ESERCIZIO 4

METHOD 1

From the definition we know:

$$\phi(k) = \int e^{-ik x} p(x) dx$$

where $p(x)$ is the multivariate Gaussian distribution:

$$p(x) = \frac{1}{(2\pi)^{M/2} |C|^{1/2}} \cdot \exp\left(-\frac{1}{2}(x-\mu)^T C^{-1} (x-\mu)\right)$$

$$\Rightarrow \phi(k) = \frac{1}{(2\pi)^{M/2} |C|^{1/2}} \int \exp\left(-ik^T x - \frac{1}{2}(x-\mu)^T C^{-1}(x-\mu)\right) dx$$

We expand the quadratic term:

$$(x-\mu)^T C^{-1} (x-\mu) = x^T C^{-1} x - 2\mu^T C^{-1} x + \mu^T C^{-1} \mu$$

$$\Rightarrow \exp\left(-ik^T x - \frac{1}{2} x^T C^{-1} x + \mu^T C^{-1} x - \frac{1}{2}\mu^T C^{-1} \mu\right)$$

Combine the terms involving x:

$$-ik^T x + \mu^T C^{-1} x = x^T(-ik + C^{-1}\mu)$$

$$\Rightarrow \phi(k) = \frac{\exp\left(-\frac{1}{2}\mu^T C^{-1}\mu\right)}{(2\pi)^{M/2} |C|^{1/2}} \int \exp\left(-\frac{1}{2} x^T C^{-1} x + x^T(-ik + C^{-1}\mu)\right) dx$$

Let $b = C^{-1}\mu - ik$. The quadratic term becomes:

$$-\frac{1}{2} x^T C^{-1} x + x^T b = -\frac{1}{2}(x - Cb)^T C^{-1} (x - Cb) + \frac{1}{2} b^T C b$$

$$\Rightarrow \phi(k) = \frac{\exp\left(-\frac{1}{2}\mu^T C^{-1}\mu\right)}{(2\pi)^{M/2} |C|^{1/2}} \exp\left(\frac{1}{2} b^T C b\right) \int \exp\left(-\frac{1}{2}(x - Cb)^T \cdot \right.$$

$$\left. \cdot C^{-1} (x - Cb)\right) dx$$

The integral over x is a standard Gaussian integral:

$$\int \exp\left(-\frac{1}{2}(x-Cb)^T C^{-1}(x-Cb)\right) dx = (2\pi)^{N/2}|C|^{1/2}$$

$$\Rightarrow \phi(k) = \exp\left(-\frac{1}{2}\mu^T C^{-1}\mu + \frac{1}{2} b^T C b\right)$$

Recall $b = C^{-1}\mu - ik$

$$b^T C b = (C^{-1}\mu - ik)^T C (C^{-1}\mu - ik) = \mu^T C^{-1}\mu - 2i\mu^T k - k^T C k$$

$$\phi(k) = \exp\left(-\frac{1}{2}\mu^T C^{-1}\mu + \frac{1}{2}(\mu^T C^{-1}\mu - 2i\mu^T k - k^T C k)\right) =$$

$$= \boxed{\exp\left(-i\mu^T k - \frac{1}{2}k^T C k\right)}$$

METHOD 2

$$C^{-1} = O^T D^{-1} O \quad \begin{cases} O = \text{orthogonal rotation matrix} \\ D = \text{diagonal matrix of eigenvalues} \end{cases}$$

Introduce variables to $\vec{y} = O\vec{x}$, $\vec{\nu} = O\vec{\mu}$, $\vec{l} = O\vec{k}$

The exponential term is:

$$-i\vec{k}\vec{x} - \frac{1}{2}(\vec{x}-\vec{\mu})^T C^{-1}(\vec{x}-\vec{\mu}) = -i\vec{k}^T O^T O\vec{x} - \frac{1}{2}(\vec{x}-\vec{\mu})^T \cdot$$

$$\cdot O^T D^{-1} O(\vec{x}-\vec{\mu}) =$$

$$= -i\vec{l}\cdot\vec{y} - \frac{1}{2}(\vec{y}-\vec{\nu})^T D^{-1}(\vec{y}-\vec{\nu}) = -\sum_i i\, l_i y_i + \frac{1}{2} d_i (y_i - \nu_i)^2$$

$$= -\sum_i i\, l_i y_i + \frac{d_i}{2} y_i^2 - d_i y_i \nu_i + \frac{d_i}{2}\nu_i^2 =$$

$$= -\sum_i \frac{d_i}{2} y_i^2 + y_i(i l_i - d_i \nu_i) + \frac{d_i}{2}\nu_i^2 =$$

$$= -\sum_i \frac{d_i}{2}\left[y_i^2 + 2y_i\left(\frac{i l_i - d_i\nu_i}{d_i}\right) + \nu_i^2\right] =$$

