Problem 1

In classic geometry, a rotation is characterized as a length-preserving and orientation-preserving linear transformation. Starting from this characterization, it can also be shown that rotations preserve angles between vectors, which is a fact that you should use to solve the following problem.

The angle φ between two vectors $\vec{x}, \vec{y}$ in a standard euclidean space (with standard scalar product) can be defined through the relation:

$$\cos\varphi = \frac{\vec{x}^{\mathrm{T}}\vec{y}}{|\vec{x}| \cdot |\vec{y}|}$$

Using this definition, show that any rotation matrix R is orthonormal, or $R \cdot R^T = I_3$.

Problem 2

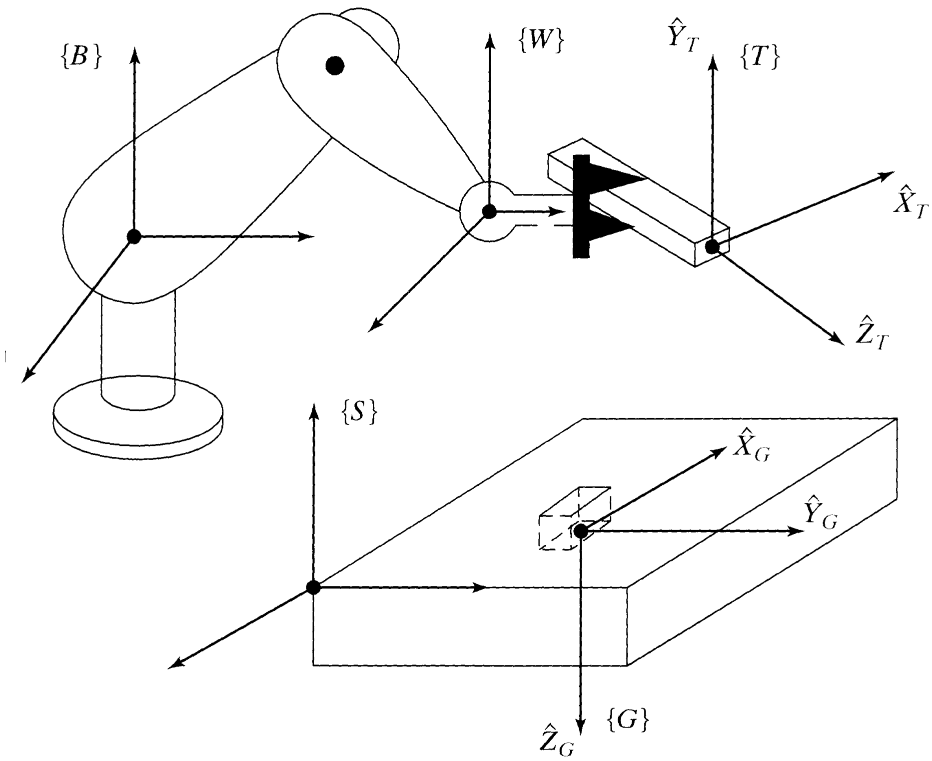


Figure 1: *Coordinate systems (Problem 2)*

Consider the situation shown in Figure 1. There are 5 coordinate frames

$$\{B\}, \{W\}, \{T\}, \{S\}, \{G\}.$$

We are interested in determining the transformation ${}^{W}_{T}\mathrm{T}$. This can be motivated as follows: Imagine that the robot has previously picked up the tool, and that the position

P1 let space matrix $V_A = (x_A, Y_A, Z_A)^T$ $\quad V_B = (x_B, Y_B, Z_B)^T$

$$V_B = R \cdot V_A$$

let $x_A = (1,0,0)^T$ $\quad Y_A = (0,1,0)^T$ $\quad Z_A = (0,0,1)^T$

$V_A = I$

$V_B = R \cdot V_A = R \cdot I = R$

so $x_B = R \cdot (1,0,0)^T$ $\quad Y_B = R \cdot (0,1,0)^T$ $\quad Z_B = R \cdot (0,0,1)^T$

$$\Rightarrow R = (x_B, Y_B, Z_B) = V_B$$

let unit vector $\vec{p}, \vec{q}$

$$\cos(\angle(\vec{p}, \vec{q})) = \vec{p}^T \cdot \vec{q} = \vec{q}^T \cdot \vec{p}$$
$$= (R \cdot \vec{p})^T \cdot (R \cdot \vec{q})$$
$$= \underset{1\times 3}{\vec{p}^T} \cdot \underset{3\times 3}{R^T} \cdot \underset{3\times 3}{R} \cdot \underset{3\times 1}{\vec{q}}$$

