FUNDAMENTALS OF ARTIFICIAL INTELLIGENCE
Exercise 10: Making Simple Decisions – Solutions Winter Semester 2023/24
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Problem 10.1:

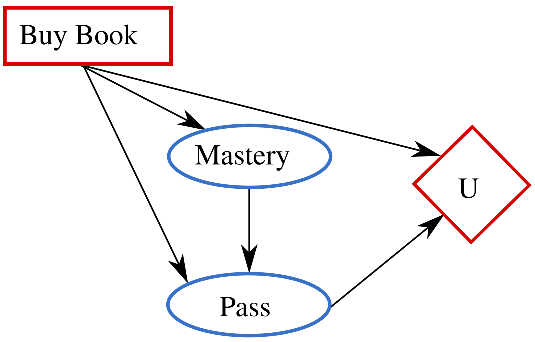
(Taken from [1] Exercise 16.15)
We have the following information:

$$P(p|b,m) = 0.9 \qquad P(p|b,\neg m) = 0.5$$
$$P(p|\neg b,m) = 0.8 \qquad P(p|\neg b,\neg m) = 0.3$$
$$P(m|b) = 0.9 \qquad P(m|\neg b) = 0.7$$

We have the following utilities:

$$U(b) = -100€, \quad U(\neg b) = 0€ \quad U(p) = 2000€, \quad U(\neg p) = 0€$$

Problem 10.1.1: Draw the decision network for the problem.



Problem 10.1.2: Compute the expected utility of buying the book and of not buying it.
General formula for expected utility:

$$EU(a|\mathbf{e}) = \sum_{s'} P(\texttt{Result}(a) = s'|a,\mathbf{e})U(s')$$

First of all, the probabilities of passing the exam after buying the book and not buying the book must be calculated. The probability of passing the exam after buying the book:

$$\begin{aligned}
P(p|b) &= \frac{\sum_{M\in\{m,\neg m\}} P(p,b,M)}{P(b)} \\
&= \frac{\sum_{M\in\{m,\neg m\}} P(p|b,M)P(M|b)P(b)}{P(b)} \\
&= \sum_{M\in\{m,\neg m\}} P(p|b,M)P(M|b) \\
&= P(p|b,m)P(m|b) + P(p|b,\neg m)P(\neg m|b) \\
&= 0.9 \cdot 0.9 + 0.5 \cdot 0.1 \\
&= 0.86
\end{aligned}$$

The probability of passing the exam without buying the book:

$$\begin{aligned}
P(p|\neg b) &= \sum_{M\in\{m,\neg m\}} P(p|\neg b, M)P(M|\neg b) \\
&= P(p|\neg b, m)P(m|\neg b) + P(p|\neg b, \neg m)P(\neg m|\neg b) \\
&= 0.8 \cdot 0.7 + 0.3 \cdot 0.3 \\
&= 0.65
\end{aligned}$$

The expected utility of buying the book:

$$\begin{aligned}
EU(b) &= \sum_{P\in\{p,\neg p\}} P(P|b)\, U(P, b) \\
&= P(p|b)\, U(p, b) + P(\neg p|b)\, U(\neg p, b) \\
&= 0.86 \cdot (2000€ - 100€) + 0.14 \cdot (-100€) \\
&= 1620€
\end{aligned}$$

The expected utility of not buying the book:

$$\begin{aligned}
EU(\neg b) &= \sum_{P\in\{p,\neg p\}} P(P|\neg b)\, U(P, \neg b) \\
&= P(p|\neg b)\, U(p, \neg b) + P(\neg p|\neg b)\, U(\neg p, \neg b) \\
&= 0.65 \cdot (2000€) + 0.35 \cdot (0€) \\
&= 1300€
\end{aligned}$$

Problem 10.1.3: What should Sam do?
Sam should definitely buy the book!

Problem 10.2:

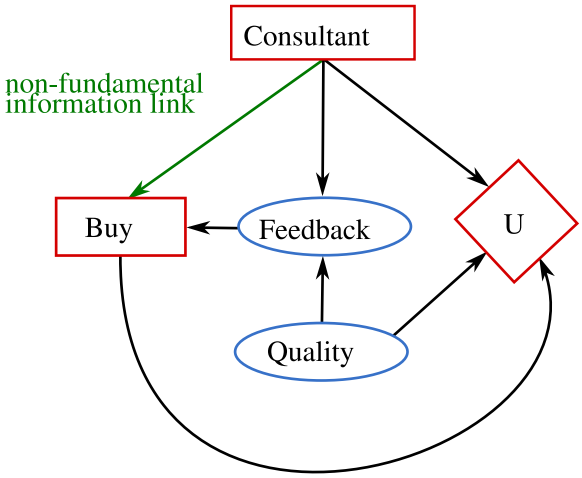
We have the following information:

$$P(q) = 0.7 \qquad P(f|q) = 0.85 \qquad P(f|\neg q) = 0.05$$
$$P(\neg q) = 0.3 \qquad P(\neg f|q) = 0.15 \qquad P(\neg f|\neg q) = 0.95$$

