Precision at next-to-leading order: Refining the NSC Subtraction Scheme with θ-Parameters

Bachelor's Degree in Physics

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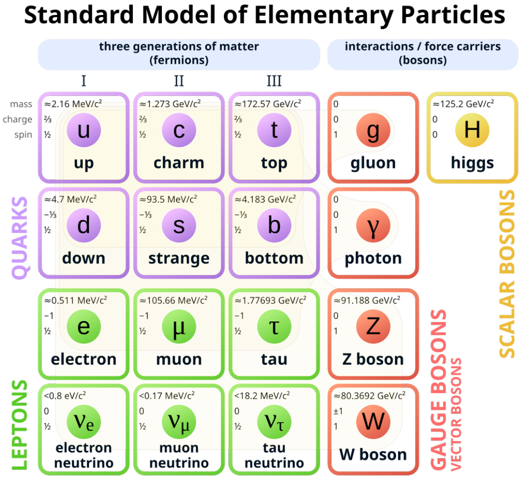
UNIVERSITÀ DEGLI STUDI DI MILANO

The Standard Model of Particle Physics

State of the art



High energy $\rightarrow$ Special Relativity
Small particles $\rightarrow$ Quantum Mechanics

$\downarrow$

Quantum Field Theory
(mathematical framework)

Evidences for physics beyond the SM

- Dark matter and dark energy
- Matter-antimatter asymmetry
- Origin of neutrino masses

Collider Physics
One of the principal methods of research

Large **H**adron **C**ollider at CERN
Proton beams accelerated to ~ 13.6 TeV

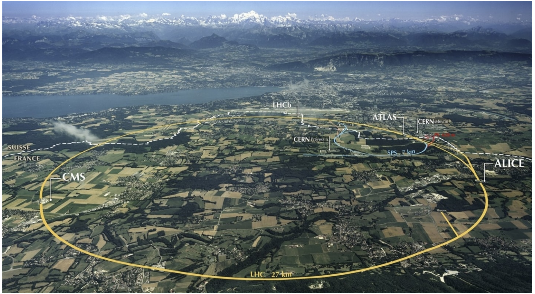


Figure: Maximilien Brice/CERN

Not sufficient to the discovery of new particles.

Solutions:

- Increase energy
  $\to$ not feasible or extremely expensive with existing technology
- Increase precision
  $\to$ both experimental and **theoretical**

Quantum Chromodynamics
Hard scattering processes

- Strong interactions are described by Quantum Chromodynamics (QCD).
- Non-Abelian gauge theory, $SU(3)$ symmetry group.
- The QCD Lagrangian is not analytically solvable.

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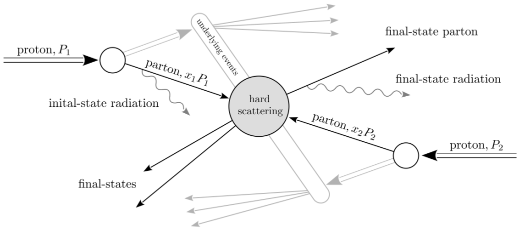


Figure: Konstantin Asteriadis/KIT

Hadron collisions

- Elastic scattering
- Diffractive dissociation
- **Hard scattering**
  (momentum exchange $\sim 100\,\text{GeV}$)

Asymptotic freedom

Interacting partons can be approximated as being nearly free

$\rightarrow$ **Perturbative description**

The collinear factorization theorem

A framework for describing hard scattering processes

Colliding hadrons are treated as **beams of partons**, carrying a fraction of the hadron's total momentum

Separation of energy scales

- SM interactions $Q \sim 100\text{GeV} - 1\text{TeV}$
- Hadronic structure $\Lambda_{\text{QCD}} \sim 100\text{MeV}$

