Tutorial Robotics IN2067

Exercise Sheet 04

P01

Problem 1

Figure 1 shows a robot with one rotational joint and one prismatic joint. The DH parameters for this robot are

i	a_{i-1}	α_{i-1}	d_i	Θ_i
1	0	0	0	Θ_1
2	l_1	$-90°$	d_2	0

The manipulator is shown for configuration $\Theta_1 = 0, d_2 \neq 0$. Gravitational force applies in negative X_0-direction, as shown. The inertia tensors are:

$$^{C_1}I_1 = \begin{pmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{pmatrix} \quad ^{C_2}I_2 = \begin{pmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{pmatrix}$$

The masses of the robot's links are m_1 and m_2, and the centers of mass of the links are located at

$$^{1}P_{C_1} = \left(\frac{l_1}{2}, 0, 0\right)^{\mathrm{T}}$$
$$^{2}P_{C_2} = (0, 0, l_2)^{\mathrm{T}}.$$

a) Determine the dynamics equations using the Newton-Euler method

b) Formulate the equations in state space (M-V-G) form

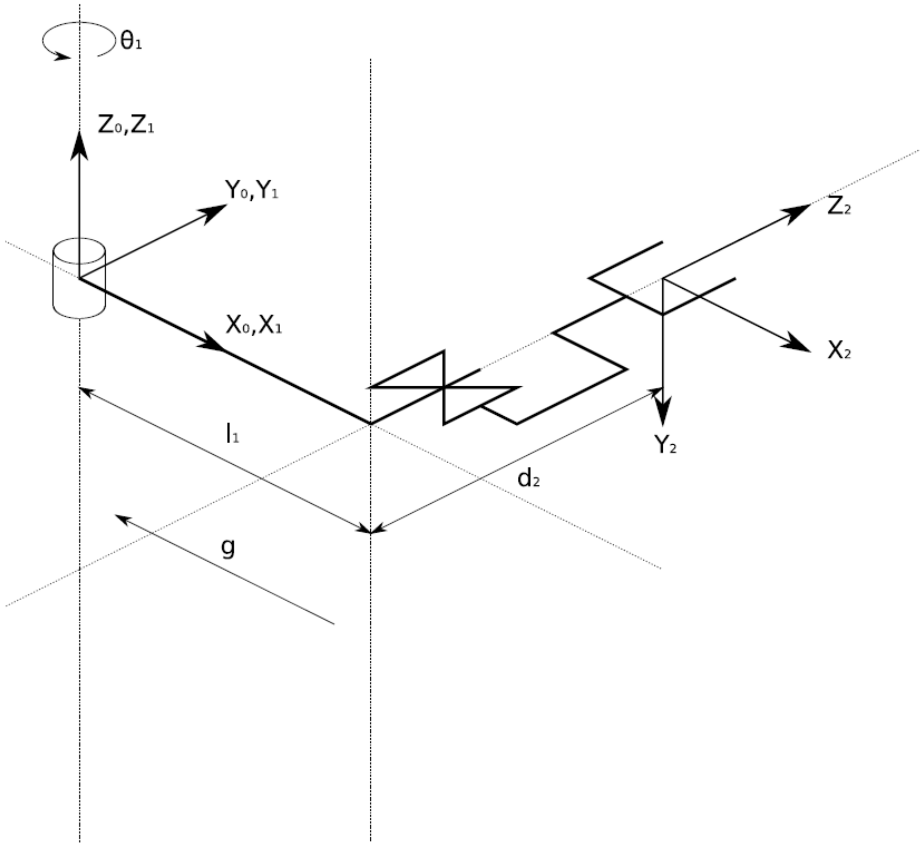

Figure 1: *RP Robot (Problem 1)*

P01

Newton-Euler Method:

Step 0: Initialization

- Assumption: the robot has N joints, $[N+1]$ is the coordinate frame of the end-effector
- Determine ${}^{0}\omega_{0}, {}^{0}\dot{\omega}_{0}, {}^{0}v_{0}, {}^{0}\dot{v}_{0}, {}^{N+1}f_{N+1}, {}^{N+1}n_{N+1}$
- Consider the direction of gravity g when writing ${}^{0}\dot{v}_{0}$

P01

Newton-Euler Method:

Step 1: Compute ${}^{i}\dot{v}_{c_i}$, ${}^{i}\omega_i$ and ${}^{i}\dot{\omega}_i$

- Forward propagation of velocities and accelerations:

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{\ \ i}R\,{}^{i}\omega_i + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{pmatrix} $$

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{\ \ i}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{\ \ i}_{i+1}t\right) + \begin{pmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{pmatrix} $$

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{\ \ i}R\,{}^{i}\dot{\omega}_i + \left({}^{i+1}_{\ \ i}R\,{}^{i}\omega_i\right) \times \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_{i+1} \end{pmatrix} $$

$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{\ \ i}R\left({}^{i}\dot{v}_i + {}^{i}\dot{\omega}_i \times {}^{\ \ i}_{i+1}t + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{\ \ i}_{i+1}t\right)\right) + $$

$$ + 2\left({}^{i+1}\omega_{i+1} \times \begin{pmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{pmatrix}\right) + \begin{pmatrix} 0 \\ 0 \\ \ddot{d}_{i+1} \end{pmatrix} $$

$$ {}^{i}\dot{v}_{c_i} = {}^{i}\dot{v}_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{c_i} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{c_i}\right) $$

P01

Newton-Euler Method:

Step 2: Compute ${}^{i}F_i$ and ${}^{i}N_i$

Using the previously computed ${}^{i}\dot{v}_{c_i}$, ${}^{i}\omega_i$ and ${}^{i}\dot{\omega}_i$

$$ {}^{i}F_i = m_i \cdot {}^{i}\dot{v}_{C_i} $$

$$ {}^{i}N_i = {}^{C_i}I_i \cdot {}^{i}\dot{\omega}_i + {}^{i}\omega_i \times {}^{C_i}I_i \cdot {}^{i}\omega_i $$

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Newton-Euler Method:

Step 3: Compute joint torques vector τ

- Backward propagation of forces and torques:

$$ {}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i $$

$$ {}^{i}n_i = {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R\,{}^{i+1}f_{i+1}\right) + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}N_i $$

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CF	α	a	d	Θ
1	0°	0	0	θ_1
2	-90°	l_1	d_2	0°
3	0°	0	0	0°

$$ {}^{0}_{1}T = \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad {}^{1}_{2}T = \begin{pmatrix} 1 & 0 & 0 & l_1 \\ 0 & 0 & 1 & d_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{2}_{3}T = I_4 $$

