Finite Differences Methods in Financial Engineering

A partial Differential Equation Approach

Daniel J. Duffy

November 19, 2019

Contents

Listings

List of Algorithms

0. Goals of this Book and Global Overview

0.1. What is this book?

0.2. Why has this book been written?

0.3. For whom is this book intended?

0.4. Why should I read this book?

0.5. The structure of this book

0.6. What this book does not cover

0.7. Contact, feedback and more information

Part I.

The continuous theory of partial differential equations

1. An Introduction to Ordinary Differential Equations

1.1. Introduction and objectives

1.2. Two-point boundary value problem

1.3. Special kinds of boundary condition

1.4. Linear boundary value problems

1.5. Initial value problems

1.6. Some special cases

1.7. Summary and conclusions

2. An Introduction to Partial Differential Equations

2.1. Introduction and objectives

2.2. Partial differential equations

2.3. Specializations

2.3.1. Elliptic equations

2.3.2. Free boundary value problems

2.4. Parabolic partial differential equations

2.4.1. Special cases

2.5. Hyperbolic equations

2.5.1. Second-order equations

2.5.2. First-order equations

2.6. Systems of equations

2.6.1. Parabolic systems

2.6.2. First-order hyperbolic systems

2.7. Equations containing integrals

2.8. Summary and conclusions

3. Second-Order Parabolic Differential Equations

3.1. Introduction and objectives

3.2. Linear parabolic equations

3.3. The continuous problem

3.4. The maximum principle for parabolic equations

3.5. A special case: one-factor generalized Black–Scholes models

3.6. Fundamental solution and the Green's function

3.7. Integral representation of the solution of parabolic PDEs

3.8. Parabolic equations in one space dimension

3.9. Summary and conclusions

4. An Introduction to the Method of Characteristics

4.1. Introduction and objectives

4.2. First-order hyperbolic equations

4.2.1. An example

4.3. Second-order hyperbolic equations

4.3.1. Numerical integration along the characteristic lines

4.4. Applications to financial engineering

4.4.1. Generalizations

4.5. Systems of equations

4.5.1. An example

4.6. Propagation of discontinuities

4.6.1. Other problems

4.7. Summary and conclusions

Part II.

