Introduction to Deep Learning (I2DL)

Exercise 2: Math Recap

Overview

Linear Algebra

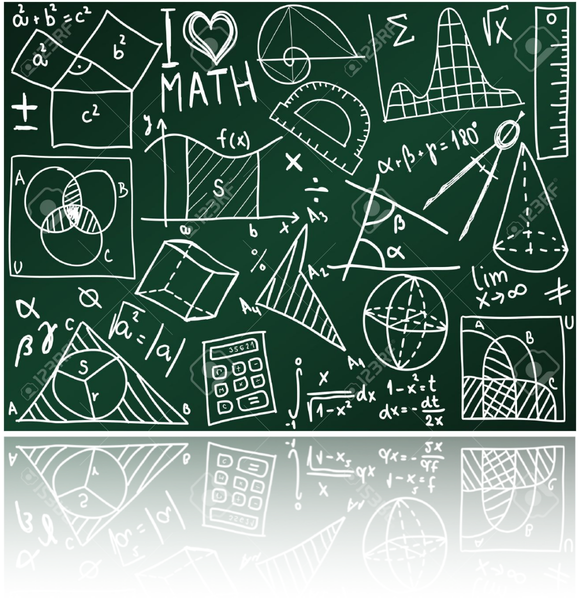
- Vectors and matrices
- Basic operations on matrices & vectors
- Tensors
- Norms, Loss functions

Calculus

- Scalar derivatives
- Gradient
- Jacobian Matrix
- Chain Rule

Probability Theory

- Probability space
- Random variables
- PMF, PDF, CDF
- Mean, variance
- Standard probability distributions



Linear Algebra

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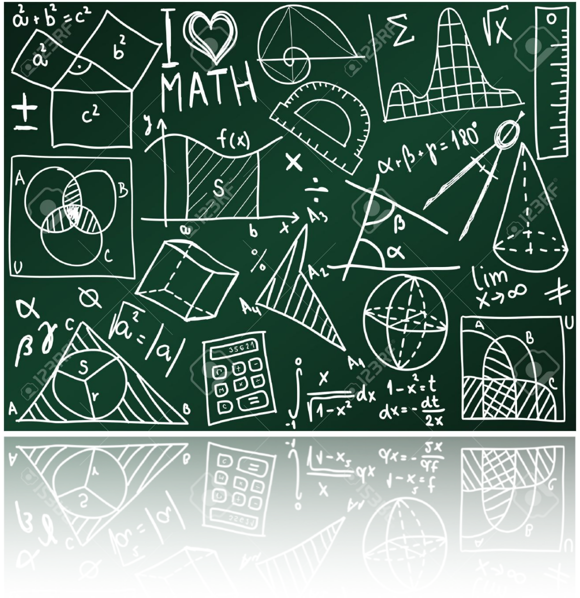


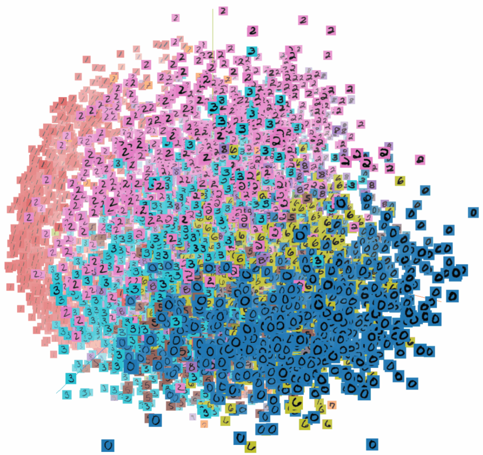
Basic Notation

- **Vector:** We call an element of $\mathbb{R}^n$ a vector with n entries.
- **Elements of a vector:** The ith element of a vector $v \in \mathbb{R}^n$ is denoted by $v_i \in \mathbb{R}$.
- **Matrix:** We call an element of $\mathbb{R}^{n \times m}$ a matrix with n rows and m columns.
- **Elements of a matrix:** For $A \in \mathbb{R}^{n \times m}$, we denote the element at the ith row and jth column by $A_{ij} \in \mathbb{R}$.
- **Transpose:** The transpose of a matrix results from "flipping" rows and columns. We denote the transpose of a matrix $A \in \mathbb{R}^{n \times m}$ by $A^T \in \mathbb{R}^{m \times n}$. Similarly, we use transposed vectors.

Vector

An n-dimensional vector describes an element in an n-dimensional space

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$$



Vector Operations:

Addition	Subtraction	Scalar Multiplication	Dot Product

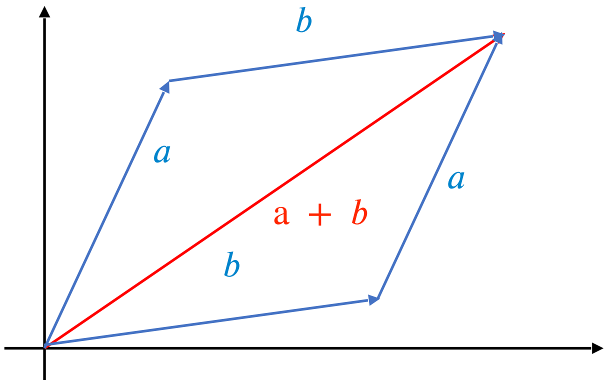
https://projector.tensorflow.org/

Vector Operations

Vector Operations: **Addition** | Subtraction | Scalar Multiplication | Dot Product

For $a, b \in \mathbb{R}^n$ we have

$$a + b = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix} \in \mathbb{R}^n$$

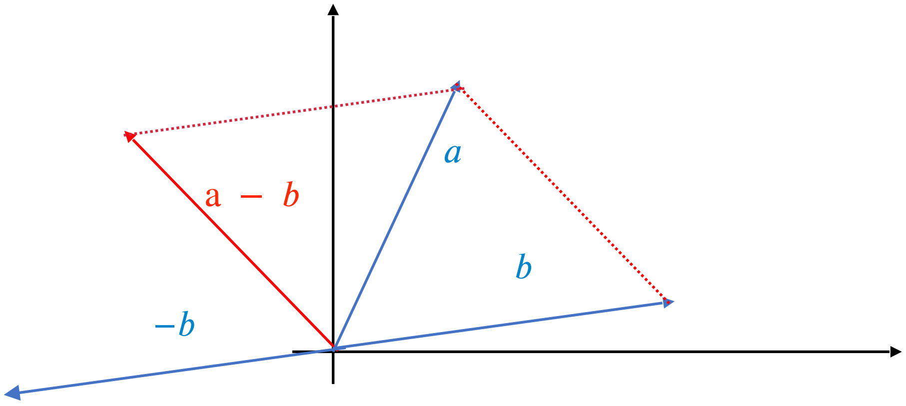


Vector Operations

Vector Operations: Addition | **Subtraction** | Scalar Multiplication | Dot Product

For $a, b \in \mathbb{R}^n$ we have

$$a - b = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ \vdots \\ a_n - b_n \end{pmatrix} \in \mathbb{R}^n$$

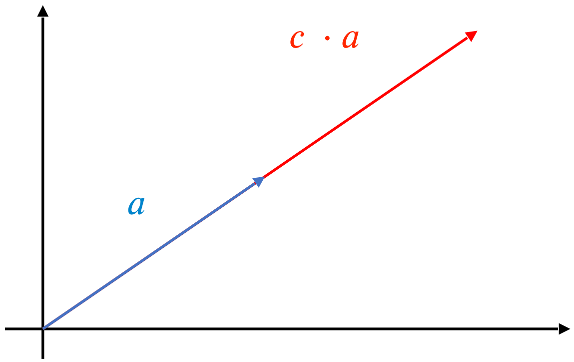


Vector Operations

Vector Operations: Addition | Subtraction | **Scalar Multiplication** | Dot Product

