DL-Lite Full: a sub-Language of OWL 2 Full for Powerful Meta-modeling (Appendix)

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Appendix: Proofs of the theorems and lemmas in the manuscript

In the following, **for simplification**, for a role S and $x, y \in \mathsf{V} \cup \mathsf{N}$, we define $S(x, y) = P(x, y)$ if $S = P$ and $P \in \mathsf{N}$ and define $S(x, y) = P(y, x)$ if $S = P^-$ and $P \in \mathsf{N}$.

Proof of Proposition 1

Proof. (1. $\Rightarrow$) If $\mathcal{K}$ is ν-satisfiable then obviously, $\mathcal{K}'$ is ν-satisfiable. For each axiom α with the form:

$$B \sqsubseteq_c \neg\exists\mathsf{type}, \;\; B \sqsubseteq_c \neg\exists\mathsf{type}^-, \;\; P \sqsubseteq_r \neg\mathsf{type}, \;\; P \sqsubseteq \neg\mathsf{type}^- \qquad (\spadesuit)$$

$\mathsf{ans}_\nu(\delta(\alpha), \mathcal{K}) = \emptyset$ holds. $\mathsf{ans}_\nu(\delta(\alpha), \mathcal{K}') \subseteq \mathsf{ans}_\nu(\delta(\alpha), \mathcal{K})$ holds trivially. Thus $\mathsf{ans}_\nu(\delta(\alpha), \mathcal{K}) = \emptyset$ holds.

(1. $\Leftarrow$) We show that each ν-model of $\mathcal{K}'$ is also a ν-model of $\mathcal{K}$. $\mathcal{K}'$ is ν-satisfiable. Let $\mathcal{V}$ be an arbitrary ν-model of $\mathcal{K}'$. For each disjoint axiom $\alpha : B \sqsubseteq_c \neg C$, where C is $\exists\mathsf{type}$ or $\exists\mathsf{type}^-$, $\mathsf{ans}_\nu(\delta(\alpha), \mathcal{K}') = \emptyset$. Thus $\mathcal{V} \models \alpha$. For each disjoint axiom $\beta : S \sqsubseteq \neg R$ where R is type or type^-, we can similarly prove that $\mathcal{V} \models_\nu \beta$ holds. Thus $\mathcal{V}$ is a ν-model of $\mathcal{K}$. Thus $\mathcal{K}$ is satisfiable.

(2) Obviously, each ν-model of $\mathcal{K}$ is also a ν-model of $\mathcal{K}'$. According to (1. $\Leftarrow$), each ν-model of $\mathcal{K}'$ is also a ν-model of $\mathcal{K}$. Thus, for each MQ Q, $\mathsf{ans}_\nu(Q, \mathcal{K}) = \mathsf{ans}_\nu(Q, \mathcal{K}')$ holds.

□

Proof of Lemma 1

Proof. (1) $\mathcal{K}$ is ν-satisfiable, thus it has a ν-model $\mathcal{V}$. From $\mathcal{V}$, we construct an interpretation $\mathcal{I}$ of $\tau(\mathcal{K})$ by setting (a) $\Delta^\mathcal{I} = \Delta^\mathcal{V}$; (b) for each $a \in \mathsf{N}$, $a^\mathcal{I} = a^\mathcal{V}$; (c) for each $A \in \mathsf{C}$, $A^\mathcal{I} = \mathcal{C}^\mathcal{V}(v_c^-(A)^\mathcal{V})$; and (d) for each $P \in \mathsf{R}$, $P^\mathcal{I} = \mathcal{R}^\mathcal{V}(v_r^-(A)^\mathcal{V})$, where v_c^- and v_r^- denote the inverse functions of v_c and v_r, respectively. DL-Lite Full and DL-Lite$_\mathcal{R}$ follow the same principles to interpret the constructors of classes and roles. Thus we can get that $(\spadesuit)$ $\mathcal{C}^\mathcal{V}(C^\mathcal{V}) = \tau_c(C)^\mathcal{I}$ for each DL-Lite Full class C and $\mathcal{R}^\mathcal{V}(R^\mathcal{V}) = \tau_r(R)^\mathcal{I}$ for each DL-Lite Full role R. Then we can further obtain that $\mathcal{I}$ satisfies all the axioms and assertions in $\tau(\mathcal{K})$. Thus $\tau(\mathcal{K})$ is satisfiable.

(2) If $\mathcal{K}$ is not ν-satisfiable then this conclusion holds directly. Next, we assume $\mathcal{K}$ is ν-satisfiable. Thus $\tau(\mathcal{K})$ is satisfiable. Let $\vec{u} \in \mathsf{ans}(\tau(Q), \tau(\mathcal{K}))$. Let $\mathcal{V}$ be an arbitrary ν-model of $\mathcal{K}$. By the approach described in (1), a model $\mathcal{I}$ of $\tau(\mathcal{K})$ can be constructed. Thus there exists a binding π of $\tau(Q)$ over $\mathcal{I}$ such that $\mathcal{I}, \pi \models \tau(Q(\vec{u}))$ holds. From π, a binding π' of $Q(\vec{u})$ over $\mathcal{V}$ can be constructed by setting $\pi'(x) = \pi(x)$ for each variable x of $Q(\vec{u})$ and $\pi'(a) = \pi(a)$ for each name a in $Q(\vec{u})$. Then by $(\spadesuit)$, we can further obtain that $\mathcal{V}, \pi' \models Q(\vec{u})$. Thus $\vec{u} \in \mathsf{ans}_\nu(Q, \mathcal{K})$ holds. Therefore, this direction holds. □

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Preprint submitted to Elsevier *August 26, 2021*

Proof of Theorem 1

Proof. Let $\tau^{\mathsf{m}}(\mathcal{K}) = (\tau(\mathcal{T}), \tau(\mathcal{A}) \cup \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3)$ where $\mathcal{N}_1 - \mathcal{N}_3$ are the sets of assertions obtained from the Step 1-3 respectively. Let $\mathcal{A}_2 - \mathcal{A}_3$ be the sets of assertions such that $\tau(\mathcal{A}_i) = \mathcal{N}_i$ for $i \in \{2, 3\}$. Suppose:

$$\begin{aligned}\mathcal{A}_2 = &\quad \{A_1(a_1), \cdots, A_k(a_k)\}\\ \mathcal{A}_3 = &\quad \{P_1(b_1, c_1), c_1(b_1), \cdots, P_m(b_m, c_m), c_m(b_m)\} \quad \text{then:}\end{aligned}$$

$$\begin{aligned}\mathcal{N}_2 = &\quad \{v_c(A_1)(a_1), \cdots, v_c(A_k)(a_k)\}\\ \mathcal{N}_3 = &\quad \{v_r(P_1)(b_1, c_1), v_c(c_1)(b_1), \cdots, v_r(P_m)(b_m, c_m), v_c(c_m)(b_m)\}\end{aligned}$$

And let $\mathcal{K}' = (\mathcal{T}, \mathcal{A} \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3)$. Then $\tau^{\mathsf{m}}(\mathcal{K}) = \tau(\mathcal{K}')$ holds. And for each $1 \le i \le k$, $\tau(\mathcal{K}) \models \exists v_r(\mathsf{type})^-(A_i)$, and for each $1 \le j \le m$, $\tau(\mathcal{K})$ entails $\exists v_r(P_j)^-(c_j)$ and $v_r(P_j) \sqsubseteq v_r(\mathsf{type})$. And the corresponding individual assertions in $\mathcal{N}_1$ are entailed by $\mathcal{K}$. For example, if $A(a) \in \mathcal{N}_1$ then $\mathcal{K}$ v-entails $v_c^-(A)(a)$, since $\tau(\mathcal{K})$ entails $v_r(\mathsf{type})(a, v_c^-(A))$. Thus here we do not give the concrete assertions in $\mathcal{N}_1$.

