Machine Learning for Graphs and Sequential Data Exercise Sheet 06

Autoregressive Models, Markov Chains, Hidden Markov Models

Exercises marked with a $(*)$ will be discussed in the in-person exercise session.

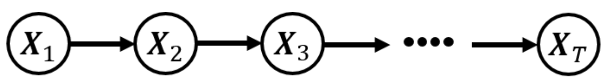
Problem 1: Consider the stationary AR(p) process $X_t = c + \sum_{i=1}^{p} \phi_i X_{t-i} + \epsilon$ with $\epsilon \sim \mathcal{N}(0, \sigma^2)$. We denote by μ the mean $E[X_t]$ and by γ_i the autocovariance $Cov(X_t, X_{t-i})$. Show:

1. $\mu = \frac{c}{1-\sum_{i=1}^{p} \phi_i}$, for all t
2. $\gamma_0 = \sum_{j=1}^{p} \phi_j \gamma_{-j} + \sigma^2$
3. $\gamma_i = \sum_{j=1}^{p} \phi_j \gamma_{i-j}$, for all $t, i \in [1, p]$

Problem 2: $(*)$ Let $\mathbf{X}_t$ be a 2-D random vector:

$$\mathbf{X}_t = \begin{bmatrix} u_t \\ v_t \end{bmatrix} \quad \text{where } u_t, v_t \in \{1, 2, \dots, K\}. \tag{1}$$

Consider the following Markov chain.



Model parameters are as follows:

- initial distribution $\boldsymbol{\pi}_x \in \mathbb{R}^{K \times K}$ that parametrizes $\Pr(\mathbf{X}_1)$:

$$\Pr\left(\mathbf{X}_1 = \begin{bmatrix} i \\ j \end{bmatrix}\right) = \boldsymbol{\pi}_x(i, j). \tag{2}$$

- transition probability matrix $\mathbf{A}_x \in \mathbb{R}^{K \times K \times K \times K}$ that parametrizes $\Pr(\mathbf{X}_{t+1} | \mathbf{X}_t)$:

$$\Pr\left(\mathbf{X}_{t+1} = \begin{bmatrix} i_{t+1} \\ j_{t+1} \end{bmatrix} \middle| \mathbf{X}_t = \begin{bmatrix} i_t \\ j_t \end{bmatrix}\right) = \mathbf{A}_x(i_t, j_t, i_{t+1}, j_{t+1}). \tag{3}$$

Because of the Markov property of $\mathbf{X}_t$, the joint probability can be factorized as

$$\Pr\left(\mathbf{X}_1, \dots, \mathbf{X}_T\right) = \Pr\left(\mathbf{X}_1\right) \prod_{t=1}^{T-1} \Pr\left(\mathbf{X}_{t+1} | \mathbf{X}_t\right).$$

In this task, we refer to this model as *"2-D first-order Markov chain"*.

a) Does the sequence $[u_1, \dots, u_T]$ (where $u_t \in \{1, 2, \dots, K\}$ is defined in Eq. (1)) have the first-order Markov property? Why or why not?

b) Let $[Y_1, \dots, Y_T] \in \{1, 2\}^T$ be a first-order Markov chain with initial probability distribution $\boldsymbol{\pi}_y \in \mathbb{R}^2$ and transition probabilities $\mathbf{A}_y \in \mathbb{R}^{2 \times 2}$.

Problem 1: Consider the stationary AR(p) process $X_t = c + \sum_{i=1}^{p} \phi_i X_{t-i} + \epsilon$ with $\epsilon \sim \mathcal{N}(0, \sigma^2)$. We denote by μ the mean $E[X_t]$ and by γ_i the autocovariance $Cov(X_t, X_{t-i})$. Show:

1. $\mu = \frac{c}{1-\sum_{i=1}^{p} \phi_i}$, for all t
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3. $\gamma_i = \sum_{j=1}^{p} \phi_j \gamma_{i-j}$, for all $t, i \in [1, p]$

$E[\cdot\cdot] = E[\cdot \sim\sim \cdot\cdot]+ \cdot\cdot\ E(\varepsilon) = 0$

1\. $E[X_t] = c + \sum_{i=1}^{p} \phi_i E[X_{t-i}] + E[\varepsilon]$

$\because$ stationary $\Rightarrow E[X_t] = E[X_{t-i}] = \mu$

$\therefore \mu = c + \sum_{i=1}^{p} \phi_i \mu$

$\mu = \frac{c}{1-\sum_{i=1}^{p} \phi_i}$

$cov(\cdot\cdot,\cdot\cdot) = cov(\cdot\cdot\cdot\cdot) + \cdot\cdot\cdot$

2\. $cov(X_t, X_t) = cov(c, X_{t-i}) + \sum_{i=1}^{p} \phi_i\, cov(X_{t-i}, X_t) + cov(\varepsilon, X_t)$

