

# Continuous-time stochastic processes

## Homework 1

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**Exercise 1.** A Poisson process with rate  $\lambda > 0$  can be defined as a counting process  $\{N(t) : t \geq 0\}$  with the following properties:

- (i)  $N(0) = 0$ .
- (ii)  $N(t)$  has independent and stationary increments.
- (iii) Let  $\Delta N(t) = N(t + \Delta t) - N(t)$  with  $\Delta t > 0$ . The following relations hold:

$$\begin{aligned}P[\Delta N(t) = 0] &= 1 - \lambda\Delta t + o(\Delta t), \\P[\Delta N(t) = 1] &= \lambda\Delta t + o(\Delta t), \\P[\Delta N(t) \geq 2] &= o(\Delta t).\end{aligned}$$

From this definition show that

$$P[N(t) = n] = \frac{1}{n!} \lambda^n t^n e^{-\lambda t}. \quad (1)$$

To this end, set up a system of differential equations for the quantities  $P[N(t) = 0]$  and  $P[N(t) = n]$  with  $n \geq 1$ . Then verify that Eq. (1) satisfies the differential equations derived.

Illustrate the validity of the derivation by comparing the empirical distribution obtained in a simulation of the Poisson process and the theoretical distribution of  $P[N(t) = n]$  given by Eq. (1) for the values  $\lambda = 10$  and  $t = 2$ .

*Solution.* We begin by deriving the differential equation for  $P[N(t) = 0]$ . Given  $\Delta t > 0$ , we have:

$$\begin{aligned}P[N(t + \Delta t) = 0] &\stackrel{(ii)}{=} P[N(t) = 0]P[\Delta N(t) = 0] \\&\stackrel{(iii)}{=} P[N(t) = 0](1 - \lambda\Delta t + o(\Delta t)) \\&= P[N(t) = 0] - P[N(t) = 0]\lambda\Delta t + o(\Delta t).\end{aligned}$$

Rearranging and dividing both sides by  $\Delta t$ , we get

$$\frac{P[N(t + \Delta t) = 0] - P[N(t) = 0]}{\Delta t} = -\lambda P[N(t) = 0] + \frac{o(\Delta t)}{\Delta t}.$$

Finally, we obtain the desired differential equation by letting  $\Delta t \rightarrow 0^+$ :

$$\frac{d}{dt}P[N(t) = 0] = -\lambda P[N(t) = 0],$$

where the last term vanishes by definition of  $o(\cdot)$ :

$$\lim_{\Delta t \rightarrow 0^+} \frac{o(\Delta t)}{\Delta t} = 0.$$

It is well-known that the solution to this differential equation with initial condition  $P[N(0) = 0] = 1$  is

$$P[N(t) = 0] = e^{-\lambda t}.$$

We can now tackle the general case. Firstly, for a fixed  $n \geq 1$  we notice that the event  $N(t + \Delta t) = n$  can happen in three different ways:

1. There are no events between times  $t$  and  $\Delta t$ :  $N(t) = n$  and  $\Delta N(t) = 0$ .
2. There is one event between  $t$  and  $\Delta t$ :  $N(t) = n - 1$  and  $\Delta N(t) = 1$ .
3. There is more than one event between  $t$  and  $\Delta t$ :  $N(t + \Delta t) = n$  and  $\Delta N(t) \geq 2$ .