(1. $\Rightarrow$) $\mathcal{K}$ is v-satisfiable, thus it has a v-model $\mathcal{V}$. Obviously, $\mathcal{V}$ satisfies the assertions in $\mathcal{A}_1$. The satisfiability of $\tau^m(\mathcal{K})$ can be proved by showing that $\mathcal{K}'$ is satisfiable. Next, based on $\mathcal{V}$, we construct a model for $\mathcal{K}'$. $a_1, \cdots, a_k$ and $b_1, \cdots, b_m$ are names not occurring in $\mathcal{K}$. So we can assume that the class and role extensions of the interpretations of $a_1, \cdots, a_k$ and $b_1, \cdots, b_m$ are empty set, and the interpretations of $a_1, \cdots, a_k$ and $b_1, \cdots, b_m$ do not occur in the class and role extensions of other elements. Otherwise, we can make the following operation. Let $\{a'_1, \cdots, a'_k, b'_1, \cdots, b'_m\}$ be $k + m$ new elements not occurring in $\Delta^{\mathcal{V}} \cup \mathsf{N}$. In $\mathcal{V}$, set $\Delta^{\mathcal{V}} = \Delta^{\mathcal{V}} \cup \{a'_1, \cdots, a'_k, b'_1, \cdots, b'_m\}$, $a_i^{\mathcal{V}} = a'_i$ and $b_j^{\mathcal{V}} = b'_j$ as well as $\mathcal{C}^{\mathcal{V}}(o) = \mathcal{R}^{\mathcal{V}}(o) = \emptyset$ for $1 \le i \le k$, $1 \le j \le m$ and $o \in \{a'_1, \cdots, a'_k, b'_1, \cdots, b'_m\}$, while keeping the others unchanged. Then $\mathcal{V}$ is still a v-model of $\mathcal{K}$.

Next, based on $\mathcal{V}$, we construct a v-model of $\mathcal{K}'$ by first making a copy of $\mathcal{V}$ for each fresh name a_i and b_j and then merging these copies. For each $A_i(a_i)$, we know $\mathcal{K}$ entails $\exists\mathsf{type}^-(A_i)$. Thus there exists $(e_i, A_i^{\mathcal{V}}) \in \mathcal{R}^{\mathcal{V}}(\mathsf{type}^{\mathcal{V}})$. Similarly, for each $P_j(b_j, c_j)$ and $c_j(b_j)$, there exists $o_j \in \Delta^{\mathcal{V}}$ such that $(o_j, b_j^{\mathcal{V}}) \in \mathcal{R}^{\mathcal{V}}(P_j^{\mathcal{V}})$ and $(o_j, b_j^{\mathcal{V}}) \in \mathcal{R}^{\mathcal{V}}(\mathsf{type}^{\mathcal{V}})$. For each $\gamma \in \{a_1, \cdots, a_k, b_1, \cdots, b_m\}$, we make a copy $\mathcal{V}_\gamma = (\Delta^{\mathcal{V}_\gamma}, \cdot^{\mathcal{V}_\gamma}, \mathcal{C}^{\mathcal{V}_\gamma}, \mathcal{R}^{\mathcal{V}_\gamma})$ of $\mathcal{V}$ where (a1) $|\Delta^{\mathcal{V}_\gamma}| = |\Delta^{\mathcal{V}}|$, $\Delta^{\mathcal{V}_\gamma} \cap \Delta^{\mathcal{V}} = \emptyset$ and there exists a bijective function $\delta_\gamma : \Delta^{\mathcal{V}_\gamma} \to \Delta^{\mathcal{V}}$; (b1) for each $a \in \mathsf{N} - \{\gamma\}$, set $a^{\mathcal{V}_\gamma} = \delta_\gamma^-(a^{\mathcal{V}})$; and if $\gamma = a_i$ then set $a^{\mathcal{V}_\gamma} = \delta_\gamma^-(e_i^{\mathcal{V}})$, and if $\gamma = b_j$ then set $a^{\mathcal{V}_\gamma} = \delta_\gamma^-(o_j^{\mathcal{V}})$. and (c1) for each $o \in \Delta^{\mathcal{V}_\gamma}$, $\mathcal{C}^{\mathcal{V}_\gamma}(o) = \{e|\delta_r(e) \in \mathcal{C}^{\mathcal{V}}(\delta_\gamma(o))\}$ and $\mathcal{R}^{\mathcal{V}_\gamma}(o) = \{(e, e')|(\delta(e), \delta(e)') \in \mathcal{R}^{\mathcal{V}}(\delta_\gamma(o))\}$. Then ($\spadesuit$) $\mathcal{V}_\gamma$ is also a v-model of $\mathcal{K}$. For every different elements γ and γ' in $\{a_1, \cdots, a_k, b_1, \cdots, b_m\}$, we assume ($\clubsuit$) $\Delta^{\mathcal{V}_\gamma} \cap \Delta^{\mathcal{V}_{\gamma'}} = \emptyset$.

From $\mathcal{V}, \mathcal{V}_{a_1}, \cdots, \mathcal{V}_{a_k}, \mathcal{V}_{b_1}, \cdots, \mathcal{V}_{b_m}$, we construct a new v-interpretation $\mathcal{V}'$ by setting:

(a2) $\Delta^{\mathcal{V}'} = \Delta^{\mathcal{V}} \cup (\bigcup_{i=1}^k \Delta^{\mathcal{V}_{a_i}}) \cup (\bigcup_{j=1}^m \Delta^{\mathcal{V}_{b_j}})$;

(b2) For each $a \in \mathsf{N} - \{a_1, \cdots, a_k, b_1, \cdots, b_m\}$, $a^{\mathcal{V}'} = a^{\mathcal{V}}$; for each a_i, $a_i^{\mathcal{V}'} = a_i^{\mathcal{V}_{a_i}}$; and for each b_j, $b_j^{\mathcal{V}'} = b_j^{\mathcal{V}_{b_j}}$ for $1 \le i \le k$ and $1 \le j \le m$

(c2) For each $o \in \Delta^{\mathcal{V}'}$, if there exists $a \in \mathsf{N}$ such that $o = a^{\mathcal{V}'}$ then $\mathcal{C}^{\mathcal{V}'}(o) = \mathcal{C}^{\mathcal{V}}(a^{\mathcal{V}}) \cup (\bigcup_{i=1}^k \mathcal{C}^{\mathcal{V}_{a_i}}(a^{\mathcal{V}_{a_i}})) \cup (\bigcup_{j=1}^m \mathcal{C}^{\mathcal{V}_{b_j}}(a^{\mathcal{V}_{b_j}}))$ and $\mathcal{R}^{\mathcal{V}'}(o) = \mathcal{R}^{\mathcal{V}}(a^{\mathcal{V}}) \cup (\bigcup_{i=1}^k \mathcal{R}^{\mathcal{V}_{a_i}}(a^{\mathcal{V}_{a_i}})) \cup (\bigcup_{j=1}^m \mathcal{R}^{\mathcal{V}_{b_j}}(a^{\mathcal{V}_{b_j}}))$; otherwise, there exists $V \in \{\mathcal{V}, \mathcal{V}_{a_1}, \cdots, \mathcal{V}_{a_k}, \mathcal{V}_{b_1}, \cdots, \mathcal{V}_{b_m}\}$ such that $o \in \Delta^V$, then set $\mathcal{C}^{\mathcal{V}'}(o) = \mathcal{C}^V(o)$ and $\mathcal{R}^{\mathcal{V}'}(o) = \mathcal{C}^V(o)$.

