Computer Vision II: Multiple View Geometry (IN2228)

Chapter 01 Mathematical Background

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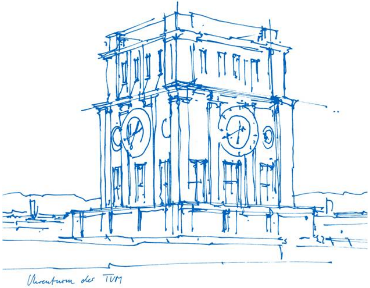
20 April 2023 11:00-11:45



Announcements

- **Exam**

The exam dates and locations are determined centrally by the Department of Studies. It will take a while until the dates are visible to us. We will provide any update in time.

- **Registration**

If you need us to register you in Moodle, please send me an email with your name and TUM ID.

- **Slides**

I will upload slides before each class to both course website and Moodle.

Outline

- Vector Operations
- Vector Space
- Matrices and Transformation
- Matrix Properties
- Matrix Decomposition

Vector Operations

- Dot Product

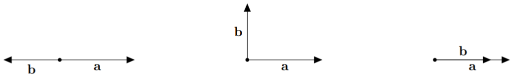
Definition

$$\mathbf{a} \cdot \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Geometric illustration

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$$

$\cos\pi = -1$ $\qquad$ $\cos\frac{\pi}{2} = 0$ $\qquad$ $\cos 0 = 1$



Normalized vectors

The dot product measures how similar two normalized vectors are.

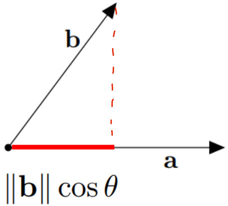
Vector Operations

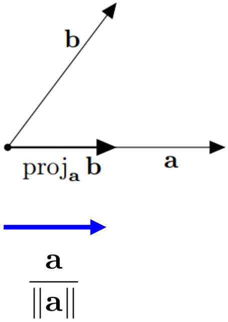
- Dot Product

Geometric illustration

The projection of **b** onto **a**

$$\mathrm{proj}_{\mathbf{a}}\,\mathbf{b} = \underbrace{\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}}_{\|\mathbf{b}\|\cos\theta}\ \underbrace{\frac{\mathbf{a}}{\|\mathbf{a}\|}}_{\text{Unit vector along }\mathbf{a}}$$





Vector Operations

➢ Cross Product

Definition

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

An alternative way to remember the definition using the **determinant** of a matrix

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} + (-1)(a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

anti-diagonal

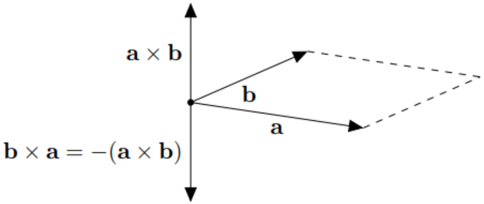
diagonal

$$= \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

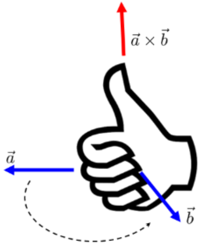
Vector Operations

➢ Cross Product

Geometric illustration



a $\times$ **b** is a vector that is orthogonal to both **a** and **b**.



Direction is determined by the right hand rule.

✓ Make your fingers sweep from one vector to the other

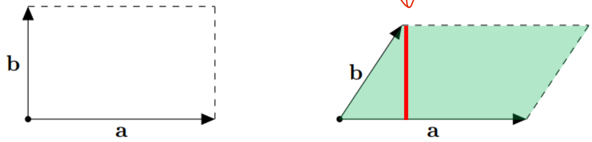
✓ The cross product direction is where your thumb points

Vector Operations

➢ Cross Product

Geometric illustration

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \boxed{\|\mathbf{b}\| \sin\theta}$$ 计算高



Area of parallelogram spanned by **a** and **b**.