$\rightarrow$ **Decoupling** the motions of partons from proton's dynamics

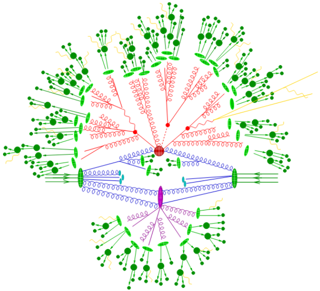


Figure: Stefan Höche/SLAC National Accelerator Laboratory

$$\mathrm{d}\sigma = \sum_{a,b} \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 f_a(x_1, \mu_\mathrm{F}) f_b(x_2, \mu_\mathrm{F})\, \mathrm{d}\hat{\sigma}_{a,b}(x_1, x_2, \mu_\mathrm{F}, \mu_\mathrm{R}; \mathcal{O}) \left(1 + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{Q}\right)^n\right), \; n \geq 1$$

The partonic cross section

A perturbative description

The partonic cross section can be expanded in powers of the strong and the electroweak coupling constants, $\alpha_S \approx 0.1$ and $\alpha \approx 1/137$

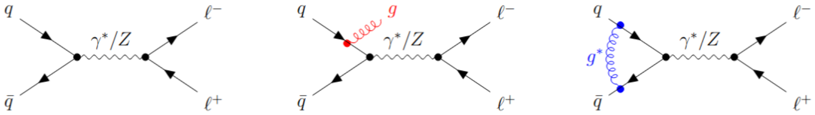
$$\mathrm{d}\hat{\sigma}_{a,b} = \mathrm{d}\hat{\sigma}^{(0,0)}_{a,b} + \alpha_s \mathrm{d}\hat{\sigma}^{(1,0)}_{a,b} + \alpha_s^2 \mathrm{d}\hat{\sigma}^{(2,0)}_{a,b} + \alpha_s^3 \mathrm{d}\hat{\sigma}^{(3,0)}_{a,b} + \alpha \mathrm{d}\hat{\sigma}^{(0,1)}_{a,b} + \alpha\alpha_s \mathrm{d}\hat{\sigma}^{(1,1)}_{a,b} + \dots$$

We focus on NLO QCD corrections

Account for short distance, high-energy effects

They consist of three terms

$$\mathrm{d}\hat{\sigma}^{\mathrm{NLO}}_{a,b} = \mathrm{d}\hat{\sigma}^{\mathrm{R}}_{a,b} + \mathrm{d}\hat{\sigma}^{\mathrm{V}}_{a,b} + \mathrm{d}\hat{\sigma}^{\mathrm{pdf}}_{a,b}$$



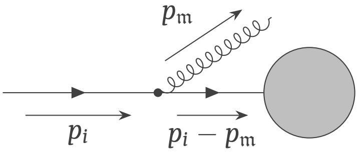
UV and IR divergences

A considerable obstacle

The treatment of real and virtual corrections is **non-trivial**

① **Ultraviolet** (UV) singularities in virtual contributions
$\longrightarrow$ Renormalization

② **Infrared** (IR) singularities in both real and virtual contributions
$\longrightarrow$ Low-momentum (soft) and small-angle (collinear) kinematic regions



$$\sim \frac{1}{(p_i - p_{\mathfrak{m}})^2} = \frac{1}{2E_i E_{\mathfrak{m}}\,(1-\cos\theta_{i\mathfrak{m}})} \xrightarrow[E_{\mathfrak{m}},\theta_{i\mathfrak{m}}\to 0]{} \infty$$

UV and IR divergences

Dimensional regularization and pole cancellation

Dimensional regularization

$$d = 4 - 2\epsilon \qquad \epsilon \in \mathbb{C}\,, \mathrm{Re}(\epsilon) < 0$$

Divergences appear as poles in $1/\epsilon$

- Virtual $\rightarrow$ explicit
- Real $\rightarrow$ phase space integration needed 1

- Singularities signal the presence of long-distance (low-energy) effects
- The division into real and virtual terms is therefore a computational tool

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Bloch-Nordsieck and Kinoshita-Lee-Nauenberg theorems

Infrared divergences are **guaranteed** to cancel when we sum real and virtual corrections