By the setting $(b1)$, we can get that $\mathcal{V}'$ satisfies all the individual assertions in $\mathcal{A}_2$ and $\mathcal{A}_3$. Then by ($\spadesuit$) and ($\clubsuit$) as well as the settings $(a2) - (c2)$, we can get that $\mathcal{V}'$ satisfies the axioms in $\mathcal{T}$ and assertions in $\mathcal{A}_1$. Thus $\mathcal{V}'$ is a v-model of $\mathcal{K}'$. From $\mathcal{V}'$, a model $\mathcal{I}$ can be constructed for $\tau^{\mathsf{m}}(\mathcal{K})$ using the approach in the proof of Lemma 1. Thus $\tau^{\mathsf{m}}(\mathcal{K})$ is satisfiable.

(1. $\Leftarrow$) $\tau^{\mathsf{m}}(\mathcal{K})$ is satisfiable. Thus it has a canonical model $\mathcal{I}$ [1]. Let $(e_1, a_1), \cdots, (e_k, a_k)$ be all the elements in $v_r(\mathsf{type})^{\mathcal{I}}$ such that $e_i \notin \mathsf{N}$ and $a_i \in \mathsf{N}$. Let $\mathcal{I}_{e_i}$ be a canonical model of $(\tau(\mathcal{T}), \{v_c(a_i)(e_i)\})$ (here e_i is treated as a name) for $1 \le i \le k$, where ($\blacklozenge$) $(\Delta^{\mathcal{I}} \cap \Delta^{\mathcal{I}_{e_i}}) - (\mathsf{N} - \{e_i\}) = \emptyset$ and $(\Delta^{\mathcal{I}_{e_i}} \cap \Delta^{\mathcal{I}_{e_j}}) - \mathsf{N} = \emptyset$ when $i \neq j$. From $\mathcal{I}$ and $\mathcal{I}_{e_1} - \mathcal{I}_{e_k}$, we construct another interpretation $\mathcal{I}'$ by setting (a1) $\Delta^{\mathcal{I}'} = \Delta^{\mathcal{I}} \cup (\bigcup_{i=1}^m \Delta^{\mathcal{I}_{e_i}})$; (b1) for each $a \in \mathsf{N}$, $a^{\mathcal{I}'} = a$; and (c1) for each $A \in \mathsf{C}$, $A^{\mathcal{I}'} = A^{\mathcal{I}} \cup (\bigcup_{i=1}^m A^{\mathcal{I}_{e_i}})$, and for each $P \in \mathsf{R}$, $P^{\mathcal{I}'} = P^{\mathcal{I}} \cup (\bigcup_{i=1}^m P^{\mathcal{I}_{e_i}})$. Obviously, $\mathcal{I}'$

[1]As shown in [7], a satisfiable DL-Lite$_{\mathcal{R}}$ KB O has a canonical interpretation $\mathcal{I}$ such that for each individual a, $a^{\mathcal{I}} = a$ holds, and for each CQ q, $\vec{u} \in \mathsf{ans}(q, O)$ iff $\mathcal{I} \models q(\vec{u})$.

satisfies the individual assertions as well as positive inclusion axioms in $\tau^{\mathsf{m}}(\mathcal{K})$. Next, we show $\mathcal{I}'$ also satisfies the negative axioms in $\tau(\mathcal{T})$. Actually, we just need to check class disjoint axioms. For each $C \sqsubseteq \neg D \in \tau(\mathcal{T})$, suppose ($\spadesuit$) there exists $o \in C^{\mathcal{I}'} \cap D^{\mathcal{I}'}$. Then by the construction of $\mathcal{I}'$, we can get that there exists e_i such that $o = e_i$, and $e_i \in C^{\mathcal{I}}$ and $e_i \in D^{\mathcal{I}_{e_i}}$ holds, or $e_i \in C^{\mathcal{I}_{e_i}}$ and $e_i \in D^{\mathcal{I}}$ holds, since $C^{\mathcal{I}} \cap D^{\mathcal{I}} = C^{\mathcal{I}_{e_i}} \cap D^{\mathcal{I}_{e_i}} = \emptyset$. Suppose $e_i \in C^{\mathcal{I}} \cap D^{\mathcal{I}_{e_i}}$. Then we can get that $\tau(\mathcal{K})$ entails $v_c(c_i) \sqsubseteq D$ and there exists role S such that $(e_i, c_i) \in S^{\mathcal{I}}$ and $\tau(\mathcal{T})$ entails $\exists S \sqsubseteq C$ and $S \sqsubseteq v_r(\mathsf{type})$. Then there exists $S(b_i, c_i), v_c(c_i)(b_i)$ in $\mathcal{A}_2$. And this contradicts with the fact that $\mathcal{K}_o = (\tau(\mathcal{T}), \{S(b_i, c_i), v_c(c_i)(b_i)\})$ is satisfiable, since $\mathcal{K}_o$ entails $C(b_i)$ and $D(b_i)$. Thus assumption ($\spadesuit$) does not hold. Thus $\mathcal{I}'$ satisfies all the negative axioms in $\tau(\mathcal{T})$. Thus $\mathcal{I}'$ is a model of $\tau^{m}(\mathcal{K})$. From $\mathcal{I}'$, we construct a ν-interpretation $\mathcal{V}$ for $\mathcal{K}$ by setting $(a2)$ $\Delta^{\mathcal{V}} = \Delta^{\mathcal{I}'}$; $(b2)$ for each $a \in \mathsf{N}$, $a^{\mathcal{V}} = a$; $(c2)$ for each $o \in \Delta^{\mathcal{V}}$, if $o \in \mathsf{N}$, then $\mathcal{C}^{\mathcal{V}}(o) = o^{\mathcal{I}'}$, otherwise $\mathcal{C}^{\mathcal{V}}(o) = \{e | (e, o) \in \mathsf{type}^{\mathcal{I}'}\}$; and $(d2)$ for each $o \in \Delta^{\mathcal{V}}$, if $o \in \mathsf{N} - \{\mathsf{type}\}$, then $\mathcal{R}^{\mathcal{V}}(o) = o^{\mathcal{I}'}$, otherwise if $o = \mathsf{type}$ then $\mathcal{R}^{\mathcal{V}}(o) = o^{\mathcal{I}'} \cup \{(e, e') | e' \in \Delta^{\mathcal{V}} \wedge e \in \mathcal{C}^{\mathcal{V}}(e')\}$, otherwise $\mathcal{R}^{\mathcal{V}}(o) = \emptyset$. Next, we show $\mathcal{V}$ is correctly defined by checking the interpretation of type, i.e., ($\clubsuit$) $\mathcal{R}^{\mathcal{V}}(\mathsf{type}) = \{(e, o) | o \in \Delta^{\mathcal{V}}, e \in \mathcal{C}^{\mathcal{V}}(o)\}$ holds. For each $(e, o) \in \mathcal{R}^{\mathcal{V}}(\mathsf{type})$, if $o \in \mathsf{N}$ then $e \in v_c(o)^{\mathcal{I}'}$ holds, thus $(e, o) \in \{(e, o) | o \in \Delta^{\mathcal{V}}, e \in \mathcal{C}^{\mathcal{V}}(o)\}$ holds; otherwise $(e, o) \in \{(e, o) | o \in \Delta^{\mathcal{V}}, e \in \mathcal{C}^{\mathcal{V}}(o)\}$ holds directly. The other direction of ($\clubsuit$) holds directly. Thus $\mathcal{V}$ is a ν-model of $\mathcal{K}$. $\mathcal{V}$ and $\mathcal{I}'$ follow the same principles to interpret class and role constructors. Then by the construction of $\mathcal{V}$, we can get that $\mathcal{V}$ satisfies all the axioms and assertions in $\mathcal{K}$. Thus $\mathcal{K}$ is ν-satisfiable.