Vector Operations

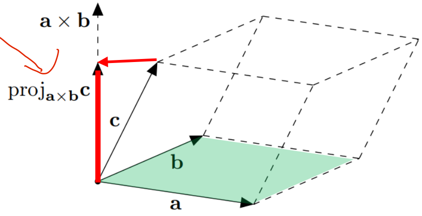
➢ Triple Product

Geometric illustration

$$
\begin{aligned}
V &= \|\mathbf{a}\times\mathbf{b}\| \boxed{\|\mathrm{proj}_{\mathbf{a}\times\mathbf{b}}\mathbf{c}\|} \\
&= \|\mathbf{a}\times\mathbf{b}\| \boxed{\frac{|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|}{\|\mathbf{a}\times\mathbf{b}\|}} = |(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|
\end{aligned}
$$

Absolute value for positive result

(introduced in dot product)



Volume of the parallelepiped spanned by vectors **a**, **b**, and **c**.

Vector Operations

- Kronecker Product

Definition

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 2\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \\ 3\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 4\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \end{bmatrix}$$

An example

Vector Space

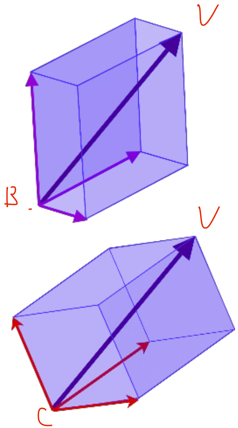
➢ Vector Space and Basis

Definition

A set ***B*** of vectors in a vector space ***V*** is called a **basis** if *every* element of ***V*** may be written in a unique way as a **finite linear combination** of elements of ***B***.

$$B = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$$ ⟺ Basis ⟹ express any vector in vector space

$$\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$$



The same vector can be represented in two **different bases** (purple and red arrows).

Vector Space

➢ Vector Space and Basis

Linear Span

Let S be a linear space. Let $x_1,\dots,x_n \in S$ be n vectors. The linear span of $x_1,\dots,x_n$, denoted by $\boxed{\mathrm{span}(x_1,\dots,x_n)}$ **contains** all the linear combinations

A vector set

$$\boxed{x = \alpha_1 x_1 + \dots + \alpha_n x_n}$$

where $\alpha_1, \dots, \alpha_n$ are arbitrary scalars.

Vector Space

➢ Vector Space and Basis

An Example

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad x_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Does x_3 belong to the linear span of x_1 and x_2?

All the linear combinations

$$\begin{aligned} s &= \alpha_1 x_1 + \alpha_2 x_2 \\ &= \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} \alpha_1 + 2\alpha_2 \\ \alpha_1 + 2\alpha_2 \end{bmatrix} \\ &= (\alpha_1 + 2\alpha_2) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

$\mathrm{span}(x_1, x_2)$

contains [1, 1]’, [2, 2]’, [1.5, 1.5]’ …

x_3

does not belong to it.

Vector Space

➢ Linear Independence

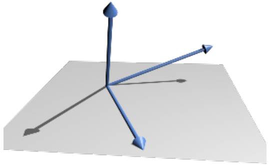
Definition

A set of vectors {v1,v2,...,vk} is **linearly independent** if the vector equation

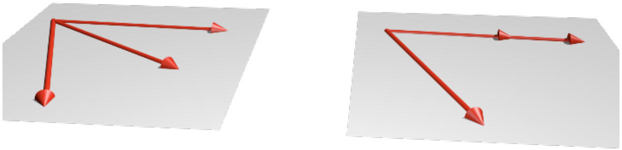
$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0$$

coefficiency = 0

has only the trivial solution $x_1 = x_2 = \cdots = x_k = 0.$



Linearly independent vectors



Linearly dependent vectors in a plane

Vector Space

➢ Linear Independence

An example

$$\overbrace{\begin{bmatrix}0\\0\\1\end{bmatrix},\begin{bmatrix}0\\2\\-2\end{bmatrix},\begin{bmatrix}1\\-2\\1\end{bmatrix}}^{\text{independent}}$$

$$\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix},\begin{bmatrix}0\\2\\-2\end{bmatrix},\begin{bmatrix}1\\-2\\1\end{bmatrix},\begin{bmatrix}4\\2\\3\end{bmatrix}}_{\text{dependent}}$$

