Continuous-time stochastic processes

Homework 1

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Exercise 1. *A Poisson process with rate* $\lambda > 0$ *can be defined as a counting process* $\{N(t) : t \geq 0\}$ *with the following properties:*

(i) $N(0) = 0$.

(ii) $N(t)$ *has independent and stationary increments.*

(iii) Let $\Delta N(t) = N(t + \Delta t) - N(t)$ *with* $\Delta t > 0$*. The following relations hold:*

$$\begin{aligned} P[\Delta N(t) = 0] &= 1 - \lambda \Delta t + o(\Delta t), \\ P[\Delta N(t) = 1] &= \lambda \Delta t + o(\Delta t), \\ P[\Delta N(t) \geq 2] &= o(\Delta t). \end{aligned}$$

From this definition show that

$$P[N(t) = n] = \frac{1}{n!} \lambda^n t^n e^{-\lambda t}. \tag{1}$$

To this end, set up a system of differential equations for the quantities $P[N(t) = 0]$ *and* $P[N(t) = n]$ *with* $n \geq 1$*. Then verify that Eq. (1) satisfies the differential equations derived.*

Illustrate the validity of the derivation by comparing the empirical distribution obtained in a simulation of the Poisson process and the theoretical distribution of $P[N(t) = n]$ *given by Eq. (1) for the values* $\lambda = 10$ *and* $t = 2$.

Solution. We begin by deriving the differential equation for $P[N(t) = 0]$. Given $\Delta t > 0$, we have:

$$\begin{aligned} P[N(t + \Delta t) = 0] &\overset{(ii)}{=} P[N(t) = 0] P[\Delta N(t) = 0] \\ &\overset{(iii)}{=} P[N(t) = 0](1 - \lambda \Delta t + o(\Delta t)) \\ &= P[N(t) = 0] - P[N(t) = 0] \lambda \Delta t + o(\Delta t). \end{aligned}$$

Rearranging and dividing both sides by Δt, we get

$$\frac{P[N(t + \Delta t) = 0] - P[N(t) = 0]}{\Delta t} = -\lambda P[N(t) = 0] + \frac{o(\Delta t)}{\Delta t}.$$

Finally, we obtain the desired differential equation by letting $\Delta t \to 0^+$:

$$\frac{d}{dt} P[N(t) = 0] = -\lambda P[N(t) = 0],$$

where the last term vanishes by definition of $o(\cdot)$:

$$\lim_{\Delta t \to 0^+} \frac{o(\Delta t)}{\Delta t} = 0.$$

It is well-known that the solution to this differential equation with initial condition $P[N(0) = 0] = 1$ is

$$P[N(t) = 0] = e^{-\lambda t}.$$

We can now tackle the general case. Firstly, for a fixed $n \geq 1$ we notice that the event $N(t+\Delta t) = n$ can happen in three different ways:

1. There are no events between times t and Δt: $N(t) = n$ and $\Delta N(t) = 0$.
2. There is one event between t and Δt: $N(t) = n - 1$ and $\Delta N(t) = 1$.
3. There is more than one event between t and Δt: $N(t + \Delta t) = n$ and $\Delta N(t) \geq 2$.

Thus, since the increments are independent, we have

$$\begin{aligned} P[N(t+\Delta t) = n] = P[N(t) = n]P[\Delta N(t) = 0] \\ + P[N(t) = n-1]P[\Delta N(t) = 1] \\ + P[N(t+\Delta t) = n \mid \Delta N(t) \geq 2]P[\Delta N(t) \geq 2]. \end{aligned} \tag{2}$$

We can expand the expressions in which $\Delta N(t)$ is involved by virtue of (iii), noting that some terms will be negligible when divided by $\Delta t \to 0^+$. In particular, since the last term in the RHS of Eq. (2) is smaller than $P[\Delta N(t) \geq 2]$ and the latter is $o(\Delta t)$, the former is $o(\Delta t)$ as a whole. Following the same reasoning as before, we have:

$$\frac{P[N(t+\Delta t) = n] - P[N(t) = n]}{\Delta t} = -\lambda P[N(t) = n] + \lambda P[N(t) = n-1] + \frac{o(\Delta t)}{\Delta t}.$$