We have the following utilities:

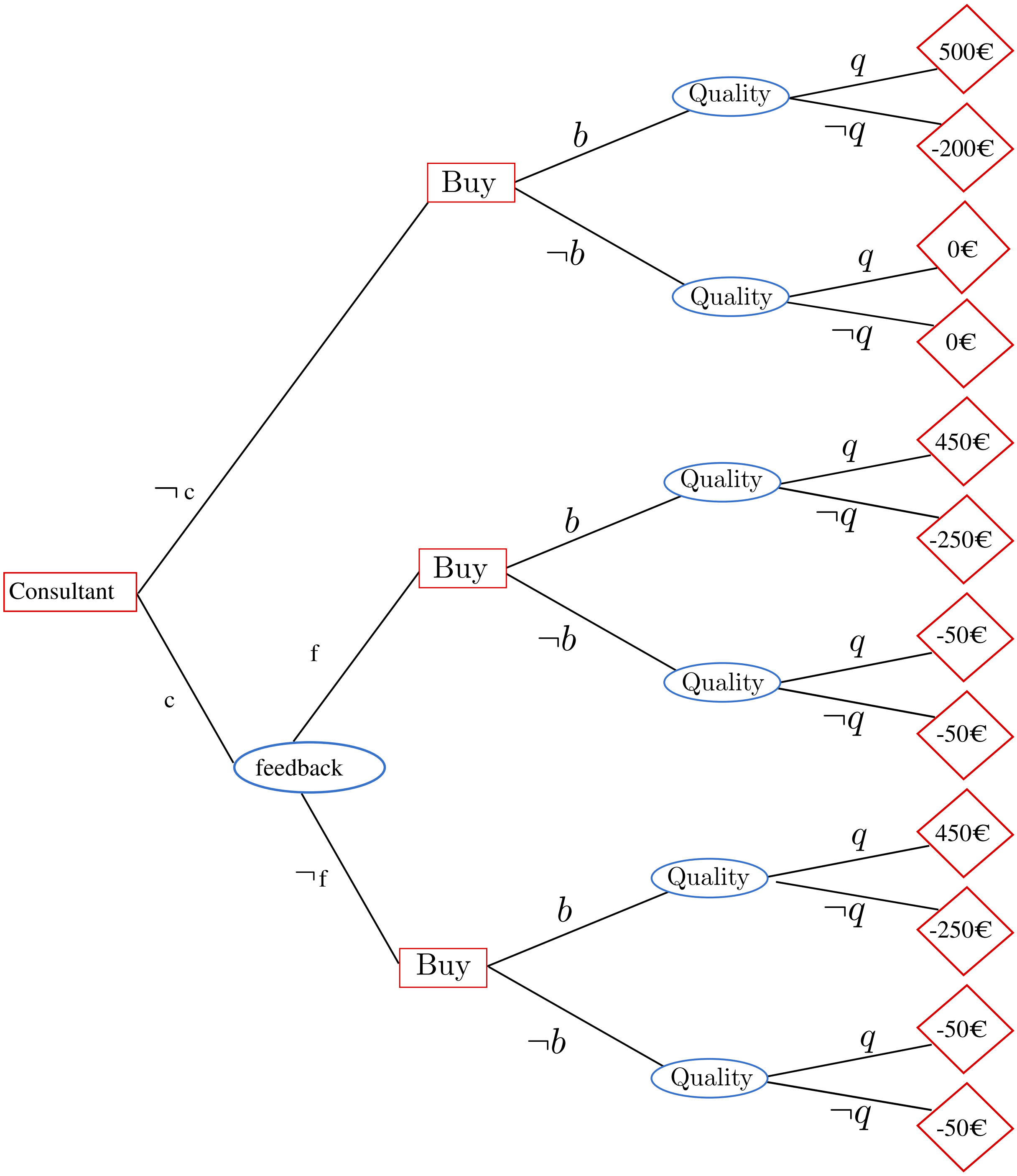
$$U(b) = -1500€, \quad U(\neg b) = 0€, \quad U(q) = 2000€ \quad U(\neg q) = 1300€, \quad U(c) = -50€$$
$$U(\neg c) = 0€$$

Problem 10.2.1: Draw the decision network that represents this problem.



