Q1(a)

The general solution for this second-order linear homogeneous ODE comes in the form of:

$$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

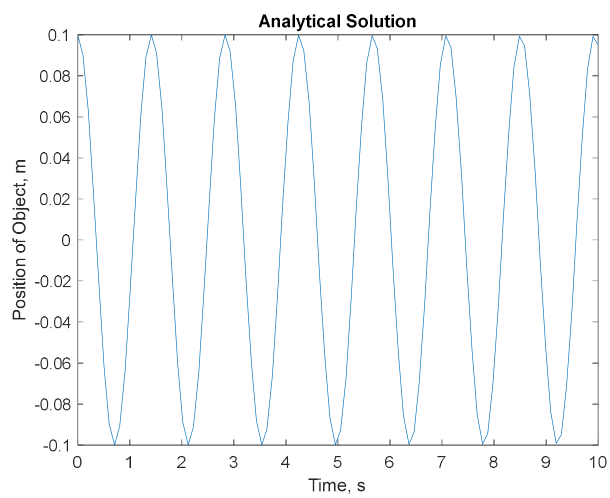
Where the values of λ are the solutions to the quadratic: $\lambda^2 + \omega^2 = 0 \rightarrow \lambda = \pm i\omega$

Using the relation $e^{i\theta} = \cos\theta + isin\,\theta$: $y(t) = Acos(\omega t) + Bsin(\omega t)$

We can solve for the constant A and B by substituting and solving for the initial conditions:

$$y(t) = 0.1\cos(\omega t)$$

Q1(b)



Q1(c)

$$\dot{Y} = AY \text{ where } Y = [y, y']^T\text{, where } A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$$

Q1(d)

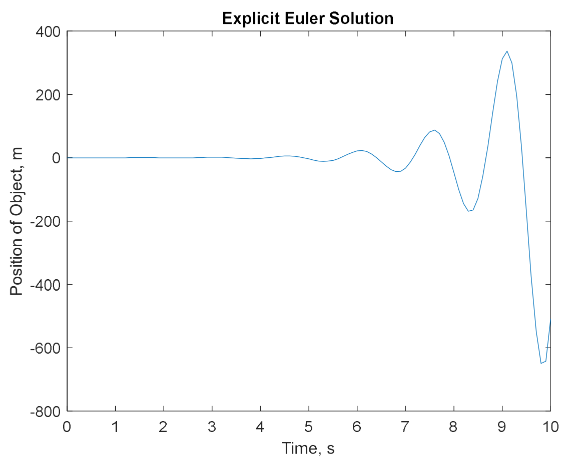
To find the eigenvalues we must solve for:

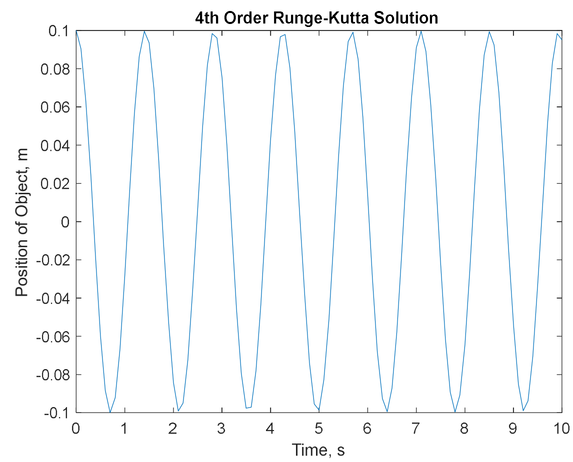
$$\det(A - I\lambda) = 0 \rightarrow \lambda = \pm i\omega$$

The eigenvalue of this matrix is the same as the solutions to λ for the first general solution we use. Additionally, the value of the eigenvalues could be used to predict the shape of the exact solution without solving for it. Specifically, the fact that the solutions are both purely imaginary shows the system will be a harmonic of equal magnitude for each phase.