Letting $\Delta t \to 0^+$, we get:

$$\frac{d}{dt} P[N(t) = n] = -\lambda P[N(t) = n] + \lambda P[N(t) = n-1].$$

Rearranging and multiplying both sided by $e^{\lambda t}$ yields:

$$e^{\lambda t}\left(\frac{d}{dt} P[N(t) = n] + \lambda P[N(t) = n]\right) = e^{\lambda t} \lambda P[N(t) = n-1].$$

Now we realize that the left hand side is the result of applying Leibniz's product differentiation rule to a certain pair of functions. Indeed, it is easy to see that the above expression is equivalent to:

$$\frac{d}{dt}\left(e^{\lambda t} P[N(t) = n]\right) = e^{\lambda t} \lambda P[N(t) = n-1]. \tag{3}$$

We will find a closed-form solution to Eq. (3) via induction. For $n = 1$, since we already know that $P[N(t) = 0] = e^{-\lambda t}$, we have:

$$\frac{d}{dt} e^{\lambda t} P[N(t) = 1] = e^{\lambda t} \lambda P[N(t) = 0] = \lambda e^{-\lambda t} e^{\lambda t} = \lambda.$$

Integrating in both sides w.r.t t, we arrive at

$$e^{\lambda t} P[N(t) = 1] = \lambda t + C \implies P[N(t) = 1] = \lambda t e^{-\lambda t} + e^{-\lambda t} C,$$

but C must be zero to fulfill the initial condition $P[N(0) = 1] = 0$. Next, we assume that the solution for $n-1$ is:

$$P[N(t) = n-1] = \frac{1}{(n-1)!}\lambda^{n-1}t^{n-1}e^{-\lambda t},$$

and we will prove the desired equality for n. Applying the induction hypothesis, we have:

$$\frac{d}{dt}e^{\lambda t}P[N(t) = n] = e^{\lambda t}\lambda P[N(t) = n-1] = \frac{1}{(n-1)!}\lambda^n t^{n-1}.$$

Integrating in both sides yields

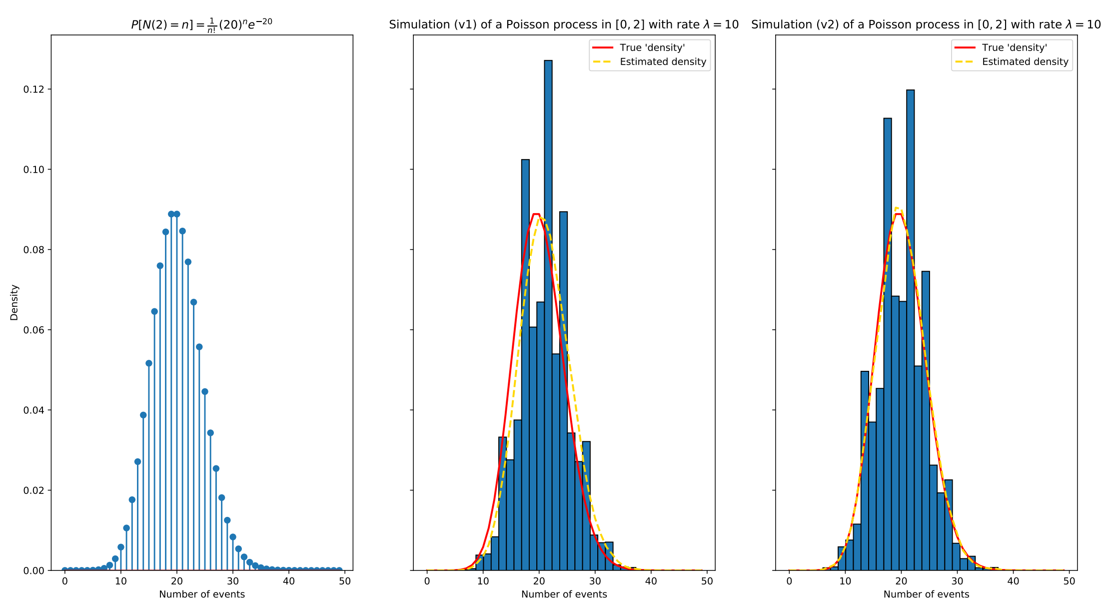
$$e^{\lambda t}P[N(t) = n] = \frac{1}{n!}\lambda^n t^n + C \implies P[N(t) = n] = \frac{1}{n!}\lambda^n t^n e^{-\lambda t} + e^{-\lambda t}C,$$

