Equations of coned granular flow systems

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I. Coned coordinate system

Map: $(x, y, z) \to (a, \theta, b)$

$$\begin{aligned} a &= \sqrt{x^2 + y^2}\cos\alpha - z\sin\alpha \\ \theta &= \arctan\frac{y}{x} \\ b &= \sqrt{x^2 + y^2}\sin\alpha + z\cos\alpha \end{aligned} \tag{1}$$

Therefore, we can get: And the Lame parameters are

Table 1 Partial derivative between two coordinate systems

Operators	a	θ	b
$\frac{\partial}{\partial x}$	$\frac{x\cos\alpha}{\sqrt{x^2+y^2}}$	$\frac{-y}{x^2+y^2}$	$\frac{x\sin\alpha}{\sqrt{x^2+y^2}}$
$\frac{\partial}{\partial y}$	$\frac{y\cos\alpha}{\sqrt{x^2+y^2}}$	$\frac{1}{x^2+y^2}$	$\frac{y\sin\alpha}{\sqrt{x^2+y^2}}$
$\frac{\partial}{\partial z}$	$-\sin\alpha$	0	$\cos\alpha$

$$\begin{aligned} h_1 &= \frac{1}{\sqrt{(\partial a/\partial x)^2 + (\partial a/\partial y)^2 + (\partial a/\partial z)^2}} = 1 \\ h_2 &= \frac{1}{\sqrt{(\partial \theta/\partial x)^2 + (\partial \theta/\partial y)^2 + (\partial \theta/\partial z)^2}} = \sqrt{x^2 + y^2} = a\cos\alpha + b\sin\alpha \\ h_3 &= \frac{1}{\sqrt{(\partial b/\partial x)^2 + (\partial b/\partial y)^2 + (\partial b/\partial z)^2}} = 1 \end{aligned} \tag{2}$$

General differential operations in orthogonal curvilinear coordinate system (ϕ is a scalar function; $\boldsymbol{F}$ is a vector):

Gradient:

$$\nabla\phi = \frac{1}{h_i}\frac{\partial\phi}{\partial x_i}\hat{\mathbf{e}}_i \tag{3}$$

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Divergence:

$$\boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{1}{\sqrt{g}}\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\frac{\sqrt{g}F_i}{h_i}\right), \quad \text{where } \sqrt{g} = h_1 h_2 h_3 \tag{4}$$

Curl:

$$\boldsymbol{\nabla}\times\boldsymbol{F} = \frac{1}{h_2 h_3}\left[\frac{\partial(h_3F_3)}{\partial x_2} - \frac{\partial(h_2F_2)}{\partial x_3}\right]\hat{\mathbf{e}}_1 + \frac{1}{h_1 h_3}\left[\frac{\partial(h_1F_1)}{\partial x_3} - \frac{\partial(h_3F_3)}{\partial x_1}\right]\hat{\mathbf{e}}_2 + \frac{1}{h_1 h_2}\left[\frac{\partial(h_2F_2)}{\partial x_1} - \frac{\partial(h_1F_1)}{\partial x_2}\right]\hat{\mathbf{e}}_3 \tag{5}$$

Laplace:

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial x_3}\right)\right] \tag{6}$$

Directional derivative:

$$(\hat{\boldsymbol{e}}_\tau\cdot\boldsymbol{\nabla})\hat{\boldsymbol{e}}_\rho = -\frac{1}{h_\tau}(\boldsymbol{\nabla}h_\rho)\delta_\tau(\rho) + \frac{1}{h_\rho h_\tau}\frac{\partial h_\tau}{\partial x_\rho}\hat{\boldsymbol{e}}_\tau \tag{7}$$

II. Governing equations in coned coordinate system

Mass conservation:

$$(\partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla})\phi + \phi\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \tag{8}$$

Momentum conservation:

$$\rho_*\phi(\partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{\nabla}\cdot\tau + \rho_*\phi\boldsymbol{g} \tag{9}$$

A. Simplification of mass conservation equation

According to Eq (4), we can get

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \partial_a u_a + \frac{1}{a\cos\alpha + b\sin\alpha}\partial_\theta u_\theta + \partial_b u_b + \frac{\cos\alpha}{a\cos\alpha + b\sin\alpha}u_a + \frac{\sin\alpha}{a\cos\alpha + b\sin\alpha}u_b \tag{10}$$

B. Simplification of momentum conservation equations

Based on Eq (7), we have: Therefore, we can deduce the convection term as:

Table 2

Operators	$\hat{\mathbf{e}}_a$	$\hat{\mathbf{e}}_\theta$	$\hat{\mathbf{e}}_b$
$\hat{\mathbf{e}}_a\cdot\boldsymbol{\nabla}$	0	0	0
$\hat{\mathbf{e}}_\theta\cdot\boldsymbol{\nabla}$	$-\frac{\cos\alpha\hat{\mathbf{e}}_a+\sin\alpha\hat{\mathbf{e}}_b}{a\cos\alpha+b\sin\alpha}$	$\frac{\cos\alpha\hat{\mathbf{e}}_\theta}{a\cos\alpha+b\sin\alpha}$	$\frac{\sin\alpha\hat{\mathbf{e}}_\theta}{a\cos\alpha+b\sin\alpha}$
$\hat{\mathbf{e}}_b\cdot\boldsymbol{\nabla}$	0	0	0

$$
\begin{aligned}
(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} &= \left[(u_a\hat{\mathbf{e}}_a + u_\theta\hat{\mathbf{e}}_\theta + u_b\hat{\mathbf{e}}_b)\cdot\boldsymbol{\nabla}\right](u_a\hat{\mathbf{e}}_a + u_\theta\hat{\mathbf{e}}_\theta + u_b\hat{\mathbf{e}}_b) \\
&= (u_a\hat{\mathbf{e}}_a\cdot\boldsymbol{\nabla})(u_a\hat{\mathbf{e}}_a + u_\theta\hat{\mathbf{e}}_\theta + u_b\hat{\mathbf{e}}_b) + (u_\theta\hat{\mathbf{e}}_\theta\cdot\boldsymbol{\nabla})(u_a\hat{\mathbf{e}}_a + u_\theta\hat{\mathbf{e}}_\theta + u_b\hat{\mathbf{e}}_b) + (u_b\hat{\mathbf{e}}_b\cdot\boldsymbol{\nabla})(u_a\hat{\mathbf{e}}_a + u_\theta\hat{\mathbf{e}}_\theta + u_b\hat{\mathbf{e}}_b) \\
&= (\mathrm{d}_t u_a + \frac{-\cos\alpha\cdot u_\theta^2}{a\cos\alpha + b\sin\alpha})\hat{\mathbf{e}}_a + (\mathrm{d}_t u_a + \frac{\sin\alpha\cdot u_\theta u_b}{a\cos\alpha + b\sin\alpha} + \frac{\cos\alpha\cdot u_\theta u_a}{a\cos\alpha + b\sin\alpha})\hat{\mathbf{e}}_\theta + (\mathrm{d}_t u_b + \frac{-\sin\alpha\cdot u_\theta^2}{a\cos\alpha + b\sin\alpha})\hat{\mathbf{e}}_b
\end{aligned} \tag{11}
$$

where $\mathrm{d}_t \equiv \partial_t + u_a\partial_a + u_\theta/(a\cos\alpha + b\sin\alpha)\partial_\theta + u_b\partial_b$.

Since $\boldsymbol{g} = g\sin\alpha\hat{\mathbf{e}}_a - g\cos\alpha\hat{\mathbf{e}}_b$ in our case, we can expand the governing equations in new coordinate system as:

$$
\rho_*\phi\left(\mathrm{d}_t u_a + \frac{-\cos\alpha\cdot u_\theta^2}{a\cos\alpha + b\sin\alpha}\right) = (\boldsymbol{\nabla}\cdot\tau)_a + \rho_*\phi g\sin\alpha \tag{12}
$$

$$
\rho_*\phi\left(\mathrm{d}_t u_a + \frac{\sin\alpha\cdot u_\theta u_b}{a\cos\alpha + b\sin\alpha} + \frac{\cos\alpha\cdot u_\theta u_a}{a\cos\alpha + b\sin\alpha}\right) = (\boldsymbol{\nabla}\cdot\tau)_\theta \tag{13}
$$

$$
\rho_*\phi\left(\mathrm{d}_t u_b + \frac{-\sin\alpha\cdot u_\theta^2}{a\cos\alpha + b\sin\alpha}\right) = (\boldsymbol{\nabla}\cdot\tau)_b - \rho_*\phi g\cos\alpha \tag{14}
$$

C. Shallow water equations

Now, we apply shallow water assumption ($\partial_b u_a = \partial_b u_\theta = 0$, $a \gg b$), and integrate along $\hat{\mathbf{e}}_b$ direction to simplify the original system.