$\gamma_0 = 0 + \sum_{i=1}^{p} \phi_i \gamma_{-i} + \sigma^2$

3\. $cov(X_t, X_{t-i}) = cov(c, X_{t-i}) + \sum_{\boxed{j=1}}^{p} \phi_j\, cov(X_{t-j}, X_{t-i}) + cov(\varepsilon_t, X_{t-i})$

$\gamma_i = 0 + \sum_{j=1}^{p} \phi_j \gamma_{j-i} + 0$ time noise independ

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Problem 2: (∗) Let $\mathbf{X}_t$ be a 2-D random vector:

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Consider the following Markov chain.

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow \cdots\cdots \rightarrow X_T$$

Model parameters are as follows:

- initial distribution $\boldsymbol{\pi}_x \in \mathbb{R}^{K \times K}$ that parametrizes $\Pr(\mathbf{X}_1)$:

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$$\Pr\left(\mathbf{X}_{t+1} = \begin{bmatrix} i_{t+1} \\ j_{t+1} \end{bmatrix} \,\middle|\, \mathbf{X}_t = \begin{bmatrix} i_t \\ j_t \end{bmatrix}\right) = \mathbf{A}_x(i_t, j_t, i_{t+1}, j_{t+1}). \tag{3}$$

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In this task, we refer to this model as *"2-D first-order Markov chain"*.

a) Does the sequence $[u_1, \ldots, u_T]$ (where $u_t \in \{1, 2, \ldots, K\}$ is defined in Eq. (1)) have the first-order Markov property? Why or why not?

b) Let $[Y_1, \ldots, Y_T] \in \{1, 2\}^T$ be a first-order Markov chain with initial probability distribution $\boldsymbol{\pi}_y \in \mathbb{R}^2$ and transition probabilities $\mathbf{A}_y \in \mathbb{R}^{2\times 2}$.
   - Briefly explain why the sequence $\begin{bmatrix} Y_2 \\ Y_1 \end{bmatrix}, \begin{bmatrix} Y_3 \\ Y_2 \end{bmatrix}, \ldots, \begin{bmatrix} Y_T \\ Y_{T-1} \end{bmatrix}$ is a 2-D first-order Markov chain.
   - Compute initial and transition probabilities, $\boldsymbol{\pi}_x$ and $\mathbf{A}_x$ (defined in Eqs. (2) and (3)) for the sequence $\begin{bmatrix} Y_2 \\ Y_1 \end{bmatrix}, \begin{bmatrix} Y_3 \\ Y_2 \end{bmatrix}, \ldots, \begin{bmatrix} Y_T \\ Y_{T-1} \end{bmatrix}$.

a) $u_t = A_1 u_{t-1} + A_2 \cdot v_{t-1}$

NO

b) Y is first order

$Y_2 = A_1 \cdot Y_1$

$Y_3 = A_2 \cdot Y_2$

$\vdots$

$$\begin{bmatrix} Y_3 \\ Y_2 \end{bmatrix} = \begin{bmatrix} A_2 & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} Y_2 \\ Y_1 \end{bmatrix}$$

$\vdots$

$$\begin{bmatrix} Y_T \\ Y_{T-1} \end{bmatrix} = \begin{bmatrix} A_T & 0 \\ 0 & A_{T-1} \end{bmatrix} \begin{bmatrix} Y_{T-1} \\ Y_{T-2} \end{bmatrix}$$

$\therefore$ 2-order

$$P\left(\begin{bmatrix} Y_2 \\ Y_1 \end{bmatrix}\right) = P\left(\begin{bmatrix} j \\ i \end{bmatrix}\right) = \pi_x(i, j) = A_y(i,j) \cdot \pi_y(i)$$

$$P\left(\begin{bmatrix} Y_T \\ Y_{T-1} \end{bmatrix} \,\middle|\, \begin{bmatrix} Y_{T-1} \\ Y_{T-2} \end{bmatrix}\right) = P\left(\begin{bmatrix} i \\ j \end{bmatrix} \,\middle|\, \begin{bmatrix} i' \\ j' \end{bmatrix}\right) = \begin{cases} 0 & j \neq i' \\ A_y(j,i) & \end{cases}$$