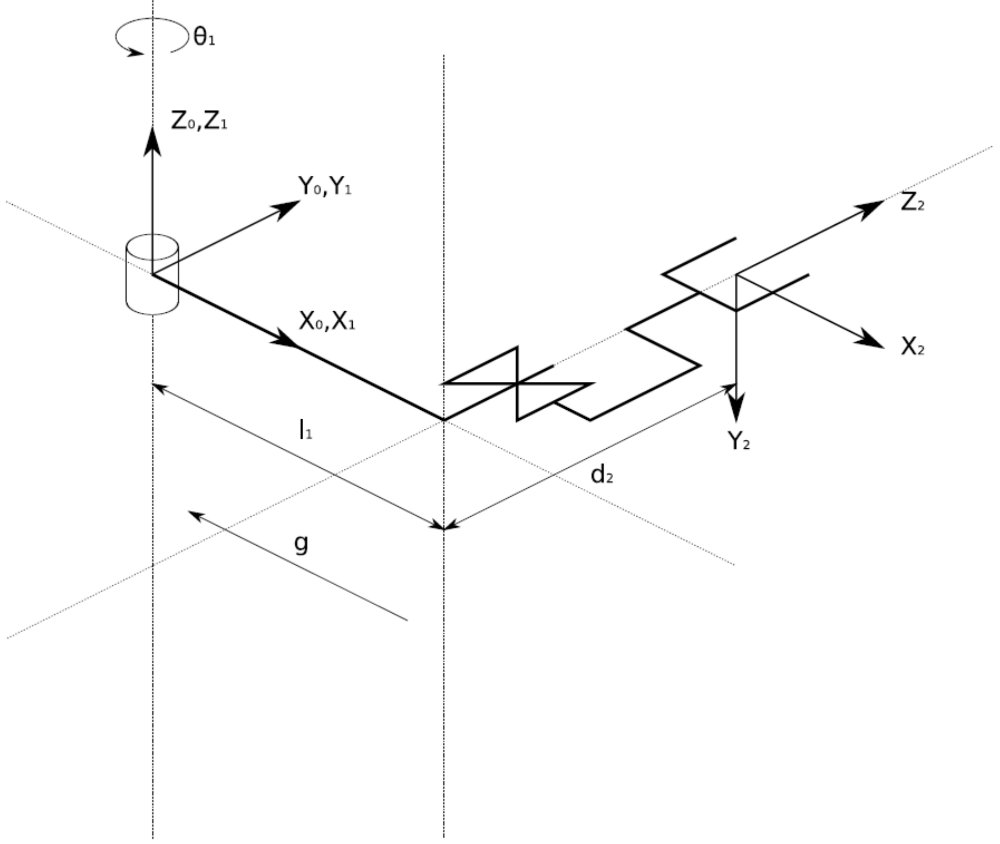

Figure 1: *RP Robot (Problem 1)*

P01

Newton-Euler Method:

Step 0: Initialization

- ${}^{0}\omega_0, {}^{0}\dot{\omega}_0, {}^{0}v_0, {}^{3}f_3, {}^{3}n_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
  - Because not otherwise given in the text
- ${}^{0}\dot{v}_0 = \begin{pmatrix} g \\ 0 \\ 0 \end{pmatrix}$

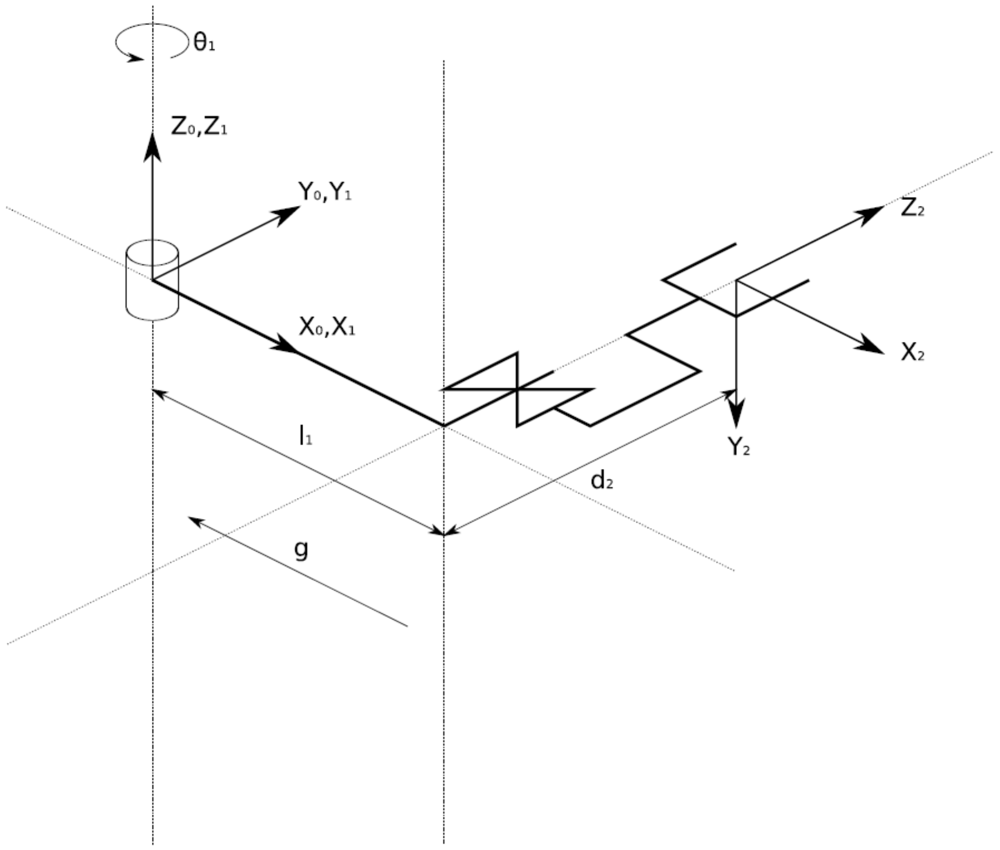

Figure 1: *RP Robot (Problem 1)*

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$$
{}^0\omega_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad {}^0\dot{\omega}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad {}^0v_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad {}^0\dot{v}_0 = \begin{pmatrix} g \\ 0 \\ 0 \end{pmatrix}
$$

Newton-Euler Method:

Step 1: Compute ${}^i\dot{v}_{c_i}$, ${}^i\omega_i$ and ${}^i\dot{\omega}_i$

$$
{}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^i\omega_i + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{pmatrix}
$$

$$
{}^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^iv_i + {}^i\omega_i \times {}^i_{i+1}t\right) + \begin{pmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{pmatrix}
$$

$$
{}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^i\dot{\omega}_i + \left({}^{i+1}_{i}R\,{}^i\omega_i\right) \times \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_{i+1} \end{pmatrix}
$$

$$
{}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left({}^i\dot{v}_i + {}^i\dot{\omega}_i \times {}^i_{i+1}t + {}^i\omega_i \times \left({}^i\omega_i \times {}^i_{i+1}t\right)\right) + 2\left({}^{i+1}\omega_{i+1} \times \begin{pmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{pmatrix}\right) + \begin{pmatrix} 0 \\ 0 \\ \ddot{d}_{i+1} \end{pmatrix}
$$

$$
{}^i\dot{v}_{c_i} = {}^i\dot{v}_i + {}^i\dot{\omega}_i \times {}^iP_{c_i} + {}^i\omega_i \times \left({}^i\omega_i \times {}^iP_{c_i}\right)
$$