For $a \in \mathbb{R}^n, c \in \mathbb{R}$ we have

$$c \cdot a = \begin{pmatrix} c \cdot a_1 \\ c \cdot a_2 \\ \vdots \\ c \cdot a_n \end{pmatrix} \in \mathbb{R}^n$$



Vector Operations

Vector Operations: | Addition | Subtraction | Scalar Multiplication | **Dot Product**

Definition: For $a, b \in \mathbb{R}^n$, the dot product is defined as follows:

$$\begin{aligned} a \cdot b &= a^T \cdot b \\ &= a_1 \cdot b_1 + a_2 \cdot b_2 + \ldots + a_n \cdot b_n \\ &= \sum_{i=1}^{n} a_i \cdot b_i \in \mathbb{R} \end{aligned}$$

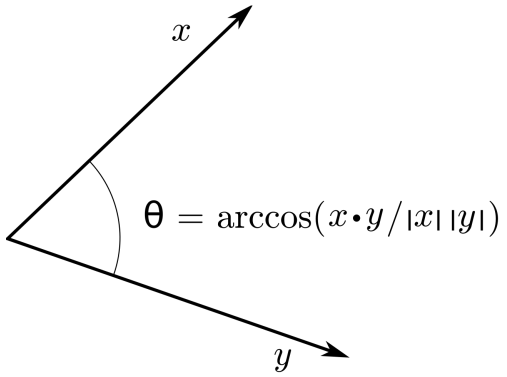
Vector Operations

Vector Operations:

Addition	Subtraction	Scalar Multiplication	**Dot Product**

Properties:

- Commutative: $a \cdot b = b \cdot a$
- Geometric interpretation: $a \cdot b = ||a|| \cdot ||b|| \cdot \cos(\theta)$
- Orthogonality: Two non-zero vectors are orthogonal to each other $\iff$ $a \cdot b = 0$



Vector Operations

Vector Operations: | Addition | Subtraction | Scalar Multiplication | **Dot Product**

Properties:

- Commutative: $a \cdot b = b \cdot a$
- Geometric interpretation:
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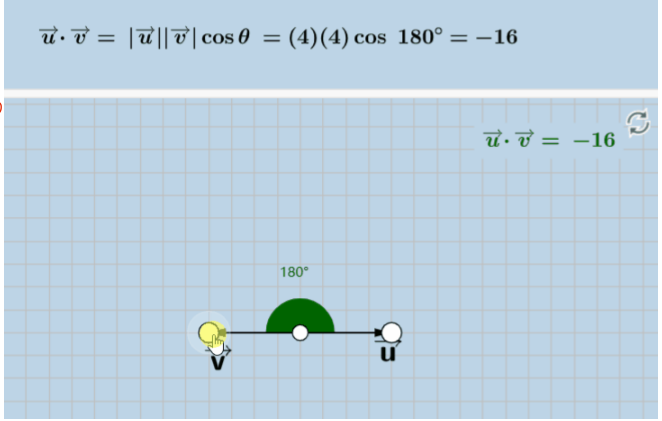
$a \cdot b < 0 \quad \theta > 90$

$a \cdot b = 0 \quad \theta = 90$

$a \cdot b > 0 \quad \theta < 90$



https://www.geogebra.org/m/Yu6869By

Matrix

A matrix $A \in \mathbb{R}^{n \times m}$ is denoted as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \in \mathbb{R}^{n \times m}$$

Matrix Operations:

Matrix-vector Multiplication	Matrix-matrix Multiplication	Hadamard Product

Matrix

Matrix Operations: **Matrix-vector Multiplication** | Matrix-matrix Multiplication | Hadamard Product

- Multiplication of matrix with a vector is defined as follows:

$$\text{For } A \in \mathbb{R}^{n \times m}, b \in \mathbb{R}^m \text{: } A \cdot b = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} a_{11} \cdot b_1 + a_{12} \cdot b_2 + \dots + a_{1m} \cdot b_m \\ a_{21} \cdot b_1 + a_{22} \cdot b_2 + \dots + a_{2m} \cdot b_m \\ \vdots \\ a_{n1} \cdot b_1 + a_{n2} \cdot b_2 + \dots + a_{nm} \cdot b_m \end{pmatrix} \in \mathbb{R}^n$$

- **Attention:** The respective dimension have to fit, otherwise the multiplication is not well-defined.

$$\Longrightarrow \underbrace{A}_{n \times m} \cdot \underbrace{b}_{m \times 1} = \underbrace{c}_{n \times 1}$$

- **Example:** $A \in \mathbb{R}^{3 \times 2}, b \in \mathbb{R}^2$ with $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 18 \\ 28 \end{pmatrix}$

Matrix Operations

Matrix Operations: | Matrix-vector Multiplication | Matrix-matrix Multiplication | Hadamard Product

- Similar, the multiplication of two matrices with each other is defined as follows:
  For $A \in \mathbb{R}^{n \times m}, B \in \mathbb{R}^{m \times l}$ we have

$$A \cdot B = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1l} \\ b_{21} & b_{22} & \dots & b_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{ml} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1l} \\ c_{21} & c_{22} & \dots & c_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nl} \end{pmatrix} \in \mathbb{R}^{n \times l} \text{ where}$$

$$c_{ij} = \sum_{k=1}^{m} a_{ik} \cdot b_{kj} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \dots + a_{im} \cdot b_{mj}$$

- Attention: Matrix Multiplication is in general not commutative, i.e. for two matrices $A \in \mathbb{R}^{n \times m}, B \in \mathbb{R}^{m \times n}$ we have $A \cdot B \neq B \cdot A$

Matrix Operations

Matrix Operations: | Matrix-vector Multiplication | Matrix-matrix Multiplication | **Hadamard Product**

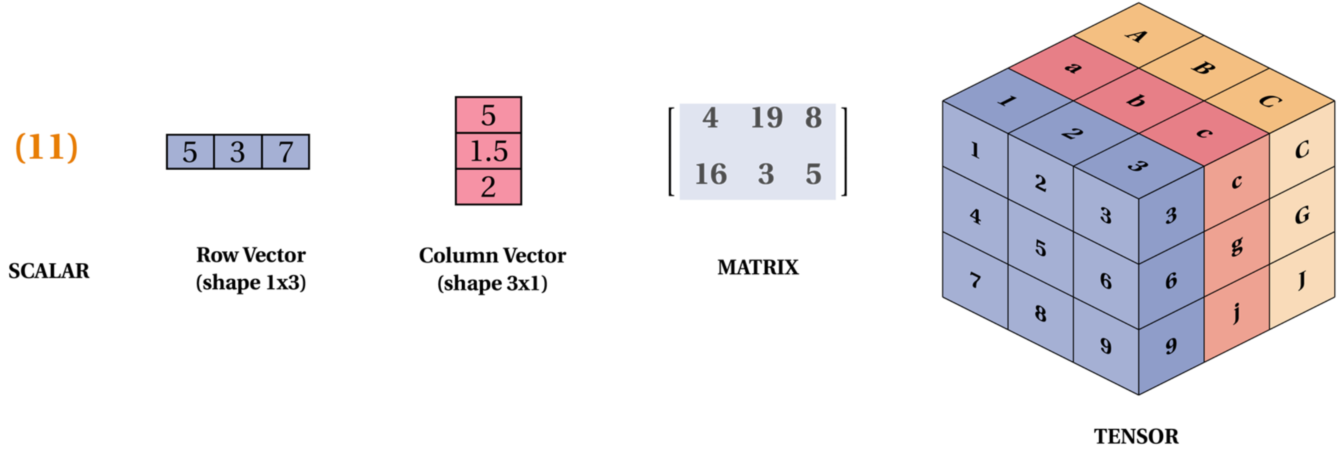
- The Hadamard product is the element wise product of two matrices. For two matrices of the same dimension $A, B \in \mathbb{R}^{n \times m}$ it is given by

$$A \odot B = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ a_{21} & \dots & a_{2m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \dots & b_{1m} \\ b_{21} & \dots & b_{2m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nm} \end{pmatrix} = \begin{pmatrix} a_{11} \cdot b_{11} & \dots & a_{1m} \cdot b_{1m} \\ a_{21} \cdot b_{21} & \dots & a_{2m} \cdot b_{2m} \\ \vdots & \ddots & \vdots \\ a_{n1} \cdot b_{n1} & \dots & a_{nm} \cdot b_{nm} \end{pmatrix} \in \mathbb{R}^{n \times m}$$

→ *For all matrix operations, it is important to check the dimensions!*

Tensor

- Definition: A tensor is a multidimensional array and a generalization of the concepts of a vector and a matrix.

