Chapter Twenty

Cost Minimization

Cost Minimization

- A firm is a cost-minimizer if it produces any given output level $y \geq 0$ at smallest possible total cost.
- $c(y)$ denotes the firm's smallest possible total cost for producing y units of output.
- $c(y)$ is the firm's **total cost function**.

Cost Minimization

- When the firm faces given input prices $w = (w_1,w_2,\ldots,w_n)$ the total cost function will be written as

$$c(w_1,\ldots,w_n,y).$$

The Cost-Minimization Problem

- Consider a firm using two inputs to make one output.
- The production function is
  $$y = f(x_1,x_2).$$
- Take the output level $y \geq 0$ as given.
- Given the input prices w_1 and w_2, the cost of an input bundle (x_1,x_2) is $w_1x_1 + w_2x_2$.

The Cost-Minimization Problem

- For given w_1, w_2 and y, the firm's cost-minimization problem is to solve $$\min_{x_1,x_2 \geq 0} w_1x_1 + w_2x_2$$

  subject to $f(x_1,x_2) = y.$

The Cost-Minimization Problem

- The levels $x_1^*(w_1,w_2,y)$ and $x_1^*(w_1,w_2,y)$ in the least-costly input bundle are the firm's **conditional demands for inputs 1 and 2**.
- The (smallest possible) total cost for producing y output units is therefore

$$c(w_1,w_2,y) = w_1 x_1^*(w_1,w_2,y) + w_2 x_2^*(w_1,w_2,y).$$

Conditional Input Demands

- Given w_1, w_2 and y, how is the least costly input bundle located?
- And how is the total cost function computed?

Iso-cost Lines

- A curve that contains all of the input bundles that cost the same amount is an iso-cost curve.
- E.g., given w_1 and w_2, the \$100 iso-cost line has the equation

$$w_1x_1 + w_2x_2 = 100.$$

Iso-cost Lines

- Generally, given w_1 and w_2, the equation of the \$c iso-cost line is

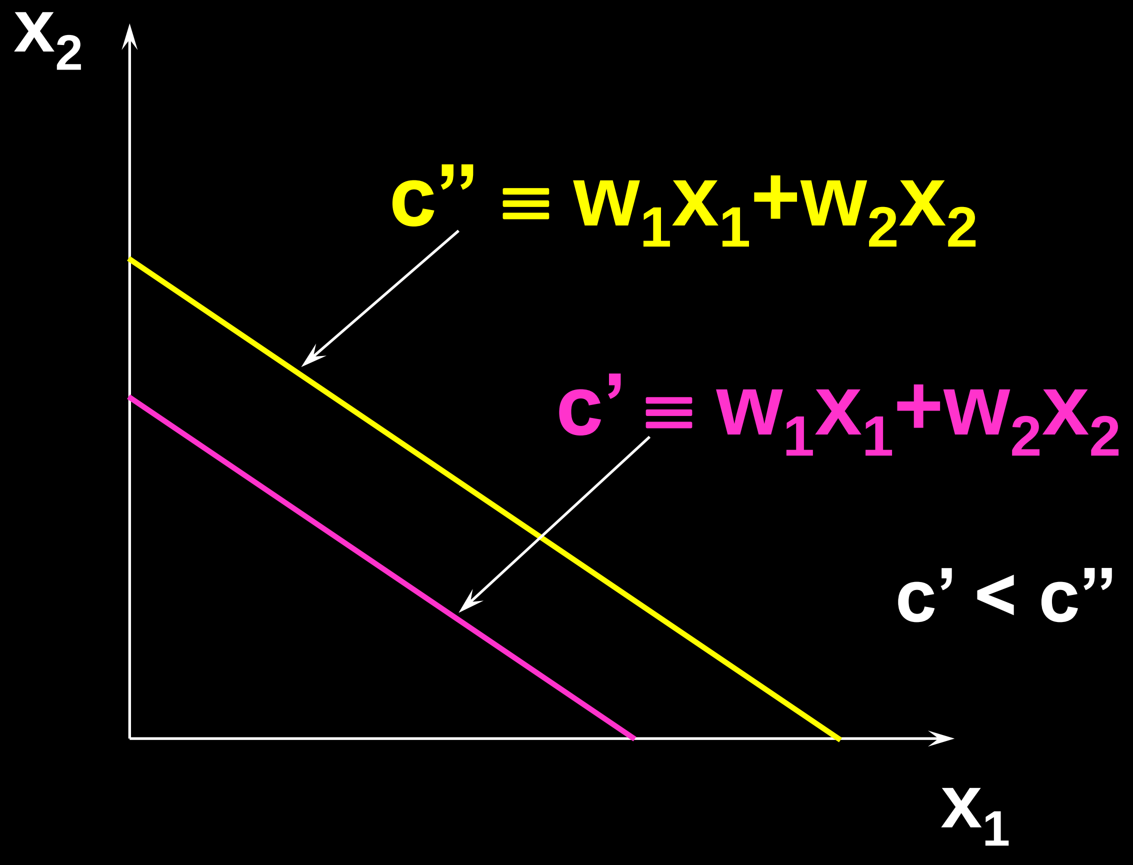
$$w_1 x_1 + w_2 x_2 = c$$

i.e.

$$x_2 = -\frac{w_1}{w_2} x_1 + \frac{c}{w_2}.$$

- Slope is - w_1/w_2.

Iso-cost Lines



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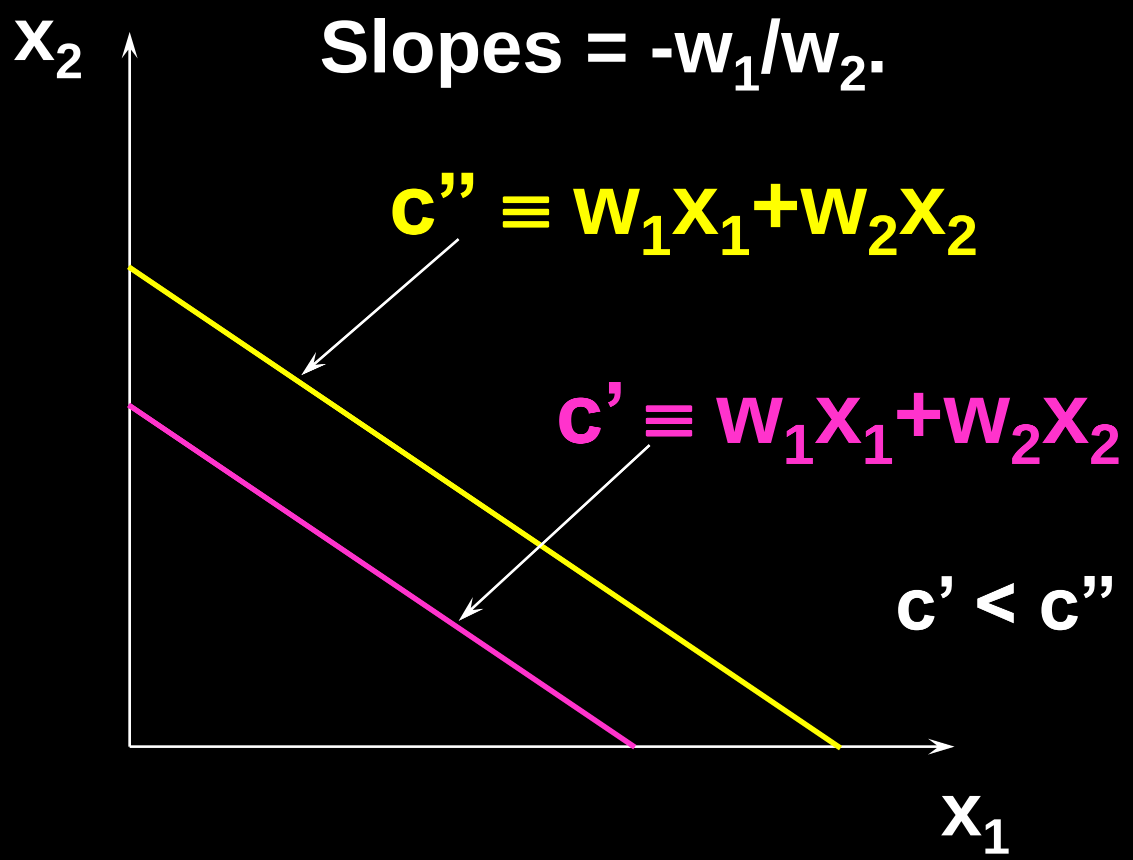
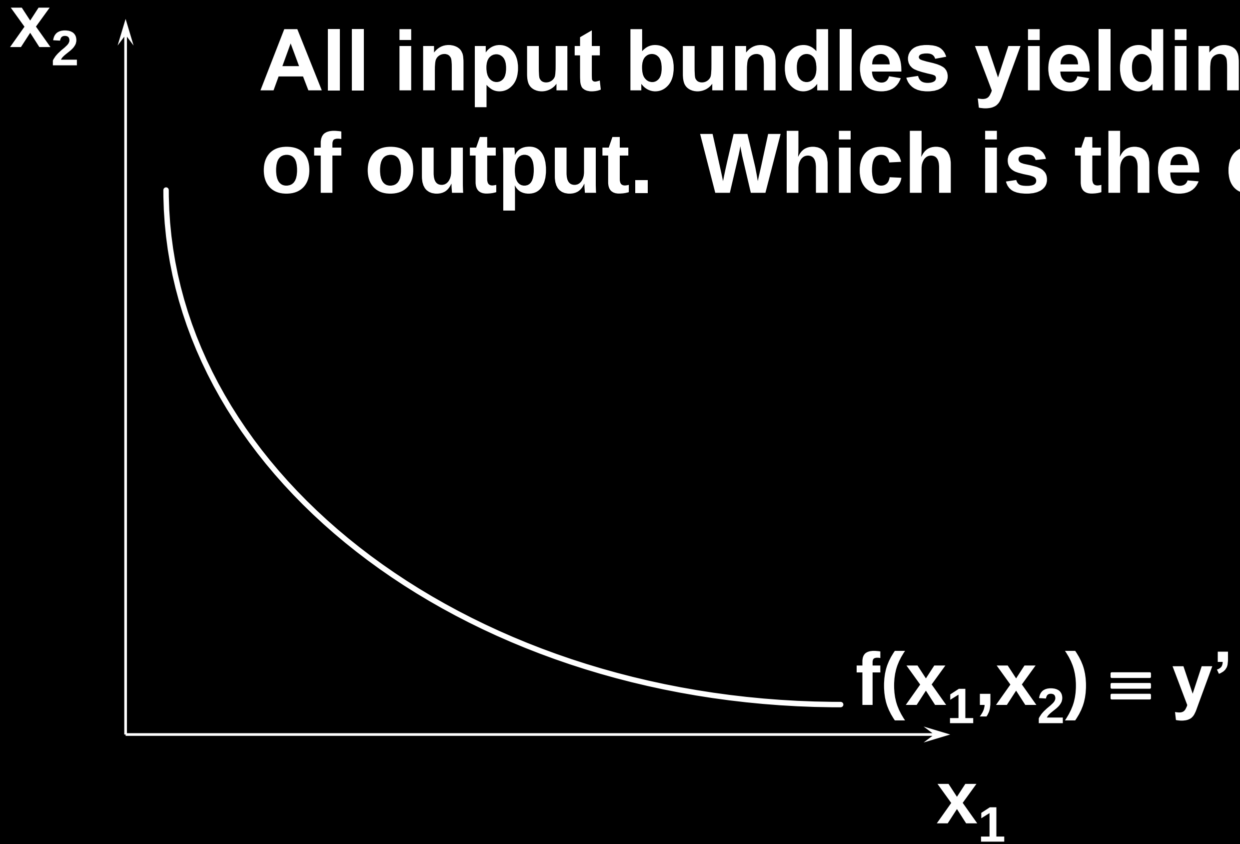


The y'-Output Unit Isoquant

All input bundles yielding y' units of output. Which is the cheapest?



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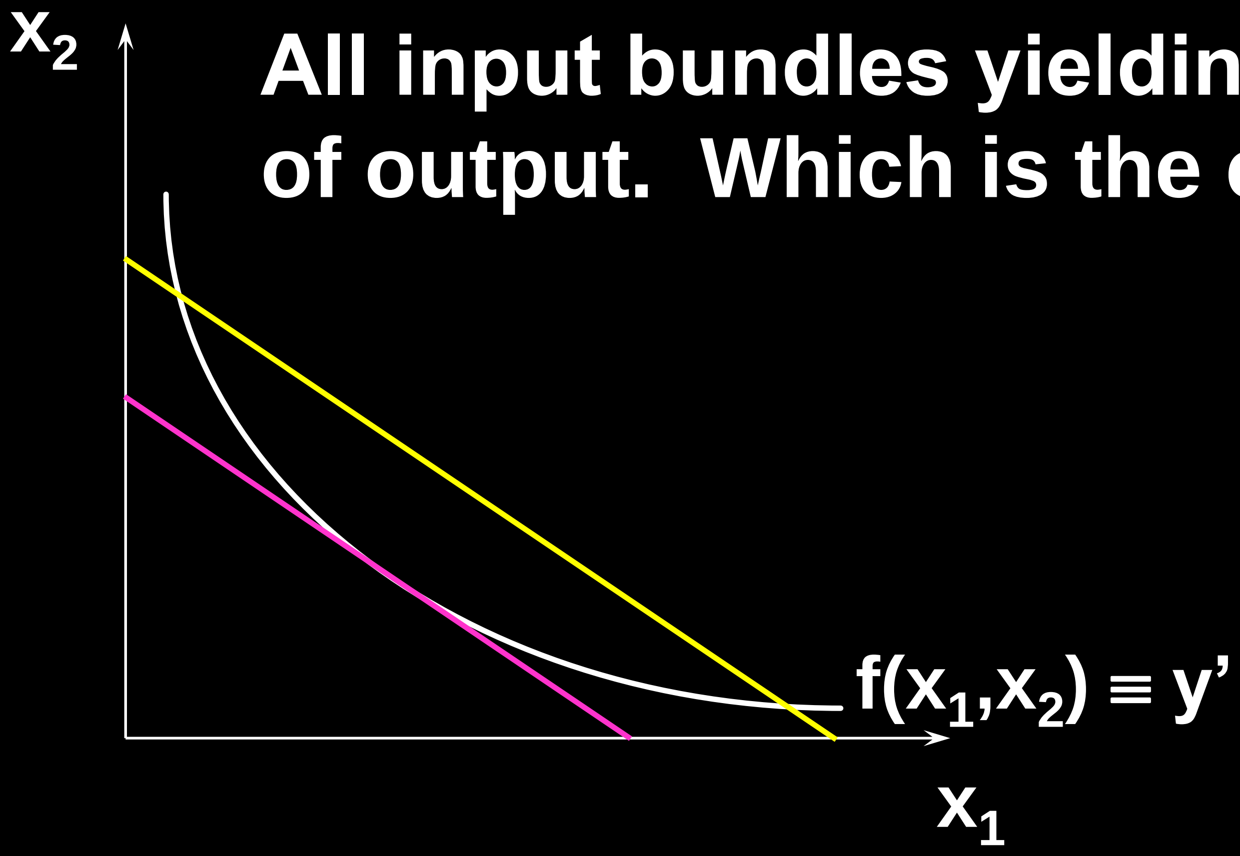

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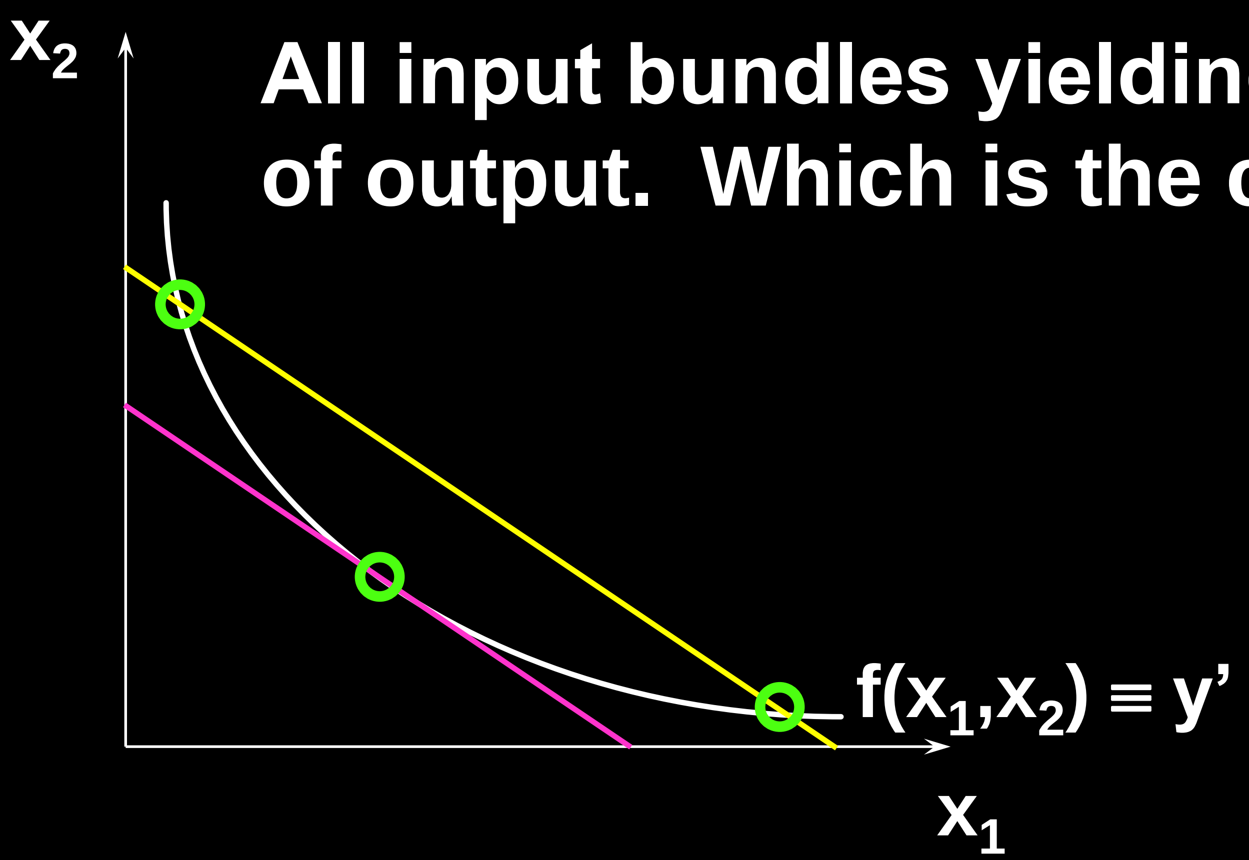