- **Mismatch** in the dimensionality of the integration domains 2

IR divergences

Subtraction scheme

Insights

- **Factorization** of the amplitudes in the soft and collinear limits
- Real emissions are **unresolved** in singular kinematics regions

We adopt a **subtraction method**

$$2s_{a,b}\mathrm{d}\hat{\sigma}^{\mathrm{R}}_{a,b} = \int [\mathrm{d}p_{\mathfrak{m}}] F_{\mathrm{LM}}(\mathfrak{m}) = \int [\mathrm{d}p_{\mathfrak{m}}](F_{\mathrm{LM}}(\mathfrak{m}) - \mathcal{S}) + \int [\mathrm{d}p_{\mathfrak{m}}]\mathcal{S}$$

$\longrightarrow$ Describes the singular behaviour of the amplitude

$\longrightarrow$ Singular behaviour removed: numerical integration with Monte Carlo methods

IR divergences

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$\rightarrow$ Describes the singular behaviour of the amplitude

$\rightarrow$ Singular behaviour removed: numerical integration with Monte Carlo methods

Restricting the integration region to the **minimal necessary volume** is crucial for improving the efficiency of the computation

Current calculations require $\sim 10^8$ CPUh[1]
$\rightarrow$ Further refinements needed for NNLO calculations

[1] Czakon et al., SciPost Phys. 11, 001 (2021), Tour de force in Quantum Chromodynamics

NSC Subtraction Scheme

Sequential extraction of singularities

FKS (Frixione, Kunszt, Signer) constructs S from the soft and collinear limits

$$S_i A := \lim_{E_i \to 0} A\,, \qquad C_{ij} A := \lim_{\rho_{ij} \to 0} A$$

The singularities are isolated and extracted **sequentially** in a nested manner

$$\langle \Delta^{(\mathfrak{m})} F_{\rm LM}(\mathfrak{m}) \rangle = \langle S_{\mathfrak{m}} F_{\rm LM}(\mathfrak{m}) \rangle + \sum_{i=1}^{N_p} \langle \bar{S}_{\mathfrak{m}} C_{i\mathfrak{m}} \Delta^{(\mathfrak{m})} F_{\rm LM}(\mathfrak{m}) \rangle + \langle \bar{S}_{\mathfrak{m}} \bar{C}_{i\mathfrak{m}} \Delta^{(\mathfrak{m})} \omega^{\mathfrak{m}i} F_{\rm LM}(\mathfrak{m}) \rangle$$

→ First, we remove the soft singularities

→ Then, the collinear and soft-collinear ones

→ **Completely finite** term, can be integrated in four dimensions

- $\bar{S}_{\mathfrak{m}} = \mathbb{1} - S_{\mathfrak{m}}$, $\bar{C}_{\mathfrak{m}} = \mathbb{1} - C_{\mathfrak{m}}$
- $\omega^{\mathfrak{m}i}$ in order to treat one collinear singularity at a time
- $\Delta^{(\mathfrak{m})}$ to select the unresolved parton

NSC Subtraction Scheme

The soft term

We integrate over the phase space of the unresolved parton $\mathfrak{m}$

$$\langle S_{\mathfrak{m}} F_{\mathrm{LM}}(\mathfrak{m})\rangle = \int \mathrm{d}E_{\mathfrak{m}} E_{\mathfrak{m}}^{d-5}\,\theta(E_{\mathfrak{m}} < E_{\max}) \int \frac{\mathrm{d}\Omega_{d-1}}{2(2\pi)^{d-1}}(-g_{s,b}^2) \sum_{(i\neq j)}^{N_p} \frac{\rho_{ij}}{\rho_{im}\rho_{jm}}(\vec{T}_i \cdot \vec{T}_j) F_{\mathrm{LM}}\,,$$

