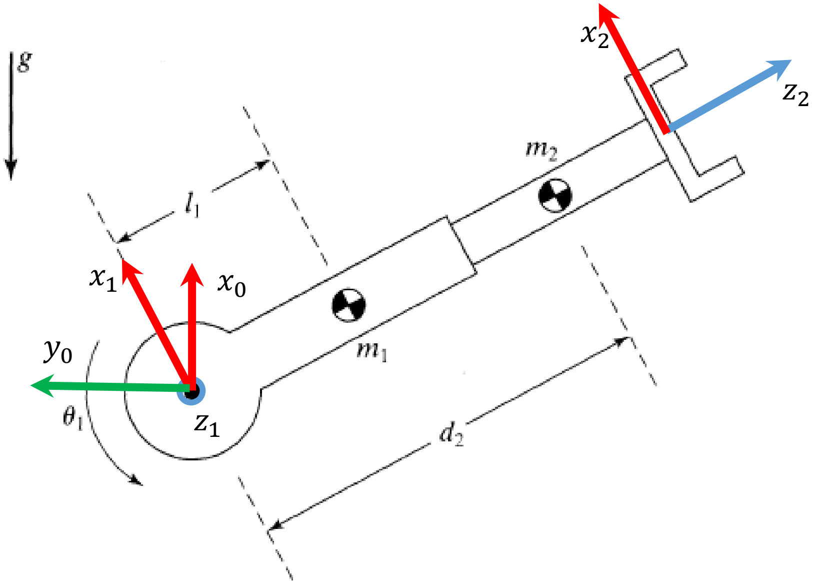

- Lagrange Method – Step 1:
  Compute kinetic and potential energies (for every link)

$$k_i = \frac{1}{2} m_i \, {}^{j}v_{c_i}^{T} \, {}^{j}v_{c_i} + \frac{1}{2} \, {}^{i}\omega_i^{T} \cdot {}^{c_i}I_i \cdot {}^{i}\omega_i \quad , \forall \{j\} \text{ coordinate frame}$$

$$u_i = - m_i \, {}^{0}g^{T} \cdot {}^{0}P_{c_i} + u_{ref,i}$$

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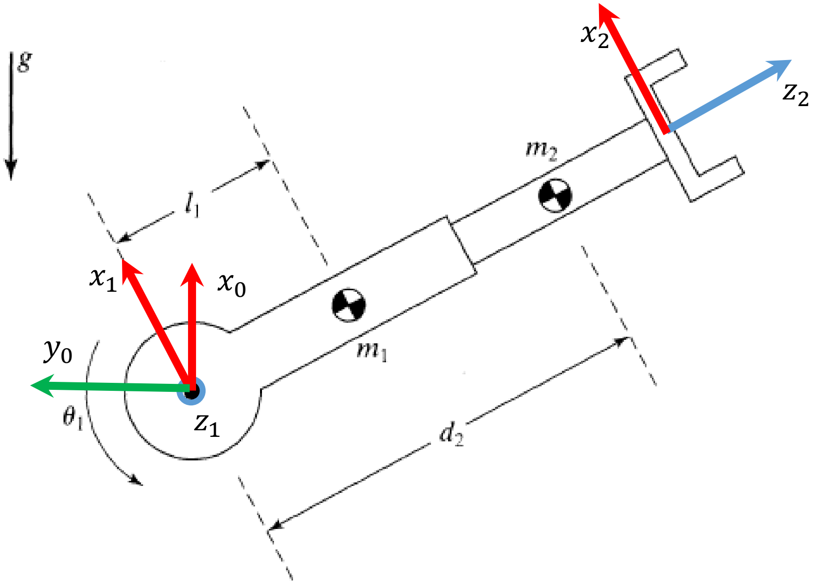

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$$u_i = -m_i \, {}^{0}g^{T} \cdot {}^{0}P_{c_i} + u_{ref,i}$$

$$k_1 = \frac{m_1}{2} l_1^2 \dot{\theta}_1^2 + \frac{I_{zz1}}{2} \dot{\theta}_1^2 \qquad u_1 = m_1 g l_1 s_1$$

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- Lagrange Method – Step 1:
  Compute kinetic and potential energies (for every link)

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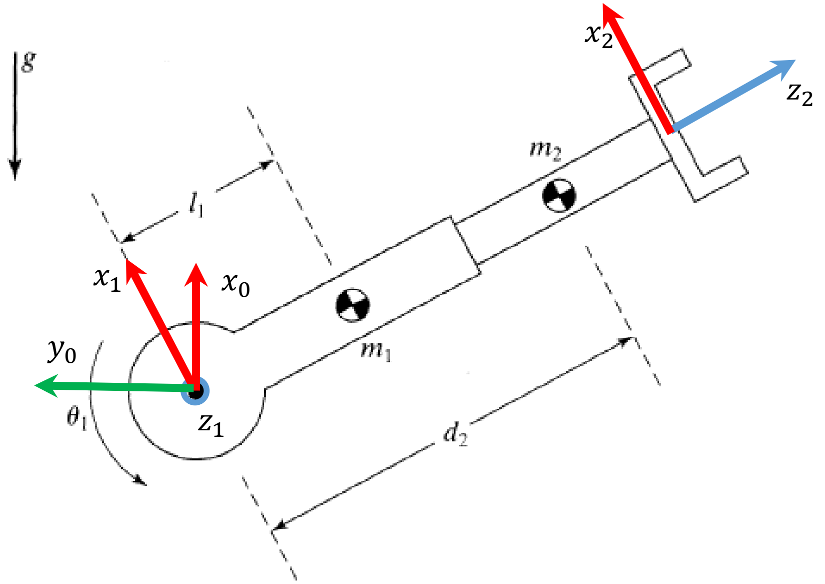
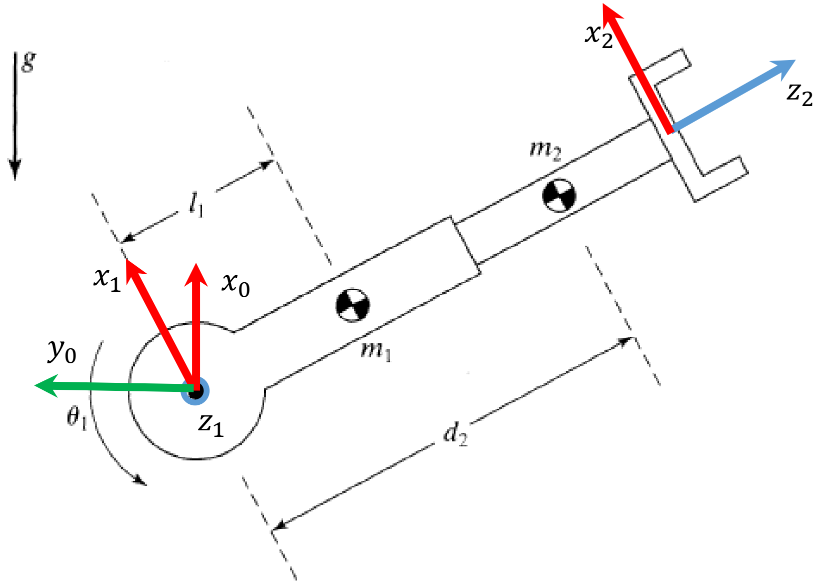
$$k_1 = \frac{m_1}{2} l_1^2 \dot{\theta}_1^2 + \frac{I_{zz1}}{2} \dot{\theta}_1^2 \qquad u_1 = m_1 g l_1 s_1$$

$$k_2 = \frac{m_2}{2} \left(d_2^2 \dot{\theta}_1^2 + \dot{d}_2^2 \right) + \frac{I_{yy2}}{2} \dot{\theta}_1^2 \qquad u_2 = m_2 g d_2 s_1$$





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- **Lagrange Method – Step 1:**
  Compute kinetic and potential energies (for every link)

$$k_i = \frac{1}{2} m_i \, {}^j v_{c_i}^T \, {}^j v_{c_i} + \frac{1}{2} \, {}^i\omega_i^T \cdot {}^{c_i}I_i \cdot {}^i\omega_i \quad , \ \forall \{j\} \text{ coordinate frame}$$

$$u_i = - m_i \, {}^0 g^T \cdot {}^0 P_{c_i} + u_{ref,i}$$

$$k_1 = \frac{m_1}{2} l_1^2 \dot\theta_1^2 + \frac{I_{zz1}}{2} \dot\theta_1^2 \qquad u_1 = m_1 g l_1 s_1$$

$$k_2 = \frac{m_2}{2} \left(d_2^2 \dot\theta_1^2 + \dot d_2^2 \right) + \frac{I_{yy2}}{2} \dot\theta_1^2 \qquad u_2 = m_2 g d_2 s_1$$

$$k = \frac{1}{2} \dot\theta_1^2 \left(I_{zz1} + I_{yy2} + m_1 l_1^2 \right) + \frac{m_2}{2} \left(\dot d_2^2 + d_2^2 \dot\theta_1^2 \right)$$