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Newton-Euler Method:

Step 1: Compute ${}^i\dot{v}_{c_i}$, ${}^i\omega_i$ and ${}^i\dot{\omega}_i$

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{pmatrix}$$

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}_{i+1}t\right) + \begin{pmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{pmatrix}$$

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + \left({}^{i+1}_{i}R\,{}^{i}\omega_i\right) \times \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_{i+1} \end{pmatrix}$$

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{v}_i + {}^{i}\dot{\omega}_i \times {}^{i}_{i+1}t + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}_{i+1}t\right)\right) + 2\left({}^{i+1}\omega_{i+1} \times \begin{pmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{pmatrix}\right) + \begin{pmatrix} 0 \\ 0 \\ \ddot{d}_{i+1} \end{pmatrix}$$

$$^{i}\dot{v}_{c_i} = {}^{i}\dot{v}_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{c_i} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{c_i}\right)$$

$$^{0}\omega_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad {}^{0}\dot{\omega}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad {}^{0}v_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad {}^{0}\dot{v}_0 = \begin{pmatrix} g \\ 0 \\ 0 \end{pmatrix}$$

$$^{1}\omega_1 = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} \quad {}^{1}\dot{\omega}_1 = \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{pmatrix}$$

$$^{1}v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad {}^{1}\dot{v}_1 = \begin{pmatrix} gc_1 \\ -gs_1 \\ 0 \end{pmatrix}$$

$$^{1}\dot{v}_{c_1} = \begin{pmatrix} gc_1 - \frac{1}{2}l_1\dot{\theta}_1^2 \\ -gs_1 + \frac{1}{2}l_1\ddot{\theta}_1 \\ 0 \end{pmatrix}$$

P01

Newton-Euler Method:

Step 1: Compute ${}^i\dot{v}_{c_i}$, ${}^i\omega_i$ and ${}^i\dot{\omega}_i$

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \begin{pmatrix}0\\0\\\dot{\theta}_{i+1}\end{pmatrix}$$

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}_{i+1}t\right) + \begin{pmatrix}0\\0\\\dot{d}_{i+1}\end{pmatrix}$$

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + \left({}^{i+1}_{i}R\,{}^{i}\omega_i\right)\times\begin{pmatrix}0\\0\\\dot{\theta}_{i+1}\end{pmatrix} + \begin{pmatrix}0\\0\\\ddot{\theta}_{i+1}\end{pmatrix}$$

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{v}_i + {}^{i}\dot{\omega}_i \times {}^{i}_{i+1}t + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}_{i+1}t\right)\right) + 2\left({}^{i+1}\omega_{i+1} \times \begin{pmatrix}0\\0\\\dot{d}_{i+1}\end{pmatrix}\right) + \begin{pmatrix}0\\0\\\ddot{d}_{i+1}\end{pmatrix}$$

$$^{i}\dot{v}_{c_i} = {}^{i}\dot{v}_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{c_i} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{c_i}\right)$$

$$^0\omega_0 = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad {}^0\dot{\omega}_0 = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad {}^0v_0 = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad {}^0\dot{v}_0 = \begin{pmatrix}g\\0\\0\end{pmatrix}$$

$$^1\omega_1 = \begin{pmatrix}0\\0\\\dot{\theta}_1\end{pmatrix} \quad {}^1\dot{\omega}_1 = \begin{pmatrix}0\\0\\\ddot{\theta}_1\end{pmatrix}$$

$$^1v_1 = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad {}^1\dot{v}_1 = \begin{pmatrix}gc_1\\-gs_1\\0\end{pmatrix}$$

$$^1\dot{v}_{c_1} = \begin{pmatrix}gc_1 - \frac{1}{2}l_1\dot{\theta}_1^2\\-gs_1 + \frac{1}{2}l_1\ddot{\theta}_1\\0\end{pmatrix}$$

$$^2\omega_2 = \begin{pmatrix}0\\-\dot{\theta}_1\\0\end{pmatrix} \quad {}^2\dot{\omega}_2 = \begin{pmatrix}0\\-\ddot{\theta}_1\\0\end{pmatrix}$$

$$^2v_2 = \begin{pmatrix}-d_2\dot{\theta}_1\\0\\\dot{d}_2 + l_1\dot{\theta}_1\end{pmatrix} \quad {}^2\dot{v}_2 = \begin{pmatrix}-d_2\ddot{\theta}_1 - 2\dot{d}_2\dot{\theta}_1 - l_1\dot{\theta}_1^2 + gc_1\\0\\-d_2\dot{\theta}_1^2 + \ddot{d}_2 + l_1\ddot{\theta}_1 - gs_1\end{pmatrix}$$

$$^2\dot{v}_{c_2} = \begin{pmatrix}-(d_2+l_2)\ddot{\theta}_1 - 2\dot{d}_2\dot{\theta}_1 - l_1\dot{\theta}_1^2 + gc_1\\0\\-(d_2+l_2)\dot{\theta}_1^2 + \ddot{d}_2 + l_1\ddot{\theta}_1 - gs_1\end{pmatrix}$$

P01

Newton-Euler Method:

Step 2: Compute ${}^{i}F_i$ and ${}^{i}N_i$

$$ {}^{i}F_i = m_i \cdot {}^{i}\dot{v}_{C_i} $$
$$ {}^{i}N_i = {}^{C_i}I_i \cdot {}^{i}\dot{\omega}_i + {}^{i}\omega_i \times {}^{C_i}I_i \cdot {}^{i}\omega_i $$

P01

Newton-Euler Method:

Step 2: Compute ${}^{i}F_i$ and ${}^{i}N_i$

$$ {}^{i}F_i = m_i \cdot {}^{i}\dot{v}_{C_i} $$
$$ {}^{i}N_i = {}^{C_i}I_i \cdot {}^{i}\dot{\omega}_i + {}^{i}\omega_i \times {}^{C_i}I_i \cdot {}^{i}\omega_i $$

$$ {}^{1}F_1 = \begin{pmatrix} m_1\left(g c_1 - \frac{1}{2} l_1 \dot{\theta}_1^2\right) \\ m_1\left(-g s_1 + \frac{1}{2} l_1 \ddot{\theta}_1\right) \\ 0 \end{pmatrix} \qquad {}^{1}N_1 = \begin{pmatrix} 0 \\ 0 \\ I_{zz1}\ddot{\theta}_1 \end{pmatrix} $$

P01

Newton-Euler Method:

