

# **Finite Differences Methods in Financial Engineering**

**A partial Differential Equation Approach**

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