$$u = g s_1 \left(m_1 l_1 + m_2 d_2 \right)$$

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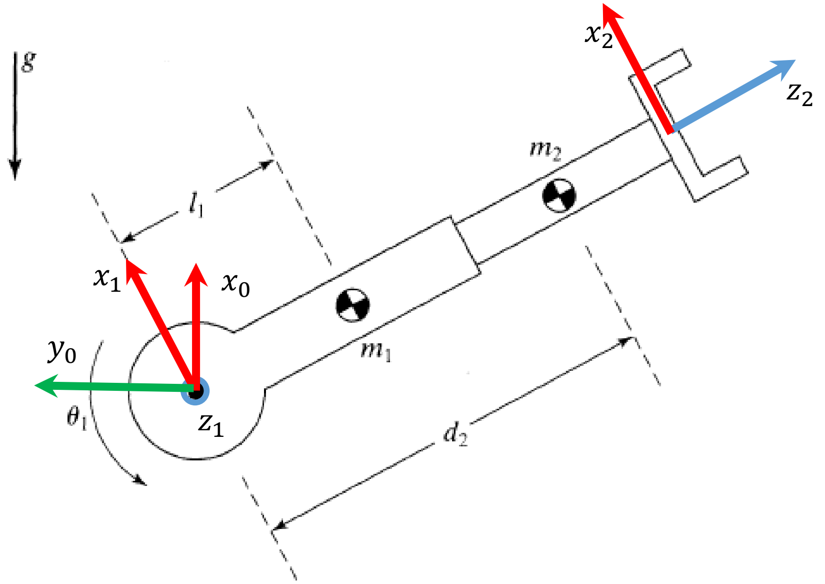
- Lagrange Method – Step 2:
  Compute energy derivatives

$$\frac{\partial u}{\partial \theta_i}$$



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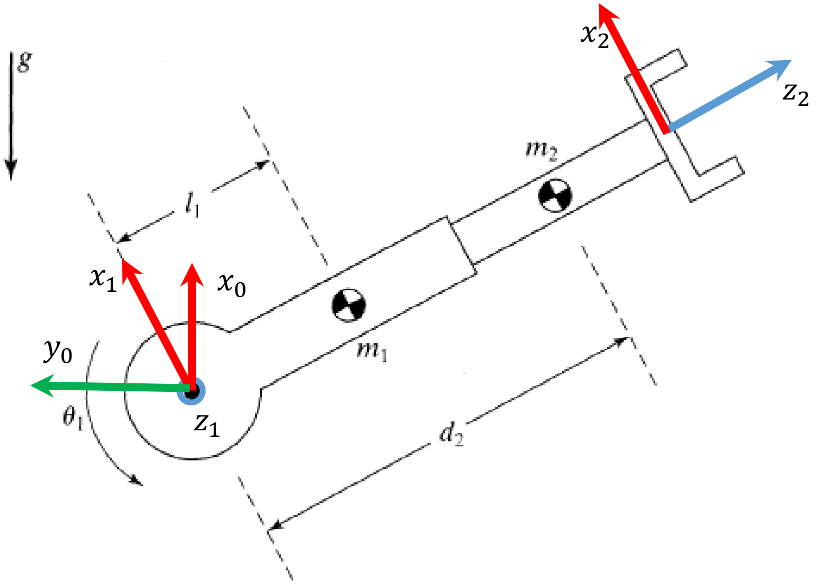

- **Lagrange Method – Step 2:**
  Compute energy derivatives

$$\frac{\partial u}{\partial \theta_i}$$

$$\frac{\partial k}{\partial \theta_i}$$

$$\frac{\partial u}{\partial \theta_1} = g c_1 (m_1 l_1 + m_2 d_2)$$

$$\frac{\partial u}{\partial \theta_2} = g s_1 m_2$$

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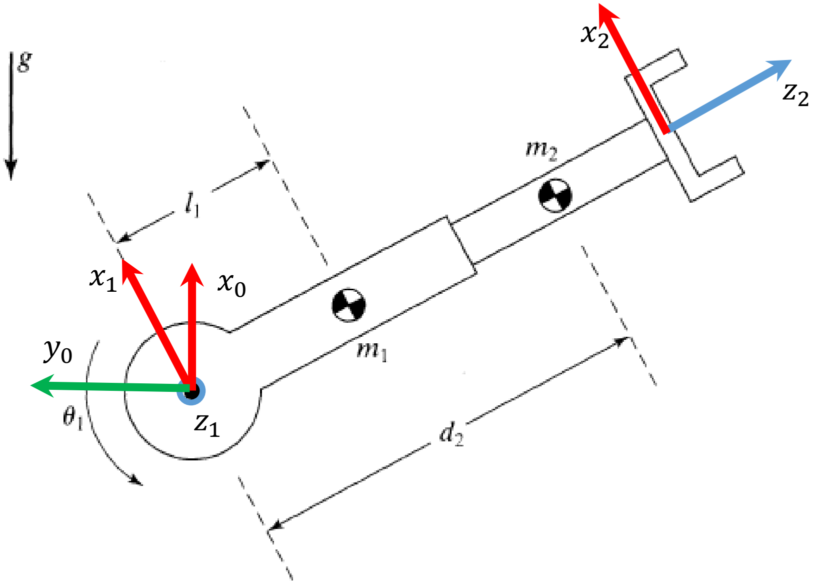

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$$\frac{\partial k}{\partial \theta_i}$$

$$\frac{\partial k}{\partial \theta_1} = 0 \qquad \frac{\partial k}{\partial \theta_2} = m_2 d_2 \dot{\theta}_1^2$$

$$\frac{\partial k}{\partial \dot{\theta}_i}$$

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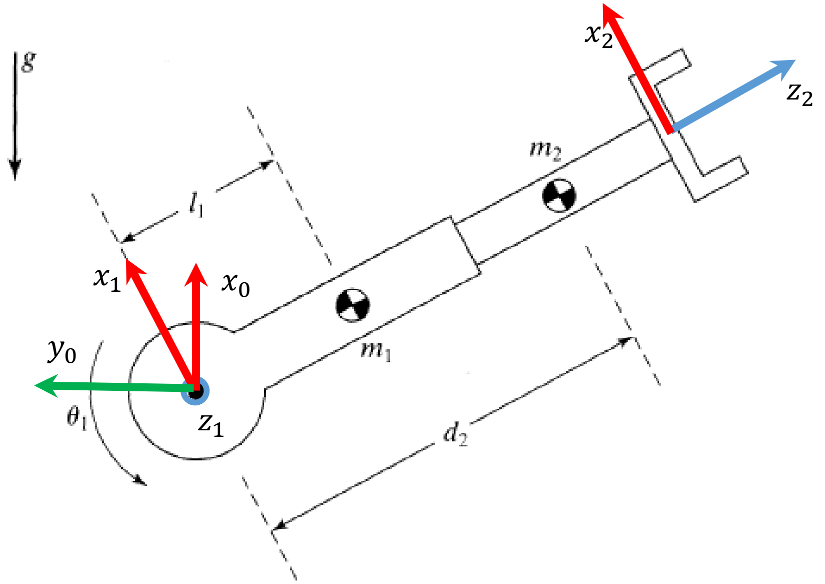

- Lagrange Method – Step 2:
  Compute energy derivatives

$$\frac{\partial u}{\partial \theta_i}$$

$$\frac{\partial k}{\partial \theta_i}$$

$$\frac{\partial k}{\partial \dot{\theta}_i}$$

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_i}$$

$$\frac{\partial u}{\partial \theta_1} = g c_1 (m_1 l_1 + m_2 d_2) \qquad \frac{\partial u}{\partial \theta_2} = g s_1 m_2$$

$$\frac{\partial k}{\partial \theta_1} = 0 \qquad \frac{\partial k}{\partial \theta_2} = m_2 d_2 \dot{\theta}_1^2$$

$$\frac{\partial k}{\partial \dot{\theta}_1} = \dot{\theta}_1 (I_{zz1} + I_{yy2} + m_1 l_1^2 + m_2 d_2^2) \qquad \frac{\partial k}{\partial \dot{\theta}_2} = m_2 \dot{d}_2$$

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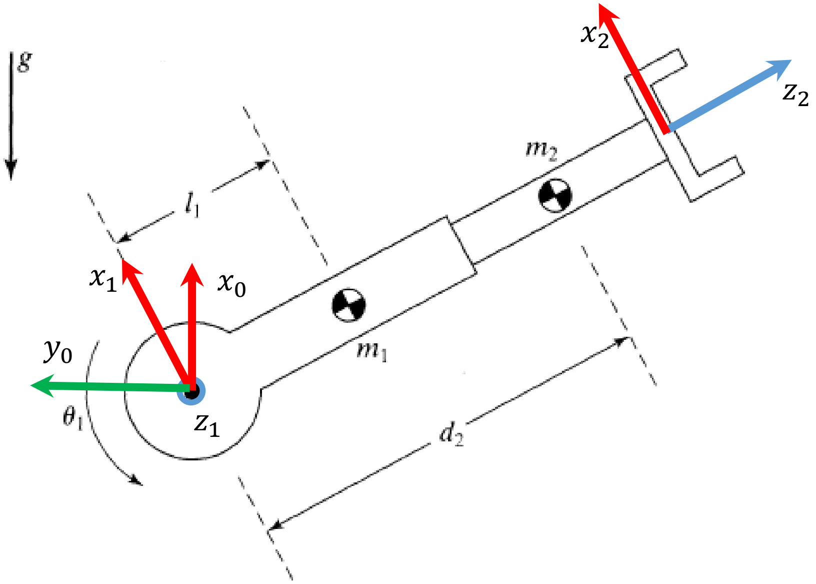

