# Recapitulation: Jacobians

In mathematics, the Jacobi matrix (less formally the "Jacobian") plays an important role and is encountered quite frequently, in a much more general manner than here within robotics. The Jacobian is the generalization of the differentiation to multi-dimensional functions.

The Jacobian of a general multi-dimensional function $f : \mathbb{R}^n \to \mathbb{R}^m$ is the matrix of all partial derivatives of the components $f_1, f_2, \ldots, f_m$ of $f$ with respect to its $n$ parameters (which are here denoted by $x_1, x_2, \ldots, x_n$). This can be written down explicitely as follows:

$$J_f = \frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

An example (taken from Wikipedia):

$$f : \mathbb{R}^3 \to \mathbb{R}^2, f(x, y, z) = \begin{pmatrix} x^2 + y^2 + z \cdot \sin(x) \\ z^2 + z \cdot \sin(y) \end{pmatrix}$$

The Jacobian of this function is then:

$$J_f(x, y, z) = \begin{pmatrix} 2x + z \cdot \cos(x) & 2y & \sin(x) \\ 0 & z \cdot \cos(y) & 2z + \sin(y) \end{pmatrix}$$

One important usage of the Jacobian is the approximation of a function $f$ in a sufficiently small neighborhood about a point $x \in \mathbb{R}^n$, known as Taylor's theorem:

$$f(x + \delta x) \approx f(x) + \frac{\partial f}{\partial x} \cdot \delta x = f(x) + J_f(x) \cdot \delta x$$

The Jacobian also appears when applying the multi-dimensional chain rule. The one-dimensional chain rule reads as follows:

$$(f \circ g)'(t) = f'(g(t))g'(t).$$

The multi-dimensional chain rule simply replaces the scalar derivatives $f'$ and $g'$ with Jacobi matrices:

$$J_{\mathbf{a}}(f \circ g) = J_{g(\mathbf{a})}(f) J_{\mathbf{a}}(g)$$

with $\mathbf{a} = (x, y, z)^{\mathrm{T}}$.

## The Jacobian as Derivative of the Position Representation

Now we'll explain how and why the Jacobian is relevant in robotics applications. The first exercise sheet discussed kinematics of a robot quite extensively: We have seen how we can determine positions and orientations of links and end effectors, given concrete values for robot parameters. Often times however, we are not only interested in position and orientation themselves, but also how they are affected by changes in robot parameters. These changes can be either small changes (leading to approximation via Taylor's theorem), or velocities (specified as derivatives of a joint trajectory).

In the following, we assume that we are given a position description of a robot coordinate system (for example the end effector system) in form of a function $f$ of type

$$f : \mathbb{R}^n \to \mathbb{R}^6, (x_1, x_2, \ldots, x_n) \mapsto (y_1, y_2, \ldots, y_6),$$

where $f$ has as parameters the joint positions of the robot (formulated generally for $n$ joints here) and these are mapped to a 6-dimensional position and orientation representation. Here, we assume that we are using one of the many possible three-parameter rotation representations.

Theoretically, one can use an arbitrary rotation representation to describe coordinate system orientations. This includes, e.g., quaternions, rotation matrices, all variants of Euler angle conventions, etc. Obviously, one will end up with a very different Jacobian depending on the chosen rotation representation.

However, the representation underlying Craig's derivation of rotational velocities is another one: He uses a so-called rotation vector, which is simply a vector $v \in \mathbb{R}^3$ that represents the axis of rotation through its direction (or the normalized vector $\frac{v}{|v|}$) and the angle of rotation through its length $|v|$. The 0 vector represents the identity. In Craig's book, it is never explicitely mentioned that this is the underlying representation of rotations that leads to his preferred representation of angular velocity. However, it is possible (and not very difficult) to show that differentiating rotation vectors $v$ w.r.t. joint angles leads exactly to the angular velocity vectors $\omega$ that are used in his book.

## Jacobians and Velocities

The Jacobian has the nice property of defining a relation between velocities in joint space and velocities in cartesian space. Intuitively, we are interested in derivatives of $f$ with respect to time.