(2. $\subseteq$) Let $\vec{u} \in \mathsf{ans}_\nu(Q, \mathcal{K})$. Obviously, $\vec{u}$ does not contain the extra names in $\mathcal{N}_2 \cup \mathcal{N}_3$. Let $\mathcal{I}$ be a canonical model of $\tau^{\mathsf{m}}(\mathcal{K})$. From $\mathcal{I}$, a ν-model $\mathcal{V}$ can be constructed from an extension $\mathcal{I}'$ of $\mathcal{I}$ by the approach described in (1. $\Leftarrow$). Thus there exists a binding π of Q over $\mathcal{V}$ such that $\mathcal{V}, \pi \models_\nu Q(\vec{u})$. From π, a binding π' of $\tau(Q(\vec{u}))$ over $\mathcal{I}$ can be constructed by setting $\pi'(x) = \pi(x)$ for each variable x in $\tau(Q)$ and $\pi'(a) = \pi(a)$ for each individual a in $\tau(Q)$. Then, $\mathcal{I}', \pi' \models \tau(Q(\vec{u}))$ holds. In the construction of $\mathcal{I}'$ ((1. $\Leftarrow$)), for each $v_c(a_i)(e_i)$, there exists assertion $v_c(a_i)(o) \in \mathcal{N}_2$ or $v_r(P_j)(b_j, a_i), v_c(a_i)(b_j) \in \mathcal{N}_3$ such that $\tau(\mathcal{T}) \models \exists v_r(\mathsf{type})^-(a_i)$ or $\tau(\mathcal{T}) \models \exists v_r(P_j)^-(a_i)$ and $v_r(P_j) \sqsubseteq v_r(\mathsf{type})$. This means that ($\bigstar$) there exists a mapping δ_{e_i} from $\Delta^{\mathcal{I}_{e_i}}$ to $\Delta^{\mathcal{I}}$ such that for each class $A \in \mathsf{C}$, $\{\delta(o) | o \in A^{\mathcal{I}_{e_i}}\} \subseteq A^{\mathcal{I}}$ and for each role $P \in \mathsf{R}$, $\{(\delta(o), \delta(e)) | (o, e) \in P^{\mathcal{I}_{e_i}}\} \subseteq P^{\mathcal{I}}$. Then from π' and the functions $\mathcal{I}_{e_1} - \mathcal{I}_{e_k}$, a binding π'' of $\tau(Q(\vec{u}))$ over $\mathcal{I}$ can be constructed by the settings that (a) for each individual a in $\tau(Q(\vec{u}))$, then set $\pi''(a) = a$; (b) for each variable x in $\tau(Q(\vec{u}))$, if $\pi'(x) \in \Delta^{\mathcal{I}}$ then set $\pi''(x) = \pi'(x)$, otherwise there exists e_i such that $\pi'(x) \in \Delta^{\mathcal{I}_{e_i}}$ then set $\pi''(x) = \delta_{e_i}(\pi'(x))$. Then based on ($\blacklozenge$) (in (1. $\Leftarrow$)) and ($\bigstar$), $\mathcal{I}, \pi'' \models \tau(Q(\vec{u}))$ holds. Thus $\vec{u} \in \mathsf{ans}(Q, \tau^{\mathsf{m}}(\mathcal{K}))$ holds.

(2. $\supseteq$) Let $\vec{u} \in \mathsf{ans}(\tau(Q), \tau^{\mathsf{m}}(\mathcal{K}))$. $\vec{u}$ solely contains the names occurring in $\mathcal{K}$. Next, we show that $\vec{u} \in \mathsf{ans}_\nu(Q, \mathcal{K})$. Let $\mathcal{V}$ be an arbitrary model of $\mathcal{K}$. From $\mathcal{V}$, we can construct a model $\mathcal{V}'$ of $\mathcal{K}'$ using the approach in (1. $\Rightarrow$). Then a model $\mathcal{I}$ of $\tau^{\mathsf{m}}(\mathcal{K})$ can be constructed from $\mathcal{V}'$ using the approach described in (1. $\Rightarrow$). Thus there exists a binding π of $\tau(Q(\vec{u}))$ over $\mathcal{I}$ such that $\mathcal{I}, \pi \models \tau(Q\vec{u})$. From π, a binding π' of $Q(\vec{u})$ over $\mathcal{V}'$ can be constructed by setting $\pi'(a) = a^{\mathcal{V}'}$ for each name a in $Q(\vec{u})$ and $\pi'(x) = \pi(x)$ for each variable x in $Q(\vec{u})$. Then by the construction of $\mathcal{I}$, $\mathcal{V}', \pi' \models Q(\vec{u})$ holds. $\vec{u}$ does not contain the names not occurring in $\mathcal{K}$. Then by the construction of $\mathcal{V}'$ (illustrated in (1. $\Rightarrow$)), we can construct a binding π'' of $Q(\vec{u})$ over $\mathcal{V}$ by the setting that for name or variable x in $Q(\vec{u})$, if $\pi'(x) \in \Delta^{\mathcal{V}}$ then set $\pi''(x) = \pi(x)$, otherwise there exists $\gamma \in \{a_1, \cdots, a_k, b_1, \cdots, b_m\}$ such that $\pi'(x) \in \Delta^{\mathcal{V}_\gamma}$ then set $\pi''(x) = \delta_\gamma^-(\pi(x))$. Then by the construction of these $\mathcal{V}_\gamma$, $\mathcal{V}, \pi'' \models Q(\vec{u})$ holds. From the arbitrary feature of $\mathcal{V}$, $\vec{u} \in \mathsf{ans}_\nu(Q, \mathcal{K})$ holds. □

Proof of Theorem 2

Proof. Suppose $\tau^{\mathsf{m}}(\mathcal{K}) = (\tau(\mathcal{T}), \tau(\mathcal{A}) \cup \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3)$. Similar to the proof of Theorem 1, Let $\mathcal{A}_1 - \mathcal{A}_3$ be the sets of assertions such that $\tau(\mathcal{A}_i) = \mathcal{N}_i$ for $i \in \{1, 2, 3\}$. Let $\mathcal{K}' = (\mathcal{T}, \mathcal{A} \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3)$. Then $\tau^{\mathsf{m}}(\mathcal{K}) = \tau(\mathcal{K}')$. It holds trivially that for each conjunctive query Q' and DL-Lite Full ABox $\mathcal{A}$, $\mathsf{ans}_\nu(Q', \mathcal{A}) = \mathsf{ans}(\tau(Q'), \tau(\mathcal{A}))$ holds.

(1. $\Rightarrow$) Suppose there exists $q \in \mathsf{Violates}_\nu(\mathcal{T})$ such that $\mathsf{ans}_\nu(q, \mathcal{A}) \neq \emptyset$. Then there exists a disjoint axiom $\gamma \sqsubseteq_l \neg\gamma'$ ($l \in \{r, c\}$) in $\mathcal{T}$, such that $q \in \mathsf{PerfectRef}_\nu(Q, \mathcal{T})$, where Q is the query corresponding to $\gamma \sqsubseteq_l \neg\gamma'$. For example, if $\gamma \sqsubseteq_l \neg\gamma'$ is $A \sqsubseteq_c \neg B$, then Q is $A(?x) \wedge B(?x) \rightarrow q()$ where A and B are names. According to the algorithm $\mathsf{PerfectRef}_\nu$, $\mathsf{ans}_\nu(q, \mathcal{A}) \subseteq \mathsf{ans}_\nu(Q, \mathcal{K})$ holds trivially. Thus $\mathsf{ans}_\nu(Q, \mathcal{K}) \neq \emptyset$. This means $\mathcal{K}$ entails that γ and γ' share some individuals. This contradicts with that $\mathcal{K}$ is ν-satisfiable. Thus for each $q \in \mathsf{Violate}_\nu(\mathcal{T})$, $\mathsf{ans}_\nu(q, \mathcal{A}) = \emptyset$.

