

Kan extensions

The extension problem

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & \dashrightarrow & \\ C & \dashrightarrow & \end{array}$$

Many universal construction
can be encoded as
"solution to an extension
problem".

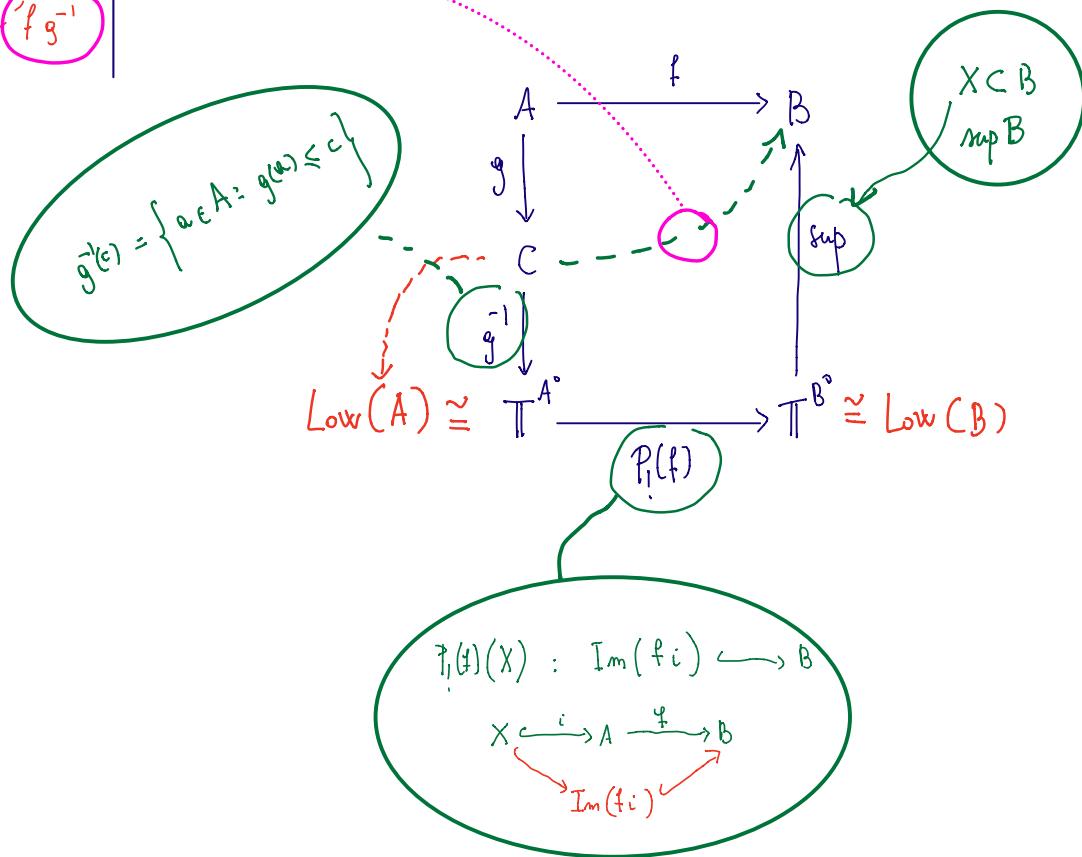
We will use this
theory to construct
adjoints (Adjoint functor
theorem).

Intuition

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \dashrightarrow & \\ C & \dashrightarrow & \end{array}$$

What would you do for posets?

(If B has supreme)



So, if we went to follow this program,
we need to give a definition of several
ingredients for cats -

- $\mathcal{P}(A)$, a substitute for Th^{op}
- g^{-1}
- $\mathcal{P}_1(f)$
- $\sup ?$

① $\underline{\mathcal{P}(A)}$, (the category of "small" presheaves)

Def

Given a locally small category A
we define $\underline{\mathcal{P}(A)}$ the full subcategory
of $\text{Set}^{A^{\text{op}}}$ of those presheaves
that are small columns of representable

We will
see where this
is relevant

Remark $f_A : A \longrightarrow \underline{\mathcal{P}(A)}$ (Yoneda)

(1.1)

the Grothendieck construction

In the analogy that we are pushing,
every $f: \mathcal{P}^{\text{op}} \rightarrow \mathbb{I}$ induces a lower
set " $f(1) \hookrightarrow \mathcal{P}$ ".

