Multiple View Geometry: Exercise Sheet 2

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1. Write down the matrices $M \in SE(3) \subset \mathbb{R}^{4\times 4}$ representing the following transformations:
   (a) Translation by the vector $T \in \mathbb{R}^3$.
   (b) Rotation by the rotation matrix $R \in \mathbb{R}^{3\times 3}$.
   (c) Rotation by R followed by the translation T.
   (d) Translation by T followed by the rotation R.

2. Let $M_1, M_2 \in \mathbb{R}^{3\times 3}$. Please prove the following:

$$\begin{array}{ccc} \mathbf{x}^\top M_1 \mathbf{x} = \mathbf{x}^\top M_2 \mathbf{x} & \text{iff} & M_1 - M_2 \text{ is skew-symmetric} \\ \text{for all } \ \mathbf{x} \in \mathbb{R}^3 & & (\text{i.e. } M_1 - M_2 \in so(3)) \end{array}$$

   Info: The group $SO(3)$ is called a **Lie group**.
   The space $so(3) = \{\hat{\omega} \mid \omega \in \mathbb{R}^3\}$ of skew-symmetric matrices is called its **Lie algebra**.

3. Consider a vector $\omega \in \mathbb{R}^3$ with $\|\omega\| = 1$ and its corresponding skew-symmetric matrix $\hat{\omega}$.
   (a) Show that $\hat{\omega}^2 = \omega\omega^\top - I$ and $\hat{\omega}^3 = -\hat{\omega}$.
   (b) Following the result of (a), find simple rules for the calculation of $\hat{\omega}^n$ and proof your result. Distinguish between odd and even numbers n.
   (c) Derive the Rodrigues' formula for a skew-symmetric matrix $\hat{\omega}$ corresponding to an arbitrary vector $\omega \in \mathbb{R}^3$ (i.e. $\|\omega\|$ does not have to be equal to 1):

$$e^{\hat{\omega}} = I + \frac{\hat{\omega}}{\|\omega\|}\sin(\|\omega\|) + \frac{\hat{\omega}^2}{\|\omega\|^2}(1 - \cos(\|\omega\|))$$

   Hint: Combine your result from (b) with

$$e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!} \quad \text{and} \quad \sin(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} \quad \text{and} \quad 1-\cos(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{(2n)!}$$

1. Write down the matrices $M \in SE(3) \subset \mathbb{R}^{4\times4}$ representing the following transformations:

   (a) Translation by the vector $T \in \mathbb{R}^3$.

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(a) $M = \begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $M = \begin{bmatrix} R & \begin{matrix}0\\0\\0\end{matrix} \\ 0\,0\,0 & 1 \end{bmatrix}$

(c) $M = \begin{bmatrix} I & T \\ 0 & 1 \end{bmatrix}\begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$

(d) $M = \begin{bmatrix} R & \begin{matrix}0\\0\\0\end{matrix} \\ 000 & 1 \end{bmatrix}\begin{bmatrix} I & T \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R & RT \\ 0 & 1 \end{bmatrix}$

$[a]_\times b = a \times b$

2. Let $M_1, M_2 \in \mathbb{R}^{3\times3}$. Please prove the following:

$$\mathbf{x}^\top M_1 \mathbf{x} = \mathbf{x}^\top M_2 \mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^3 \quad \text{iff} \quad M_1 - M_2 \text{ is skew-symmetric (i.e. } M_1 - M_2 \in so(3))$$

$\Longleftrightarrow M_1 - M_2 = (M_2 - M_1)^T$

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$x^T(M_1 - M_2)x = 0 \implies M_1 - M_2 = (M_2 - M_1)^T$

$x^T M_1^T x = x^T M_2^T x$

$x^T(M_1^T - M_2^T)x = 0$

$x^T(M_1 - M_2)^T x = 0$

$x^T(M_2 - M_1)^T x = 0$

3. Consider a vector $\omega \in \mathbb{R}^3$ with $\|\omega\| = 1$ and its corresponding skew-symmetric matrix $\hat{\omega}$.

   (a) Show that $\hat{\omega}^2 = \omega\omega^\top - I$ and $\hat{\omega}^3 = -\hat{\omega}$.