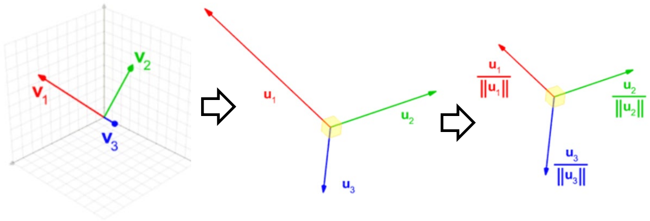
$$9 * \begin{bmatrix}0\\0\\1\end{bmatrix} + 5* \begin{bmatrix}0\\2\\-2\end{bmatrix} + 4* \begin{bmatrix}1\\-2\\1\end{bmatrix} = \begin{bmatrix}4\\2\\3\end{bmatrix}$$

Vector Space

➢ Gram–Schmidt Process

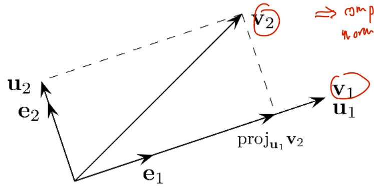
Definition

The Gram–Schmidt process is a method for ortho-normalizing a set of vectors.



An example in 3D space

$\Rightarrow$ compute u_1, u_2
normalize e_1, e_2



The first two steps of the Gram–Schmidt process

Vector Space

➢ Gram–Schmidt Process

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|} \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

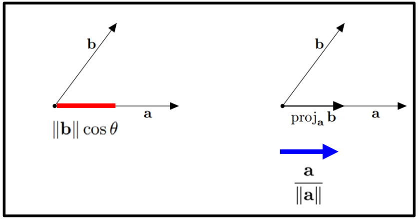
$\|\mathbf{b}\| \cos\theta$ Unit vector along **a**

Definition

We define the projection operator

内积

$$\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}$$



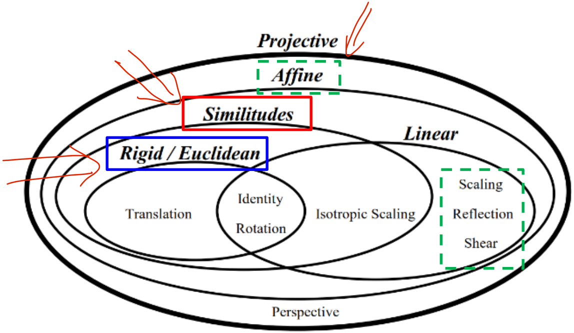
The Gram–Schmidt process then works as follows:

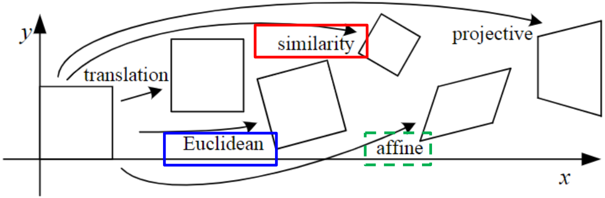
$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1, \\
\mathbf{u}_2 &= \mathbf{v}_2 - \operatorname{proj}_{\mathbf{u}_1}(\mathbf{v}_2), \\
\mathbf{u}_3 &= \mathbf{v}_3 - \operatorname{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \operatorname{proj}_{\mathbf{u}_2}(\mathbf{v}_3), \\
\mathbf{u}_4 &= \mathbf{v}_4 - \operatorname{proj}_{\mathbf{u}_1}(\mathbf{v}_4) - \operatorname{proj}_{\mathbf{u}_2}(\mathbf{v}_4) - \operatorname{proj}_{\mathbf{u}_3}(\mathbf{v}_4), \\
&\ \ \vdots \\
\mathbf{u}_k &= \mathbf{v}_k - \sum_{j=1}^{k-1} \operatorname{proj}_{\mathbf{u}_j}(\mathbf{v}_k),
\end{aligned}$$