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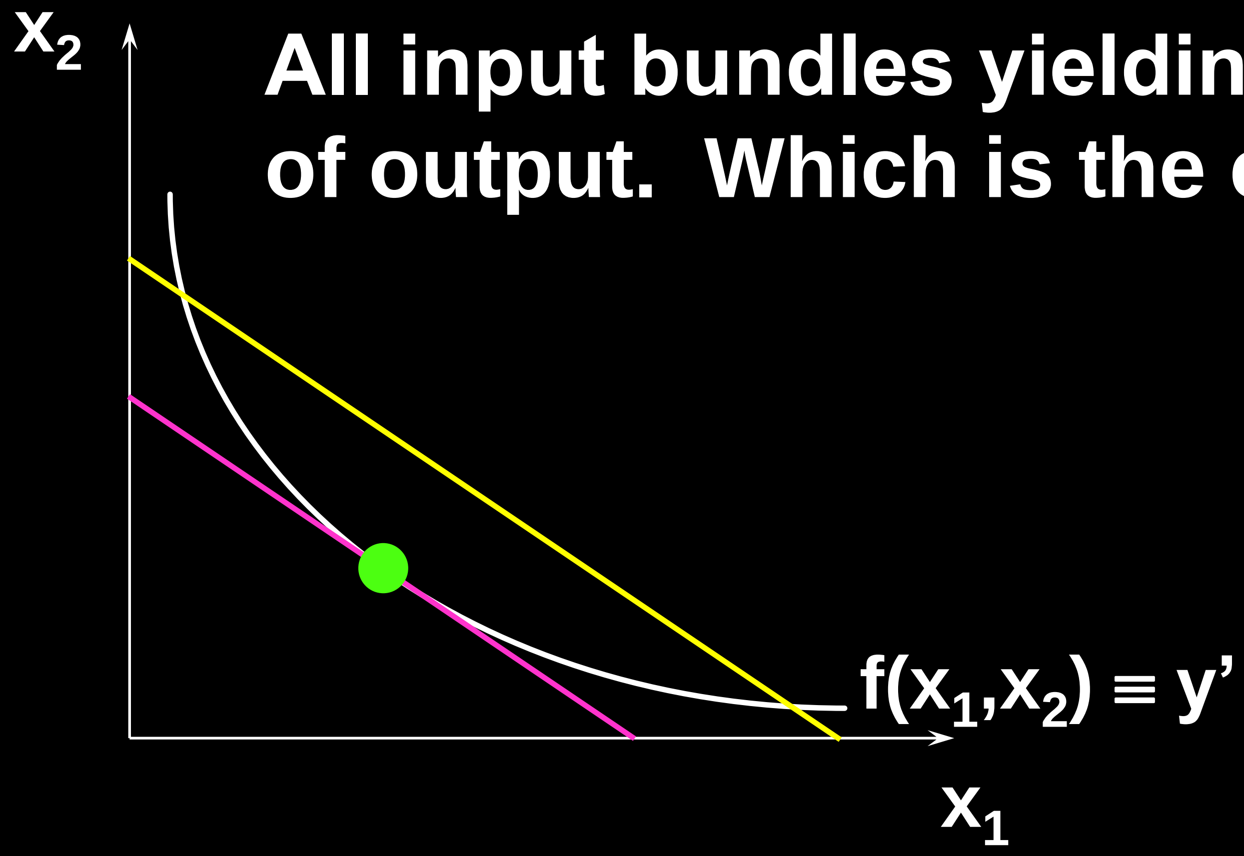

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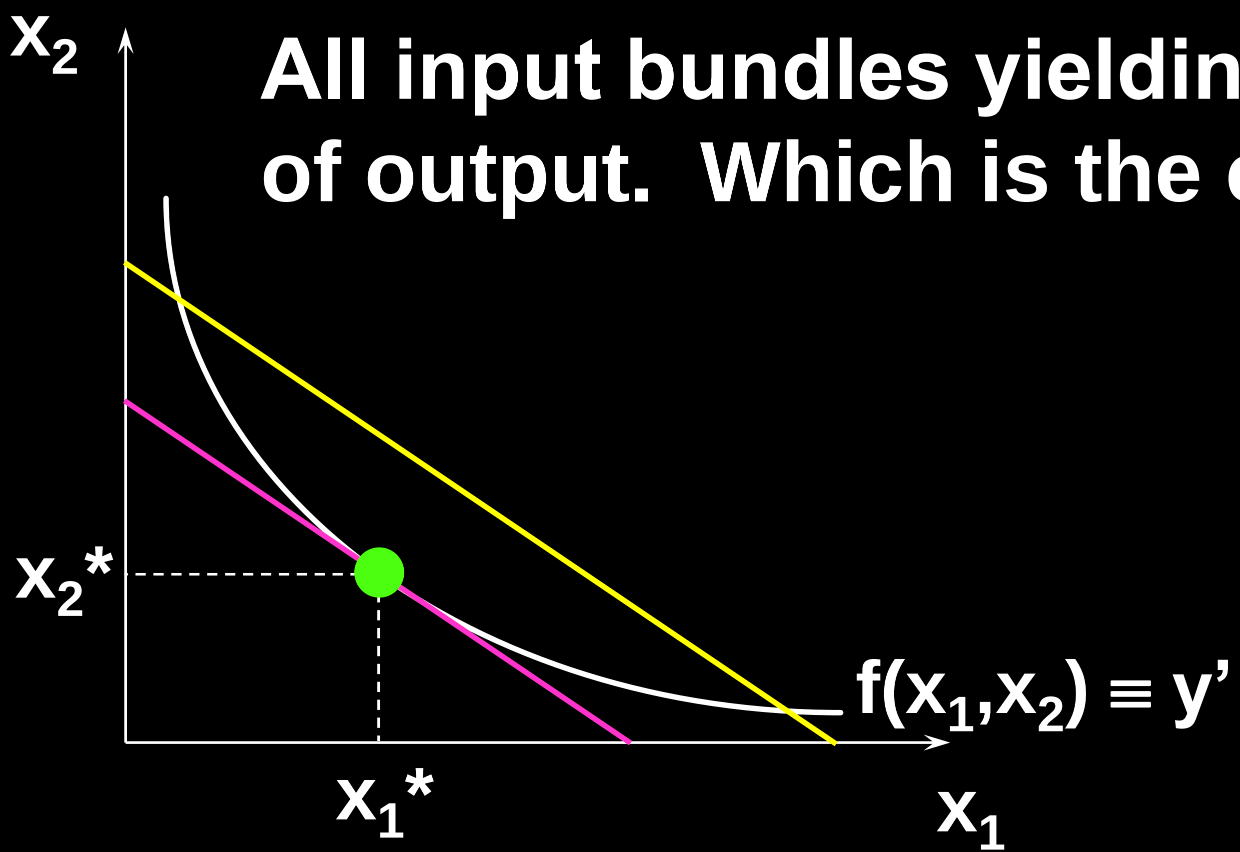
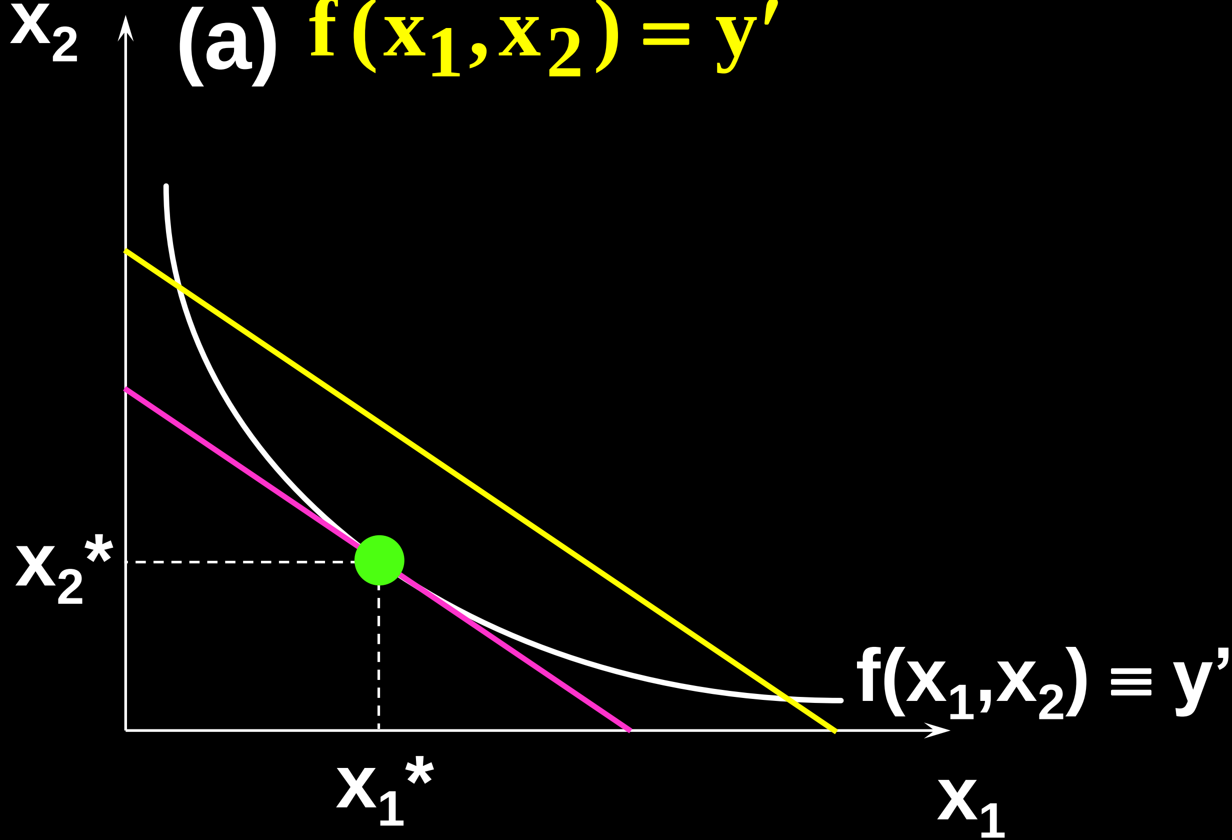


The Cost-Minimization Problem

At an interior cost-min input bundle:

(a) $f(x_1^*, x_2^*) = y'$

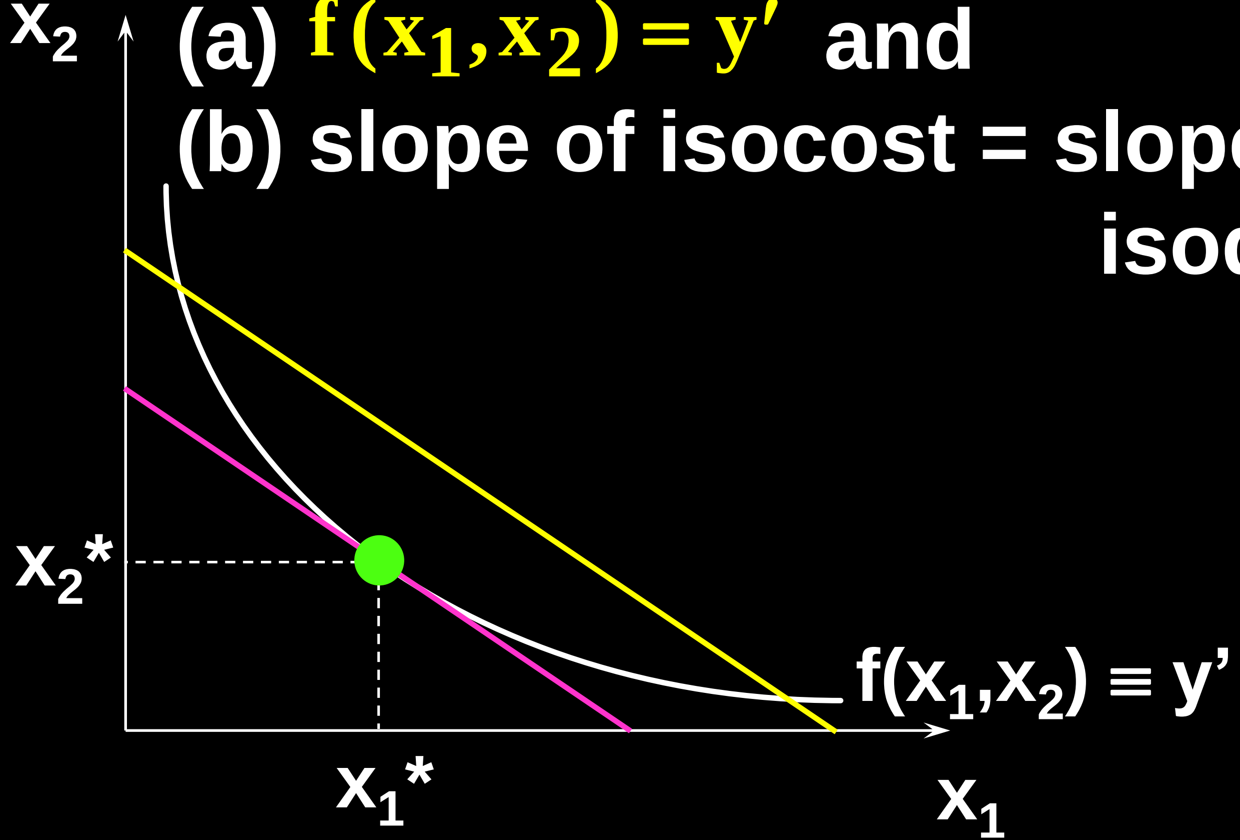


The Cost-Minimization Problem

At an interior cost-min input bundle:

(a) $f(x_1^*, x_2^*) = y'$ and

(b) slope of isocost = slope of isoquant



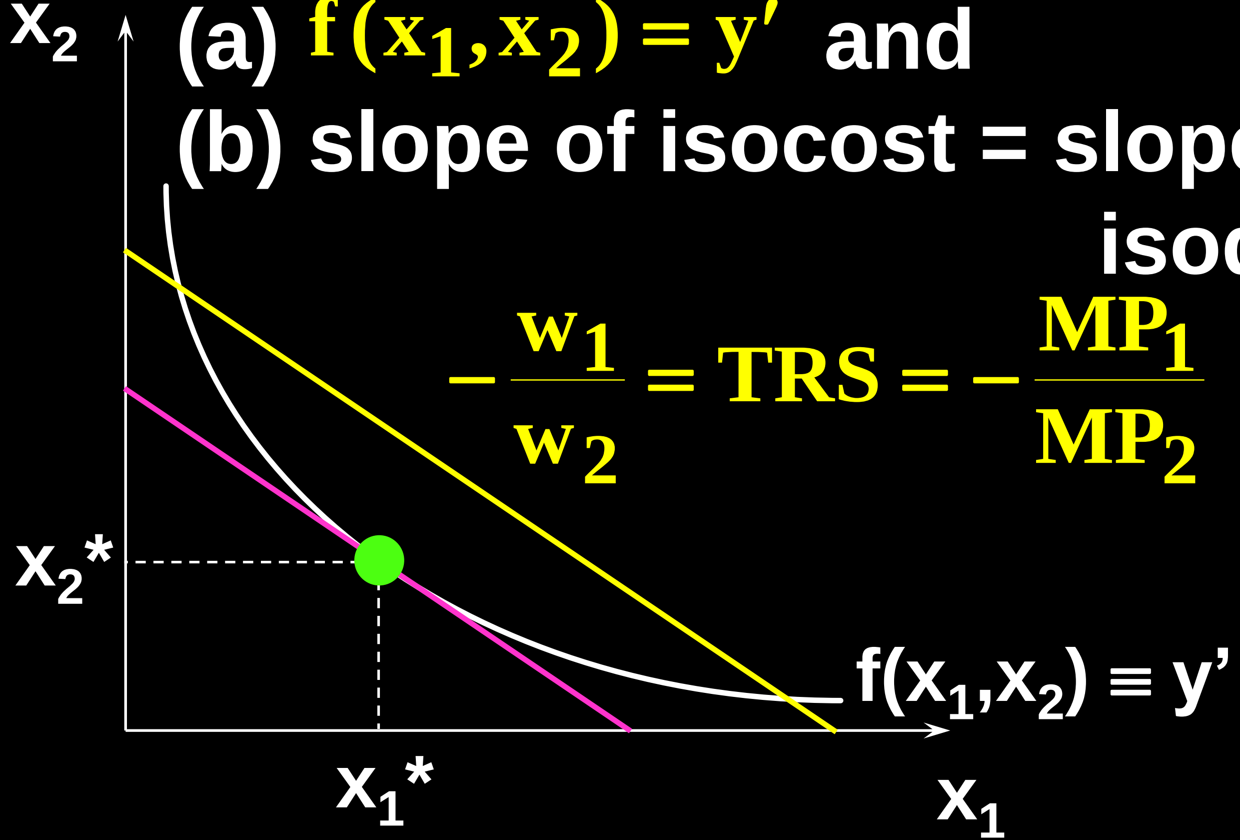
The Cost-Minimization Problem

At an interior cost-min input bundle:

(a) $f(x_1^*, x_2^*) = y'$ and

(b) slope of isocost = slope of isoquant; i.e.

$$-\frac{w_1}{w_2} = TRS = -\frac{MP_1}{MP_2} \text{ at } (x_1^*, x_2^*).$$



A Cobb-Douglas Example of Cost Minimization

- A firm's Cobb-Douglas production function is
$$y = f(x_1, x_2) = x_1^{1/3} x_2^{2/3}.$$
- Input prices are w_1 and w_2.
- What are the firm's conditional input demand functions?