$$= -\sum_i \frac{d_i}{2}\left[y_i^2 + 2y_i\left(\frac{i l_i - d_i\nu_i}{d_i}\right) - \left(\frac{i l_i - d_i\nu_i}{d_i}\right)^2 + \left(\frac{i l_i - d_i\nu_i}{d_i}\right)^2 + \nu_i^2\right] =$$

$$= -\sum_i \frac{d_i}{2}\left[\left(y_i - \frac{i\ell_i - d_i\nu_i}{d_i}\right)^2 - \left(\frac{i\ell_i - d_i\nu_i}{d_i}\right)^2 + \nu_i^2\right] =$$

$$= \sum_i -\frac{d_i}{2}\left(y_i - \frac{i\ell_i - d_i\nu_i}{d_i}\right)^2 + \left(\frac{(i\ell_i - d_i\nu_i)^2}{2d_i} - \frac{d_i}{2}\nu_i^2\right)$$

Changing variables to $\vec{y}$, with $d^n x = d^n y$ (since $\det O = 1$).

$$\phi(K) = \frac{1}{(2\pi)^{n/2}\sqrt{\det C}} \int d^n x \, e^{-i\vec{k}\vec{x} - \frac{1}{2}(\vec{x}-\mu)^T C^{-1}(\vec{x}-\mu)} =$$

$$= \frac{1}{(2\pi)^{n/2}\sqrt{\det C}} \int d^n y \; e^{-i\vec{\ell}\vec{y} - \frac{1}{2}(\vec{y}-\nu)^T D^{-1}(\vec{y}-\nu)}$$

The integral over y_i becomes:

$$\int \exp\left(-\frac{d_i}{2}\left(y_i - \frac{\ell_i - d_i\nu_i}{d_i}\right)^2\right) dy_i = \sqrt{\frac{2\pi}{d_i}}$$

$$\Rightarrow \phi(K) = \exp\left(\sum_i \frac{(\ell_i - d_i\nu_i)^2}{2d_i} - \frac{d_i}{2}\nu_i^2\right) \cdot \prod_i \sqrt{\frac{2\pi}{d_i}} \cdot \frac{1}{(2\pi)^{n/2}\sqrt{\det C}}$$

Using $\det C = \prod_i d_i^{-1}$, this simplifies to:

$$\phi(K) = \exp\left(-\frac{1}{2}\ell^T D \ell - \ell \cdot \nu\right)$$

We know $\vec{\ell} = O\vec{k}$ and $\nu = O\mu$:

$$\boxed{\phi(K) = \exp\left(-\frac{1}{2}k^T C k - i k^T \mu\right)}$$

ESERCIZIO 5

For a multivariate Gaussian distribution $N(\mu, \Sigma)$, the characteristic function is :

$$\phi(k) = \exp\left(-\frac{1}{2} k^T \Sigma k - i k^T \mu\right)$$

MEAN

$$\mathbb{E}[x_\alpha] = \left.\frac{\partial \phi(k)}{\partial(-ik_\alpha)}\right|_{k=0}$$

$$\frac{\partial \phi(k)}{\partial(-ik_\alpha)} = \phi(k) \cdot \frac{\partial}{\partial(-ik_\alpha)}\left(-\frac{1}{2} k^T \Sigma k - i k^T \mu\right) =$$

$$= \phi(k) \cdot \Bigg[\underbrace{\frac{\partial}{\partial(-ik_\alpha)}\left(-\frac{1}{2}\sum_{i,j} k_i \Sigma_{ij} k_j\right)}_{-\frac{1}{2}\sum_j (\Sigma_{\alpha j} k_j + \Sigma_{j\alpha} k_j)} + \underbrace{\frac{\partial}{\partial(-ik_\alpha)}(-ik^T\mu)}_{\mu_\alpha} \Bigg]$$

Since Σ is symmetric $\Sigma_{\alpha j} = \Sigma_{j\alpha} \Rightarrow -\frac{1}{2}\sum_j (\Sigma_{\alpha j} k_j + \Sigma_{j\alpha} k_j) =$

$$= -\sum_j \Sigma_{\alpha j} k_j$$

$$\mathbb{E}[x_\alpha] = \left.\frac{\partial \phi(k)}{\partial(-ik_\alpha)}\right|_{k=0} = \underbrace{\phi(k)}_{=1}\left.\Big(\underbrace{-\sum_j \Sigma_{\alpha j} k_j}_{=0} + \mu_\alpha\Big)\right|_{k=0} = \mu_\alpha$$

COVARIANCE

$$\mathbb{E}[x_\alpha x_\beta] = \left.\frac{\partial^2 \phi(k)}{\partial(-ik_\alpha)\partial(-ik_\beta)}\right|_{k=0}$$

$$\frac{\partial^2 \phi(k)}{\partial(-ik_\alpha)\partial(-ik_\beta)} = \frac{\partial}{\partial(-ik_\beta)}\left[\phi(k) \cdot \frac{\partial}{\partial(-ik_\alpha)}\left(-\frac{1}{2} k^T \Sigma k - i k^T \mu\right)\right] =$$

$$= \frac{\partial}{\partial(-ik_\beta)}\left[\phi(k)\left(-\sum_j \Sigma_{\alpha j} k_j + \mu_\alpha\right)\right] = \phi(k)(-\Sigma_{\alpha\beta})$$