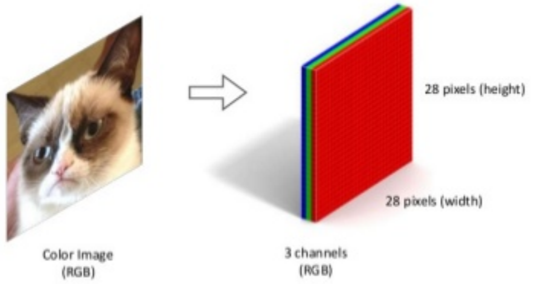


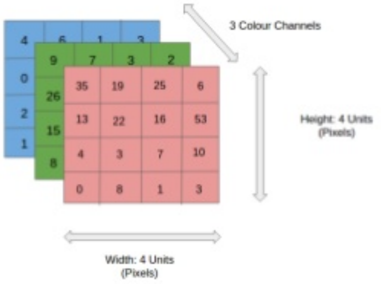
Tensors in Computer Vision

color image is 3rd-order tensor

Tensors are used to represent RGB images.

$$H \times W \times RGB$$





Source: https://www.slideshare.net/BertonEarnshaw/a-brief-survey-of-tensors

Norm

- **Norm:** measure of the “length” of a vector
- **Definition:** A norm is a non-negative function $|| \cdot || : V \to \mathbb{R}$ which is defined by the following the properties for elements $v, w \in V$:
  1. Triangle inequality: $||v + w|| \leq ||v|| + ||w||$
  2. $||a \cdot v|| = a \cdot ||v||$ for a scalar a
  3. $||v|| = 0$ if and only if $v = 0$

(*V is a vector space over a field $\mathbb{F}$; in our case we have $V = \mathbb{R}^n$)

- **Remark:** Every such function defines a norm on the vector space.
- **Examples:** L1-norm, L2-norm

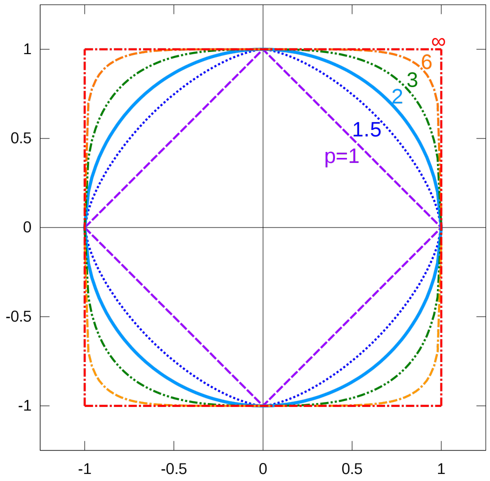
L1-Norm

- **Norm:** measure of the "length" of a vector
- **L1-Norm:** We denote the L1-norm with $\|\cdot\|_1 : \mathbb{R}^n \to \mathbb{R}$ such that for a vector $v = (v_1, v_2, \ldots, v_n)$

$$\|v\|_1 = \sum_{i=1}^{n} |v_i|$$

- **Example:** Let $v = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \in \mathbb{R}^3$, then

$$\|v\|_1 = (1 + 3 + 2) = 6$$



https://en.wikipedia.org/wiki/Lp_space

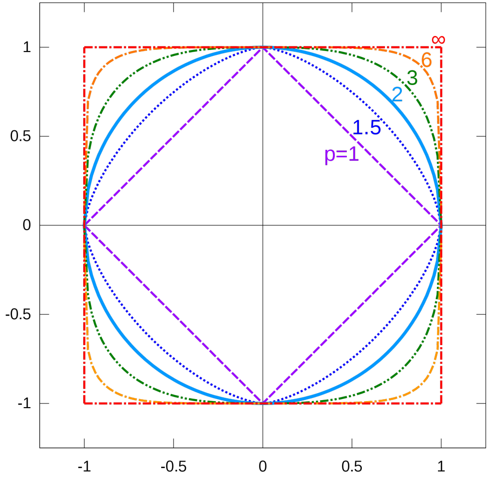
L2-Norm

- **Norm:** measure of the "length" of a vector
- **L2-Norm:** We denote the L2-norm with $\|\cdot\|_2 : \mathbb{R}^n \to \mathbb{R}$ such that for a vector $v = (v_1, v_2, \ldots, v_n)$

$$\|v\|_2 = \sqrt{\sum_{i=1}^{n} (v_i)^2}$$

- **Example:** Let $v = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \in \mathbb{R}^3$, then

$$\|v\|_2 = \sqrt{(1^2 + (-3)^2 + 2^2)} = \sqrt{14}$$



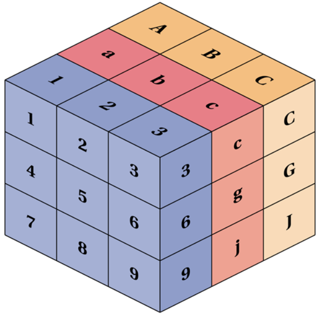
https://en.wikipedia.org/wiki/Lp_space

Loss functions

- A loss function is a function that takes as input two vectors and as output measures the distance between these two
  $\rightarrow$ uses a norm to measure the distance
  $\rightarrow$ L1-Loss uses the L1-norm, L2-Loss uses the L2-norm
- **L1-Loss:** The L1-Loss between two vectors $v, w \in \mathbb{R}^n$ is defined as $L_1(v, w) = \|v - w\|_1 = \Sigma_{i=1}^{n} |v_i - w_i|$
- **L2-Loss:** The L2-Loss between two vectors $v, w \in \mathbb{R}^n$ is defined as

$$L_2(v, w) = \|v - w\|_2 = \sqrt{(v_1 - w_1)^2 + \ldots + (v_n - w_n)^2}$$

Outlook



TENSOR

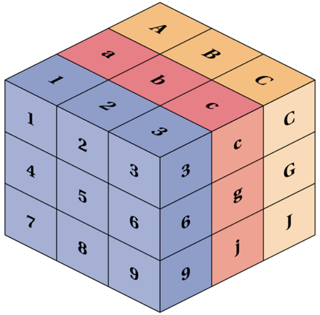
Neural Network

17% cat
13% dog
70% deer

Loss

(http://www.isfpga.org/fpga2017/slides/D2_S1_02.pdf)

Outlook



$$W \cdot x$$

$$W \in \mathbb{R}^{n \times m}$$

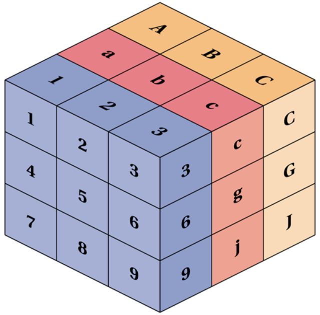
17% cat
13% dog
70% deer

Loss

The elements of the matrix W are called weights and they determine the prediction of our network.