but again $C = 0$ since $P[N(0) = n] = 0$. Thus the inductive step is completed, and the proof is concluded.

□

To illustrate the result, we compare the theoretical distribution given by Eq (1) (which is a Poisson distribution with parameter λt) and the result of many simulations of a Poisson process. In particular, we use two strategies for simulating such a process: one that leverages the fact that interarrival times follow an exponential distribution; and one that uses the order statistics of a certain uniform distribution (see Exercise 3).

Below are the results for 1000 independent simulations with a fixed value of $\lambda = 10$ and $t = 2$. That is, we are interested in counting the number of events up to the time instant $t = 2$ in a Poisson process governed by a rate $\lambda = 10$. The code employed is available in the file `ex1.ipynb`.

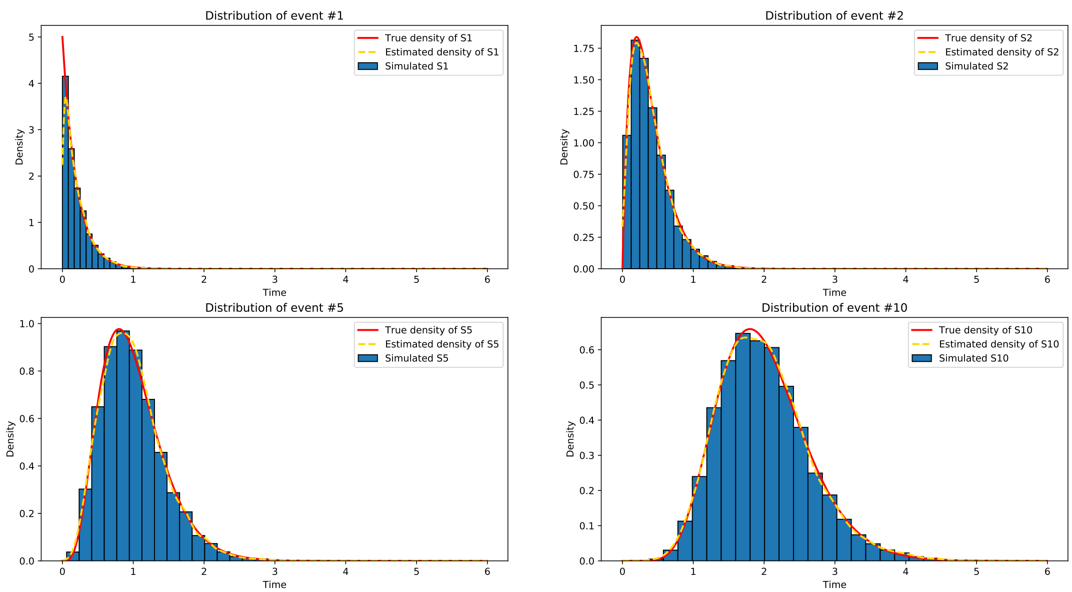


The leftmost graph shows the theoretical distribution, and the other two are the result of the simulations with each strategy. We also superimpose the estimated kernel density (via a Gausian kernel) and the "true" density, that is, the theoretical p.m.f. of the Poisson distribution in which the points have been joined in a continuous line. We can see that the simulations accurately represent the theoretical distribution.