(1. $\Leftarrow$) Suppose ($\spadesuit$) $\mathcal{K}$ is not ν-satisfiable. Then $\tau^{\mathsf{m}}(\mathcal{K})$ is not satisfiable. Then there exists a disjoint axiom in $\mathsf{cl}(\tau(\mathcal{T}))$ (the negative closure of $\tau(\mathcal{T})$ defined in [7]), suppose it is $A \sqsubseteq \neg B$, such that ($\clubsuit$) $\mathsf{ans}(q, \tau(\mathcal{A}) \cup \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3) \neq \emptyset$, where q is the query $A(?x) \wedge B(?x) \rightarrow q()$. From the algorithm $\mathsf{Violates}_\nu$, we can get that there exists $Q \in \mathsf{Violates}_\nu(\mathcal{T})$ such that $\tau(Q) = q$. From ($\clubsuit$), we can also get that (a) $\mathsf{ans}(q, \tau(\mathcal{A})) \neq \emptyset$; (bi) $\mathsf{ans}(q, \mathcal{N}_i) \neq \emptyset$

where $i = 1, 2, 3$, since $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$ do not share any individuals; (c) $\mathsf{ans}(q, \tau(\mathcal{A}) \cup \mathcal{N}_1) \neq \emptyset$. If (a) holds, then obviously $\mathsf{ans}_\nu(Q, \mathcal{A}) \neq \emptyset$. This contradicts with the condition that (C) for each $Q_o \in \mathsf{Violates}_\nu(\mathcal{T})$, $\mathsf{ans}_\nu(Q_o, \mathcal{A}) \neq \emptyset$. If (b1) holds, i.e., $\mathsf{ans}(q, \mathcal{N}_1) \neq \emptyset$. Then there exists individual a such that $A(a)$ and $B(a)$ in $\mathcal{N}_1$. Then from the construction of $\mathcal{N}_1$, $\tau(\mathcal{K}) \models \nu_r(\mathsf{type})(a, \nu_c^-(A))$ and $\tau(\mathcal{K}) \models \nu_r(\mathsf{type})(a, \nu_c^-(B))$ hold. Thus there exists $P_1(a, \nu_c^-(A))$ and $P_2(a, \nu_c^-(B))$ in $\tau(\mathcal{A})$ such that $\tau(\mathcal{T}) \models P_1 \sqsubseteq \nu_r(\mathsf{type})$ and $\tau(\mathcal{T}) \models P_2 \sqsubseteq \nu_r(\mathsf{type})$. According to $\mathsf{PerfectRef}_\nu$, we can get that $q : \nu_r^-(P_1)(?x, \nu_c^-(A)) \wedge \nu_r^-(P_2)`(?x, \nu_c^-(B)) \to q()$ in $\mathsf{Violates}_\nu(\mathcal{T})$. Obviously, $\mathsf{ans}_\nu(q, \mathcal{A}) \neq \emptyset$. This also contradicts with (C), thus (b1) does not hold.If (b2) holds, then $A = B$ must holds, since the assertion in $\mathcal{A}_2$ do not share individuals. From the construction of $\mathcal{N}_2$, $\tau(\mathcal{K}) \models \exists \nu_r(\mathsf{type})^-(\nu_c^-(A))$. Then we can get that (d1) there exists $P(a, \nu_c^-(A)) \in \tau(\mathcal{A})$ such that $\tau(\mathcal{T}) \models \exists P^- \sqsubseteq \exists \nu_r(\mathsf{type})^-$; or (d2) there exists $A'(\nu_c^-(A)) \in \tau(\mathcal{A})$ such that $\tau(\mathcal{T}) \models A' \sqsubseteq \exists \nu_c^-(\mathsf{type})$. Then $Q' \in \mathsf{PerfectRef}_\nu(\nu_c^-(A)(x) \to q(), \mathcal{T})$ where Q' is $\nu_r^-(P)(x, \nu_c^-(A)) \to q()$ if case (d1) holds and Q' is $\nu_c^-(A')(\nu_c^-(A)) \to q()$ if case (d2) holds. And $\mathsf{ans}_\nu(Q', \mathcal{A}) \neq \emptyset$. This also contradicts with condition (C). Thus (b2) does not hold. Condition (b3) and (c) as well as other forms of the disjoint axioms can be validated similarly. Thus the assumption (♠) does not holds. Therefore $\mathcal{K}$ is satisfiable.

(2. $\supseteq$) This direction holds directly according to the following two observations. (O_1) For two conjunctive queries Q' and Q'', if Q'' is obtained from Q' by applying an inclusion axiom over an atom of Q' via the rule described in Algorithm 1, then it holds trivially that $\mathsf{ans}_\nu(Q'', \mathcal{K}) \subseteq \mathsf{ans}_\nu(Q', \mathcal{K})$ holds. (O_2) For each conjunctive query Q', $\mathsf{ans}_\nu(Q', \mathcal{A}) \subseteq \mathsf{ans}_\nu(Q', \mathcal{K})$. Combing ($\mathsf{O}_1$) and ($\mathsf{O}_2$), this direction holds trivially.

(2. $\subseteq$) Let $\vec{u} \in \mathsf{ans}_\nu(Q, \mathcal{K})$. Next, we show that (♣) there exists $Q' \in \mathsf{PerfectRef}_\nu(Q, \mathcal{T})$ such that $\vec{u} \in \mathsf{ans}_\nu(Q', \mathcal{A})$. By Theorem 1, $\vec{u} \in \mathsf{ans}(\tau(Q), \tau^{\mathsf{m}}(\mathcal{K}))$ holds. Thus there exists $q \in \mathsf{PerfectRef}(\tau(Q), \tau(\mathcal{T}))$ such that $\vec{u} \in \mathsf{ans}(q, \tau(\mathcal{A}) \cup \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3)$. Let Q' be the query such that $\tau(Q') = q$. Then $Q' \in \mathsf{PerfectRef}_\nu(Q, \mathcal{T})$. If $\vec{u} \in \mathsf{ans}(q, \tau(\mathcal{A}))$ holds, then $\vec{u} \in \mathsf{ans}_\nu(Q', \mathcal{A})$ holds directly. Otherwise let $\mathcal{N}_1'$, $\mathcal{N}_2'$ and $\mathcal{N}_3'$ be the smallest sets of $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$ respectively such that $\vec{u} \in \mathsf{ans}(q, \tau(\mathcal{A}) \cup \mathcal{N}_1' \cup \mathcal{N}_2' \cup \mathcal{N}_3')$. Suppose:

$$
\begin{aligned}
\mathcal{N}_1' &= \{A_1(a_1), \cdots, A_k(a_k)\} \\
\mathcal{N}_2' &= \{B_1(b_1), \cdots, B_m(b_m)\} \\
\mathcal{N}_3' &= \{P_1(c_1, d_1), \nu_c(d_1)(c_1), \cdots, P_l(c_l, d_l), \nu_c(d_l)(c_l)\} \cup \{C_1(o_1), \cdots, C_n(o_n)\} \cup \{S_1(e_1, f_1), \cdots, S_h(e_h, f_h)\}
\end{aligned}
$$

where (♣) b_i, c_j, d_l, o_h, e_d, and f_w are all different names and do not occur in $\tau(\mathcal{K})$. Then q is :

$$
(\bigwedge_{i=1}^{j} \alpha_i) \wedge (\bigwedge_{i=1}^{k} A_i(x_i^1)) \wedge (\bigwedge_{i=1}^{m} B_i(x_i^2)) \wedge (\bigwedge_{i=1}^{l} (P_i(x_i^3, y_i^3) \wedge \nu_c(d_i)(x'^3_i))) \wedge (\bigwedge_{i=1}^{n} C_i(x_i^4)) \wedge (\bigwedge_{i=1}^{l} S_i(x_i^5, y_i^5)) \to q(\vec{x})
$$