A Cobb-Douglas Example of Cost Minimization

At the input bundle (x_1^*,x_2^*) which minimizes the cost of producing y output units:

(a)
$$y = (x_1^*)^{1/3}(x_2^*)^{2/3}$$
and

(b)
$$-\frac{w_1}{w_2} = -\frac{\partial y/\partial x_1}{\partial y/\partial x_2} = -\frac{(1/3)(x_1^*)^{-2/3}(x_2^*)^{2/3}}{(2/3)(x_1^*)^{1/3}(x_2^*)^{-1/3}}$$
$$= -\frac{x_2^*}{2x_1^*}.$$

A Cobb-Douglas Example of Cost Minimization

(a) $y = (x_1^*)^{1/3}(x_2^*)^{2/3}$ (b) $\frac{w_1}{w_2} = \frac{x_2^*}{2x_1^*}.$

A Cobb-Douglas Example of Cost Minimization

(a) $y = (x_1^*)^{1/3}(x_2^*)^{2/3}$ **(b)** $\dfrac{w_1}{w_2} = \dfrac{x_2^*}{2x_1^*}.$

From (b), $x_2^* = \dfrac{2w_1}{w_2} x_1^*.$

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Now substitute into (a) to get

$$y = (x_1^*)^{1/3}\left(\frac{2w_1}{w_2}x_1^*\right)^{2/3}$$

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$$y = (x_1^*)^{1/3}\left(\frac{2w_1}{w_2}x_1^*\right)^{2/3} = \left(\frac{2w_1}{w_2}\right)^{2/3} x_1^*.$$

So $x_1^* = \left(\dfrac{w_2}{2w_1}\right)^{2/3} y$ is the firm's conditional demand for input 1.

A Cobb-Douglas Example of Cost Minimization

Since $x_2^* = \frac{2w_1}{w_2} x_1^*$ and $x_1^* = \left(\frac{w_2}{2w_1}\right)^{2/3} y$

$$x_2^* = \frac{2w_1}{w_2}\left(\frac{w_2}{2w_1}\right)^{2/3} y = \left(\frac{2w_1}{w_2}\right)^{1/3} y$$

is the firm's conditional demand for input 2.

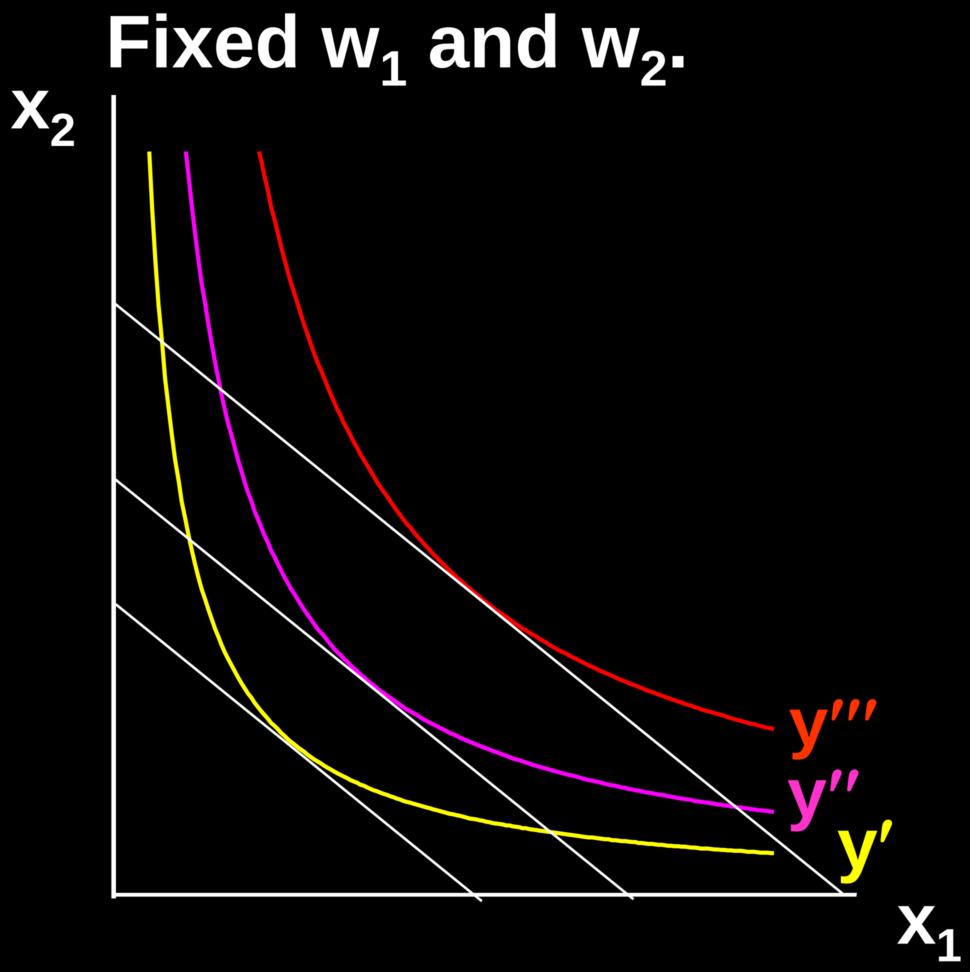
A Cobb-Douglas Example of Cost Minimization

So the cheapest input bundle yielding y output units is

$$\left(x_1^*(w_1,w_2,y),x_2^*(w_1,w_2,y)\right)$$

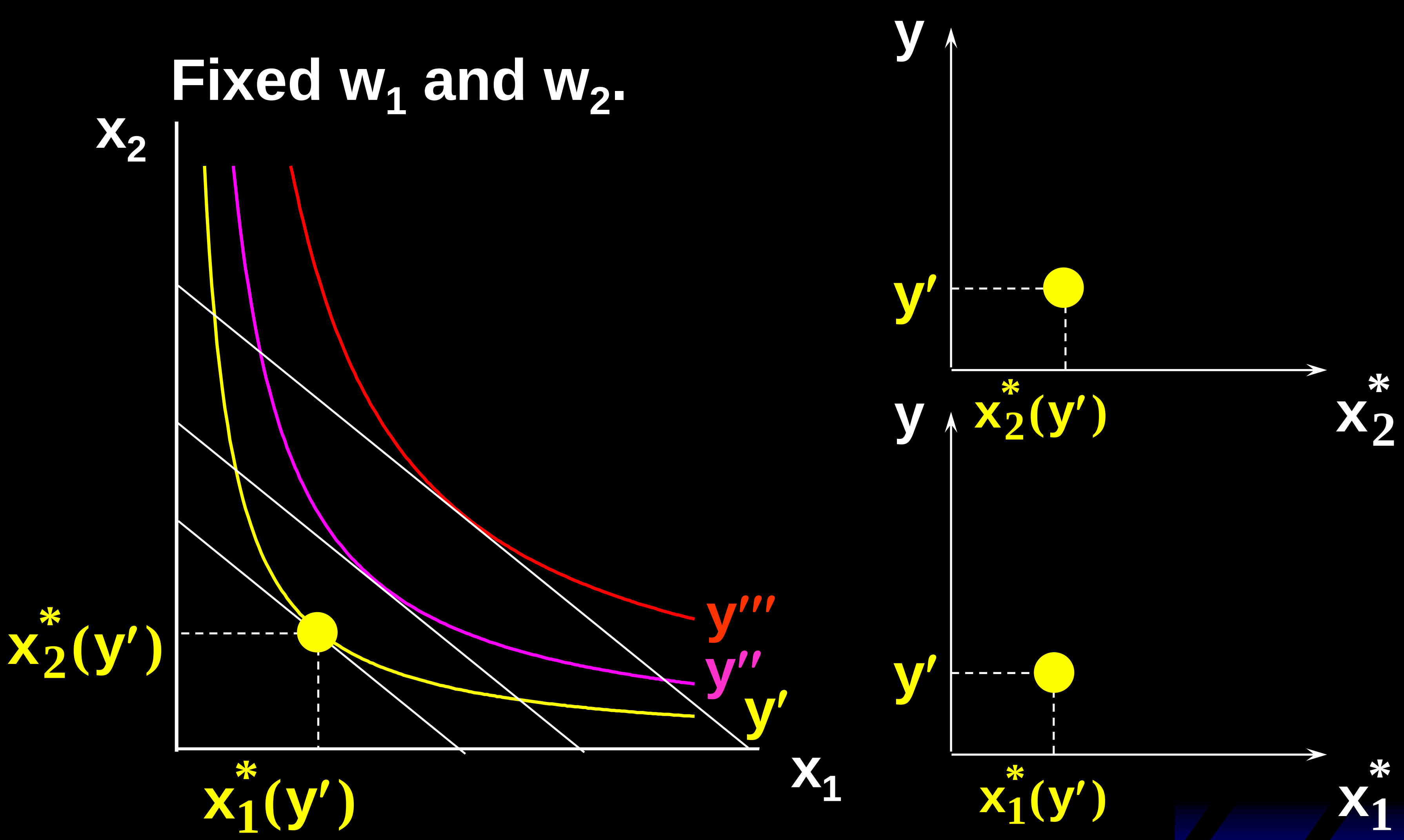
$$=\left(\left(\frac{w_2}{2w_1}\right)^{2/3}y,\left(\frac{2w_1}{w_2}\right)^{1/3}y\right).$$

Conditional Input Demand Curves



Conditional Input Demand Curves

Fixed w_1 and w_2.



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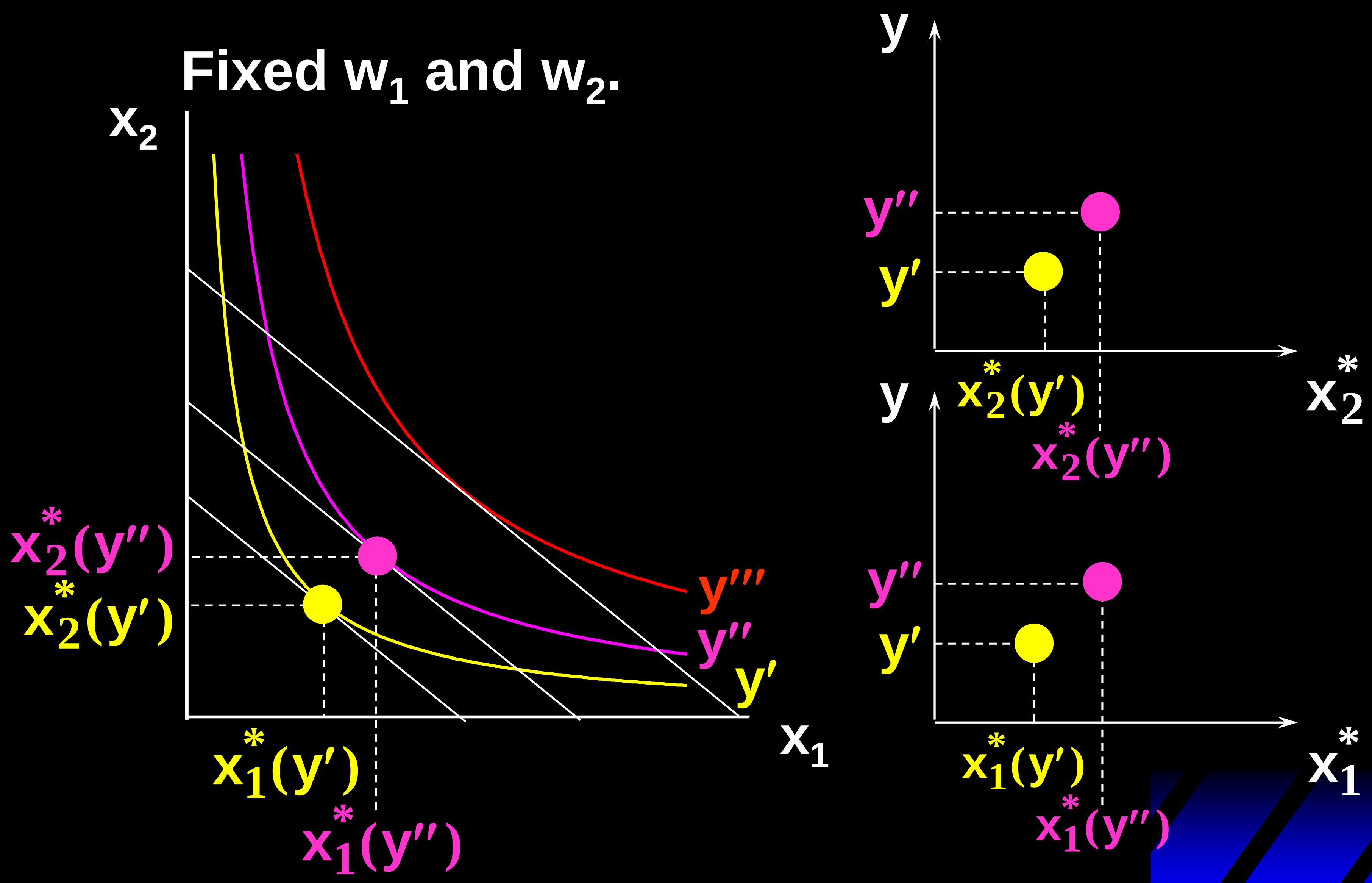

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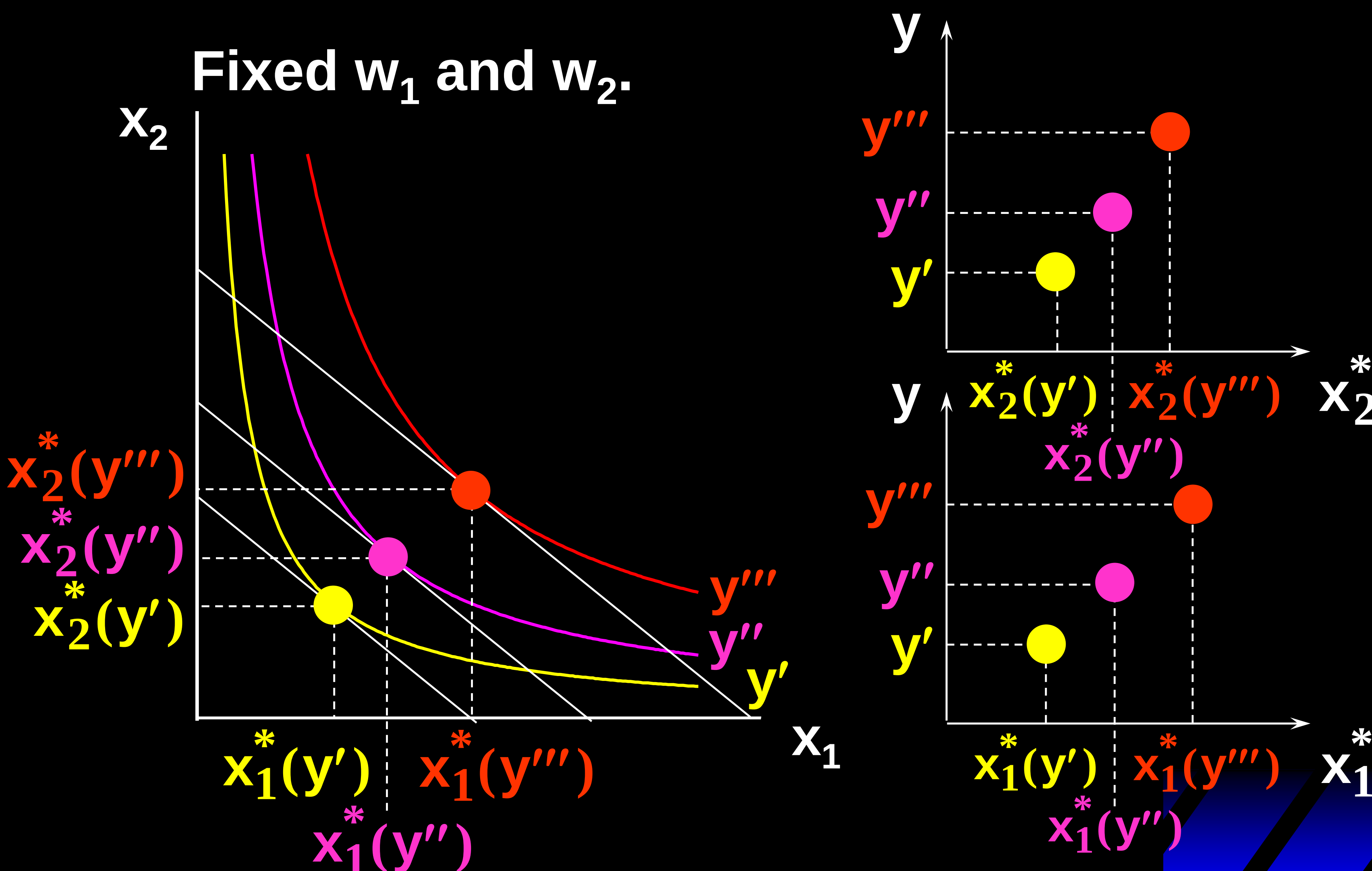

