A卷答案

一. 事件A：先抛者吃到苹果.

H_i：第i次抛掷中首次出现了正面.

$P(A|H_1)=\frac{2}{3}$　　$P(H_1)=\frac{3}{4}$ （正正，正反，反正）

$P(A|H_2)=\frac{2}{3}$　　$P(H_2)=\frac{1}{2^2}\times\frac{3}{4}$ （第一轮：反反 第二轮：正正.正反.反正）

$P(A|H_3)=\frac{2}{3}$　　$P(H_3)=\frac{1}{2^2}\times\frac{1}{2^2}\times\frac{3}{4}$

$\vdots$　　$\vdots$

$P(A|H_i)=\frac{2}{3}$　　$P(H_i)=\left(\frac{1}{4}\right)^{i-1}\times\frac{3}{4}$

$$P(A)=\sum_{i=1}^{\infty}P(A|H_i)P(H_i)=\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^{2i-2}\times\frac{1}{2}=\sum_{i=1}^{\infty}\frac{1}{2^{2i-1}}=\frac{2}{3}$$

或方法二：

令A_i为第i次抛掷中先抛者吃到苹果.

$P(A_1)=\frac{1}{2}$ （正反~~正~~）

$P(A_2)=\frac{1}{2^2}\times\frac{1}{2}=\frac{1}{2^3}$ （第一轮.反反 第二轮：正反.正正）

$P(A_3)=\frac{1}{2^2}\times\frac{1}{2^2}\times\frac{1}{2}=\frac{1}{2^5}$ （第一轮：反反 第二轮：反反 第三轮：正反.正正）

$\vdots$

$P(A_i)=\frac{1}{2^{2i-1}}$

$$P(A)=P(A_1)+P(A_2)+\cdots+P(A_\infty)=\sum_{i=1}^{\infty}\frac{1}{2^{2i-1}}=\frac{2}{3}$$

二 (1) X_1为农场1的鸡蛋. X_2为农场2的鸡蛋.

$X_1\sim U(50,60)$，$X_2\sim U(55,70)$

$$P(X\le x)=P(X\le x|X\in X_1)P(X_1)+P(X\le x|X\in X_2)P(X_2)$$

$$\begin{array}{lll}
=0 & +\ 0 & ,\ x<50 \\
=\frac{x-50}{10}\times 0.4 & +\ 0 & ,\ 50\le x<55 \\
=\frac{x-50}{10}\times 0.4 & +\ \frac{x-55}{15}\times 0.6 & ,\ 55\le x<60 \\
=1\times 0.4 & +\ \frac{x-55}{15}\times 0.6 & ,\ 60\le x<70 \\
=1\times 0.4 & +\ 1\times 0.6=1 & ,\ x\ge 70
\end{array}$$

$$\therefore\quad F(x)=\begin{cases}0 & x<50\\ 0.04x-2 & 50\le x<55\\ 0.08x-\cancel{2.22}\ 4.2 & 55\le x<60\\ 0.04x+\cancel{0.18}\ 2.2 & 60\le x<70\\ 1 & x\ge 70\end{cases}$$

$$(2)\ P(X\in X_1 \mid X<60)=\frac{P(X\in X_1,\ X<60)}{P(X<60)}=\frac{P(X<60\mid X\in X_1)P(X_1)}{P(X<60)}$$

$$=\frac{0.4^{2}}{F(60)}=\frac{\cancel{0.40}\ 0.4}{\cancel{2.58}\ 0.6}\approx\cancel{0.16}\ \frac{2}{3}$$

三. (1) X边缘:

X	-1	2
P	0.6	0.4

Y边缘:

Y	-1	1	2
P	0.3	0.3	0.4

(2) $P(Y=2)=0.4$

$$P(X=-1\mid Y=2)=\frac{P(X=-1,Y=2)}{P(Y=2)}=\frac{0.3}{0.4}=\frac{3}{4}$$

$$P(X=2\mid Y=2)=\frac{P(X=2,Y=2)}{P(Y=2)}=\frac{0.1}{0.4}=\frac{1}{4}$$

(3) $E[X]=-1\times0.6+2\times0.4=0.2$

$E[X^2]=(-1)^2\times0.6+2^2\times0.4=2.2$

$E[Y]=-1\times0.3+1\times0.3+2\times0.4=0.8$

$E[Y^2]=(-1)^2\times0.3+1^2\times0.3+2^2\times0.4=2.2$

$\Rightarrow$ $D(X)=E[X^2]-(E[X])^2=2.16$

$D(Y)=E[Y^2]-(E[Y])^2=1.56$

$$E[XY]=(-1)\times(-1)\times 0.1+(-1)\times 1\times 0.2+(-1)\times 2\times 0.3+2\times(-1)\times 0.2$$
$$+2\times 1\times 0.1+2\times 2\times 0.1=-0.5$$

