Relación de ejercicios 3

Procesos estocásticos

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Exercise 1. *A Poisson process with rate λ can be defined as a counting process $\{N(t);\, t \geq 0\}$ with the following properties:*

1. $N(0) = 0$
2. $N(t)$ *has independent and stationary increments.*
3. *Let $\Delta N(t) = N(t + \Delta t) - N(t)$. The following relations hold:*

$$
\begin{aligned}
P[\Delta N(t) = 0] &= 1 - \lambda \Delta t + o(\Delta t),\\
P[\Delta N(t) = 1] &= \lambda \Delta t + o(\Delta t),\\
P[\Delta N(t) \geq 2] &= o(\Delta t).
\end{aligned}
$$

From this definition show that

$$
P[N(t) = n] = \frac{1}{n!}\lambda^n t^n e^{-\lambda t}. \tag{1}
$$

To this end, set up a system of differential equations for the quantities $P[N(t) = 0]$ and $P[N(t) = n]$ with $n \geq 1$. Then verify that Equation 1 satisfies the differential equations derived.

Illustrate the validity of the derivation by comparing the empirical distribution obtained in a simulation of the Poisson process and the theoretical distribution of $P[N(t) = n]$ given by Equation 1 for the values $\lambda = 10$ and $t = 2$.

Solution. For instance, the differential equation for $P[N(t) = 0]$ can be derived from the fact that

$$
\begin{aligned}
P[N(t + \Delta t) = 0] &= P[N(t) = 0]P[\Delta N(t) = 0]\\
&= P[N(t) = 0](1 - \lambda \Delta t + o(\Delta t))\\
&= P[N(t) = 0] - P[N(t) = 0]\lambda \Delta t + o(\Delta t).
\end{aligned}
$$

Passing $P[N(t) = 0]$ to the left and dividing in both sides by Δt,

$$
\frac{P[N(t + \Delta t) = 0] - P[N(t) = 0]}{\Delta t} = -\lambda P[N(t) = 0].
$$

Where the following property of $o(\cdot)$ is used:

$$
\lim_{\Delta t \to 0^+} \frac{o(\Delta t)}{\Delta t} = 0
$$

The corresponding differential equation is obtained in the limit $\Delta t \to 0^+$

$$
\frac{d}{dt}P[N(t) = 0] = -\lambda P[N(t) = 0].
$$

The solution is this differential equation with initial condition $P[N(0)=0]=1$ is

$$P[N(t)=0]=e^{-\lambda t}.$$

Now, given a fixed $n \geq 1$, $P[N(t+\Delta t)=n]$ can be split into three different successes:

1. There is no increment in Δt: $P[N(t)=n]$ and $P[\Delta N(t)=0]$.
2. There is one increment in $Deltat$: $P[N(t)=n-1]$ and $P[\Delta N(t)=1]$
3. There is more than 1 increment in Δt: $P[N(t+\Delta t)=n]$ and $P[\Delta N(t)\geq 2]$.

Thus,

$$\begin{aligned} P[N(t+\Delta t)=n] &= P[N(t)=n]P[\Delta N(t)=0] \\ &+ P[N(t)=n-1]P[\Delta N(t)=1] \\ &+ P[N(t+\Delta t)=n]P[\Delta N(t)\geq 2] \end{aligned}$$

Where the last term is $o(\Delta t)$, following the same procedure done before:

$$\frac{P[N(t+\Delta t)=n]-P[N(t)=n]}{\Delta t} = -\lambda P[N(t)=n]+\lambda P([N(t)=n-1])+\frac{o(\Delta t)}{\Delta t}$$

Limiting in $\Delta t \to 0^+$:

$$\frac{d}{dt}P[N(t)=n] = -\lambda P[N(t)=n]+\lambda P([N(t)=n-1])$$

Multiplying by $e^{\lambda t}$

$$e^{\lambda t}\left(\frac{d}{dt}P[N(t)=n]-+\lambda P[N(t)=n]\right)=e^{\lambda t}\lambda P([N(t)=n-1])$$

$$\downarrow$$

$$\frac{d}{dt}e^{\lambda t}P[N(t)=n]=e^{\lambda t}\lambda P([N(t)=n-1]).$$

Evaluating in $n=1$ and using that $P([N(t)=0])=e^{-\lambda t}$

$$\frac{d}{dt}e^{\lambda t}P[N(t)=1]=e^{\lambda t}\lambda P([N(t)=0])=\lambda e^{-\lambda t}e^{\lambda t}=\lambda.$$

Using an inductive procedure, and supposing that $P[N(t)=n-1]=\frac{1}{(n-1)!}\lambda^{n-1}t^{n-1}e^{-\lambda t}$.

$$\frac{d}{dt}e^{\lambda t}P[N(t)=n]=e^{\lambda t}\lambda P([N(t)=n-1])=\frac{\lambda}{(n-1)!}\lambda^{n-1}t^{n-1}$$

$$e^{\lambda t}P[N(t)=n]=\frac{1}{n!}\lambda^n t^n+c.$$

Where $c=0$ given that $P[N(t)=n]=0$.

