Lazy Math Instructor

Proseminar: Selected Fun Problems of the ACM Programming Contest

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Introduction to the problem

The lazy math instructor needs to check arbitrary terms for equivalence.
Example term:

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(3*a) - c + (b*b)*8
```

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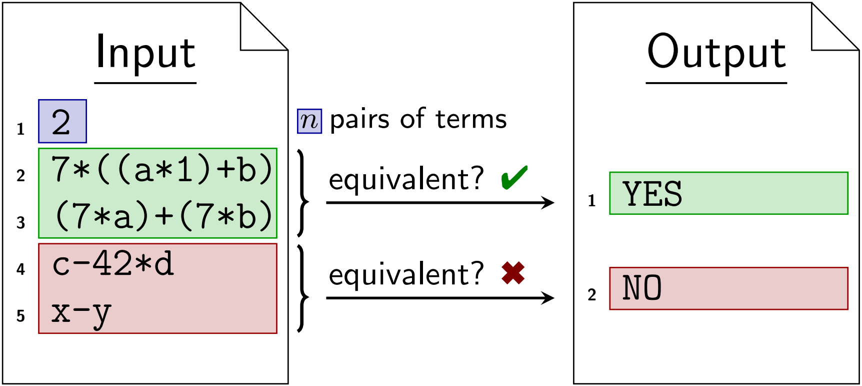
More example terms:

- `c - 42 * d` (left to right evaluation $\Rightarrow (c - 42) * d$)
- `7 * ((a*1)+b)`
- `(7*a) + (7*b)`

$\left.\begin{array}{l}\texttt{7 * ((a*1)+b)}\\ \texttt{(7*a) + (7*b)}\end{array}\right\}$ equivalent

Problem specification

Input: number n and $2n$ terms (n pairs)
Output: n lines: "YES" or "NO" to the equivalence of each pair



Plan of attack

How to check arbitrary terms for equivalence?

- Normalize the term into a polynomial (e.g. $3a^2bd - 6b^2c^5 + 42$)
- Check if the resulting polynomials are identical
  - Unique ordering of sub-terms inside a polynomial

Plan of attack

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Example:

Term:	`(b * a * b) - (3 * 7) - (a * b * b) + (a * a * a)`
Parsed:	$a^3 + -21$
Term:	`(a * a * a) - 21`
Parsed:	$a^3 + -21$

Parsed polynomials are identical $\iff$ Terms are equivalent.

Polynomial representation

Example polynomial: $3a^2bd - 6b^2c^5 + 42$

Polynomial

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Polynomial

Coefficients:

VariableProduct I a^2bd	3
VariableProduct II b^2c^5	-6
VariableProduct III	42

Polynomial representation

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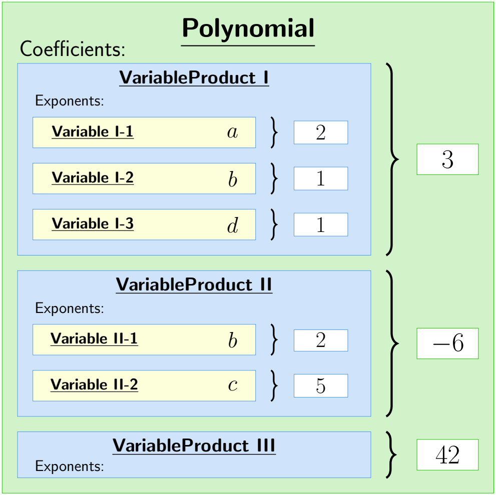

Ordering inside a polynomial

Unique ordering of summands allows for efficient polynomial comparison.

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- Ordering of variable products:
  - By exponent of a primarily (higher exponent first)
  - By exponent of b secondarily
  - ...
  - By exponents of z lastly
  - $a^3 < a^2b^2 < a^2b < a^2 < b^2 < b <$ (None)

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- Establishes a unique order for sub-terms of the polynomial
- Equivalent terms are represented as identical polynomials

Term parsing: Left-to-right? Right-to-left!

Terms should be evaluated left-to-right: `a + b * c` $\equiv$ `(a + b) * c`
Parse sub-terms right-to-left!

Parse(`"a + b * c"`)

Term strings are `blue`, polynomial objects are $green$.

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Parse(`"a + b * c"`) | Split off right sub-term "c"
Parse(`"a + b"`) * Parse(`"c"`)

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Parse(`"a + b"`) * Parse(`"c"`)	Parse variable "c"
Parse(`"a + b"`) * c	

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Parse("a + b * c")	Split off right sub-term "c"
Parse("a + b") * Parse("c")	Parse variable "c"
Parse("a + b") * c	Split off right sub-term "b"
(Parse("a") + Parse("b")) * c	

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Parse(`"a + b"`) * c	Split off right sub-term "b"
(Parse(`"a"`) + Parse(`"b"`)) * c	Parse variable "b"
(Parse(`"a"`) + b) * c	

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Parse(`"a + b"`) * Parse(`"c"`)	Parse variable "c"
Parse(`"a + b"`) * c	Split off right sub-term "b"