$$Cov(XY)=E[XY]-E[X]E[Y]=-0.66$$

$$\rho_{XY}=\frac{Cov(X,Y)}{\sqrt{D(X)}\sqrt{D(Y)}}=-0.36$$

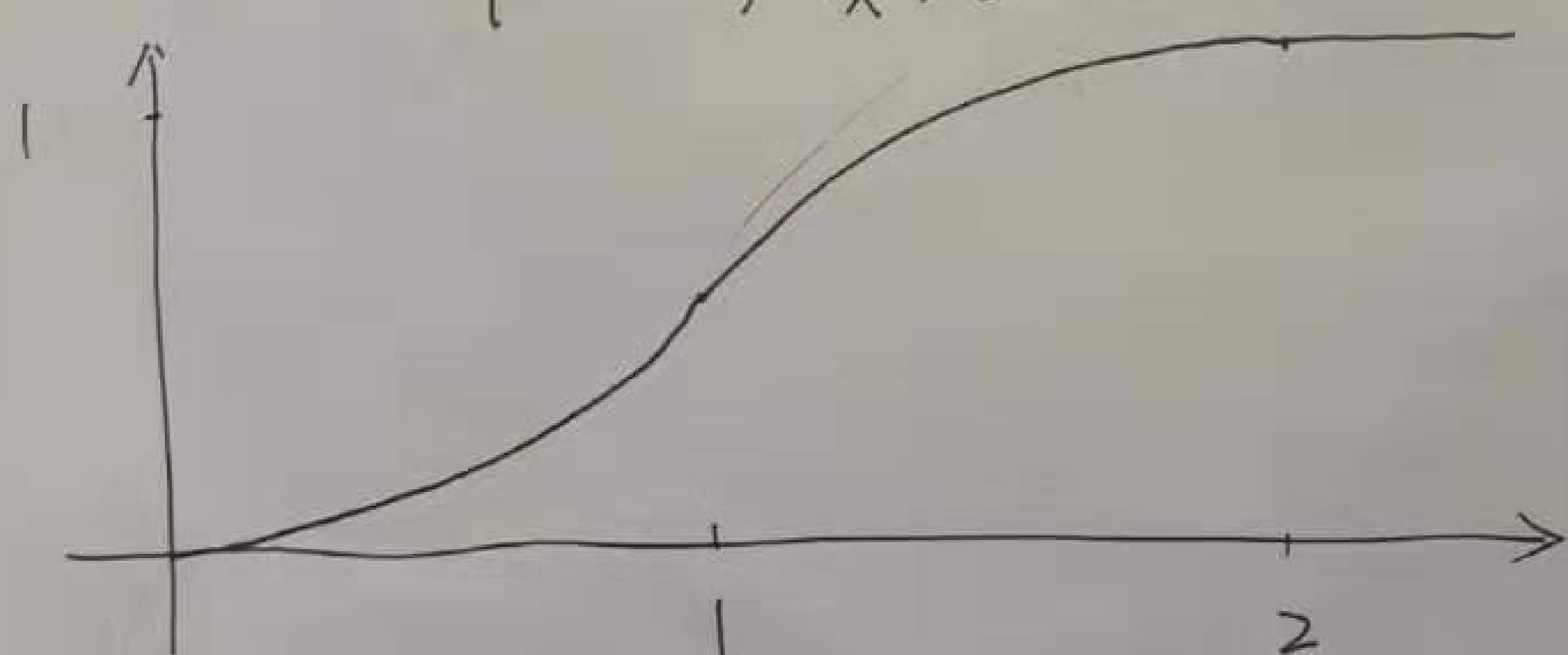
Ⅳ (1) $F(x)=0, \quad x\le 0$

$$F(x)=\int_0^x t\,dt=\frac{1}{2}x^2,\ 0<x\le 1$$

$$F(x)=\int_0^1 t\,dt+\int_1^x (2-t)\,dt, \quad 1<x\le 2$$

$$=1-\frac{1}{2}(x-2)^2$$

$$F(x)=\begin{cases}0, & x\le 0\\ \frac{1}{2}x^2, & 0<x\le 1\\ 1-\frac{1}{2}(x-2)^2, & 1<x\le 2\\ 1, & x>2\end{cases}$$



(2) $P(X<0.5)=F(0.5)=\frac{1}{8}$

$$P(0.5<X<1.5)=F(1.5)-F(0.5)=\frac{3}{4}$$

(3) $F_Y(Y\le y)=F(\log x\le y)=F(X\le e^y)=F_X(e^y)$

$$F_Y(y)=\begin{cases}\frac{1}{2}e^{2y}, & y\le 0\\ 1-\frac{1}{2}(e^{y}-2)^2, & 0<y\le \log 2\\ 1, & y>\log 2\end{cases}$$

$$f_Y(y)=\begin{cases}e^{2y}, & y\le 0\\ e^{-2y}+2e^{y}, & 0<y\le \log 2\\ 0, & y>\log 2\end{cases}$$