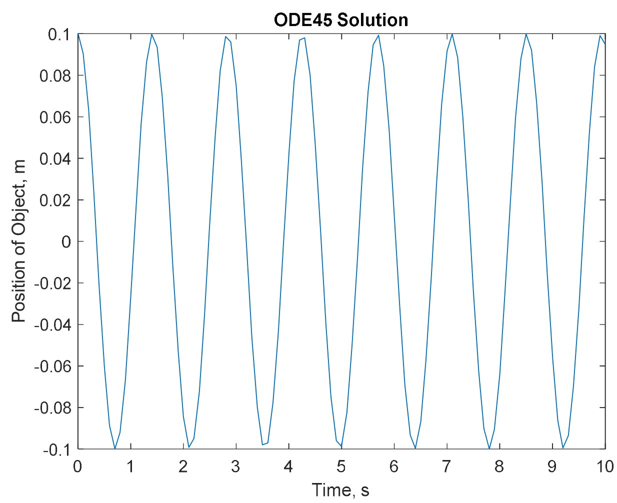
Q2(a)

Q2(b)

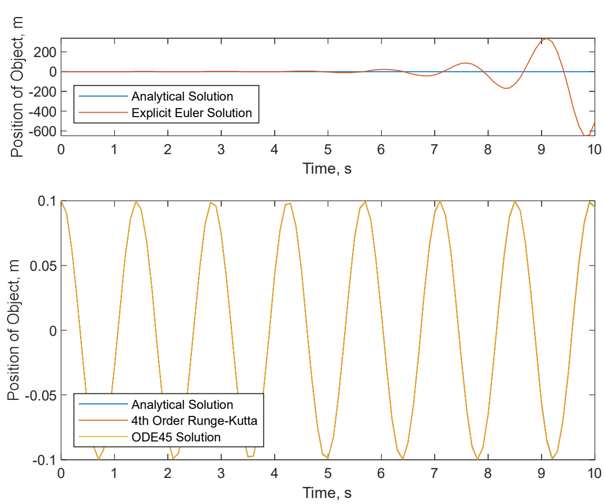




Q2(c)

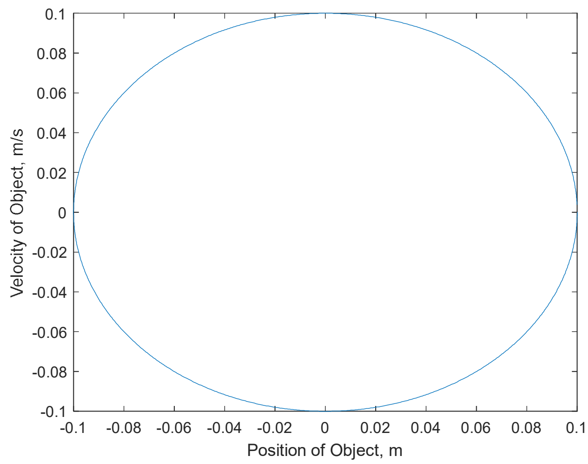


Q2(d)



The Explicit Euler produces an unstable solution that diverges while the 4th order Runge-Kutta and build in ODE45 produce accurate stable solutions nearly indistinguishable from the exact solution.

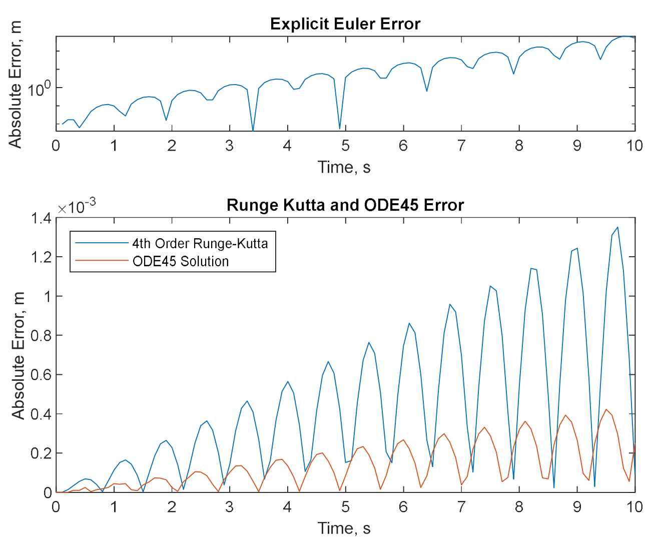
Q2(e)



Physically the velocity is the derivate of the position with respect to time. Since the displacement can be modeled with a cosine wave, the velocity will be a sine wave. With consistent amplitude, which this model has, this can simply be modeled as a circle