(http://www.isfpga.org/fpga2017/slides/D2_S1_02.pdf)

Outlook



TENSOR

$W \cdot x$

$W \in \mathbb{R}^{n \times m}$

17% cat
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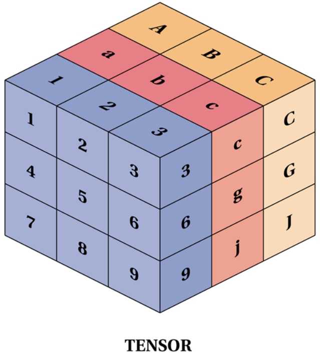
Loss

How can we get an accurate matrix W to minimize the loss?

(http://www.isfpga.org/fpga2017/slides/D2_S1_02.pdf)

Outlook



下降

$$W \cdot x$$

$$W \in \mathbb{R}^{n \times m}$$

17% cat
13% dog
70% deer

Loss

Gradient Descent: Method to approximate the best values for the weights

Calculus

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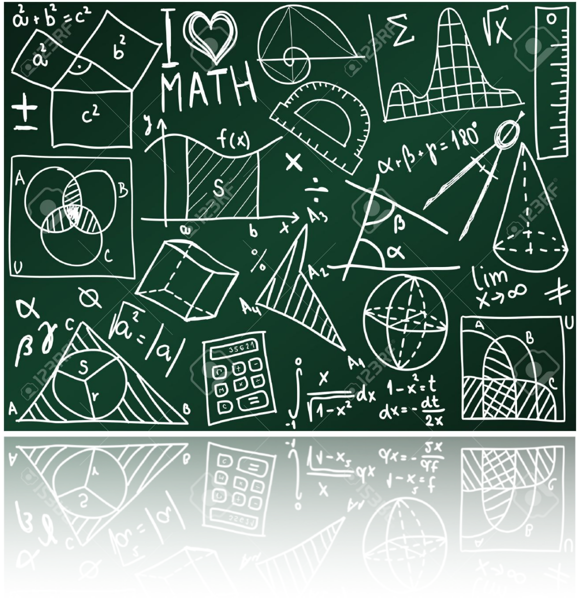


Derivatives

标量导数

- **Well known:** Scalar derivatives, i.e. derivatives of functions $f : \mathbb{R} \to \mathbb{R}$
- **Matrix calculus:** Extension of calculus to higher dimensional setting, i.e. functions like $f : \mathbb{R}^n \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}^n, f : \mathbb{R}^n \to \mathbb{R}^m$ and $f : \mathbb{R}^{n \times m} \to \mathbb{R}$ for $n, m \in \mathbb{N}$
- Actual calculus we use is relatively trivial, but the notation can often make things look much more difficult than they are.

'我们使用的实际微积分是相对微不足道的，但符号往往会使事情看起来比实际要困难得多。'

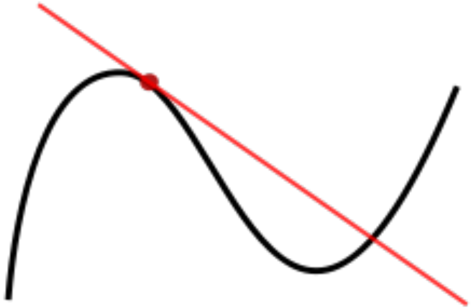
Overview

Setting	Derivative	Notation
$f: \mathbb{R} \rightarrow \mathbb{R}$	Scalar derivative	$f'(x)$
$f: \mathbb{R}^n \rightarrow \mathbb{R}$	Gradient	$\nabla f(x)$
$f: \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$	Gradient	$\nabla f(x)$
$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$	Jacobian	J_f

Scalar derivatives

- **Setting:** $f : \mathbb{R} \to \mathbb{R}$
- **Notation:** $f'(x)$ or $\dfrac{\mathrm{d}f}{\mathrm{d}x}$
- **Derivative:** Derivative of a function at a chosen input value is the slope of the tangent line to the graph of the function at that point.

一个函数在一个选定的输入值处的导数是该点的函数图形的切线的斜率。



https://en.wikipedia.org/wiki/Derivative#/media/File:Tangent_to_a_curve.svg

Derivation Rules

Common functions	Derivative
$f(x) = c$ for $c \in \mathbb{R}$	$f'(x) = 0$
$f(x) = x$	$f'(x) = 1$
$f(x) = x^n$ for $n \in \mathbb{N}$	$f'(x) = n \cdot x^{n-1}$
$f(x) = e^x$	$f'(x) = e^x$
$f(x) = ln(x)$	$f'(x) = \frac{1}{x}$
$f(x) = sin(x)$	$f'(x) = cos(x)$
$f(x) = cos(x)$	$f'(x) = -sin(x)$

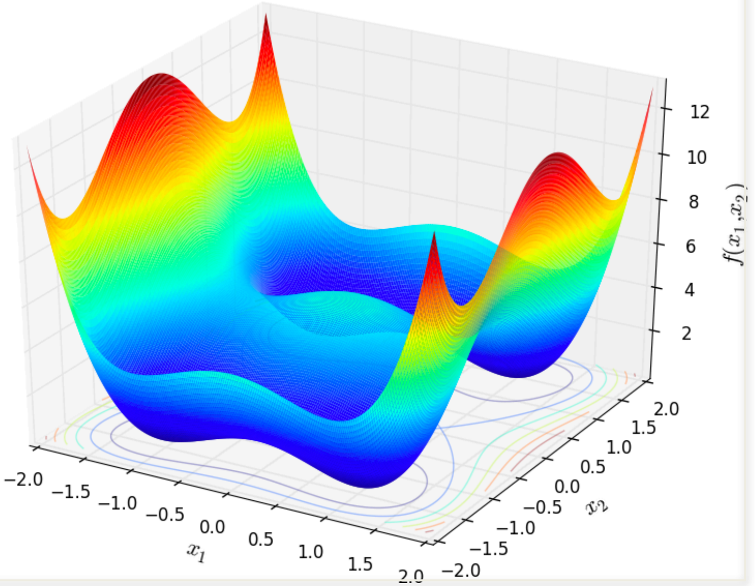
Derivation Rules

Rule	Function	Derivative
Sum rule	$f(x) + g(x)$	$f'(x) + g'(x)$
Difference rule	$f(x) - g(x)$	$f'(x) - g'(x)$
Multiplication by constant	$c \cdot f(x)$	$c \cdot f'(x)$
Product rule	$f(x) \cdot g(x)$	$f'(x) \cdot g(x) + f(x) \cdot g'(x)$
Quotient rule	$\frac{f(x)}{g(x)}$	$\frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$
Chain rule	$f(g(x))$	$f'(g(x)) \cdot g'(x)$

Multivariate functions $f : \mathbb{R}^n \to \mathbb{R}$

Multivariate Function

$$f : \mathbb{R}^n \to \mathbb{R}$$



Gradient

$$\nabla f : \mathbb{R}^n \to \mathbb{R}^n$$

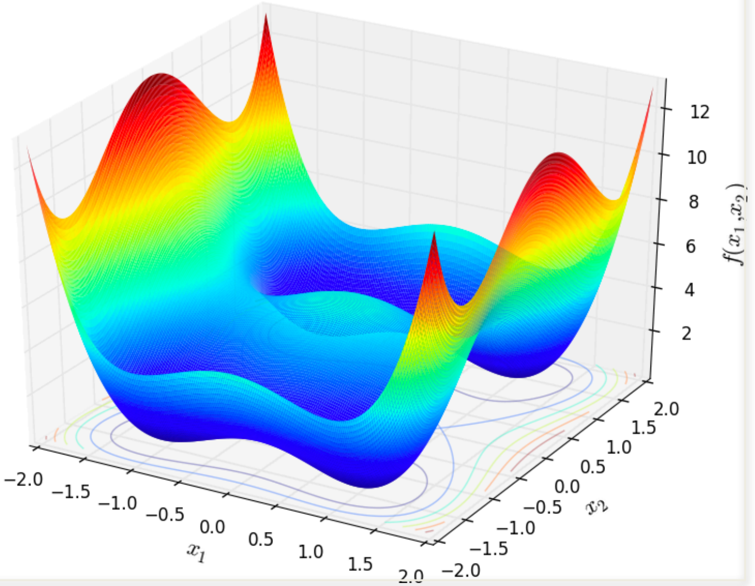
Partial derivative

$$\nabla f : x \to \nabla f(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{pmatrix}$$