Matrices and Transformation

➢ Overview

Overview

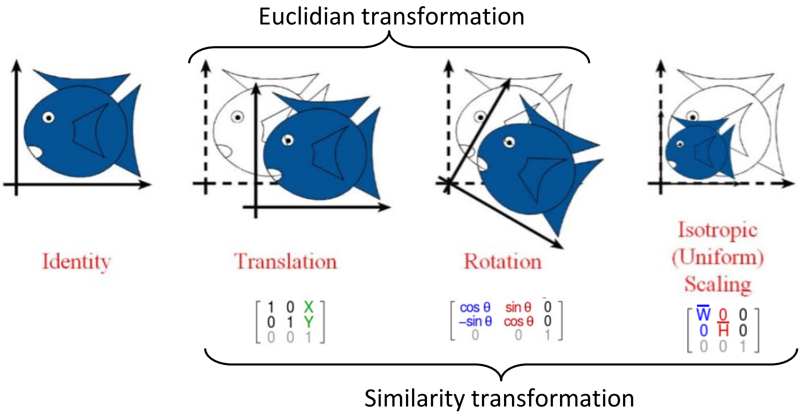




Matrices and Transformation

➢ Overview

Euclidian transformation and Similarity transformation



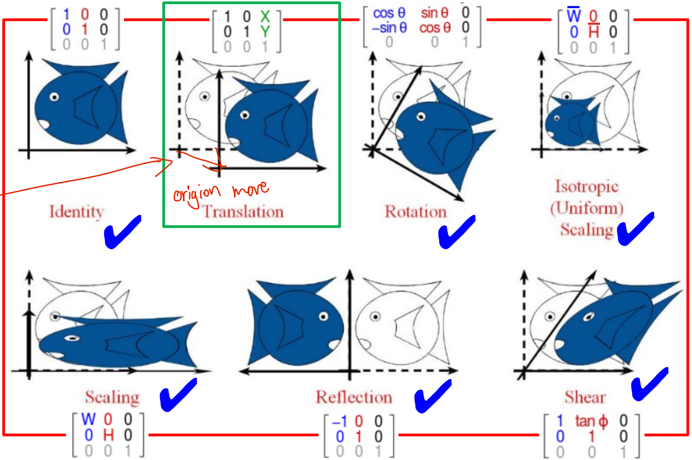
Matrices and Transformation

- Overview

Linear transformation and Affine transformation

Linear: Origin remains unchanged

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Affine = Linear + Translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Matrix Properties

➢ Transpose

Definition

Formally, the *i*-th row, *j*-th column element of $\mathbf{A}^{\mathrm{T}}$ is the *j*-th row, *i*-th column element of **A**:

$$[\mathbf{A}^{\mathrm{T}}]_{ij} = [\mathbf{A}]_{ji}$$

Example

$$\begin{bmatrix} \boxed{\begin{matrix} 1 \\ 3 \\ 5 \end{matrix}} & \begin{matrix} 2 \\ 4 \\ 6 \end{matrix} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \boxed{1 \quad 3 \quad 5} \\ 2 \quad 4 \quad 6 \end{bmatrix}$$

Matrix Properties

➢ Rank

Definition

The rank of a matrix **A** is the dimension of the vector space spanned by its columns/rows. It corresponds to the **maximal number of linearly independent** columns of **A**.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 6 \\ 6 & 9 & 8 \end{bmatrix} \qquad \begin{aligned} a_1 &= \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \\ a_2 &= \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \\ a_3 &= \begin{bmatrix} 6 & 9 & 8 \end{bmatrix} \end{aligned} \qquad 2a_1 + a_2 = a_3$$

For matrix **A**, rank is 2 (row vector a1 and a2 are linearly independent).

Matrix Properties

- Trace

Definition

The trace of an n $\times$ n square matrix **A** is defined as

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$$

where a_{ii} denotes the entry on the *i*th row and *i*th column of **A**.