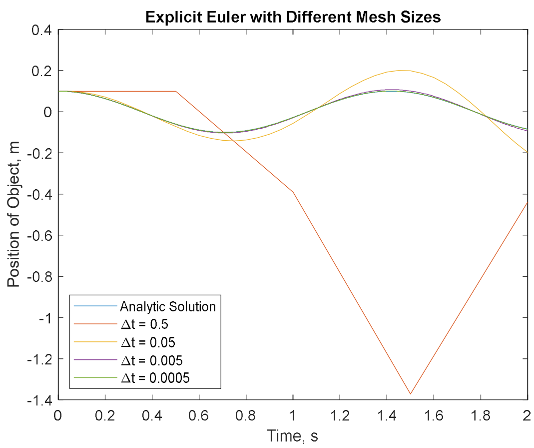
Q2(f) - The 4th Order Runge-Kutta function and ODE45 function produce accurate resualts while the explicit Euler does not. Looking at how each of these methods work we can clearly see that the Explicit Euler is by far the most efficient. This is because the Runge-Kutta method essential solves and averages multiple Explicit Euler solutions to produce a more accurate and stable solution. In this case our 4th Order Runge-Kutta is 4 times as complex than our Explicit Euler, on the contrary the ODE45 function is more efficient than both methods despite being more accurate. This is a result of dynamic step sizing greatly increasing efficiency.

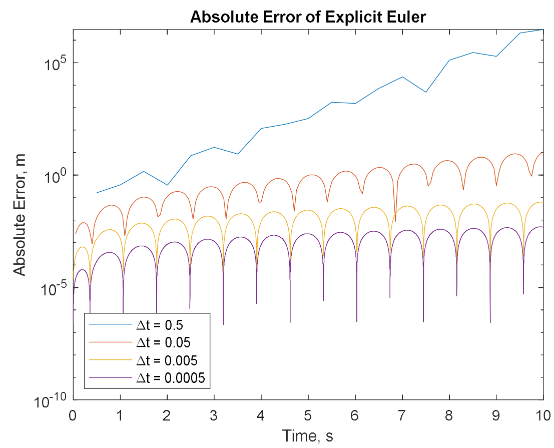
Q3(a)



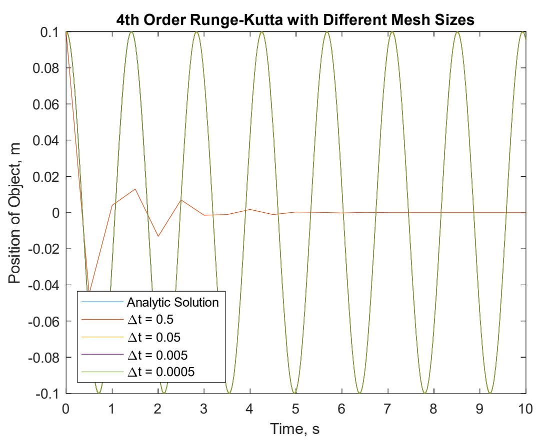
We can see the error of each of the methods, especially the unstable Explicit Euler method which has an error that grows exponentially as time increases (Note the logarithmic y-axis). On the other hand, the Runge-Kutta and ODE45 are stable with errors that increase linearly as time increases. You can also see how the ODE45, which takes the average of a 4th and 5th order Runge-Kutta method is more accurate.

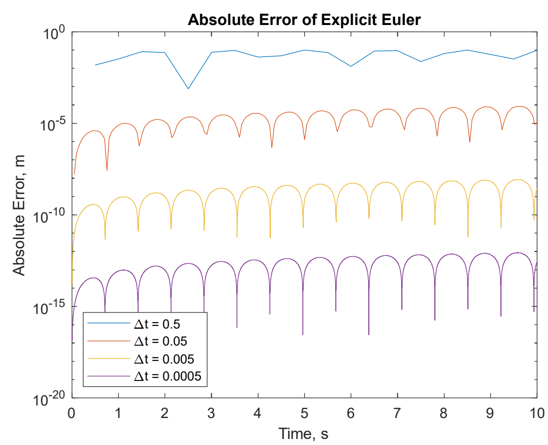
Q3(b) You can see clearly how the absolute error of the Explicit Euler method seems to become linear as the step size decreases, meaning the systems becomes stable. This is consistent with our understanding stability for a stiff system (such as our system which has a $\frac{|\lambda|_{min}}{|\lambda|_{max}}$ value of 1).