Note: A information link is fundamental if its removal would change the partial ordering. The *no forgetting assumption* means that all past decisions and revealed variables are available at the current decision. When the *no forgetting assumption* hold, we do not need to draw non-fundamental information links.

Problem 10.2.2: Draw the decision tree that represents this problem.



Note: The utility depends on the Consultation C and Buy B as well as on the Quality Q of the stock: $U(C, B, Q)$, e.g. $U(c, b, \neg q) = -50€ - 1500€ + (2000 - 700)€ = -250€$.

Problem 10.2.3: Calculate the expected utility of buying the stock, without asking the consultant.

$$
\begin{aligned}
EU(b, \neg c) &= \sum_{Q \in \{q, \neg q\}} P(Q|b, \neg c) U(Q, b, \neg c) \\
&= \sum_{Q \in \{q, \neg q\}} P(Q) U(Q, b, \neg c) \\
&= P(q)U(q, b, \neg c) + P(\neg q)U(\neg q, b, \neg c) \\
&= 0.7 \cdot (2000€ - 1500€ - 0€) + 0.3 \cdot (1300€ - 1500€ - 0€) \\
&= 290€
\end{aligned}
$$

Problem 10.2.4: Derive an optimal conditional plan for the investor. Start with determining the optimal decisions whether to buy the stock given no consultation, a positive feedback or a negative feedback. Calculate the value of information of the consultation.
We have the following utilities:

$$
\begin{aligned}
U(q, b) &= 2000€ - 1500€ = 500€ \\
U(\neg q, b) &= 1300€ - 1500€ = -200€ \\
U(q, \neg b) &= 0€ \\
U(\neg q, \neg b) &= 0€
\end{aligned}
$$

We have to find a decision maximizing the expected utility for each decision variable:

$$
\pi^*(d_i|\mathbf{e}) = \underset{d_i}{\operatorname{argmax}}\, EU(d_i|\mathbf{e})
$$

$$
\pi^*(d_i|x_{1:i-1}, d_{1:i-1}) = \underset{d_i}{\operatorname{argmax}}\, EU(d_i|x_{1:i-1}, d_{1:i-1})
$$

$$
MEU(d_{1:n}) = \max_{d_1} \sum_{x_1} \dots \max_{d_n} \sum_{x_n} \prod_{i=1}^{n} P(x_i|x_{1:i-1}, d_{1:i}) U(x_{1:n}, d_{1:n})
$$

In our case, there are four nodes including two decisions and two variables. The partial ordering of decisions and variables is: Consultant $<$ Feedback $<$ Buy $<$ Quality.
So, $i = 2$. $d_1 = C$, $x_1 = F$, $d_2 = B$, $x_2 = Q$.

$$
\begin{aligned}
MEU(C, B) &= \max_{C} \sum_{F} P(F|C) \max_{B} \sum_{Q} P(Q|F, C, B) U(F, Q, C, B) \\
&= \max_{C} \sum_{F} P(F|C) \max_{B} \sum_{Q} P(Q|F, C) U(Q, C, B)
\end{aligned} \tag{1}
$$

We simplify it because utility not directly depends on the feedback and the quality of stock has nothing to do with whether this investor buys it or not.

We solve the problem backwards, first consider the last decision, whether the investor should buy the stock. And then consider whether the investor should ask the consultant.

The **last decision** is: whether the investor should **buy the stock?**

There are three situations:

- Case1: Not asking the consultant.
- Case2: Ask the consultant, the feedback is positive. (The consultant considers the stock as a high quality stock).
- Case3: Ask the consultant, the feedback is negative.