Thus, since the increments are independent, we have

$$\begin{aligned} P[N(t + \Delta t) = n] &= P[N(t) = n]P[\Delta N(t) = 0] \\ &\quad + P[N(t) = n - 1]P[\Delta N(t) = 1] \\ &\quad + P[N(t + \Delta t) = n \mid \Delta N(t) \geq 2]P[\Delta N(t) \geq 2]. \end{aligned} \tag{2}$$

We can expand the expressions in which  $\Delta N(t)$  is involved by virtue of (iii), noting that some terms will be negligible when divided by  $\Delta t \rightarrow 0^+$ . In particular, since the last term in the RHS of Eq. (2) is smaller than  $P[\Delta N(t) \geq 2]$  and the latter is  $o(\Delta t)$ , the former is  $o(\Delta t)$  as a whole. Following the same reasoning as before, we have:

$$\frac{P[N(t + \Delta t) = n] - P[N(t) = n]}{\Delta t} = -\lambda P[N(t) = n] + \lambda P[N(t) = n - 1] + \frac{o(\Delta t)}{\Delta t}.$$

Letting  $\Delta t \rightarrow 0^+$ , we get:

$$\frac{d}{dt}P[N(t) = n] = -\lambda P[N(t) = n] + \lambda P[N(t) = n - 1].$$

Rearranging and multiplying both sides by  $e^{\lambda t}$  yields:

$$e^{\lambda t} \left( \frac{d}{dt}P[N(t) = n] + \lambda P[N(t) = n] \right) = e^{\lambda t} \lambda P[N(t) = n - 1].$$

Now we realize that the left hand side is the result of applying Leibniz's product differentiation rule to a certain pair of functions. Indeed, it is easy to see that the above expression is equivalent to:

$$\frac{d}{dt} \left( e^{\lambda t} P[N(t) = n] \right) = e^{\lambda t} \lambda P[N(t) = n - 1]. \tag{3}$$

We will find a closed-form solution to Eq. (3) via induction. For  $n = 1$ , since we already know that  $P[N(t) = 0] = e^{-\lambda t}$ , we have:

$$\frac{d}{dt} e^{\lambda t} P[N(t) = 1] = e^{\lambda t} \lambda P[N(t) = 0] = \lambda e^{-\lambda t} e^{\lambda t} = \lambda.$$

Integrating in both sides w.r.t  $t$ , we arrive at

$$e^{\lambda t} P[N(t) = 1] = \lambda t + C \implies P[N(t) = 1] = \lambda t e^{-\lambda t} + e^{-\lambda t} C,$$

but  $C$  must be zero to fulfill the initial condition  $P[N(0) = 1] = 0$ . Next, we assume that the solution for  $n - 1$  is:

$$P[N(t) = n - 1] = \frac{1}{(n - 1)!} \lambda^{n-1} t^{n-1} e^{-\lambda t},$$

and we will prove the desired equality for  $n$ . Applying the induction hypothesis, we have:

$$\frac{d}{dt} e^{\lambda t} P[N(t) = n] = e^{\lambda t} \lambda P[N(t) = n - 1] = \frac{1}{(n - 1)!} \lambda^n t^{n-1}.$$