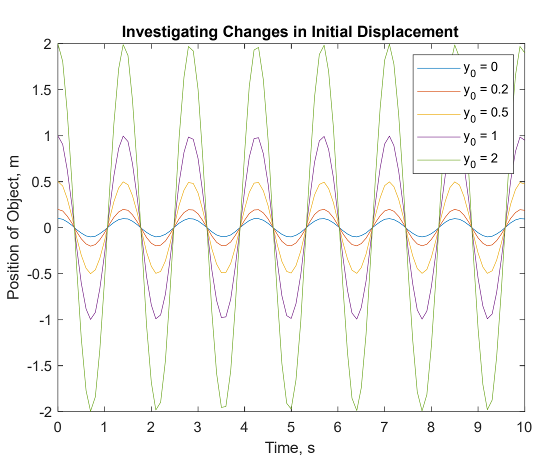


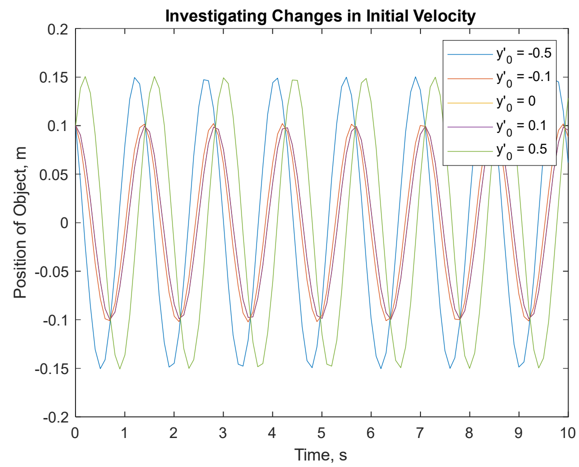
Q3(c) As the Runge-Kutta can be treated as a finer meshed version of the Explicit Euler it can also become unstable if the step size was too large. However, instead of diverging to infinity the Runge-Kutta solution instead converges to 0. This means when unstable the error will look smaller than the Explicit Euler despite providing equally inaccurate results.





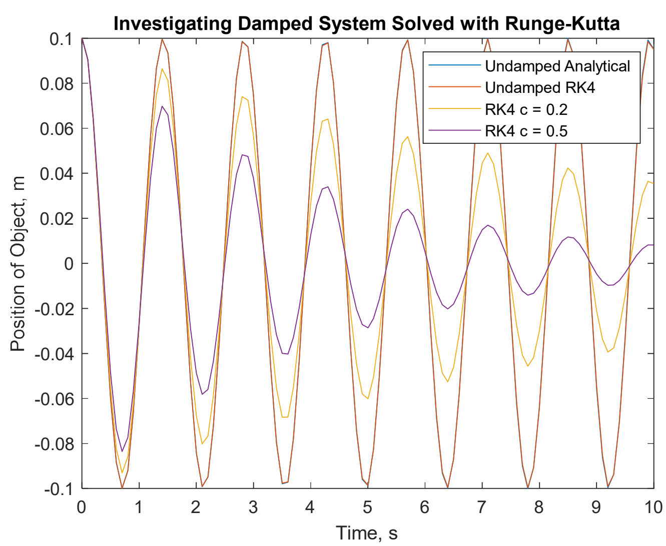
Q3(d) - Changing the initial position and the initial velocity directly affects the amplitude and phase of the resulting graph. This can be best evaluated by considering the problem itself. At maximum displacement the velocity will always be equal to zero, on the other hand, when the displacement is zero, the velocity will be at its maximum. The object sinking can defined as when the object is completely submerged (ie. y(t) > L/2 = 0.5m), considering the problem. This will always happen when the velocity is greater than 0.5ω.





Q4(a) - The main difference between a buoyant model is the natural frequency, while this is a factor in practice it makes a very small difference when finding exact and numerical solutions. One key difference is what physically happens when the object is fully submerged. When this happens, the force acting on the object becomes constant (instead of displacement dependent), while this will change the shape of the curve, but will maintain the cyclical nature.

Q4(b)



Damping changes equation as an oscillation with constant amplitude to one with decaying amplitude as shown in the figure. Unlike the undamped system will generally be less stiff as the Eigenvalues of this system are $\lambda = \frac{-c \pm \sqrt{c^2 - 4\omega^2}}{2}$. This means that the system is generally more stable if it is damped especially when you considering that the Runge-Kutta method decays to 0 naturally when its unstable.