- Lagrange Method – Step 2:
  Compute energy derivatives

$$\frac{\partial u}{\partial \theta_i}$$

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$$\frac{\partial k}{\partial \dot{\theta}_i}$$

$$\frac{\partial k}{\partial \dot{\theta}_1} = \dot{\theta}_1 (I_{zz1} + I_{yy2} + m_1 l_1^2 + m_2 d_2^2) \qquad \frac{\partial k}{\partial \dot{\theta}_2} = m_2 \dot{d}_2$$

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_i}$$

$$\frac{d}{dt}\left(\frac{\partial k}{\partial \dot{\theta}_1}\right) = \ddot{\theta}_1 (I_{zz1} + I_{yy2} + m_1 l_1^2 + m_2 d_2^2) + 2 m_2 \dot{\theta}_1 d_2 \dot{d}_2 \qquad \frac{d}{dt}\left(\frac{\partial k}{\partial \dot{\theta}_2}\right) = m_2 \ddot{d}_2$$

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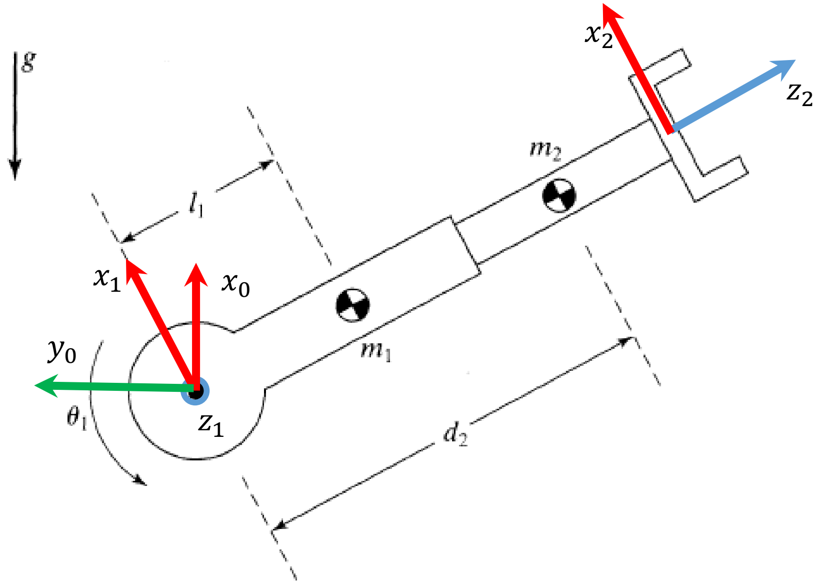

- Lagrange Method – Step 3:
  Compute joint torques vector τ

$$\tau_i = \frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_i} - \frac{\partial k}{\partial \theta_i} + \frac{\partial u}{\partial \theta_i}$$

$$\tau_1 = \ddot{\theta}_1 \left(I_{zz1} + I_{yy2} + m_1 l_1^2 + m_2 d_2^2\right) + 2 m_2 \dot{\theta}_1 d_2 \dot{d}_2 + g c_1 \left(m_1 l_1 + m_2 d_2\right)$$

$$\tau_2 = m_2 \ddot{d}_2 - m_2 d_2 \dot{\theta}_1^2 + m_2 g s_1$$

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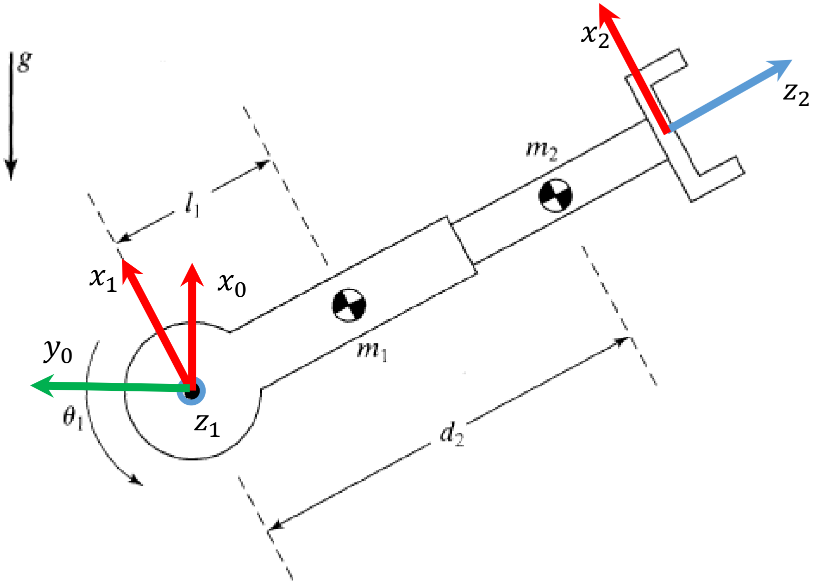
- M, V, G

$$M(\theta) = \begin{pmatrix} I_{zz1} + I_{yy2} + m_1 l_1^2 + m_2 d_2^2 & 0 \\ 0 & m_2 \end{pmatrix}$$

$$V(\theta, \dot{\theta}) = \begin{pmatrix} 2 m_2 d_2 \dot{d}_2 \dot{\theta}_1 \\ - m_2 d_2 \dot{\theta}_1^2 \end{pmatrix}$$

$$G(\theta) = \begin{pmatrix} g c_1 (m_1 l_1 + m_2 d_2) \\ m_2 g s_1 \end{pmatrix}$$



$$\tau_1 = \ddot{\theta}_1 (I_{zz1} + I_{yy2} + m_1 l_1^2 + m_2 d_2^2) + 2 m_2 \dot{\theta}_1 d_2 \dot{d}_2 + g c_1 (m_1 l_1 + m_2 d_2)$$

$$\tau_2 = m_2 \ddot{d}_2 - m_2 d_2 \dot{\theta}_1^2 + m_2 g s_1$$

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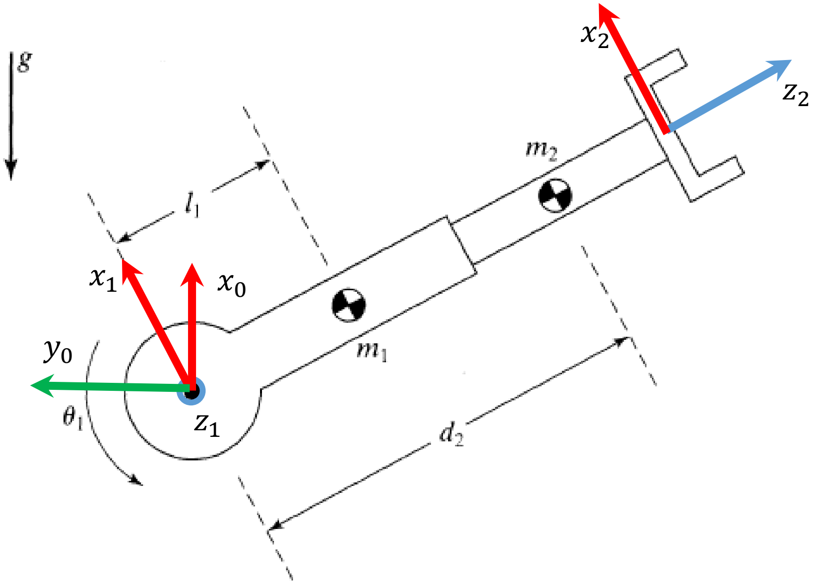
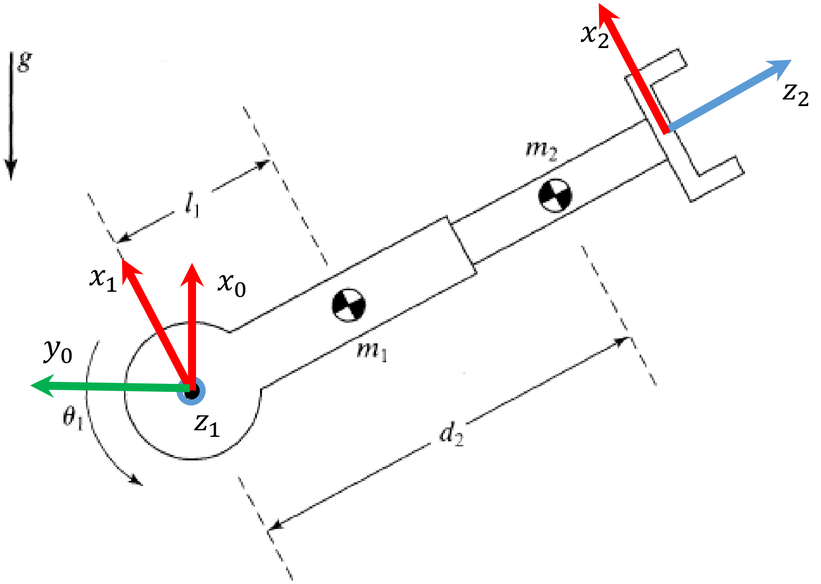