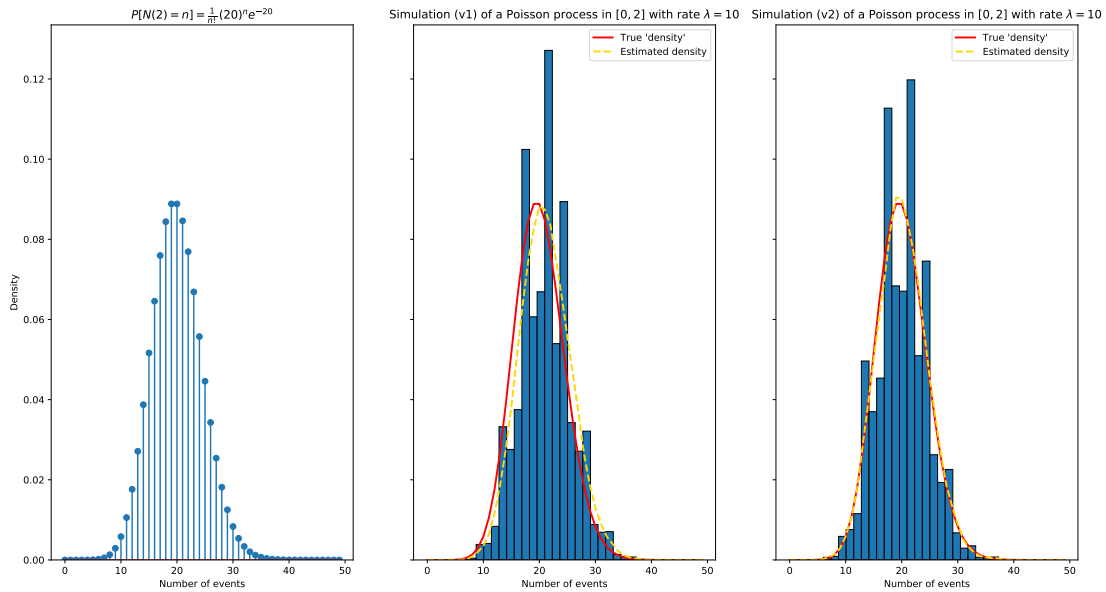
Integrating in both sides yields

$$e^{\lambda t} P[N(t) = n] = \frac{1}{n!} \lambda^n t^n + C \implies P[N(t) = n] = \frac{1}{n!} \lambda^n t^n e^{-\lambda t} + e^{-\lambda t} C,$$

but again  $C = 0$  since  $P[N(0) = n] = 0$ . Thus the inductive step is completed, and the proof is concluded.  $\square$

To illustrate the result, we compare the theoretical distribution given by Eq (1) (which is a Poisson distribution with parameter  $\lambda t$ ) and the result of many simulations of a Poisson process. In particular, we use two strategies for simulating such a process: one that leverages the fact that interarrival times follow an exponential distribution; and one that uses the order statistics of a certain uniform distribution (see Exercise 3).

Below are the results for 1000 independent simulations with a fixed value of  $\lambda = 10$  and  $t = 2$ . That is, we are interested in counting the number of events up to the time instant  $t = 2$  in a Poisson process governed by a rate  $\lambda = 10$ . The code employed is available in the file `ex1.ipynb`.

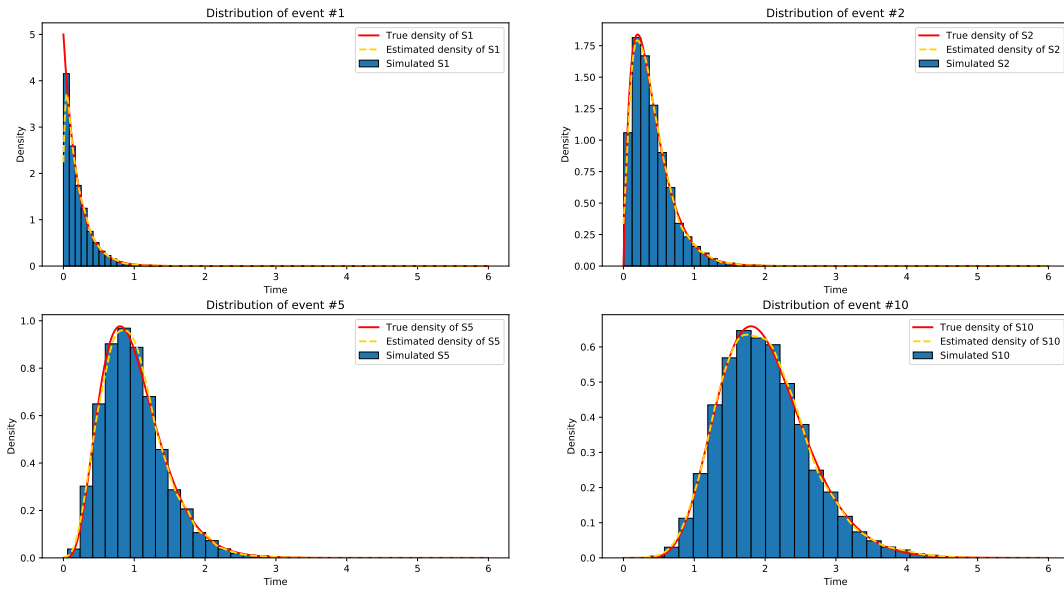


The leftmost graph shows the theoretical distribution, and the other two are the result of the simulations with each strategy. We also superimpose the estimated kernel density (via a Gaussian kernel) and the “true” density, that is, the theoretical p.m.f. of the Poisson distribution in which the points have been joined in a continuous line. We can see that the simulations accurately represent the theoretical distribution.