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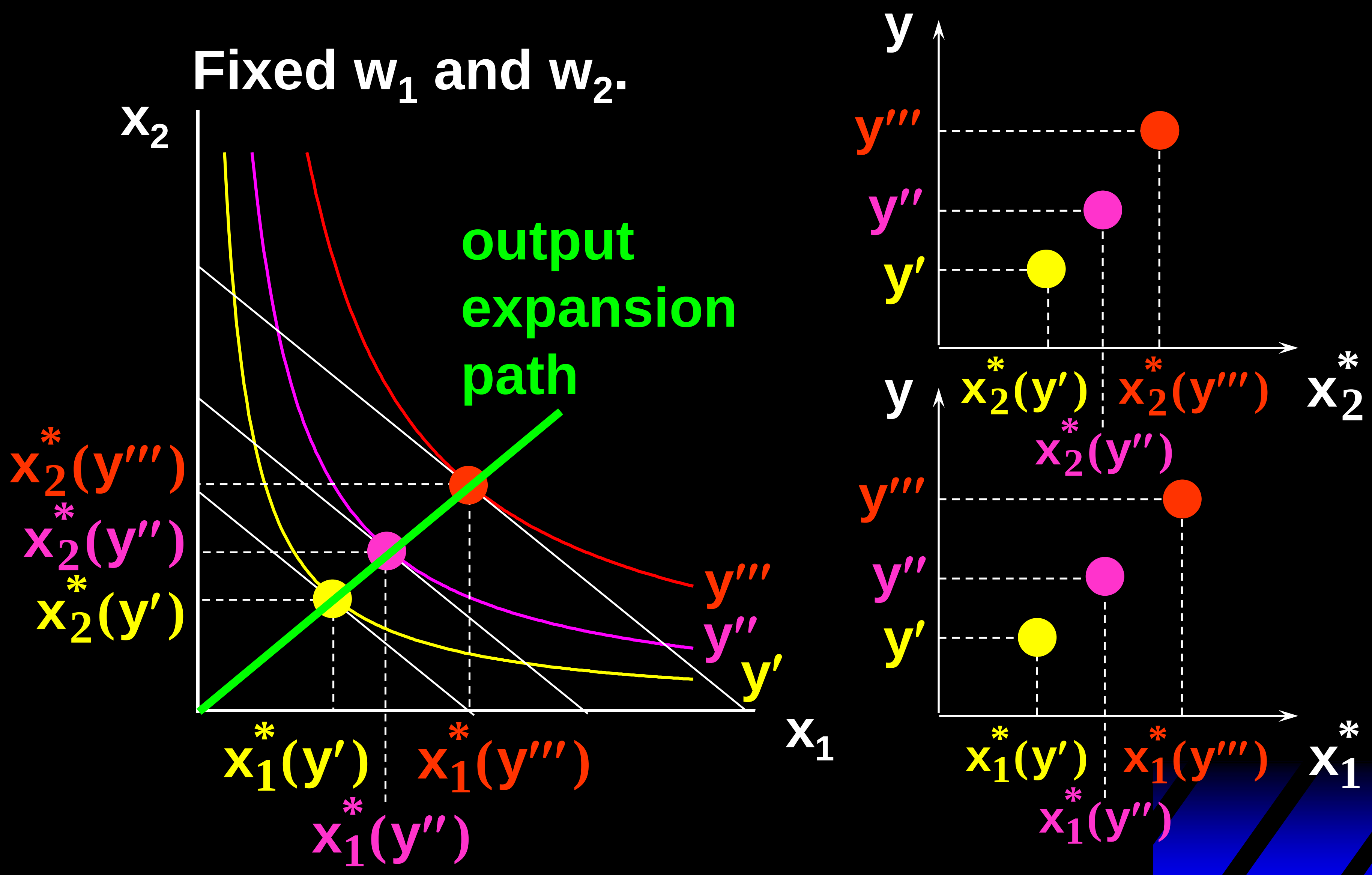

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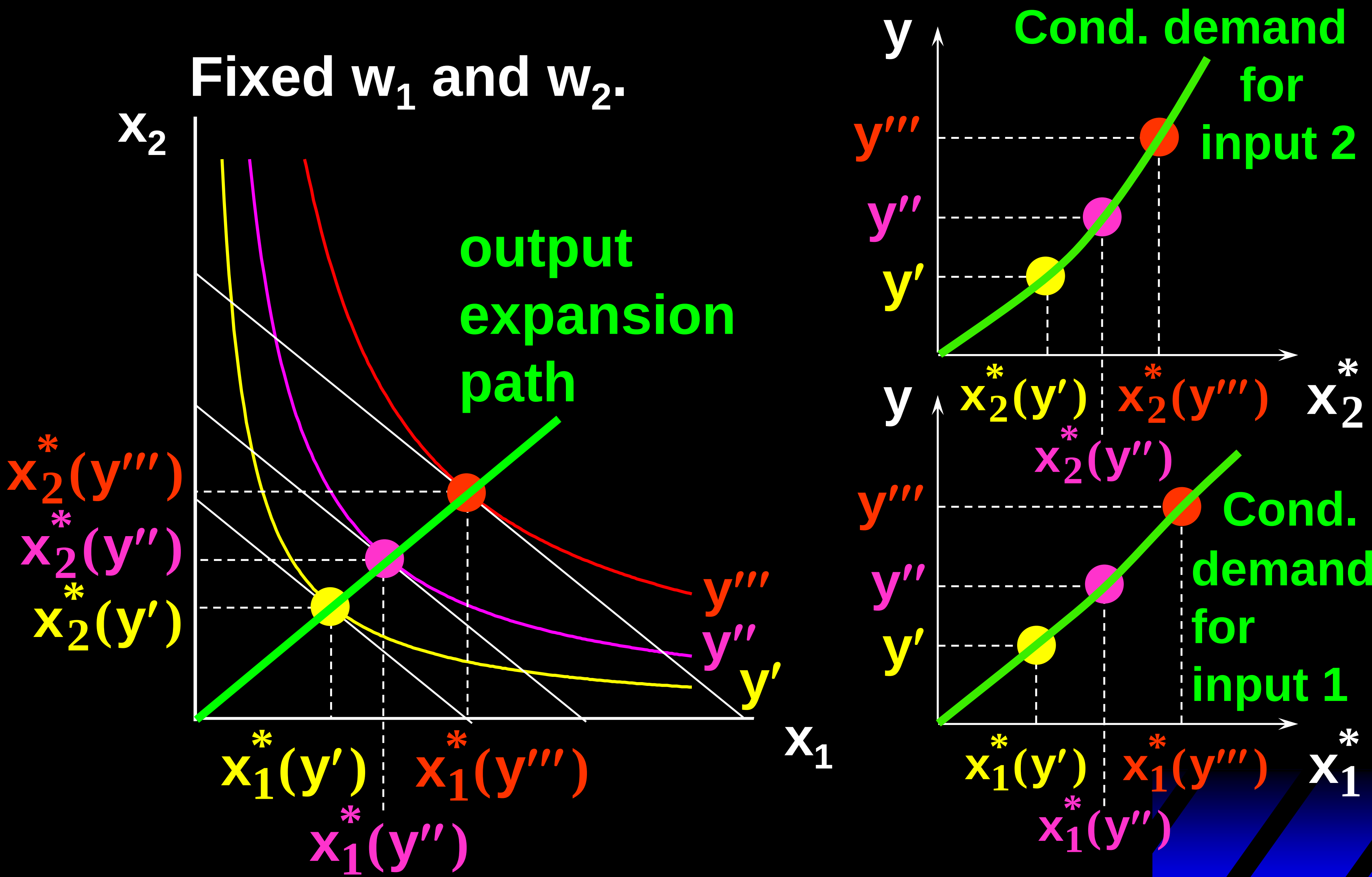

A Cobb-Douglas Example of Cost Minimization

For the production function

$$y = f(x_1, x_2) = x_1^{1/3} x_2^{2/3}$$

the cheapest input bundle yielding y output units is

$$\left(x_1^*(w_1, w_2, y), x_2^*(w_1, w_2, y)\right)$$

$$= \left(\left(\frac{w_2}{2w_1} \right)^{2/3} y, \left(\frac{2w_1}{w_2} \right)^{1/3} y \right).$$

A Cobb-Douglas Example of Cost Minimization

- **So the firm's total cost function is**

$$c(w_1, w_2, y) = w_1 x_1^*(w_1, w_2, y) + w_2 x_2^*(w_1, w_2, y)$$

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$$= w_1 \left(\frac{w_2}{2w_1}\right)^{2/3} y + w_2 \left(\frac{2w_1}{w_2}\right)^{1/3} y$$

$$= \left(\frac{1}{2}\right)^{2/3} w_1^{1/3} w_2^{2/3} y + 2^{1/3} w_1^{1/3} w_2^{2/3} y$$

A Cobb-Douglas Example of Cost Minimization

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$$= \left(\frac{1}{2}\right)^{2/3} w_1^{1/3}w_2^{2/3}y + 2^{1/3}w_1^{1/3}w_2^{2/3}y$$

$$= 3\left(\frac{w_1w_2^2}{4}\right)^{1/3} y.$$

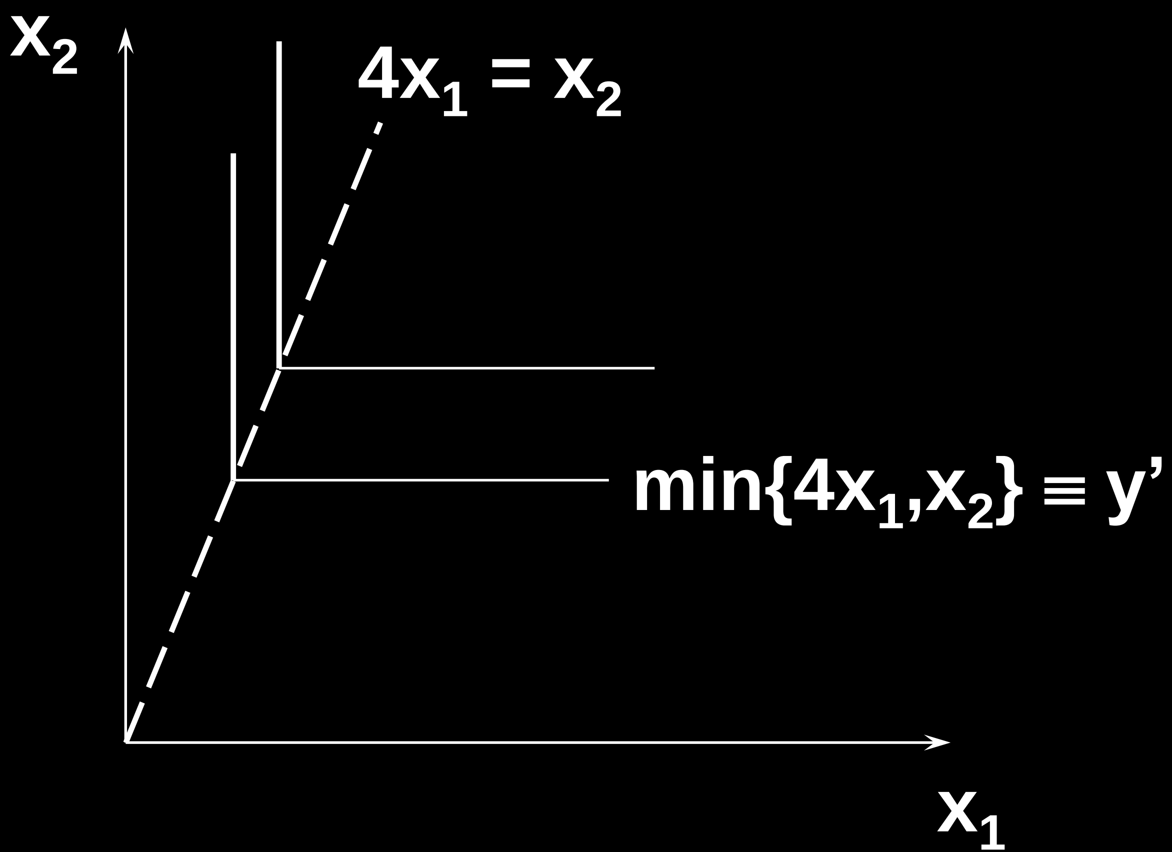
A Perfect Complements Example of Cost Minimization

- The firm's production function is

$$y = \min\{4x_1, x_2\}.$$

- Input prices w_1 and w_2 are given.
- What are the firm's conditional demands for inputs 1 and 2?
- What is the firm's total cost function?

A Perfect Complements Example of Cost Minimization



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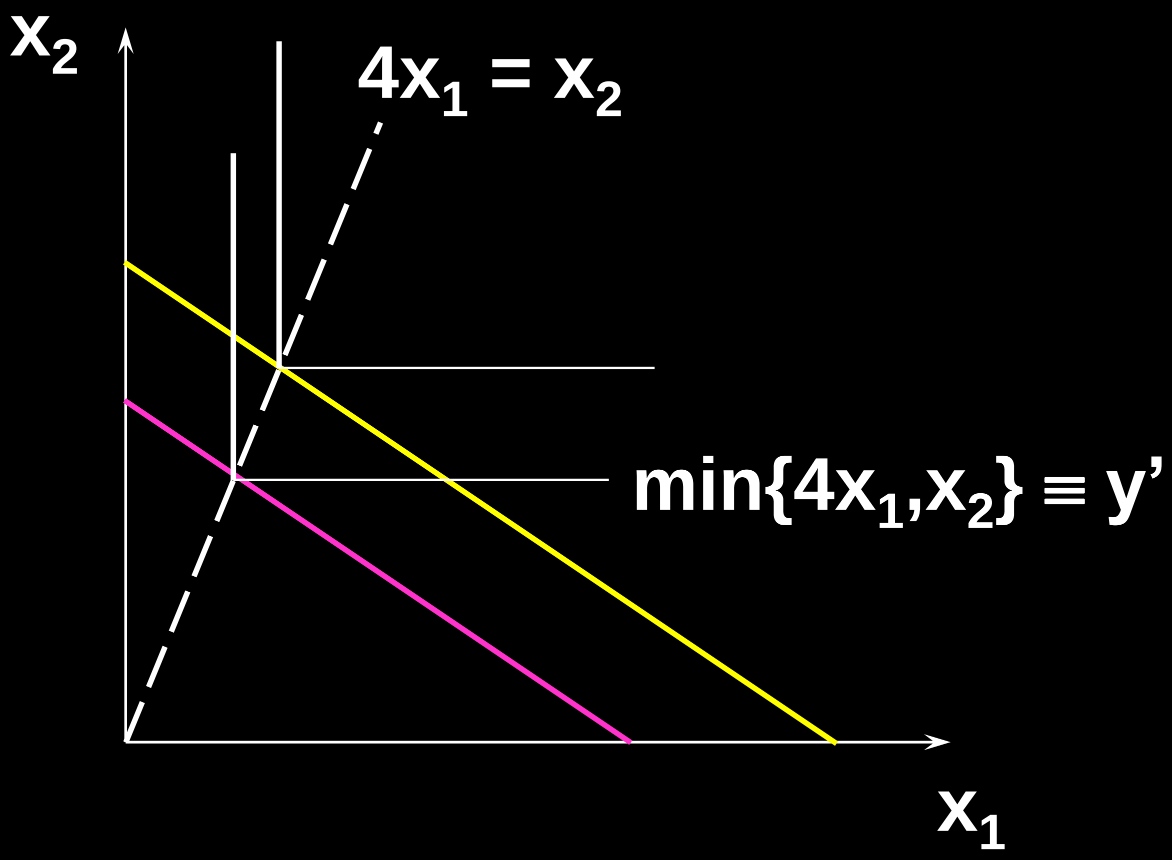

