Fundamentals of Artificial Intelligence
Exercise 5: First Order Logic

Florian Lercher

Technical University of Munich

Tweedbach

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Problem 5.1: Syntax of first-order logic

Recall the formal syntax of first-order logic, in particular the definition of terms and sentences.

Reminder: Basic Elements

Syntactic element	Examples
Variables	$x, y, a, b, \ldots$ ← placeholders for objects
Connectives	$\land, \lor, \neg, \Rightarrow, \Leftrightarrow$
Equality	$=$
Quantifiers	$\forall, \exists$

NEW!

Reminder: Basic Elements

Syntactic element	Representation of	Examples
Constants	Objects	*KingJohn*, 2, *TUM*, ...
Predicates	Relations	*Brother*, $>$, ...
Functions	Functions	*Sqrt*, *LeftLegOf*, ...

Predicates:

Object

$\downarrow$

True/False

Functions:

Objects

$\downarrow$

Object

True and False are not objects in FOL

Reminder: Terms and sentences

Backus-Naur Form of terms

$$\mathit{Term} ::= \mathit{Function}(\mathit{Term}, \ldots) \mid \mathit{Constant} \mid \mathit{Variable}$$

↳ evaluates to an object

Backus-Naur Form of atomic sentences

$$\mathit{AtomicSentence} ::= \mathit{Predicate} \mid \mathit{Predicate}(\mathit{Term}, \ldots) \mid \mathit{Term} = \mathit{Term}$$

↳ evaluates to True / False

Backus-Naur Form of complex sentences

$$\begin{aligned}\mathit{ComplexSentence} ::= & (\mathit{Sentence}) \mid [\mathit{Sentence}] \mid \neg \mathit{Sentence} \mid \\ & \mathit{Sentence} \land \mathit{Sentence} \mid \mathit{Sentence} \lor \mathit{Sentence} \mid \\ & \mathit{Sentence} \Rightarrow \mathit{Sentence} \mid \mathit{Sentence} \Leftrightarrow \mathit{Sentence} \mid \\ & \mathit{Quantifier}\ \mathit{Variable}, \ldots\ \mathit{Sentence}\end{aligned}$$

$\forall x,y \quad \alpha$

$\exists x \quad \alpha$

↳ evaluates to True / False

Problem 5.1: Syntax of first-order logic

Problem 5.1.1: Let $\mathcal{F} = \{d, f, g\}$ be the set of symbols, where d is a constant, f a function symbol with two arguments, and g a function symbol with three arguments. Which of the following expressions are terms over $\mathcal{F}$? If an expression is not a term, give a reason why. (You may assume that x, y, z are variables.)

1. $g(d, d)$ × $\rightsquigarrow g(d, d, d)$ ✓
2. $f(x, g(y, z), d)$ ×
3. $g(x, f(y, z), d)$ ✓
4. $g(x, k(y, z), d)$ ×

Problem 5.1: Syntax of first-order logic

Problem 5.1.2: Let m be a constant, h a function symbol with one argument, and S and B two predicate symbols, each with two arguments. Which of the following expressions are sentences in first-order logic? If an expression is not a sentence, give a reason why. (You may assume that x, y, z are variables.)

terms

1. $S(m, x)$ ✓
2. $B(m, h(m))$ ✓ term?
3. $B(B(m, x), y)$ ×
4. $B(x, y) \Rightarrow [\exists z\, S(z, y)]$ ✓
5. $S(x, y) \Rightarrow S(y, h(h(x)))$ ✓

Red (x) → True iff x is red

Red (Red(x))

→ True/False

Problem 5.2: Universal and existential quantifier

$\forall x \ \alpha$ $\exists x \ \alpha$

Let us abbreviate the predicate "x is taking the Bus" by $B(x)$, and "x has a Ticket" by $T(x)$. Suppose that we only consider the universe of discourse where there are only three people: Alice, Bob, and Charlie.

$B(x) \Rightarrow T(x)$ x is free

$\forall x \ B(x) \Rightarrow T(x)$ x is bound

$\{x \mapsto Alice\}$

$\{x \mapsto Bob\}$

x	$B(x)$	$T(x)$
Alice	*True*	*True*
Bob	*False*	*True*
Charlie	*False*	*False*

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Problem 5.2.1: For each of our protagonists, the table above lists whether they have a ticket, and whether they take the bus. Suppose that we want to formalize the sentence "All people who take the bus have a ticket." Given the characteristics in the table above, do you think the sentence evaluates to true? Or false?