If we use coordinary A to discribe B

then we get ${}^A x_B = (x_B \cdot x_A \quad x_B \cdot Y_A \quad x_B \cdot Z_A)^T$

$$\text{so } {}^A_B R = ({}^A x_B, {}^A Y_B, {}^A Z_B)$$
$$= \begin{array}{c} \begin{pmatrix} x_B \cdot x_A & Y_B \cdot x_A & Z_B \cdot x_A \\ x_B \cdot Y_A & Y_B \cdot Y_A & Z_B \cdot Y_A \\ x_B \cdot Z_A & Y_B \cdot Z_A & Z_B \cdot Z_A \end{pmatrix} \end{array} \begin{array}{l} \to {}^B x_A \\ \to {}^B Y_A \\ \to {}^B Z_A \end{array}$$
$$= ({}^B x_A, {}^B Y_A, {}^B Z_A)^T$$
$$= {}^B_A R^T$$

0 1 0
1 0 0

$${}^A_B R^T \cdot {}^A_B R = \begin{pmatrix} {}^A x_B^T \\ {}^A Y_B^T \\ {}^A Z_B^T \end{pmatrix} \cdot ({}^A x_B, {}^A Y_B, {}^A Z_B) = \begin{pmatrix} {}^A x_B^T \cdot {}^A x_B & \cdots \\ {}^A Y_B^T \cdot {}^A x_B & \\ {}^A Z_B^T \cdot {}^A x_B & \end{pmatrix} = I_3$$

$$\therefore {}^A_B R^T = {}^A_B R^{-1}$$

$$\Rightarrow {}^A_B R = {}^B_A R^T = {}^B_A R^{-1}$$

P2 $\quad {}^B_S T \quad {}^S_G T / {}^S_T T \quad {}^B_W T$

$${}^3_0 T = {}^3_2 T \cdot {}^2_1 T \cdot {}^1_0 T$$

$${}^W \vec{p} = {}^W_T T \cdot {}^T \vec{p}$$

$${}^B \vec{p} = {}^B_S T \cdot {}^S_T T \cdot {}^T \vec{p}$$

$${}^B \vec{p} = {}^B_W T \cdot {}^W \vec{T}$$

$${}^B \cancel{\vec{p}} \cdot {}^B_W T^{-1} = {}^W_T T \cdot {}^B \cancel{\vec{p}} \cdot {}^S_B T^{-1} \, {}^T_S T^{-1}$$

$${}^B_W T^{-1} \cdot {}^B_S T \cdot {}^S_T T = {}^W_T T$$

of the tool in the gripper is not known very well, but we would like to know it exactly. However, the robot might be equipped with a force sensor and it could try to "feel" around to fit the tool to the goal. Assuming that the transformations ${}^{B}_{S}\mathrm{T}, {}^{S}_{G}\mathrm{T}, {}^{B}_{W}\mathrm{T}$ are known, and that the coordinate frames $\{G\}$ (goal) and $\{T\}$ (tool) are "calibrated", i.e., coincident, how can we determine the unknown transformation ${}^{W}_{T}\mathrm{T}$?

