Basic Morse Theory

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1 Introduction

Morse theory is mainly about how Morse functions can reflect the topology of a manifold. Here, I will start with basic definitions, then show there are abundant Morse functions by the theorem of existence. Finally, I will talk about handle decomposition, which, from my point of view, is the heart of Morse theory.

2 Morse Function

2.1 Basic Definition
Definition (Critical points)
Let $f : \mathcal{M} \to \mathbf{R}$ be a smooth function, where $\mathcal{M}$ is a smooth manifold with dimension m. A point p_0 is called critical point of f if

$$\frac{\partial f}{\partial x^1} = 0, \frac{\partial f}{\partial x^2} = 0, \ldots, \frac{\partial f}{\partial x^m} = 0$$

where $(x^1, x^2, \cdots, x^m)$ is a local coordinate system.

There are 2 kinds of critical points, which can be determined by the Hessian matrix $H_f(p) = [\frac{\partial^2 f}{\partial x^i \partial x^j}](p)$. A critical point p_0 is called non-degenerate if $H_f(p_0) \neq 0$, degenerate if $H_f(p_0) = 0$.

Definition (Morse function)
$f : \mathcal{M} \to \mathbf{R}$ is a Morse function if every critical point of f is non-degenerate.

The reason why we need non-degenerate critical points is because of stability. Generally speaking, we can always perturb a smooth function to make degenerate critical points vanish. For example, $x = 0$ is the degenerate critical point for $f(x) = x^3$. If we perturb this function linearly $f(x) = x^3 + \epsilon x$, then either there is no critical point($\epsilon > 0$), or split into 2 non-degenerate critical points $x = \pm\sqrt{\frac{-\epsilon}{3}}$ ($\epsilon < 0$).

2.2 Morse Lemma
Morse functions have a nice behavior locally, here is an example.
Theorem (Morse lemma)
Let p_0 be a non-degenerate critical point of $f : \mathcal{M} \to \mathbf{R}$, then we can choose a local coordinate system $(x^1, ..., x^m)$ such that f locally has the standard form:

$$f = -(x^1)^2 - (x^2)^2 - \cdots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \ldots (x^m)^2 + c$$

where p_0 correspond the origin and $c = f(p_0)$.
We then call λ the index of p_0, which is an integer between 0 and m.

This theorem means f locally looks like a quadratic form near a critical point. Here I give a brief proof.
Proof.
It will be easy if we assume f is replaced by $f - f(p_0)$. i.e. $f(p_0) = 0$. Moreover, let $p_0 = (0...0) = \overline{0}$.
For $f(0...0) = 0$, there exist m smooth functions g_i with local coordinate $x^1...x^m$ s.t.

$$f(x^1...x^m) = \sum_{i=1}^{m} x^i g_i(x^1...x^m) \quad \& \quad \frac{\partial f}{\partial x^i}(\overline{0}) = g_i(\overline{0})$$

Since p_0 is critical point, then $g^i(\overline{0}) = 0$, which means we can apply the same procedure:

$$g_i(x^1...x^m) = \sum_{j=1}^{m} x^j h_{ij}(x^1...x^m) \quad \& \quad \frac{\partial g_i}{\partial x^j}(\overline{0}) = h_{ij}(\overline{0})$$

Thus, we have

$$f(x^1...x^m) = \sum_{i,j=1}^{m} x^i x^j h_{ij}(x^1...x^m)$$

.

Since p_0 is non-degenerate, $\det H_f(\overline{0}) \neq 0$. WLOG, we can assume $\frac{\partial^2 f}{\partial x^1 \partial x^1}(\overline{0}) \neq 0$ (we can do this since $\exists j$ such that $\frac{\partial^2 f}{\partial x^1 \partial x^j}(\overline{0}) \neq 0$, then just reorder the index). Let $H_{ij} = \frac{h_{ij}+h_{ji}}{2}$, can check $H_{11}(\overline{0}) \neq 0$. Since H is continuous, we can find a neighborhood of $\overline{0}$ s.t. $H_{11} \neq 0$.