Multivariate functions $f: \mathbb{R}^{n\times m} \to \mathbb{R}$

Multivariate Function

$$f: \mathbb{R}^{n\times m} \to \mathbb{R}$$

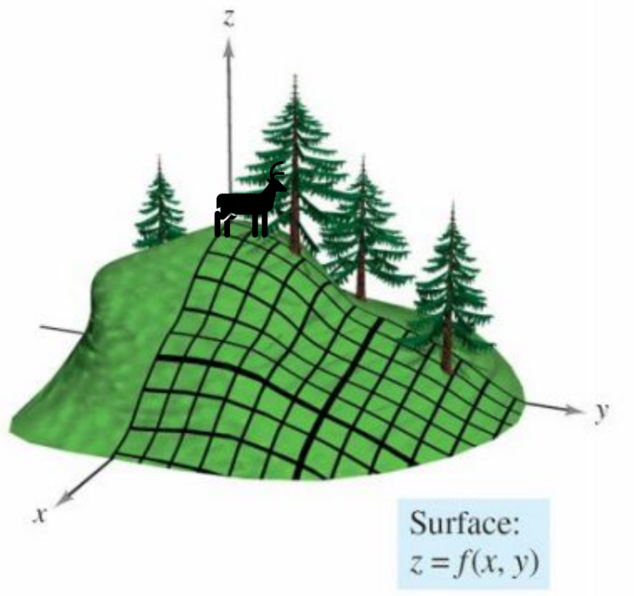


Gradient

$$\nabla f: \mathbb{R}^{n\times m} \to \mathbb{R}^{n\times m}$$

$$\nabla f: x \to \nabla f(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_{11}} & \frac{\partial f(x)}{\partial x_{12}} & \cdots & \frac{\partial f(x)}{\partial x_{1m}} \\ \frac{\partial f(x)}{\partial x_{21}} & \frac{\partial f(x)}{\partial x_{22}} & \cdots & \frac{\partial f(x)}{\partial x_{2m}} \\ \vdots & & & \\ \frac{\partial f(x)}{\partial x_{n1}} & \frac{\partial f(x)}{\partial x_{n2}} & \cdots & \frac{\partial f(x)}{\partial x_{nm}} \end{pmatrix}$$

Gradient – Example 1



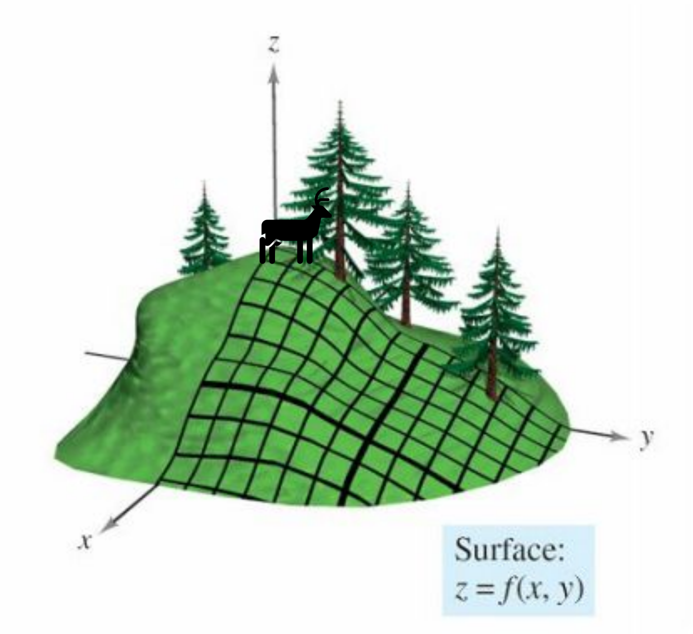
$$f(x,y) = 3x^2y \qquad \nabla f(x,y) = \left[\frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y}\right]$$

$$\frac{\partial}{\partial x}3yx^2 = 3y\frac{\partial}{\partial x}x^2 = 3y2x = 6yx$$

$$\frac{\partial}{\partial y}3x^2y = 3x^2\frac{\partial}{\partial y}y = 3x^2\frac{\partial y}{\partial y} = 3x^2 \times 1 = 3x^2$$

$$\nabla f(x,y) = \left[\frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y}\right] = [6yx,\ 3x^2]$$

Gradient – Example 2



$$g(x, y) = 2x + y^8$$

$$\frac{\partial g(x,y)}{\partial x} = \frac{\partial 2x}{\partial x} + \frac{\partial y^8}{\partial x} = 2\frac{\partial x}{\partial x} + 0 = 2 \times 1 = 2$$

$$\frac{\partial g(x,y)}{\partial y} = \frac{\partial 2x}{\partial y} + \frac{\partial y^8}{\partial \mathrm{y}} = 0 + 8y^7 = 8y^7$$

$$\nabla g(x, y) = [2, 8y^7]$$

https://www.zhihu.com/question/36301367

Vector-valued functions

Vector-Valued function

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

$$f : x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \longrightarrow \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix}$$

Jacobian Matrix

$$J_f : \mathbb{R}^n \to \mathbb{R}^{m \times n}$$

$$x \to J_f(x) = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \cdots & \frac{\partial f_2(x)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m(x)}{\partial x_1} & \frac{\partial f_m(x)}{\partial x_2} & \cdots & \frac{\partial f_m(x)}{\partial x_n} \end{pmatrix}$$

Jacobian Matrix – Example 3

Assume that $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix}$ where $f_1(x, y) = 3x^2y$ and $f_2(x, y) = 2x + y^8$.

Calculate Jacobian matrix:

$$J_f(x) = \begin{pmatrix} \dfrac{\partial f_1(x, y)}{\partial x} & \dfrac{\partial f_1(x, y)}{\partial y} \\ \dfrac{\partial f_2(x, y)}{\partial x} & \dfrac{\partial f_2(x, y)}{\partial y} \end{pmatrix} = \begin{pmatrix} 6xy & 3x^2 \\ 2 & 8y^7 \end{pmatrix}$$

Single Variable Chain Rule

Setting: We are given the function h(x) = f(g(x)).

Task: Compute the derivative of this function with chain rule.

1. **Introduce the intermediate variable:** Let $u = g(x)$ be the intermediate variable.
2. **Compute individual derivatives:** $\frac{\mathrm{d}f}{\mathrm{d}u}$ and $\frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}$
3. **Chain rule:** $\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$
4. **Substitute intermediate variables back**

Single Variable Chain Rule: Example

Example: Let $h(x) = sin(x^2)$.

Task: Compute the derivative of this function with chain rule.

Observation: Here, $h(x) = f(g(x))$ with $f(x) = sin(x)$ and $g(x) = x^2$.