Problem 3

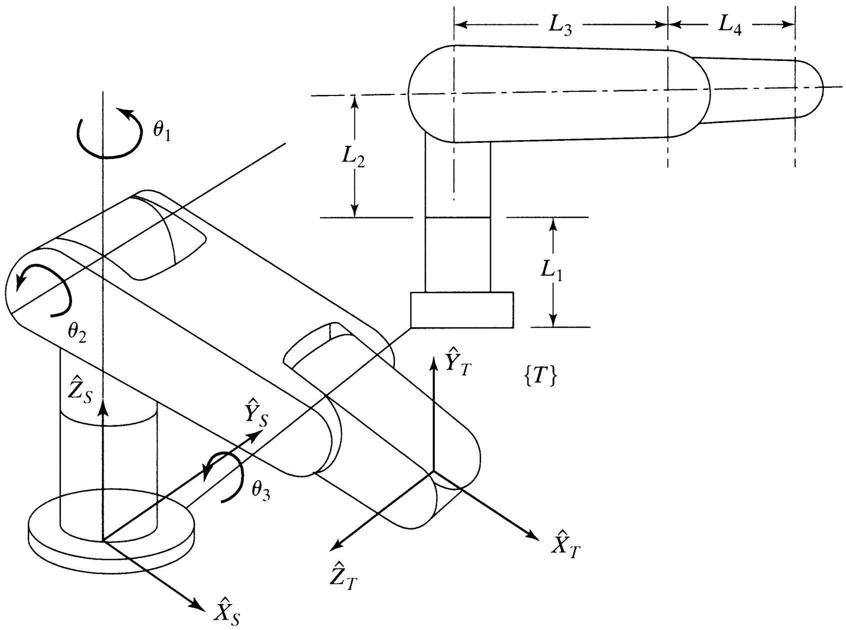


Figure 2: *Nonplanar 3R Robot (Problem* 3*)*

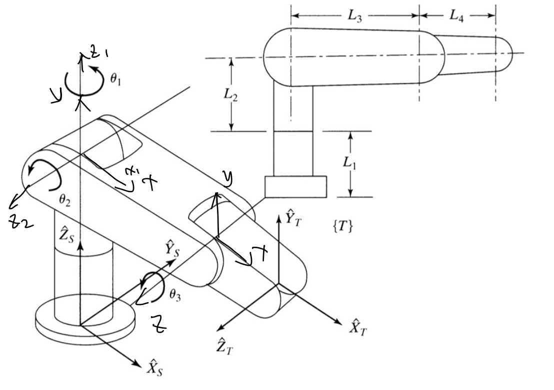
Figure 2 shows a robot arm with three rotational axes, thus three degrees of freedom. As shown in the diagram, all robot joints are in zero position. Determine the positions of the robot arm coordinate systems according to the Denavit-Hartenberg convention and derive the corresponding coordinate transformations.

P3

$${}^{i-1}\boldsymbol{A}_i = \boldsymbol{T}(0,0,d_i)\cdot\boldsymbol{R}(z,\theta_i)\cdot\boldsymbol{T}(a_i,0,0)\cdot\boldsymbol{R}(x,\alpha_i)$$

$${}^{i-1}\boldsymbol{A}_i = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&d_i\\0&0&0&1\end{pmatrix}\cdot\begin{pmatrix}C\theta_i&-S\theta_i&0&0\\S\theta_i&C\theta_i&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\cdot\begin{pmatrix}1&0&0&a_i\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0\\0&C\alpha_i&-S\alpha_i&0\\0&S\alpha_i&C\alpha_i&0\\0&0&0&1\end{pmatrix}$$

$${}^{i-1}\boldsymbol{A}_i = \begin{pmatrix}C\theta_i&-C\alpha_i\cdot S\theta_i&S\alpha_i\cdot S\theta_i&a_i\cdot C\theta_i\\S\theta_i&C\alpha_i\cdot C\theta_i&-S\alpha_i\cdot C\theta_i&a_i\cdot S\theta_i\\0&S\alpha_i&C\alpha_i&d_i\\0&0&0&1\end{pmatrix}$$



$${}^0A_1 = \begin{pmatrix}C\theta_1&-S\theta_1&0&0\\S\theta_1&C\theta_1&0&0\\0&0&1&l_1+l_2\\0&0&0&1\end{pmatrix}$$

J	θ_i	α_i	a_i	d_i
1	θ_1	0	0	l_1+l_2
2	θ_2	90	0	0
3	θ_3	0	L_3	0
4.	0	0	L_4	0

$${}^1A_2 = \begin{pmatrix}C\theta_2&0&S\theta_2&0\\S\theta_2&0&-C\theta_2&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$$

$${}^2A_3 = \begin{pmatrix}C\theta_3&-S\theta_3&0&l_3\cdot C\theta_3\\S\theta_3&C\theta_3&0&L_3\cdot S\theta_3\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$${}^0A_2 = \begin{pmatrix}C(\theta_1+\theta_2)&0&S(\theta_1+\theta_2)&0\\S(\theta_1+\theta_2)&0&-C(\theta_1+\theta_2)&0\\0&1&0&l_1+l_2\\0&0&0&1\end{pmatrix}$$

$${}^0A_3 \begin{pmatrix}C(\theta_1+\theta_2)\cdot C\theta_3&-C(\theta_1+\theta_2)\cdot S\theta_3&S(\theta_1+\theta_2)&L_3\cdot C(\theta_1+\theta_2)\cdot C\theta_3\\S(\theta_1+\theta_2)\cdot C\theta_3&-S(\theta_1+\theta_2)\cdot S\theta_3&-C(\theta_1+\theta_2)&L_3\ S(\theta_1+\theta_2)\cdot S\theta_3\\S\theta_3&C\theta_3&0&L_3\ S\theta_3+L_1+L_2\\0&0&0&1\end{pmatrix}$$