Step 2: Compute ${}^{i}F_i$ and ${}^{i}N_i$

$$ {}^{i}F_i = m_i \cdot {}^{i}\dot{v}_{C_i} $$

$$ {}^{i}N_i = {}^{C_i}I_i \cdot {}^{i}\dot{\omega}_i + {}^{i}\omega_i \times {}^{C_i}I_i \cdot {}^{i}\omega_i $$

$$ {}^{1}F_1 = \begin{pmatrix} m_1 \left(g c_1 - \frac{1}{2} l_1 \dot{\theta}_1^2\right) \\ m_1 \left(-g s_1 + \frac{1}{2} l_1 \ddot{\theta}_1\right) \\ 0 \end{pmatrix} \quad {}^{1}N_1 = \begin{pmatrix} 0 \\ 0 \\ I_{zz1} \ddot{\theta}_1 \end{pmatrix} $$

$$ {}^{2}F_2 = \begin{pmatrix} m_2 \left(-(d_2+l_2)\ddot{\theta}_1 - 2\dot{d}_2\dot{\theta}_1 - l_1 \dot{\theta}_1^2 + g c_1\right) \\ 0 \\ m_2 \left(-(d_2+l_2)\dot{\theta}_1^2 + \ddot{d}_2 + l_1 \ddot{\theta}_1 - g s_1\right) \end{pmatrix} \quad {}^{2}N_2 = \begin{pmatrix} 0 \\ -I_{yy2} \ddot{\theta}_1 \\ 0 \end{pmatrix} $$

P01

Newton-Euler Method:

Step 3: Compute joint torques vector τ

$$f_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad n_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$^{i}f_i = {}^{i}_{i+1}R \, {}^{i+1}f_{i+1} + {}^{i}F_i$$

$$^{i}n_i = {}^{i}_{i+1}R \, {}^{i+1}n_{i+1} + {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R \, {}^{i+1}f_{i+1}\right) + {}^{i}P_{c_i} \times {}^{i}F_i + {}^{i}N_i$$

P01

Newton-Euler Method:

Step 3: Compute joint torques vector τ

$$ {}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i $$

$$ {}^{i}n_i = {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R\,{}^{i+1}f_{i+1}\right) + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}N_i $$

$$ f_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad n_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} $$

$$ f_2 = \begin{pmatrix} m_2(-(d_2+l_2)\ddot{\theta}_1 - 2\dot{d}_2\dot{\theta}_1 - l_1\dot{\theta}_1^2 + gc_1) \\ 0 \\ m_2(-(d_2+l_2)\dot{\theta}_1^2 + \ddot{d}_2 + l_1\ddot{\theta}_1 - gs_1 \end{pmatrix} $$

$$ n_2 = \begin{pmatrix} 0 \\ -I_{yy2}\ddot{\theta}_1 - l_2 m_2((d_2+l_2)\ddot{\theta}_1 + 2\dot{d}_2\dot{\theta}_1 + l_1\dot{\theta}_1^2 - gc_1 \\ 0 \end{pmatrix} $$

P01

Newton-Euler Method:

Step 3: Compute joint torques vector τ

$$ {}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i $$

$$ {}^{i}n_i = {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R\,{}^{i+1}f_{i+1}\right) + {}^{i}P_{c_i} \times {}^{i}F_i + {}^{i}N_i $$

$$ f_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad n_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} $$

$$ f_2 = \begin{pmatrix} m_2\left(-(d_2+l_2)\ddot\theta_1 - 2\dot d_2\dot\theta_1 - l_1\dot\theta_1^2 + gc_1\right) \\ 0 \\ m_2\left(-(d_2+l_2)\dot\theta_1^2 + \ddot d_2 + l_1\ddot\theta_1 - gs_1\right) \end{pmatrix} $$

$$ n_2 = \begin{pmatrix} 0 \\ -I_{yy2}\ddot\theta_1 - l_2 m_2\left((d_2+l_2)\ddot\theta_1 + 2\dot d_2\dot\theta_1 + l_1\dot\theta_1^2 - gc_1\right) \\ 0 \end{pmatrix} $$

$$ f_1 = \begin{pmatrix} m_1 gc_1 - m_1\frac{l_1}{2}\dot\theta_1^2 - m_2\left((d_2+l_2)\ddot\theta_1 + 2\dot d_2\dot\theta_1 + l_1\dot\theta_1^2 - gc_1\right) \\ -m_1 gs_1 + m_1 l_1\ddot\theta_1\cdot\frac{1}{2} - m_2\left((d_2+l_2)\dot\theta_1^2 - \ddot d_2 - l_1\ddot\theta_1 + gs_1\right) \\ 0 \end{pmatrix} $$

$$ n_1 = \begin{pmatrix} 0 \\ 0 \\ (I_{zz1}+I_{yy2})\ddot\theta_1 - l_1 m_1\left(2gs_1 - l_1\ddot\theta_1\right)\frac{1}{4} + l_2 m_2\left((d_2+l_2)\ddot\theta_1 + 2\dot d_2\dot\theta_1 + l_1\dot\theta_1^2 - gc_1\right) + \\ + m_2\left((d_2^2 + d_2 l_2 + l_1^2)\ddot\theta_1 + 2d_2\dot d_2\dot\theta_1 + \ddot d_2 l_1 - l_1 l_2\dot\theta_1^2 - d_2 gc_1 - l_1 gs_1\right) \end{pmatrix} $$

P01

Newton-Euler Method:

Step 3: Compute joint torques vector τ

$$^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i$$

$$^{i}n_i = {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}_{i+1}t \times \left({}^{i}_{i+1}R\,{}^{i+1}f_{i+1}\right) + {}^{i}P_{c_i} \times {}^{i}F_i + {}^{i}N_i$$

$$\tau = \begin{pmatrix} n_{1z} \\ f_{2z} \end{pmatrix}$$

$$f_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad n_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$f_2 = \begin{pmatrix} m_2\left(-(d_2+l_2)\ddot{\theta}_1 - 2\dot{d}_2\dot{\theta}_1 - l_1\dot{\theta}_1^2 + gc_1\right) \\ 0 \\ m_2\left(-(d_2+l_2)\dot{\theta}_1^2 + \ddot{d}_2 + l_1\ddot{\theta}_1 - gs_1\right) \end{pmatrix}$$

$$n_2 = \begin{pmatrix} 0 \\ -I_{yy2}\ddot{\theta}_1 - l_2 m_2\left((d_2+l_2)\ddot{\theta}_1 + 2\dot{d}_2\dot{\theta}_1 + l_1\dot{\theta}_1^2 - gc_1\right) \\ 0 \end{pmatrix}$$

$$f_1 = \begin{pmatrix} m_1 gc_1 - m_1\frac{l_1}{2}\dot{\theta}_1^2 - m_2\left((d_2+l_2)\ddot{\theta}_1 + 2\dot{d}_2\dot{\theta}_1 + l_1\dot{\theta}_1^2 - gc_1\right) \\ -m_1 gs_1 + m_1 l_1\ddot{\theta}_1\cdot\frac{1}{2} - m_2\left((d_2+l_2)\dot{\theta}_1^2 - \ddot{d}_2 - l_1\ddot{\theta}_1 + gs_1\right) \\ 0 \end{pmatrix}$$

$$n_1 = \begin{pmatrix} 0 \\ 0 \\ (I_{zz1}+I_{yy2})\ddot{\theta}_1 - l_1 m_1\left(2gs_1 - l_1\ddot{\theta}_1\right)\frac{1}{4} + l_2 m_2\left((d_2+l_2)\ddot{\theta}_1 + 2\dot{d}_2\dot{\theta}_1 + l_1\dot{\theta}_1^2 - gc_1\right) + \\ + m_2\left((d_2^2 + d_2 l_2 + l_1^2)\ddot{\theta}_1 + 2d_2\dot{d}_2\dot{\theta}_1 + \ddot{d}_2 l_1 - l_1 l_2\dot{\theta}_1^2 - d_2 gc_1 - l_1 gs_1\right) \end{pmatrix}$$