Now change the coordinate, let

$$y^1 = \sqrt{|H_{11}|}(x^1 + \sum_{i=2}^{m} x^i \frac{H_{1i}}{H_{11}})$$

We can check the determinant of Jacobian of $(y^1, x^2, ..., x^m)$ is nonzero, thus it's a local coordinate system. Combine the formula of y^1 with $f(x^1...x^m) = \sum_{i,j=1}^{m} x^i x^j H_{ij}(x^1...x^m)$, we get:

$$f = \pm(y^1)^2 + \sum_{i,j=2}^{m} x^i x^j H_{ij} - (\sum_{i=2}^{m} x^i \frac{H_{1i}}{H_{11}^2})^2$$

. Then we can do the coordinate transform of $y^2, ..., y^m$ by induction. In the case of 2 dimension, $y^2 = \sqrt{|\frac{H_{11}H_{22}-H_{12}^2}{H_{11}}|}x^2$. Q.E.D.

2.3 Existence of Morse Function

Till now, what we did is defining Morse function. However, to make Morse theory useful, we wish we can always find a Morse function given a smooth manifold $\mathcal{M}$.

Theorem (Existence of Morse function)

Let $\mathcal{M}$ be a smooth compact closed manifold and $f : \mathcal{M} \to \mathbf{R}$ is a smooth function. Then there exist a Morse function $g : \mathcal{M} \to \mathbf{R}$, and g is a (C^2, ϵ) approximation of f. (i.e. under C^2 topology)

Definition ((C^2, ϵ) approximation)

f, g are defined on a compact set K, then f is a (C^2, ϵ) approximation of g if for $\forall \epsilon > 0$

$$|f^{(k)}(p) - g^{(k)}(p)| < \epsilon$$

$\forall p \in K, \forall 0 \leq k \leq 2$, where k is the order of derivation.

This theorem means whenever there is a smooth function $\mathcal{M} \to \mathbf{R}$, there is a Morse function. Here is a sketch of proof.

Proof. First, we need a lemma:

Lemma Let U be an open set in $\mathbf{R}^m$, and $f : U \to \mathbf{R}$ is smooth. Then exist $a_1, ..., a_m$ such that $g(x^1...x^m) = f(x^1...x^m) - (a_1 x^1 + \cdots + a_m x^m)$ is a Morse function. And $|a_1|, .., |a_m|$ can be arbitrarily small.

To prove the lemma, define $h : U \to \mathbf{R}^m \quad (x^1, ..., x^m) \mapsto (\frac{\partial f}{\partial x^1} ... \frac{\partial f}{\partial x^m})$. Then the Jacobian of h becomes the Hessian of f. We know that p_0 is the critical point of h if and only if the Jacobian is 0. i.e. $\det H_f(p_0) = 0$. By Sard's Theorem, the set of critical points of h has measure zero, which suggests we can always choose $(a_1, ..., a_m)$ not a critical value of h with arbitrarily small absolute value.

Suppose p_0 is the critical point of g, then can check $h(p_0) = (a_1...a_m)$. Choose $(a_1...a_m)$ s.t. p_0 is not a critical point of h. i.e. $H_f(p_0) \neq 0$. Then $H_g(p_0) = H_f(p_0) \neq 0$. Thus p_0 is a non-degenerate critical point of g. This completes the proof of lemma.

Now since $\mathcal{M}$ is compact, we can choose finite charts $U_1...U_k$ that cover $\mathcal{M}$.

We prove this theorem inductively. Suppose f_{i-1} is a Morse function defined on C_{i-1}, where $C_i = U_1 \cup ... \cup U_i$. Define:

$$f_i = f_{i-1} + (a_1 x^1 + \cdots + a_m x^m) h_i(x^1...x^m)$$

where h_i is a smooth function defined on C_i, $0 \leq h_i \leq 1$. For two neighborhood V, K of $C_{i-1}(C_{i-1} \subset V \subset K \subset C_i)$, $h_i = 1$ in V and $h_i = 0$ outside K. Thus, f_i is smooth on C_i.