$$\mathbb{I}^{\mathcal{P}^{\text{op}}} \rightsquigarrow \text{Low } \mathcal{P}$$

Also the small presheaf construction
has this property (some how) -

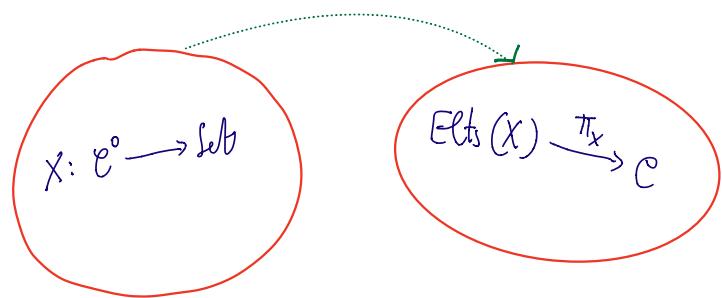
Def For a presheaf $X: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$
we define its **category of elements**

$\text{Elt}_s(X) \rightsquigarrow \text{obj } (c, a_c): c \in \mathcal{C}, a_c \in X(c).$

where $f: c \rightarrow d \in \mathcal{C}(c, d)$ such
that $X(f)(a_c) = (a_d)$ -

Rmk There is a natural projection

$$\text{Elt}_s(X) \xrightarrow{\pi_X} \mathcal{C}$$

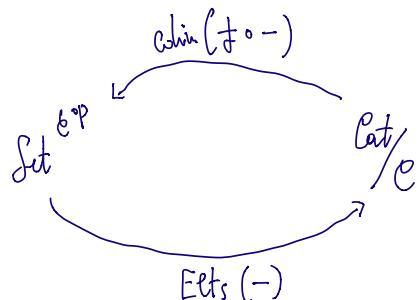


Rew π_x is not a full functor!

Rew We could characterize the image, but we do not \Rightarrow -
(discrete opfibration)-

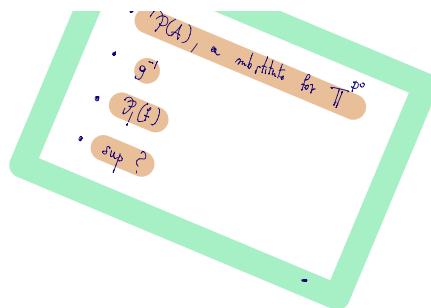
Rew One can also go in the other direction

$$\begin{array}{c} \pi: D \xrightarrow{x} C \\ \Downarrow \\ \pi: D \xrightarrow{x} C \xrightarrow{f} \text{Set}^C \\ \hline P := \text{colim}(f \circ -) \in \text{Set}^{C^\text{op}}. \end{array}$$



Thm $\text{colim}(f \circ \pi_p) \cong \emptyset$

② " g^{-1} ".



Def the "verse" of g is
the functor

$$B(g-, -) : B \longrightarrow \text{Set}^{A^{\text{op}}}$$

$$b \mapsto B(g-, b)$$

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ f \downarrow & & \swarrow "g^{-1}" \\ \text{Set}^{A^{\circ}} & \xleftarrow{\quad} & \end{array}$$

Rank this is well defined, indeed

$$B(g-, b) : A^{\text{op}} \longrightarrow \text{Set}$$

$$a \longmapsto B(ga, b)$$

$$\begin{array}{c} A \xrightarrow{g} B \\ a \longmapsto ga \end{array}$$

Def A functor is admissible
if for every $b \in B$,
 $B(g-, b)$ is a small presheaf.

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \nearrow ! & \downarrow B(g-, -) \\ \mathcal{X}(A) & \longrightarrow & \text{Set}^{A^{\circ}} \end{array}$$

Rank functors with domain
small categories are admissible

Rank functors with arity one
admissible.