$$= P\left(\begin{bmatrix} i \\ j \end{bmatrix} \,\middle|\, \begin{bmatrix} j \\ j' \end{bmatrix}\right) = P(Y_T = i \mid Y_{T-1} = j, Y_{T-2} = j') \underbrace{P(Y_{T-1} = j \mid Y_{T-1} = j, Y_{T-2} = j')}_{=1}$$

$$= A_y(j,i)$$

- Briefly explain why the sequence $\begin{bmatrix} Y_2 \\ Y_1 \end{bmatrix}, \begin{bmatrix} Y_3 \\ Y_2 \end{bmatrix}, \dots, \begin{bmatrix} Y_T \\ Y_{T-1} \end{bmatrix}$ is a 2-D first-order Markov chain.
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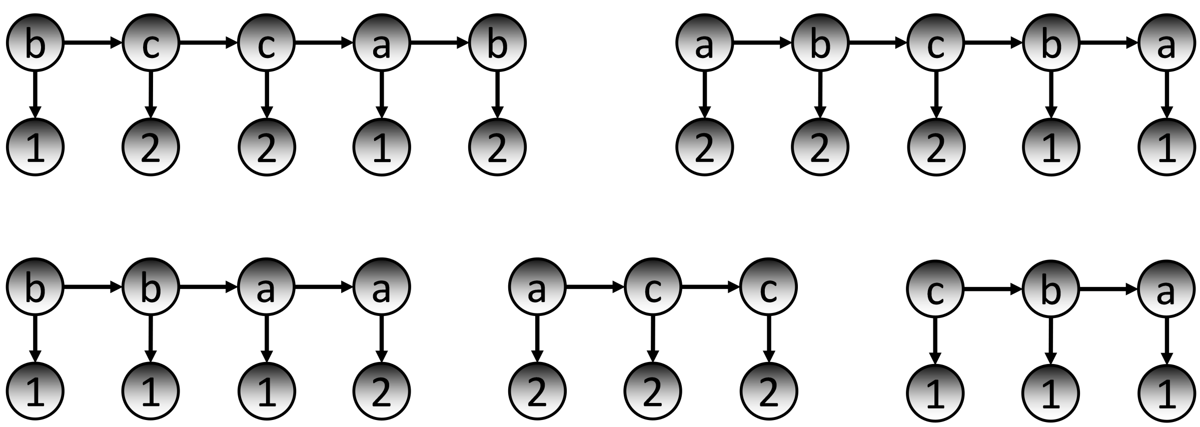
Problem 3: $(*)$ Consider an HMM where hidden variables are in $\{1, 2\}$ and observed variables are in $\{a, b, c\}$. Let the model parameters be as follows:

$$A = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix} \end{array} \qquad B = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 0.2 & 0 & 0.8 \\ 0.4 & 0.6 & 0 \end{bmatrix} \end{array} \qquad \pi = \begin{array}{c c} \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \end{array}$$

Assume that the sequence $X_{1:5} = [cabac]$ is observed.

1. Filtering: find the distribution $P(Z_3 | X_{1:3})$.
2. Smoothing: find the distribution $P(Z_3 | X_{1:5})$.
3. Viterbi algorithm: find the most probable sequence $[Z_1, \dots, Z_5]$.

Problem 4: Consider an HMM where states Z_t are in $\{a, b, c\}$ and emissions X_t are in $\{1, 2\}$. Given is the following set of fully-observed instances (two sequences of length 5, one sequence of length 4, and two sequences of length 3):



Learn the parameters of the HMM (i.e. $\pi \in \mathbb{R}^3, \mathbf{A} \in \mathbb{R}^{3\times3}$, and $\mathbf{B} \in \mathbb{R}^{3\times2}$) using maximum-likelihood estimation.

Problem 3: $(*)$ Consider an HMM where hidden variables are in $\{1, 2\}$ and observed variables are in $\{a, b, c\}$. Let the model parameters be as follows:

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Assume that the sequence $X_{1:5} = [cabac]$ is observed.