$${}^{i-1}T = \begin{bmatrix}c\theta_i&-s\theta_i&0&a_{i-1}\\s\theta_i c\alpha_{i-1}&c\theta_i c\alpha_{i-1}&-s\alpha_{i-1}&-s\alpha_{i-1}d_i\\s\theta_i s\alpha_{i-1}&c\theta_i s\alpha_{i-1}&c\alpha_{i-1}&c\alpha_{i-1}d_i\\0&0&0&1\end{bmatrix}$$

$${}^0_1T = \begin{pmatrix}C\theta_1&-S\theta_1&0&0\\S\theta_1&C\theta_1&0&0\\0&0&1&L_1+L_2\\0&0&0&1\end{pmatrix}$$

$${}^0_2T = \begin{pmatrix}C\theta_1\cdot C\theta_2&-C\theta_1\cdot S\theta_2&S\theta_1&0\\S\theta_1\cdot C\theta_2&-S\theta_1\cdot S\theta_2&-C\theta_1&0\\S\theta_2&C\theta_2&0&L_1+L_2\\0&0&0&1\end{pmatrix}$$

$${}^1_2T = \begin{pmatrix}C\theta_2&-S\theta_2&0&0\\0&0&-1&0\\S\theta_2&C\theta_2&0&0\\0&0&0&1\end{pmatrix}$$

$${}^0_3T = \begin{pmatrix}C\theta_1\cdot C\theta_2\cdot C\theta_3-C\theta_1 S\theta_2\cdot S\theta_3&-C\theta_1 C\theta_2\cdot S\theta_3-C\theta_1 S\theta_2\cdot C\theta_3&S\theta_1&L_3\cdot C\theta_1 C\theta_2\\S\theta_1\cdot C\theta_2\cdot S\theta_3-S\theta_1 S\theta_2\cdot S\theta_3&-S\theta_1\cdot C\theta_2\cdot S\theta_3-S\theta_1 S\theta_2 C\theta_3&-C\theta_1&L_3 S\theta_1\cdot C\theta_2\\S\theta_2\cdot C\theta_3+C\theta_2\cdot S\theta_3&-S\theta_2\cdot S\theta_3+C\theta_2\cdot C\theta_3&0&L_3\cdot S\theta_2+L_1+L_2\\0&0&0&1\end{pmatrix}$$

$${}^2_3T = \begin{pmatrix}C\theta_3&-S\theta_3&0&L_3\\S\theta_3&C\theta_3&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$${}^0_4T = \begin{pmatrix}C\theta_1\cdot C\theta_2\cdot C\theta_3-C\theta_1 S\theta_2\cdot S\theta_3&-C\theta_1 C\theta_2\cdot S\theta_3-C\theta_1 S\theta_2\cdot C\theta_3&S\theta_1&L_4\cdot C\theta_1\cdot C(\theta_2+\theta_3)+L_3\cdot C\theta_1\cdot C\theta_2\\S\theta_1\cdot C\theta_2\cdot S\theta_3-S\theta_1 S\theta_2\cdot S\theta_3&-S\theta_1\cdot C\theta_2\cdot S\theta_3-S\theta_1 S\theta_2 C\theta_3&-C\theta_1&L_4\cdot S\theta_1\ C(\theta_2+\theta_3)+L_3\cdot S\theta_1\cdot C\theta_2\\S\theta_2\cdot C\theta_3+C\theta_2\cdot S\theta_3&-S\theta_2\cdot S\theta_3+C\theta_2\cdot C\theta_3&0&L_4\cdot S(\theta_2+\theta_3)+L_3\cdot S\theta_2\cdot L_1+L_2\\0&0&0&1\end{pmatrix}$$

$${}^3_4T = \begin{pmatrix}1&0&0&L_4\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