1. **Introduce the intermediate variable:** Let $u = x^2$ be the intermediate variable.
2. **Compute individual derivatives:** $\dfrac{\mathrm{d}f}{\mathrm{d}u} = cos(u)$ and $\dfrac{\mathrm{d}g}{\mathrm{d}x} = \dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x$
3. **Chain rule:** $\dfrac{\mathrm{d}h}{\mathrm{d}x} = \dfrac{\mathrm{d}f}{\mathrm{d}u} \cdot \dfrac{\mathrm{d}u}{\mathrm{d}x} = cos(u) \cdot 2x$
4. **Substitute intermediate variables back:** $\dfrac{\mathrm{d}h}{\mathrm{d}x} = cos(u) \cdot 2x = cos(x^2) \cdot 2x$

Total Derivative Chain Rule

General Formalism:

$$\frac{\partial f(x, u_1(x), \ldots, u_n(x))}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial x} + \frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial x} + \ldots + \frac{\partial f}{\partial u_n}\frac{\partial u_n}{\partial x}$$

$$= \frac{\partial f}{\partial x} + \sum_{i=1}^{n} \frac{\partial f}{\partial u_i}\frac{\partial u_i}{\partial x}$$

References

- https://en.wikipedia.org/wiki/Matrix_calculus
- http://parrt.cs.usfca.edu/doc/matrix-calculus/index.html
- https://arxiv.org/pdf/1802.01528.pdf
- https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives
- https://explained.ai/matrix-calculus/
- http://www.deeplearningbook.org/contents/part_basics.html
- https://towardsdatascience.com/calculating-gradient-descent-manually-6d9bee09aa0b

Probability Theory

Overview

Linear Algebra

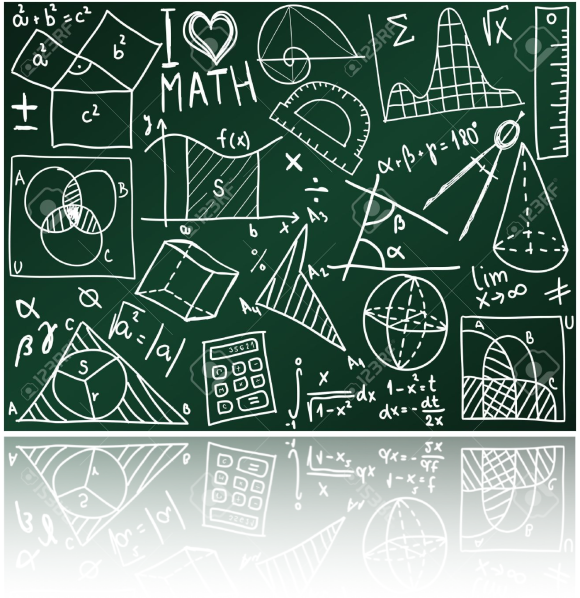
- Vectors and matrices
- Basic operations on matrices & vectors
- Tensors
- Norm & Loss functions

Calculus

- Scalar derivatives
- Gradient
- Jacobian Matrix
- Chain Rule

Probability Theory

- Probability space
- Random variables
- PMF, PDF, CDF
- Mean, variance
- Standard probability distributions



Probability space $(\Omega, \mathscr{F}, \mathbb{P})$

A probability space consist of three elements $(\Omega, \mathscr{F}, \mathbb{P})$:

- **Sample space Ω:** The set of all outcomes of a random experiment.
- **Event Space $\mathscr{F}$:** A set whose elements $A \in \mathscr{F}$ (called events) are subsets of Ω.
- **Probability measure $\mathbb{P}$:** A function $\mathbb{P} : \mathscr{F} \to [0,\ 1]$ that satisfies the following three properties:
    1. $\mathbb{P}(A) \geq 0$ for all $A \in \mathscr{F}$
    2. $\mathbb{P}(\Omega) = 1$
    3. $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i)$ for $n \in \mathbb{N}$ and disjoint events $A_1, A_2, \ldots A_n \in \mathscr{F}$

→ The probability space provides a formal model of a random experiment.

Probability space: Example

A probability space consists of three elements: $(\Omega, \mathscr{F}, \mathbb{P})$

- **Sample space Ω:** The set of all outcomes of a random experiment.
- **Event Space $\mathscr{F}$:** A set whose elements $A \in \mathscr{F}$ (called events) are subsets of Ω.
- **Probability measure $\mathbb{P}$:** A function $\mathbb{P} : \mathscr{F} \rightarrow [0,\ 1]$ that satisfies the following three properties: (…)

Example: Tossing a six-sided die

- **Sample space:** $\Omega = \{1,2,3,4,5,6\}$
- **Event space:** $\mathscr{F}_1 = \{\varnothing, \Omega\}, \mathscr{F}_2 = \mathscr{P}(\Omega)$, $\mathscr{F}_3 = \{\varnothing, A_1 = \{1,3,5\}, A_2 = \{2,4,6\}, \Omega = \{1,2,3,4,5,6\}\}$
- **Probability measure** $\mathbb{P} : \mathscr{F} \rightarrow \mathbb{R}$ with $\mathbb{P}(\varnothing) = 0$, $\mathbb{P}(\Omega) = 1$ and in the case of $\mathscr{F}_3$ we know that $\mathbb{P}(A_1) + \mathbb{P}(A_2) = 1$.
- **Example event space** $\mathscr{F}_3$: Possible probability measure are
  1. $\mathbb{P}_1(A_1) = \frac{1}{2} = \mathbb{P}_1(A_2)$
  2. $\mathbb{P}_2(A_1) = \frac{1}{4}$ and $\mathbb{P}_2(A_2) = \frac{3}{4}$.

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Random variable

- A random variable is a function defined on the probability space which maps from the sample space to the real numbers, i.e.

$$X : \Omega \to \mathbb{R}.$$

- We distinguish between **discrete** and **continuous** random variables.

Random variable

- A random variable is a function defined on the probability space which maps from the sample space to the real numbers, i.e. $X : \Omega \to \mathbb{R}$.

Example: Tossing a fair six-sided die

- **Underlying experiment:** $\Omega = \{1,2,3,4,5,6\}$, $\mathscr{F} = \mathscr{P}(\Omega)$, $\mathbb{P}(\{x\}) = \frac{1}{6}\ \forall x \in \Omega$
- **Random variable** X: Number that appears on the die, $X : \Omega \to \{1,2,3,4,5,6\}$
  $\Longrightarrow$ discrete random variable
- **Example:** One element in Ω is $\omega = 4$. Then $X(\omega) = 4$.
- **Probability measure** $\mathbb{P}$:

$$\mathbb{P}(X = 4) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = \omega = 4\}) = \mathbb{P}(\{4\}) = \frac{1}{6}$$

discrete

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Random variable

- A random variable is a function defined on the probability space which maps from the sample space to the real numbers, i.e. $X : \Omega \to \mathbb{R}$.

Example: Flipping a fair coin two times

- **Underlying experiment:** $\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}(\{\omega\}) = \frac{1}{4}\, \forall \omega \in \Omega$
- **Random variable X:** number of heads that appeared in the two flips, $X : \Omega \to \{0,1,2\}$ $\Longrightarrow$ discrete random variable
- **Example:** One element in Ω is $\omega = (T,H)$. Then $X(\omega) = 1$.
- **Probability measure $\mathbb{P}$:**

$$\mathbb{P}(X = 1) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = 1\}) = \mathbb{P}(\{(H,T),(T,H)\}) = \frac{1}{2}$$

discrete

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Random variable

- A random variable is a function defined on the probability space which maps from the sample space to the real numbers, i.e. $X : \Omega \to \mathbb{R}$.

