

[illegible]
$$\begin{aligned} &= \frac{4\pi \cdot \frac{1}{2} \cdot \frac{1}{2} (R^2 - r^2)^{\frac{3}{2}}}{\frac{4\pi}{3}} \\ &= \frac{8\pi}{3} \end{aligned}$$

$$I = \int_0^1 \int_0^{2\pi} \int_0^1 (1 + \cos \theta + \cos \phi) r^2 dr d\theta d\phi$$

八、(8分) 设有函数 $f(x, y, z) = \frac{1}{4} (x^2 + y^2 + z^2) + \frac{1}{4} (x^2 + y^2 + z^2)^2$ ，试计算下列积分。

$$I = \iiint_{\Omega} f(x, y, z) dx dy dz$$

解：设 $x+1 = 2r \sin \theta \cos \phi$
 $x-y+2 = 3r \sin \theta \sin \phi$
 $z+1 = r \cos \theta$
 $\Rightarrow \Omega: r \leq 1, \theta \in [0, \pi], \phi \in [0, 2\pi]$
 $x = \frac{1}{2} + r \sin \theta \cos \phi + \frac{1}{2} r \sin \theta \cos \phi$
 $y = \frac{1}{2} + r \sin \theta \cos \phi - \frac{1}{2} r \sin \theta \cos \phi$
 $z = r \cos \theta - 1$

$$\Rightarrow dx dy dz = 6 r^2 \sin \theta dr d\theta d\phi$$

$$\begin{pmatrix} \sin \theta \cos \phi & r \cos \theta & r \sin \theta \cos \phi \\ \sin \theta \sin \phi & -r \cos \theta & r \sin \theta \sin \phi \\ \cos \theta & -r \sin \theta & 0 \end{pmatrix}$$

$$= \cos \theta - r \cos \theta \sin \theta \left(-\frac{1}{2} \sin^2 \theta - \frac{1}{2} \sin^2 \theta - \frac{1}{4} \sin^2 \theta \cos^2 \theta \right) - \left(\frac{1}{2} \sin^2 \theta + \frac{1}{2} \sin^2 \theta - \frac{1}{4} \sin^2 \theta \cos^2 \theta \right)$$

$$+ r \sin^2 \theta \left(-\frac{1}{2} \cos^2 \theta - \frac{1}{2} \cos^2 \theta - \frac{1}{4} \cos^2 \theta \right) - \left(\frac{1}{2} \cos^2 \theta + \frac{1}{2} \cos^2 \theta - \frac{1}{4} \cos^2 \theta \right)$$

$$= r \cos \theta \sin \theta (-3) + r \sin^2 \theta (-3) = -6 r^2 \sin \theta$$

$$= \frac{1}{4} \cdot 2\pi = \frac{\pi}{2}$$

$$\therefore I = \int_0^1 \int_0^{2\pi} \int_0^1 (r \cos \theta - 1) r^2 dr d\theta d\phi$$

$$= 2\pi \cdot \int_0^1 \int_0^{2\pi} (r \cos \theta - 1) r^2 dr d\theta$$

$$= 4\pi \int_0^1 \int_0^{2\pi} (r^3 \cos \theta - r^2) dr d\theta$$

$$= 4\pi \left[\frac{1}{4} r^4 \cos \theta - r^3 \right]_0^1$$

$$= 4\pi \left(\frac{1}{4} \cos \theta - 1 \right)$$

$$= 4\pi \int_0^{2\pi} \left(\frac{1}{4} \cos \theta - 1 \right) d\theta$$

$$= 4\pi \left(\frac{1}{4} \sin \theta - \theta \right) \Big|_0^{2\pi} = 4\pi (-2\pi) = -8\pi$$

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$$\begin{cases} u = x+y+1 \\ v = x-y+2 \\ w = z+1 \end{cases}$$

$$\Rightarrow \Omega: u^2 + v^2 + w^2 \leq 1$$

$$x = \frac{u+v-1}{2}, y = \frac{u-v-1}{2}, z = w-1$$

$$\Rightarrow dx dy dz = \frac{1}{2} du dv dw$$

$$I = \frac{1}{2} \iiint_{\Omega} (u-1)^2 du dv dw$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$