Problem 4

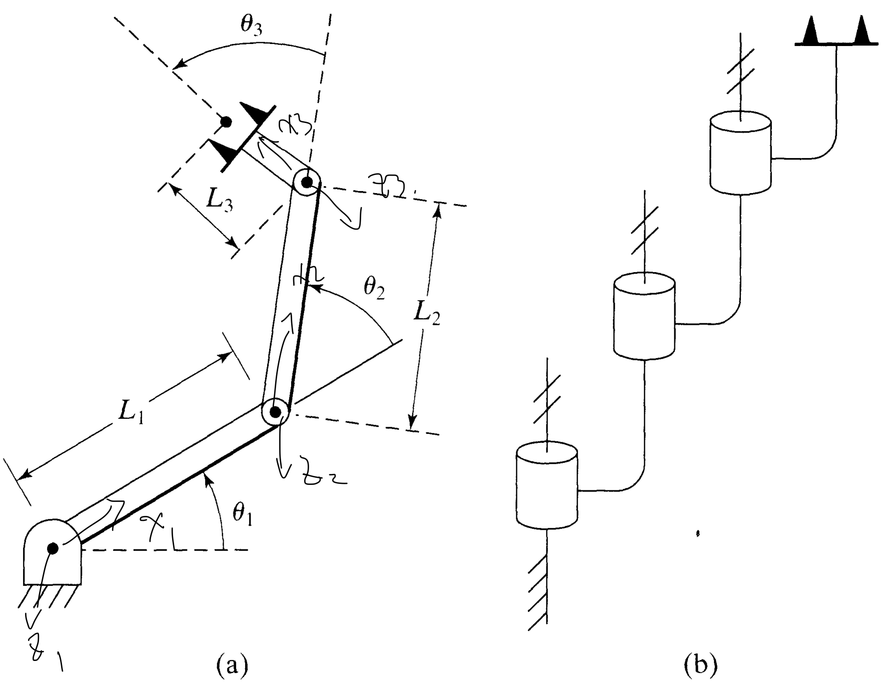


Figure 3: *Planar 3R Robot (Problem 4)*

Consider the planar 3R robot shown in Figure 3. All axes of rotation are parallel, as indicated by the hash marks.

a) Identify the DH parameters of this system.

b) Determine its forward kinematics.

c) Given a position (cartesian coordinates of gripper center and gripper angle), can you think of a simple way to find out if such a position is reachable for the robot?

?

a)

j	θ_i	α_i	a_i	d_i
1	θ_1	0	0	0
2	θ_2	0	L_1	0
3	θ_3	0	L_2	0
4	0	0	L_3	0

$$^0_1T = \begin{pmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$^1_2T = \begin{pmatrix} c\theta_2 & -s\theta_2 & 0 & L_1 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$^2_3T = \begin{pmatrix} c\theta_3 & -s\theta_3 & 0 & L_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$^3_4T = \begin{pmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_1 \cdot c\theta_1 \\ L_1 \cdot s\theta_1 \end{pmatrix} + \begin{pmatrix} L_2 \cdot c(\theta_1+\theta_2) \\ L_2 \; s(\theta_1+\theta_2) \end{pmatrix} + \begin{pmatrix} L_3 \cdot c(\theta_1+\theta_2+\theta_3) \\ L_3 \cdot s(\theta_1+\theta_2+\theta_3) \end{pmatrix}$$

Problem 5

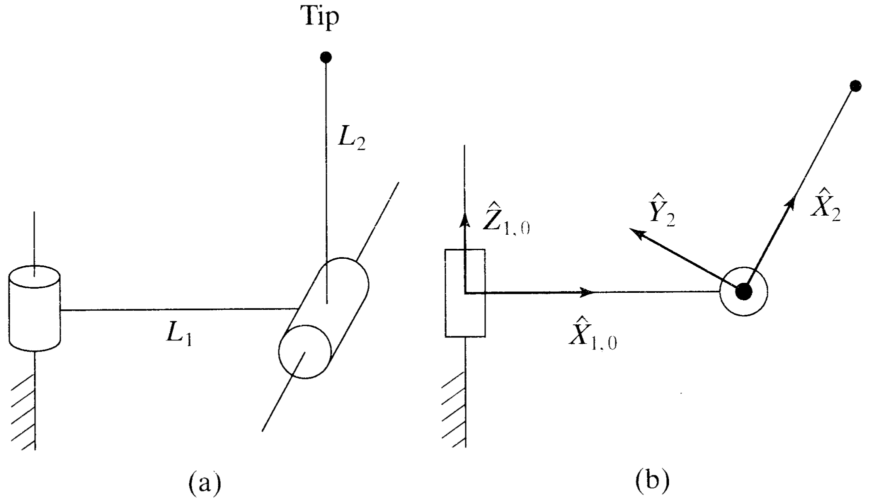


Figure 4: *2R Robot, problem 5*

Consider the 2R robot shown in Figure 4. The link transformation matrices have been determined, and the coordinate transformation for the second link is:

$$
{}^0_2\mathrm{T} = \begin{pmatrix} c_1c_2 & -c_1s_2 & s_1 & l_1c_1 \\ s_1c_2 & -s_1s_2 & -c_1 & l_1s_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

The link-frame assignments are shown in Figure 4. Note that the frames $\{0\}$ and $\{1\}$ are coincident when $\Theta_1 = 0$. The length of the second link is l_2. Find an expression for the vector ${}^0P_{\text{tip}}$, which locates the tip of the arm relative to the $\{0\}$ frame.