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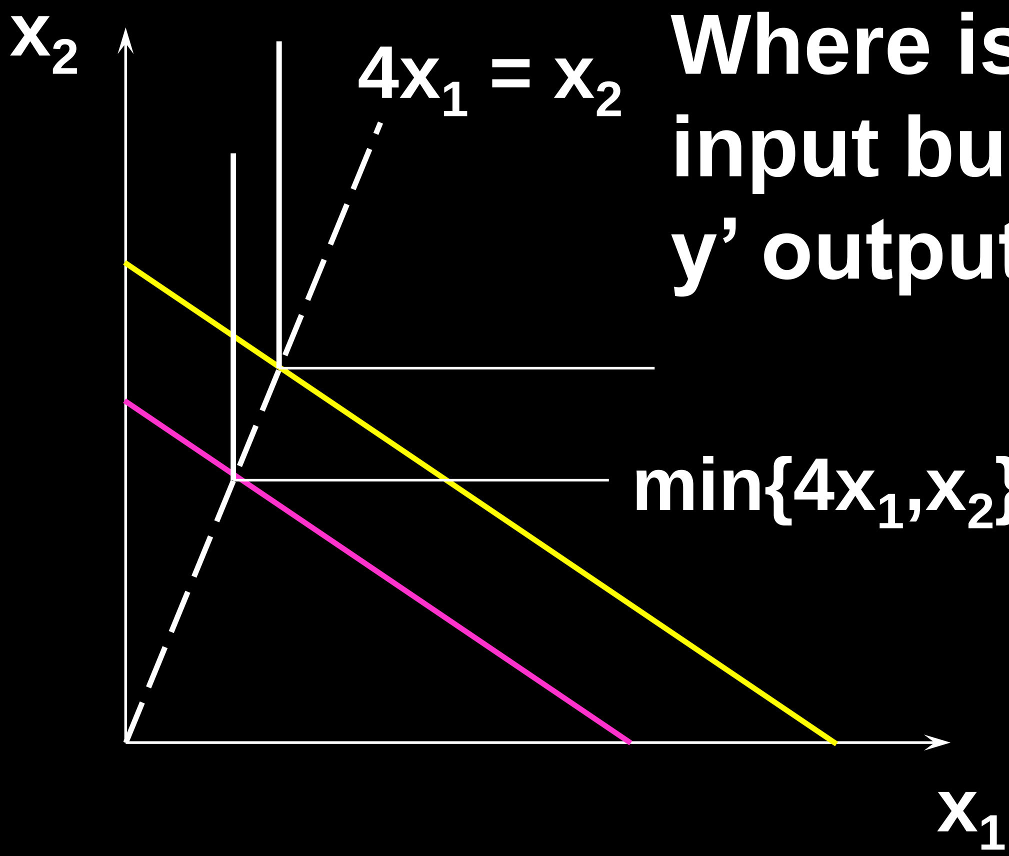
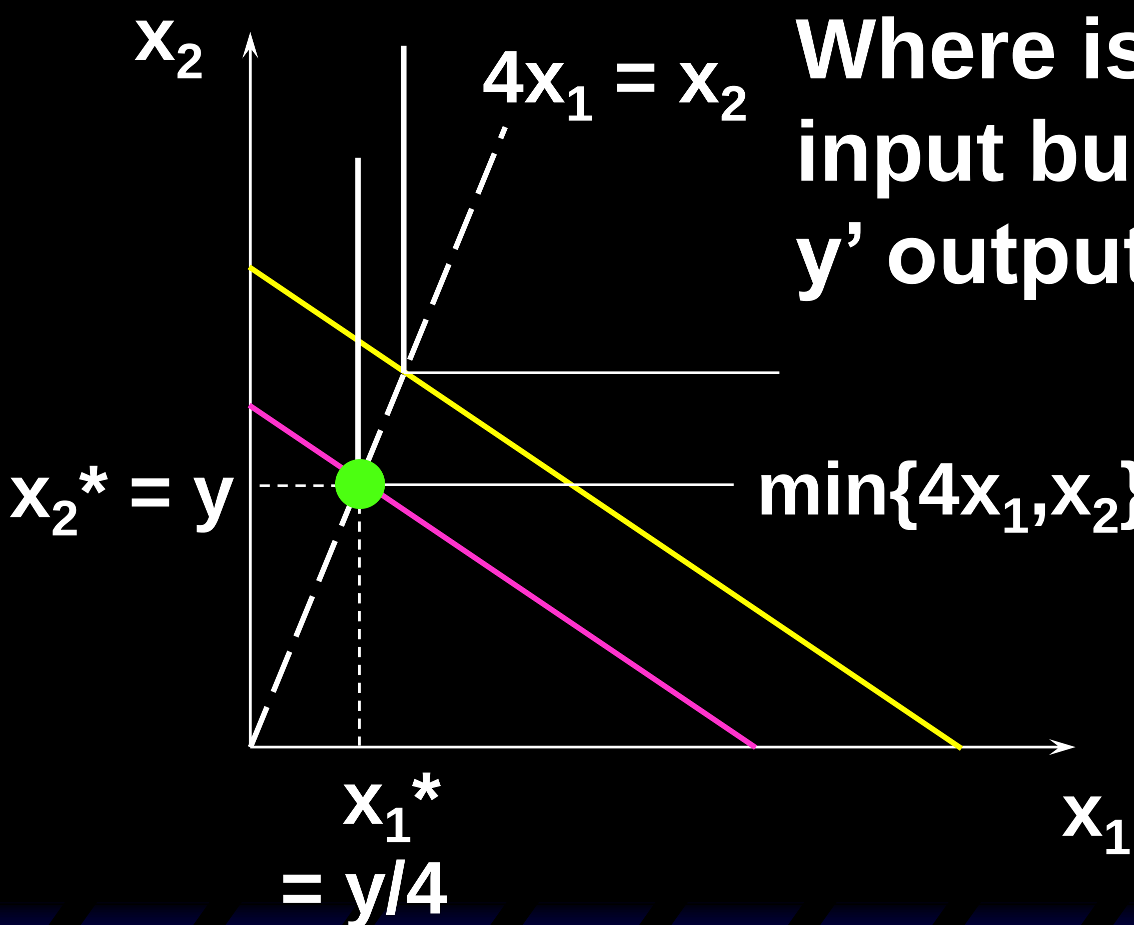


Where is the least costly input bundle yielding y’ output units?

A Perfect Complements Example of Cost Minimization



Where is the least costly input bundle yielding y' output units?

A Perfect Complements Example of Cost Minimization

- The firm's production function is
$$y = \min\{4x_1, x_2\}$$
and the conditional input demands are
$$x_1^*(w_1, w_2, y) = \frac{y}{4} \quad \text{and} \quad x_2^*(w_1, w_2, y) = y.$$

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The firm's production function is

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So the firm's total cost function is

$$c(w_1, w_2, y) = w_1 x_1^*(w_1, w_2, y) + w_2 x_2^*(w_1, w_2, y)$$

A Perfect Complements Example of Cost Minimization

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$$\begin{aligned} c(w_1, w_2, y) &= w_1 x_1^*(w_1, w_2, y) \\ &\quad + w_2 x_2^*(w_1, w_2, y) \\ &= w_1 \frac{y}{4} + w_2 y = \left(\frac{w_1}{4} + w_2\right) y. \end{aligned}$$

Average Total Production Costs

- For positive output levels y, a firm's average total cost of producing y units is

$$AC(w_1, w_2, y) = \frac{c(w_1, w_2, y)}{y}.$$

Returns-to-Scale and Av. Total Costs

- The returns-to-scale properties of a firm's technology determine how average production costs change with output level.
- Our firm is presently producing y' output units.
- How does the firm's average production cost change if it instead produces 2y' units of output?

Constant Returns-to-Scale and Average Total Costs

- **If a firm's technology exhibits constant returns-to-scale then doubling its output level from y' to 2y' requires doubling all input levels.**

Constant Returns-to-Scale and Average Total Costs

- If a firm’s technology exhibits constant returns-to-scale then doubling its output level from y’ to 2y’ requires doubling all input levels.
- Total production cost doubles.

Constant Returns-to-Scale and Average Total Costs

- **If a firm's technology exhibits constant returns-to-scale then doubling its output level from y' to 2y' requires doubling all input levels.**
- **Total production cost doubles.**
- **Average production cost does not change.**

Decreasing Returns-to-Scale and Average Total Costs

- If a firm's technology exhibits decreasing returns-to-scale then doubling its output level from y' to 2y' requires more than doubling all input levels.

Decreasing Returns-to-Scale and Average Total Costs

- If a firm's technology exhibits decreasing returns-to-scale then doubling its output level from y' to 2y' requires more than doubling all input levels.
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Decreasing Returns-to-Scale and Average Total Costs

- If a firm’s technology exhibits decreasing returns-to-scale then doubling its output level from y’ to 2y’ requires more than doubling all input levels.
- Total production cost more than doubles.
- Average production cost increases.

Increasing Returns-to-Scale and Average Total Costs

- **If a firm's technology exhibits increasing returns-to-scale then doubling its output level from y' to 2y' requires less than doubling all input levels.**

Increasing Returns-to-Scale and Average Total Costs

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Increasing Returns-to-Scale and Average Total Costs

- If a firm's technology exhibits increasing returns-to-scale then doubling its output level from y' to 2y' requires less than doubling all input levels.
- Total production cost less than doubles.
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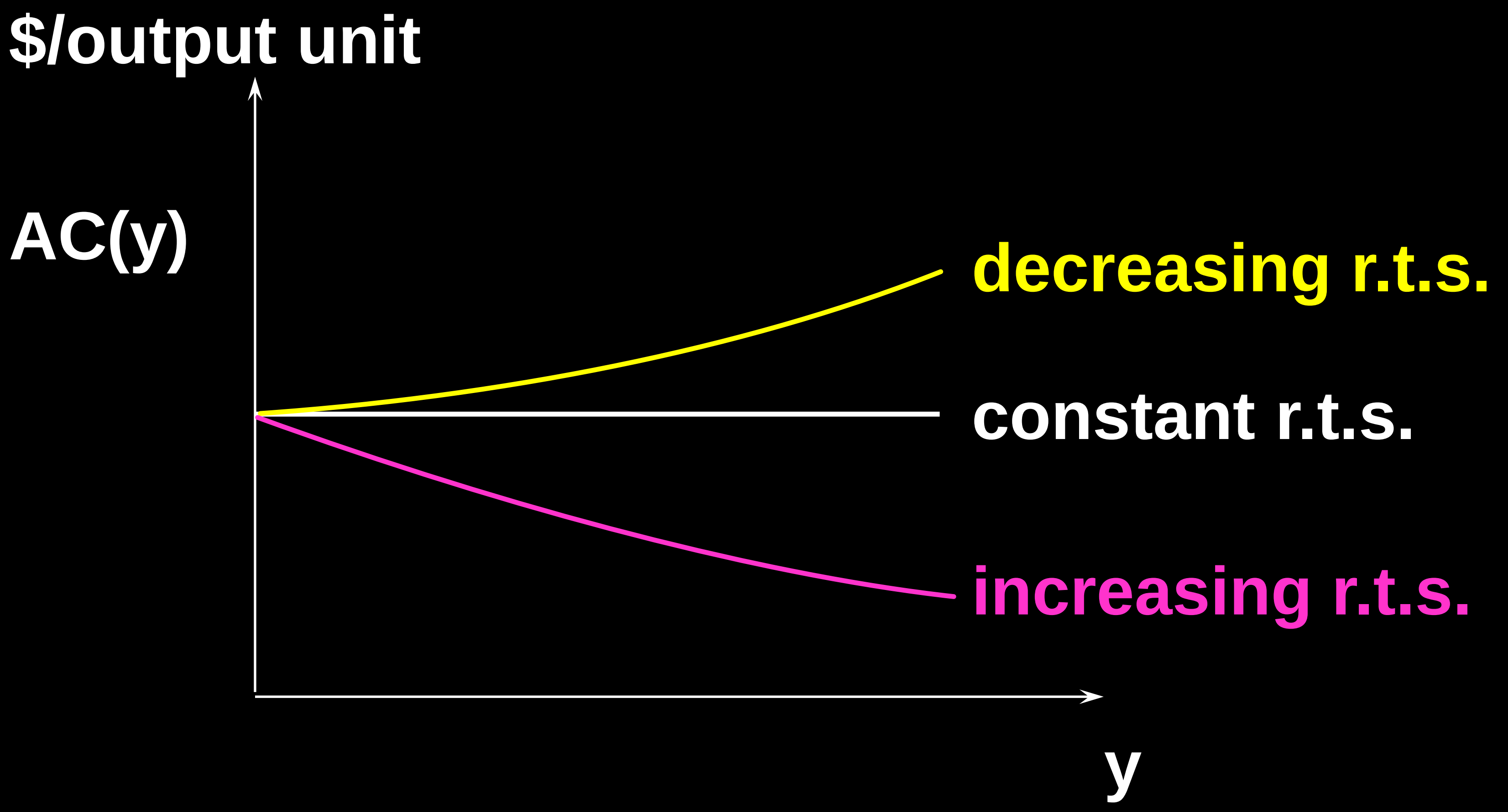
Returns-to-Scale and Av. Total Costs

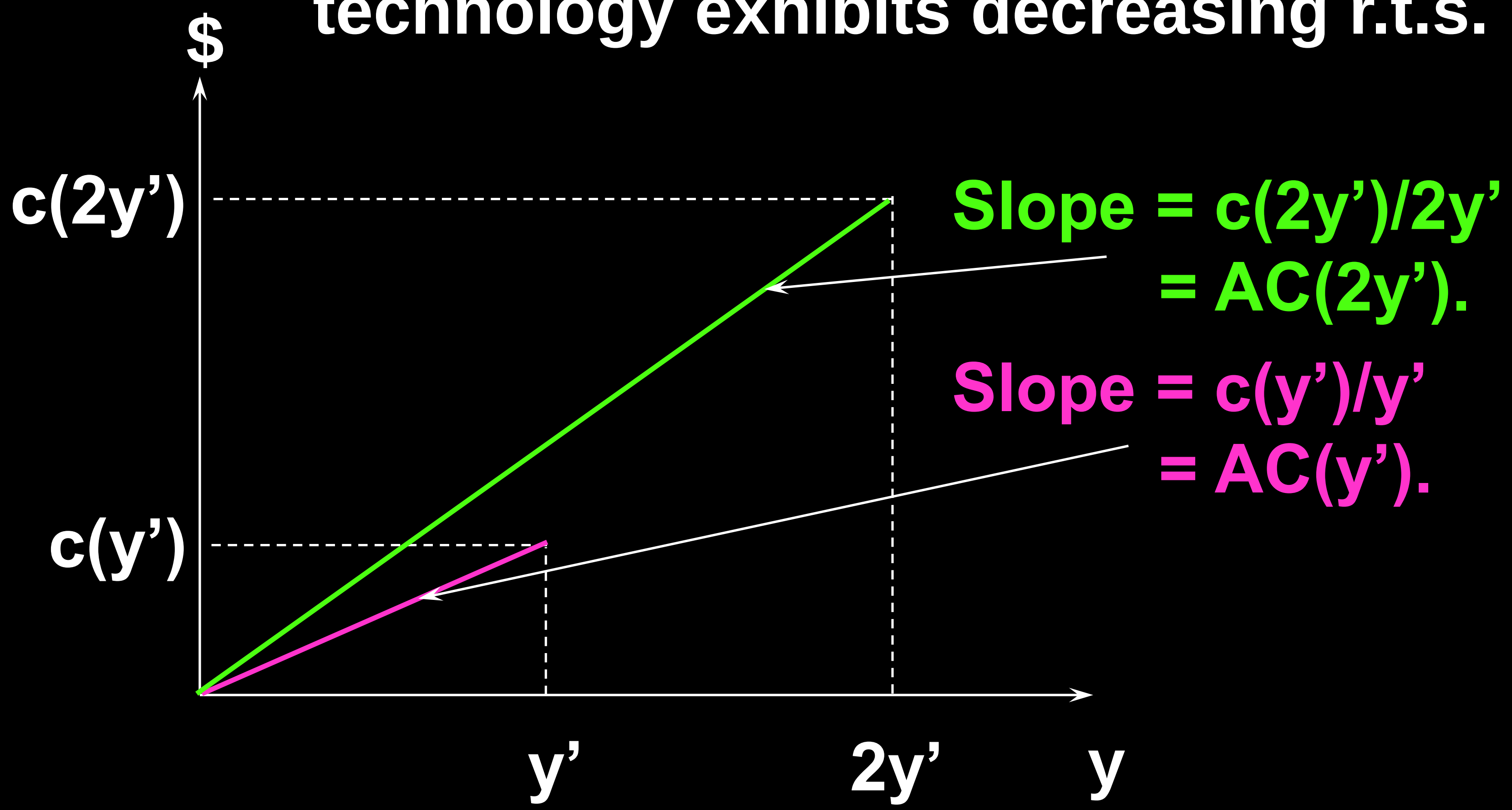


Returns-to-Scale and Total Costs

- **What does this imply for the shapes of total cost functions?**

Returns-to-Scale and Total Costs

Av. cost increases with y if the firm's technology exhibits decreasing r.t.s.



