FUNDAMENTALS OF ARTIFICIAL INTELLIGENCE

Exercise 8: Bayesian networks – Solutions — Winter Semester 2023/24

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Problem 8.1:

a. ii, iii, iv and v (see slides 29-34 of lecture 9).
   For v., consider the Markov blanket of D.
   Remember that a node in the Bayesian network is conditionally independent from all other nodes given its parents, children and children's parents (this evidence is the **Markov blanket** of the node).

b. $\mathbf{P}(D,TP,CC,UP,PP,W) = \mathbf{P}(D)\mathbf{P}(TP)\mathbf{P}(CC)\mathbf{P}(UP|D,TP)\mathbf{P}(PP|CC,UP)\mathbf{P}(W|CC,UP,PP)$.

c. To calculate these values we can directly use the equation derived in **b.**. In fact, we have to evaluate it using the requested values of the random variables:
   $P(\neg d,tp,cc,\neg up,pp,w) = P(\neg d)P(tp)P(cc)P(\neg up|\neg d,tp)P(pp|cc,\neg up)P(w|cc,\neg up,pp) =$
   $= 0.75\cdot 0.35\cdot 0.3\cdot 0.9\cdot 0.9\cdot 0.9 = 0.05740875$.
   $P(\neg d,tp,cc,\neg up,\neg pp,w) = P(\neg d)P(tp)P(cc)P(\neg up|\neg d,tp)P(\neg pp|cc,\neg up)P(w|cc,\neg up,\neg pp) =$
   $= 0.75\cdot 0.35\cdot 0.3\cdot 0.9\cdot 0.1\cdot 0.5 = 0.00354375$.

d. The required probability is $P(w|tp,\neg d,pp)$. To calculate this, we can use enumeration.
   The conditional probability can be rewritten as (see slide 13, lecture 9):
   $P(w|tp,\neg d,pp) = \frac{P(w,tp,\neg d,pp)}{P(tp,\neg d,pp)} = \alpha P(w,tp,\neg d,pp)$
   where the normalization constant α can be computed with
   $\alpha = \frac{1}{P(tp,\neg d,pp)} = \frac{1}{P(w,tp,\neg d,pp)+P(\neg w,tp,\neg d,pp)}$.

   To compute $\mathbf{P}(W,tp,\neg d,pp)$, we use the joint probability from **b.** and sum over the unknown variables UP and CC.:
   $\mathbf{P}(W,tp,\neg d,pp) = \sum_{UP}\sum_{CC} P(\neg d)P(tp)\mathbf{P}(UP|\neg d,tp)\mathbf{P}(CC)\mathbf{P}(pp|CC,UP)\mathbf{P}(W|CC,UP,pp)$.

   Moving the probabilities which do not depend on UP and CC out of the corresponding sum leads to:
   $\mathbf{P}(W,tp,\neg d,pp) = P(\neg d)P(tp)\sum_{UP}\mathbf{P}(UP|\neg d,tp)\sum_{CC}\mathbf{P}(CC)\mathbf{P}(pp|CC,UP)\mathbf{P}(W|CC,UP,pp)$.
   For $W = w$, we obtain:

$$\begin{aligned}
P(w,tp,\neg d,pp) =& P(\neg d)P(tp)\{P(up|\neg d,tp)[P(cc)P(pp|cc,up)P(w|cc,up,pp) \\
&+P(\neg cc)P(pp|\neg cc,up)P(w|\neg cc,up,pp)] \\
&+P(\neg up|\neg d,tp)[P(cc)P(pp|cc,\neg up)P(w|cc,\neg up,pp) \\
&+P(\neg cc)P(pp|\neg cc,\neg up)P(w|\neg cc,\neg up,pp)]\} \\
=& 0.75\cdot 0.35\{0.1[0.3\cdot 0.5\cdot 0.4+0.7\cdot 0.05\cdot 0.05]+0.9[0.3\cdot 0.9\cdot 0.9+0.7\cdot 0.1\cdot 0.15]\} \\
\approx& 0.0615.
\end{aligned}$$