Exercise 2. *Simulate a Poisson process with $\lambda = 5{,}0$. From these simulations show for different values of $n = 1, 2, 5$ and 10 that the probability density of the n^{th} arrival is*

$$f_{S_n}(t) = \frac{1}{(n-1)!}\lambda^n t^{n-1} e^{-\lambda t}.$$

Exercise 3. *Assume that we have a sample $\{u_i\}_{i=1,\dots,n}$ of n i.i.d $U_i \sim U[0,t]$ variables. The probability density of the order statistics $\{U_{(1)} < U_{(2)} < \cdots < U_{(n)}\}$ is*

$$f_{U_{(1)},\dots,U_{(n)}}(u_{(1)},\dots,u_{(n)}) = \frac{n!}{t^n}.$$

Let $\{N(t);\, t \geq 0\}$ be a Poisson process with rate λ. show that conditioned on $N(t) = n$, the distribution of the arrival times $\{0 < S_1 < S_2 < \dots < S_n\}$ coincides with the distribution of order statistics of n i.i.d $U[0,t]$ random variables

$$f_{S_1,\dots,S_n|N(t)}(s_1,\dots,s_n \mid N(t) = n) = \frac{n!}{t^n}.$$

Solo si $t \geq s_n$, en otro caso es 0 Cambiarla por:.

$$f_{S_1,\dots,S_n|N(t)}(s_1,\dots,s_n \mid N(t) = n) = \begin{cases} \frac{n!}{t^n} & t \geq s_n \\ 0 & \text{otherwise} \end{cases}.$$

Solution. Using the Bayes theorem:

$$f(s_1,\dots,s_{n+1} \mid N(t) = n) = \frac{f(N(t) = n \mid s_1,\dots,s_{n+1}) f(s_1,\dots,s_{n+1})}{f(N(t) = n)}.$$

Where subindexes are removed in order to clarify the structure. We may use that $N(t) = n \iff s_n \leq t < s_{n+1}$ in order to write

$$f(N(t) = n \mid s_1,\dots,s_{n+1}) = \begin{cases} 1 & s_n \leq t < s_{n+1} \\ 0 & \text{otherwise} \end{cases}.$$

Given this distinction, the previous density verifies

$$f(s_1,\dots,s_{n+1} \mid N(t) = n) \neq 0 \iff s_n \leq t < s_{n+1}.$$

Which makes sense: *the probability of having $s_1,\dots,s_{n+1}$ arrival times is zero if after s_n there are not n events.*

We are now focusing in the case $s_n \leq t < s_{n+1}$, where the density verifies

$$f(s_1,\dots,s_{n+1} \mid N(t) = n) = \frac{f(s_1,\dots,s_{n+1})}{f(N(t) = n)} = \frac{\lambda^{n+1}\exp(-\lambda s_{n+1})}{\frac{1}{n!}(\lambda t)^n \exp(-\lambda t)} = \frac{n! \exp(-\lambda(s_{n+1} - t))}{t^n}.$$

Using the *memoryless* property for $s_{n+1} > t$

$$f(s_1,\dots,s_{n+1} \mid n) = f(s_{n+1} \mid s_1,\dots,s_n,n) f(s_1,\dots,s_n \mid n) = f(s_{n+1} \mid n) f(s_1,\dots,s_n \mid n)$$

Substituting in the above formula:

$$f(s_1,\dots,s_n \mid n) = \frac{n!}{t^n}\frac{\exp(-\lambda(s_{n+1} - t))}{P(s_{n+1} \mid n)} = \frac{n!}{t^n}.$$

Conditional on $N(t) = n$, S_{n+1} is the first arrival event after t, whose probability is

Exercise 4. *Two teams A and B play a soccer match. The number of goals scored by Team A is modeled by a Poisson process $N_1(t)$ with rate $\lambda_1 = 0{,}02$ goals per minute. The number of goals scored by Team B is modeled by a Poisson process $N_2(t)$ with rate $\lambda_2 = 0{,}03$ goals per minute. The two processes are assumed to be independent. Let $N(t)$ be the total number of goals in the game up to and including time t. The game lasts for 90 minutes.*

1. *Find the probability that no goals are scored.*
2. *Find the probability that at least two goals are scored in the game.*
3. *Find the probability of the final score being $A:1$, $B:2$.*
4. *Find the probability that they draw.*
5. *Find the probability that Team B scores the first goal.*

Confirm your results by writing a Python program that simulates the process and estimate the answers from the simulations.

In this problem, the series representation of the modifier Bessel function of order ν can be useful

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu}$$

Solution.

1. The probability that no goals are scored equals
$$P[N(90) = 0] = P[N_1(90)+N_2(90) = 0] = P[N_1(90) = 0]P[N_2(90) = 0] = e^{-\lambda_1 90}e^{-\lambda_2 90} = 0{,}0111090\ldots$$

2. The probability that at least two goals are scored in the game is
$$\begin{aligned} P[N(90) > 1] = P[N_1(90) + N_2(90) > 1]& \\ =&1 - P[N_1(90) = 0]P[N_2(90) = 0] \\ &- P[N_1(90) = 1]P[N_2(90) = 0] - P[N_1(90) = 0]P[N_2(90) = 1] \\ =&1 - e^{-\lambda_1 90 - \lambda_2 90} - \lambda_1 90 e^{-\lambda_1 90 - \lambda_2 90} - \lambda_2 90 e^{-\lambda_1 90 - \lambda_2 90} \\ =& \end{aligned}$$

3. The probability of finishing with a score of $A:1$ and $B:2$:
$$P[N_1(90) = 1]P[N_2(90) = 2] = \lambda_1 90 e^{-\lambda_1 90}\frac{1}{2}\lambda_2^2 90^2 e^{-\lambda_2 90}$$

4. The probability that they draw is
$$\sum_{n=1}^{\infty} P[N_1(90) = n]P[N_2(90) = n] = \sum_{n=1}^{\infty} \frac{1}{n!^2}\lambda_1^n\lambda_2^n 90^{2n} e^{-\lambda_1 90 - \lambda_2 90}$$

5. The probability that team B scores the first goal is
$$\sum_{t=0}^{90} P[N_1(t) = 0]P[N_2(t) = 1] = \sum_{t=0}^{90} \lambda_2 t e^{-\lambda_2 t - \lambda_1 t}$$