**Exercise 2.** Simulate a Poisson process with  $\lambda = 5.0$ . From these simulations show for different values of  $n = 1, 2, 5$  and 10 that the probability density of the  $n^{\text{th}}$  arrival is

$$f_{S_n}(t) = \frac{1}{(n-1)!} \lambda^n t^{n-1} e^{-\lambda t}. \quad (4)$$

*Solution.* We simulate 10000 Poisson processes with rate  $\lambda = 5$ , using the code available in the file `ex2.ipynb`. From these simulations we extract the times of the first, second, fifth and tenth events, and we create a histogram of them. We also depict the estimated kernel density, and the true density of these arrival times. We know that the  $n$ th arrival time follows an Erlang distribution with shape parameter  $n$  and rate  $\lambda$ , whose p.d.f. is given by Eq. (4). After running the simulations we obtain the following graphs:



As we can see, the simulations agree with the theoretical distribution for each arrival time.

**Exercise 3.** Assume that we have a sample  $\{U_i\}_{i=1,\dots,n}$  of  $n$  i.i.d  $U_i \sim U[0, t]$  random variables. The probability density of the order statistics  $\{U_{(1)} < U_{(2)} < \dots < U_{(n)}\}$  is

$$f_{U_{(1)}, \dots, U_{(n)}}(u_{(1)}, \dots, u_{(n)}) = \frac{n!}{t^n}.$$

Let  $\{N(t) : t \geq 0\}$  be a Poisson process with rate  $\lambda$ . show that conditioned on  $N(t) = n$ , the distribution of the arrival times  $\{0 < S_1 < S_2 < \dots < S_n\}$  coincides with the distribution of order statistics of  $n$  i.i.d  $U[0, t]$  random variables, i.e.:

$$f_{S_1, \dots, S_n | N(t)}(s_1, \dots, s_n | N(t) = n) = \frac{n!}{t^n}.$$

*Solution.* Consider a fixed sample  $\{s_1, \dots, s_n\}$  of arrival times, where necessarily  $t \geq s_n$ . From now on, we will drop the subindexes when possible to avoid cluttering the notation, and we will use “ $f$ ” to represent both a p.d.f. and a p.m.f. interchangeably. We split up the proof in several steps.

Firstly, using Bayes' theorem we have:

$$f(s_1, \dots, s_{n+1} \mid N(t) = n) = \frac{f(N(t) = n \mid s_1, \dots, s_{n+1})f(s_1, \dots, s_{n+1})}{f(N(t) = n)}. \quad (5)$$

Now we may use the self-evident fact that  $N(t) = n \iff s_n \leq t < s_{n+1}$  in order to write

$$f(N(t) = n \mid s_1, \dots, s_{n+1}) = \begin{cases} 1 & s_n \leq t < s_{n+1} \\ 0 & \text{otherwise} \end{cases}.$$

From this distinction and looking at Eq. (5) it follows that

$$f(s_1, \dots, s_{n+1} \mid N(t) = n) \neq 0 \iff s_n \leq t < s_{n+1},$$

which makes perfect sense: *the probability of observing  $n+1$  events at times  $s_1, \dots, s_{n+1}$ , having observed  $n$  events at time  $t$ , is positive if and only if the  $n$  events were observed just at or after the second-to-last arrival time and strictly before the last one.*