Returns-to-Scale and Total Costs

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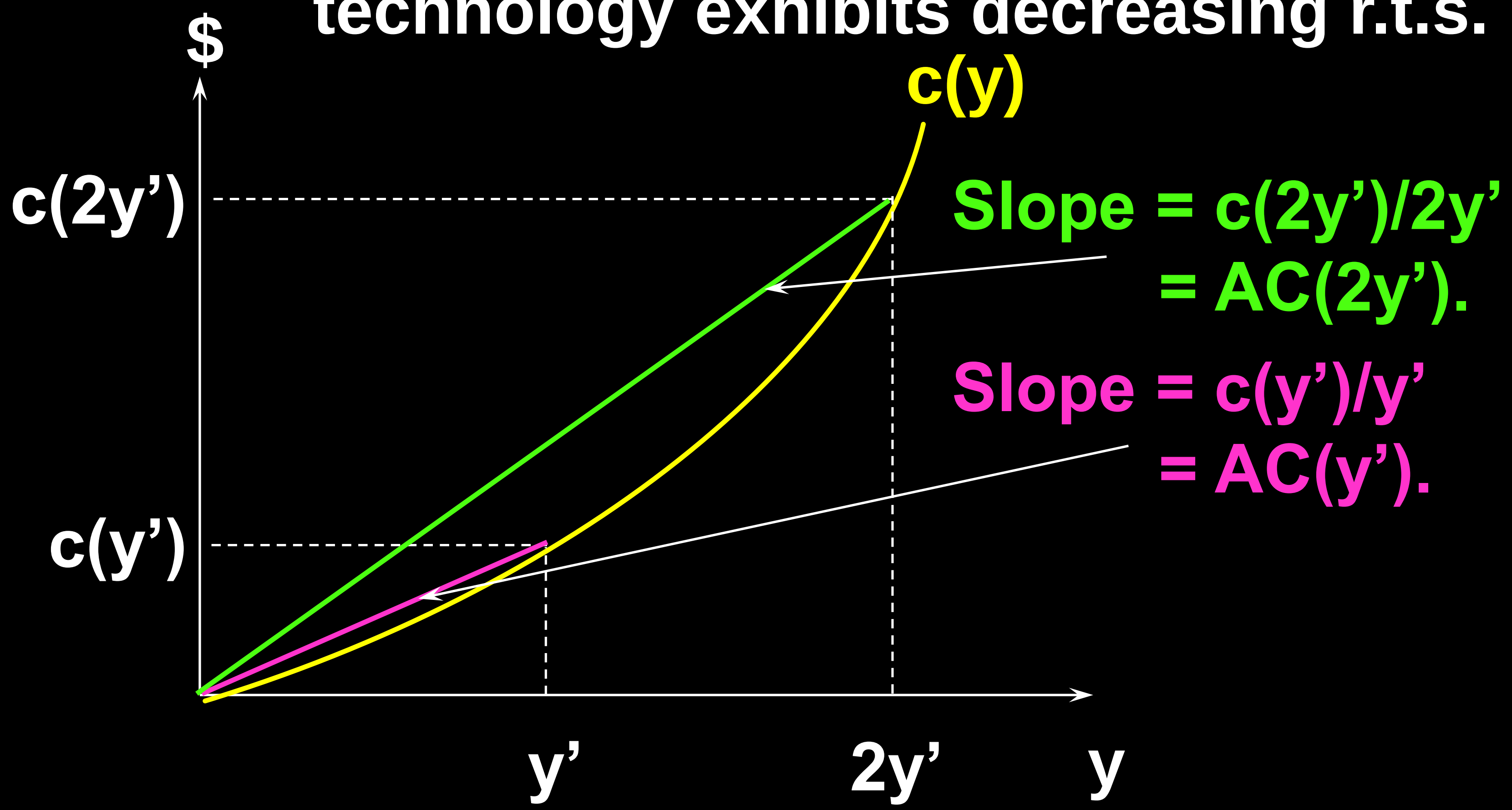

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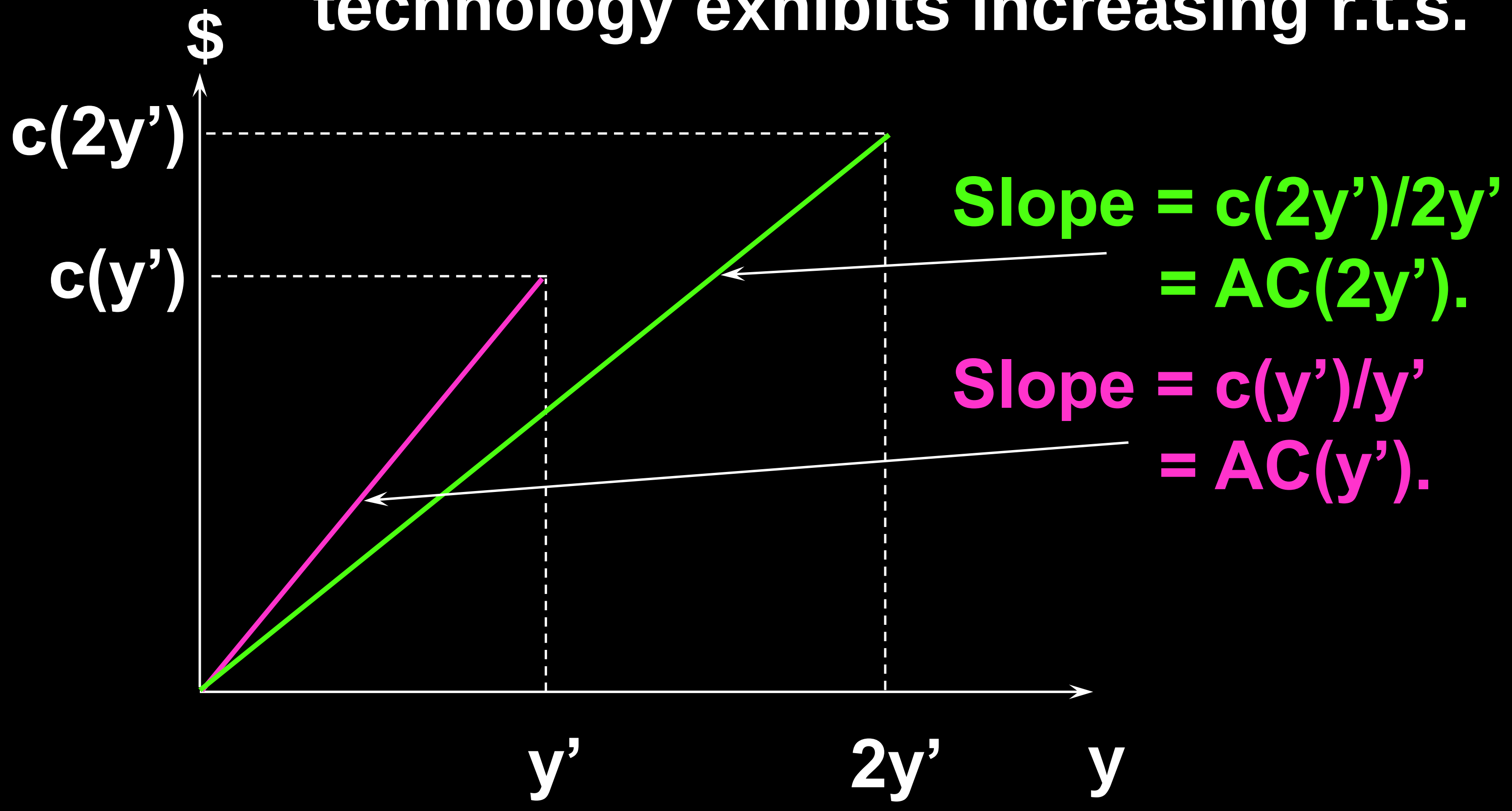

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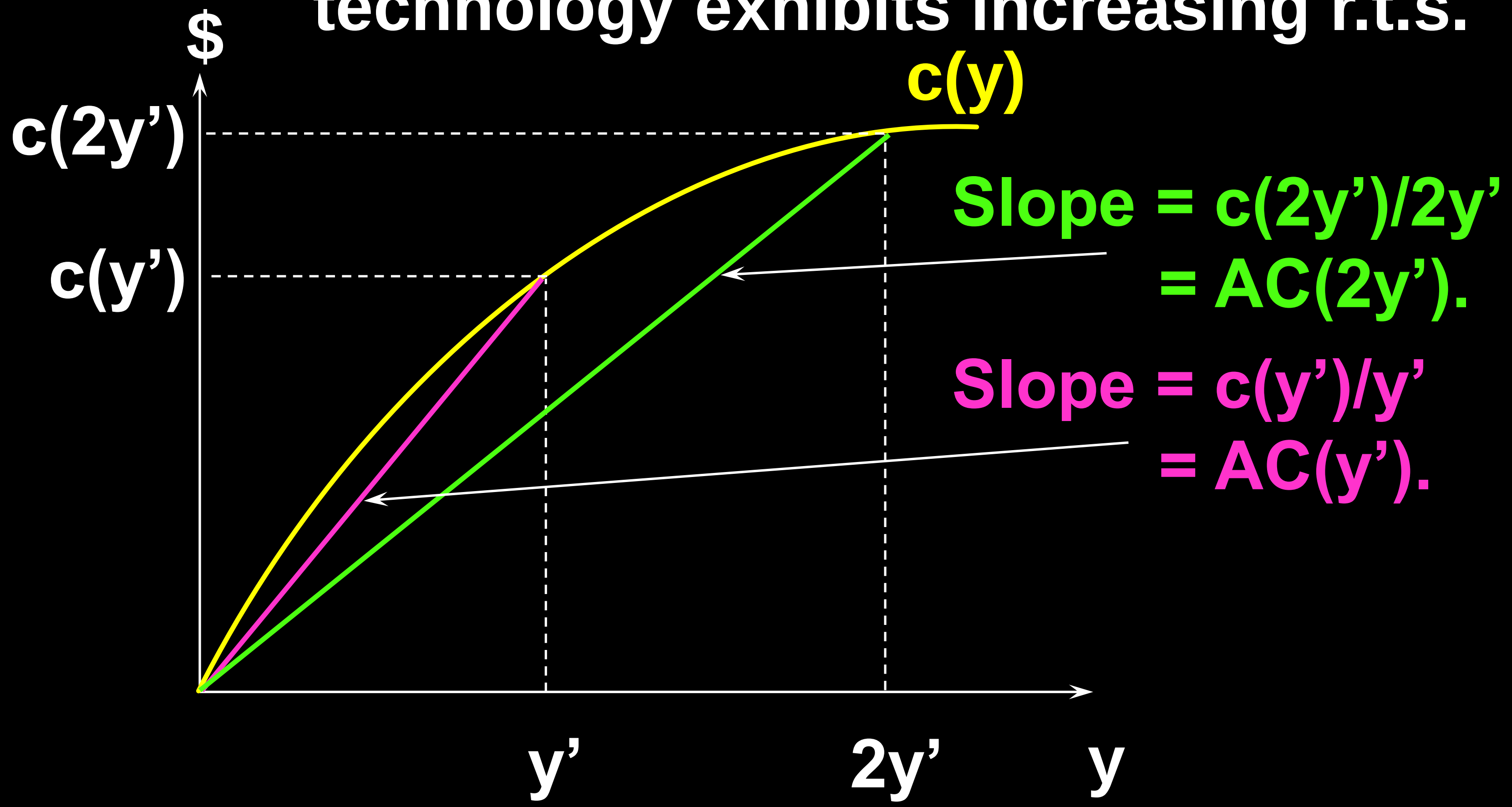
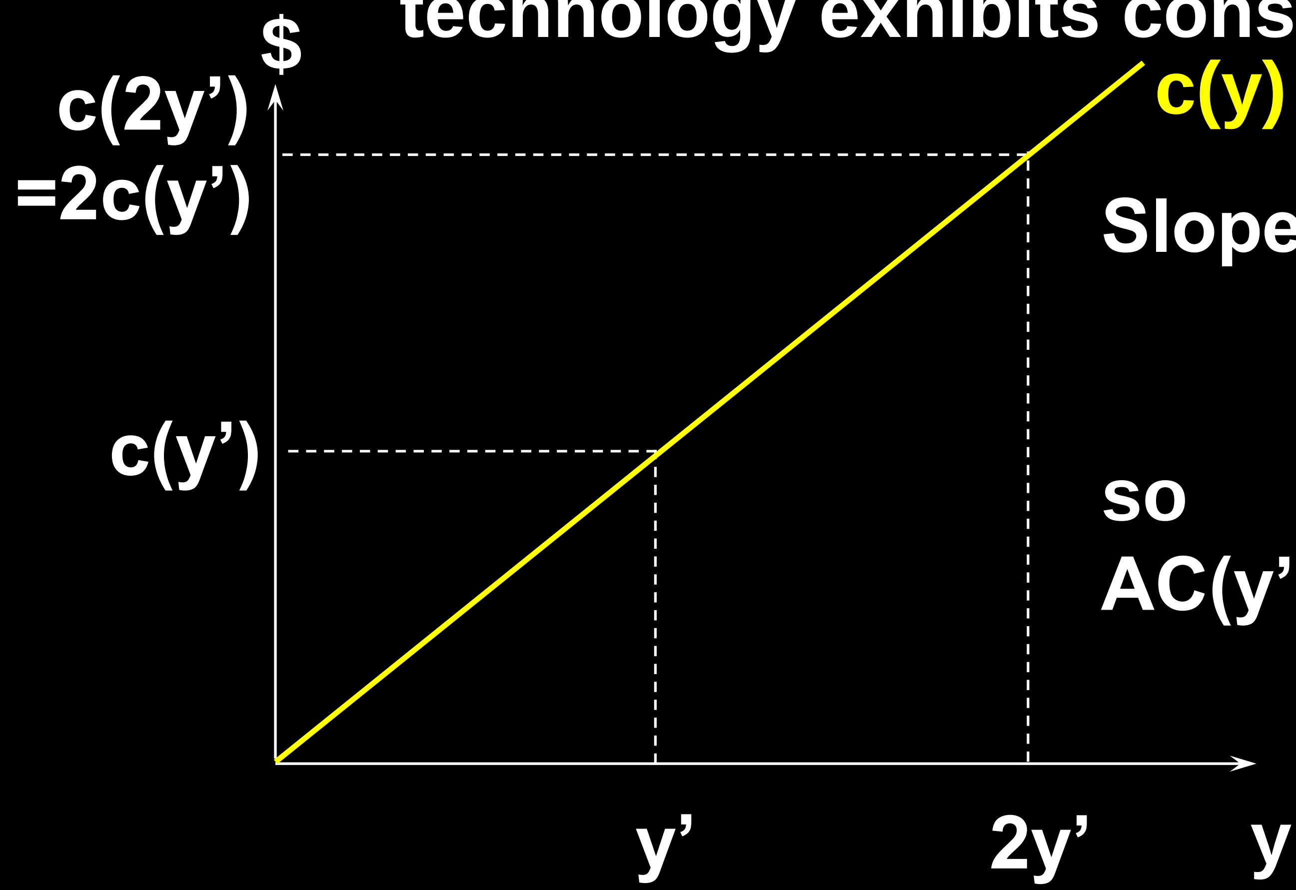


Returns-to-Scale and Total Costs

Av. cost is constant when the firm's technology exhibits constant r.t.s.



Slope = c(2y')/2y'
= 2c(y')/2y'
= c(y')/y'
so
AC(y') = AC(2y').

Short-Run & Long-Run Total Costs

- In the long-run a firm can vary all of its input levels.
- Consider a firm that cannot change its input 2 level from x_2' units.
- How does the short-run total cost of producing y output units compare to the long-run total cost of producing y units of output?

Short-Run & Long-Run Total Costs

- The long-run cost-minimization problem is $\min_{x_1,x_2 \geq 0} \; w_1x_1 + w_2x_2$
  subject to $f(x_1,x_2) = y.$
- The short-run cost-minimization problem is $\min_{x_1 \geq 0} w_1x_1 + w_2x_2'$
  subject to $f(x_1,x_2') = y.$

Short-Run & Long-Run Total Costs

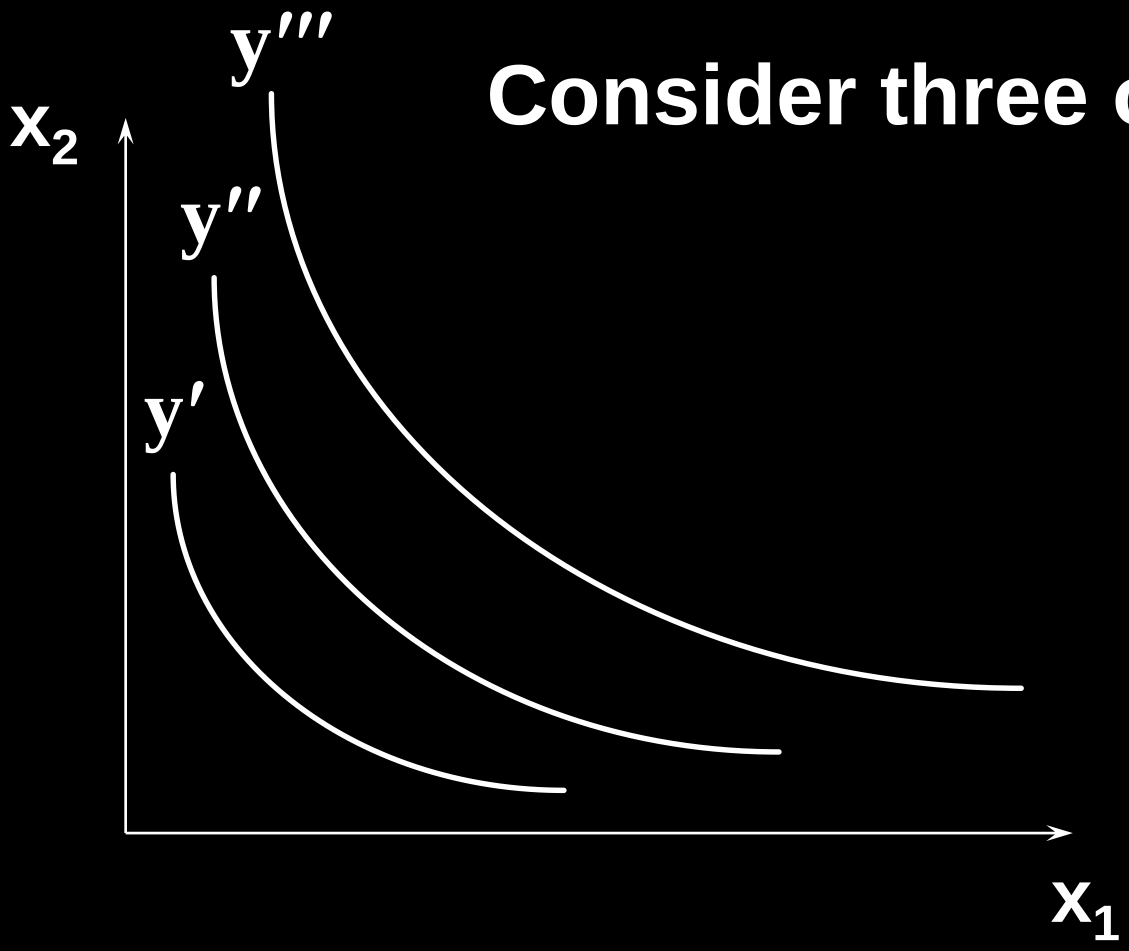
- **The short-run cost-min. problem is the long-run problem subject to the extra constraint that $x_2 = x_2'$.**
- **If the long-run choice for x_2 was x_2' then the extra constraint $x_2 = x_2'$ is not really a constraint at all and so the long-run and short-run total costs of producing y output units are the same.**