五. (1) 均匀分布: $k=\frac{1}{圆的面积}=\frac{1}{\pi}$

(2) $f_Y(y)=\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\frac{1}{\pi}dx=\frac{2}{\pi}\sqrt{1-y^2}\quad -1\le y\le 1$

$$f(x|y)=\begin{cases}\frac{f(x,y)}{f_Y(y)}=\frac{\frac{1}{\pi}}{\frac{2}{\pi}\sqrt{1-y^2}}=\frac{1}{2\sqrt{1-y^2}}, & -\sqrt{1-y^2}\le x\le\sqrt{1-y^2},\ -1\le y\le 1\\ 0 & 其他\end{cases}$$

$$P(X<0.5|Y=0.5)=\int_{-\frac{\sqrt{3}}{2}}^{0.5}\frac{1}{2\sqrt{1-y^2}}dx=\frac{1}{\sqrt{3}}x\Big|_{-\frac{\sqrt{3}}{2}}^{0.5}=\frac{3+\sqrt{3}}{6}$$

(3) 显然 $E[X]=E[Y]=0$

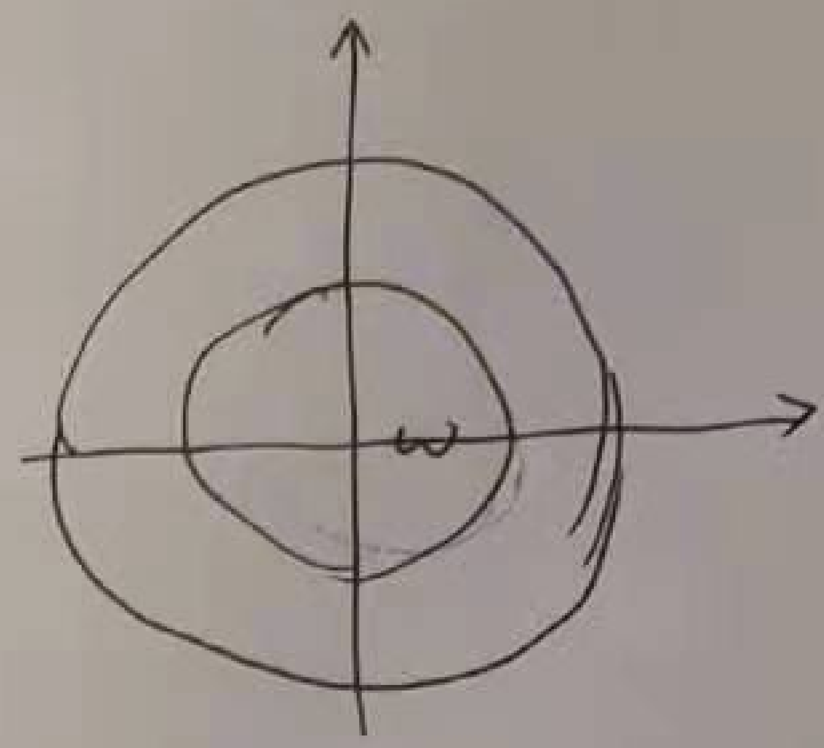
$E[XY]=\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}xy\,dx\,dy=0$

$Cov(X,Y)=E[XY]-E[X]E[Y]=0 \Rightarrow \rho_{XY}=0$

由(2)可知 $f(x|y)\ne f_X(x)$,因此 X,Y 不独立.

(4). $P_W(\sqrt{x^2+y^2} \le w) = P(x^2+y^2 \le w^2)$

即 X,Y 落在半经为 w 的圆内的概率.



X,Y 是均匀落在半经为1的圆内的二维随机变量,(X,Y)落在半径为 w 的小圆内的概率可由几何概型求得:

$$P(x^2+y^2 \le w^2) = \begin{cases} \dfrac{S_{小圆}}{S_{大圆}} = \dfrac{\pi w^2}{\pi} = w^2, & 0<w<1 \\ 1 & w \ge 1 \end{cases}$$

$$F_W(x) = \begin{cases} x^2, & 0<x<1 \\ 1, & x \ge 1 \end{cases}$$

$$f_W(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & 其他 \end{cases}$$

六.(1) $(0.9998)^{2000}$

(2) X: 行李丢失的件数.

$X \sim Poi(0.4)$

$P(X=1) = \dfrac{0.4}{1!}e^{-0.4} = 0.4e^{-0.4}$.

(3) $X \sim b(2000, 0.0002)$

中心极限: $E[X] = 0.4$, $D(X) = 0.0002 \times 0.9998 \times 2000 = 0.4$

$X \sim N(0.4, 0.4)$

即求 $P(1000X > 800) = P(X > 0.8) = 1 - P(X \le 0.8)$

$= 1 - \Phi\left(\dfrac{0.8-0.4}{\sqrt{0.4}}\right)$

$= 1 - \Phi(\sqrt{0.4})$

七. $T = \sum_{i=1}^{n} X_i \qquad X_i \sim \exp(0.1^{-1})$

$E[X_i] = 0.1 \qquad D(X_i) = \left((0.1)^{-1}\right)^{-2} = 0.01$

$E[T] = 0.1n \qquad D(T) = 0.01n$

$T \sim N(0.1n,\ 0.01n)$

要求 $P(T > 350) \geq 0.95$

即 $1 - P(T \leq 350) \geq 0.95$

$P(T \leq 350) \leq 0.05$

$\Phi\left(\frac{350 - 0.1n}{0.1\sqrt{n}}\right) \leq 0.05$