As for incompressible granular flow, i.e.

$$
\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0
$$

We first integrate the above equation along $\hat{\mathbf{e}}_b$, and we can get:

$$
\begin{aligned}
&\int_0^h \left(\partial_a u_a + \frac{1}{a\cos\alpha + b\sin\alpha}\partial_\theta u_\theta + \partial_b u_b + \frac{\cos\alpha}{a\cos\alpha + b\sin\alpha}u_a + \frac{\sin\alpha}{a\cos\alpha + b\sin\alpha}u_b\right)\mathrm{d}b \\
&= \partial_a(u_a h) + \frac{\partial_\theta(u_\theta h)}{a\cos\alpha + h\sin\alpha} + \frac{\cos\alpha\cdot u_a h}{a\cos\alpha + h\sin\alpha} - u_a\partial_a h + u_b\Big|_{b=0}^{h} - \partial_\theta h\frac{u_\theta}{a\cos\alpha + h\sin\alpha} \\
&= 0
\end{aligned}
$$

where h is the local height of the sand flow. And we have applied the following approximation:

$$
\frac{1}{a\cos\alpha + b\sin\alpha} \approx \frac{1}{a\cos\alpha + h\sin\alpha} \tag{15}
$$

$$
\int_0^h u_b\mathrm{d}b = 0 \tag{16}
$$

Considering the boundary conditions that

$$
\begin{aligned}
&u_b(b=0)=0,\\
&\frac{\mathrm{D}h}{\mathrm{Dt}}=\mathrm{d}_t h+\boldsymbol{u}\cdot\boldsymbol{\nabla}h=u_b,\ \text{on the free surface } (b=h)
\end{aligned}
$$

by using the boundary conditions, we can finally get the integral form of continuity equation:

$$
(continuity)\quad \partial_t h+\partial_a(u_a h)+\frac{\partial_\theta(u_\theta h)}{a\cos\alpha+h\sin\alpha}+\frac{\cos\alpha\cdot(u_a h)}{a\cos\alpha+h\sin\alpha}=0 \tag{17}
$$

Next, we can look at the momentum equation along $\hat{\mathbf{e}}_b$ direction (neglect acceleration along $\hat{\mathbf{e}}_b$):

$$
\rho_*\phi\left(\mathrm{d}_t u_b+\frac{-sin\alpha\cdot u_\theta^2}{a\cos\alpha+b\sin\alpha}\right)=0=(\boldsymbol{\nabla}\cdot\boldsymbol{T})_b-\rho_*\phi g\cos\alpha
$$

, and we can expand the above equation as:

$$
\begin{aligned}
0&=\left(\partial_a\tau_{ab}+\frac{\partial_\theta\tau_{\theta b}}{a\cos\alpha+b\sin\alpha}+\partial_b\tau_{bb}-\partial_b p\right)-\rho_*\phi g h\cos\alpha\\
&=0-\partial_b p-\rho_*\phi g\cos\alpha\\
&=-\partial_b p-\rho_*\phi g h\cos\alpha
\end{aligned}
$$

Therefore, we can get the formula for p:

$$
p(b)=\rho_*\phi g\cos\alpha(b-h) \tag{18}
$$

In summary, the original system can be simplified as:

$$
\partial_t h+\partial_a(u_a h)+\frac{\partial_\theta(u_\theta h)}{a\cos\alpha+h\sin\alpha}+\frac{\cos\alpha\cdot(u_a h)}{a\cos\alpha+h\sin\alpha}=0 \tag{19}
$$

$$
\rho_*\phi\left(\mathrm{d}_t u_a+\frac{-\cos\alpha\cdot u_\theta^2}{a\cos\alpha+h\sin\alpha}\right)=(\boldsymbol{\nabla}\cdot\boldsymbol{T})_a+\rho_*\phi\sin\alpha \tag{20}
$$

$$
\rho_*\phi\left(\mathrm{d}_t u_\theta+\frac{\sin\alpha\cdot u_b u_\theta+\cos\alpha\cdot u_a u_\theta}{a\cos\alpha+h\sin\alpha}\right)=(\boldsymbol{\nabla}\cdot\boldsymbol{T})_\theta \tag{21}
$$

$$
p=\rho_*\phi g\cos\alpha(b-h) \tag{22}
$$

where

$$
\mathrm{d}_t=\partial_t+u_a\partial_a+\frac{u_\theta\partial_\theta}{a\cos\alpha+h\sin\alpha}
$$