Example: radioactive decay

- **Underlying experiment:** $\Omega = \mathbb{R}_{\geq 0}$, $\mathscr{F} = \mathscr{B}(\Omega)$, $\mathbb{P} = \lambda$ is the Lebesgue measure
- **Random variable** X**:** indicating amount of time that it takes for a radioactive particle to decay, $X : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \Longrightarrow$ continuous random variable
- **Probability measure** $\mathbb{P}$**:** is defined on the set of events $\mathscr{F}$ and is now used for random variables as follows:
  $\mathbb{P}(a \leq X \leq b) = \mathbb{P}(\{\omega \in \Omega : a \leq X(\omega) \leq b\})$

Continuous

Probability measures

$\Longrightarrow$ specify the probability measures with alternative functions (CDF, PDF and PMF)

Random Variable		
Discrete	Cumulative distribution function (CDF) $F_X(x) = \mathbb{P}(X \leq x)$	Probability mass function (PMF) $p_X(x) = \mathbb{P}(X = x)$
Continuous	Cumulative distribution function (CDF) $F_X(x) = \mathbb{P}(X \leq x)$	Probability distribution function (PDF)

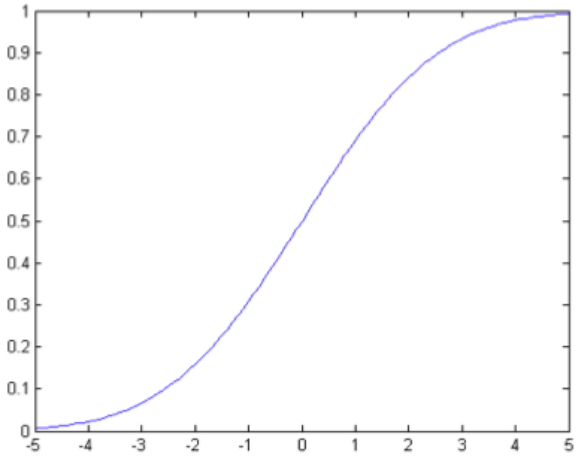
Cumulative Distribution Function

- A **cumulative distribution function** (CDF) of a random variable X is a function $F_X : \mathbb{R} \rightarrow [0,1]$ which is defined as

$$F_X(x) = \mathbb{P}(X \leq x)$$

- **Properties:** Per definition, it satisfies the following properties:

1. $0 \leq F_X(x) \leq 1$
2. $\lim_{x \rightarrow -\infty} F_X(x) = 0$
3. $\lim_{x \rightarrow \infty} F_X(x) = 1$
4. $\forall x \leq y \implies F_X(x) \leq F_X(y)$



A sample CDF

[Figure: http://cs229.stanford.edu/section/cs229-prob.pdf]

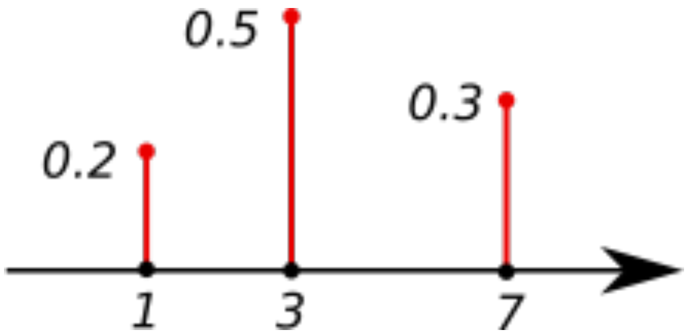
Discrete Case: Probability Mass Function

- The **probability mass function** of a random variable is a function $p_X : \Omega \to \mathbb{R}$ defined as

$$p_X(x) = \mathbb{P}(X = x)$$

- **Properties:** Again, we can derive some properties:

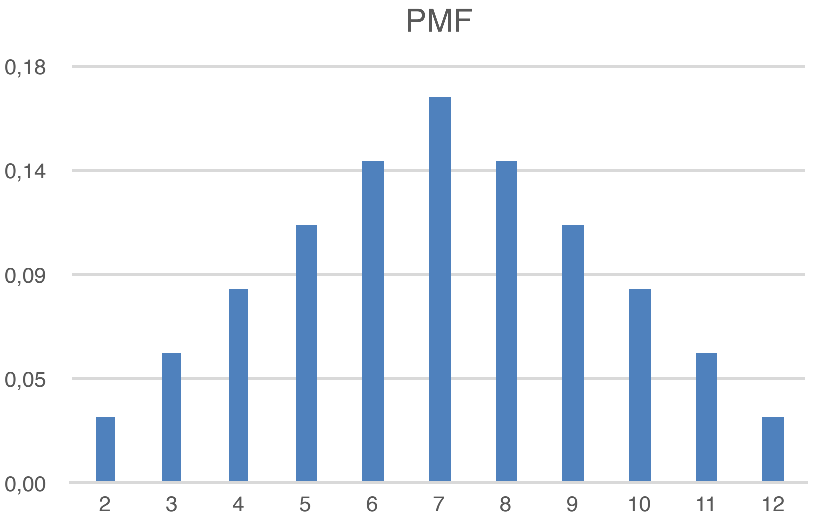
1. $0 \leq p_X(x) \leq 1$
2. $\sum_{x \in \Omega} p_X(x) = 1$

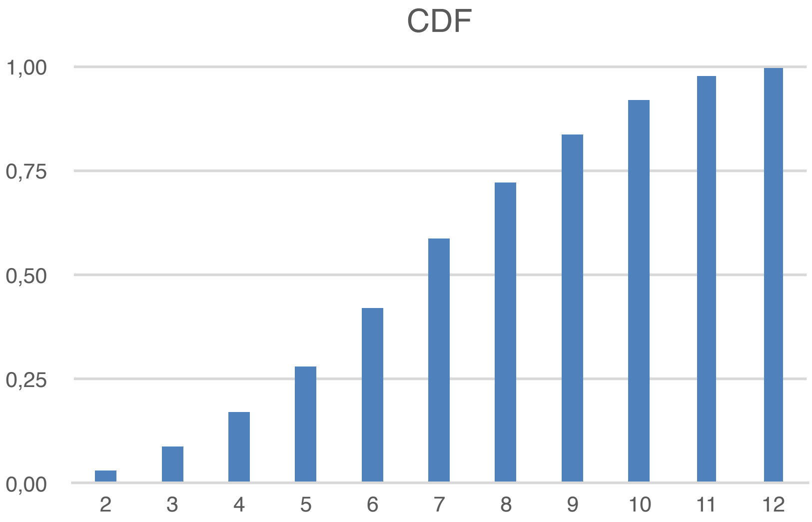


A sample PMF

[Figure: https://en.wikipedia.org/wiki/Probability_mass_function]

Discrete Example: Sum of 2 Dice Rolls





Continuous case: Probability Density Function

- **Continuous case:** For some continuous random variables, the CDF $F_X(x)$ is differentiable everywhere. Then we define the probability density function as the function $f_X(x) : \Omega \rightarrow \mathbb{R}$ with

$$f_X(x) = \frac{\mathrm{d}F_X(x)}{\mathrm{d}x}$$

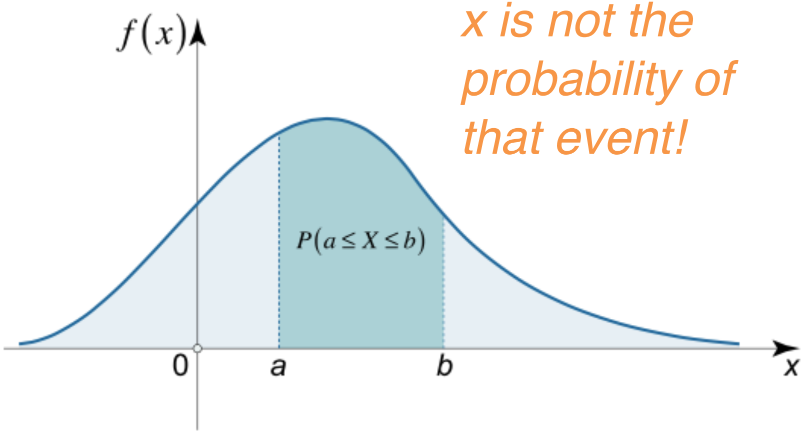
Note: the value of a PDF at any given point x is not the probability of that event!