Therefore Q' is the query:

$$
(\bigwedge_{i=1}^{j} \beta_i) \wedge (\bigwedge_{i=1}^{k} \nu_c^-(A_i)(x_i^1)) \wedge (\bigwedge_{i=1}^{m} \nu_c^-(B_i)(x_i^2)) \wedge (\bigwedge_{i=1}^{l} (\nu_r^-(P_i)(x_i^3, y_i^3) \wedge d_i(x'^3_i))) \wedge (\bigwedge_{i=1}^{n} \nu_c^-(C_i)(x_i^4)) \wedge (\bigwedge_{i=1}^{l} \nu_r^-(S_i)(x_i^5, y_i^5)) \to q(\vec{x})
$$

where $\tau(\beta_i) = \alpha_i$. Q does not contain the names not occurring in $\mathcal{K}$. Then according to (♣), we can get that (♠1) x_i^2, x_j^3, x_l^5 and x_h^5 are non-bind variables; (♠2) y_i^3 is either d_i or is a non-bind variable; (♠3) $x_i^3 = x'^3_i$ or x_i^3 and x'^3_i are both non-bind variable; (♠4) y_i^5 is either f_i or is a non-bind variable. Next, we construct a rewriting Q'' of Q'. Let $Q'' = Q'$. Then do the following operations:

(a) For each $A_i(a_i)$, $\tau(\mathcal{K}) \models \nu_r(\mathsf{type})(a_i, \nu_c^-(A_i))$. Thus, there exists $R_i(a_i, \nu_c^-(A_i)) \in \tau(\mathcal{A})$ such that $\tau(\mathcal{T}) \models R_i \sqsubseteq \nu_r(\mathsf{type})$. Then replace $\nu_c^-(A_i)(x_i^1)$ in Q' with $\nu_r^-(R_i)(x_i^1, \nu_c^-(A_i))$;
(b) For each $P_i(c_i, d_i)$, $\nu_c(d_i)(c_i)$, there exists $P(e, d_i)$ (or $A(d_i)$) in $\tau(\mathcal{A})$ such that $\tau(\mathcal{T})$ entails $\exists P^- \sqsubseteq \exists P_i^-$ (or $A \sqsubseteq \exists P_i^-$) and $P_i \sqsubseteq \nu_r(\mathsf{type})$. Then replace $\nu_r^-(P_i)(x_i^3, y_i^3) \wedge d_i(x'^3_i)$ in Q' with $\nu_r^-(P)(-, d_i)$ (or $\nu_c^-(A)(d_i)$); We explain one situation. Suppose there exists $P(e, d_i) \in \tau(\mathcal{A})$ such that $\tau(\mathcal{T})$ entails $\exists P^- \sqsubseteq \exists P_i^-$ and $P_i \sqsubseteq \nu_r(\mathsf{type})$. Then the expression $\nu_r^-(P_i)(x_i^3, y_i^3) \wedge d_i(x'^3_i)$ can be rewritten into $\nu_r^-(P_i)(x_i^3, y_i^3) \wedge \nu_r^-(p_i)(x'^3_i, d_i)$. Then according to (♠2) and (♠3), the rewritten expression can be unified into $\nu_r^-(p_i)(x_i^3, d_i)$. Then according to $\tau(\mathcal{T})$ entailing $\exists P^- \sqsubseteq \exists P_i^-$, thus $\nu_r^-(p_i)(x_i^3, d_i)$ can be further rewritten into $\nu_r^-(P)(-, d_i)$.
(c) For each $C_i(o_i)$, there also exists $P(e, \nu_c^-(C_i))$ (or $A(\nu_c^-(C_i))$) in $\tau(\mathcal{A})$ and role S such that $\tau(\mathcal{T})$ entails $\exists P^- \sqsubseteq \exists S^-$ (or $A \sqsubseteq \exists S^-$) and $S \sqsubseteq \nu_r(\mathsf{type})$. Then replace $\nu_c^-(C_i)(x_i^4)$ in Q' with $\nu_r^-(P)(-, \nu_c^-(C_i))$ or $\nu_c^-(A)(\nu_c^-(C_i))$;

(d) For each $S_i(e_i, f_i)$, there also exists $P(e, f_i)$ (or $A(f_i)$) in $\tau(\mathcal{A})$ such that $\tau(\mathcal{T})$ entails $\exists P^- \sqsubseteq \exists S_i^-$ (or $A \sqsubseteq \exists S_i^-$) and $S_i \sqsubseteq v_r(\mathsf{type})$. Then replace $v_r^-(S_i)(x_i^5, y_i^5)$ with $v_r^-(P)(-, y_i^5)$ (or $v_c^-(A)(y_i^5)$)

Obviously, $Q'' \in \mathsf{PerfectRef}_v(Q', \mathcal{T})$ and $\vec{u} \in \mathsf{ans}_v(Q'', \mathcal{A})$. $Q' \in \mathsf{PerfectRef}_v(Q, \mathcal{T})$, thus $\mathsf{PerfectRef}_v(Q', \mathcal{T}) \subseteq \mathsf{PerfectRef}_v(Q, \mathcal{T})$ holds, and further $Q'' \in \mathsf{PerfectRef}_v(Q, \mathcal{T})$ holds. Thus this direction holds. □

Proof of Theorem 3

Proof. By Theorem 2, this theorem holds. □

Proof of Theorem 4

Proof. Let $\tau^{\mathsf{m}}(\mathcal{K}) = (\tau(\mathcal{T}), \tau(\mathcal{A}) \cup \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3)$ where $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$ are the sets of assertions obtained from Steps 1-3, respectively.

($\subseteq$) Let $\vec{u} \in \mathsf{ans}_v(Q, \mathcal{K})$. Next, we show there exists $\theta \in \mathsf{MVB}(Q, \mathcal{K})$ and $Q' \in \mathsf{PerfectRef}_v(Q\theta, \mathcal{T})$ without class variables such that $\vec{u} \in \mathsf{ans}_v(Q', \mathcal{A})$ holds. Let $\mathcal{V}$ be a v-model of $\mathcal{K}$ constructed from a canonical model $\mathcal{I}$ of $\tau^{\mathsf{m}}(\mathcal{K})$ using the approach in the (1. $\Leftarrow$) direction of the proof of Theorem 1. Then there exists a binding π of $Q(\vec{u})$ over $\mathcal{V}$ such that $V, \pi \models Q(\vec{u})$. Based on π, we construct a MV-Binding $\theta = (\theta_r, \theta_c)$ of Q over $\mathcal{K}$. For each role variable $?x$ of Q, if $?x$ occurs in the i-th position of $\vec{u}$, then set $\theta_r(?x) = \vec{u}[i]$, otherwise, we know $\pi(?x)$ is a name, then set $\theta_r = \pi(?x)$. For each class variable $?y$ of $[Q\theta_r]$, if $?y$ occurs in the i-th position of $\vec{u}$ then set $\theta_c(?y) = \vec{u}[i]$, otherwise if $\pi(?y)$ is a name occurring in $\mathcal{K}$ then set $\theta_c(?y) = \pi(?y)$. Let q be the query obtained by replacing each atom $P(x, y)$ in $Q\theta$ with $v_r(P)(x, y)$, each $A(x)$ in $Q\theta$ with $v_c(A)(x)$, and each $?c(x)$ in $Q\theta$ with $v_r(\mathsf{type}(x, y)$, where $?c$ is a variable. Then $\vec{u} \in \mathsf{ans}(q, \tau^{\mathsf{m}}(\mathcal{K}))$ holds. Thus there exists $q' \in \mathsf{PerfectRef}(q, \tau(\mathcal{T}))$ such that $\vec{u} \in \mathsf{ans}(q', \tau(\mathcal{A}) \cup \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3)$. According to the algorithm $\mathsf{PerfectRef}_v$, we can get that there exists $Q' \in \mathsf{PerfectRef}_v(Q\theta, \mathcal{T})$ such that $\tau(Q') = q'$. Similar to the proof in the (2. $\supseteq$) direction of the proof of Theorem 2, we can further get that there exists $Q'' \in \mathsf{PerfectRef}_v(Q', \mathcal{T})$ such that $\vec{u} \in \mathsf{ans}_v(Q'', \mathcal{A})$. $\mathsf{PerfectRef}_v(Q', \mathcal{T}) \subseteq \mathsf{PerfectRef}_v(Q\theta, \mathcal{T})$ holds, since $Q' \in \mathsf{PerfectRef}_v(Q\theta, \mathcal{T})$. Thus $Q'' \in \mathsf{PerfectRef}_v(Q\theta, \mathcal{T})$ holds. Thus this direction holds.