Exercise 2. *Simulate a Poisson process with $\lambda = 5.0$. From these simulations show for different values of $n = 1, 2, 5$ and 10 that the probability density of the n^{th} arrival is*

$$f_{S_n}(t) = \frac{1}{(n-1)!}\lambda^n t^{n-1} e^{-\lambda t}. \tag{4}$$

Solution. We simulate 10000 Poisson processes with rate $\lambda = 5$, using the code available in the file `ex2.ipynb`. From these simulations we extract the times of the first, second, fifth and tenth events, and we create a histogram of them. We also depict the estimated kernel density, and the true density of these arrival times. We know that the nth arrival time follows an Erlang distribution with shape parameter n and rate λ, whose p.d.f. is given by Eq. (4). After running the simulations we obtain the following graphs:



As we can see, the simulations agree with the theoretical distribution for each arrival time.

Exercise 3. *Assume that we have a sample $\{U_i\}_{i=1,\ldots,n}$ of n i.i.d $U_i \sim U[0,t]$ random variables. The probability density of the order statistics $\{U_{(1)} < U_{(2)} < \cdots < U_{(n)}\}$ is*

$$f_{U_{(1)},\ldots,U_{(n)}}(u_{(1)},\ldots,u_{(n)}) = \frac{n!}{t^n}.$$

Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ. show that conditioned on $N(t) = n$, the distribution of the arrival times $\{0 < S_1 < S_2 < \cdots < S_n\}$ coincides with the distribution of order statistics of n i.i.d $U[0,t]$ random variables, i.e.:

$$f_{S_1,\ldots,S_n|N(t)}(s_1,\ldots,s_n \mid N(t) = n) = \frac{n!}{t^n}.$$

Solution. Consider a fixed sample $\{s_1,\ldots,s_n\}$ of arrival times, where necessarily $t \geq s_n$. From now on, we will drop the subindexes when possible to avoid cluttering the notation, and we will use "f" to represent both a p.d.f. and a p.m.f. interchangeably. We split up the proof in several steps.

Firstly, using Bayes' theorem we have:

$$f(s_1,\dots,s_{n+1}\mid N(t)=n)=\frac{f(N(t)=n\mid s_1,\dots,s_{n+1})f(s_1,\dots,s_{n+1})}{f(N(t)=n)}. \tag{5}$$

Now we may use the self-evident fact that $N(t)=n \iff s_n\le t<s_{n+1}$ in order to write

$$f(N(t)=n\mid s_1,\dots,s_{n+1})=\begin{cases}1 & s_n\le t<s_{n+1}\\ 0 & \text{otherwise}\end{cases}.$$

From this distinction and looking at Eq. (5) it follows that

$$f(s_1,\dots,s_{n+1}\mid N(t)=n)\neq 0 \iff s_n\le t<s_{n+1},$$

which makes perfect sense: *the probability of observing $n+1$ events at times $s_1,\dots,s_{n+1}$, having observed n events at time t, is positive if and only if the n events were observed just at or after the second-to-last arrival time and strictly before the last one.*

For this reason we will be focusing on the case $s_n\le t<s_{n+1}$, in which Eq. (5) translates to[1]

$$f(s_1,\dots,s_{n+1}\mid N(t)=n)=\frac{f(s_1,\dots,s_{n+1})}{f(N(t)=n)}=\frac{\lambda^{n+1}\exp(-\lambda s_{n+1})}{\frac{1}{n!}(\lambda t)^n\exp(-\lambda t)}=\frac{n!\lambda\exp(-\lambda(s_{n+1}-t))}{t^n}. \tag{6}$$

Next, we will use the fact that the conditional probability factorizes as:

$$f(s_1,\dots,s_{n+1}\mid N(t)=n)=f(s_{n+1}\mid s_1,\dots,s_n,N(t)=n)f(s_1,\dots,s_n\mid N(t)=n),$$

and also the *memoryless* property for $s_{n+1}>t$:

$$f(s_{n+1}\mid s_1,\dots,s_n,N(t)=n)=f(s_{n+1}\mid N(t)=n).$$

Combining these two properties, we have:

$$f(s_1,\dots,s_n\mid N(t)=n)=\frac{f(s_1,\dots,s_{n+1}\mid N(t)=n)}{f(s_{n+1}\mid N(t)=n)}.$$