- Control Law Partitioning

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta)$$



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- Control Law Partitioning

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta)$$

$$\tau = \alpha\ddot{\theta} + \beta = \alpha\tau' + \beta$$

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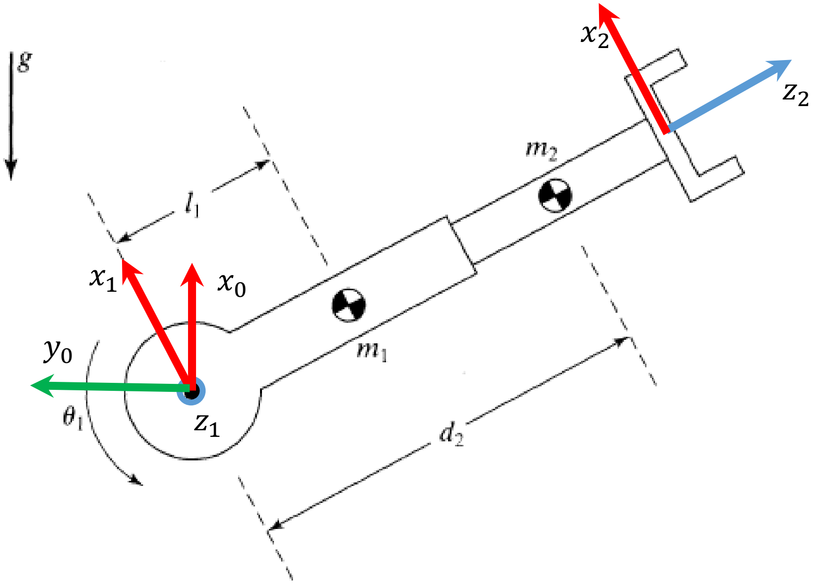

- Control Law Partitioning

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta)$$

$$\tau = \alpha\ddot{\theta} + \beta = \alpha\tau' + \beta \Rightarrow \alpha = M(\theta) \; ; \; \beta = V(\theta,\dot{\theta}) + G(\theta)$$

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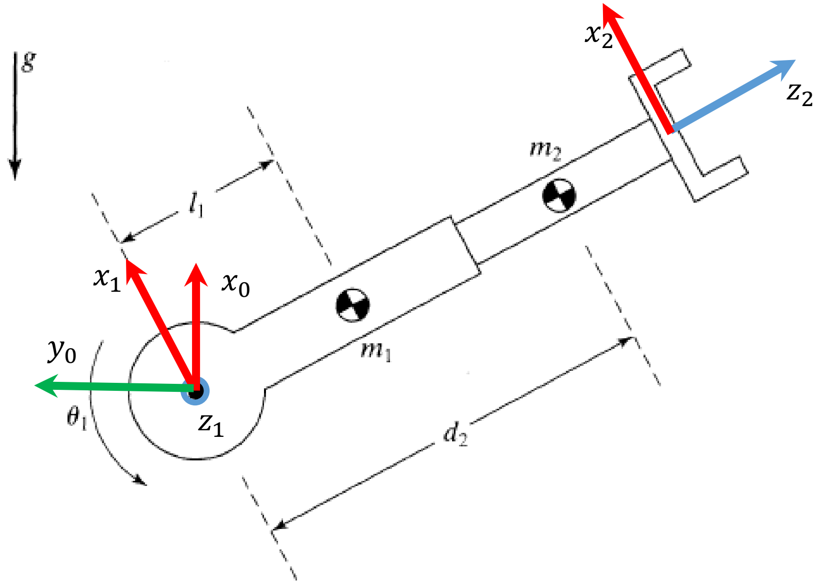

- Control Law Partitioning

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\ddot{\theta}) + G(\theta)$$

$$\tau = \alpha\ddot{\theta} + \beta = \alpha\tau' + \beta \Rightarrow \alpha = M(\theta)\ ;\ \beta = V(\theta,\dot{\theta}) + G(\theta)$$

$$\tau' = \ddot{\theta}_d + K_v\dot{e} + K_p e \quad ;\ K_v = \begin{pmatrix} k_{v1} & 0 \\ 0 & k_{v2} \end{pmatrix} \quad K_p = \begin{pmatrix} k_{p1} & 0 \\ 0 & k_{p2} \end{pmatrix}$$

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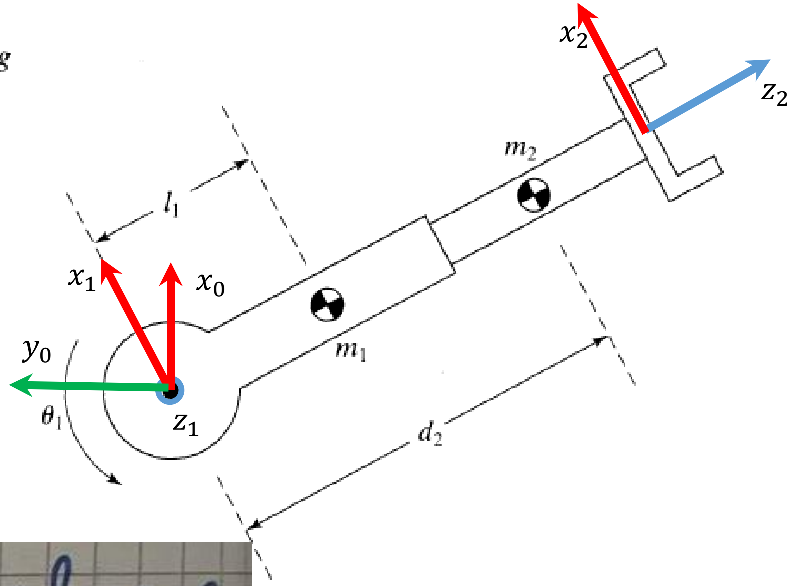
- Critically Damping

$$\ddot{\theta} = \tau' = \ddot{\theta}_d + K_p\dot{e} + K_p e \Leftrightarrow \ddot{e} + K_v\dot{e} + K_p e = 0 \Leftrightarrow \begin{cases} \ddot{e}_1 + k_{v1}\dot{e}_1 + k_{p1} e_1 = 0 \\ \ddot{e}_2 + k_{v2}\dot{e}_2 + k_{p2} e_2 = 0 \end{cases}$$



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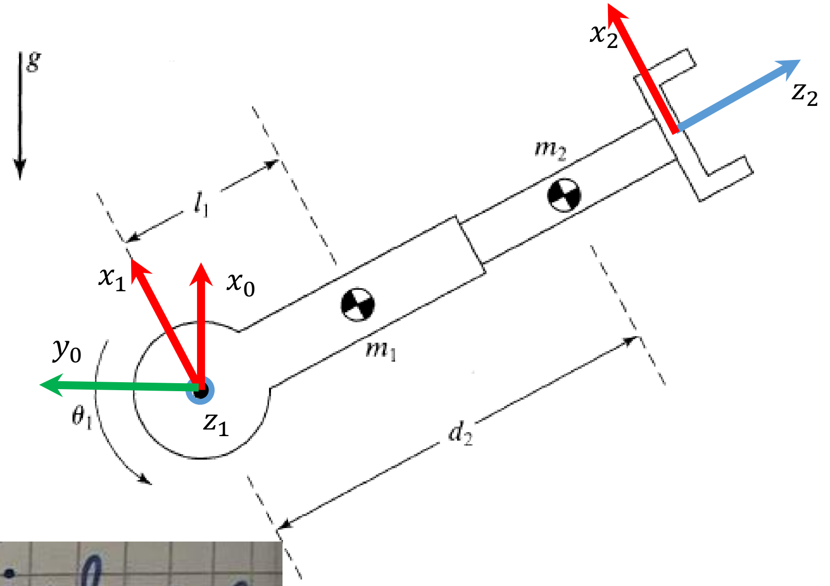
- Critically Damping



$$\ddot{\theta} = \tau' = \ddot{\theta}_d + K_p\dot{e} + K_p e \Leftrightarrow \ddot{e} + K_v\dot{e} + K_p e = 0 \Leftrightarrow \begin{cases} \ddot{e}_1 + k_{v1}\dot{e}_1 + k_{p1}e_1 = 0 \\ \ddot{e}_2 + k_{v2}\dot{e}_2 + k_{p2}e_2 = 0 \end{cases}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

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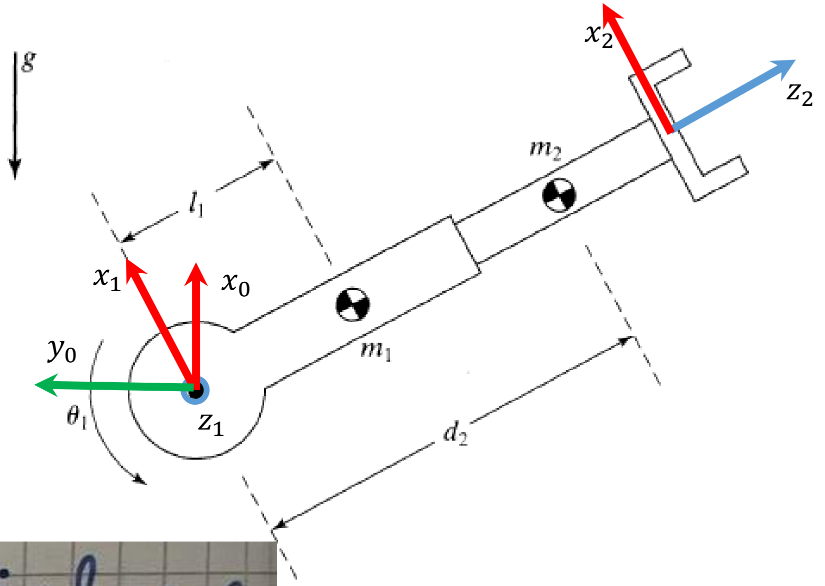