(Parse(`"a"`) $+$ Parse(`"b"`)) * c	Parse variable "b"
(Parse(`"a"`) $+$ b) * c	Parse variable "a"
$(a + b)$ * c	

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Parse(`"a + b"`) * Parse(`"c"`)	Parse variable "c"
Parse(`"a + b"`) * c	Split off right sub-term "b"
(Parse(`"a"`) $+$ Parse(`"b"`)) * c	Parse variable "b"
(Parse(`"a"`) $+$ b) * c	Parse variable "a"
(a $+$ b) * c	Add polynomials a and b
$(a+b)$ * c	

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Parse(`"a + b"`) * Parse(`"c"`)	Parse variable "c"
Parse(`"a + b"`) * c	Split off right sub-term "b"
(Parse(`"a"`) + Parse(`"b"`)) * c	Parse variable "b"
(Parse(`"a"`) + b) * c	Parse variable "a"
(a + b) * c	Add polynomials a and b
$(a + b)$ * c	Multiply polynomials $a + b$ and c
$ac + bc$	

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How to handle brackets and integers

Brackets can be handled the same way: Split the entire bracket off and perform a recursive call to parse that sub-term first. Integers are treated like variables.

Parse(`"a + (17 * c)"`)

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Brackets can be handled the same way: Split the entire bracket off and perform a recursive call to parse that sub-term first. Integers are treated like variables.

Parse(`"a + (17 * c)"`) | Split off right sub-term "(17 ∗ c)"
Parse(`"a"`) + Parse(`"17 * c"`)

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How to handle brackets and integers

Brackets can be handled the same way: Split the entire bracket off and perform a recursive call to parse that sub-term first. Integers are treated like variables.

Parse(`"a + (17 * c)"`)	Split off right sub-term "(17 ∗ c)"
Parse(`"a"`) + Parse(`"17 * c"`)	Split off right sub-term "c"
Parse(`"a"`) + (Parse(`"17"`) * Parse(`"c"`))	Parse variable "c"
Parse(`"a"`) + (Parse(`"17"`) * c)	Parse integer "17"
Parse(`"a"`) + (17 * c)	Multiply polynomials 17 and c
Parse(`"a"`) + $17c$	Parse variable "a"
a + $17c$	Add polynomials a and $17c$
$a + 17c$	

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Parsing algorithm: Alternatives

I used a recursive algorithm to parse the terms into polynomials.
Other ideas?

- Iterative instead of recursive
  - Go through the term left-to-right
  - Harder to handle brackets: Would need to store parsed sub-terms in a more complex data structure
- Don't parse into polynomials at all
  - Substitute concrete values for each variable and evaluate
    $\Rightarrow$ Very hard to be *sure* that the terms are equivalent
- Other approaches possible

Implementation in C#

Representing polynomials and variable products as objects:

- Custom addition, subtraction, multiplication operators
- Custom equality checks
  $\Rightarrow$ Efficient use of built-in types such as dictionaries
- Custom ordering for variable products
  $\Rightarrow$ Efficient use of *sorted* dictionaries

Allows for readable code like this:

C# example on how to use the classes

```
Polynomial term1 = Polynomial.Parse("a + b");
Polynomial term2 = Polynomial.Parse("2*a");
Polynomial diff = term1 - term2;            // diff is -a + b

Polynomial term3 = Polynomial.Parse("b - a");

// diff and term3 are both -a + b and are treated as identical
Debug.Assert(diff.Equals(term3));
```

Any questions?

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