$X \in \mathbb{R}^{m\times n}$

- $r=1$. $\quad \operatorname{rank}(X)=1 \iff \exists\, u, v$, s.t. $X = \underset{\in\mathbb{R}^m}{u}\,\underset{\in\mathbb{R}^n}{v^T} \in \mathbb{R}^{m\times n}$

$\Downarrow$

$$\min_{u,v,E} \ \frac{1}{2}\|A - uv^T - E\|_F^2 + \lambda\|E\|_1 \qquad (1)$$

$$\text{s.t.} \quad u \geq 0 \qquad u_i \geq 0,\ i=1,\dots,m$$
$$\qquad v \geq 0 \qquad v_i \geq 0,\ i=1,\dots,n$$

$\|u-E\|_F^2 = \|E-u\|_F^2$

$$\min_{u\geq 0,\ v\geq 0} \left\{ \min_E \left\{ \frac{1}{2}\|\underbrace{A - uv^T}_{u} - E\|_F^2 + \underbrace{\lambda\|E\|_1}_{f(E)} \right\} \right\}$$

$$= \min_{u\geq 0,\ v\geq 0} \left\{ \frac{1}{2}\|A - uv^T - \operatorname{prox}_{\lambda f}(A-uv^T)\|_F^2 + \lambda\|\operatorname{prox}_{\lambda f}(A-uv^T)\|_1 \right\}$$

$$= \min_{u\geq 0,\ v\geq 0} \left\{ \lambda \cdot M_f^{\lambda}(A - uv^T) \right\} \qquad (2)$$

关于 u, v 的函数.

$$E = \operatorname{prox}_{\lambda f}(A - uv^T)$$

$$\begin{cases} H(u,v) \triangleq \lambda M_f^{\lambda}(A - uv^T); \\ \delta_{\mathbb{R}_+^m}(u) = \begin{cases} 0, & u\geq 0 \\ +\infty, & \text{否则} \end{cases}, \quad \delta_{\mathbb{R}_+^n}(v) = \begin{cases} 0, & v\geq 0 \\ +\infty, & \text{否则} \end{cases} \end{cases}$$

$\mathbb{R}_+^m = \{x\in\mathbb{R}^m \mid x_i \geq 0,\ i=1,2,\dots,m\}$ 指示函数

$$\min_{u,v} H(u,v), \quad s.t \ u \geq 0, \ v \geq 0$$ 约束条件

$$\min_{u,v} \delta_{R_+^m}(u) + \delta_{R_+^n}(v) + H(u,v)$$ PALM.

$$u^{k+1} = \arg\min_u \left\{ \delta_{R_+^m}(u) + \langle \nabla_u H(u^k, v^k), u - u^k \rangle + \frac{c_k}{2} \| u - u^k \|_2^2 \right\}$$

$$= \arg\min_u \left\{ \underbrace{\delta_{R_+^m}(u)}_{\text{即 } u \geq 0} + \frac{c_k}{2} \| u - u^k + \frac{1}{c_k} \nabla_u H(u^k, v^k) \|_2^2 \right\}$$

$$= \arg\min_{u \geq 0} \left\{ \frac{c_k}{2} \| u - (u^k - \frac{1}{c_k} \nabla_u H(u^k, v^k)) \|_2^2 \right\}$$

$$= \Pi_+ \left[u^k - \frac{1}{c_k} \nabla_u H(u^k, v^k) \right] \in \mathbb{R}^m$$

(对每个分量与0取最大值)

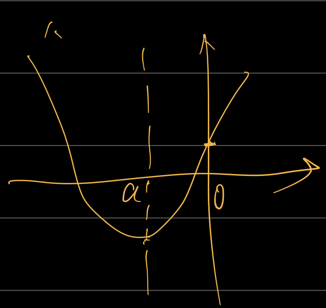
$$v^{k+1} = \Pi_+ \left[v^k - \frac{1}{d_k} \nabla_v H(u^{k+1}, v^k) \right] \in \mathbb{R}^n$$

$\times \ \max\{\Phi, 0\}$

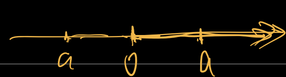
D.