$\boxed{\text{True}}$

Problem 5.2: Universal and existential quantifier

Problem 5.2.2: Now, consider the formula

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$$\forall x \quad \underline{B(x) \wedge T(x)}.$$

By using the extended interpretation for universal quantifiers (in other words, the transformation to propositional logic), decide whether this formula is true or false in our universe of discourse. Do you think this formula is a faithful formalization of "All people who take the bus have a ticket"? If not, what would be a better formula?

$$\forall x \quad B(x) \wedge T(x)$$

$$\equiv [B(Alice) \wedge T(A)] \wedge [B(B) \wedge T(B)] \wedge [B(C) \wedge T(C)]$$

False

$$\equiv \text{False}$$

$\rightsquigarrow$ Not a faithful formalization $\qquad \forall x \quad B(x) \Rightarrow T(x)$

Problem 5.2: Universal and existential quantifier

x	$B(x)$	$T(x)$
Alice	*False*	*True*
Bob	*False*	*True*
Charlie	*False*	*False*

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Problem 5.2.3: Now consider the changed truth assignments in the table above. Suppose that we want to formalize the sentence "Some people who take the bus have a ticket"[1]. Given the characteristics in the table above, do you think the sentence evaluates to true? Or false?

False

[1] This has to be understood as "There is **at least** one person that takes the bus, who has a ticket".

Problem 5.2: Universal and existential quantifier

Problem 5.2.4: Now, consider the formula

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$$\exists x \quad \underline{B(x)} \Rightarrow \underline{T}(x).$$

By using the extended interpretation for existential quantifiers, decide whether this formula is true or false in our universe of discourse. Do you think this formula is a faithful formalization of "Some people who take the bus have a ticket"? If not, what would be a better formula?

$\exists x \quad B(x) \Rightarrow T(x)$

$\equiv \underbrace{[\underset{\text{False}}{B(A)} \Rightarrow T(A)]}_{\text{True}} \vee [B(B) \Rightarrow T(B)] \vee [B(C) \Rightarrow T(C)]$

$\equiv$ True

$\rightsquigarrow$ not a faithful formalization

$\exists x \quad B(x) \wedge T(x)$

Learnings from Problem 5.2

Limiting the domain of quantification

To limit a quantifier to objects satisfying a predicate P, use ...

- $\forall x \quad P(x) \Rightarrow \ldots$ for universal quantification, and $\forall x \in X \ \ldots$
- $\exists x \quad P(x) \land \ldots$ for existential quantification. $\exists x \in X \ \ldots$

$\forall x \quad x \in X \Rightarrow \ldots$

Example:

- "Some $\underbrace{\text{doors}}_{\text{domain}}$ are locked" $\rightsquigarrow \exists x \quad Door(x) \land Locked(x)$
- "Everything $\underbrace{\text{that glitters}}_{\text{domain}}$ is gold" $\rightsquigarrow \forall x \quad Glitters(x) \Rightarrow Gold(x)$

Problem 5.3: Formalization to First Order Logic

Problem 5.3.1: Formalize the next sentences using the following predicates and constant:

$A(x, y)$: x admires y

$P(x)$: x is a professor

d : Dan

1. Dan admires every $\underbrace{\text{professor}}_{\text{domain}}$.

   $\forall x \; P(x) \Rightarrow A(d, x)$

2. Some $\underbrace{\text{professor}}_{\text{domain}}$ admires Dan.

   $\exists x \; P(x) \wedge A(x, d)$

3. Dan admires himself.

   $A(d, d)$

Problem 5.3: Formalization to First Order Logic

Problem 5.3.2: Formalize the next sentences using the following predicates:

$A(x, y)$: x attended y

$S(x)$: x is a student

$L(x)$: x is a lecture

$\forall x\ \alpha \equiv \neg\exists x\ \neg\alpha$

$\neg\forall x\ \alpha \equiv \exists x\ \neg\alpha$

1. No student attended every lecture.

   $\neg\exists x\ S(x) \wedge \forall y\ L(y) \Rightarrow A(x,y)$

   or $\forall x\ S(x) \Rightarrow \exists y\ L(y) \wedge \neg A(x,y)$

2. No lecture was attended by every student.

   $\neg\exists x\ L(x) \wedge \forall y\ S(y) \Rightarrow A(y,x)$

   $\forall x\ \exists y\ L(x) \Rightarrow (S(y) \wedge \neg A(y,x))$

3. No lecture was attended by any student.

   $\neg\exists x\ L(x) \wedge \exists y\ S(y) \wedge A(y,x)$

   or $\forall x\ \forall y\ (L(x) \wedge S(y)) \Rightarrow \neg A(y,x)$

Problem 5.3.3: Sherlock Holmes and Professor Moriarty

Let the constant h stand for Holmes (Sherlock Holmes) and m for Moriarty (Professor Moriarty). Let us abbreviate "x can trap y" with the predicate $T(x, y)$. Give symbolic rendition of the following statements:

1. Holmes can trap everyone who can trap Moriarty.
2. Holmes can trap everyone whom Moriarty can trap.
3. Holmes can trap everyone who can be trapped by Moriarty.
4. If someone can trap Moriarty, then Holmes can.
5. If everyone can trap Moriarty, then Holmes can.
6. Anyone who can trap Holmes can trap Moriarty.
7. No one can trap Holmes unless that person can trap Moriarty.
8. Everyone can trap someone who cannot trap Moriarty.
9. Everyone who can trap Holmes can trap everyone whom Holmes can trap.