NSC Subtraction Scheme

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We introduce the parameter θ_s, which limits the energy of unresolved partons

$$\langle S_{\mathfrak{m}}^{\theta_s} F_{\rm LM}(\mathfrak{m})\rangle = \int \mathrm{d}E_{\mathfrak{m}} E_{\mathfrak{m}}^{d-5}\,\theta(E_{\mathfrak{m}} < \theta_s E_{\rm max}) \int \frac{\mathrm{d}\Omega_{d-1}}{2(2\pi)^{d-1}}(-g_{s,b}^2)\sum_{(i\neq j)}^{N_p}\frac{\rho_{ij}}{\rho_{im}\rho_{jm}}(\vec{T}_i\cdot\vec{T}_j)F_{\rm LM}$$

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And we obtain $\langle S_{\mathfrak{m}}^{\theta_s} F_{\mathrm{LM}}(\mathfrak{m})\rangle = [\alpha_s]\langle I_{\mathrm{S}} \cdot F_{\mathrm{LM}}\rangle$

$$I_{\mathrm{S}}(\epsilon, \theta_s) = -\frac{1}{\epsilon^2}\left(\frac{2E_{\max}}{\mu}\right)^{-2\epsilon} \theta_s^{-2\epsilon} \sum_{i \neq j}^{N_p} \eta_{ij}^{-\epsilon} K_{ij} \left(\vec{T}_i \cdot \vec{T}_j\right)$$

NSC Subtraction Scheme

The collinear term - initial state

We integrate over the phase space of the unresolved parton $\mathfrak{m}$

$$\int [\mathrm{d}p_{\mathfrak{m}}]\, C_{a\mathfrak{m}} F_{\mathrm{LM}}(\mathfrak{m}) = \frac{1}{8\pi^2}\frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)} 2^{-2\epsilon} \int_0^1 \mathrm{d}\eta_{a\mathfrak{m}} \left(\eta_{a\mathfrak{m}}(1-\eta_{a\mathfrak{m}})\right)^{-\epsilon} \times$$
$$\times \int_0^{E_{\max}} \mathrm{d}E_{\mathfrak{m}} E_{\mathfrak{m}}^{1-2\epsilon} \frac{g_{s,b}^2}{(1-z)E_a^2\eta_{a\mathfrak{m}}} \frac{P_{f_{[a\mathfrak{m}]}f_a}(z)}{z} F_{\mathrm{LM}}(z\cdot a)\,,$$

NSC Subtraction Scheme

The collinear term - initial state

We integrate over the phase space of the unresolved parton $\mathfrak{m}$

$$\int [dp_{\mathfrak{m}}]\, C_{a\mathfrak{m}} F_{\mathrm{LM}}(\mathfrak{m}) = \frac{1}{8\pi^2} \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)} 2^{-2\epsilon} \int_0^1 d\eta_{a\mathfrak{m}} (\eta_{a\mathfrak{m}}(1-\eta_{a\mathfrak{m}}))^{-\epsilon} \times \\ \times \int_0^{E_{\max}} dE_{\mathfrak{m}} E_{\mathfrak{m}}^{1-2\epsilon} \frac{g_{s,b}^2}{(1-z) E_a^2 \eta_{a\mathfrak{m}}} \frac{P_{f_{[a\mathfrak{m}]} f_a}(z)}{z} F_{\mathrm{LM}}(z \cdot a)\,,$$

We introduce the parameter θ_a, which controls the angular separation of unresolved partons

$$\int [dp_{\mathfrak{m}}]\, C_{a\mathfrak{m}}^{\theta_a} F_{\mathrm{LM}}(\mathfrak{m}) = \frac{1}{8\pi^2} \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)} 2^{-2\epsilon} \int_0^{\theta_a} d\eta_{a\mathfrak{m}} (\eta_{a\mathfrak{m}}(1-\eta_{a\mathfrak{m}}))^{-\epsilon} \times \\ \times \int_0^{E_{\max}} dE_{\mathfrak{m}} E_{\mathfrak{m}}^{1-2\epsilon} \frac{g_{s,b}^2}{(1-z) E_a^2 \eta_{a\mathfrak{m}}} \frac{P_{f_{[a\mathfrak{m}]} f_a}(z)}{z} F_{\mathrm{LM}}(z \cdot a)\,,$$

