Numerical Analysis Final Report

Using Cholesky Decomposition to solve positive definite linear equations system

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https://youtu.be/dT_EnCPeC3g

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Introduction

Cholesky Decomposition: Basic Explanation

First using LU Decomposition for matrix $\mathbf{A}$:

$$\mathbf{A} = \mathbf{L}\mathbf{U} \tag{1}$$

the upper triangular matrix $\mathbf{U}$ is the same that Gaussian Elimination generates, and the lower triangular matrix $\mathbf{L}$ has all its diagonal elements l_{ii} and the element l_{ij} is the number of the i_{th} row we use to eliminate the j_{th} row.

afterward, $\mathbf{L}\mathbf{U}$ is modified to $\mathbf{L}\mathbf{D}\mathbf{U}'$, where $\mathbf{D}$ is the diagonal of $\mathbf{U}$ and $\mathbf{U}'$ is the divided $\mathbf{U}$, so:

$$\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{U}' \tag{2}$$

Cholesky Decomposition: Basic Explanation

$\lambda > 0$

If $\mathbf{A}$ is positive definite, $\mathbf{U}'$ will be the transpose of $\mathbf{L}$, i.e.:

$$\mathbf{A} = \mathbf{LDU}' = \mathbf{LDL^T} \tag{3}$$

If we decompose $\mathbf{D}$ into $\mathbf{D}^{\frac{1}{2}}\mathbf{D}^{\frac{1}{2}}$, then:

$$\mathbf{A} = \mathbf{LDL^T} = (\mathbf{LD}^{\frac{1}{2}})(\mathbf{D}^{\frac{1}{2}}\mathbf{L^T}) = \mathbf{R^T R} \tag{4}$$

where $\mathbf{R} = \mathbf{D}^{\frac{1}{2}}\mathbf{L^T}$

Example

$$\underset{\mathbb{A}}{\begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}} = \underset{\mathbb{L}}{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}} \underset{\mathbb{U}}{\begin{pmatrix} 2 & 2 \\ 0 & 3 \end{pmatrix}} = \underset{\mathbb{L}}{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}} \underset{\mathbb{D}}{\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}} \underset{\mathbb{H}'}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{pmatrix}^2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \underset{\substack{\mathbb{R}^T \\ \mathbb{L}}}{\begin{pmatrix} \sqrt{2} & 0 \\ \sqrt{2} & \sqrt{3} \end{pmatrix}} \underset{\mathbb{R}}{\begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{pmatrix}}$$

Cholesky Decomposition: Application

Because the function by Numpy (np.linalg.cholesky()) involves the code written in C++, which makes us difficult to compare the operation time, so we ask ChatGPT to write the functions without using Numpy. The function output a lower triangular matrix $\mathbf{L}$, which is equal to $\mathbf{R^T} = \mathbf{LD^{\frac{1}{2}}}$ in the derivation of the last part.

$$\begin{aligned} \mathbf{Ax} &= \mathbf{LL^T x} = \mathbf{y} \\ \mathbf{L^T x} &= \mathbf{L^{-1} y} \Rightarrow \text{Backward Substitution} \end{aligned} \tag{5}$$

Upp. Tri.

Gaussian Elimination

- Transforms a system of linear equations into a triangular form.
- Row operations such as swapping, scaling, and adding/subtracting rows are used.
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- We have done the implementation in NA class. But in this project, we use a Numpy-free version generated by ChatGPT

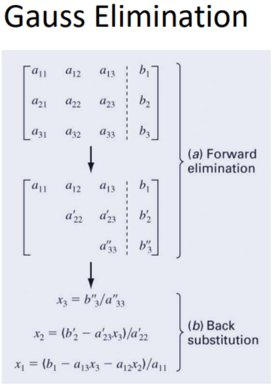

Figure: from L4.pptx of NA class.

Comparison of the Two Methods

Table: Comparison of Two Methods

	Gauss Elimination	Cholesky Decomposition
Time Complexity	$O(n^3)$	$O(n^3)$
Efficiency	Takes more time for larger systems.	Generally more efficient than Gaussian elimination.
Stability	Pivot elements cannot be zero or close to zero.	Only stable for positive definite matrices.
Flexibility	Can handle most general systems of equations.	Matrices should be symmetric and positive definite.

Experiment

By the instruction of ChatGPT, we build a large Positive Definite Matrix with rank 1000 (n1000.ipynb) and 10000 (n10000.ipynb).
Then, we use the two methods and compare their operation time without using any package in the algorithm.

Comparison of Operation Time

We compare the operation time between Gauss elimination and Cholesky decomposition in solving linear equations systems.

Table: Comparison of Operation Time between Gauss and Cholesky

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Method	Time(100*100)	Time(1000*1000)	Predicted Time(10000*10000)
Cholesky	52.4 ms	48.6 sec	5×10^4 sec
Gauss	157 ms	156 sec	1.5×10^5 sec

$O(\frac{n^3}{3})$

10^3

$O(n^3)$

Using Numpy to Solve

We compare the operation time between Gauss elimination and Cholesky decomposition in solving linear equations systems.

Table: Comparison of Operation Time between Gauss and Cholesky

Method	Time(10000*10000)
Cholesky	239 sec
Gauss	414 sec

Conclusion

- In summary, Gaussian elimination is a versatile method for general linear systems but can be computationally expensive and less stable.
- Cholesky decomposition is specialized for symmetric positive definite matrices, providing efficiency and numerical stability but with specific requirements.

Thanks for your attention!