For this reason we will be focusing on the case  $s_n \leq t < s_{n+1}$ , in which Eq. (5) translates to<sup>1</sup>

$$f(s_1, \dots, s_{n+1} \mid N(t) = n) = \frac{f(s_1, \dots, s_{n+1})}{f(N(t) = n)} = \frac{\lambda^{n+1} \exp(-\lambda s_{n+1})}{\frac{1}{n!} (\lambda t)^n \exp(-\lambda t)} = \frac{n! \lambda \exp(-\lambda(s_{n+1} - t))}{t^n}. \quad (6)$$

Next, we will use the fact that the conditional probability factorizes as:

$$f(s_1, \dots, s_{n+1} \mid N(t) = n) = f(s_{n+1} \mid s_1, \dots, s_n, N(t) = n) f(s_1, \dots, s_n \mid N(t) = n),$$

and also the *memoryless* property for  $s_{n+1} > t$ :

$$f(s_{n+1} \mid s_1, \dots, s_n, N(t) = n) = f(s_{n+1} \mid N(t) = n).$$

Combining these two properties, we have:

$$f(s_1, \dots, s_n \mid N(t) = n) = \frac{f(s_1, \dots, s_{n+1} \mid N(t) = n)}{f(s_{n+1} \mid N(t) = n)}.$$

The numerator in the RHS of the previous expression is given by Eq. (6). The denominator can be computed if we realize that, conditional on  $N(t) = n$ , the time instant  $s_{n+1}$  is the *first arrival time after time  $t$* . In other words, it follows the same distribution as the first arrival time if the origin had been put at time  $t$ , which is an *Erlang*(1,  $\lambda$ ) shifted by the location parameter  $t$ . Putting it all together we arrive at the desired result:

$$f(s_1, \dots, s_n \mid N(t) = n) = \frac{n! \lambda \exp(-\lambda(s_{n+1} - t))}{t^n \lambda \exp(-\lambda(s_{n+1} - t))} = \frac{n!}{t^n}.$$

□

**Exercise 4.** Two teams  $A$  and  $B$  play a soccer match. The number of goals scored by Team  $A$  is modeled by a Poisson process  $N_1(t)$  with rate  $\lambda_1 = 0.02$  goals per minute. The number of goals scored by Team  $B$  is modeled by a Poisson process  $N_2(t)$  with rate  $\lambda_2 = 0.03$  goals per minute. The two processes are assumed to be independent. Let  $N(t)$  be the total number of goals in the game up to and including time  $t$ . The game lasts for 90 minutes.

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<sup>1</sup>The expression for  $f(s_1, \dots, s_{n+1})$  can be derived from the fact that the time increments  $T_i = S_i - S_{i-1}$  are identically (exponentially) distributed and independent:  $f(s_1, \dots, s_{n+1}) = f(T_1 = s_1)f(T_2 = s_2 - s_1) \cdots f(T_{n+1} = s_{n+1} - s_n)$ .

- (i) Find the probability that no goals are scored.
- (ii) Find the probability that at least two goals are scored in the game.
- (iii) Find the probability of the final score being Team A: 1, Team B: 2.
- (iv) Find the probability that they draw.
- (v) Find the probability that Team B scores the first goal.

Confirm your results by writing a Python program that simulates the process and estimate the answers from the simulations.

*Solution.* We know that the sum of two independent Poisson processes is also a Poisson process with rate equal to the sum of the rates, so we can write  $N(t) \sim \text{Poisson}(0.05)$ . We will make repeated use of the expression of the p.m.f. of a Poisson process (see Eq. (1)).