An example

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 2 & 3 & 4 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{aligned} \mathrm{tr}(A) &= A_{11} + A_{22} + A_{33} \\ &= 2 + 3 + 0 \\ &= 5 \end{aligned}$$

Matrix Properties

➢ $\boxed{\text{Determinant}}$

Definition

A scalar value that is a function of the entries of a square matrix

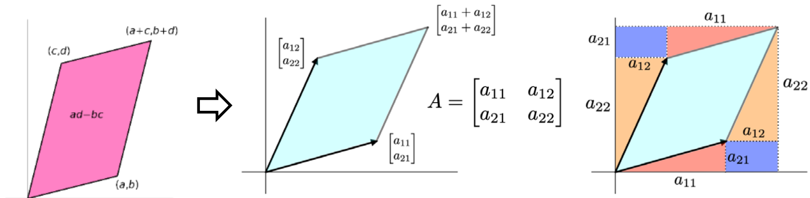
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

Matrix Properties

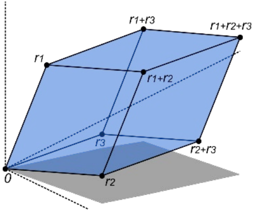
➢ Determinant

Geometric meaning



$$\begin{aligned} \text{area of parallelogram} &= (a_{11}+a_{12})(a_{21}+a_{22}) - a_{12}a_{22} - a_{11}a_{21} - 2a_{21}a_{12} \\ &= \cancel{a_{11}a_{21}} + a_{11}a_{22} + a_{12}a_{21} + \cancel{a_{12}a_{22}} - \cancel{a_{12}a_{22}} - \cancel{a_{11}a_{21}} - 2a_{21}a_{12} \\ &= a_{11}a_{22} - a_{21}a_{12} \end{aligned}$$

2D case: The area of the parallelogram is the absolute value of the determinant of the matrix.



3D case: The volume of this parallelepiped is the absolute value of the determinant of the matrix.

Matrix Properties

Independent $\Longleftrightarrow$ unique solution $\Longleftrightarrow$ zero solution. $\Longleftrightarrow$ det $\neq 0$

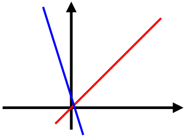
Dependent $\Longleftrightarrow$ infinite number $\Longleftrightarrow$ non zero solution $\Longleftrightarrow$ det $= 0$

➢ Determinant

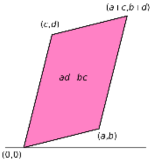
Applications

A homogeneous system of linear equations has a unique solution (the **trivial, i.e., zero solution**) if and only if its **determinant is non-zero**.

Independent $\begin{cases} 2x+y=0 \\ x-y=0 \end{cases}$



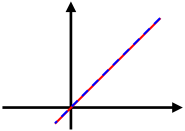
$$\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = 2(-1) - 1(1) = -3$$



If this **determinant is zero**, then the system has an infinite number of solutions (**non-zero solutions**).

$\Updownarrow$

dependent



Matrix Properties

- Kernel or Null Space

A denotes a matrix. Kernel of **A** is a set of vectors {**x**} satisfying

$$\mathrm{N}(A) = \mathrm{Null}(A) = \ker(A) = \{\mathbf{x} \in K^n \mid A\mathbf{x} = \mathbf{0}\}.$$

Null space is non-empty because it clearly contains the zero vector: **x** = **0** always satisfies **Ax**=**0**. However, we are interested in non-trivial solution in practice.

$$A\mathbf{x} = \mathbf{0} \quad \Leftrightarrow \quad \begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0. \end{array}$$

The kernel of **A** is the same as the solution set to the above homogeneous equations.

Matrix Properties

➢ Skew-symmetric Matrix

Definitions

$$\mathbf{a} = (a_1 \; a_2 \; a_3)^\mathsf{T} \qquad [\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

Non-diagonal elements

Application to cross product

$$\mathbf{b} = (b_1 \; b_2 \; b_3)^\mathsf{T} \qquad \mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

Matrix Properties

➢ Eigenvalues and Eigenvectors

Definition

Eigenvector

$$A\mathbf{v} = \lambda\mathbf{v}$$

Eigenvalue

Computation

$$(A - \lambda I)\,\mathbf{v} = \mathbf{0}$$

Equation has a **nonzero solution v** if and only if the determinant of the coefficient matrix is zero (vectors are linear dependent).