For the robot to be able to move at all, it is obviously necessary for the joint angles to vary over time. This means that we assume that the joint positions $\Theta_1, \Theta_2, \ldots, \Theta_n$ also change over time, so that they are actually functions depending on time $t$. This means that we would have to write $\Theta_1(t), \Theta_2(t), \ldots, \Theta_n(t)$, but the parameter $t$ is usually simply omitted.

All in all, we can thus define one function $\Theta(t)$ as follows:

$$\Theta(t) : \mathbb{R} \to \mathbb{R}^n, t \mapsto (\Theta_1(t), \Theta_2(t), \ldots, \Theta_n(t))^T$$

$\Theta$ is a vector-valued function that generates for each time $t$ a column vector containing the joint parameters at that time. All in all the position at time $t$ of a system can now be computed as:

$$f(\Theta(t)).$$

Note that $f(\Theta(t)) = f \circ \Theta$ is a vector-valued function of type $\mathbb{R} \to \mathbb{R}^6$. To compute the derivative of that function with respect to time, we need to apply the chain rule:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial(x_1, \ldots, x_n)} \frac{\partial \Theta}{\partial t}$$

Here $\frac{\partial f}{\partial(x_1,\ldots,x_n)}$ is the Jacobian of $f$, and $\frac{\partial \Theta}{\partial t}$ is the derivative of $\Theta$ with respect to $t$, meaning

$$\frac{\partial \Theta}{\partial t} = \left(\frac{\partial \Theta_1}{\partial t}, \frac{\partial \Theta_2}{\partial t}, \ldots, \frac{\partial \Theta_n}{\partial t}\right)^T$$

Keep in mind that the Jacobian depends on the following circumstances:

- How are positions/orientations represented? A Jacobian that is based on the axis-angle representation will look different than one based on Yaw-Pitch-Roll angles.
- What is the reference coordinate system for the Jacobian?
- How is the current joint configuration?

In physics, it is common to use the symbol $\cdot$ to indicate the derivative of a function with respect to time. Using this notation, we can write above considerations concisely as

$$\dot{f} = J \cdot \dot{\Theta}$$

## The Jacobian for approximating very small ("infinitesimal") Movements

To see how the position of the end effector changes whenever small changes are made to robot parameters, Taylor's theorem can be utilized. This useful theorem states that a function behaves, in a sufficiently small environment, like its derivative - It is thus possible to approximate the function through its derivative up to a small error. Applying Taylor's theorem, we immediately obtain:

$$f(x+\delta x) \approx f(x) + \frac{\partial f}{\partial x} \cdot \delta x$$

The term $\delta x$ denotes a small change in $x$, or a small change in the robot parameters. The approximation is better for "smaller" values of $\delta x$. Rearranging terms, we end up with something that looks similar to the equations in Craig's book:

$$f(x+\delta x) - f(x) \approx \frac{\partial f}{\partial x} \cdot \delta x$$

The left hand side of above equation would correspond to the term $\delta Y$ in the book, the right hand side would be $\frac{\partial F}{\partial X}\delta X$. In Craig's notation, the equation symbol is used instead of the symbol $\approx$, which is justified by talking about "infinitesimal" motions, or using the notion of differentials.

Using above derivation, it is now possible to relate a small change $\delta x$ in joint parameters to a small change $f(x+\delta x) - f(x)$ in the position. But also the other direction is possible: Given a desired small step $f(x+\delta x) - f(x)$, we can compute a small change $\delta x$ in joint parameters that leads approximately to the desired change in workspace parameters. To achieve this, we need to rearrange the equations as follows:

$$\delta x \approx J^{-1}(f(x+\delta x) - f(x))$$

This is the foundation of incremental inverse kinematics. It also shows one reason why singular configurations should be avoided: As the robot approaches a singular configuration, $J$ becomes ill-conditioned. This means, in turn, that the joint rates $\delta_x$ that one computes using above formula might approach infinity. At the singular configuration itself, it becomes impossible to directly invert above equations.