- (i) The probability that no goals are scored equals:

$$P[N(90) = 0] = \frac{1}{0!} (0.05 \cdot 90)^0 e^{-0.05 \cdot 90} = e^{-4.5} \approx 0.0111.$$

- (ii) The probability that at least two goals are scored in the game is:

$$\begin{aligned} P[N(90) \geq 2] &= 1 - P[N(90) \leq 1] = 1 - (P[N(90) = 0] + P[N(90) = 1]) \\ &= 1 - (e^{-4.5} + 0.05 \cdot 90 e^{-4.5}) \approx 0.9389. \end{aligned}$$

- (iii) Since  $N_1(t)$  and  $N_2(t)$  are independent, the probability of finishing with a score of Team A: 1 and Team B: 2 is:

$$\begin{aligned} P[N_1(90) = 1, N_2(90) = 2] &= P[N_1(90) = 1] P[N_2(90) = 2] \\ &= 0.02 \cdot 90 e^{-0.02 \cdot 90} \frac{1}{2} \cdot 0.03^2 \cdot 90^2 e^{-0.03 \cdot 90} \approx 0.0729. \end{aligned}$$

- (iv) The probability that they draw is given by the expression:

$$P[N_1(90) = N_2(90)] = \sum_{n=0}^{\infty} P[N_1(90) = n] P[N_2(90) = n] = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} 0.02^n 0.03^n 90^{2n} e^{-90(0.03+0.02)}.$$

We could try to sum this infinite series, but we are better off using the fact that the difference of two independent Poisson variables follows a **Skellam distribution**. Indeed, if  $V_1$  and  $V_2$  are two independent Poisson-distributed random variables with means  $\lambda_1$  and  $\lambda_2$  respectively, the p.m.f. for the difference  $V = V_1 - V_2$  is given by:

$$p(\nu; \lambda_1, \lambda_2) = P[V = \nu] = e^{-(\lambda_1 + \lambda_2)} \left( \frac{\lambda_1}{\lambda_2} \right)^{\nu/2} I_{\nu}(2\sqrt{\lambda_1 \lambda_2}), \quad (7)$$

where  $I_{\nu}(x)$  is the modified Bessel function of the first kind of order  $\nu$ , i.e.:

$$I_{\nu}(x) = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n + \nu + 1)} \left( \frac{x}{2} \right)^{2n + \nu}.$$

Now we can compute the desired probability with the aid of Python, either by evaluating the p.m.f. of a  $\text{Skellam}(90 \cdot 0.02, 90 \cdot 0.03)$  at  $\nu = 0$  with `scipy.stats.skellam`, or by substituting the appropriate values in Eq. (7) and evaluating  $I_0$  in the corresponding point via `scipy.special.iv`. Either way, we have:

$$P[N_1(90) - N_2(90) = 0] = e^{-90 \cdot 0.05} I_0(2 \cdot 90 \sqrt{0.02 \cdot 0.03}) \approx 0.1793.$$

- (v) Let  $X$  model the time of the first goal scored by Team B, and let  $Y$  be the number of goals scored by Team A before Team B scores. On the one hand we have that, conditional on  $X = t$ , the variable  $Y$  is counting the number of events (goals of Team A) up to time  $t$ , so it may be viewed as a Poisson variable with mean  $0.02t$ :

$$P[Y = n \mid X = t] = \frac{1}{n!} (0.02t)^n e^{-0.02t}.$$

On the other hand, the distribution of  $X$  is that of the first arrival time of the Poisson process  $N_2(t)$ , which is known to be exponentially distributed:

$$P[X = t] = 0.03e^{-0.03t}.$$

With this notation, we are interested in computing the probability of  $Y = 0$ , given the restrictions that  $0 \leq X \leq 90$  (that is, we are requiring that Team B scores at least once). Combining the above expressions and using the *law of total probability*, we get:

$$\begin{aligned} P[Y = 0 \mid 0 \leq X \leq 90] &= \int_0^{90} P[Y = 0 \mid X = t] P[X = t] dt \\ &= \int_0^{90} 0.03e^{-0.05t} dt = -\frac{0.03}{0.05} \left[ e^{-0.05t} \right]_0^{90} \approx 0.5933. \end{aligned}$$

The code which contains the simulations that illustrate the correctness of these results can be consulted in `ex4.ipynb`.