Finite difference methods: The Fundamentals

5. An Introduction to the Finite Difference Method

5.1. Introduction and objectives

5.2. Fundamentals of numerical differentiation

5.3. Caveat: accuracy and round-off errors

5.4. Where are divided differences used in instrument pricing?

5.5. Initial value problems

5.5.1. Padé matrix approximations

5.5.2. Extrapolation

5.6. Nonlinear initial value problems

5.6.1. Predictor–corrector methods

5.7. Runge–Kutta methods

5.8. Scalar initial value problems

5.8.1. Exponentially fitted schemes

5.9. Summary and conclusions

6. An Introduction to the Method of Lines

6.1. Introduction and objectives

6.2. Classifying semi-discretisation methods

6.3. Semi-discretisation in space using FDM

6.3.1. A test case

6.3.2. Toeplitz matrices

6.3.3. Semi-discretisation for convection-diffusion problems

6.3.4. Essentially positive matrices

6.4. Numerical approximation of first-order systems

6.4.1. Fully discrete schemes

6.4.2. Semi-linear problems

6.5. Summary and conclusions

7. General Theory of the Finite Difference Method

7.1. Introduction and objectives

7.2. Some fundamental concepts

7.2.1. Consistency

7.2.2. Stability

7.2.3. Convergence

7.3. Stability and the Fourier transform

7.4. The discrete Fourier transform

7.4.1. Some other examples

7.5. Stability for initial boundary value problems

7.5.1. Gerschgorin's circle theorem

7.6. Summary and conclusions

8. Finite Difference Schemes for First-Order Partial Differential Equations

8.1. Introduction and objectives

8.2. Scoping the problem

8.3. Why first-order equations are different: Essential difficulties

8.3.1. Discontinuous initial conditions

8.4. A simple explicit scheme

8.5. Some common schemes for initial value problems

8.5.1. Some other schemes

8.6. Some common schemes for initial boundary value problems

8.7. Monotone and positive-type schemes

8.8. Extensions, generalizations and other applications

8.8.1. General linear problems

8.8.2. Systems of equations

8.8.3. Nonlinear problems

8.8.4. Several independent variables

8.9. Summary and conclusions

9. FDM for the One-Dimensional Convection–Diffusion Equation

9.1. Introduction and objectives

9.2. Approximation of derivatives on the boundaries

9.3. Time-dependent convection–diffusion equations

9.4. Fully discrete schemes

9.5. Specifying initial and boundary conditions

9.6. Semi-discretization in space

9.7. Semi-discretization in time

9.8. Summary and conclusions

10. Exponentially Fitted Finite Difference Schemes

10.1. Introduction and objectives

10.2. Motivating exponential fitting

10.2.1. “Continuous” exponential approximation

10.2.2. “Discrete” exponential approximation

10.2.3. Where is exponential fitting being used?

10.3. Exponential fitting and time-dependent convection-diffusion

10.4. Stability and convergence analysis

10.5. Approximating the derivative of the solution

10.6. Special limiting cases

10.7. Summary and conclusions

Part III.

Applying FDM to one-factor instrument pricing

11. Exact Solutions and Explicit Finite Difference Method for One-Factor Models

11.1. Introduction and objectives

11.2. Exact solutions and benchmark cases

11.3. Perturbation analysis and risk engines

11.4. The trinomial method: Preview

11.4.1. Stability of the trinomial method

11.5. Using exponential fitting with explicit time marching

11.6. Approximating the Greeks

11.7. Summary and conclusions

11.8. Appendix: the formula for Vega

12. An Introduction to the Trinomial Method

12.1. Introduction and objectives

12.2. Motivating the trinomial method

12.3. Trinomial method: Comparisons with other methods

12.3.1. A general formulation

12.4. The trinomial method for barrier options

12.5. Summary and conclusions

13. Exponentially Fitted Difference Schemes for Barrier Options

13.1. Introduction and objectives

13.2. What are barrier options?

13.3. Initial boundary value problems for barrier options

13.4. Using exponential fitting for barrier options

13.4.1. Double barrier call options

13.4.2. Single barrier call options

13.5. Time-dependent volatility

13.6. Some other kinds of exotic options

13.6.1. Plain vanilla power call options

13.6.2. Capped power call options

13.7. Comparisons with exact solutions

13.8. Other schemes and approximations

13.9. Extensions to the model

13.10. Summary and conclusions

14. Advanced Issues in Barrier and Lookback Option Modelling

14.1. Introduction and objectives

14.2. Kinds of boundaries and boundary conditions

14.3. Discrete and continuous monitoring

14.3.1. What is discrete monitoring?

14.3.2. Finite difference schemes and jumps in time

14.3.3. Lookback options and jumps

14.4. Continuity corrections for discrete barrier options

14.5. Complex barrier options

14.6. Summary and conclusions

15. Extending the Black–Scholes Model: Jump Processes

15.1. Introduction and objectives

15.2. Jump–diffusion processes

15.2.1. Convolution transformations

15.3. Partial integro-differential equations and financial applications

15.4. Numerical solution of PIDE: Preliminaries

15.5. Techniques for the numerical solution of PIDEs

15.6. Implicit and explicit methods

15.7. Implicit–explicit Runge–Kutta methods

15.8. Using operator splitting

15.9. Splitting and predictor–corrector methods

15.10. Summary and conclusions

Part IV.