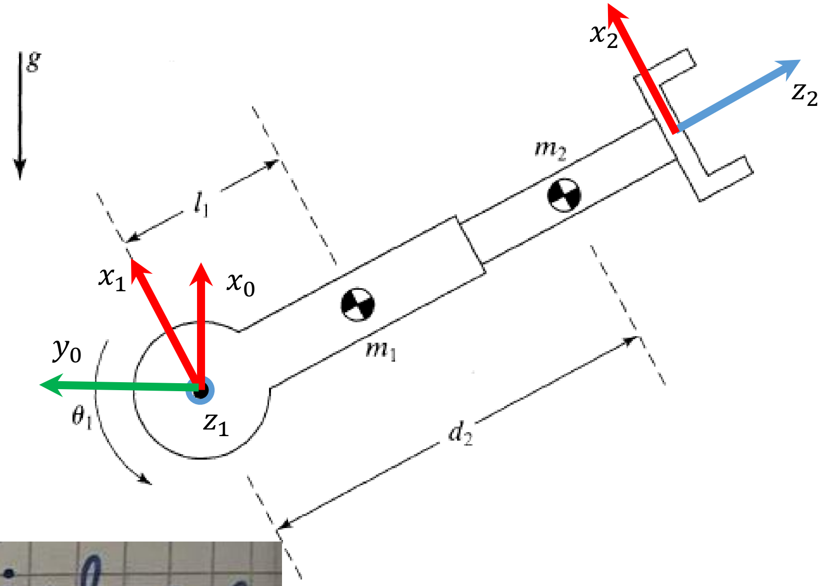
- Critically Damping

$$\ddot{\theta} = \tau' = \ddot{\theta}_d + K_p\dot{e} + K_p e \Leftrightarrow \ddot{e} + K_v\dot{e} + K_p e = 0 \Leftrightarrow \begin{cases} \ddot{e}_1 + k_{v1}\dot{e}_1 + k_{p1}e_1 = 0 \\ \ddot{e}_2 + k_{v2}\dot{e}_2 + k_{p2}e_2 = 0 \end{cases}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\omega_n^2 = k_{pi}$$

$\Delta = 0$ for critically damped $\Leftrightarrow k_{vi}^2 - 4k_{pi} = 0 \Leftrightarrow k_{vi} = 2\sqrt{k_{pi}}$



P01

- Critically Damping

$$\ddot{\theta} = \tau' = \ddot{\theta}_d + K_p\dot{e} + K_p e \Leftrightarrow \ddot{e} + K_v\dot{e} + K_p e = 0 \Leftrightarrow \begin{cases} \ddot{e}_1 + k_{v1}\dot{e}_1 + k_{p1}e_1 = 0 \\ \ddot{e}_2 + k_{v2}\dot{e}_2 + k_{p2}e_2 = 0 \end{cases}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\omega_n^2 = k_{pi}$$

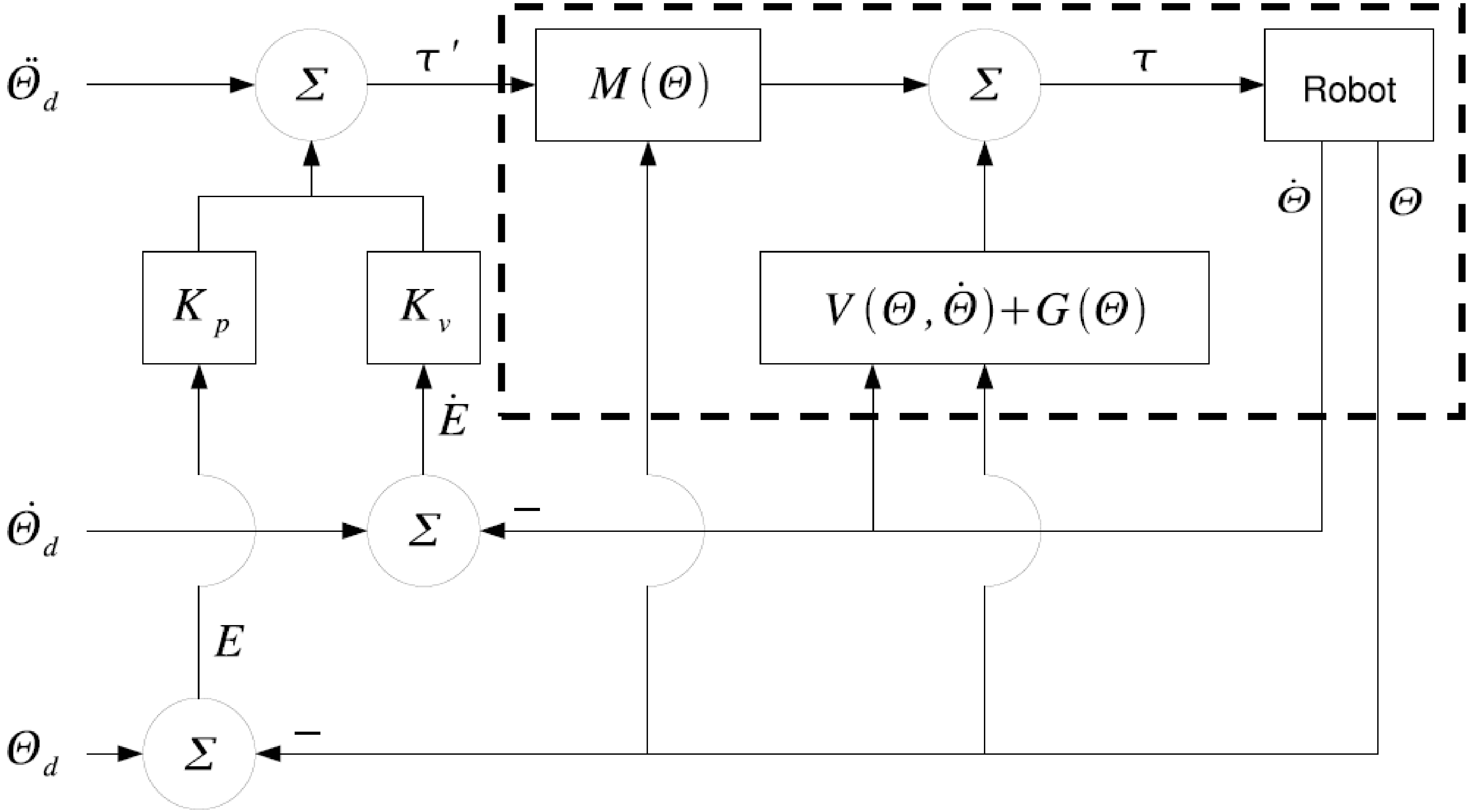
$\Delta = 0$ for critically damped $\Leftrightarrow k_{vi}^2 - 4k_{pi} = 0 \Leftrightarrow k_{vi} = 2\sqrt{k_{pi}}$

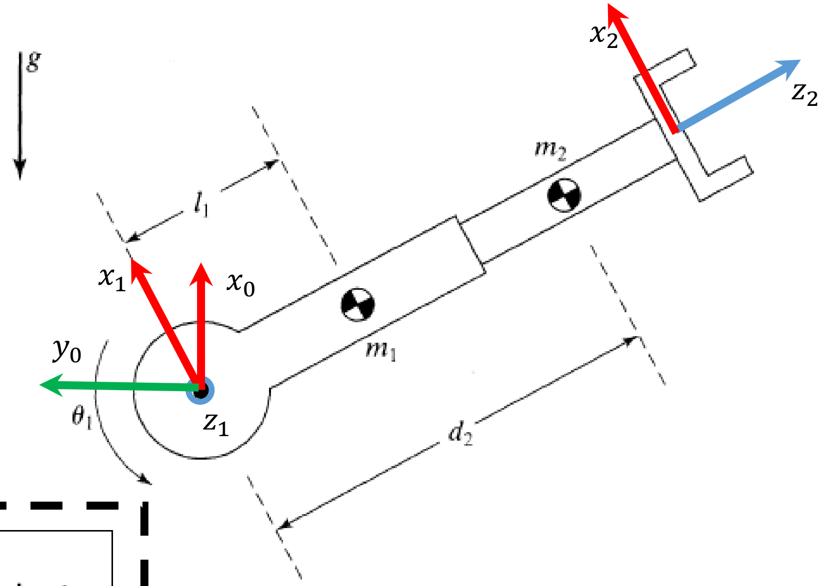
$\omega_{n1} = 20$ rad/s $\Rightarrow k_{p1} = 400$ $\quad k_{v1} = 40$

$\omega_{n2} = 25$ rad/s $\Rightarrow k_{p2} = 625$ $\quad k_{v2} = 50$

P01

- Block Diagram





P02

Problem 2

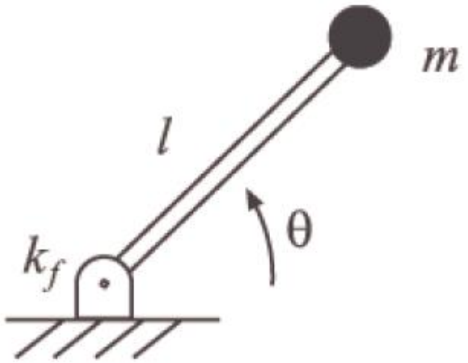


Figure 2: *Simple Robot with mass at distal end of link.*

Consider the robot shown in Figure 2. The robot has only one joint and one link with lenght l, and at the distal end of the link there is a point mass m. The mass of the link is neglected, thus, the center of mass is also located at the end of the link. The joint is affected by friction with a friction constant k_f. The inertia tensor associated with the link is denoted by I_m. You do not need to consider gravity.

a) Determine the equations of motion for this system. The computation of the inertia tensor can be performed easily if the following formula for an accumulation of point-shaped masses is used:

$$I = \sum_i m_i \begin{pmatrix} y_i^2 + z_i^2 & -x_i y_i & -x_i z_i \\ -y_i x_i & x_i^2 + z_i^2 & -y_i z_i \\ -z_i x_i & -z_i y_i & x_i^2 + y_i^2 \end{pmatrix}$$

b) Assume that a desired position Θ_d has been specified. Design a closed-loop controller that uses only $\Theta(t), \dot{\Theta}(t)$ and receives Θ_d as input.

c) Draw a block diagram of the controller.

