

Limits

- Products
- Equalizers
- Pullbacks
- General definitions

Colimits

- Coproducts
- Cequalizers
- Pushouts
- General def

- \times of sets
- \times in a poset coincide w/ \wedge .
- Vect $\text{Ker}(f)$ is an equalizer.
-

Morphs λ Epis

- Morphs in Set
- Epis in Set.
- $\mathbb{Z} \hookrightarrow \oplus$ (in Rings).

Thm

all limits \Leftrightarrow  +  equalizers.

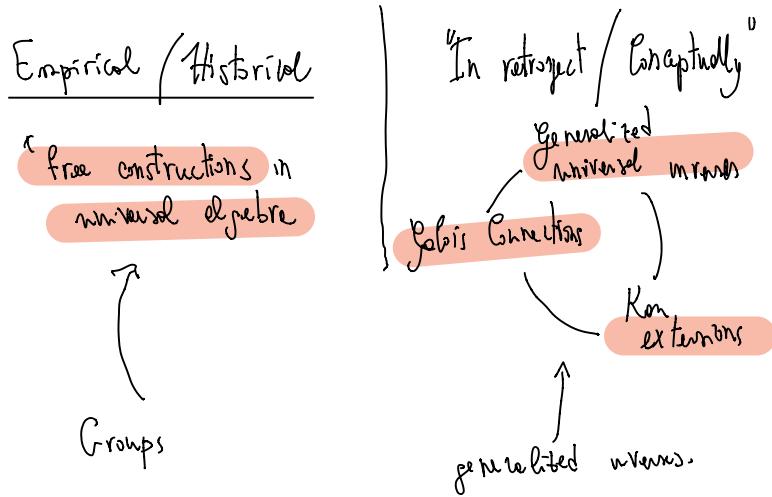
ω limits \Leftrightarrow  +  coequalizers.

terminal object \Leftrightarrow 

(with) pullbacks.

Adjunctions

two approaches

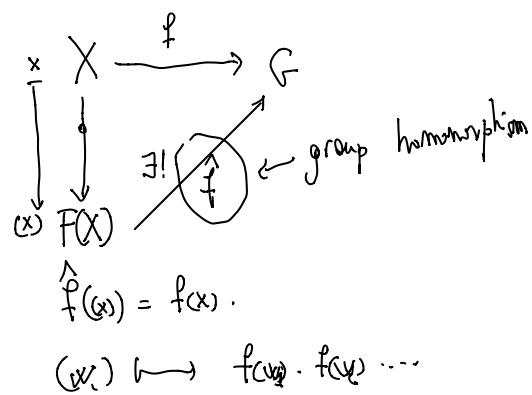


Groups "the free group on a set X "

$\underbrace{\quad}_{\{ w_1 w_2 w_3 \dots \mid w_i \in X \}}$

$\underbrace{F(X)}_{w_i} \quad \underbrace{F_X}_{w_i^{-1}} \quad \underbrace{F_n}_{w_i^1}$

$w_i^{-1} \cdot w_i = \text{empty word.}$



So there is a functor

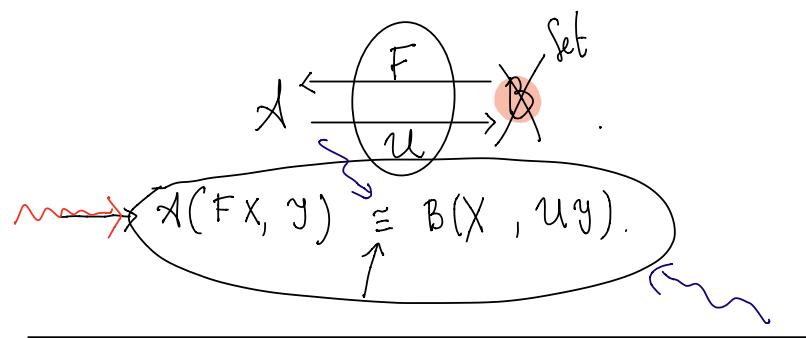
$$\begin{array}{ccc} \text{Set} & \xrightarrow{F} & \text{Grp} \\ X & \longmapsto & F(X) \\ f \downarrow & \sim\!\!\! \sim & \uparrow f \downarrow \\ y & & F(y) \end{array}$$

$$\begin{array}{ccc} \text{Grp} & \xrightarrow{u} & \text{Set} \\ G & \longmapsto & \{G\}. \end{array}$$