Rank accessible functors are admissible

Rmk (sanity check, the nerve
in posets is a weak
counterimage)

$$\begin{array}{ccc} p & \xrightarrow{f} & q \\ \downarrow f & \nearrow & \downarrow \mathcal{Q}(f, -) \\ \mathbb{T}^{\text{op}} & \xleftarrow{\mathcal{Q}(f, -)} & \end{array}$$

$$\mathcal{Q}(f, q) : \mathbb{P}^{\text{op}} \longrightarrow \mathbb{T}$$

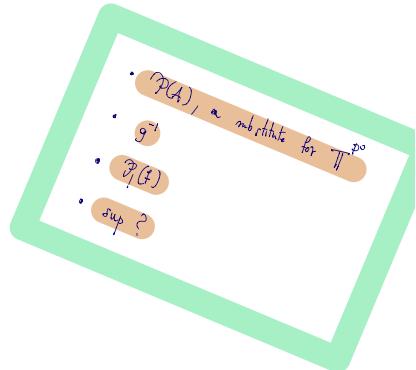
$$p \mapsto \begin{cases} 1 & f_p \leq q \\ 0 & \text{otherwise} \end{cases}$$

So the corresponding lower set
to $\mathcal{Q}(f, q)$ is

$$f^*(q) = \{ p \in P : f_p \leq q \}$$

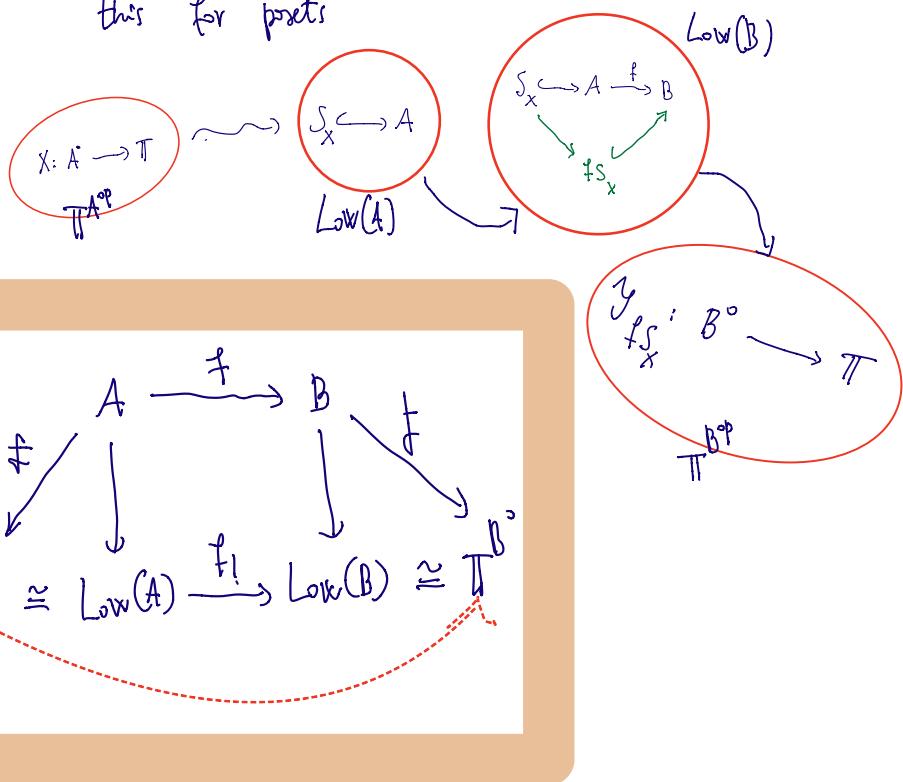
(3) $P_!$

So, now, given a functor
 $f : A \longrightarrow B$ we would like
to construct $P_!(f) : \mathbb{P}(A) \longrightarrow \mathbb{P}(B)$.

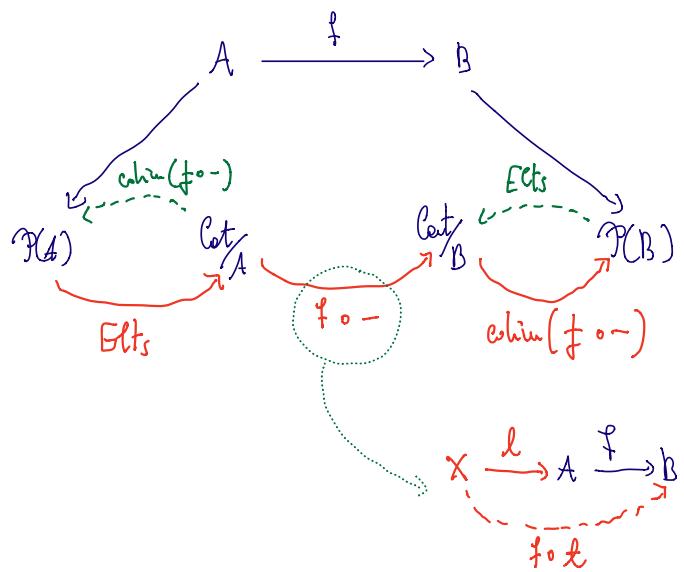


$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow f & & \downarrow f \\ \mathbb{P}(A) & \xrightarrow{\quad} & \mathbb{P}(B) \\ & P_!(f) & \end{array}$$