The numerator in the RHS of the previous expression is given by Eq. (6). The denominator can be comptued if we realize that, conditional on $N(t)=n$, the time instant s_{n+1} is the *first arrival time after time t*. In other words, it follows the same distribution as the first arrival time if the origin had been put at time t, whichs is an *Erlang*$(1,\lambda)$ shifted by the location parameter t. Putting it all together we arrive at the desired result:

$$f(s_1,\dots,s_n\mid N(t)=n)=\frac{n!}{t^n}\frac{\lambda\exp(-\lambda(s_{n+1}-t))}{\lambda\exp(-\lambda(s_{n+1}-t))}=\frac{n!}{t^n}.$$

□

Exercise 4. *Two teams A and B play a soccer match. The number of goals scored by Team A is modeled by a Poisson process $N_1(t)$ with rate $\lambda_1=0.02$ goals per minute. The number of goals scored by Team B is modeled by a Poisson process $N_2(t)$ with rate $\lambda_2=0.03$ goals per minute. The two processes are assumed to be independent. Let $N(t)$ be the total number of goals in the game up to and including time t. The game lasts for 90 minutes.*

[1] The expression for $f(s_1,\dots,s_{n+1})$ can be derived from the fact that the time increments $T_i=S_i-S_{i-1}$ are identically (exponentially) distributed and independent: $f(s_1,\dots,s_{n+1})=f(T_1=s_1)f(T_2=s_2-s_1)\cdots f(T_{n+1}=s_{n+1}-s_n)$.

(i) Find the probability that no goals are scored.

(ii) Find the probability that at least two goals are scored in the game.

(iii) Find the probability of the final score being Team A: 1, Team B: 2.

(iv) Find the probability that they draw.

(v) Find the probability that Team B scores the first goal.

Confirm your results by writing a Python program that simulates the process and estimate the answers from the simulations.

Solution. We know that the sum of two independent Poisson processes is also a Poisson process with rate equal to the sum of the rates, so we can write $N(t) \sim \mathit{Poisson}(0.05)$. We will make repeated use of the expression of the p.m.f. of a Poisson process (see Eq. (1)).

(i) The probability that no goals are scored equals:

$$P[N(90) = 0] = \frac{1}{0!}(0.05 \cdot 90)^0 e^{-0.05 \cdot 90} = e^{-4.5} \approx 0.0111.$$

(ii) The probability that at least two goals are scored in the game is:

$$\begin{aligned} P[N(90) \geq 2] &= 1 - P[N(90) \leq 1] = 1 - (P[N(90) = 0] + P[N(90) = 1]) \\ &= 1 - (e^{-4.5} + 0.05 \cdot 90 e^{-4.5}) \approx 0.9389. \end{aligned}$$

(iii) Since $N_1(t)$ and $N_2(t)$ are independent, the probability of finishing with a score of Team A: 1 and Team B: 2 is:

$$\begin{aligned} &P[N_1(90) = 1, N_2(90) = 2] = P[N_1(90) = 1]P[N_2(90) = 2] \\ &= 0.02 \cdot 90 e^{-0.02 \cdot 90} \frac{1}{2} \cdot 0.03^2 \cdot 90^2 e^{-0.03 \cdot 90} \approx 0.0729. \end{aligned}$$

(iv) The probability that they draw is given by the expression:

$$P[N_1(90) = N_2(90)] = \sum_{n=0}^{\infty} P[N_1(90) = n]P[N_2(90) = n] = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} 0.02^n 0.03^n 90^{2n} e^{-90(0.03+0.02)}.$$