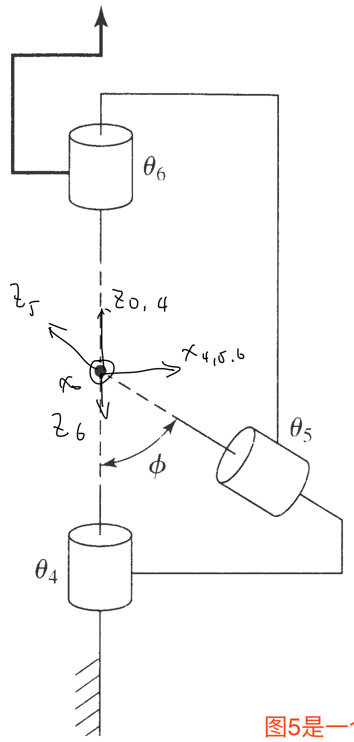
$$
{}^0P_{tip} = {}^0_2T \cdot {}^2P_{tip}
$$

$$
{}^2P_{tip} = \begin{pmatrix} L_2 \\ 0 \\ 0 \\ 1 \end{pmatrix}
$$

$$
\Rightarrow {}^0P_{tip} = \begin{pmatrix} C_1C_2 & -C_1S_2 & S_1 & L_1C_1 \\ S_1C_2 & -S_1S_2 & -C_1 & L_1S_1 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} L_2 \\ 0 \\ 0 \\ 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} C_1C_2 \cdot L_2 + L_1 \cdot C_1 \\ S_1C_2L_2 + L_1S_1 \\ S_2 \cdot L_2 \\ 1 \end{pmatrix}
$$

Problem 6



J	θ_i	α_i	a_i	d_i
6	0	0	0	0.
4	-90.	0	0	0.
5	0	ϕ	0	0.
6	6	(90-ϕ)	0	0

图5是一个有三个非正交轴的手腕的示意图。给手腕分配链接框架（就像一个普通的3R操纵器一样），并给出链接参数。

Figure 5: *3R nonorthogonal-axis robot (problem 6)*

Figure 5 shows a schematic of a wrist that has three intersecting axes that are not orthogonal. Assign link frames to the wrist (as if it were a regular 3R manipulator) and give the link parameters.

Problem 7

The formula for determining the transformation between consecutive frames within the Denavit-Hartenberg-Convention is given as:

$$
{}^{i-1}_{\ i}\mathrm{T} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_{i-1} \\ \sin\theta_i\cos\alpha_{i-1} & \cos\theta_i\cos\alpha_{i-1} & -\sin\alpha_{i-1} & -\sin\alpha_{i-1}d_i \\ \sin\theta_i\sin\alpha_{i-1} & \cos\theta_i\sin\alpha_{i-1} & \cos\alpha_{i-1} & \cos\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

Is is possible to express arbitrary rigid body transformations this way?

是否可以用这种方式来表达任意刚体变换?

$p = (x, y, z, \alpha, \beta, \gamma)$

In DH, there is only 4 parameters are discribed: x Rotation, x Transfon, z Rotation and z Transform.

$\Rightarrow$ So, no.