## Frame of reference of a Jacobian

As has been mentioned above, the Jacobian depends on the choice of the reference coordinate system. This is obvious, since the position description $f$ (both for position and angle) also depends on the reference frame. Given a Jacobian in reference system $B$, the following relation holds:

$$\begin{pmatrix} {}^{B}v \\ {}^{B}\omega \end{pmatrix} = {}^{B}J(\Theta)\dot{\Theta}$$

This formula relates joint velocities to cartesian and angular velocities. How about the Jacobian with respect to a different coordinate system $A$? We know:

$$\begin{pmatrix} {}^{A}v \\ {}^{A}\omega \end{pmatrix} = \begin{pmatrix} {}^{A}_{B}R & \mathbf{0} \\ \mathbf{0} & {}^{A}_{B}R \end{pmatrix} \begin{pmatrix} {}^{B}v \\ {}^{B}\omega \end{pmatrix}$$

Note that the $\mathbf{0}$-entries represent $3 \times 3$ zero-valued sub-matrices. Above relation holds because:

$${}^{A}v = {}^{A}_{B}R\,{}^{B}v, \quad {}^{A}\omega = {}^{A}_{B}R\,{}^{B}\omega.$$

All in all, we can now derive:

$${}^{A}J(\Theta)\dot{\Theta} = \begin{pmatrix} {}^{A}_{B}R & \mathbf{0} \\ \mathbf{0} & {}^{A}_{B}R \end{pmatrix} {}^{B}J(\Theta)\dot{\Theta} \quad \Rightarrow \quad {}^{A}J(\Theta) = \begin{pmatrix} {}^{A}_{B}R & \mathbf{0} \\ \mathbf{0} & {}^{A}_{B}R \end{pmatrix} {}^{B}J(\Theta)$$

# Solution 1

## a)

We are already familiar with the manipulator considered in this problem from problem 4 of the first sheet. We are going to reuse the results computed back then, and we can immediately specify a minimal position description of the end effector:

$$
{}^0p(\Theta_1, \Theta_2, \Theta_3) = {}^0p(\Theta) = \begin{pmatrix} l_3\cos(\Theta_3+\Theta_2+\Theta_1) + l_2\cos(\Theta_2+\Theta_1) + l_1\cos(\Theta_1) \\ l_3\sin(\Theta_3+\Theta_2+\Theta_1) + l_2\sin(\Theta_2+\Theta_1) + l_1\sin(\Theta_1) \\ \Theta_3+\Theta_2+\Theta_1 \end{pmatrix}
$$

## b)

Here, we simply need to compute the Jacobian of the position description specified above:

$$
{}^0J(\Theta) = \begin{pmatrix} \frac{\partial p_1}{\partial \Theta_1} & \frac{\partial p_1}{\partial \Theta_2} & \frac{\partial p_1}{\partial \Theta_3} \\ \frac{\partial p_2}{\partial \Theta_1} & \frac{\partial p_2}{\partial \Theta_2} & \frac{\partial p_2}{\partial \Theta_3} \\ \frac{\partial p_3}{\partial \Theta_1} & \frac{\partial p_3}{\partial \Theta_2} & \frac{\partial p_3}{\partial \Theta_3} \end{pmatrix}
$$

The partial derivatives can be evaluated quite simply, and the Jacobian is then:

$$
{}^0J(\Theta) = \begin{pmatrix} -l_3\mathrm{s}_{123} - l_2\mathrm{s}_{12} - l_1\mathrm{s}_1 & -l_3\mathrm{s}_{123} - l_2\mathrm{s}_{12} & -l_3\mathrm{s}_{123} \\ l_3\mathrm{c}_{123} + l_2\mathrm{c}_{12} + l_1\mathrm{c}_1 & l_3\mathrm{c}_{123} + l_2\mathrm{c}_{12} & l_3\mathrm{c}_{123} \\ 1 & 1 & 1 \end{pmatrix}
$$