NSC Subtraction Scheme

The collinear term - final state

We integrate over the phase space of the unresolved parton $\mathfrak{m}$

$$\int [\mathrm{d}p_i] \int [\mathrm{d}p_{\mathfrak{m}}]\, C_{i\mathfrak{m}} \Delta^{(\mathfrak{m})} F_{\mathrm{LM}} = \int \frac{\mathrm{d}\Omega^{d-2}}{2(2\pi)^{d-1}} 2^{-2\epsilon} \int_0^1 \frac{\mathrm{d}\eta_{i\mathfrak{m}}}{(\eta_{i\mathfrak{m}}(1-\eta_{i\mathfrak{m}}))^{\epsilon}} \times \int [\mathrm{d}\Omega_i] \int \mathrm{d}E_{\mathfrak{m}} E_{\mathfrak{m}}^{1-2\epsilon} \int \mathrm{d}E_i E_i^{1-2\epsilon} \frac{g_{s,b}^2 P_{f_{[i\mathfrak{m}]}f_i} F_{\mathrm{LM}}(i\mathfrak{m})}{(1-z) E_{i\mathfrak{m}}^2 \eta_{i\mathfrak{m}}}.$$

NSC Subtraction Scheme

The collinear term - final state

We integrate over the phase space of the unresolved parton $\mathfrak{m}$

$$\int [\mathrm{d}p_i] \int [\mathrm{d}p_\mathfrak{m}] \, C_{i\mathfrak{m}} \Delta^{(\mathfrak{m})} F_{\mathrm{LM}} = \int \frac{\mathrm{d}\Omega^{d-2}}{2(2\pi)^{d-1}} 2^{-2\epsilon} \int_0^1 \frac{\mathrm{d}\eta_{i\mathfrak{m}}}{(\eta_{i\mathfrak{m}}(1-\eta_{i\mathfrak{m}}))^\epsilon} \times \int [\mathrm{d}\Omega_i] \int \mathrm{d}E_\mathfrak{m} E_\mathfrak{m}^{1-2\epsilon} \int \mathrm{d}E_i E_i^{1-2\epsilon} \frac{g_{s,b}^2 P_{f_{[i\mathfrak{m}]}f_i} F_{\mathrm{LM}}(i\mathfrak{m})}{(1-z) E_{i\mathfrak{m}}^2 \eta_{i\mathfrak{m}}} .$$

We introduce the parameter θ_i, which controls the angular separation of unresolved partons

$$\int [\mathrm{d}p_i] \int [\mathrm{d}p_\mathfrak{m}] \, C_{i\mathfrak{m}}^{\theta_i} \Delta^{(\mathfrak{m})} F_{\mathrm{LM}} = \int \frac{\mathrm{d}\Omega^{d-2}}{2(2\pi)^{d-1}} 2^{-2\epsilon} \int_0^{\theta_i} \frac{\mathrm{d}\eta_{i\mathfrak{m}}}{(\eta_{i\mathfrak{m}}(1-\eta_{i\mathfrak{m}}))^\epsilon} \times \int [\mathrm{d}\Omega_i] \int \mathrm{d}E_\mathfrak{m} E_\mathfrak{m}^{1-2\epsilon} \int \mathrm{d}E_i E_i^{1-2\epsilon} \frac{g_{s,b}^2 P_{f_{[i\mathfrak{m}]}f_i} F_{\mathrm{LM}}(i\mathfrak{m})}{(1-z) E_{i\mathfrak{m}}^2 \eta_{i\mathfrak{m}}} .$$