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Hint: Combine your result from (b) with

$$e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!} \quad \text{and} \quad \sin(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} \quad \text{and} \quad 1 - \cos(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{(2n)!}$$

(1) $w = (w_1, w_2, w_3)^T$

$\hat{w} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$

$\hat{w}^2 = \hat{w}\hat{w} = \begin{bmatrix} -(w_3^2 + w_2^2) & w_1 w_2 & w_1 w_3 \\ w_1 w_2 & -(w_3^2 + w_1^2) & w_2 w_3 \\ w_1 w_3 & w_2 w_3 & -(w_1^2 + w_2^2) \end{bmatrix}$

$\hat{w}^3 = \begin{pmatrix} 0 & w_3(w_3^2 + w_2^2 + w_1^2) & w_2(-w_3^2 - w_2^2 - w_1^2) \\ w_3(-w_3^2 - w_1^2 - w_2^2) & 0 & w_1(w_2^2 + w_1^2 + w_3^2) \\ w_2(w_3^2 + w_1^2 + w_2^2) & w_1(-w_3^2 - w_1^2 - w_2^2) & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{pmatrix} = -\hat{w}$

$w \cdot w^T = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \cdot (w_1 \; w_2 \; w_3) = \begin{pmatrix} w_1^2 & w_1 w_2 & w_1 w_3 \\ w_1 w_2 & w_2^2 & w_2 w_3 \\ w_1 w_3 & w_2 w_3 & w_3^2 \end{pmatrix}$

$w \cdot w^T - I = \begin{pmatrix} w_1^2 - 1 & w_1 w_2 & w_1 w_3 \\ w_1 w_2 & w_2^2 - 1 & w_2 w_3 \\ w_1 w_3 & w_2 w_3 & w_3^2 - 1 \end{pmatrix}$

$\because \|w\| = 1$

$\therefore w_1^2 + w_2^2 + w_3^2 = 1$

$= \left(-(w_2^2 + w_3^2) \quad -(w_1 + \cdots \right.$

$\therefore \hat{w}^2 = w \cdot w^T - I$

(2) $\hat{w}$

$\hat{w}^2 = ww^T - I$

$\hat{w}^3 = -\hat{w}$

$\hat{w}^4 = -\hat{w}^2$

$\hat{w}^5 = -\hat{w}^3 = \hat{w}$

$\hat{w}^6 = -\hat{w}^4 = \hat{w}^2$

$\hat{w}^n = \begin{cases} (-1)^{n/2+1}\,\hat{w} \\ (-1)^{n/2+1}\,\hat{w}^2 \end{cases}$

$\hat{w}^{2n} = (-1)^{n+1}\hat{w}^2 \quad n \geq 1$

$\hat{w}^{2n+1} = (-1)^n \hat{w} \quad n \geq 0$

(c) Derive the Rodrigues' formula for a skew-symmetric matrix $\hat{\omega}$ corresponding to an arbitrary vector $\omega \in \mathbb{R}^3$ (i.e. $||\omega||$ does not have to be equal to 1):

$$e^{\hat{\omega}} = I + \frac{\hat{\omega}}{||\omega||}\sin(||\omega||) + \frac{\hat{\omega}^2}{||\omega||^2}(1 - \cos(||\omega||))$$

$\hat{w}^{2n} = (-1)^{n+1}\hat{w}^2 \quad n \geq 1$

$\hat{w}^{2n+1} = (-1)^n \hat{w} \quad n \geq 0$

Hint: Combine your result from (b) with

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assume $v = \frac{w}{||w||} \quad t = ||w|| \quad w = vt \quad \hat{w} = \hat{v}t$

$$e^{\hat{w}} = e^{\hat{v}t} = \sum_{n=0}^{\infty} \frac{(\hat{v}t)^n}{n!}$$

$$= I + \sum_{n=0}^{\infty} \frac{(\hat{v}t)^{2n+1}}{(2n+1)!} + \sum_{n=1}^{\infty} \frac{(\hat{v}t)^{2n}}{(2n)!}$$

$$= I + \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} \cdot \hat{v}^{2n+1} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{(2n)!} \hat{v}^{2n}$$

$$= I + \sin(||w||) \cdot \hat{v} + (1-\cos(||w||)) \cdot \hat{v}^2$$

$$= I + \frac{\hat{w}}{||w||} \cdot \sin(||w||) + (1-\cos(||w||)) \cdot \frac{\hat{w}^2}{||w||^2}$$