Convex optimization
Exercises

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Exercise 1. *Show that if S is an open set, its complement S^c is closed, and viceversa.*

Solution. Given that S is open, we got that $\forall x \in S\ \exists \delta > 0\ :\ B(x, \delta) \subset S$. By definition $S^c \subset \bar{S^c}$, where

$$\bar{S^c} = \{x\ :\ \forall \delta\ B(x, \delta) \cap S^c \neq \emptyset\}.$$

Let $x \in \bar{S^c}$, then $\forall \delta > 0$,

$$B(x, \delta) \cap S^c \neq \emptyset \implies B(x, \delta) \not\subset S \implies x \notin S \implies x \in S^c.$$

Therefore, we have shown that $\bar{S^c} \subset S^c \implies \bar{S^c} = S^c$.

Exercise 2. *If S_1, s_2 are convex subsets, prove that $S_1 \cup S_2, S_1 + S_2$ and $S_1 - S_2$ are also convex sets.*

Solution. Let us begin with $S_1 \cup S_2$, let $x, x' \in S_1 \cup S_2$ and $\lambda \in (0, 1)$. We got that

$$x, x' \in S_1 \cup S_2 \implies \lambda x + (1 - \lambda)x' \in S_1 \text{ and } \lambda x + (1 - \lambda)x' \in S_2$$

Therefore, $\lambda x + (1 - \lambda)x' \in S_1 \cup S_2$, meaning the set is convex.

Consider now $x, x' \in S_1 + S_2$, we can decompose them as $x = x_1 + x_2$ and $x' = x'_1 + x'_2$, such that

$$\lambda x + (1 - \lambda)x' = \lambda x_1 + (1 - \lambda x_2) + \lambda x'_1 + (1 - \lambda x'_2) \in S_1 + S_2.$$

Lastly, consider the set $S_1 - S_2$, using the same reasoning used in the previous set, it is convex.

Exercise 3. *If $f : S \to \mathbb{R}$ is a convex function on the convex set S, the set $\{x :\ x$ is a minimum of $f\}$ is a convex set.*

Solution. Let $A = \{x \in S :\ x \text{ is a minimum of f}\}$ and consider $x, x' \in A$, $\lambda \in (0, 1)$, since f is convex, there is only one minimal value, ie, $f(x) = f(x') = c$. Using this, $x \in A \iff f(x) = c$ and $\nexists x \in S$ such that $f(x) < c$. Therefore

$$f(\lambda x + (1 - \lambda)x') \leq \lambda f(x) + (1 - \lambda)f(x') = f(x) \implies \lambda x + (1 - \lambda)x' \in A.$$

As a result A is a convex subset of S.

Exercise 4. *Let $f : S \subset \mathbb{R}^d \to \mathbb{R}$ be a convex function on the convex set S and we extend it to an $\hat{f} : \mathbb{R}^d \to \mathbb{R}$ as*

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in S, \\ +\infty & \text{if } x \notin S \end{cases}$$

Show that $\hat{f}$ is a convex function on $\mathbb{R}^D$.

Solution: Let $x, x' \in \mathbb{R}^d$ and $\lambda \in (0, 1)$,

- If $x, x' \in S$, $\hat{f}(\lambda x + (1-\lambda)x') \leq \lambda \hat{f}(x) + (1-\lambda)\hat{f}(x') = f(x)$.
- If $x \notin S$, $\lambda \hat{f}(x) + (1-\lambda)\hat{f}(x) = +\infty$ and the save happens for $x' \notin S$.

Exercise 5. *If Q is a symmetric, positive definite $d \times d$ matrix, show that $f(x) = x^T Q x,\ x \in \mathbb{R}^d$, is a convex function.*

Solution. We know that f is convex given that its Hessian its semi-definite positive

$$\nabla^2 f(x) = 2Q \geq 0.$$