## c)

Joint velocities relate to cartesian and angular velocities as follows:

$$
{}^0\dot{p}(\Theta, \dot{\Theta}) = {}^0J(\Theta) \cdot \begin{pmatrix} \dot{\Theta}_1 \\ \dot{\Theta}_2 \\ \dot{\Theta}_3 \end{pmatrix}
$$

Thus, the cartesian and angular velocities are:

$$
{}^0\dot{p}(\Theta, \dot{\Theta}) = \begin{pmatrix} (-l_3\mathrm{s}_{123} - l_2\mathrm{s}_{12} - l_1\mathrm{s}_1)\dot{\Theta}_1 + (-l_3\mathrm{s}_{123} - l_2\mathrm{s}_{12})\dot{\Theta}_2 + (-l_3\mathrm{s}_{123})\dot{\Theta}_3 \\ (l_3\mathrm{c}_{123} + l_2\mathrm{c}_{12} + l_1\mathrm{c}_1)\dot{\Theta}_1 + (l_3\mathrm{c}_{123} + l_2\mathrm{c}_{12})\dot{\Theta}_2 + (l_3\mathrm{c}_{123})\dot{\Theta}_3 \\ \dot{\Theta}_1 + \dot{\Theta}_2 + \dot{\Theta}_3 \end{pmatrix}
$$

## d)

Determining singular positions can be done through evaluating the determinant of the Jacobian:

$$
\begin{aligned}
\det(J) &= \begin{vmatrix} -l_3\mathrm{s}_{123} - l_2\mathrm{s}_{12} - l_1\mathrm{s}_1 & -l_3\mathrm{s}_{123} - l_2\mathrm{s}_{12} & -l_3\mathrm{s}_{123} \\ l_3\mathrm{c}_{123} + l_2\mathrm{c}_{12} + l_1\mathrm{c}_1 & l_3\mathrm{c}_{123} + l_2\mathrm{c}_{12} & l_3\mathrm{c}_{123} \\ 1 & 1 & 1 \end{vmatrix} \\
&= \begin{vmatrix} -l_1\mathrm{s}_1 & -l_2\mathrm{s}_{12} & -l_3\mathrm{s}_{123} \\ l_1\mathrm{c}_1 & l_2\mathrm{c}_{12} & l_3\mathrm{c}_{123} \\ 0 & 0 & 1 \end{vmatrix} \\
&= -l_1l_2\mathrm{s}_1\mathrm{c}_{12} + l_1l_2\mathrm{s}_{12}\mathrm{c}_1
\end{aligned}
$$

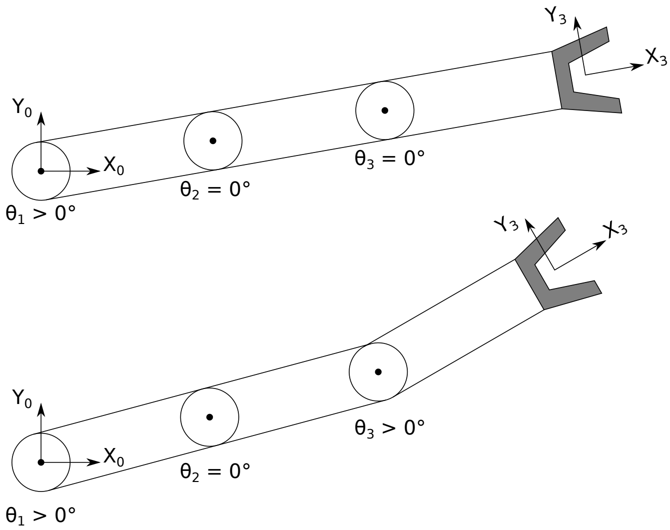


**Figure 1**: *Singular configurations: At workspace boundary (upper), and workspace interior (lower).*

To find singular positions, we need to solve:

$$\begin{aligned} -l_1 l_2 \mathrm{s}_1 \mathrm{c}_{12} + l_1 l_2 \mathrm{s}_{12} \mathrm{c}_1 &= 0 \quad \Leftrightarrow \\ \mathrm{s}_{12}\mathrm{c}_1 - \mathrm{s}_1 \mathrm{c}_{12} &= 0 \quad \Leftrightarrow \end{aligned}$$