P01

Express τ as a sum of multiple Matrix-Vector products

necessary for controlling a robot (future tutorial)

$$\tau = \begin{pmatrix} n_{1z} \\ f_{2z} \end{pmatrix} = M(\theta) \cdot \ddot{\theta} + V(\theta, \dot{\theta}) + G(\theta)$$

P01

Express τ as a sum of multiple Matrix-Vector products

necessary for controlling a robot (future tutorial)

$$\tau = \begin{pmatrix} n_{1z} \\ f_{2z} \end{pmatrix} = M(\theta) \cdot \ddot{\theta} + V(\theta, \dot{\theta}) + G(\theta)$$

$$M(\theta) = \begin{pmatrix} I_{zz1} + I_{yy2} + m_2 (d_2^2 + 2 d_2 l_2 + l_2^2 + l_1^2) + \frac{m_1}{4} l_1^2 & l_1 m_2 \\ l_1 m_2 & m_2 \end{pmatrix}$$

$$V(\theta, \dot{\theta}) = \begin{pmatrix} 2 m_2 \dot{d}_2 \dot{\theta}_1 (d_2 + l_2) \\ - m_2 \dot{\theta}_1^2 (d_2 + l_2) \end{pmatrix}$$

$$G(\theta) = \begin{pmatrix} g \left(- m_2 c_1 (d_2 + l_2) - l_1 s_1 \left(\frac{m_1}{2} + m_2 \right) \right) \\ - g m_2 s_1 \end{pmatrix}$$

P02

Problem 2

The manipulator shown in Figure 2 has the following properties:

- Masses of the three links are m_1, m_2, m_3
- Inertia tensors are

$$
{}^{C_1}I_1 = \begin{pmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{pmatrix} \quad {}^{C_2}I_2 = \begin{pmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{pmatrix} \quad {}^{C_3}I_3 = \begin{pmatrix} I_{xx3} & 0 & 0 \\ 0 & I_{yy3} & 0 \\ 0 & 0 & I_{zz3} \end{pmatrix}
$$

- The positions of the centers of mass for the three links are:

$$
{}^0P_{C_1} = \begin{pmatrix} 0 \\ 0 \\ d_1 - l_1 \end{pmatrix}, \quad {}^0P_{C_2} = \begin{pmatrix} \frac{l_2}{2}\mathrm{c}_2 \\ \frac{l_2}{2}\mathrm{s}_2 \\ d_1 \end{pmatrix}, \quad {}^0P_{C_3} = \begin{pmatrix} l_2\mathrm{c}_2 + \frac{l_3}{2}\mathrm{c}_{23} \\ l_2\mathrm{s}_2 + \frac{l_3}{2}\mathrm{s}_{23} \\ d_1 \end{pmatrix}
$$

Furthermore, gravity applies in negative Z_0-direction, as shown. Using the Lagrangian approach to robot dynamics, determine the manipulator dynamic equations in state space (M-V-G) and configuration space (M-B-C-G) form.

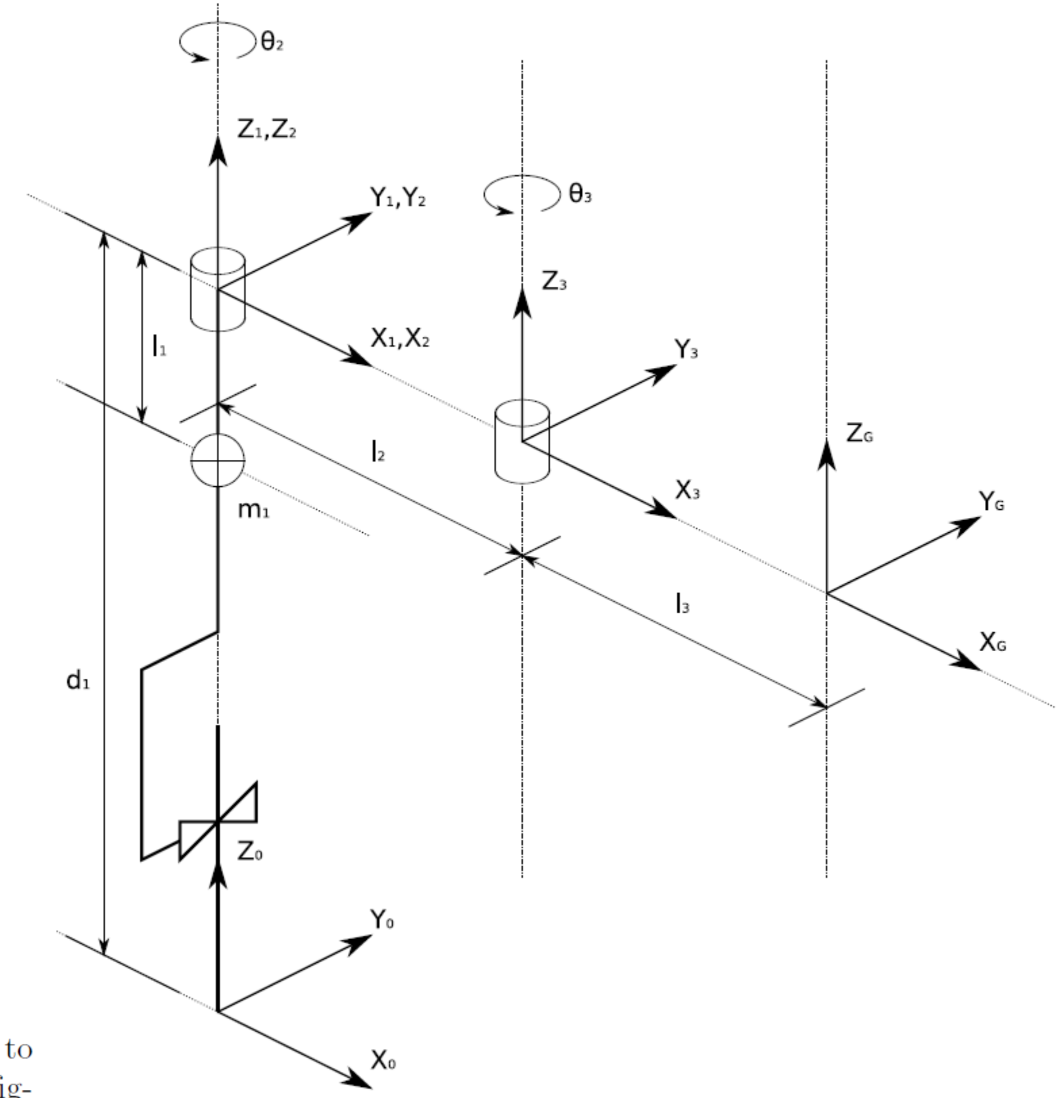

Figure 2: *PRR Robot (Problem 2)*

P02

Lagrange Method:

Step 1: Compute kinetic and potential energies (for every link)

$$k_i = \frac{1}{2} m_i \, {}^{j}\dot{v}_{c_i}^{T} \, {}^{j}\dot{v}_{c_i} + \frac{1}{2} \, {}^{i}\omega_i^{T} \cdot {}^{c_i}I_i \cdot {}^{i}\omega_i \quad , \quad \forall \{j\} \text{ coordinate frame}$$

$$u_i = -m_i \, {}^{0}g^{T} \cdot {}^{0}P_{c_i} + u_{ref,i}$$

$$k = \sum_i k_i \quad ; \quad u = \sum_i u_i$$

P02

Lagrange Method:

Step 2: Compute energy derivatives

$$\frac{\partial u}{\partial \theta_i}$$

$$\frac{\partial k}{\partial \theta_i}$$

$$\frac{\partial k}{\partial \dot{\theta}_i}$$

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_i}$$

Pay VERY much attention when computing the time derivative of $\frac{\partial k}{\partial \dot{\theta}_i}$

Besides differentiating the $\dot{\theta}$ terms, you must also differentiate the terms that contain θ!

P02

Lagrange Method:

Step 2: Compute energy derivatives

$$\boxed{\frac{\partial u}{\partial \theta_i}}$$

$$\boxed{\frac{\partial k}{\partial \theta_i}}$$

$$\frac{\partial k}{\partial \dot{\theta}_i}$$

$$\boxed{\frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_i}}$$

Pay VERY much attention when computing the time derivative of $\frac{\partial k}{\partial \dot{\theta}_i}$

Besides differentiating the $\dot{\theta}$ terms, you must also differentiate the terms that contain θ!

P02

Lagrange Method:

Step 3: Compute joint torques vector τ

$$\tau = \frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}} - \frac{\partial k}{\partial \theta} + \frac{\partial u}{\partial \theta}$$

$$\tau_i = \frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_i} - \frac{\partial k}{\partial \theta_i} + \frac{\partial u}{\partial \theta_i}$$

P02

$$-\dot{\theta}_2 \cdot l_2 s_2 - \frac{l_3}{2} s_{23} \cdot \dot{\theta}_2 - \frac{l_3}{2} s_{23} \cdot \dot{\theta}_3$$

$${}^0v_{C_1} = \frac{d}{dt}\,{}^0P_{C_1} = \frac{d}{dt}\begin{pmatrix}0\\0\\d_1-l_1\end{pmatrix} = \begin{pmatrix}0\\0\\\dot{d}_1\end{pmatrix}$$

$${}^0v_{C_2} = \frac{d}{dt}\,{}^0P_{C_2} = \frac{d}{dt}\begin{pmatrix}l_2c_2/2\\l_2s_2/2\\d_1\end{pmatrix} = \begin{pmatrix}-l_2s_2\dot{\theta}_2/2\\l_2c_2\dot{\theta}_2/2\\\dot{d}_1\end{pmatrix}$$

$${}^0v_{C_3} = \frac{d}{dt}\,{}^0P_{C_3} = \frac{d}{dt}\begin{pmatrix}l_2c_2+l_3c_{23}/2\\l_2s_2+l_3s_{23}/2\\d_1\end{pmatrix} = \begin{pmatrix}-\dot{\theta}_2(2l_2s_2+l_3s_{23})/2-\dot{\theta}_3 l_3 s_{23}/2\\\dot{\theta}_2(2l_2c_2+l_3c_{23})/2+\dot{\theta}_3 l_3 c_{23}/2\\\dot{d}_1\end{pmatrix}$$

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{\ \ i}R\,{}^i\omega_i + \begin{pmatrix}0\\0\\\dot{\theta}_{i+1}\end{pmatrix}, \quad {}^0\omega_0 = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$${}^1\omega_1 = \begin{pmatrix}0\\0\\0\end{pmatrix}; \quad {}^2\omega_2 = \begin{pmatrix}0\\0\\\dot{\theta}_2\end{pmatrix}; \quad {}^3\omega_3 = \begin{pmatrix}0\\0\\\dot{\theta}_2+\dot{\theta}_3\end{pmatrix}$$

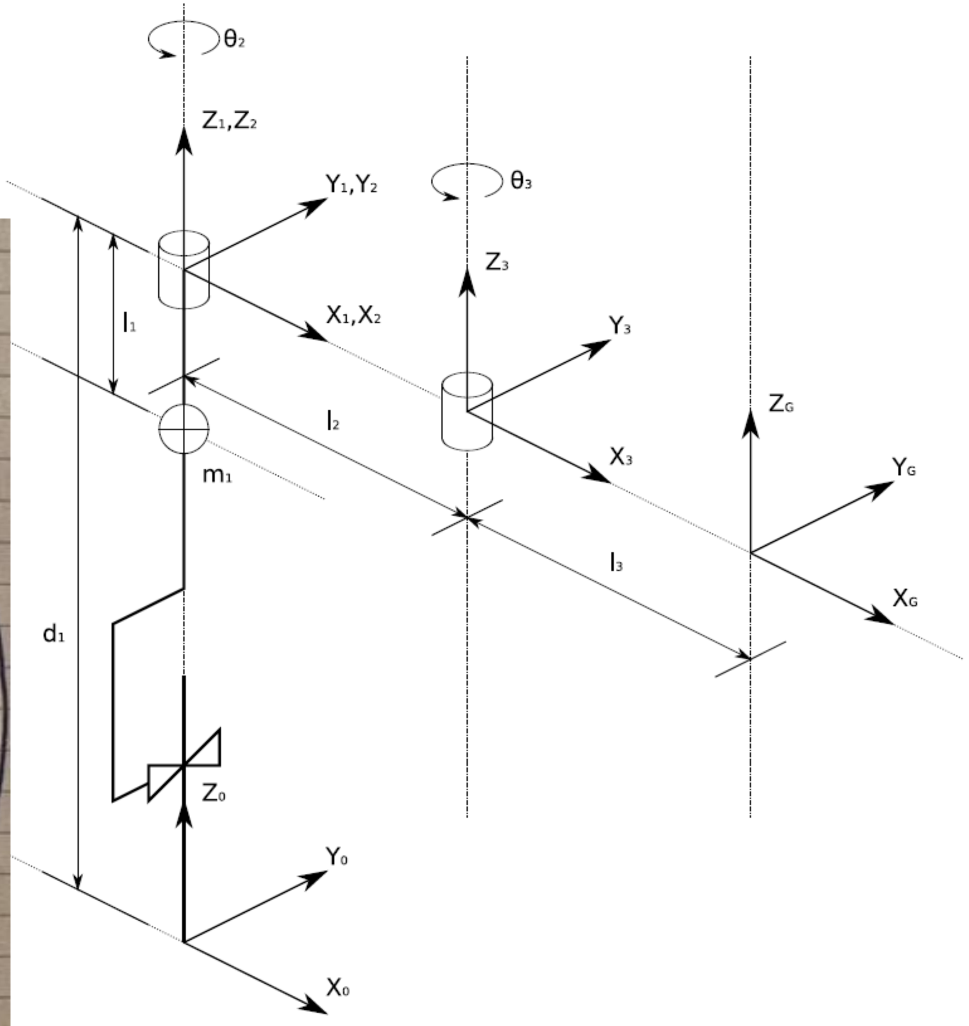

Figure 2: *PRR Robot (Problem 2)*

P02

Lagrange Method:

$$k_i = \frac{1}{2} m_i \, {}^{j}v_{c_i}^{T} \, {}^{j}v_{c_i} + \frac{1}{2} \, {}^{i}\omega_i^{T} \cdot {}^{c_i}I_i \cdot {}^{i}\omega_i \quad , \quad \forall \{j\} \text{ coordinate frame}$$

$$u_i = - m_i \, {}^{0}g^{T} \cdot {}^{0}P_{c_i} + u_{ref,i}$$

Step 1: Compute kinetic and potential energies (for every link)