$$\min_{x \in \mathbb{R}} \frac{c_k}{2} (x - a)^2 \quad s.t \ x \geq 0$$, $c_k > 0$

$$\rightarrow x^* = \begin{cases} a, & \text{if } a \geq 0 \\ 0, & \text{if } a < 0 \end{cases}$$



$\max\{a, 0\}$ $\quad \Pi_+[a]$



$\lambda = 1,\ 0.5,\ 2$

0.01

② $\nabla_u H(u,v)$ & $\nabla_v H(u,v)$

$H(u,v) \triangleq \lambda M_f^{\lambda}(A - uv^T)$;

- $M_f^{\lambda}(u) = \min_E \left\{ \frac{1}{2\lambda}\|u - E\|_F^2 + \|E\|_1 \right\}$, $\quad E^* = \mathrm{prox}_{\lambda f}(u)$

- $\nabla M_f^{\lambda}(u) = \frac{1}{\lambda}\left(u - \mathrm{prox}_{\lambda f}(u)\right)$. ※!

chain rule:

$A \ominus uv^T$

- $\nabla_u H(u,v) = \lambda \cdot \nabla_u M_f^{\lambda}(A - uv^T)$

$$= \lambda \cdot \left[-\frac{1}{\lambda}\left(A - uv^T - \mathrm{prox}_{\lambda f}(A - uv^T)\right) v \right]$$

$$= -\left(A - uv^T - \mathrm{prox}_{\lambda f}(A - uv^T)\right) v \in \mathbb{R}^{m\times 1}$$

$m\times n \quad n\times 1$

$\nabla_v H(u,v) = \lambda \nabla_v M_f^{\lambda}(A - uv^T)$

$$= \lambda \left[-\frac{1}{\lambda}\left(A^T - vu^T - \mathrm{prox}_{\lambda f}(A^T - vu^T)\right) \right] u$$

$$= -\left(A^T - vu^T - \mathrm{prox}_{\lambda f}(A^T - vu^T)\right) u \in \mathbb{R}^n$$

$n\times m \quad m\times 1$

$$\mathrm{prox}_{\lambda f}(A - uv^T) = \mathrm{sgn}(A - uv^T) \odot \max\left\{ |A - uv^T| - \lambda,\ 0 \right\}$$

③ c_k & d_k $\quad \gamma_1\ L_1 \leftarrow \nabla_u H(u^k, v^k) \leftarrow \|v^k\|_2$

$\gamma_2\ L_2 \leftarrow \nabla_v H(u^{k+1}, v^k) \leftarrow \|u^{k+1}\|_2$

$c_k = \gamma_1 \cdot \|v^k\|_2 \qquad d_k = \gamma_2 \cdot \|u^{k+1}\|_2, \qquad \gamma_1 = \gamma_2 = 1.01,\ 1.02\ 1.05$

每个 k，计算 A_u^k & A_v^k，若 $\|A_u^k\|_2 < \varepsilon$ 以及 $\|A_v^k\|_2 < \varepsilon$，stop. 返回 (u^k, v^k)

③ stop criterion: norm(x,2)

$E = \mathrm{prox}_{\lambda f}(A - u^k (v^k)^T)$

$$A_u^k = c_{k-1}(u^{k-1} - u^k) + \nabla_u H(u^k, v^k) - \nabla_u H(u^{k-1}, v^{k-1}) \in \mathbb{R}^m$$

$$A_v^k = d_{k-1}(v^{k-1} - v^k) + \nabla_v (u^k, v^k) - \nabla_v H(u^k, v^{k-1}) \in \mathbb{R}^n$$

$A = X + Y + N.$

$X = uv^T$

```
m = 256, n = 20;
u* = rand(m, 1)
v* = rand(n, 1);
Y = zeros(m,n); ∈ R^{m×n}
rind = randperm(m*n);
srind = rind(1:floor(m*n*0.1));
Y(srind) = 1;
Y = Y + randn(m, n)*0.1;
A = u* * (v*)^T + Y;
```

全Y有 $\boxed{\text{floor}(mn * 0.1)}$ 非零元 取1.

输入

输出 $u, v, E.$

$$\| uv^T - u^* v^{*T} \|_F$$

norm(A, 'fro')

help norm