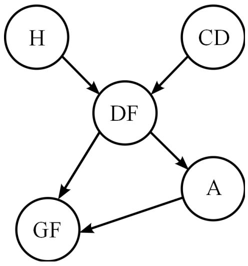
   For $W = \neg w$, we have:

$$\begin{aligned}
P(\neg w,tp,\neg d,pp) =& P(\neg d)P(tp)\{P(up|\neg d,tp)[P(cc)P(pp|cc,up)P(\neg w|cc,up,pp) \\
&+P(\neg cc)P(pp|\neg cc,up)P(\neg w|\neg cc,up,pp)] \\
&+P(\neg up|\neg d,tp)[P(cc)P(pp|cc,\neg up)P(\neg w|cc,\neg up,pp) \\
&+P(\neg cc)P(pp|\neg cc,\neg up)P(\neg w|\neg cc,\neg up,pp)]\} \\
=& 0.75\cdot 0.35\{0.1[0.3\cdot 0.5\cdot 0.6+0.7\cdot 0.05\cdot 0.95]+0.9[0.3\cdot 0.9\cdot 0.1+0.7\cdot 0.1\cdot 0.85]\} \\
\approx& 0.0237.
\end{aligned}$$

   This leads to
   $\alpha = \frac{1}{P(w,tp,\neg d,pp)+P(\neg w,tp,\neg d,pp)} = \frac{1}{0.0615+0.0237} \approx 11.7371$,
   $P(w|tp,\neg d,pp) = \alpha P(w,tp,\neg d,pp) \approx 11.7371\cdot 0.0615 \approx 0.7218$.

Problem 8.2:

a. The corresponding network is the following:



b. Using enumeration:

$$\mathbf{P}(CD|\neg a, gf) = \alpha \mathbf{P}(CD) \sum_{DF} \mathbf{P}(gf|\neg a, DF)\, \mathbf{P}(\neg a|DF) \sum_{H} \mathbf{P}(DF|CD,H)\, \mathbf{P}(H).$$

$$\begin{aligned} P(cd|\neg a, gf) &= \alpha P(cd)\{P(gf|df,\neg a)P(\neg a|df)[P(df|cd,h)P(h) + P(df|cd,\neg h)P(\neg h)] \\ &\quad + P(gf|\neg df,\neg a)P(\neg a|\neg df)[P(\neg df|cd,h)P(h) + P(\neg df|cd,\neg h)P(\neg h)]\} \\ &= \alpha \cdot 0.6\{0.4\cdot 0.3[0.15\cdot 0.5 + 0.01\cdot 0.5] + 0.05\cdot 0.75[0.85\cdot 0.5 + 0.99\cdot 0.5]\} \\ &\approx \alpha \cdot 0.0265, \end{aligned}$$

$$\begin{aligned} P(\neg cd|\neg a, gf) &= \alpha P(\neg cd)\{P(gf|df,\neg a)P(\neg a|df)[P(df|\neg cd,h)P(h) + P(df|\neg cd,\neg h)P(\neg h)] \\ &\quad + P(gf|\neg df,\neg a)P(\neg a|\neg df)[P(\neg df|\neg cd,h)P(h) + P(\neg df|\neg cd,\neg h)P(\neg h)]\} \\ &= \alpha \cdot 0.4\{0.4\cdot 0.3[0.99\cdot 0.5 + 0.1\cdot 0.5] + 0.05\cdot 0.75[0.01\cdot 0.5 + 0.9\cdot 0.5]\} \\ &\approx \alpha \cdot 0.0330, \end{aligned}$$

$\alpha = \frac{1}{0.0265+0.0330} \approx 16.8067,$

$\mathbf{P}(CD|\neg a, gf) \approx 16.8067 \cdot \begin{bmatrix} 0.0265 \\ 0.0330 \end{bmatrix} \approx \begin{bmatrix} 0.4454 \\ 0.5546 \end{bmatrix}.$

c. Using variable elimination:

$$\mathbf{P}(CD|\neg a, gf) = \alpha \underbrace{P(CD)}_{f_1(CD)} \sum_{DF} \underbrace{P(gf|\neg a, DF)}_{f_2(DF)} \underbrace{P(\neg a|DF)}_{f_3(DF)} \sum_{H} \underbrace{P(DF|CD,H)}_{f_4(DF,CD,H)} \underbrace{P(H)}_{f_5(H)}.$$