We can use the trigonometric identities

$$\sin x \ \cos y = \frac{1}{2}\Big( \sin(x-y) + \sin(x+y)\Big)$$

to obtain:

$$\begin{aligned} \mathrm{s}_{12}\mathrm{c}_1 &= \frac{1}{2}(\sin(\Theta_2) + \sin(2\Theta_1 + \Theta_2)) \\ \mathrm{s}_1\mathrm{c}_{12} &= \frac{1}{2}(\sin(-\Theta_2) + \sin(2\Theta_1 + \Theta_2)) \end{aligned}$$

And finally, we see that

$$\sin \Theta_2 = 0$$

must hold for a singular position. Thus, the solution is $\Theta_2 \in \{0°, 180°\}$.

## e)

Figure 1 shows the robot in two singular positions, for $\Theta_2 = \Theta_3 = 0$, and $\Theta_2 = 0$. The first of both configurations is also called a workspace boundary singularity. It is often the case that workspace boundary singularities are easier to identify and explain than workspace interior singularities.

For the first of both configurations, it is clear immediately that no movement in ${}^3x$-direction is possible. This can also be seen directly by looking at the Jacobian for a specific configuration:

$${}^3J(0°, 0°, 0°) = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

with $l_1 = l_2 = l_3 = 1$.

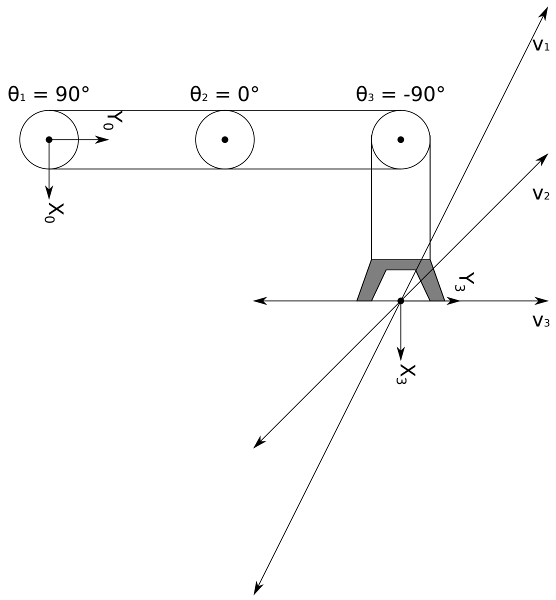


**Figure 2**: *Geometric explanation for the $\Theta_2 = 0$ singularity. The vectors $v_1, v_2, v_3$ represent linear velocities achieved by setting $\dot{\Theta}_1 = 1, \dot{\Theta}_2 = 1, \dot{\Theta}_3 = 1$, respectively.*

For the second configuration, it is not so simple to determine which kind of movement not possible.

Let's look at a concrete example of the Jacobian for that configuration, with, let's say, $\Theta_1 = 90°, \Theta_2 = 0°, \Theta_3 = -90°$, and all lengths are equal 1. The Jacobian looks like this:

$$^3J(90°, 0°, -90°) = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

So we see that the last two rows of the Jacobian are linearly dependent, and if we imagine multiplying any vector $\dot{\Theta}$ with that matrix, we see that the angular rate and the $y$ velocity will be coupled directly, meaning that we cannot choose $\dot{\Theta}$ such that arbitrary rotation speed and $y$ speed can be achieved! Figure 2 shows a geometric explanation for this situation.

In that figure, the cartesian speeds corresponding to the upper two rows of the Jacobian are shown. These speeds are, in ${}^3X, {}^3Y$ direction, $(-2, 1)^T$, $(-1, 1)^T$, $(0, 1)^T$, denoted by $v_1, v_2, v_3$. It is not really possible, however, to show the angular velocities as well! So, even though arbitrary speeds in ${}^3X, {}^3Y$ are possible in this configuration, the ${}^3Y$ speed and the angular rate will always be coupled directly. Thus, it is not possible in this configuration to achieve independent rotation and $y$ speeds.