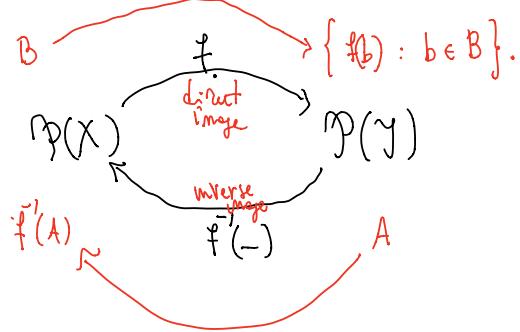
the universal property of F

$$(\text{Grp}(FX, G) \underset{\text{Set}(X, u(G))}{\approx} \text{Set}(X, u(G)))$$

↗ f ↙ $\sim\!\!\! \sim f: X \rightarrow G$
 ↗ $\sim\!\!\! \sim$ Set function



$f: X \rightarrow Y$ is a set function.



$P(X)$ is a poset.
 $\mathcal{F}(X)$ is a category - $A \rightarrow B \Leftrightarrow A \subseteq B$.

f and f^{-1} are functors
 $\hookrightarrow A \subseteq B \Rightarrow f(A) \subseteq f(B)$.

Rem

$$\begin{array}{c} \textcircled{f^{-1}A} \downarrow \\ \supseteq B \Leftrightarrow A \supseteq fB \\ \rightsquigarrow \mathcal{P}(X)(A, \textcircled{f^{-1}B}) \cong \mathcal{P}(Y)(f(A), B). \end{array}$$

Grp

f^{-1}

Def Let $L: \mathcal{A} \rightleftarrows \mathcal{B}: R$ be functors between categories. We say that L is left adjoint to R (R is right adjoint to F) ($L \dashv R$) if there exist

$$\eta : \mathbb{1}_{\mathcal{A}} \Rightarrow RL$$

$$\varepsilon : LR \Rightarrow \mathbb{1}_{\mathcal{B}}$$

such that

$\forall b \in \mathcal{B}$

$$LR(b) \xrightarrow{\varepsilon_b} b \in \mathbb{1}_{\mathcal{B}}(b).$$

$R(x)$
 $\eta_{R(x)}$
 $RLR(x)$
 $R(\varepsilon_x)$

L
 (η_y)
 $LR(L)$
 ε_L

$\forall b \in \mathcal{B}$
 $LR(b) \xrightarrow{\varepsilon_b} b \in \mathbb{1}_{\mathcal{B}}(b).$