Now of course we know how to do
this for posets



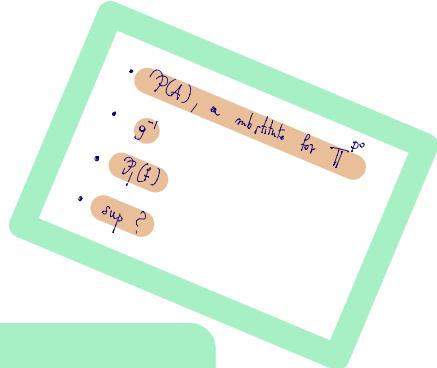
Now we just do the same!



Run the idea is:

$$\mathcal{P}_f(f)(X) = \mathcal{P}_f(f)\left(\bigcup_{x \in X} \{x\}\right) = \bigcup_{x \in X} \{f(x)\}.$$

(3) sup

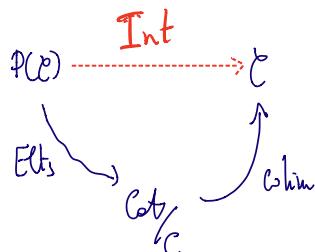


thus If \mathcal{C} is a small-cocomplete category, the Jónsson embedding has a left adjoint

$$\text{Int} : \mathcal{P}(\mathcal{C}) \rightleftarrows \mathcal{C} : f$$

Proof

We build Int.



Run this is where we use the preservers are small, to cut down the size of the colimit

Now we prove that Int is left adjoint. I want to show that...

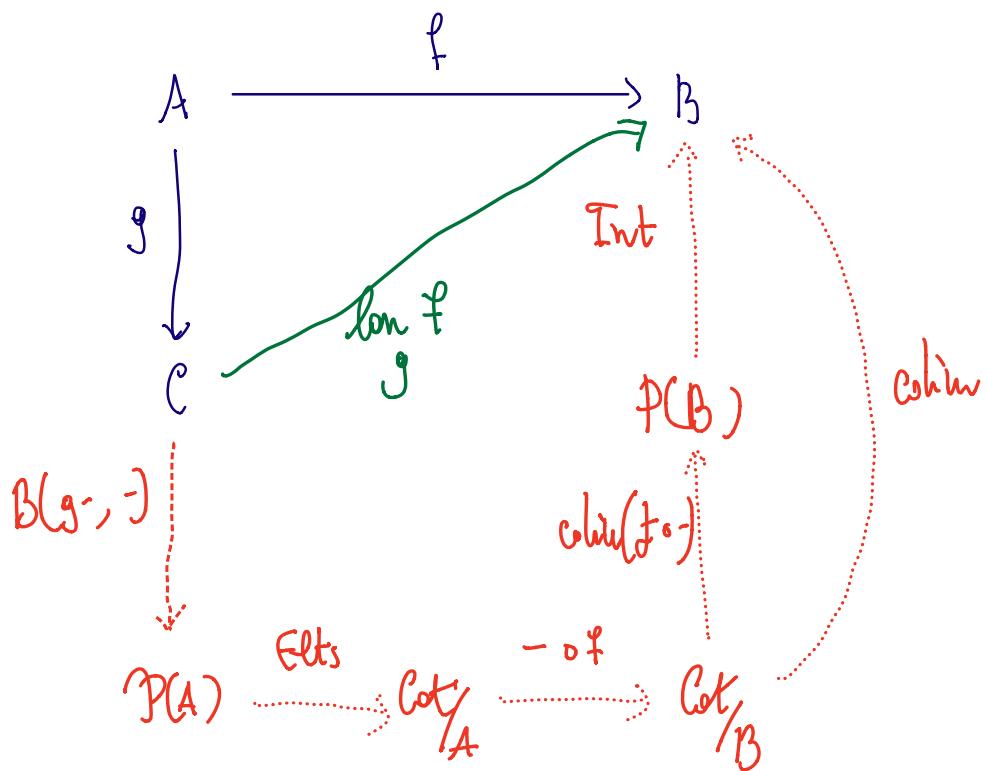
$$C(\text{Int } X, d) \cong \mathcal{P}(C)(X, fd)$$