In general, with this robot, it is quite difficult to explain geometrically which degrees of freedom are lost. Looking at the Jacobian, one can see that it always is rank-deficient and thus some dependencies between velocities that can be achieved are always present. However, these dependencies are for general configurations quite tricky to analyze and not easy to explain intuitively.

## Recapitulation: Velocities

Here is an overview of formulas for computing angular and linear velocities:

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R \cdot {}^i\omega_i + \dot{\Theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1} \tag{1}$$

${}^iZ_i = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^iv_i + {}^i\omega_i \times {}^iP_{i+1}) \tag{2}$$

${}^0\omega_0$ usually $\to 0$

${}^0V_{EE} = {}^0\dot{P}_{EE} = \dot{f}(\theta)$

${}^AV_{EE} = {}^A_0R\,{}^0V_{EE} = {}^A_0R\,{}_0\dot{f}(\theta)$

$= \underbrace{{}^A_0R \cdot {}^0J(\theta)}_{{}^AJ(\theta)} \cdot \dot{\theta} = {}^AJ(\theta)\dot{\theta}$

These formulas can also be used to determine the Jacobian indirectly: Since we know that

$$ {}^{n}J\dot{\Theta} = \begin{pmatrix} {}^{n}v_n \\ {}^{n}\omega_n \end{pmatrix}, $$

the shape of the Jacobian on the left hand side can be determined from the shape of the linear-angular velocity vector on the right hand side.

## Solution 2

To see how the formulas for the velocities can be used to determine the Jacobian, we are going to compute the Jacobian (which is already specified on the problem sheet) again in this way. First of all, we need the transformation matrices:

$$ {}^{0}_{1}\mathrm{T} = \begin{pmatrix} \mathrm{c}_1 & -\mathrm{s}_1 & 0 & 0 \\ \mathrm{s}_1 & \mathrm{c}_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^{1}_{2}\mathrm{T} = \begin{pmatrix} \mathrm{c}_2 & -\mathrm{s}_2 & 0 & l_1 \\ \mathrm{s}_2 & \mathrm{c}_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, {}^{2}_{3}\mathrm{T} = \begin{pmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

Now we want to determine the quantities ${}^{3}\omega_3$ and ${}^{3}v_3$. We simply use the formulas:

$$ {}^{1}\omega_1 = {}^{1}_{0}R \cdot {}^{0}\omega_0 + \dot{\Theta}_1 {}^{1}\hat{Z}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\Theta}_1 \end{pmatrix} $$

$$ {}^{1}v_1 = {}^{1}_{0}R({}^{0}v_0 + {}^{0}\omega_0 \times {}^{0}P_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} $$

$$ {}^{2}\omega_2 = \begin{pmatrix} \mathrm{c}_2 & \mathrm{s}_2 & 0 \\ -\mathrm{s}_2 & \mathrm{c}_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \dot{\Theta}_1 \end{pmatrix} + \dot{\Theta}_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\Theta}_1 + \dot{\Theta}_2 \end{pmatrix} $$

$$ {}^{2}v_2 = \begin{pmatrix} \mathrm{c}_2 & \mathrm{s}_2 & 0 \\ -\mathrm{s}_2 & \mathrm{c}_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \left( \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\Theta}_1 \end{pmatrix} \times \begin{pmatrix} l_1 \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} l_1\mathrm{s}_2\dot{\Theta}_1 \\ l_1\mathrm{c}_2\dot{\Theta}_1 \\ 0 \end{pmatrix} $$

$$ {}^{3}\omega_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \dot{\Theta}_1 + \dot{\Theta}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\Theta}_1 + \dot{\Theta}_2 \end{pmatrix} $$

$$ {}^{3}v_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \left( \begin{pmatrix} l_1\mathrm{s}_2\dot{\Theta}_1 \\ l_1\mathrm{c}_2\dot{\Theta}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\Theta}_1 + \dot{\Theta}_2 \end{pmatrix} \times \begin{pmatrix} l_2 \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} l_1\mathrm{s}_2\dot{\Theta}_1 \\ l_1\mathrm{c}_2\dot{\Theta}_1 + (\dot{\Theta}_1 + \dot{\Theta}_2)l_2 \\ 0 \end{pmatrix} $$