P02

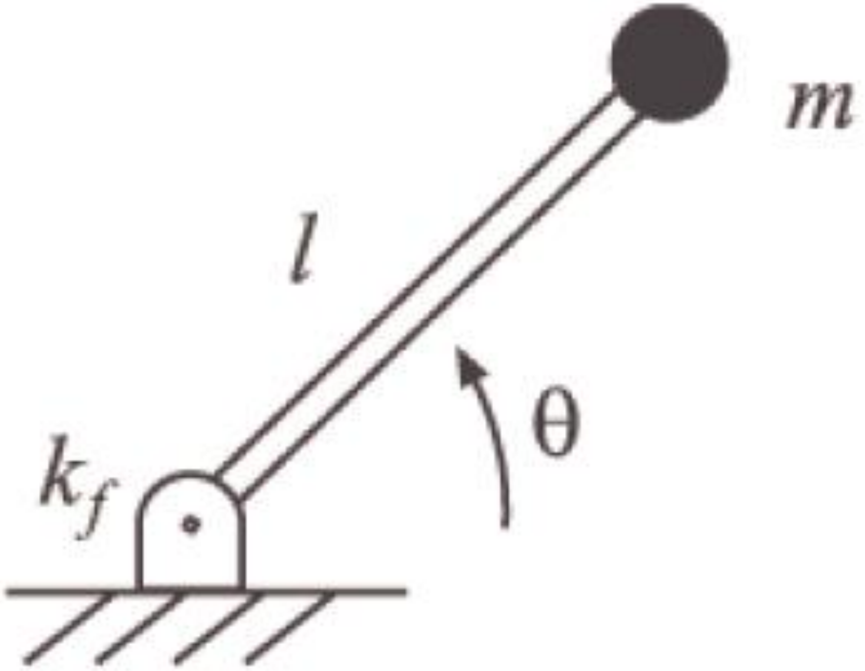


Figure 2: *Simple Robot with mass at distal end of link.*

P02

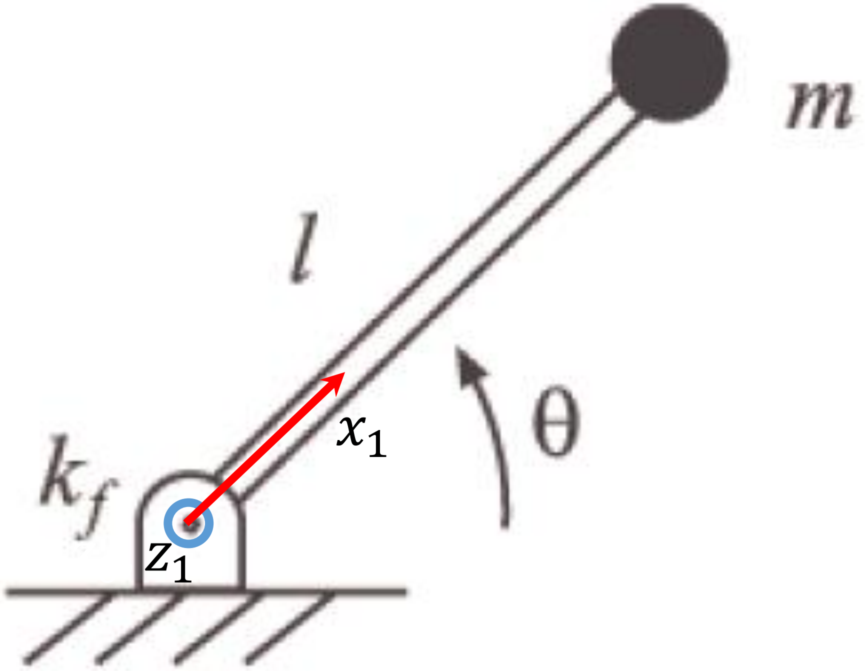


Figure 2: *Simple Robot with mass at distal end of link.*

P02

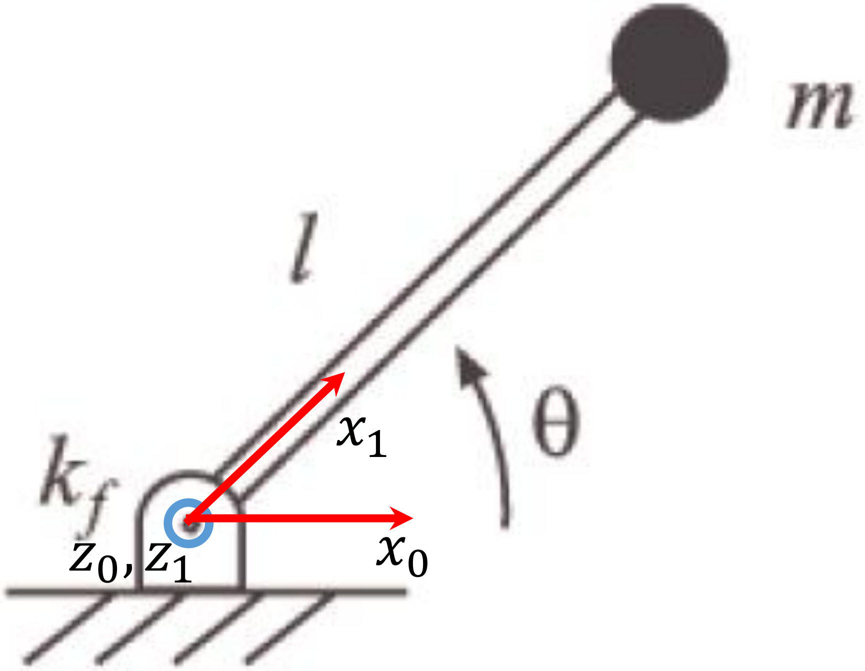


Figure 2: *Simple Robot with mass at distal end of link.*

P02

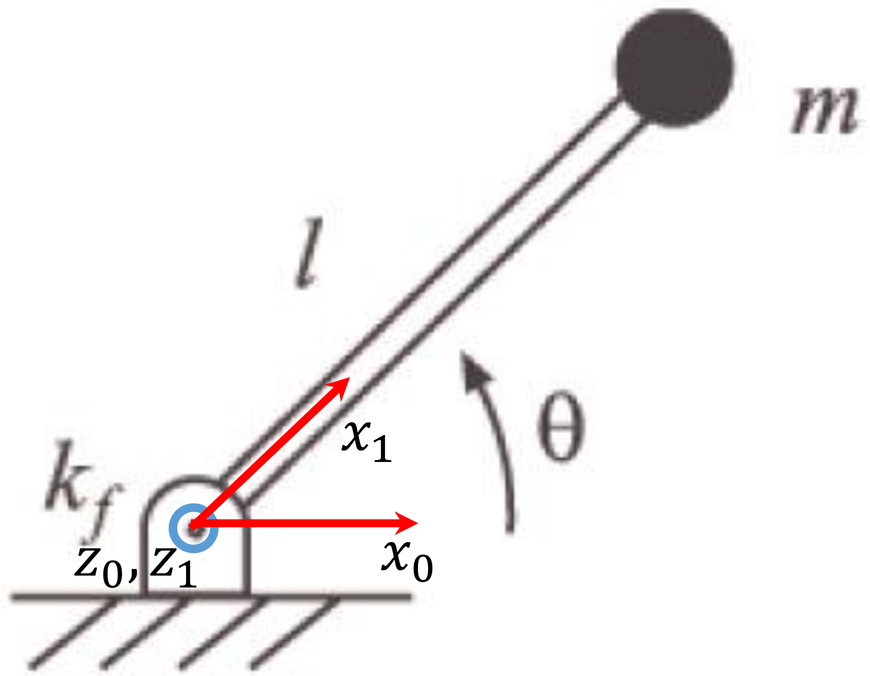
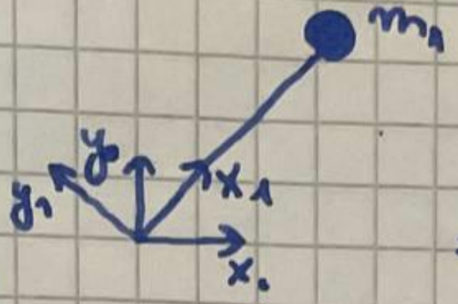