P02

Lagrange Method:

$$k_i = \frac{1}{2} m_i \, {}^j v_{c_i}^T \, {}^j v_{c_i} + \frac{1}{2} \, {}^i \omega_i^T \cdot {}^{c_i} I_i \cdot {}^i \omega_i \quad , \quad \forall \{j\} \text{ coordinate frame}$$

$$u_i = - m_i \, {}^0 g^T \cdot {}^0 P_{c_i} + u_{ref,i}$$

Step 1: Compute kinetic and potential energies (for every link)

$$k_1 = \frac{m_1}{2} \dot{d}_1^2$$

$$u_1 = m_1 g (d_1 - l_1)$$

P02

Lagrange Method:

$$k_i = \frac{1}{2} m_i \, {}^j v_{c_i}^T \, {}^j v_{c_i} + \frac{1}{2} \, {}^i\omega_i^T \cdot {}^{c_i}I_i \cdot {}^i\omega_i \quad , \forall \{j\} \text{ coordinate frame}$$

$$u_i = - m_i \, {}^0 g^T \cdot {}^0 P_{c_i} + u_{ref,i}$$

Step 1: Compute kinetic and potential energies (for every link)

$$k_1 = \frac{m_1}{2} \dot{d}_1^2$$

$$k_2 = \frac{1}{2} I_{zz2} \dot{\theta}_2^2 + \frac{m_2}{2} \left(\dot{d}_1^2 + \frac{1}{4} l_2^2 \dot{\theta}_2^2 \right)$$

$$u_1 = m_1 g (d_1 - l_1)$$

$$u_2 = m_2 g d_1$$

P02

$$k_i = \frac{1}{2} m_i \, {}^{j}v_{c_i}^{T} \, {}^{j}v_{c_i} + \frac{1}{2} \, {}^{i}\omega_i^{T} \cdot {}^{c_i}I_i \cdot {}^{i}\omega_i \quad , \forall \{j\} \text{ coordinate frame}$$

$$u_i = -m_i \, {}^{0}g^{T} \cdot {}^{0}P_{c_i} + u_{ref,i}$$

Lagrange Method:

Step 1: Compute kinetic and potential energies (for every link)

$$k_1 = \frac{m_1}{2} \dot{d}_1^2$$

$$k_2 = \frac{1}{2} I_{zz2} \dot{\theta}_2^2 + \frac{m_2}{2}\left(\dot{d}_1^2 + \frac{1}{4} l_2^2 \dot{\theta}_2^2\right)$$

$$k_3 = \frac{1}{2} I_{zz3} (\dot{\theta}_2 + \dot{\theta}_3)^2 + \frac{m_3}{2}\left(\dot{d}_1^2 + \dot{\theta}_2^2\left(l_2^2 + l_2 l_3 c_3 + \frac{1}{4} l_3^2\right) + \dot{\theta}_2 \dot{\theta}_3 \left(l_2 l_3 c_3 + \frac{1}{2} l_3^2\right) + \dot{\theta}_3^2 \cdot \frac{1}{4} l_3^2\right)$$

$$u_1 = m_1 g (d_1 - l_1)$$

$$u_2 = m_2 g d_1$$

$$u_3 = m_3 g d_1$$

P02

Lagrange Method:

Step 2: Compute energy derivatives

$$\frac{\partial u}{\partial \theta_i}$$

$$\frac{\partial u}{\partial \theta_1} = g(m_1 + m_2 + m_3) \qquad \frac{\partial u}{\partial \theta_2} = 0 \qquad \frac{\partial u}{\partial \theta_3} = 0$$

P02

Lagrange Method:

Step 2: Compute energy derivatives

$$\frac{\partial u}{\partial \theta_i}$$

$$\frac{\partial k}{\partial \theta_i}$$

P02

$$\frac{\partial u}{\partial \theta_1} = g(m_1+m_2+m_3) \qquad \frac{\partial u}{\partial \theta_2} = 0 \qquad \frac{\partial u}{\partial \theta_3} = 0$$

$$\frac{\partial k}{\partial \theta_1} = 0 \qquad \frac{\partial k}{\partial \theta_2} = 0 \qquad \frac{\partial k}{\partial \theta_3} = -m_3 l_2 l_3 \dot{\theta}_2 s_3 (\dot{\theta}_2+\dot{\theta}_3)\frac{1}{2}$$

Lagrange Method:

Step 2: Compute energy derivatives

$$\frac{\partial u}{\partial \theta_i}$$

$$\frac{\partial k}{\partial \theta_i}$$

$$\frac{\partial k}{\partial \dot{\theta}_i}$$

P02

Lagrange Method:

Step 2: Compute energy derivatives

$$\frac{\partial u}{\partial \theta_i}$$

$$\frac{\partial k}{\partial \theta_i}$$

$$\frac{\partial k}{\partial \dot{\theta}_i}$$

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_i}$$

$$\frac{\partial u}{\partial \theta_1} = g(m_1+m_2+m_3) \qquad \frac{\partial u}{\partial \theta_2} = 0 \qquad \frac{\partial u}{\partial \theta_3} = 0$$

$$\frac{\partial k}{\partial \theta_1} = 0 \qquad \frac{\partial k}{\partial \theta_2} = 0 \qquad \frac{\partial k}{\partial \theta_3} = -m_3 l_2 l_3 \dot{\theta}_2 s_3 (\dot{\theta}_2+\dot{\theta}_3)\frac{1}{2}$$

$$\frac{\partial k}{\partial \dot{\theta}_1} = \dot{d}_1(m_1+m_2+m_3)$$

$$\frac{\partial k}{\partial \dot{\theta}_2} = \dot{\theta}_2\left(I_{zz2}+I_{zz3}+\frac{m_2}{4}l_2^2+m_3\left(l_2^2+l_2l_3c_3+\frac{1}{4}l_3^2\right)\right)+\dot{\theta}_3\left(I_{zz3}+m_3\left(\frac{1}{2}l_2l_3c_3+\frac{1}{4}l_3^2\right)\right)$$

$$\frac{\partial k}{\partial \dot{\theta}_3} = \dot{\theta}_2\left(I_{zz3}+m_3\left(\frac{1}{2}l_2l_3c_3+\frac{1}{4}l_3^2\right)\right)+\dot{\theta}_3\left(I_{zz3}+\frac{m_3}{4}l_3^2\right)$$