Note that the third transformation is merely a translation. Thus, the parameter $\dot{\Theta}_3$ is always 0, and we can leave the corresponding term out in the computation of ${}^{3}\omega_3$. Based on the velocity ${}^{3}v_3$, the Jacobian has to look like this:

$$ {}^{3}J = \begin{pmatrix} l_1\mathrm{s}_2 & 0 \\ l_1\mathrm{c}_2 + l_2 & l_2 \end{pmatrix} $$

This is determined by collecting the factors of terms that are multiples of $\dot{\Theta}_1$ and $\dot{\Theta}_2$.

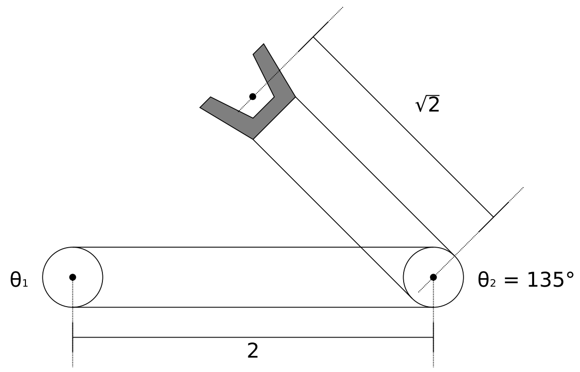


**Figure 3**: *Robot from problem 2 in isotropic configuration.*

Now on to the actual problem. Since the column vectors have to be orthogonal, we have:

$$\begin{aligned}(l_1 \mathrm{s}_2, l_1 \mathrm{c}_2 + l_2) \cdot (0, l_2)^T = 0 \quad &\Leftrightarrow \\ l_2(l_1 \mathrm{c}_2 + l_2) = 0 \quad &\Leftrightarrow \\ \mathrm{c}_2 = -\frac{l_2}{l_1}\end{aligned}$$

Also, the vectors have to be of equal length, so we have:

$$\begin{aligned}l_1^2 \mathrm{s}_2^2 + (l_1 \mathrm{c}_2 + l_2)^2 = l_2^2 \quad &\Leftrightarrow \\ l_1^2 \mathrm{s}_2^2 + l_1^2 \mathrm{c}_2^2 + 2 l_1 \mathrm{c}_2 l_2 + l_2^2 = l_2^2 \quad &\Leftrightarrow \\ l_1^2 + 2 l_1 \mathrm{c}_2 l_2 = 0 \quad &\Leftrightarrow \\ \mathrm{c}_2 = -\frac{l_1}{2 l_2}\end{aligned}$$

Combining both constraints, we conclude:

$$\begin{aligned}-\frac{l_1}{2l_2} = -\frac{l_2}{l_1} \quad &\Leftrightarrow \\ l_1^2 = 2 l_2^2 \quad &\Rightarrow \\ l_1 = \sqrt{2} l_2\end{aligned}$$

Which yields the solution $\mathrm{c}_2 = \frac{-1}{\sqrt{2}} \Rightarrow \Theta_2 = \pm 135°$. The Jacobian in this position looks like this:

$$^3J = \begin{pmatrix} \frac{l_1}{\sqrt{2}} & 0 \\ -\frac{l_1}{\sqrt{2}} + \frac{l_1}{\sqrt{2}} & l_2 \end{pmatrix} = \begin{pmatrix} l_2 & 0 \\ 0 & l_2 \end{pmatrix}$$

One can see now that $\dot{\Theta}_1$ maps directly to a velocity in $^3x$ direction, and $\dot{\Theta}_2$ maps directly to a velocity in $^3y$ direction. The robot in this configuration is shown in 3.