Figure 2: *Simple Robot with mass at distal end of link.*

DH Table

	α	a	d	θ
1	$0°$	0	0	θ_1



$$\Rightarrow {}^0_1R = \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

P02

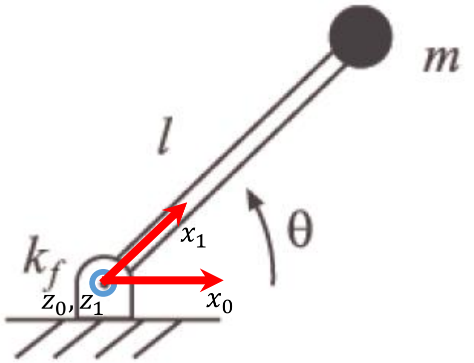
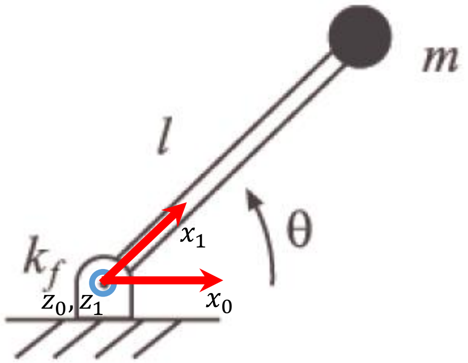


Figure 2: *Simple Robot with mass at distal end of link.*

- Newton-Euler Method – Step 0:
  Initialization

$$
{}^1P_{c_1} = \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix} \qquad {}^0v_0 = {}^0\dot{v}_0 = {}^0\omega_0 = {}^0\dot{\omega}_0 = {}^2f_2 = {}^2n_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

P02



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- Newton-Euler Method – Step 0:
  Initialization

$$ {}^1P_{c_1} = \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix} \qquad {}^0v_0 = {}^0\dot{v}_0 = {}^0\omega_0 = {}^0\dot{\omega}_0 = {}^2f_2 = {}^2n_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} $$

$$ {}^{c_1}I_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} $$ because link mass is collected into a point

P02

- **Newton-Euler Method – Step 1:**
  Compute ${}^i\dot{v}_{c_i}$, ${}^i\omega_i$ and ${}^i\dot{\omega}_i$

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{pmatrix}$$

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}_{i+1}t\right) + \begin{pmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{pmatrix}$$

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + \left({}^{i+1}_{i}R\,{}^{i}\omega_i\right) \times \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_{i+1} \end{pmatrix}$$

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{v}_i + {}^{i}\dot{\omega}_i \times {}^{i}_{i+1}t + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}_{i+1}t\right)\right) + 2\left({}^{i+1}\omega_{i+1} \times \begin{pmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{pmatrix}\right) + \begin{pmatrix} 0 \\ 0 \\ \ddot{d}_{i+1} \end{pmatrix}$$

$$^{i}\dot{v}_{C_i} = {}^{i}\dot{v}_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{C_i} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{C_i}\right)$$

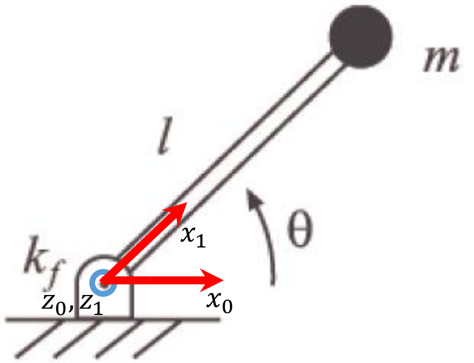


Figure 2: *Simple Robot with mass at distal end of link.*

P02

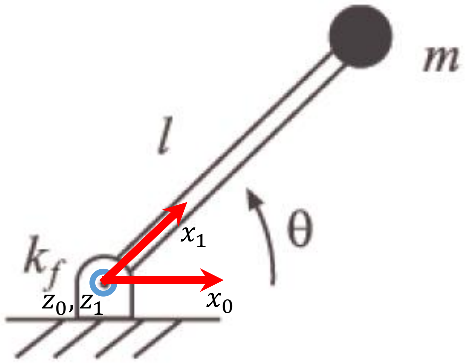


Figure 2: *Simple Robot with mass at distal end of link.*

- Newton-Euler Method – Step 1:
  Compute ${}^i\dot{v}_{c_i}$, ${}^i\omega_i$ and ${}^i\dot{\omega}_i$

$$^{i+1}\omega_{i+1} = {}^{i+1}_{\;\;i}R\,{}^i\omega_i + \begin{pmatrix}0\\0\\\dot\theta_{i+1}\end{pmatrix}$$

$$^{i+1}v_{i+1} = {}^{i+1}_{\;\;i}R\left({}^iv_i + {}^i\omega_i \times {}^i_{i+1}t\right) + \begin{pmatrix}0\\0\\\dot d_{i+1}\end{pmatrix}$$

$$^{i+1}\dot\omega_{i+1} = {}^{i+1}_{\;\;i}R\,{}^i\dot\omega_i + \left({}^{i+1}_{\;\;i}R\,{}^i\omega_i\right)\times\begin{pmatrix}0\\0\\\dot\theta_{i+1}\end{pmatrix} + \begin{pmatrix}0\\0\\\ddot\theta_{i+1}\end{pmatrix}$$

$$^{i+1}\dot v_{i+1} = {}^{i+1}_{\;\;i}R\left({}^i\dot v_i + {}^i\dot\omega_i\times{}^i_{i+1}t + {}^i\omega_i\times\left({}^i\omega_i\times{}^i_{i+1}t\right)\right) + \\ + 2\left({}^{i+1}\omega_{i+1}\times\begin{pmatrix}0\\0\\\dot d_{i+1}\end{pmatrix}\right) + \begin{pmatrix}0\\0\\\ddot d_{i+1}\end{pmatrix}$$

$$^i\dot v_{C_i} = {}^i\dot v_i + {}^i\dot\omega_i\times{}^iP_{C_i} + {}^i\omega_i\times\left({}^i\omega_i\times{}^iP_{C_i}\right)$$

$$^1\omega_1 = \begin{pmatrix}0\\0\\\dot\theta_1\end{pmatrix} \qquad ^1V_1 = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$^1\dot\omega_1 = \begin{pmatrix}0\\0\\\ddot\theta_1\end{pmatrix} \qquad ^1\dot v_1 = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$^1\dot v_{C_1} = \begin{pmatrix}-l_1\dot\theta_1^2\\ l_1\ddot\theta_1\\ 0\end{pmatrix}$$

P02

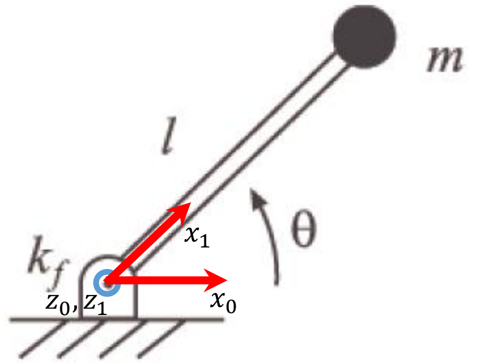


Figure 2: *Simple Robot with mass at distal end of link.*

- Newton-Euler Method – Step 2:
  Compute ${}^{i}F_i$ and ${}^{i}N_i$

$$
{}^{i}F_i = m_i \cdot {}^{i}\dot{v}_{C_i}
$$
$$
{}^{i}N_i = {}^{C_i}I_i \cdot {}^{i}\dot{\omega}_i + {}^{i}\omega_i \times {}^{C_i}I_i \cdot {}^{i}\omega_i
$$

P02

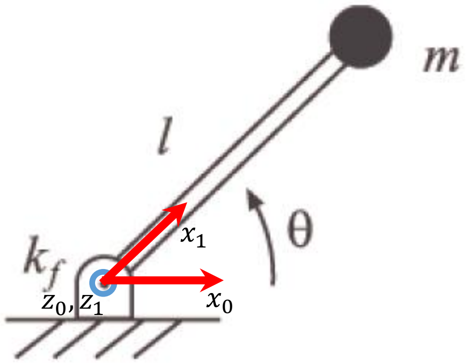


Figure 2: *Simple Robot with mass at distal end of link.*

- Newton-Euler Method – Step 2:
  Compute ${}^iF_i$ and ${}^iN_i$

$$ {}^iF_i = m_i \cdot {}^i\dot{v}_{C_i} $$
$$ {}^iN_i = {}^{C_i}I_i \cdot {}^i\dot{\omega}_i + {}^i\omega_i \times {}^{C_i}I_i \cdot {}^i\omega_i $$

$$ {}^1F_1 = \begin{pmatrix} -m_1 l_1 \dot{\theta}_1^2 \\ m_1 l_1 \ddot{\theta}_1 \\ 0 \end{pmatrix} $$
$$ {}^1N_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} $$