Case1: Not asking the consultant:

If the investor buy it:

$$\begin{aligned} EU(b|\neg c) &= P(q)U(q,b) + P(\neg q)U(\neg q, b) \\ &= 0.7 * 500 + 0.3 * (-200) = 290 \end{aligned}$$

If the investor doesn't buy it:

$$\begin{aligned} EU(\neg b|\neg c) &= P(q)U(q,\neg b) + P(\neg q)U(\neg q, \neg b) \\ &= 0.7 * 0 + 0.3 * 0 = 0 \end{aligned}$$

The optimal decision is to buy it.

$$\pi^*(B|\neg c) = b$$

Case2: Ask the consultant, the feedback is positive:

$$\begin{aligned} P(Q|c,f) &= \alpha P(Q,c,f) \\ &= \alpha \langle P(q)P(f|q,c), P(\neg q)P(f|\neg q, c)\rangle \\ &= \alpha \langle 0.7 * 0.85, 0.3 * 0.05\rangle = \langle 0.975, 0.025\rangle \end{aligned}$$

If the investor buys it:

$$\begin{aligned} EU(b|c,f) &= P(q|c,f)U(q,b) + P(\neg q|c,f)U(\neg q,b) - 50 \\ &= 0.975 * 500 + 0.025 * (-200) - 50 = 432.5 \end{aligned}$$

If the investor doesn't buy it:

$$\begin{aligned} EU(\neg b|c,f) &= P(q|c,f)U(q,\neg b) + P(\neg q|c,f)U(\neg q,\neg b) - 50 \\ &= 0.975 * 0 + 0.025 * 0 - 50 = -50 \end{aligned}$$

The optimal decision is to buy it.

$$\pi^*(B|c,f) = \operatorname*{argmax}_B EU(B|c,f) = b$$

Case3: Ask the consultant, the feedback is negative.

$$\begin{aligned} P(Q|c,\neg f) &= \alpha P(Q,c,\neg f) \\ &= \alpha \langle P(q)P(\neg f|q,c), P(\neg q)P(\neg f|\neg q, c)\rangle \\ &= \alpha \langle 0.7 * 0.15, 0.3 * 0.95\rangle = \langle 0.269, 0.731\rangle \end{aligned}$$

If the investor buy it:

$$\begin{aligned} EU(b|c,\neg f) &= P(q|c,\neg f)U(q,b) + P(\neg q|c,\neg f)U(\neg q,b) - 50 \\ &= 0.269 * 500 + 0.731 * (-200) - 50 = -61.7 \end{aligned}$$

If the investor doesn't buy it:

$$\begin{aligned} EU(\neg b|c,\neg f) &= P(q|c,\neg f)U(q,\neg b) + P(\neg q|c,\neg f)U(\neg q,\neg b) - 50 \\ &= 0.269 * 0 + 0.731 * 0 - 50 = -50 \end{aligned}$$

The optimal decision is not to buy it.

$$\pi^*(B|c,\neg f) = \neg b$$

The **second decision** is: whether the investor should **ask the consultant?**

There are two situations:

- Case1: Asking the consultant.
- Case2: Not asking the consultant.

Case1: Asking the consultant. Let us summarize the information in the table below. $P(F,Q|c) = P(Q) * P(F|Q,c)$

	$P(F,Q\|c)$	$\pi^*(B\|c,F)$	$U(\pi^*(B\|c,F),Q,c)$
f,q	$0.7 * 0.85 = 0.595$	b	$500 - 50 = 450$
$\neg f,q$	$0.7 * 0.15 = 0.105$	$\neg b$	$0 - 50 = -50$
$f,\neg q$	$0.3 * 0.05 = 0.015$	b	$-200 - 50 = -250$
$\neg f,\neg q$	$0.3 * 0.95 = 0.285$	$\neg b$	$0 - 50 = -50$

From equation 1 we get:

$$\begin{aligned} EU(c) &= \sum_F P(F|c) \max_B \sum_Q P(Q|F,c)U(Q,c,B) \\ &= \sum_F \sum_Q P(F|c)P(Q|F,c)U(\pi^*(B|c,F),Q,c) \\ &= 0.595 * 450 + 0.105 * (-50) + 0.015 * (-250) + 0.285 * (-50) = 244.5 \end{aligned}$$

Case2: Not asking the consultant. The optimal decision which we obtained before is to buy the stock:

$$\begin{aligned} EU(\neg c) &= \sum_Q P(Q|\neg c)U(\pi^*(B|\neg c),Q) \\ &= P(q|\neg c)U(q,b) + P(\neg q|\neg c)U(\neg q,b) \\ &= 0.7 * 500 + 0.3 * (-200) = 290 \end{aligned}$$

Finally, we can calculate the value of information of the consultation.

Value of information = expected utility given the information at no charge - expected utility without the information

$$VOI_{\mathbf{e}}(E_j) = EU(c) + 50 - EU(\neg c) = 244.5 + 50 - 290 = 4.5$$

References

[1] S. Russell and P. Norvig, *Artificial Intelligence: A Modern Approach*. Prentice Hall, 2010.