triangle equation

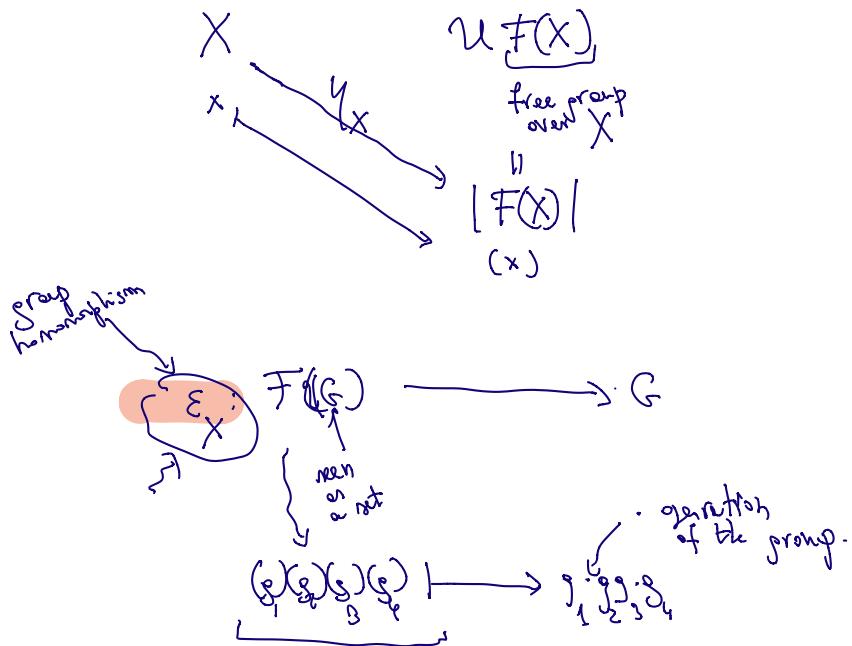
① In the case of groups

$$F \rightarrow UL$$

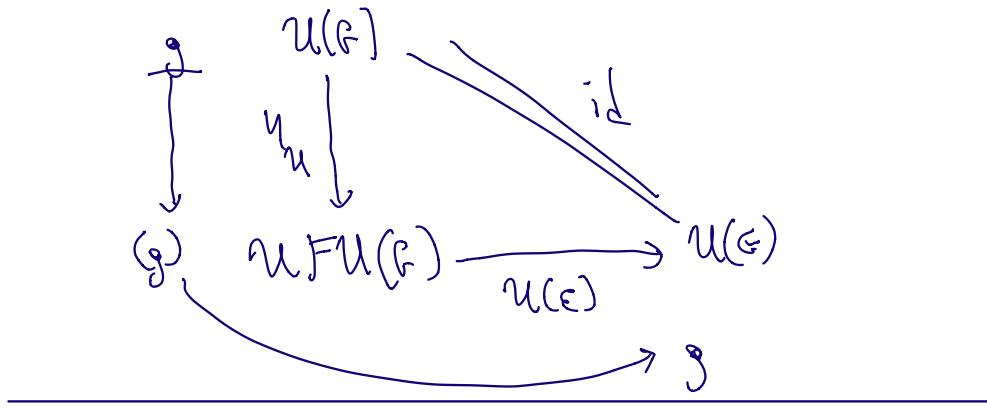
→ ② We try to recover the hom-set definition from the definition above.

1! $F: \text{Set} \rightleftarrows \text{Grp} : U$

$\sim \gamma_X: X \rightarrow UL(X)$ "maps enter in the right adjoint"
 $\varepsilon_X: UL(X) \rightarrow X$



We check one triangle equality



then $L : A \rightleftarrows B : R$, then

$$\varphi : \underbrace{A(X, Ry)}_{\cong} \xrightarrow{\quad} \underbrace{B(LX, y)}_{\cong} : \varphi^{-1}$$

Prof

$$\varphi : \begin{matrix} X & & L(X) \\ \downarrow f & \nearrow L(\epsilon) & \downarrow \\ Ry & \xrightarrow{\quad} & L(X) \xrightarrow{g \circ -} y \\ & \downarrow & \downarrow \\ & & LR(y) \\ & & \downarrow \varepsilon_y \\ & & y \end{matrix}$$

$$\varphi: \mathcal{A}(X, RY) \longrightarrow \mathcal{B}(LX, Y)$$

$$f \longmapsto \xi_y \circ L(f)$$

$$\varphi^{-1}: \mathcal{B}(L(X), Y) \longrightarrow \mathcal{A}(X, R(OY))$$

$$g \longmapsto R(g) \circ \eta_X.$$

$$\begin{array}{ccccc}
 & X & & X & \\
 & \downarrow \eta_X & & \downarrow \eta_X & \\
 LX & \xrightarrow{RL} & RL(X) & \xrightarrow{R(\xi) \circ \eta_X} & RY \\
 g \downarrow & & R(g) \downarrow & & \\
 Y & & RY & &
 \end{array}$$