Since the 1st and 2nd derivative of h is bounded, we can take arbitrarily small $(a_1, ..., a_m)$ to make f_i is a (C^2, ϵ) approximation of f_{i-1}. By induction, $f_k = g$ is the Morse function we need. Q.E.D.

3 Handle Decomposition

We have discussed some properties of Morse functions. Now, it's time to see how they are related to the topology of manifolds. In this section, we will show a manifold could be built with some basic blocks.

Theorem Let $f : \mathcal{M} \to \mathbf{R}$ be a Morse function and $\mathcal{M}$ is a closed manifold. If f has no critical values in [a,b], then $\mathcal{M}_{[a,b]} \cong f^{-1}(a) \times [0,1]$. ($\mathcal{M}_{[a,b]} = \{p \in \mathcal{M} | a \le f(p) \le b\}$)
Note that two manifolds $\mathcal{N} \cong \mathcal{M}$ means $\exists$ diffeomorphism $f : \mathcal{N} \to \mathcal{M}$.
Before beginning our proof, I give a theorem about gradient-like vector field without proof.
Theorem(Gradient-like vector field)
$f : \mathcal{M} \to \mathbf{R}$ is Morse function on compact manifold $\mathcal{M}$, then there exist a gradient like vector field X for f. Where
(i)$Xf > 0$ if not at critical points
(ii)$X = -2x^1 \frac{\partial}{\partial x^1} - \dots 2x^\lambda \frac{\partial}{\partial x^\lambda} + \dots + 2x^m \frac{\partial}{\partial x^m}$ on a neighborhood of critical points.
Proof. Def $Y = \frac{X}{Xf}$ since there is no critical value in $[a,b]$. $\forall p \in f^{-1}(a)$, we can always find a curve $c_p(t)$ with $\frac{d}{dt} f(c_p(t)) = \frac{dc}{dt} f = Y_{c(t)} f = 1$. We can check $c_p(t)$ is the diffeomorphism between $f^{-1}(a) \times [0, b-a]$ and $M_{[a,b]}$ since different $c_p(t)$ will never meet.

Corollary Let $f : \mathcal{M} \to \mathbf{R}$ be a Morse function and $\mathcal{M}$ is a closed manifold. If f has no critical values in [a,b], then $\mathcal{M}_a \cong \mathcal{M}_b$. ($\mathcal{M}_t = \{p \in \mathcal{M} | f(p) \le t\}$, called the sublevel set).
Proof. Similarly, since there is no critical value in $[a,b]$, then we can let the curve flow along the gradient-like vector field. And after a finite time, $\mathcal{M}_a$ meets $\mathcal{M}_b$.
This telles us if the shape of $\mathcal{M}_t$ wouldn't change unless t pass a critical value. But how does the shape change exactly? We then need the theorem of handle decomposition.

Definition(λ handle)
A λ handle is the Cartesian product of a λ disk and $m - \lambda$ disk: $D^\lambda \times D^{m-\lambda}$. Here, the λ is the index of critical points and m is the dimension of manifold. D represents disk.
Theorem
Let c_i be critical value with index λ. Then $\mathcal{M}_{c_i+\epsilon} \cong \mathcal{M}_{c_i-\epsilon} \cup (D^\lambda \times D^{m-\lambda})$.

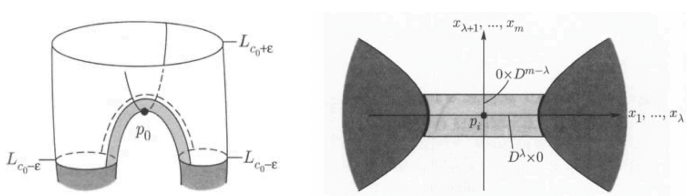

Figure1.[1] $\mathcal{M}_{c_i-\epsilon}$ attached with a λ handle

An intuitive way of thinking is given the quadratic form of Morse function near critical point, the 'bridge' area could be thought as

$$(x^1)^2 + \dots + (x^\lambda)^2 - (x^{\lambda+1})^2 - \dots (x^m)^2 \le \epsilon$$
$$(x^{\lambda+1})^2 + \dots + (x^m)^2 \le \delta \le \epsilon$$