FDM for multidimensional problems

16. Finite Difference Schemes for Multidimensional Problems

16.1. Introduction and objectives

16.2. Elliptic equations

16.2.1. A self-adjoint elliptic operator

16.2.2. Solving the matrix systems

16.2.3. Exact solutions to elliptic problems

16.3. Diffusion and heat equations

16.3.1. Exact solutions to the heat equation

16.4. Advection equation in two dimensions

16.4.1. Initial boundary value problems

16.5. Convection–diffusion equation

16.6. Summary and conclusions

17. An Introduction to Alternating Direction Implicit and Splitting Methods

17.1. Introduction and objectives

17.2. What is ADI, really?

17.3. Improvements on the basic ADI scheme

17.3.1. The D'Yakonov scheme

17.3.2. Approximate factorization of operators

17.3.3. ADI classico for two-factor models

17.3.4. ADI classico for two-factor models

17.4. ADI for first-order hyperbolic equations

17.5. ADI classico and three-dimensional problems

17.6. The Hopscotch method

17.7. Boundary conditions

17.8. Summary and conclusions

18. Advanced Operator Splitting Methods: Fractional Steps

18.1. Introduction and objectives

18.2. Initial examples

18.3. Problems with mixed derivatives

18.4. Predictor–corrector methods (approximation correctors)

18.5. Partial integro-differential equations

18.6. More general results

18.7. Summary and conclusions

19. Modern Splitting Methods

19.1. Introduction and objectives

19.2. Systems of equations

19.2.1. ADI and splitting for parabolic systems

19.2.2. Compound and chooser options

19.2.3. Leveraged knock-in options

19.3. A different kind of splitting: The IMEX schemes

19.4. Applicability of IMEX schemes to Asian option pricing

19.5. Summary and conclusions

Part V.

Applying FDM to multi-factor instrument pricing

20. Options with Stochastic Volatility: The Heston Model

20.1. Introduction and objectives

20.2. An introduction to Ornstein–Uhlenbeck processes

20.3. Stochastic differential equations and the Heston model

20.4. Boundary conditions

20.4.1. Standard european call option

20.4.2. European put options

20.4.3. Other kinds of boundary conditions

20.5. Using finite difference schemes: Prologue

20.6. A detailed example

20.7. Summary and conclusions

21. Finite Difference Methods for Asian Options and Other "Mixed" Problems

21.1. Introduction and objectives

21.2. An introduction to Asian options

21.3. My first PDE formulation

21.4. Using operator splitting methods

21.4.1. For sake of completeness: ADI methods for asian option PDEs

21.5. Cheyette interest models

21.6. New developments

21.7. Summary and conclusions

22. Multi-Asset Options

22.1. Introduction and objectives

22.2. A taxonomy of multi-asset options

22.2.1. Exchange options

22.2.2. Rainbow options

22.2.3. Basket options

22.2.4. The best and worst

22.2.5. Quotient options

22.2.6. Foreign equity options

22.2.7. Quanto options

22.2.8. Spread options

22.2.9. Dual-strike options

22.2.10. Out-perfomance options

22.3. Common framework for multi-asset options

22.4. An overview of finite difference schemes for multi-asset problems

22.5. Numerical solution of elliptic equations

22.6. Solving multi-asset Black–Scholes equations

22.7. Special guidelines and caveats

22.8. Summary and conclusions

23. Finite Difference Methods for Fixed-Income Problems

23.1. Introduction and objectives

23.2. An introduction to interest rate modelling

23.3. Single-factor models

23.4. Some specific stochastic models

23.4.1. The Merton model

23.4.2. The Vasicek model

23.4.3. Cox, Ingersoll and Ross (CIR)

23.4.4. The Hull–White model

23.4.5. Lognormal models

23.5. An introduction to multidimensional models

23.6. The thorny issue of boundary conditions

23.6.1. One-factor models

23.6.2. Multi-factor models

23.7. Introduction to approximate methods for interest rate models

23.7.1. One-factor models

23.7.2. Many-factor models

23.8. Summary and conclusions

Part VI.