$$\underbrace{\varphi^{-1}\varphi(f)}_f = f.$$

$$\begin{aligned}
 \varphi^{-1}(\xi_y \circ L(f)) &= R(\xi_y \circ L(f)) \circ \eta_X \\
 &= R(\xi_y) \circ L(f) \circ \eta_X = f - \\
 &\quad \underbrace{\text{triangular equat.}}_{\text{triangular equation}}
 \end{aligned}$$

More is true the bijection is
natural in X and Y

Adjunctions $(L, R, \eta, \varepsilon)$.
 $\{$
 $[-, -] \cong [-, R-]$

1 example Groups

Examples.

Cluster "free constructions"

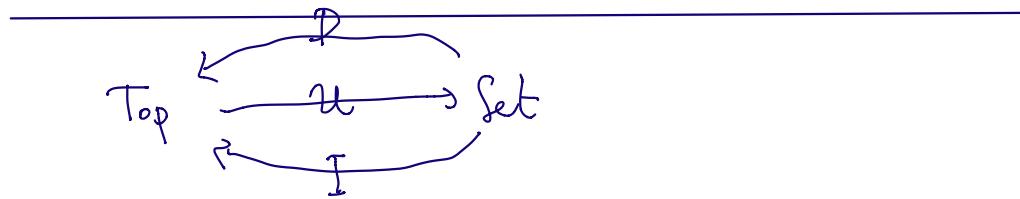
$$\begin{array}{c} \text{Set} \xleftarrow{\eta} \text{Vect} \\ \dashv \quad \vdash F \quad \text{Finitely supported} \\ X \longmapsto R(X) \end{array}$$

$$\begin{array}{c} \text{Set} \xleftarrow{\eta} \text{Grp} \\ \dashv \quad \vdash F \\ X \longmapsto R(X) \end{array}$$

$$\begin{array}{c} \text{Set} \xleftarrow{\eta} \text{Ab} \\ \dashv \quad \vdash F \\ X \longmapsto R[X] = \left\{ \sum n_i x_i \mid n_i \in \mathbb{Z} \right\}. \end{array}$$

$$\begin{array}{c} \text{Set} \xleftarrow{\eta} \text{R-Mod} \\ \dashv \quad \vdash F \\ X \longmapsto R[X]. \end{array}$$

$$\text{Set} \xleftarrow{\eta} \text{Gra}$$



$$\boxed{D \dashv u \dashv I.}$$

$$\begin{aligned} \text{Top}(D(x), y) &\cong \text{Set}(X, \underline{u(y)}) \\ \uparrow f &\quad \uparrow f \\ \text{Set}(u(x), y) &\cong \text{Top}(X, I(y)) \\ \downarrow f &\quad \downarrow f. \end{aligned}$$

Cartesian closed categories.

8, 5, 7 numbers

$$2^{(5 \times 7)} = (2^5)^7.$$

$$\begin{aligned} \text{Set}(5 \times 7, 2) &\cong \text{Set}(7, \text{Set}(5, 2)) \\ \rightsquigarrow \text{Set}(X \times Y, Z) &\cong \text{Set}(Y, \text{Set}(X, Z)) \end{aligned}$$

$$\begin{array}{ccc} & X_x(-) & \\ \text{Set} & \xrightarrow{\quad} & \text{Set} \\ y & \longmapsto & X_x y \\ & (-)^X & \\ \text{Set} & \xleftarrow{\quad} & \text{Set} \\ Z^X & \longleftrightarrow & Z \end{array}$$

In the category of sets $X_x(-) \dashv (-)^X$.