NSC Subtraction Scheme

The collinear term

$$\sum_{i\in\mathcal{H}_f} \langle C_{i\mathfrak{m}}^{\theta_i} \bar{S}_{\mathfrak{m}}^{\theta_s} \Delta^{\mathfrak{m}} F_{\mathrm{LM}}(\mathfrak{m})\rangle = [\alpha_s]\langle I_{\mathrm{C}}(\epsilon,\theta_s,\theta_i)\cdot F_{\mathrm{LM}}\rangle$$

$$+ [\alpha_s]\Big\langle[\mathrm{d}\eta_{a\mathfrak{m}\,\theta_a}] \left(\frac{2E_a}{\mu}\right)^{-2\epsilon} \int_0^1 \mathrm{d}z \left[\hat{P}_{aa}^{(0)} - \epsilon\mathcal{P}_{aa}^{\mathrm{fin}}\right] \frac{1}{z} F_{\mathrm{LM}}(z\cdot a)\Big\rangle$$

$$+ [\alpha_s]\Big\langle[\mathrm{d}\eta_{b\mathfrak{m}\,\theta_b}] \left(\frac{2E_b}{\mu}\right)^{-2\epsilon} \int_0^1 \mathrm{d}z \left[\hat{P}_{bb}^{(0)} - \epsilon\mathcal{P}_{bb}^{\mathrm{fin}}\right] \frac{1}{z} F_{\mathrm{LM}}(z\cdot b)\Big\rangle\,,$$

$$I_{\mathrm{C}}(\epsilon,\theta_s,\theta_i) = \sum_{i\in\mathcal{H}} \Bigg(\frac{1}{\epsilon}\left(\gamma_i + 2\,L_i^{\theta_s}\,\vec{T}_i^2\right) - 2\log\left(\frac{2E_i}{\mu}\right)\gamma_i - 4\,\vec{T}_q^2\,L_i^{\theta_s}\log\left(\frac{2E_i}{\mu}\right)$$

$$- \log(\theta_i)\,\gamma_i - 2\log(\theta_i)\,\vec{T}_i^2\,L_i^{\theta_s}\Bigg)$$

Cancellation of poles

Unaffected by the θ-parameters

$$I_S(\epsilon, \theta_s) = \frac{1}{\epsilon^2}\sum_{i=1}^{N_p} T_i^2 + \frac{1}{\epsilon}\left[-\sum_{i=1}^{N_p}\left(2L_i^{\theta_s} + 2\log\left(\frac{2E_i}{\mu}\right)\right)\vec{T}_i^2 - \sum_{i\neq j}^{N_p} f_1(\eta_{ij})(\vec{T}_i\cdot\vec{T}_j)\right]$$

$$+\sum_{i=1}^{N_p} 2\log^2\left(\frac{2E_{\max}\theta_s}{\mu}\right)\vec{T}_i^2 - \sum_{i\neq j}^{N_p}\left[f_2\left(2E_{\max}\theta_s\right) f_1(\eta_{ij}) + K_{ij}^{(2)}\right](\vec{T}_i\cdot\vec{T}_j) + \mathcal{O}(\epsilon)$$

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$$I_C(\epsilon,\theta_s,\theta_i) = \sum_{i\in\mathcal{H}}\left(\frac{1}{\epsilon}\left(\gamma_i + 2\,L_i^{\theta_s}\,\vec{T}_i^2\right) - 2\log\left(\frac{2E_i}{\mu}\right)\gamma_i - 4\,\vec{T}_q^2\,L_i^{\theta_s}\log\left(\frac{2E_i}{\mu}\right)\right.$$

$$\left. - \log\left(\theta_i\right)\gamma_i - 2\log\left(\theta_i\right)\vec{T}_i^2\,L_i^{\theta_s}\right)$$