Linear dependent

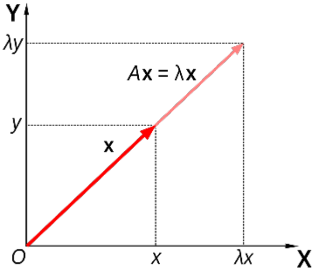
$$|A - \lambda I| = 0$$ characteristic polynomial

Matrix Properties

➢ Eigenvalues and Eigenvectors

Geometric Illustration

- Eigenvector: changes at most by a scalar factor
- Eigenvalue: the factor by which the eigenvector is scaled.



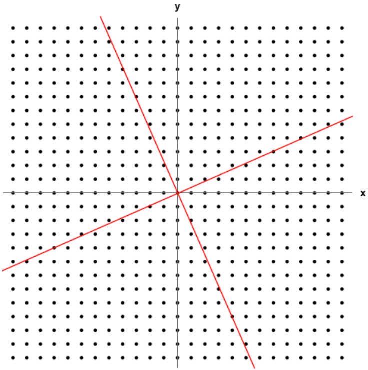
Find “invariance” from variable observations

Matrix **A** acts by stretching the vector **x**, not changing its direction, so **x** is an eigenvector of **A**.

Matrix Properties

- Eigenvalues and Eigenvectors

Geometric Illustration



An example of a 2×2 symmetric matrix. The **eigenvectors** are the two special directions such that every point on them will just **slide** on them.

Matrix Properties

➢ Eigenvalues and Eigenvectors

Application to Inverse of Matrix

✓ Definition of Invertible matrix

An n-by-n square matrix A is called invertible, if there exists an n-by-n square matrix B such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$

✓ If matrix **A** can be eigen-decomposed, and if none of its eigenvalues are zero, then **A is invertible**. The inverse of matrix is given by

$$\mathbf{A}^{-1} = \mathbf{Q}\mathbf{\Lambda}^{-1}\mathbf{Q}^{-1}$$

where **Q** is the square (N $\times$ N) matrix whose i-th column is the eigenvector of **A**, and **Λ** is the diagonal matrix whose diagonal elements are the corresponding eigenvalues.

Matrix Decomposition

- Singular Value Decomposition (SVD)

Definition

$$\boldsymbol{A}_{m\times n} = \boldsymbol{U}_{m\times m}\boldsymbol{\Sigma}_{m\times n}\boldsymbol{V}^T_{n\times n} = \boldsymbol{U}_{m\times m}\begin{pmatrix} \boldsymbol{D}_{r\times r} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix}_{m\times n}\boldsymbol{V}^T_{n\times n}$$

$$\boldsymbol{D}_{r\times r} = \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_r} \end{pmatrix}_{r\times r}$$

$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0$ are the eigen values of $\boldsymbol{A}^T\boldsymbol{A}$

What is the **geometric meaning** of SVD?

Matrix Decomposition

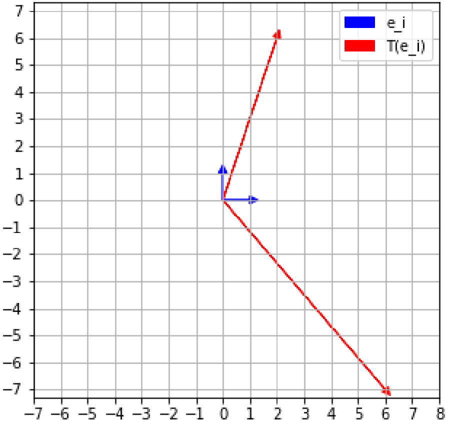
➢ Singular Value Decomposition (SVD)

Geometric meaning

A 2*2 matrix represents a linear map T: $R^2 \rightarrow R^2$

$$T = \begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix}$$

$\mathbf{T}\ [e_1, e_2] = [b_1\ b_2]$



The basis (e_1, e_2) is orthogonal, but the transformed basis $(b_1\ b_2)$ is non-orthogonal.