Problem 8

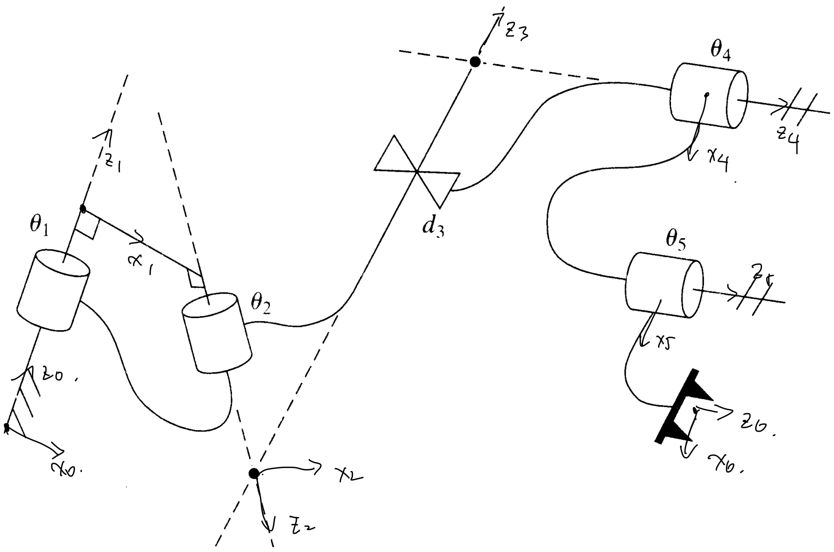


Figure 6: *Schematic of a 2RP2R manipulator (problem 8).*

Show the attachment of link frames for the manipulator shown in Figure 6.

Problem 9

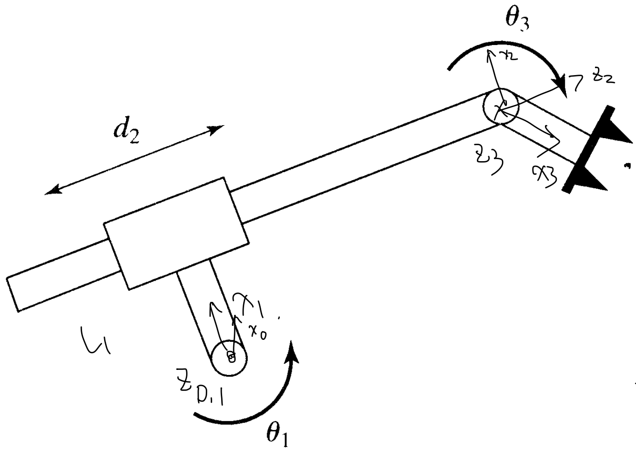
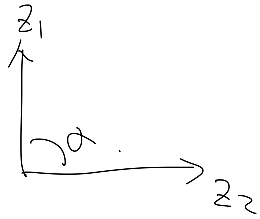


Figure 7: *RPR planar robot (problem 9)*

J	θ_i	α_i	a_i	d_i
0	0	0	0	0
1	θ_1	0	0	0
2.	0	90	4	d_2
3	θ_3	90	0	0

Assign link frames to the robot shown in Figure 7, and give approximate linkage parameters.



Problem 10

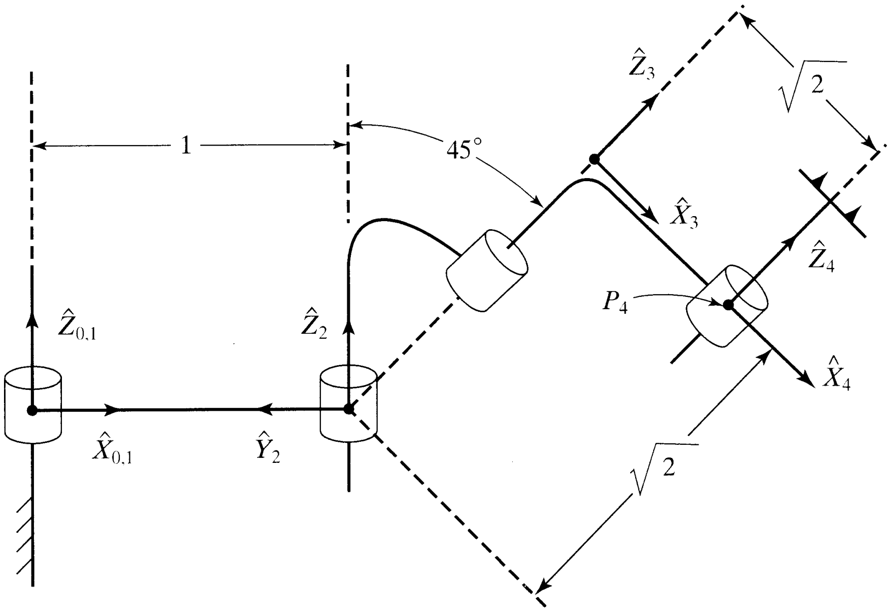


Figure 8: *4R Robot (Problem 10)*

For the 4R manipulator shown in Figure 8, we have the following set of DH parameters:

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0°	0	θ_1
2	1	0°	0	θ_2
3	0	45°	$\sqrt{2}$	θ_3
4	$\sqrt{2}$	0°	0	0

The manipulator is shown for the joint configuration

$$\Theta = [0°, 90°, -90°, 0°].$$

We ignore collisions between arms, so we assume that all joints can move freely within the range $[-180°, 180°]$. Determine values of θ_3 such that positions

$$^0\mathrm{P}_{4\mathrm{ORG}} = [\bullet, \bullet, 1.707]^T$$

are reachable for the fourth frame.