In the following notation we represent a $2\times 2\times 2$ matrix as $\{\,[2\times 2]\ [2\times 2]\,\}$.

$$\begin{aligned} \mathbf{f_4}(DF,CD,H) &= \{\ \mathbf{P}(DF|CD,h) \quad \mathbf{P}(DF|CD,\neg h)\ \} \\ &= \left\{ \begin{bmatrix} P(df|cd,h) & P(df|\neg cd,h) \\ P(\neg df|cd,h) & P(\neg df|\neg cd,h) \end{bmatrix} \begin{bmatrix} P(df|cd,\neg h) & P(df|\neg cd,\neg h) \\ P(\neg df|cd,\neg h) & P(\neg df|\neg cd,\neg h) \end{bmatrix} \right\} \\ &= \left\{ \begin{bmatrix} 0.15 & 0.99 \\ 0.85 & 0.01 \end{bmatrix} \begin{bmatrix} 0.01 & 0.1 \\ 0.99 & 0.9 \end{bmatrix} \right\}, \end{aligned}$$

$\mathbf{f_1}(CD) = \begin{bmatrix} P(cd) & P(\neg cd) \end{bmatrix} = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix},\ \mathbf{f_2}(DF) = \begin{bmatrix} P(gf|\neg a, df) \\ P(gf|\neg a,\neg df) \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.05 \end{bmatrix},$

$\mathbf{f_3}(DF) = \begin{bmatrix} P(\neg a|df) \\ P(\neg a|\neg df) \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.75 \end{bmatrix},\ \mathbf{f_5}(H) = \{\begin{bmatrix} P(h) \end{bmatrix} \begin{bmatrix} P(\neg h) \end{bmatrix}\} = \{\begin{bmatrix} 0.5 \end{bmatrix} \begin{bmatrix} 0.5 \end{bmatrix}\}.$

After defining the factors, we can write the conditional probability as:
$\mathbf{P}(CD|\neg a, gf) = \alpha \mathbf{f_1}(CD) \times \sum_{DF} \mathbf{f_2}(DF) \times \mathbf{f_3}(DF) \times \sum_{H} \mathbf{f_4}(DF,CD,H) \times \mathbf{f_5}(H).$

Going from right to left, we first use the pointwise product to compute $f_4 \times f_5$:
$\mathbf{f_4}(DF,CD,H) \times \mathbf{f_5}(H) = \left\{ \begin{bmatrix} 0.075 & 0.495 \\ 0.425 & 0.005 \end{bmatrix} \begin{bmatrix} 0.005 & 0.050 \\ 0.495 & 0.450 \end{bmatrix} \right\},$

Next, we compute the sum over H, i.e., we add the elements which correspond to the same values of DF and CD but different values of H (e.g., we add the element corresponding to df, $\neg cd$ and h to the element corresponding to df, $\neg cd$ and $\neg h$):

$$\mathbf{f_6}(DF,CD) = \sum_H \mathbf{f_4}(DF,CD,H) \times \mathbf{f_5}(H) = \sum_H \left\{ \begin{bmatrix} 0.075 & 0.495 \\ 0.425 & 0.005 \end{bmatrix} \begin{bmatrix} 0.005 & 0.050 \\ 0.495 & 0.450 \end{bmatrix} \right\} = \begin{bmatrix} 0.080 & 0.545 \\ 0.920 & 0.455 \end{bmatrix},$$

We compute the pointwise product of f_2, f_3 and the new factor f_6:

$$\mathbf{f_2}(DF) \times \mathbf{f_3}(DF) \times \mathbf{f_6}(DF,CD) = \begin{bmatrix} 0.0096 & 0.0654 \\ 0.0345 & 0.0171 \end{bmatrix},$$