$\Rightarrow \mathbb{E}[x_\alpha x_\beta] = \Sigma_{\alpha\beta} + \mu_\alpha \mu_\beta$ (because we are at $k=0$)

$\Rightarrow \mathrm{Cov}[x_\alpha x_\beta] = \mathbb{E}[x_\alpha x_\beta] - \mu_\alpha \mu_\beta = \Sigma_{\alpha\beta}$

ESERCIZIO 8

The best-fit amplitudes are determined by maximizing the likelihood, which corresponds to minimizing the chi-squared statistic:

$$\chi^2 = (\vec{d} - \vec{A}^T T)^T C^{-1} (\vec{d} - \vec{A}^T T)$$

where:

- $\vec{d}$ is the data vector (length N)
- T is the $N \times \pi$ matrix of templates
- C is the covariance matrix of noise
- $\vec{A}$ is the M-dimensional vector of unknown amplitudes

We aim to find $\vec{A}$ by solving $\frac{\partial \chi^2}{\partial A_k} = 0 \quad \forall k \in \{1, 2, \dots \pi\}$

$$\chi^2 = (\vec{d} - \vec{A}^T T)^T C^{-1} (\vec{d} - \vec{A}^T T) = \vec{d}^T C^{-1} \vec{d} - 2\vec{A}^T T^T C^{-1} \vec{d} + \\ + \vec{A}^T T^T C^{-1} T \vec{A}$$

$$\frac{\partial \chi^2}{\partial A_k} = -2 \sum_j T_{kj} (C^{-1}\vec{d})_j + 2 \sum_{ij} A_i T_{ik} (C^{-1} T)_{ij} =$$

$$= -2T^T C^{-1} \vec{d} + 2T^T C^{-1} T \vec{A} = 0$$

$$\Rightarrow \quad T^T C^{-1} \vec{d} = T^T C^{-1} T \vec{A}$$

$$\boxed{\vec{A} = (T^T C^{-1} T)^{-1} T^T C^{-1} \vec{d}}$$

ESERCIZIO 9

The model $\vec{d}$ is $d_i = w x_i + b + \varepsilon_i$ where $\varepsilon_i \sim N(0, \sigma^2)$

The likelihood of the data is:

$$\mathcal{L}(w,b) \propto \exp\left(-\frac{1}{2\sigma^2} \underbrace{\sum_{i=1}^{N} (d_i - (w x_i + b))^2}_{\text{we need to minimize this to minimize the likelihood}}\right)$$

We rewrite in form of matrix

$$T = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \qquad \vec{A} = \begin{bmatrix} w \\ b \end{bmatrix} \qquad \vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{bmatrix} \qquad C = \sigma_d^2$$

$$\Rightarrow \vec{d} = T\vec{A} + \varepsilon$$

MAP ESTIMATE

The MAP estimate minimizes the chi-squared:

$$\chi^2 = (\vec{d} - T\vec{A})^T C^{-1} (\vec{d} - T\vec{A})$$

where $C = \sigma_d^2 \mathbb{I}$ and $C^{-1} = \frac{1}{\sigma_d^2}\mathbb{I}$:

$$\Rightarrow \vec{A} = (T^T T)^{-1} T^T \vec{d} \qquad \text{(from exercise 8)}$$

$$T^T T = \begin{bmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & N \end{bmatrix} \qquad (T^T T)^{-1} = \frac{\sigma_d^2}{\Delta}\begin{bmatrix} 1 & -\vec{x} \\ -\vec{x} & \vec{x}^2 \end{bmatrix}$$

where $\Delta = N \sum x_i^2 - (\sum x_i)^2$ and $\vec{x} = \frac{\sum x_i}{N}$

$$T^T \vec{d} = \begin{bmatrix} \sum x_i d_i \\ \sum d_i \end{bmatrix}$$

Substitute :

$$\vec{A} = \frac{1}{\Delta}\begin{bmatrix} N \sum x_i d_i - \sum x_i \sum d_i \\ \sum x_i^2 \sum d_i - \sum x_i \sum x_i d_i \end{bmatrix}$$

The result for $\vec{A} = \begin{bmatrix} w \\ b \end{bmatrix}$ is:

$$w = \frac{N \sum x_i d_i - \sum x_i \sum d_i}{\Delta}$$

$$b = \frac{\sum x_i^2 \sum d_i - \sum x_i \sum x_i d_i}{\Delta}$$

COVARIANCE MATRIX

$$Cov(\vec{A}) = \sigma_d^2 \, (T^T T)^{-1} =$$

$$= \frac{\sigma_d^2}{\Delta} \begin{bmatrix} N & -\sum x_i \\ -\sum x_i & \sum x_i^2 \end{bmatrix}$$