($\supseteq$) Obviously, for each MV-Binding ϑ of Q over $\mathcal{K}$, $\mathsf{ans}_v(Q\vartheta, \mathcal{K}) \subseteq \mathsf{ans}_v(Q, \mathcal{K})$. Besides, if a query Q'' is obtained from a query Q' by applying one rule (case) in Algorithm $\mathsf{PerfectRef}_v$, then $\mathsf{ans}_v(Q'', \mathcal{K}) \subseteq \mathsf{ans}_v(Q', \mathcal{K})$ holds. Moreover, for a query Q', $\mathsf{ans}_v(Q', \mathcal{A}) \subseteq \mathsf{ans}_v(Q', \mathcal{K})$ holds. Then combing these three aspects, this direction of the equation in this theorem holds directly. □

Proof of Lemma 2

Proof. Under v-semantics, it holds trivially that $\vec{u} \in \mathsf{ans}(Q, (\emptyset, \mathcal{A}))$ iff there exists a function f such that f maps each variable in $Q(\vec{u})$ to a name in $\mathcal{A}$ and all the atoms in $Q(\vec{u})f$ occur in $\mathcal{A}$. Then this lemma holds. □

Proof of Lemma 3

Proof. (1) We first prove the following equation:

$$\mathsf{MVB}(Q, \mathcal{K}) = \bigcup_{\vartheta \in \mathsf{PMVB}(Q, \mathcal{T})} \mathsf{ePMVB}(\vartheta, Q, \mathcal{K})$$

($\supseteq$) By Definition 5, i.e., the definition of extensions of partial MV-Bindings, this direction of the equation in this lemma holds. ($\subseteq$) Let θ be an arbitrary MV-Binding of Q over $\mathcal{K}$. Next, we construct a partial MV-Binding ϑ of Q over $\mathcal{T}$ such that $\theta \in \mathsf{ePMVB}(\vartheta, Q, \mathcal{K})$. For each role variable x of Q, set $\vartheta(x) = \theta(x)$ *iff* $\theta(x) \in \mathsf{N}_{\mathcal{T}}^{rr} - \{\mathsf{type}\}$, and for each class variable x of Q, set $\vartheta(x) = \theta(x)$ *iff* $\theta(x) \in \mathsf{N}_{\mathcal{T}}^{rc}$. Then by Definition 5, $\theta \in \mathsf{ePMVB}(\vartheta, Q, \mathcal{K})$ holds. Thus the direction ($\subseteq$) holds.

(2) According to the algorithm $\mathsf{PerfectRef}_v$, we can get that (♠) for a conjunctive query q and atom $A(x)$ (resp. $P(x, y)$), if A (resp. P) does not occur in the right hands of any inclusion axioms in a DL-Lite Full TBox $\mathcal{T}$, then this

query atom will not be extended by the inclusion axioms in $\mathcal{T}$ to generate new queries, i.e., it will be occur in the queries in $\mathsf{PerfectRef}(q, \mathcal{T})$. Next, we prove the following inclusions:

$$\bigcup_{\theta \in \mathsf{ePMVB}(\vartheta, Q, \mathcal{K})} \bigcup_{Q' \in \mathsf{PerfectRef}_\nu(Q\theta, \mathcal{T})} \mathsf{ans}_\nu(Q', (\emptyset, \mathcal{A})) \subseteq \bigcup_{Q' \in \mathsf{PerfectRef}_\nu(Q\vartheta, \mathcal{T})} \mathsf{ans}_\nu(Q', (\emptyset, \mathcal{A})) \subseteq \mathsf{ans}_\nu(Q, \mathcal{K})$$

($\subseteq$ .1) Let $\theta \in \mathsf{ePMVB}(\vartheta, Q, \mathcal{K})$, $Q' \in \mathsf{PerfectRef}(Q\theta, \mathcal{T})$ and $\vec{u} \in \mathsf{ans}_\nu(Q', (\emptyset, \mathcal{A}))$, next, we show that there exists $Q_o \in \mathsf{PerfectRef}_\nu(Q\vartheta, \mathcal{T})$, such that $\vec{u} \in \mathsf{ans}_\nu(Q_o, (\emptyset, \mathcal{A}))$ holds.

For $\vartheta = (\vartheta_r, \vartheta_c)$ and $\theta = (\theta_r, \theta_c)$, let ϑ'_r be the function such that $\mathsf{dom}(\vartheta'_r) = \mathsf{dom}(\theta_r) - \mathsf{dom}(\vartheta_r)$, and for each $?r \in \mathsf{dom}(\vartheta'_r)$, $\vartheta'_r(?r) = \theta_r(?r)$. And let $\mathsf{dom}(\vartheta'_c) = \mathsf{dom}(\theta_c) - \mathsf{dom}(\vartheta_c)$, and for each $?c \in \mathsf{dom}(\vartheta'_c)$, $\vartheta'_c(?c) = \theta_c(?c)$. Then $Q\theta = ((Q\vartheta)\vartheta'_r)\vartheta'_c$ holds. This means $Q\theta$ is the query obtained from $Q\vartheta$ by binding the extra class/role variables of Q occurring in $\mathsf{dom}(\vartheta'_r) \cup \mathsf{dom}(\vartheta'_c)$ to the corresponding names not occurring in the right-sides of the inclusion axioms in $\mathcal{T}$. Based on this, we can prove that (♠) for $Q' \in \mathsf{PerfectRef}(Q\theta, \mathcal{T})$ and $\vec{u} \in \mathsf{ans}_\nu(Q', (\emptyset, \mathcal{A}))$, there exists $Q_o \in \mathsf{PerfectRef}(Q\vartheta, \mathcal{T})$ such that $\vec{u} \in \mathsf{ans}_\nu(Q_o, (\emptyset, \mathcal{A}))$.

Let $\{?r_1(x_1, y_1), \cdots, ?r_k(x_k, y_k), ?c_1(z_1), \cdots, ?c_m(z_m)\}$ be all the query atoms in $Q\vartheta$ such that $?r_i \in \mathsf{dom}(\vartheta'_r)$ and $?c_j \in \mathsf{dom}(\vartheta'_c)$. When taking $Q\theta$ and $\mathcal{T}$ as input, suppose Q' is obtained by the following procedure:

$$\mathsf{Pro}: \; Q\theta \xrightarrow{(\gamma_1, \alpha_1)} Q_1 \xrightarrow{(\gamma_2, \alpha_2)} \cdots \xrightarrow{(\gamma_k, \alpha_k)} Q_k$$

where Qk is Q' and Q_i is obtained from Q_{i-1} by applying the inclusion axiom γ_{i-1} over the atom α_{i-1} according to algorithm $\mathsf{PerfectRef}_\nu$ for $1 < i < k-1$. For simplification, we do not consider the unification used in the rewriting procedure. When unification is considered, the result can be proved similarly. For each α_i, there does not exist $?r_m(x_m, y_m)$ such that $\alpha_i = ((?r_m(x_m, y_m))\vartheta'_r)\vartheta'_c$. If some α_i is $((?c_j(z_j))\vartheta'_r)\vartheta'_c$. Then α_i is an axiom such that type occurs in the right-hand of α_i. From Pro, we construct a rewriting of $Q\vartheta$ by the following procedure:

$$\mathsf{Pro}': \; Q\vartheta \xrightarrow{(\gamma_1, \alpha'_1)} Q'_1 \xrightarrow{(\gamma_2, \alpha'_2)} \cdots \xrightarrow{(\gamma_k, \alpha'_k)} Q'_k$$

where α' is the atom in $Q\vartheta$ such that $\alpha_i = (\alpha'_i\vartheta'_r)\vartheta'_c$. Then $Q'_k \in \mathsf{PerfectRef}_\nu(Q\vartheta, \mathcal{T})$. Actually, in this situation, $Q_k = ((Q'_k)\vartheta'_r)\vartheta'_c$, i.e., $Q' = ((Q'_k)\vartheta'_r)\vartheta'_c$ holds.