Matrix Decomposition

- Singular Value Decomposition (SVD)

Geometric meaning

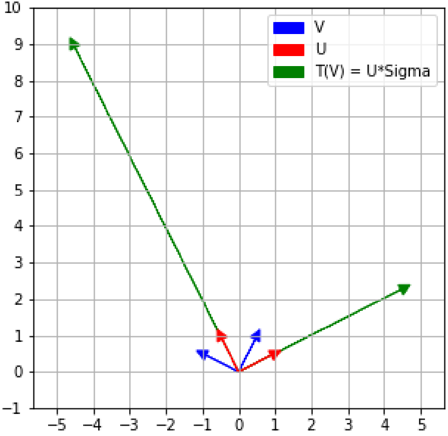
How to find an orthogonal basis that **stay orthogonal** after transformation?

$$T = \begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix}$$

T = U Σ V^{-1}

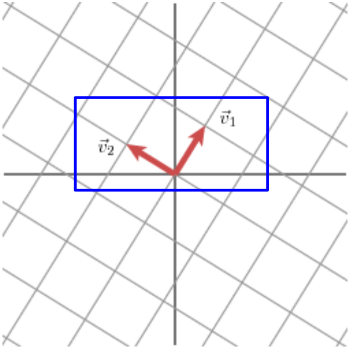
$$\begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix} \approx \boxed{\begin{pmatrix} -0.45 & 0.89 \\ 0.89 & 0.45 \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}} \boxed{\begin{pmatrix} -0.89 & 0.45 \\ 0.45 & 0.89 \end{pmatrix}}$$

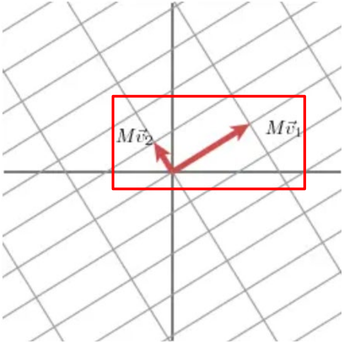
Transformed basis — Original basis



Matrix Decomposition

- Singular Value Decomposition (SVD)





Original basis

Transformed basis

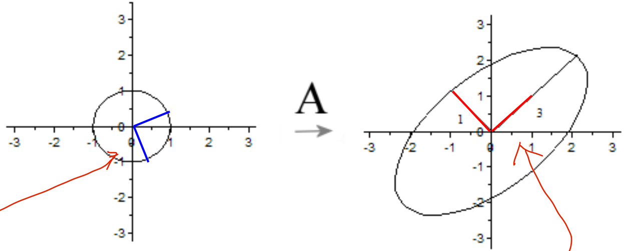
$$\boxed{M}\boxed{[v_1 \quad v_2]} = \boxed{[u_1 \quad u_2]\begin{bmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{bmatrix}\begin{bmatrix}v_1^{\mathrm{T}}\\ v_2^{\mathrm{T}}\end{bmatrix}}\boxed{[v_1 \quad v_2]} = \boxed{[u_1 \quad u_2]\begin{bmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{bmatrix}}$$

Matrix Decomposition

➢ Singular Value Decomposition (SVD)

Mapping points on a circle into points on an ellipse

$$A = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1^{\mathrm{T}} \\ v_2^{\mathrm{T}} \end{bmatrix}$$



point

Original unit basis

$$y = Ax = A \boxed{\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1^{\mathrm{T}} \\ v_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 3\xi_1 u_1 + \xi_2 u_2$$

Coefficients

New basis

Matrix Decomposition

➢ Singular Value Decomposition (SVD)

Application to the Generalized Inverse

- For a certain quadratic matrix ***A*** one can define an inverse matrix, if det(***A***) does not equal 0.
- One can also define a g $A \in \mathbb{R}^{m \times n}$ nverse (also called pseudo inverse) for an arbitrary (non-quadratic) matrix

not square

$$A^\dagger = V \Sigma^\dagger U^\top, \text{ where } \Sigma^\dagger = \begin{pmatrix} \Sigma_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}_{n \times m},$$

where Σ_1 is the diagonal matrix of non-zero singular values.