We could see such handles are building blocks of manifold. Here is the precise theorem about it:

Definition(Handle body)
Let D^m be a disk. After attaching handles one after one,

$$D^m \cup_{\varphi_1} (D^{\lambda_1} \times D^{m-\lambda_1}) \cup_{\varphi_2} \dots \cup_{\varphi_n} (D^{\lambda_n} \times D^{m-\lambda_n}) = \mathcal{H}(D^m, \varphi_1, \dots, \varphi_n)$$

Such a manifold is called m-dimensional handlebody.
Here, $\cup_{\varphi_i}$ means using a 'gluing' diffeomorphism $\varphi_i : \partial\mathcal{H}(D^m, \varphi_1, \dots, \varphi_{i-1}) \to \partial(D^{\lambda_i} \times D^{m-\lambda_i})$. Note that after attaching, the manifold is then smoothed out.
Theorem(Handle decomposition)
$f : \mathcal{M} \to \mathbf{R}$ is a Morse function defined on closed manifold $\mathcal{M}$, then a structure of handlebody is determined by f.

The index of handles corresponds to the critical points.

This theorem tells us that a closed manifold could be cut into several building blocks(handles).
Proof. By induction. For $i = 0$, $M_{c_0+\epsilon} \cong D^m$. Assume that $M_{c_{i-1}+\epsilon} \cong \mathcal{H}(D^m, \varphi_1, \ldots, \varphi_{i-1})$. Since there is no critical point in $[c_{i-1}+\epsilon, c_i - \epsilon]$, $M_{c_{i-1}+\epsilon} \cong M_{c_i-\epsilon}$. By the theorem above, $\mathcal{M}_{c_i+\epsilon} \cong \mathcal{M}_{c_i-\epsilon} \cup_{\varphi_i} (D^{\lambda_i} \times D^{m-\lambda_i})$. Thus $\mathcal{M}_{c_i+\epsilon} \cong \mathcal{H}(D^m, \varphi_1, \ldots, \varphi_i)$. This completes the proof.

Note that there may be more than one way to glue handles. For example, as shown in fig2, in $\mathbf{R}^2$, there are two ways to attach a 1-handle $D^1 \times D^1$ to a 2-handle D^2. We can check the first one is diffeomorphic to a cylinder, while the second one is diffeomorphic to a Mobius strip.

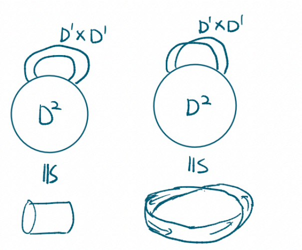

Figure2. two ways to attach a 1-handle of 2 dimension

We then show an example of handle decomposition to end this article. Fig3 shows the handle decomposition of a torus T^2.

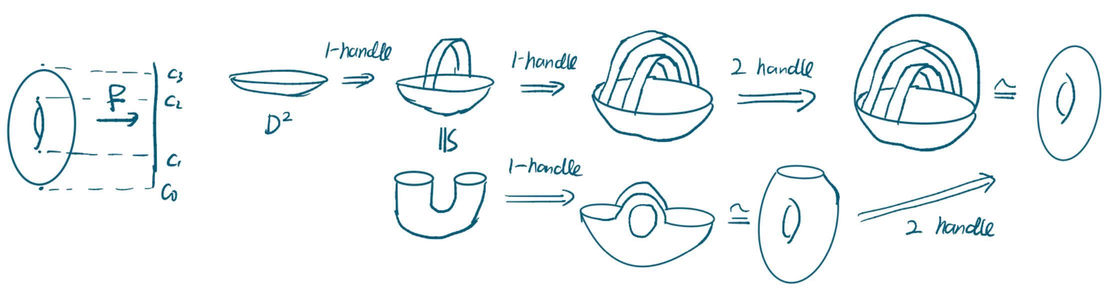

Figure3. handle decomposition of a torus T^2

4 Reference

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