Summing over DF leads to:

$$\mathbf{f_7}(CD) = \sum_{DF} \mathbf{f_2}(DF) \times \mathbf{f_3}(DF) \times \mathbf{f_6}(DF,CD) = \sum_{DF} \begin{bmatrix} 0.0096 & 0.0654 \\ 0.0345 & 0.0171 \end{bmatrix} = \begin{bmatrix} 0.0441 & 0.0825 \end{bmatrix}.$$

And finally, we obtain:

$\mathbf{P}(CD|\neg a, gf) = \alpha \mathbf{f_1}(CD) \times \mathbf{f_7}(CD) \approx \alpha \begin{bmatrix} 0.0265 & 0.0330 \end{bmatrix}$,

$\alpha = \frac{1}{0.0265+0.0330} \approx 16.8067$,

$\mathbf{P}(CD|\neg a, gf) \approx 16.8067 \cdot \begin{bmatrix} 0.0265 & 0.0330 \end{bmatrix} \approx \begin{bmatrix} 0.4454 & 0.5546 \end{bmatrix}$.

d. Besides the last calculation to complete the normalization that both methods require, we have:

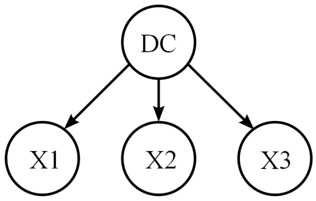
	Enumeration	Variable Elimination
additions	6	6
multiplications	18	16

In this case we do not have a significant improvement using variable elimination. The improvement becomes more significant for larger Bayesian networks.

Problem 8.3:

a. The network is composed of four random variables: The coin that has been drawn DC with domain $< a, b, c >$ and the results of the three flips $X1$, $X2$ and $X3$ with domains $< heads, tails >$.

$P(DC = a)$	$P(DC = b)$	$P(DC = c)$
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$



DC	$\mathbf{P}(X1 = heads \mid DC)$
a	0.2
b	0.6
c	0.8

DC	$\mathbf{P}(X2 = heads \mid DC)$
a	0.2
b	0.6
c	0.8

DC	$\mathbf{P}(X3 = heads \mid DC)$
a	0.2
b	0.6
c	0.8

b. The required probability is $\mathbf{P}(DC|X1 = heads, X2 = heads, X3 = tails)$.
We first write the full joint probability:
$\mathbf{P}(DC, X1, X2, X3) = \mathbf{P}(X1|DC)\,\mathbf{P}(X2|DC)\,\mathbf{P}(X3|DC)\,\mathbf{P}(DC)$.
The result is directly obtained introducing the evidence and using normalization:

$$\mathbf{P}(DC|X1 = heads, X2 = heads, X3 = tails) =$$
$$= \alpha \mathbf{P}(X1 = heads|DC)\,\mathbf{P}(X2 = heads|DC)\,\mathbf{P}(X3 = tails|DC)\,\mathbf{P}(DC) =$$
$$= \alpha \begin{bmatrix} 0.2 \\ 0.6 \\ 0.8 \end{bmatrix} \times \begin{bmatrix} 0.2 \\ 0.6 \\ 0.8 \end{bmatrix} \times \begin{bmatrix} 0.8 \\ 0.4 \\ 0.2 \end{bmatrix} \times \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \approx \alpha \begin{bmatrix} 0.0107 \\ 0.0480 \\ 0.0427 \end{bmatrix},$$

$$\alpha = \frac{1}{(0.0107 + 0.0480 + 0.0427)} \approx 9.8619329,$$

$$\mathbf{P}(DC|X1 = heads, X2 = heads, X3 = tails) \approx 9.8619329 \begin{bmatrix} 0.0107 \\ 0.0480 \\ 0.0427 \end{bmatrix} = \begin{bmatrix} 0.1055 \\ 0.4734 \\ 0.4211 \end{bmatrix}.$$

The coin that is most likely to have been drawn is b.