$$\begin{array}{c} \text{Def} \\ \uparrow f \\ \forall X \end{array} \quad \begin{array}{c} \text{cartesian} \\ \text{closed} \\ \text{has right adj.} \end{array}$$

Who is the counit in this core?

$$\begin{array}{ccc} A \times B & \xrightarrow{\epsilon_B} & B \\ \overline{(a, f)} & \longmapsto & \overline{f(a)} \end{array}$$

evalution,

Notice that

Vect is not cartesian closed !! $(A \times B = \underline{A \oplus B})$

$$(\text{Vect} \otimes) \quad V \otimes \underline{} : \text{Vect} \longrightarrow \text{Vect}$$

$$W \longmapsto V \otimes W.$$

HV has a right adjoint!

$$\text{Vect} \hookrightarrow \text{Vect}: [V, -]$$

$$\text{Vect}(A \otimes B, C) \cong \text{Vect}(A, C^B).$$

Monoidal closed
Category

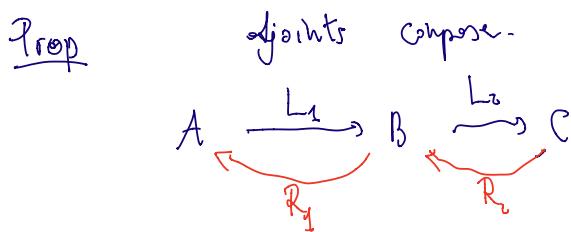
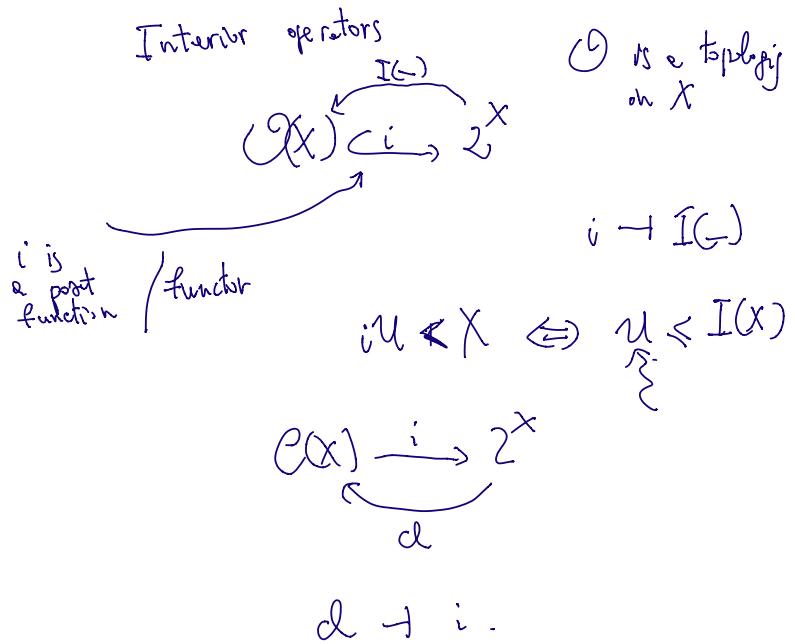
Ex not every forgetful functor has a left adjoint

A diagram illustrating a mapping or relationship between two sets. On the left, there is a box labeled "Set". On the right, there is another box labeled "Set". A horizontal double-headed arrow connects the two boxes. Above this arrow, there is a small bracket-like shape with a vertical line pointing upwards from its center, indicating a correspondence or a function between the two sets.

$$X \xleftarrow{\quad} (X \otimes) \xrightarrow{f} (y, y_0)$$

is left adjoint to u .

L: $x \longmapsto (x \amalg \{1\}, \uparrow)$.



$L_2 \circ L_1$ has a right adjoint and it is
 $R_1 \circ R_2$

Proof

$$A(-, R_1 R_2 -) \cong B(L_1 -, R_2 -) \cong C(L_2 L_1 -, -)$$

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