Short-Run & Long-Run Total Costs

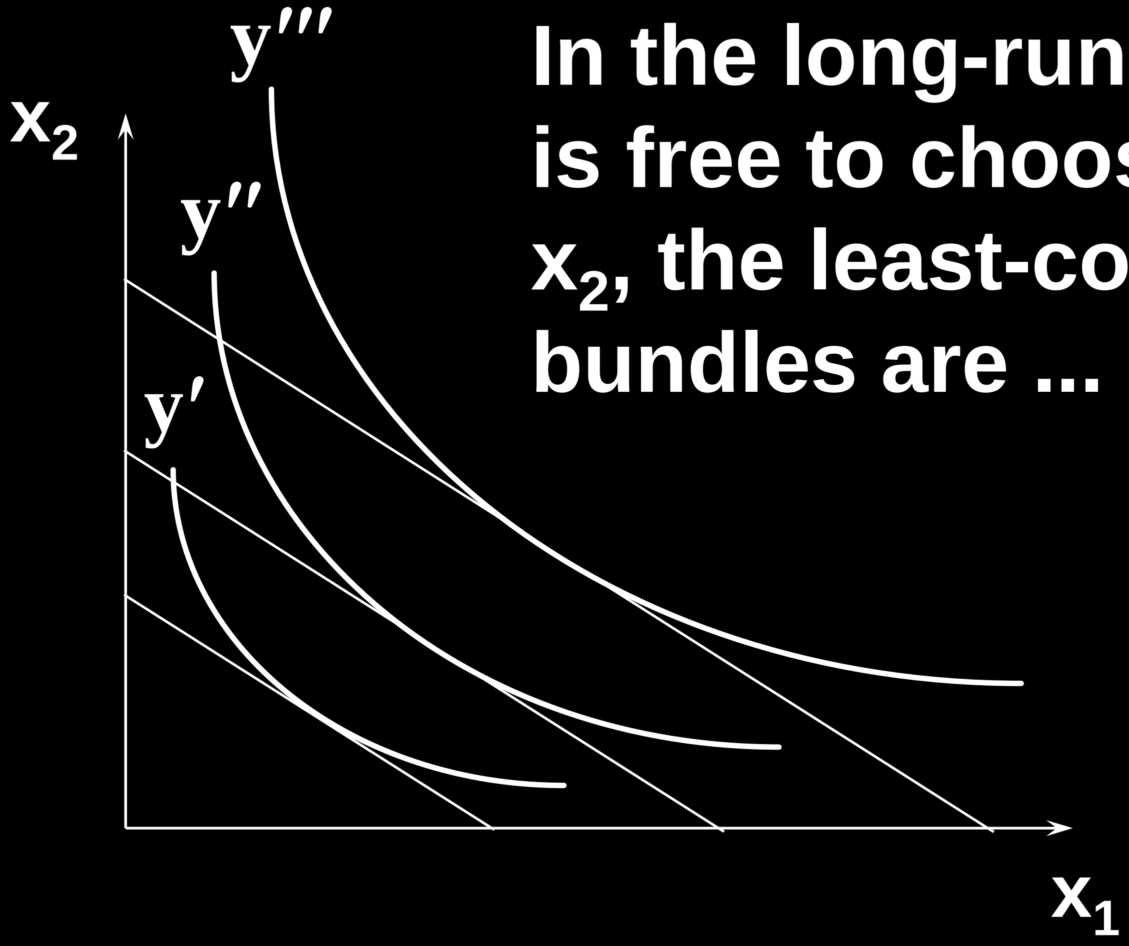
- The short-run cost-min. problem is therefore the long-run problem subject to the extra constraint that $x_2 = x_2''$.
- But, if the long-run choice for $x_2 \neq x_2''$ then the extra constraint $x_2 = x_2''$ prevents the firm in this short-run from achieving its long-run production cost, causing the short-run total cost to exceed the long-run total cost of producing y output units.

Short-Run & Long-Run Total Costs

Consider three output levels.

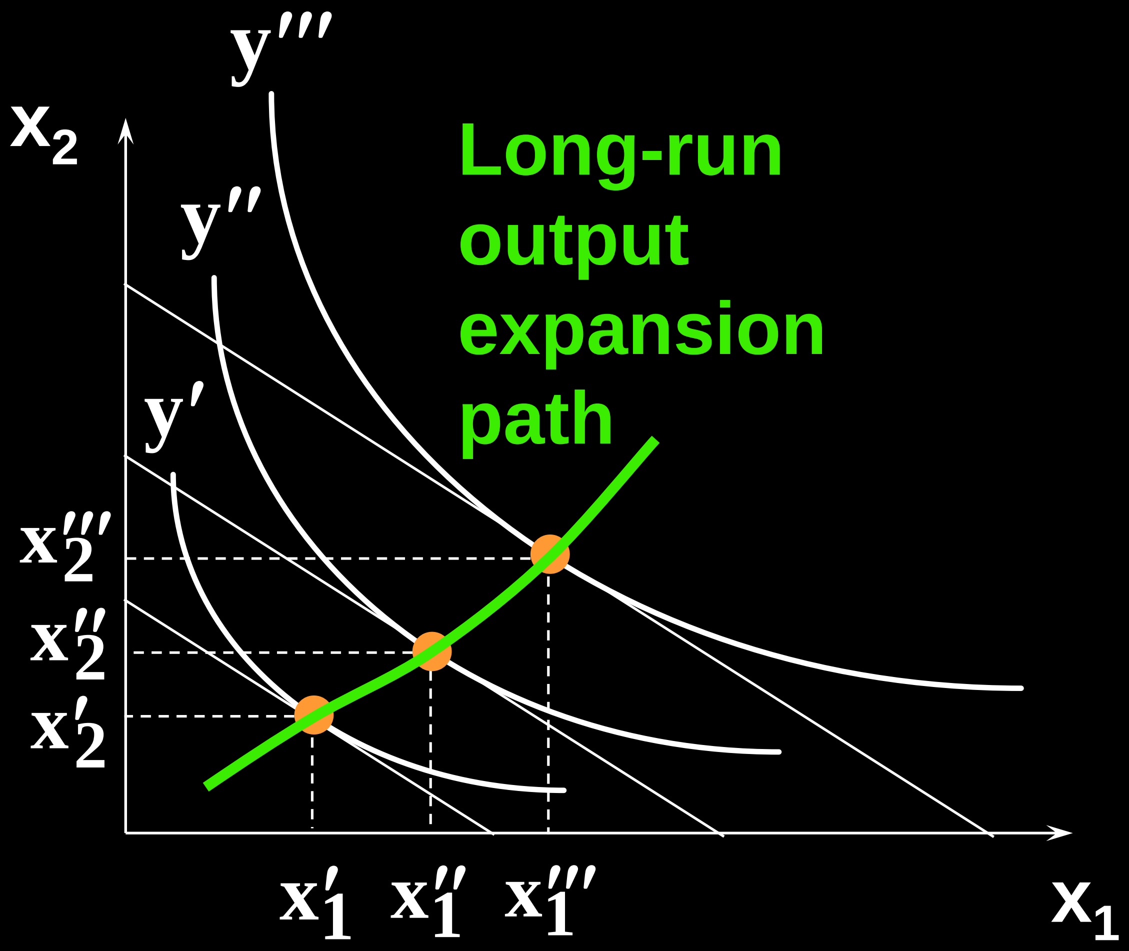


Short-Run & Long-Run Total Costs

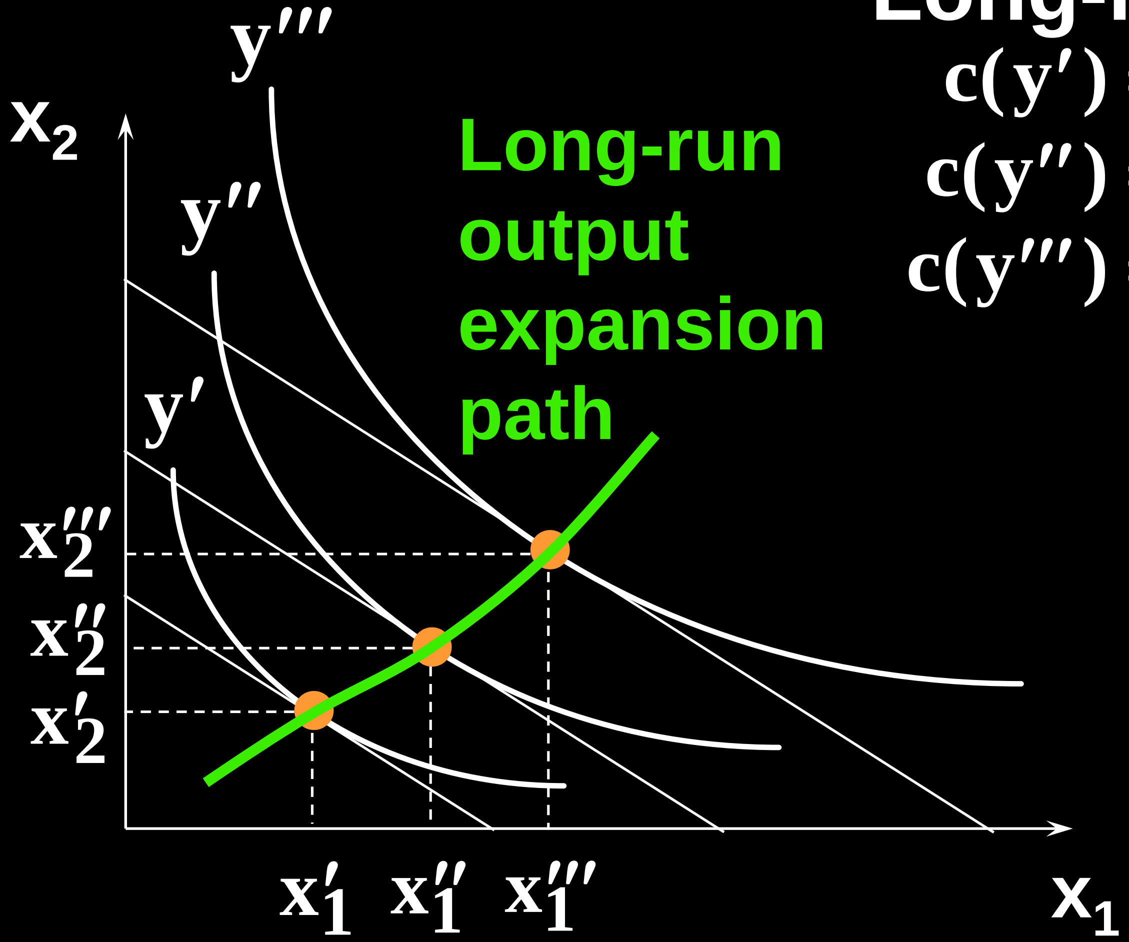


In the long-run when the firm is free to choose both x_1 and x_2, the least-costly input bundles are ...

Short-Run & Long-Run Total Costs



Short-Run & Long-Run Total Costs



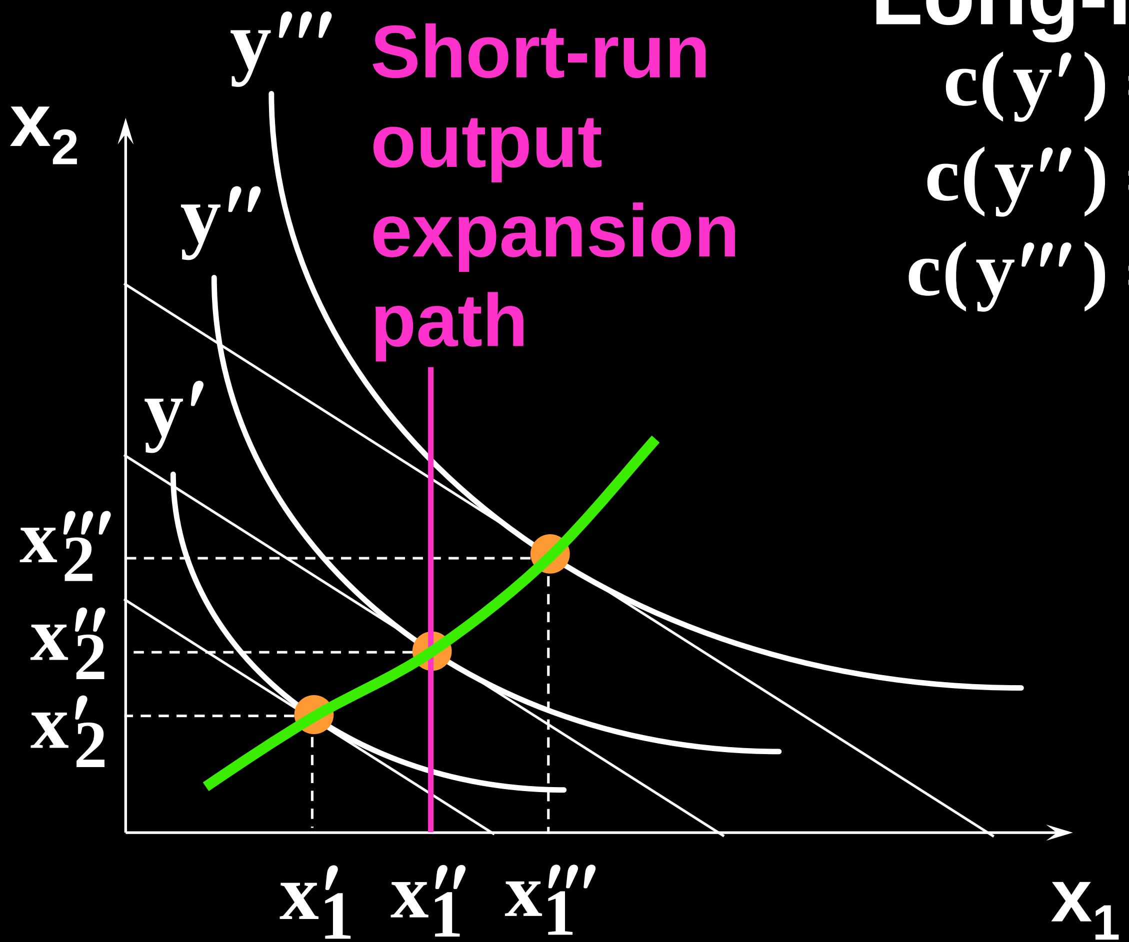
Long-run costs are:

$$c(y') = w_1x_1' + w_2x_2'$$
$$c(y'') = w_1x_1'' + w_2x_2''$$
$$c(y''') = w_1x_1''' + w_2x_2'''$$

Short-Run & Long-Run Total Costs

- **Now suppose the firm becomes subject to the short-run constraint that $x_2 = x_2$".**