- **Properties:**

1. $f_X(x) \geq 0$

2. $\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x = 1$

3. $\int_{a}^{b} f_X(x)\mathrm{d}x = F_X(b) - F_X(a)$



[Figure:https://www.math24.net/probability-density-function/]

Expectation of a random variable

- **Idea:** "weighted average" of the values that the random variable can take on
- **Discrete setting:** Assume that X is a discrete random variable with PMF $p_X(x)$. Then the expectation of X is given by

$$\mathbb{E}[X] = \sum_{x \in \Omega} x \cdot p_X(x)$$

- **Continuous setting:** Assume that X is a continuous random variable with PDF $f_X(x)$. Then the expectation of X is given by

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f_X(x) \ \mathrm{d}x$$

Expectation: Example

- **Discrete setting:** Assume that X is a discrete random variable with PMF $p_X(x)$. Then the expectation of X is given by

$$\mathbb{E}[X] = \sum_{x \in \Omega} x \cdot p_X(x)$$

Example: Tossing a six-sided die

$$\Omega = \{1,2,3,4,5,6\}$$

X: represents the outcome of the toss

$$p_X(x) = \mathbb{P}(X = x) = \frac{1}{6} \, \forall x \in \Omega$$

$$\mathbb{E}[X] = \sum_{x \in \Omega} x \cdot p_X(x) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \ldots + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5$$

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Expectation of a random variable

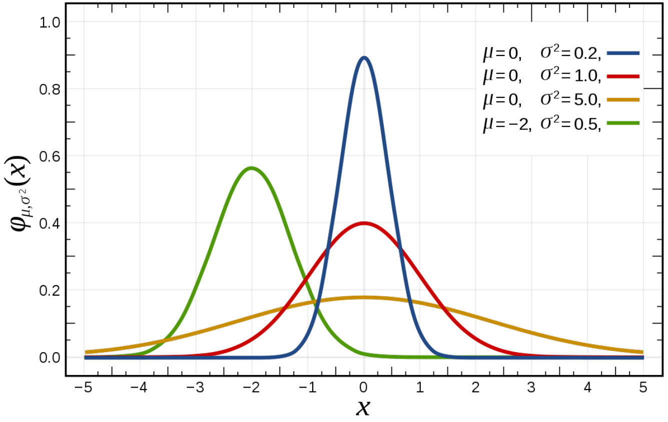
Properties: We encounter several important properties for the expectation, i.e.

1. $\mathbb{E}[a] = a$ for any constant $a \in \mathbb{R}$
2. Linearity: $\mathbb{E}[aX + bY] = a \cdot \mathbb{E}[X] + b \cdot \mathbb{E}[Y]$ for any constants $a, b \in \mathbb{R}$

Variance of a random variable

- **Idea:** The variance of a random variable is a measure how concentrated the distribution of a random variable X is around its mean.
- **Definition:** The variance is defined as

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$
$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2$$



https://en.wikipedia.org/wiki/Normal_distribution

Variance of a random variable

Definition: The variance is defined as Var$(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

Example: Tossing a fair six-sided die

$\Omega = \{1,2,3,4,5,6\}$, X: represents the outcome of the toss

$$p_X(x) = \mathbb{P}(X = x) = \frac{1}{6} \, \forall x \in \Omega$$

$$\mathbb{E}[X] = 3.5, \; \mathbb{E}[X]^2 = 12\frac{1}{4}$$

$$\mathbb{E}[X^2] = \sum_{x \in \Omega} x^2 \cdot p_X(x) = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + \ldots + 5^2 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{6} = 15\frac{1}{6}$$

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 15\frac{1}{6} - 12\frac{1}{4} = \frac{35}{12} \approx 2.91$$

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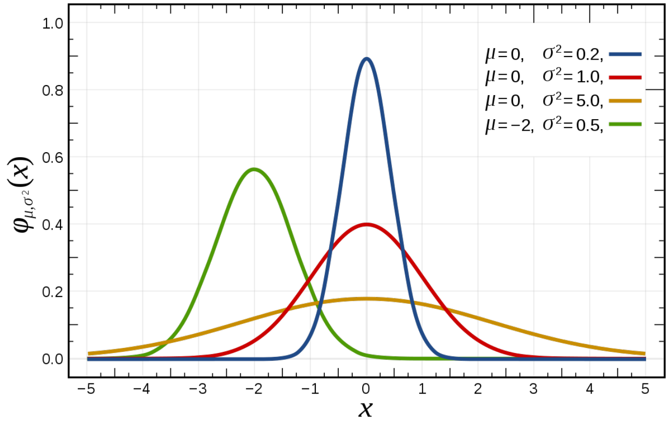
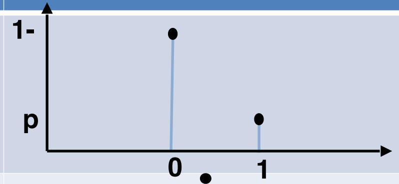
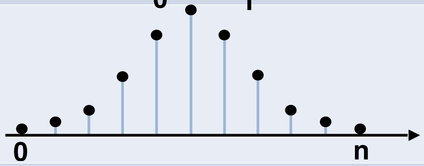
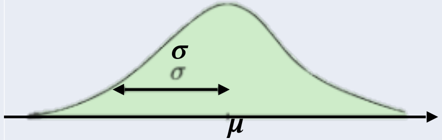
Variance of a random variable

- **Properties:** The variance has the following properties, i.e.
  1. $\text{Var}(a) = 0$ for any constant $a \in \mathbb{R}$
  2. $\text{Var}(a \cdot X + b) = a^2 \cdot \text{Var}(X)$



https://en.wikipedia.org/wiki/Normal_distribution

Standard Probability Distributions

Distribution	Parameter & Notation	PDF or PMF	Mean	Variance	Illustration
Bernoulli distribution (Discrete)	$X \sim \text{Ber}(p)$ $0 \leq p \leq 1$	$p_X(k) = p^k(1-p)^{1-k}$	$\mathbb{E}[X] = p$	$\text{Var}(X) = p(1-p)$	
Binomial distribution (Discrete)	$X \sim \text{Bin}(n,p)$ $n \in \mathbb{N}, p \in [0,1]$	$p_X(k) = \binom{n}{k} p^k(1-p)^{n-k}$	$\mathbb{E}[X] = n \cdot p$	$\text{Var}(X) = np(1-p)$	
Uniform distribution (Continuous)	$X \sim U(a,b)$ $-\infty < a < b < \infty$	$f_X(x) = \begin{cases} \frac{1}{(b-a)} & x \in [a,b] \\ 0 & \text{else} \end{cases}$	$\mathbb{E}[X] = \frac{1}{2}(a+b)$	$\text{Var}(X) = \frac{1}{12}(b-a)^2$	
Normal distribution (Continuous)	$X \sim \mathcal{N}(\mu, \sigma^2)$ $\mu \in \mathbb{R}, \sigma^2 \in \mathbb{R}_{\geq 0}$	$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	$\mathbb{E}[X] = \mu$	$\text{Var}(X) = \sigma^2$	

References

- http://cs229.stanford.edu/section/cs229-prob.pdf
  - Comprehensive Probability Review – **recommended!**
- https://stanford.edu/~shervine/teaching/cme-106/cheatsheet-probability
  - Quick Overview
- https://www.deeplearningbook.org/contents/prob.html
  - Another great resource. Also covers information theory basics.