$$\text{Now, } \mathcal{P}(\mathbb{C})(X, f^\perp) \cong \mathcal{P}(\mathbb{C})\left(\text{coker}(f \circ \pi_X), f^\perp\right)$$

$$\mathcal{E}\left(\text{Int } X, \perp\right) \stackrel{\text{y.l.}}{\cong} \lim_{\text{us}} \mathcal{P}(\mathbb{C})\left(f \circ \pi_X, f^\perp\right)$$

Wrapping up!

then If f is an admissible functor
or B is complete, we can extend!



Now let us see some properties of this construction.

(1) We have constructed **left**

hom extensions, using limits

We can construct right

hom extensions

(2) There is a more general presentation, which is useful in more abstract category theory. We are very concrete.

Then Consider a functor $g: \mathcal{A} \rightarrow \mathcal{C}$
then this induces a functor

$$[A, B] \xleftarrow{g^*} [e, B]$$
$$f \circ g \xleftarrow{\quad} f$$

then "lax": $[A, B] \xrightarrow{g}$ $\longrightarrow [e, B]$
is left adjoint to g^* .

$$\text{lax}: [A, B] \xrightleftharpoons{g} [e, B]: g^*$$

Ex try to find the unit & counit!

Prop if g is fully faithful,
then Kan extension is an extension, i.e.

$$(\lim_g f) \circ g \cong f$$

Application: AFT

Lemme (would be easy with the
more exhibitive presentation of
ex) Let $f: A \rightarrow B$ be
a continuous functor between
cocomplete categories. Then TFAE.

① f has a right adj.

② $\lim_f 1$ exists and is the right adj.

Cor
Adj
functor
theorem

Let $f: A \rightarrow B$ be
a functor between cocomplete
categories. TFAE.

- (1) f is admissible and cocontinuous
- (2) f has a right adjoint

Proof

1 \Rightarrow 2) ok, this is a corollary
of the previous part of the
lesson.

2 \Rightarrow 1) left adjoints are cocontinuous.

We need to show that it is
admissible - that is $f \dashv g$

If $b \in B$, $B(f-, b)$ is
a small colimit of representables.

Now $B(f-, b) \cong A(-, gb)$

But then

$B(f-, b) \cong f(gb)$
so it is representable!

CATEGORY THEORY

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EXERCISES

Riehl (Kan extensions have a universal property). Read section 6.1, where a Kan extensions are introduced in a more abstract way and study Thm 6.2.1 which proves that our concrete formula is explicitly computing the Kan extension, when possible.

Riehl (Concepts are Kan extensions). Read section 6.5, where it is shown that many categorical concepts can be phrased in terms of existence of Kan extensions.

Exercise 1 (■). Prove^a, when all the functors in the equations are well-defined, that

$$\text{lan}_{fg}(h) \cong \text{lan}_f(\text{lan}_g h).$$

Exercise 2 (■). Try to show that if f has a right adjoint g , then

$$\text{lan}_f(1) \cong g.$$

Exercise 3 (■). Prove, using our definition, that when g is fully faithful, then $(\text{lan}_g f) \circ g \cong f$.

^aHint. Use that Kan extensions provide left adjoints to precomposition.

- the exercises in the red group are mandatory.
- pick at least one exercise from each of the yellow groups.
- pick at least two exercises from each of the blue groups.
- nothing is mandatory in the brown box.
- The riddle of the week. It's just there to let you think about it. It is not a mandatory exercise, nor it counts for your evaluation. Yet, it has a lot to teach.
- useful to deepen your understanding. Take your time to solve it. (May not be challenging at all.)
- measures the difficulty of the exercise. Note that a technically easy exercise is still very important for the foundations of your knowledge.
- ▲ It's just too hard.

The label **Leinster** refers to the book **Basic Category Theory**, by *Leinster*.
The label **Riehl** refers to the book **Category Theory in context**, by *Riehl*.