Problem 5.3.3: Sherlock Holmes and Professor Moriarty

$T(x, m)$

1. Holmes can trap everyone who can trap Moriarty.

$\forall x \quad T(x,m) \Rightarrow T(h,x)$

Problem 5.3.3: Sherlock Holmes and Professor Moriarty

$T(m,x)$

2. Holmes can trap everyone whom Moriarty can trap.

$\forall x \; T(m,x) \Rightarrow T(h,x)$

Problem 5.3.3: Sherlock Holmes and Professor Moriarty

3. Holmes can trap everyone who can be trapped by Moriarty. $T(m,x)$

$\forall x \; T(m,x) \Rightarrow T(h,x)$

↳ same as for 2

Problem 5.3.3: Sherlock Holmes and Professor Moriarty

$$\overbrace{\text{If someone can trap Moriarty}}^{X},\ \text{then}\ \overbrace{\text{Holmes can}}^{Y}. \qquad X \Rightarrow Y$$

4. If someone can trap Moriarty, then Holmes can.

$T(h,m)$

$\exists x \; T(x,m)$

$\rightarrow \quad [\exists x \; T(x,m)] \Rightarrow T(h,m)$

$[\neg \forall x \; \neg T(x,m)] \Rightarrow T(h,m)$

Problem 5.3.3: Sherlock Holmes and Professor Moriarty

5. If everyone can trap Moriarty, then Holmes can.

$$[\forall x \ T(x, m)] \Rightarrow T(h, m)$$

Problem 5.3.3: Sherlock Holmes and Professor Moriarty

6. Anyone who can trap Holmes can trap Moriarty.

$\forall x \; T(x, h) \Rightarrow T(x, m)$

Problem 5.3.3: Sherlock Holmes and Professor Moriarty

7. No one can trap Holmes unless (but not) that person can trap Moriarty.

$$\neg \exists x \; T(x,h) \wedge \neg T(x,m)$$

or $\forall x \; \neg (T(x,h) \wedge \neg T(x,m))$

$$\equiv \forall x \; \neg T(x,h) \vee T(x,m)$$

$$\equiv \forall x \; T(x,h) \Rightarrow T(x,m)$$

$$\equiv \forall x \; \neg T(x,m) \Rightarrow \neg T(x,h)$$

Problem 5.3.3: Sherlock Holmes and Professor Moriarty

$\forall$ $\exists$

8. Everyone can trap someone who cannot trap Moriarty.

$\forall x$ $\exists y \neg T(y,m)$

$\forall x \exists y \neg T(y,m) \wedge T(x,y)$

Problem 5.3.3: Sherlock Holmes and Professor Moriarty

9. Everyone who can trap Holmes can trap everyone whom Holmes can trap.

$\forall$ $\qquad$ $\forall$

$\forall y\ T(h,y) \Rightarrow \dots$

$\forall x\ T(x,h) \Rightarrow$

$$\forall x\ T(x,h) \Rightarrow \forall y\ T(h,y) \Rightarrow T(x,y)$$

$$\equiv \forall x,y\ T(x,h) \Rightarrow (T(h,y) \Rightarrow T(x,y))$$

$$\equiv \forall x,y\ (T(x,h) \wedge T(h,y)) \Rightarrow T(x,y)$$

$$f(x,y) : X \times Y \to D$$
$$: X \to (Y \to D)$$

$$A \Rightarrow (B \Rightarrow C) \longleftrightarrow (A \wedge B) \Rightarrow C$$

$$\neg A \vee (B \Rightarrow C) \qquad \neg(A \wedge B) \vee C$$

$$\neg A \vee \neg B \vee C \longleftrightarrow \neg A \vee \neg B \vee C$$

Intuition for the Resolution Rule

$\boxed{A \lor B} \land \boxed{\neg B \lor C}$

$\boxed{A \lor C}$

"If $A \lor B$ and $\neg B \lor C$, then also $A \lor C$."

$$\frac{A \lor B \qquad \neg B \lor C}{A \lor C}$$

Premises

Conclusion

But why? Let's do a case distinction on the truth assignment of B:

Case 1: B is assigned to True

- $A \lor B$ is satisfied, because B holds
- B holds $\leadsto$ $\neg B$ does not hold
- To satisfy $\neg B \lor C$ we have to assign C to True

Case 2: B is assigned to False

- $\neg B \lor C$ is satisfied, because $\neg B$ holds
- To satisfy $A \lor B$ we have to assign A to True

$\leadsto$ Since we cannot assign B to True and False, we must assign A to True or C to True

$\hookrightarrow$ $\underline{A \lor C}$ has to hold (assigning both A and B to True is also fine)