$\vec{u} \in \mathsf{ans}_\nu(Q', (\emptyset, \mathcal{A}))$ holds. Thus, there exists a binding f of $Q'(\vec{u})$ over $\mathcal{A}$ such that for each variable $?x$ occurring in Q', if $?x$ occurs in the i-th position of $\vec{u}$ then $f(?x) = \vec{u}[i]$, otherwise $f(?x)$ is a name occurring in $\mathcal{A}$. Then $Q'f$ is satisfied by $\mathcal{A}$. From f, we construct a binding f' of $Q'_k(\vec{u})$ over $\mathcal{A}$. For each variable $?x$ occurring in $Q'_k(\vec{u})$, if $?x \in \mathsf{dom}(f)$ then $f'(?x) = f(x)$, otherwise $?x \in \mathsf{dom}(\vartheta'_r) \cup \mathsf{dom}(\vartheta'_c)$. If $?x \in \mathsf{dom}(\vartheta'_r)$ then $f'(?x) = \vartheta'_r(?x)$; and if $?x \in \mathsf{dom}(\vartheta'_c)$ then $f'(?x) = \vartheta'_c(?x)$. Then from the construction of Q'_k and f', we can get that $Q'_k(\vec{u})$ is satisfied by $\mathcal{A}$. Thus $\vec{u} \in \mathsf{ans}_\nu(Q'_k, (\emptyset, \mathcal{A}))$ holds.

From the arbitrary feature of θ, Q' and $\vec{u}$. The first inclusion holds.

($\subseteq$ .2) For the partial MV-binding ϑ, (a) $\mathsf{ans}_\nu(Q\vartheta, \mathcal{K}) \subseteq \mathsf{ans}_\nu(Q, \mathcal{K})$ holds trivially. Besides, for each query Q'_o obtained from a query Q_o by applying an inclusion axiom over one atom of Q_o or by unifying two atoms of Q_o using the way illustrated in algorithm $\mathsf{PerfectRef}_\nu$, then $\mathsf{ans}_\nu(Q'_o, \mathcal{K}) \subseteq \mathsf{ans}_\nu(Q_o, \mathcal{K})$ holds. Thus (b) for each $Q' \in \mathsf{perfectRef}_\nu(Q\vartheta, \mathcal{T})$, $\mathsf{ans}_\nu(Q', \mathcal{K}) \subseteq \mathsf{ans}_\nu(Q\vartheta, \mathcal{K})$ holds. Obviously, (c) $\mathsf{ans}_\nu(Q', (\emptyset, \mathcal{A})) \subseteq \mathsf{ans}_\nu(Q', \mathcal{K})$. Then combining (a)-(c), $\bigcup_{Q' \in \mathsf{PerfectRef}_\nu(Q\vartheta, \mathcal{T})} \mathsf{ans}_\nu(Q', (\emptyset, \mathcal{A})) \subseteq \mathsf{ans}_\nu(Q, \mathcal{K})$ holds. □

Proof of Theorem 5

Proof. For the MQ Q, KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ and partial MV-binding ϑ_o of Q over $\mathcal{T}$, Lemma 3 indicates the following relations $(a1) - (a2)$:

$$\mathsf{MVB}(Q, \mathcal{K}) = \bigcup_{\vartheta \in \mathsf{PMVB}(Q, \mathcal{T})} \mathsf{ePMVB}(\vartheta, Q, \mathcal{K}) \qquad (a1)$$

$$\bigcup_{\theta \in \mathsf{ePMVB}(\vartheta_o, Q, \mathcal{K})} \bigcup_{Q' \in \mathsf{PerfectRef}_\nu(Q\theta, \mathcal{T})} \mathsf{ans}_\nu(Q', (\emptyset, \mathcal{A})) \subseteq \bigcup_{Q' \in \mathsf{PerfectRef}_\nu(Q\vartheta_o, \mathcal{T})} \mathsf{ans}_\nu(Q', (\emptyset, \mathcal{A})) \subseteq \mathsf{ans}_\nu(Q, \mathcal{K}) \qquad (a2)$$

Then by Theorem 4, we can obtain the following equations and inclusion relations:

$$\begin{gathered}
\mathsf{ans}_v(Q,\mathcal{K}) \\
= \\
\bigcup_{\theta\in\mathsf{MVB}(Q,\mathcal{T})}\bigcup_{Q'\in\mathsf{PerfectRef}(Q\theta,\mathcal{T})\downarrow}\mathsf{ans}_v(Q',(\emptyset,\mathcal{A})) \\
\subseteq \\
\bigcup_{\theta\in\mathsf{MVB}(Q,\mathcal{T})}\bigcup_{Q'\in\mathsf{PerfectRef}(Q\theta,\mathcal{T})}\mathsf{ans}_v(Q',(\emptyset,\mathcal{A})) \\
= \\
\bigcup_{\vartheta\in\mathsf{PMVB}(Q,\mathcal{T})}\bigcup_{\theta\in\mathsf{ePMVB}(\vartheta,Q,\mathcal{K})}\bigcup_{Q'\in\mathsf{PerfectRef}(Q\theta,\mathcal{T})}\mathsf{ans}_v(Q',(\emptyset,\mathcal{A})) \\
\subseteq \\
\bigcup_{\vartheta\in\mathsf{PMVB}(Q,\mathcal{T})}\bigcup_{Q'\in\mathsf{PerfectRef}_v(Q\vartheta,\mathcal{T})}\mathsf{ans}_v(Q',(\emptyset,\mathcal{A})) \\
\subseteq \\
\mathsf{ans}_v(Q,\mathcal{K})
\end{gathered}$$

where $\mathsf{PerfectRef}(Q\theta,\mathcal{T})\downarrow$ denotes the set of queries obtained from $\mathsf{PerfectRef}(Q\theta,\mathcal{T})$ by dropping the queries with class variables. Thus the following equation holds:

$$\mathsf{ans}_v(Q,\mathcal{K}) = \bigcup_{\vartheta\in\mathsf{PMVB}(Q,\mathcal{T})}\ \bigcup_{Q'\in\mathsf{PerfectRef}_v(Q\vartheta,\mathcal{T})}\mathsf{ans}_v(Q',(\emptyset,\mathcal{A}))$$

Thus this theorem holds. □

Proof of Theorem 6

Proof. CQ answering over databases has AC_0 data complexity. By Lemma 2, we can get that MQ answering over DL-Lite Full ABox has AC_0 data complexity.

If Q has meta-variables, then it has no more than $2^{2|Q|}(2|\mathcal{T}|+2)^{2|Q|}$ partial MV-Bindings over $\mathcal{T}$. Then by Theorem 5, Definitions 4 and Lemma 4, the complexity results for meta-query answering holds. □