# Solution 3

Now we are supposed to find out if singular positions can be determined more easily if we're looking at the Jacobian with respect to a different reference frame. More specifically, we are interested in the Jacobians ${}^0J, {}^3J$. The matrix ${}^3J$ has already been determined. We will compute ${}^0J$ using a description of the position of the last coordinate system of the robot. Those coordinates, denoted by ${}^0p(\Theta)$, look like this:

$$ {}^0p(\Theta) = \begin{pmatrix} l_2\mathrm{c}_{12} + l_1\mathrm{c}_1 \\ l_2\mathrm{s}_{12} + l_1\mathrm{s}_1 \end{pmatrix} $$

This can be written down immediately, since the robot the same robot as in Problem 1, just with one link less. The Jacobian, determined by differentiating ${}^0p(\Theta)$, is then:

$$ {}^0J(\Theta) = \begin{pmatrix} -l_2\mathrm{s}_{12} - l_1\mathrm{s}_1 & -l_2\mathrm{s}_{12} \\ l_2\mathrm{c}_{12} + l_1\mathrm{c}_1 & l_2\mathrm{c}_{12} \end{pmatrix} $$

Now we'll compute the singular configurations based on ${}^0J$. The determinant is:

$$ \det({}^0J) = \begin{vmatrix} -l_2\mathrm{s}_{12} - l_1\mathrm{s}_1 & -l_2\mathrm{s}_{12} \\ l_2\mathrm{c}_{12} + l_1\mathrm{c}_1 & l_2\mathrm{c}_{12} \end{vmatrix} = \begin{vmatrix} -l_1\mathrm{s}_1 & -l_2\mathrm{s}_{12} \\ l_1\mathrm{c}_1 & l_2\mathrm{c}_{12} \end{vmatrix} = -l_1l_2\mathrm{s}_1\mathrm{c}_{12} + l_1l_2\mathrm{c}_1\mathrm{s}_{12} $$

Now we would have to use the trigonometric identities again, and the solution would be $\Theta_2 = 0$, as we have computed before. But, as we have seen, it is considerably easier to compute the singular positions for ${}^3J$:

$$ \det({}^3J) = \begin{vmatrix} l_1 \sin\Theta_2 & 0 \\ l_1 \cos\Theta_2 + l_2 & l_2 \end{vmatrix} = l_1l_2\mathrm{s}_2 $$

This directly results in $\Theta_2 \in \{0°, 180°\}$, so we see that the choice of reference frame can simplify the computation of singularities in some cases. Note that the computation of ${}^3J$ was derived from the quite simple formulas of velocity computation. In particular, we were able to compute ${}^3J$ without applying any trigonometric identities! If we had computed ${}^3J$ from ${}^0J$ using the transformation formula stated above, we would have ended up with:

$$ {}^0J(\Theta) = {}^0_3R \cdot {}^3J(\Theta) \Leftrightarrow {}^3J(\Theta) = {}^0_3R^{-1} \cdot {}^0J(\Theta) $$

We would have to use ${}^i_jR = {}^i_jR^{\mathrm{T}}$ and the trigonometric identities to compute:

$$ {}^3J(\Theta) = \begin{pmatrix} \mathrm{c}_{12} & \mathrm{s}_{12} \\ -\mathrm{s}_{12} & \mathrm{c}_{12} \end{pmatrix} \cdot \begin{pmatrix} -l_2\mathrm{s}_{12} - l_1\mathrm{s}_1 & -l_2\mathrm{s}_{12} \\ l_2\mathrm{c}_{12} + l_1\mathrm{c}_1 & l_2\mathrm{c}_{12} \end{pmatrix} = \begin{pmatrix} l_1 \sin\Theta_2 & 0 \\ l_1 \cos\Theta_2 + l_2 & l_2 \end{pmatrix} $$

So all in all, this way of computing ${}^3J$ is quite more complicated than simply using the velocity formulas.