We could try to sum this infinite series, but we are better off using the fact that the difference of two independent Poisson variables follows a Skellam distribution. Indeed, if V_1 and V_2 are two independent Poisson-distributed random variables with means λ_1 and λ_2 respectively, the p.m.f. for the difference $V = V_1 - V_2$ is given by:

$$p(\nu; \lambda_1, \lambda_2) = P[V = \nu] = e^{-(\lambda_1 + \lambda_2)} \left(\frac{\lambda_1}{\lambda_2}\right)^{\nu/2} I_\nu(2\sqrt{\lambda_1 \lambda_2}), \tag{7}$$

where $I_\nu(x)$ is the modified Bessel function of the first kind of order ν, i.e.:

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n + \nu + 1)} \left(\frac{x}{2}\right)^{2n+\nu}.$$

Now we can compute the desired probability with the aid of Python, either by evaluating the p.m.f. of a $\mathit{Skellam}(90 \cdot 0.02, 90 \cdot 0.03)$ at $\nu = 0$ with `scipy.stats.skellam`, or by substituting the appropriate values in Eq. (7) and evaluating I_0 in the corresponding point via `scipy.special.iv`. Either way, we have:

$$P[N_1(90) - N_2(90) = 0] = e^{-90 \cdot 0.05} I_0(2 \cdot 90\sqrt{0.02 \cdot 0.03}) \approx 0.1793.$$

(v) Let X model the time of the first goal scored by Team B, and let Y be the number of goals scored by Team A before Team B scores. On the one hand we have that, conditional on $X = t$, the variable Y is counting the number of events (goals of Team A) up to time t, so it may be viewed as a Poisson process with mean $0.02t$:

$$P[Y = n \mid X = t] = \frac{1}{n!}(0.02t)^n e^{-0.02t}.$$

On the other hand, the distribution of X is that of the first arrival time of the Poisson process $N_2(t)$, which is known to be exponentially distributed:

$$P[X = t] = 0.03e^{-0.03t}.$$

With this notation, we are interested in computing the probability of $Y = 0$, given the restrictions that $0 \leq X \leq 90$ (that is, we are requiring that Team B scores at least once). Combining the above expressions and using the *law of total probability*, we get:

$$P[Y = 0 \mid 0 \leq X \leq 90] = \int_0^{90} P[Y = 0 \mid X = t]P[X = t]\, dt$$
$$= \int_0^{90} 0.03e^{-0.05t}\, dt = -\frac{0.03}{0.05}\left[e^{-0.05t}\right]_0^{90} \approx 0.5933.$$

The code which contains the simulations that illustrate the correctness of these results can be consulted in `ex4.ipynb`.

Exercise 5. *Consider the process $X(t) = Z\sqrt{t}$ for $t \geq 0$ with the same value of Z for all t.*

(i) Show that the distribution of the process at time t is the same as that of a Wiener process[2]: $X(t) \sim \mathcal{N}(0, \sqrt{t})$.

(ii) What is the mathematical property that allows us to prove that this process is not Brownian?

Solution. Let $Z \sim \mathcal{N}(0, 1)$. We know that the family of normal distributions is closed under linear transformations, and more specifically, that multiplying a zero-mean normal variable by a constant $a > 0$ yields another zero-mean normally-distributed variable for which the standard deviation is the old one times a. So it is immediate to see that

$$X(t) = Z\sqrt{t} \sim \sqrt{t}\mathcal{N}(0, 1) = \mathcal{N}(0, \sqrt{t}).$$

To prove that this process is not Brownian, the key property that we have to use is that of *independence* (or lack thereof). Indeed, since the variable Z is the same for all t, given $t_2 > t_1 \geq s_2 > s_1 \geq 0$, we have

$$X(t_2) - X(t_1) = Z(\sqrt{t_2} - \sqrt{t_1}) \quad \text{and} \quad X(s_2) - X(s_1) = Z(\sqrt{s_2} - \sqrt{s_1}),$$

which are clearly not independent. Since independent increments are an essential property of Brownian processes, this process cannot be Brownian.

[2]The notation $\mathcal{N}(\mu, \sigma)$ used in this document assumes that σ is the *standard deviation* of the distribution, and not the variance.