P02

Lagrange Method:

Step 2: Compute energy derivatives

$$\frac{\partial u}{\partial \theta_i}$$

$$\frac{\partial k}{\partial \theta_i}$$

$$\frac{\partial k}{\partial \dot{\theta}_i}$$

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_i}$$

$$\frac{\partial u}{\partial \theta_1} = g(m_1+m_2+m_3) \qquad \frac{\partial u}{\partial \theta_2} = 0 \qquad \frac{\partial u}{\partial \theta_3} = 0$$

$$\frac{\partial k}{\partial \theta_1} = 0 \qquad \frac{\partial k}{\partial \theta_2} = 0 \qquad \frac{\partial k}{\partial \theta_3} = -m_3 l_2 l_3 \dot{\theta}_2 s_3 (\dot{\theta}_2+\dot{\theta}_3)\frac{1}{2}$$

$$\frac{\partial k}{\partial \dot{\theta}_1} = \dot{d}_1 (m_1+m_2+m_3)$$

$$\frac{\partial k}{\partial \dot{\theta}_2} = \dot{\theta}_2\left(I_{zz2}+I_{zz3}+\frac{m_2}{4}l_2^2+m_3\left(l_2^2+l_2 l_3 c_3+\frac{1}{4}l_3^2\right)\right)+\dot{\theta}_3\left(I_{zz3}+m_3\left(\frac{1}{2}l_2 l_3 c_3+\frac{1}{4}l_3^2\right)\right)$$

$$\frac{\partial k}{\partial \dot{\theta}_3} = \dot{\theta}_2\left(I_{zz3}+m_3\left(\frac{1}{2}l_2 l_3 c_3+\frac{1}{4}l_3^2\right)\right)+\dot{\theta}_3\left(I_{zz3}+\frac{m_3}{4}l_3^2\right)$$

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_1} = \ddot{d}_1 (m_1+m_2+m_3)$$

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_2} = \ddot{\theta}_2\left(I_{zz2}+I_{zz3}+\frac{m_2}{4}l_2^2+m_3\left(l_2^2+l_2 l_3 c_3+\frac{1}{4}l_3^2\right)\right)+\ddot{\theta}_3\left(I_{zz3}+m_3\left(\frac{1}{2}l_2 l_3 c_3+\frac{1}{4}l_3^2\right)\right) - \dot{\theta}_3(2\dot{\theta}_2+\dot{\theta}_3)\, m_3 l_2 l_3 s_3 \cdot \frac{1}{2}$$

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_3} = \ddot{\theta}_2\left(I_{zz3}+m_3\left(\frac{1}{2}l_2 l_3 c_3+\frac{1}{4}l_3^2\right)\right)+\ddot{\theta}_3\left(I_{zz3}+\frac{m_3}{4}l_3^2\right)-\dot{\theta}_2\dot{\theta}_3\, m_3 l_2 l_3 s_3 \cdot \frac{1}{2}$$

P02

Lagrange Method:

Step 3: Compute joint torques vector τ

$$\tau_i = \frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_i} - \frac{\partial k}{\partial \theta_i} + \frac{\partial u}{\partial \theta_i}$$

P02

Lagrange Method:

Step 3: Compute joint torques vector τ

$$\tau_i = \frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_i} - \frac{\partial k}{\partial \theta_i} + \frac{\partial u}{\partial \theta_i}$$

$$\tau_1 = \ddot{d}_1 (m_1 + m_2 + m_3) + g(m_1 + m_2 + m_3)$$

$$\tau_2 = \ddot{\theta}_2 \left(I_{zz2} + I_{zz3} + \frac{m_2}{4} l_2^2 + m_3\left(l_2^2 + l_2 l_3 c_3 + \frac{1}{4} l_3^2\right)\right) + \ddot{\theta}_3 \left(I_{zz3} + \frac{m_3}{4}\left(2 l_2 l_3 c_3 + \frac{1}{4} l_3^2\right)\right) -$$

$$- \dot{\theta}_3 \left(\dot{\theta}_2 + \frac{1}{2}\dot{\theta}_3\right) m_3 l_2 l_3 s_3$$

$$\tau_3 = \ddot{\theta}_2 \left(I_{zz3} + \frac{m_3}{4}\left(2 l_2 l_3 c_3 + l_3^2\right)\right) + \theta_3 \left(I_{zz3} + \frac{m_3}{4} l_3^2\right) + \frac{1}{2}\dot{\theta}_2^2 m_3 l_2 l_3 s_3 +$$

P02

Express τ as a sum of multiple Matrix-Vector products

necessary for controlling a robot (future tutorial)

$$\tau = M(\theta)\ddot{\theta} + V\left(\theta, \dot{\theta}\right) + G(\theta)$$

$$\tau = M(\theta)\ddot{\theta} + B(\theta)\left[\dot{\theta}_i\dot{\theta}_j\right] + C(\theta)\left[\dot{\theta}^2\right] + G(\theta)$$

P02

Express τ as a sum of multiple Matrix-Vector products

necessary for controlling a robot (future tutorial)

$$\tau = M(\theta)\ddot{\theta} + V\left(\theta, \dot{\theta}\right) + G(\theta)$$

$$\tau = M(\theta)\ddot{\theta} + B(\theta)\left[\dot{\theta}_i\dot{\theta}_j\right] + C(\theta)\left[\dot{\theta}^2\right] + G(\theta)$$

$$M(\theta) = \begin{pmatrix} m_1+m_2+m_3 & 0 & 0 \\ 0 & I_{zz2}+I_{zz3}+\frac{m_2}{4}l_2^2+m_3\left(l_2^2+l_2l_3c_3+\frac{1}{4}l_3^2\right) & I_{zz3}+m_3\left(\frac{1}{2}l_2l_3c_3+\frac{1}{4}l_3^2\right) \\ 0 & I_{zz3}+m_3\left(\frac{1}{2}l_2l_3c_3+\frac{1}{4}l_3^2\right) & I_{zz3}+\frac{m_3}{4}l_3^2 \end{pmatrix}$$

$$V(\theta,\dot{\theta}) = \begin{pmatrix} 0 \\ -\dot{\theta}_3\left(\dot{\theta}_2+\frac{1}{2}\dot{\theta}_3\right)m_3l_2l_3s_3 \\ \frac{1}{2}\dot{\theta}_2^2\, m_3l_2l_3s_3 \end{pmatrix}$$

$$G(\theta) = \begin{pmatrix} g(m_1+m_2+m_3) \\ 0 \\ 0 \end{pmatrix}$$

$$B(\theta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -m_3l_2l_3s_3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$C(\theta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}m_3l_2l_3s_3 \\ 0 & \frac{1}{2}m_3l_2l_3s_3 & 0 \end{pmatrix}$$