Matrix Decomposition

➢ QR Decomposition

Definition

QR decomposition is a decomposition of a matrix **A** into a product **A** = **QR** of an orthonormal matrix **Q** and an upper triangular matrix **R**.

✓ **Q** is an orthogonal matrix means that its columns are orthogonal unit vectors satisfying

$$\boxed{Q^\mathsf{T} = Q^{-1}}$$

✓ **R** is an upper triangular matrix having the form:

$$\begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ & u_{2,2} & u_{2,3} & \cdots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & u_{n-1,n} \\ 0 & & & & u_{n,n} \end{bmatrix}$$

Matrix Decomposition

- QR Decomposition

$$A = \boxed{Q}R$$

first

Computation

We first apply Gram–Schmidt process to $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{a}_1, & \mathbf{e}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \\
\mathbf{u}_2 &= \mathbf{a}_2 - \mathrm{proj}_{\mathbf{u}_1}\,\mathbf{a}_2, & \mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \\
\mathbf{u}_3 &= \mathbf{a}_3 - \mathrm{proj}_{\mathbf{u}_1}\,\mathbf{a}_3 - \mathrm{proj}_{\mathbf{u}_2}\,\mathbf{a}_3, \;\Rightarrow & \mathbf{e}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \quad \Rightarrow \quad \boxed{Q} = [\mathbf{e}_1 \quad \cdots \quad \mathbf{e}_n] \\
&\;\vdots & &\;\vdots \\
\mathbf{u}_k &= \mathbf{a}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{u}_j}\,\mathbf{a}_k, & \mathbf{e}_k &= \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}
\end{aligned}$$

Matrix Decomposition

$$\boxed{A = Q\boxed{R}}$$

- QR Decomposition

Computation

$$\boxed{Q = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix}}$$

We can now express $\mathbf{a}_i$ over the newly computed orthonormal basis $\{\mathbf{e}_i\}$:

$$\begin{aligned}
\mathbf{a}_1 &= \langle \mathbf{e}_1, \mathbf{a}_1 \rangle \mathbf{e}_1 \\
\mathbf{a}_2 &= \langle \mathbf{e}_1, \mathbf{a}_2 \rangle \mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{a}_2 \rangle \mathbf{e}_2 \\
\mathbf{a}_3 &= \langle \mathbf{e}_1, \mathbf{a}_3 \rangle \mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{a}_3 \rangle \mathbf{e}_2 + \langle \mathbf{e}_3, \mathbf{a}_3 \rangle \mathbf{e}_3 \\
&\vdots \\
\mathbf{a}_k &= \sum_{j=1}^{k} \langle \mathbf{e}_j, \mathbf{a}_k \rangle \mathbf{e}_j
\end{aligned}$$

$$\Rightarrow \quad \boxed{R} = \begin{bmatrix}
\langle \mathbf{e}_1, \mathbf{a}_1 \rangle & \langle \mathbf{e}_1, \mathbf{a}_2 \rangle & \langle \mathbf{e}_1, \mathbf{a}_3 \rangle & \cdots & \langle \mathbf{e}_1, \mathbf{a}_n \rangle \\
0 & \langle \mathbf{e}_2, \mathbf{a}_2 \rangle & \langle \mathbf{e}_2, \mathbf{a}_3 \rangle & \cdots & \langle \mathbf{e}_2, \mathbf{a}_n \rangle \\
0 & 0 & \langle \mathbf{e}_3, \mathbf{a}_3 \rangle & \cdots & \langle \mathbf{e}_3, \mathbf{a}_n \rangle \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \langle \mathbf{e}_n, \mathbf{a}_n \rangle
\end{bmatrix}.$$

Summary

- Vector Operations
- Vector Space
- Matrices and Transformation
- Matrix Properties
- Matrix Decomposition

Thank you for your listening!

If you have any questions, please come to me :-)