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1. $\alpha_{t+1}(k) = B_{k:x_{(t+1)}} \odot (A' \alpha_t(k))$

$\alpha_1(k) = \pi_k \odot B_{k\,x_1}$

$= \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}$

$\alpha_2(k) = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix} \odot \left(\begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 0.4 \\ 0 \end{bmatrix} \right)$

$= \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix} \odot \begin{bmatrix} 0.08 \\ 0.32 \end{bmatrix}$

$= \begin{bmatrix} 0.016 \\ 0.128 \end{bmatrix}$

$\alpha_3(k) = \begin{bmatrix} 0 \\ 0.6 \end{bmatrix} \odot \left(\begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 0.016 \\ 0.128 \end{bmatrix} \right)$

$= \begin{bmatrix} 0 \\ 0.6 \end{bmatrix} \odot \begin{bmatrix} \cdots \\ 0.768 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 0.04608 \end{bmatrix}$

$P = \dfrac{\alpha_3(k)}{\mathrm{sum}(\alpha_3)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

2. $\beta_t(k) = A(B_{k:x_{t+1}} \odot \beta_{t+1}(k))$ ⇐ $\beta_5 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\beta_4(k) = \begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix} \left(\begin{bmatrix} 0.8 \\ 0 \end{bmatrix} \odot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$

$= \begin{bmatrix} 0.16 \\ 0.4 \end{bmatrix}$

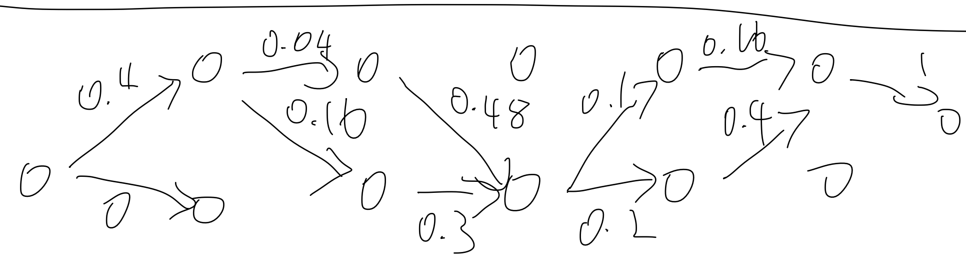
$\beta_3(k) = \begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix} \left(\begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix} \odot \begin{bmatrix} 0.16 \\ 0.4 \end{bmatrix} \right)$

$= \begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix} \begin{pmatrix} 0.032 \\ 0.16 \end{pmatrix}$

$= \begin{bmatrix} 0.1344 \\ 0.096 \end{bmatrix}$

$P = \dfrac{\alpha_3(k)\,\beta_3(k)}{\mathrm{sum}(\alpha_3 \cdot \beta_3)} = \dfrac{\begin{bmatrix} 0 \\ 0.096 \times 0.04608 \end{bmatrix}}{0.1344 \cdot 0 + 0.096 \cdot 0.04608}$ ⇐ $\mathrm{sum} = \sum_k^2 \alpha_k \cdot \beta_k$.

3.



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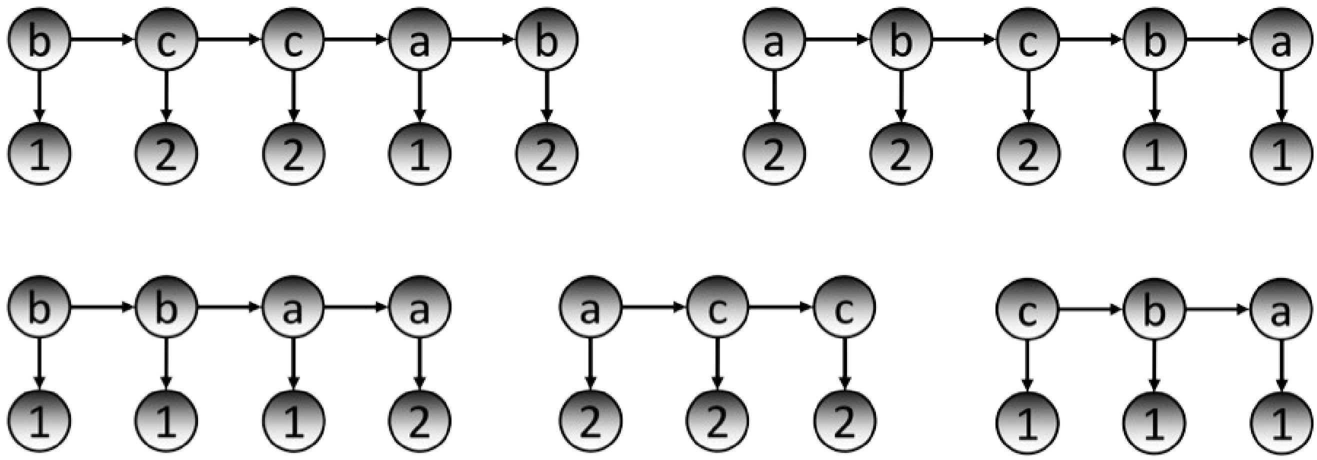

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