P02

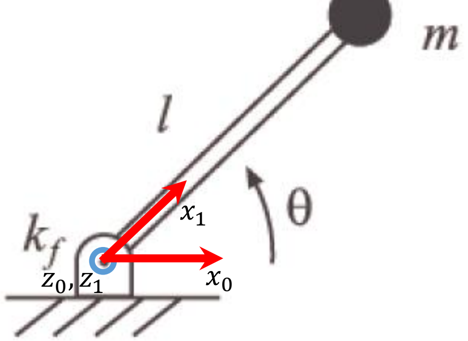


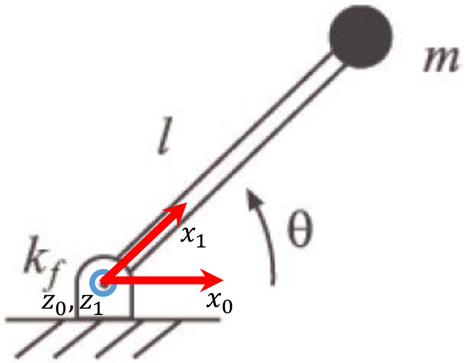
Figure 2: *Simple Robot with mass at distal end of link.*

- Newton-Euler Method – Step 3:
  Compute joint torques vector τ

$$ {}^{i}f_i = {}^{i}_{i+1}R \; {}^{i+1}f_{i+1} + {}^{i}F_i $$

$$ {}^{i}n_i = {}^{i}_{i+1}R \; {}^{i+1}n_{i+1} + {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R \; {}^{i+1}f_{i+1} \right) + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}N_i $$

P02



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- Newton-Euler Method – Step 3:
  Compute joint torques vector τ

$$ {}^{i}f_{i} = {}^{i}_{i+1}R\, {}^{i+1}f_{i+1} + {}^{i}F_{i} $$

$$ {}^{i}n_{i} = {}^{i}_{i+1}R\, {}^{i+1}n_{i+1} + {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R\, {}^{i+1}f_{i+1}\right) + {}^{i}P_{C_i} \times {}^{i}F_{i} + {}^{i}N_{i} $$

$$ {}^{1}f_{1} = \begin{pmatrix} -m_1 l_1 \dot{\theta}_1 \\ m_1 l_1 \ddot{\theta}_1 \\ 0 \end{pmatrix} \text{ and } {}^{1}n_{1} = \begin{pmatrix} 0 \\ 0 \\ m_1 l_1^2 \ddot{\theta}_1 \end{pmatrix} $$

P02

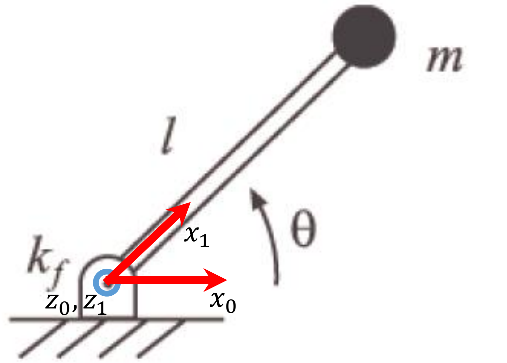

Figure 2: *Simple Robot with mass at distal end of link.*

- Newton-Euler Method – Step 3:
  Compute joint torques vector τ

$$^{i}f_i = {}^{i}_{i+1}R\, {}^{i+1}f_{i+1} + {}^{i}F_i$$

$$^{i}n_i = {}^{i}_{i+1}R\, {}^{i+1}n_{i+1} + {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R\, {}^{i+1}f_{i+1}\right) + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}N_i$$

$$^{1}f_1 = \begin{pmatrix} -m_1 l_1 \dot{\theta}_1 \\ m_1 l_1 \ddot{\theta}_1 \\ 0 \end{pmatrix} \text{ and } {}^{1}n_1 = \begin{pmatrix} 0 \\ 0 \\ m_1 l_1^2 \ddot{\theta}_1 \end{pmatrix}$$

- M, V, G

$$\tau = {}^{1}n_{1z} = m_1 l_1^2 \ddot{\theta}_1 = M(\theta)\ddot{\theta}$$

P02

$$M(\theta) = m_1 l_1^2$$

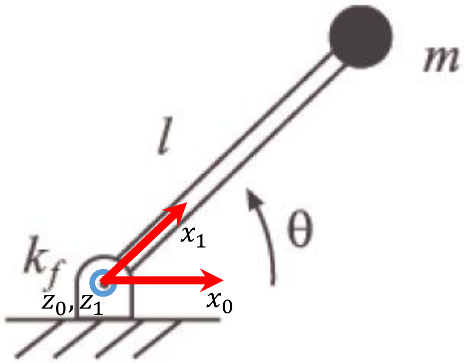

Figure 2: *Simple Robot with mass at distal end of link.*

- Consider friction force (damper in the mass-spring-damper system)

$$M(\theta)\ddot{\theta} + \underline{\underline{k_f\dot{\theta}}} = \alpha\tau + \beta$$

P02

- PD controller: Control Law Partitioning

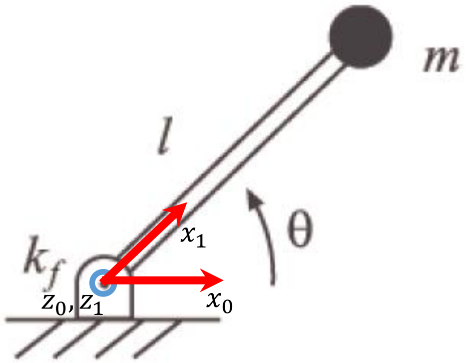

Figure 2: *Simple Robot with mass at distal end of link.*

$$M(\theta)\ddot{\theta} + \underline{\underline{k_f}}\,\dot{\theta} = \alpha\tau' + \beta$$

P02

- PD controller: Control Law Partitioning

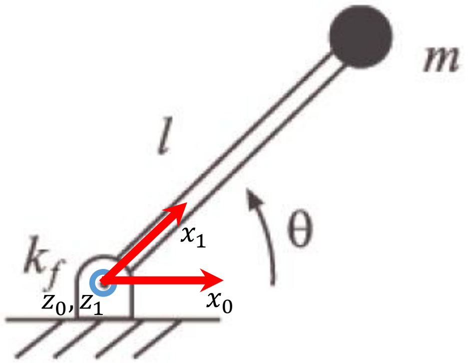

Figure 2: *Simple Robot with mass at distal end of link.*

$$M(\theta)\ddot{\theta} + \underline{\underline{k_f\dot{\theta}}} = \alpha\tau' + \beta$$

only θ_d as input $\Rightarrow$ assume $\dot{\theta}_d = \ddot{\theta}_d \overset{!}{=} 0$

P02

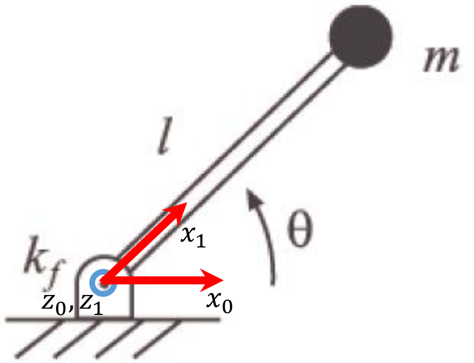

Figure 2: *Simple Robot with mass at distal end of link.*

- PD controller: Control Law Partitioning

$$\left.\begin{aligned} &M(\theta)\ddot{\theta} + \underline{\underline{k_f}}\,\dot{\theta} = \alpha\tau' + \beta \\ &\text{only } \theta_d \text{ as input} \Rightarrow \text{assume } \dot{\theta}_d = \ddot{\theta}_d \overset{!}{=} 0 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \tau' = \ddot{\theta}_d + k_v \dot{e} + k_p e = -k_v \dot{\theta} + k_p e$$

P02

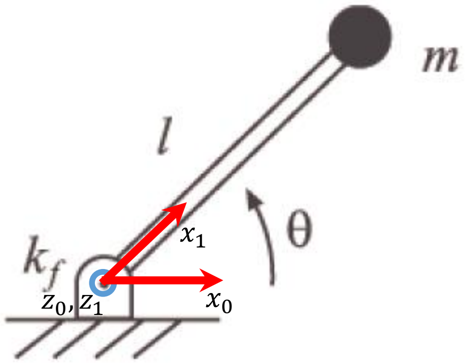

Figure 2: *Simple Robot with mass at distal end of link.*

- PD controller: Control Law Partitioning

$$\left.\begin{array}{l} M(\theta)\ddot{\theta} + \underline{\underline{k_f\dot{\theta}}} = \alpha\tau' + \beta \\ \\ \text{only } \theta_d \text{ as input} \Rightarrow \text{assume } \dot{\theta}_d = \ddot{\theta}_d \overset{!}{=} 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \tau' = \ddot{\theta}_d + k_v\dot{e} + k_p e = -k_v\dot{\theta} + k_p e$$

$$\alpha = m_1 l_1^2, \beta = k_f\dot{\theta}$$

P02

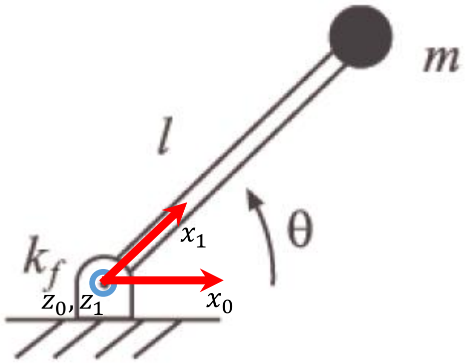

Figure 2: *Simple Robot with mass at distal end of link.*

- Block Diagram