Free and moving boundary problems

24. Background to Free and Moving Boundary Value Problems

24.1. Introduction and objectives

24.2. Notation and definitions

24.3. Some preliminary examples

24.3.1. Single-phase melting ice

24.3.2. One-factor option modelling: American exercise style

24.3.3. Two-phase melting ice

24.3.4. Two-phase melting ice

24.3.5. The inverse Stefan problem

24.3.6. Two and three space dimensions

24.3.7. Oxygen diffusion

24.4. Solutions in financial engineering: A preview

24.4.1. What kinds of early exercise features?

24.4.2. What kinds of numerical techniques?

24.5. Summary and conclusions

25. Numerical Methods for Free Boundary Value Problems: Front-Fixing Methods

25.1. Introduction and objectives

25.2. An introduction to front-fixing methods

25.3. A crash course on partial derivatives

25.4. Functions and implicit forms

25.5. Front fixing for the heat equation

25.6. Front fixing for general problems

25.7. Multidimensional problems

25.8. Front fixing and American options

25.9. Other finite difference schemes

25.9.1. The method of lines and predictor–corrector

25.10. Summary and conclusions

26. Viscosity Solutions and Penalty Methods for American Option Problems

26.1. Introduction and objectives

26.2. Definitions and main results for parabolic problems

26.2.1. Semi-continuity

26.2.2. Viscosity solutions of nonlinear parabolic problems

26.3. An introduction to semi-linear equations and penalty method

26.4. Implicit, explicit and semi-implicit schemes

26.5. Multi-asset American options

26.6. Summary and conclusions

27. Variational Formulation of American Option Problems

27.1. Introduction and objectives

27.2. A short history of variational inequalities

27.3. A first parabolic variational inequality

27.4. Functional analysis background

27.5. Kinds of variational inequalities

27.5.1. Diffusion with semi-permeable membrane

27.5.2. A one-dimensional finite element approximation

27.6. Variational inequalities using Rothe's methods

27.7. American options and variational inequalities

27.8. Summary and conclusions

Part VII.

Design and implementation in C++

28. Finding the Appropriate Finite Difference Schemes for your Financial Engineering Problem

28.1. Introduction and objectives

28.2. The financial model

28.3. The viewpoints in the continuous model

28.3.1. Payoff functions

28.3.2. Boundary conditions

28.3.3. Transformations

28.4. The viewpoints in the discrete model

28.4.1. Functional and non-functional requirements

28.4.2. Approximating the spatial derivatives in the PDE

28.4.3. Time discretisation in the PDE

28.4.4. Payoff functions

28.4.5. Boundary conditions

28.5. Auxiliary numerical methods

28.6. New Developments

28.7. Summary and conclusions

29. Design and Implementation of First-Order Problems

29.1. Introduction and objectives

29.2. Software requirements

29.3. Modular decomposition

29.4. Useful C++ data structures

29.5. One-factor models

29.5.1. Main program and output

29.6. Multi-factor models

29.7. Generalizations and applications to quantitative finance

29.8. Summary and conclusions

29.9. Appendix: Useful data structures in C++

30. Moving to Black–Scholes

30.1. Introduction and objectives

30.2. The PDE model

30.3. The FDM model

30.4. Algorithms and data structures

30.5. The C++ model

30.6. Test case: The two-dimensional heat equation

30.7. Finite difference solution

30.8. Moving to software and method implementation

30.8.1. Defining the continuous problem

30.8.2. Creating a mesh

30.8.3. Choosing a scheme

30.8.4. Termination criterion

30.9. Generalizations

30.9.1. More general PDEs

30.9.2. Other finite difference schemes

30.9.3. Flexible software solutions

30.10. Summary and conclusions

31. C++ Class Hierarchies for One-Factor and Two-Factor Payoffs

31.1. Introduction and objectives

31.2. Abstract and concrete payoff classes

31.3. Using payoff classes

31.4. Lightweight payoff classes

31.5. Super-lightweight payoff functions

31.6. Payoff functions for multi-asset option problems

31.7. Caveat: non-smooth payoff and convergence degradation

31.8. Summary and conclusions