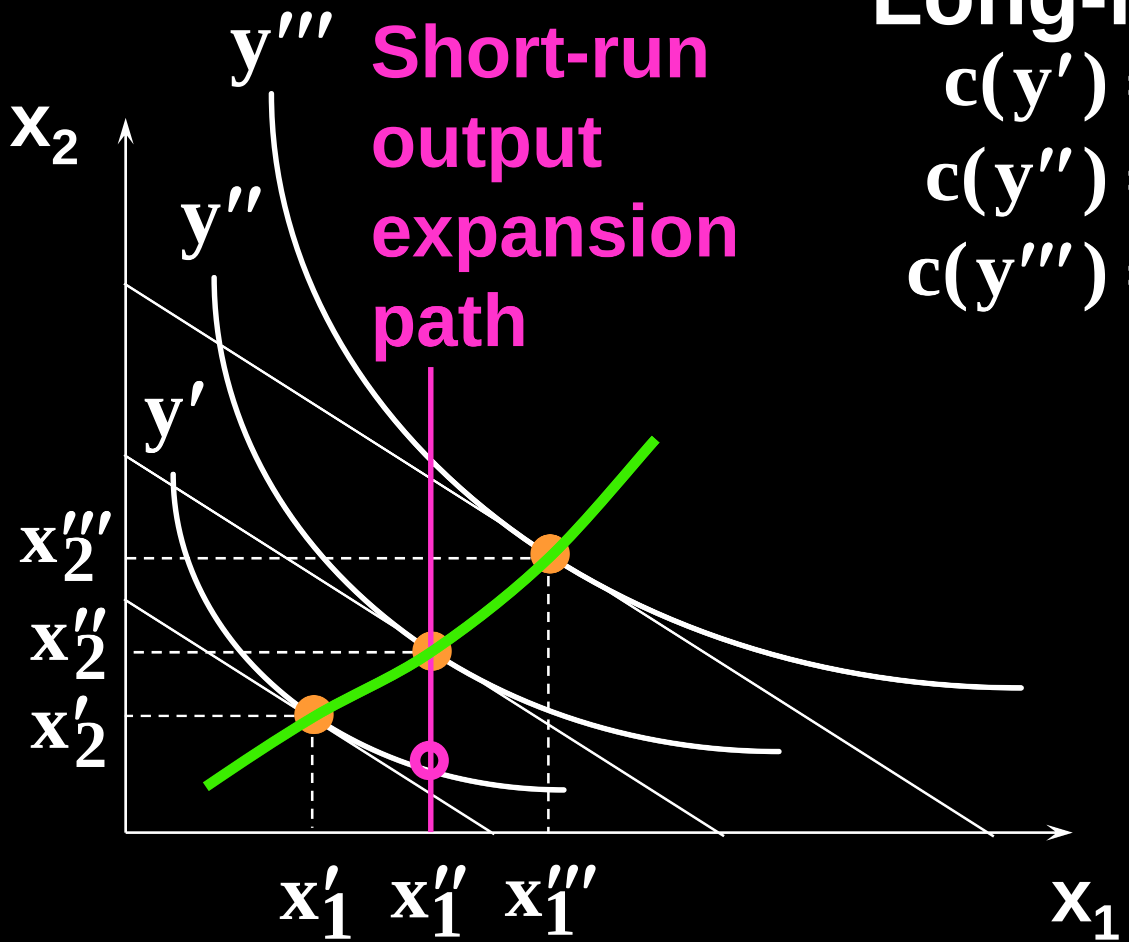
Short-Run & Long-Run Total Costs



Long-run costs are:

$$c(y') = w_1 x_1' + w_2 x_2'$$
$$c(y'') = w_1 x_1'' + w_2 x_2''$$
$$c(y''') = w_1 x_1''' + w_2 x_2'''$$

Short-Run & Long-Run Total Costs



Long-run costs are:

$$c(y') = w_1x_1' + w_2x_2'$$
$$c(y'') = w_1x_1'' + w_2x_2''$$
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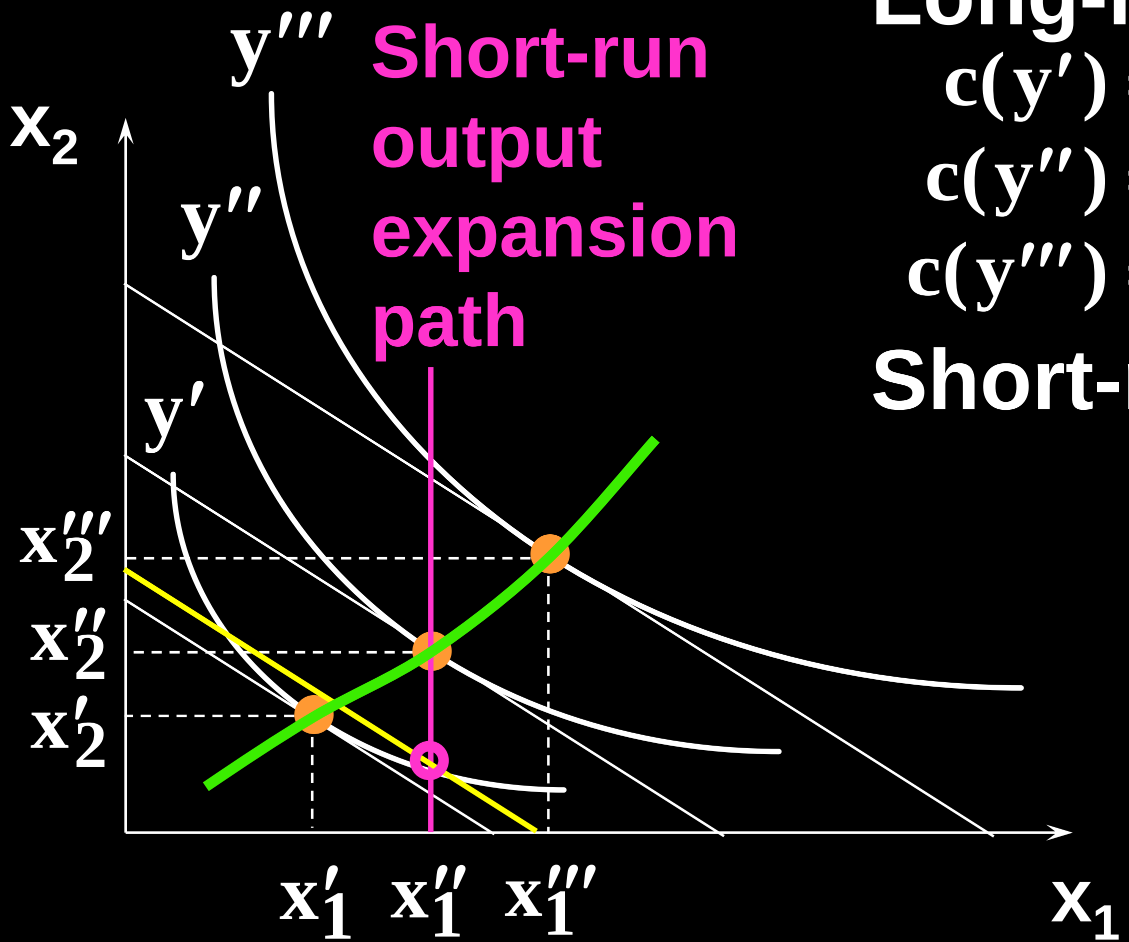
Short-Run & Long-Run Total Costs



Long-run costs are:

$$c(y') = w_1 x_1' + w_2 x_2'$$

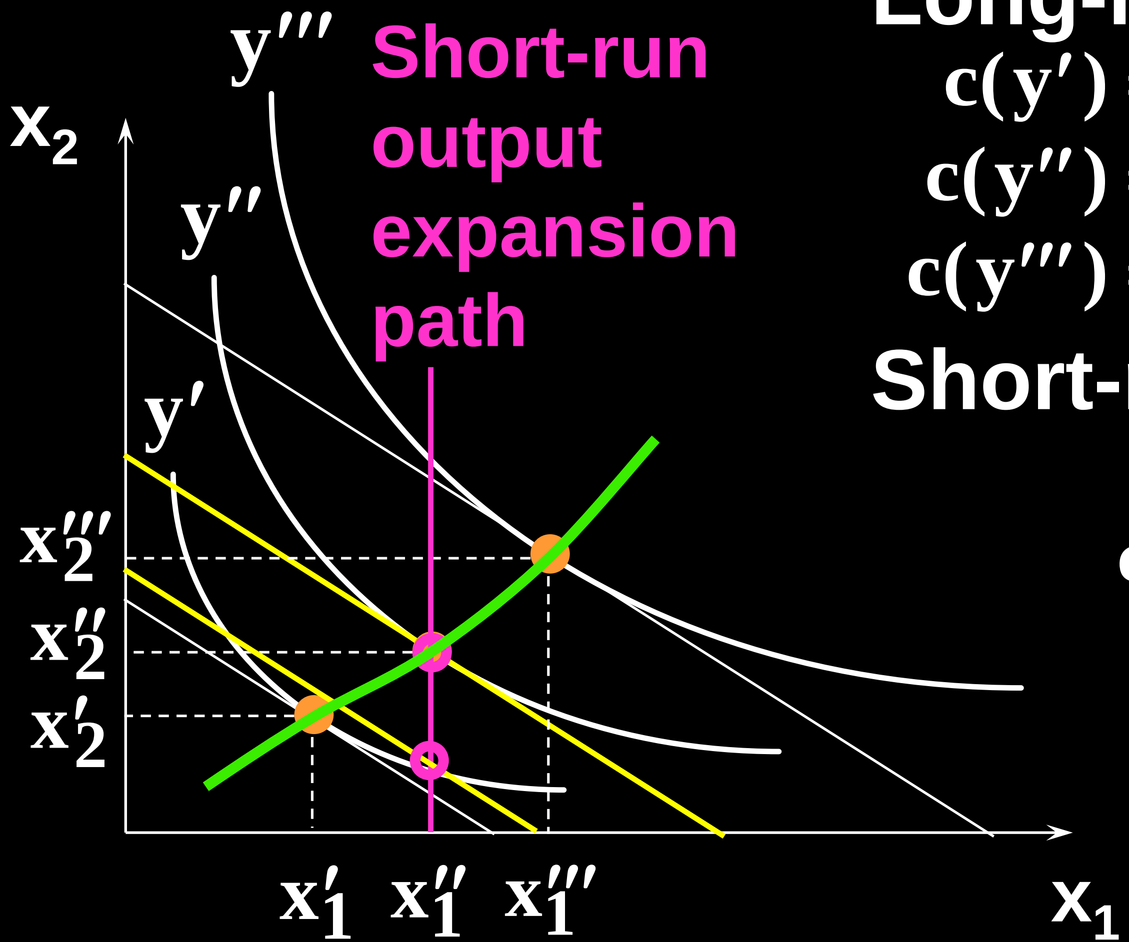
$$c(y'') = w_1 x_1'' + w_2 x_2'''$$

$$c(y''') = w_1 x_1''' + w_2 x_2'''$$

Short-run costs are:

$$c_s(y') > c(y')$$

Short-Run & Long-Run Total Costs



Long-run costs are:

$$c(y') = w_1x_1' + w_2x_2'$$

$$c(y'') = w_1x_1'' + w_2x_2''$$

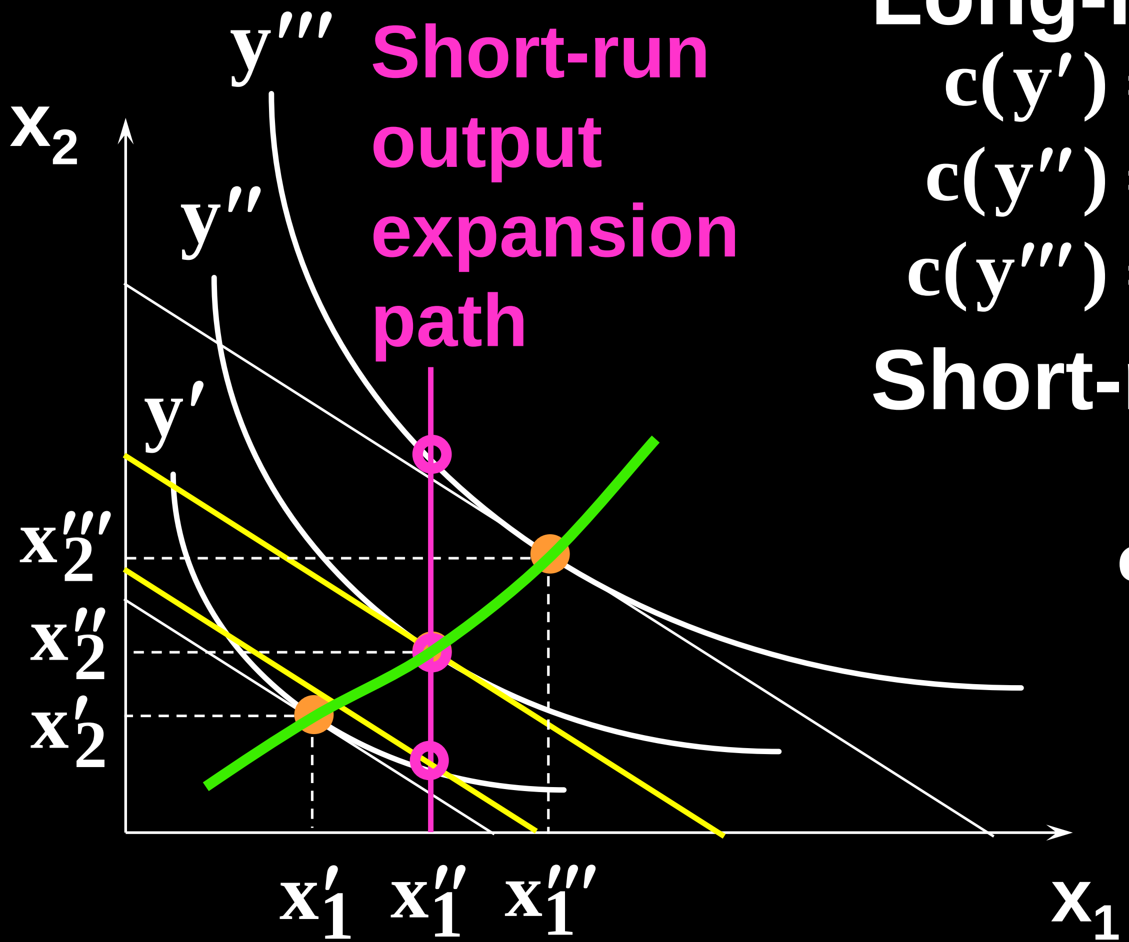
$$c(y''') = w_1x_1''' + w_2x_2'''$$

Short-run costs are:

$$c_s(y') > c(y')$$

$$c_s(y'') = c(y'')$$

Short-Run & Long-Run Total Costs



Long-run costs are:

$$c(y') = w_1 x_1' + w_2 x_2'$$

$$c(y'') = w_1 x_1'' + w_2 x_2''$$

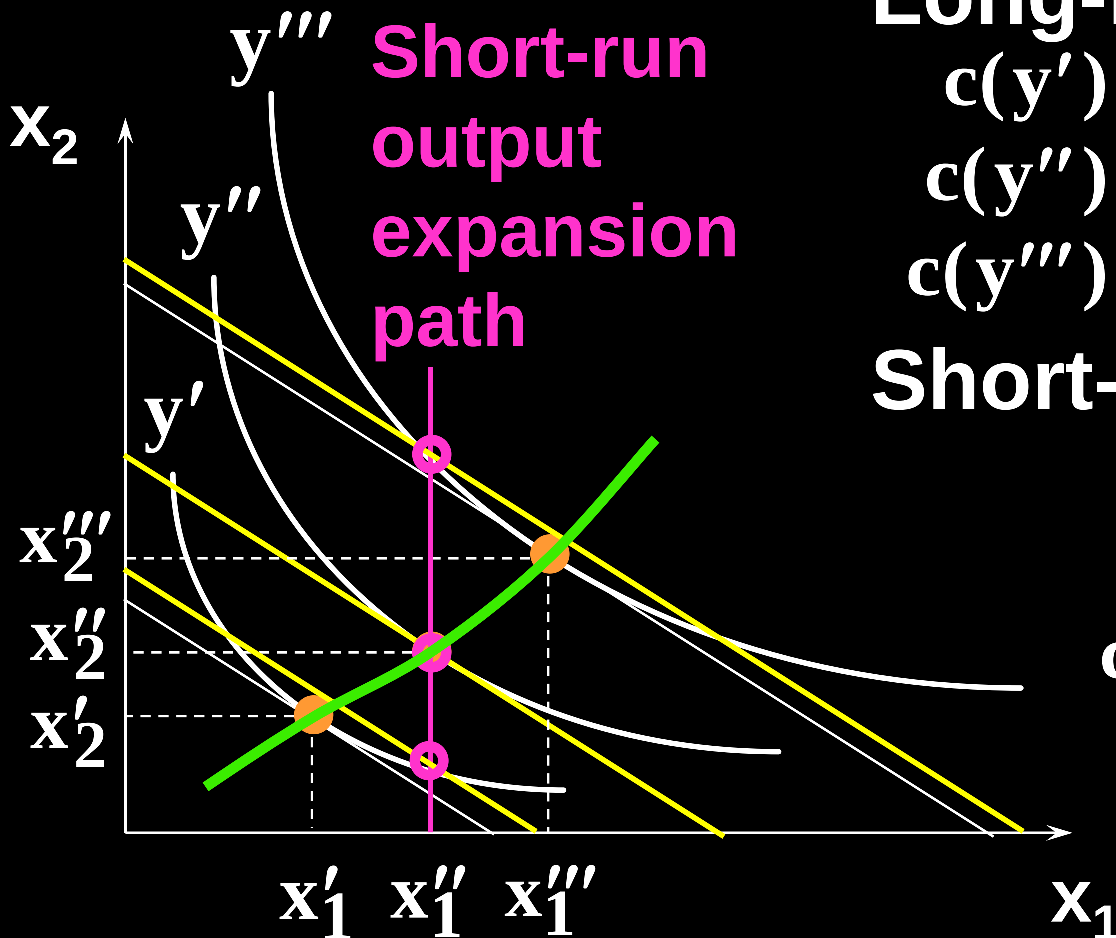
$$c(y''') = w_1 x_1''' + w_2 x_2'''$$

Short-run costs are:

$$c_s(y') > c(y')$$

$$c_s(y'') = c(y'')$$

Short-Run & Long-Run Total Costs



Long-run costs are:

$$c(y') = w_1x_1' + w_2x_2'$$
$$c(y'') = w_1x_1'' + w_2x_2''$$
$$c(y''') = w_1x_1''' + w_2x_2'''$$

Short-run costs are:

$$c_s(y') > c(y')$$
$$c_s(y'') = c(y'')$$
$$c_s(y''') > c(y''')$$

Short-Run & Long-Run Total Costs

- **Short-run total cost exceeds long-run total cost except for the output level where the short-run input level restriction is the long-run input level choice.**
- **This says that the long-run total cost curve always has one point in common with any particular short-run total cost curve.**

Short-Run & Long-Run Total Costs

A short-run total cost curve always has one point in common with the long-run total cost curve, and is elsewhere higher than the long-run total cost curve.