P10

$$ {}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{1}T = \begin{pmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{0}_{2}T = \begin{pmatrix} c(\theta_1+\theta_2) & -s(\theta_1+\theta_2) & 0 & c\theta_1 \\ s(\theta_1+\theta_2) & c(\theta_1+\theta_2) & 0 & s\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{1}_{2}T = \begin{pmatrix} c\theta_2 & -s\theta_2 & 0 & 1 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{0}_{3}T = \begin{pmatrix} c(\theta_1+\theta_2)\cdot c\theta_3 - \frac{\sqrt{2}}{2} s(\theta_1+\theta_2)\cdot s\theta_3 & -c(\theta_1+\theta_2)\cdot s\theta_3 - \frac{\sqrt{2}}{2} s(\theta_1+\theta_2)\cdot c\theta_3 & \frac{\sqrt{2}}{2} s(\theta_1+\theta_2) & s(\theta_1+\theta_2)+c\theta_1 \\ s(\theta_1+\theta_2)c\theta_3 + \frac{\sqrt{2}}{2} c(\theta_1+\theta_2)s\theta_3 & -s(\theta_1+\theta_2)\cdot s\theta_3 + \frac{\sqrt{2}}{2} c(\theta_1+\theta_2)\cdot c\theta_3 & -\frac{\sqrt{2}}{2} c(\theta_1+\theta_2) & -c(\theta_1+\theta_2)+s\theta_1 \\ \frac{\sqrt{2}}{2} s\theta_3 & \frac{\sqrt{2}}{2} c\theta_3 & \frac{\sqrt{2}}{2} & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{2}_{3}T = \begin{pmatrix} c\theta_3 & -s\theta_3 & 0 & 0 \\ \frac{\sqrt{2}}{2} s\theta_3 & \frac{\sqrt{2}}{2} c\theta_3 & -\frac{\sqrt{2}}{2} & -1 \\ \frac{\sqrt{2}}{2} s\theta_3 & \frac{\sqrt{2}}{2} c\theta_3 & \frac{\sqrt{2}}{2} & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{3}_{4}T = \begin{pmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{0}_{4}T = \begin{pmatrix} c(\theta_1+\theta_2)\cdot c\theta_3 - \frac{\sqrt{2}}{2} s(\theta_1+\theta_2)\cdot s\theta_3 & -c(\theta_1+\theta_2)\cdot s\theta_3 - \frac{\sqrt{2}}{2} s(\theta_1+\theta_2)\cdot c\theta_3 & \frac{\sqrt{2}}{2} s(\theta_1+\theta_2) & \sqrt{2}\, c(\theta_1+\theta_2)\cdot c(\theta_3) - s(\theta_1+\theta_2)\cdot s\theta_3 + s(\theta_1+\theta_2)+c\theta_1 \\ s(\theta_1+\theta_2)c\theta_3 + \frac{\sqrt{2}}{2} c(\theta_1+\theta_2)s\theta_3 & -s(\theta_1+\theta_2)\cdot s\theta_3 + \frac{\sqrt{2}}{2} c(\theta_1+\theta_2)\cdot c\theta_3 & -\frac{\sqrt{2}}{2} c(\theta_1+\theta_2) & \sqrt{2}\, s(\theta_1+\theta_2)\cdot c(\theta_3) + c(\theta_1+\theta_2)\cdot s\theta_3 - c(\theta_1+\theta_2)+s\theta_1 \\ \frac{\sqrt{2}}{2} s\theta_3 & \frac{\sqrt{2}}{2} c\theta_3 & \frac{\sqrt{2}}{2} & s\theta_3+1 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{0}P_{4ORG} = {}^{0}_{4}T \cdot {}^{4}P_{4ORG} $$

$$ 1+\frac{\sqrt{2}}{2} \qquad \begin{pmatrix} - \\ - \\ 1+\frac{\sqrt{2}}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} $$

$$ s\theta_3 + 1 = 1 + \frac{\sqrt{2}}{2} $$

$$ \theta_3 = 40^\circ \text{ or } 120^\circ $$