Cancellation of poles

Unaffected by the θ-parameters

$$I_S(\epsilon,\theta_s) = \frac{1}{\epsilon^2}\sum_{i=1}^{N_p} T_i^2 + \frac{1}{\epsilon}\left[-\sum_{i=1}^{N_p}\left(2L_i^{\theta_s} + 2\log\left(\frac{2E_i}{\mu}\right)\right)\vec{T}_i^2 - \sum_{i\neq j}^{N_p} f_1(\eta_{ij})(\vec{T}_i\cdot\vec{T}_j)\right]$$

$$+ \sum_{i=1}^{N_p} 2\log^2\left(\frac{2E_{\max}\theta_s}{\mu}\right)\vec{T}_i^2 - \sum_{i\neq j}^{N_p}\left[f_2\left(2E_{\max}\theta_s\right)f_1(\eta_{ij}) + K_{ij}^{(2)}\right](\vec{T}_i\cdot\vec{T}_j) + \mathcal{O}(\epsilon)$$

$$I_C(\epsilon,\theta_s,\theta_i) = \sum_{i\in\mathcal{H}}\left(\frac{1}{\epsilon}\left(\gamma_i + 2\,L_i^{\theta_s}\,\vec{T}_i^2\right) - 2\log\left(\frac{2E_i}{\mu}\right)\gamma_i - 4\,\vec{T}_q^2\,L_i^{\theta_s}\log\left(\frac{2E_i}{\mu}\right)\right.$$

$$\left. - \log\left(\theta_i\right)\gamma_i - 2\log\left(\theta_i\right)\vec{T}_i^2\,L_i^{\theta_s}\right)$$

The parameter θ_s appears only as a residue of the $1/\epsilon$ pole, but its dependence cancels out when summing $I_S + I_V$; both parameters, θ_i and θ_s, appear in finite terms

Cancellation of poles

Results and future outlook

The $1/\epsilon^2$ pole cancels summing $I_{\mathrm{S}} + I_{\mathrm{V}}$, the $1/\epsilon$ poles cancel summing $I_{\mathrm{S}} + I_{\mathrm{C}} + I_{\mathrm{V}}$, leaving a finite $I_{\mathrm{T}}^{\mathrm{fin}} = I_{\mathrm{S}} + I_{\mathrm{C}} + I_{\mathrm{V}}$ operator

$$I_{\mathrm{T}}^{\mathrm{fin}}(\theta_s, \theta_i) = \sum_{i=1}^{N_p} \left(2\log^2\left(\frac{2E_{\max}\theta_s}{\mu}\right)\vec{T}_i^2 - 2\log\left(\frac{2E_i}{\mu}\right)\gamma_i - 4\,\vec{T}_q^2\,L_i^{\theta_s}\log\left(\frac{2E_i}{\mu}\right) - \log(\theta_i)\,\gamma_i \right.$$
$$\left. - 2\log(\theta_i)\,\vec{T}_i^2\,L_i^{\theta_s} \right) - \sum_{i\neq j}^{N_p} \left[f_2\,(2E_{\max}\theta_s) f_1(\eta_{ij}) + K_{ij}^{(2)} \right] (\vec{T}_i \cdot \vec{T}_j) + I_{\mathrm{V}}^{\mathrm{fin}} + \mathcal{O}(\epsilon)$$

Thus the NLO QCD correction reads

$$2s_{a,b}\mathrm{d}\hat{\sigma}_{a,b}^{\mathrm{NLO}} = 2s_{a,b}\mathrm{d}\hat{\sigma}_{a,b}^{\mathrm{NLO,fin}} + [\alpha_s]\langle I_{\mathrm{T}} \cdot F_{\mathrm{LM}}\rangle + [\alpha_s]\left[\langle \mathcal{P}_{aa}^{\mathrm{NLO},\theta_{\mathrm{a}}} \otimes F_{\mathrm{LM}}\rangle + \langle F_{\mathrm{LM}} \otimes \mathcal{P}_{aa}^{\mathrm{NLO},\theta_{\mathrm{b}}}\rangle \right]$$

Conclusion

Results and future outlook

We have shown that

- The θ parameters contribute only to I_{S} and I_{C}, while leaving the pole cancellation unaffected
- They appear in the finite parts of I_{T} and the regularized amplitude

The physical result obtained by summing the finite remainders must be independent of θ_s and θ_i
$\rightarrow$ **Powerful consistency check**

Future Developments

- Numerical analysis of how θ_s and θ_i variations affect scattering processes
- Extension to NNLO and application to more complex physical scenarios