III. Modelling for stress tensor

Yield conditions:

$$\|\tau\| = Y(p, \phi, I), \qquad \nabla \cdot \boldsymbol{u} = 2f(p, \phi, I)\|\mathbf{D}\| \equiv 0 \quad (inourcase) \tag{23}$$

$$\frac{\partial Y}{\partial p} - \frac{I}{2p}\frac{\partial Y}{\partial I} = f + I\frac{\partial f}{\partial I} \tag{24}$$

where τ is stress tensor; $\mathbf{D}$ represents deviatoric strain-rate tensor; $I = 2d\|\mathbf{D}\| / \sqrt{p/\rho_*}$ is internal number. Assume:

$$\partial_I Y > 0, \qquad \partial_p f - \frac{I}{2p}\frac{\partial Y}{\partial I} < 0 \tag{25}$$

which ensures

$$\frac{\mathrm{d}f}{\mathrm{d}p} = \frac{\partial f}{\partial p} + \frac{\partial I}{\partial p}\frac{\partial f}{\partial I} = \partial_p f - \frac{I}{2p}\frac{\partial Y}{\partial I} \neq 0$$

Thus, implicit function theorem can be applied to represent pressure as $p = \mathbb{P}(\nabla \boldsymbol{u}, \phi)$. Other terms can be represented in such way as well:

$$T(\nabla \boldsymbol{u}, \phi) = Y[\mathbb{P}(\nabla \boldsymbol{u}, \phi), \phi, I(\nabla \boldsymbol{u})], \qquad I(\nabla \boldsymbol{u}, \phi) = \frac{2d\|\boldsymbol{D}\|}{\sqrt{\mathbb{P}(\nabla \boldsymbol{u}, \phi)/\rho_*}}$$

Based on these definitions, stress along one of the axises can be derived as

$$(\nabla \cdot \tau)_i = \frac{1}{h_j}\partial_j\left(\frac{T(\nabla \boldsymbol{u}, \phi)}{\|\mathbf{D}\|}D_{ij}\right) - \frac{1}{h_i}\partial_i\mathbb{P}(\nabla \boldsymbol{u}, \phi) + \rho_*\phi g_i, \quad (i, j = a, \theta) \tag{26}$$

Considering alignment requirement:

$$\frac{D_{ij}}{\|\mathbf{D}\|} = \frac{\tau_{ij}}{\|\tau\|} \tag{27}$$

Eq (26) can be simplified as

$$(\nabla \cdot \tau)_i = \frac{1}{h_j}\partial_j\left(\tau_{ij}\right) - \frac{1}{h_i}\partial_i\mathbb{P}(\nabla \boldsymbol{u}, \phi) + \rho_*\phi g_i, \quad (i, j = a, \theta) \tag{28}$$

Our case can be treated as a 2D problem, and then $\mathbf{D}$ is given as:

$$\mathbf{D} = \frac{1}{2}\begin{bmatrix} \partial_a u_a - \frac{\partial_\theta u_\theta}{a\cos\alpha + b\sin\alpha} & \partial_a u_\theta + \frac{\partial_\theta u_a}{a\cos\alpha + b\sin\alpha} \\ \partial_a u_\theta + \frac{\partial_\theta u_a}{a\cos\alpha + b\sin\alpha} & -\partial_a u_a + \frac{\partial_\theta u_\theta}{a\cos\alpha + b\sin\alpha} \end{bmatrix} \tag{29}$$

Similarly, we simplify Eq (23) as:

$$\partial_a u_a + \frac{\partial_\theta u_\theta}{a\cos\alpha + b\sin\alpha} = 2f\|\mathbf{D}\| \tag{30}$$

Since $\mathrm{tr}(\tau) = \|\tau\| / \|\mathbf{D}\|\ \mathrm{tr}(\mathbf{D}) = 0$, there exists eigenvectors and eigenvalues of τ. And we can represent τ as

$$\tau = -\|\tau\| \begin{bmatrix} \cos 2\psi & \sin 2\psi \\ \sin 2\psi & -\cos 2\psi \end{bmatrix} \tag{31}$$

where $(\cos 2\psi, \sin 2\psi)$ is an eigenvector of this matrix with eigenvalue $\|\tau\|$. Considering alignment requirement, we can get:

$$(\partial_a u_a - \frac{\partial_\theta u_\theta}{a\cos\alpha + b\sin\alpha})\sin 2\psi - (\partial_a u_\theta + \frac{\partial_\theta u_a}{a\cos\alpha + b\sin\alpha})\cos 2\psi = 0 \tag{32}$$

Finally, the governing equations can be summarized as:

$$\partial_t h + (\partial_a u_a)h + u_a\partial_a h + \frac{\partial_\theta(u_\theta h) + \cos\alpha\cdot(u_a h)}{a\cos\alpha + h\sin\alpha} = 0 \tag{33}$$

$$\rho_*\phi\left(\mathrm{d}_t u_a + \frac{-\cos\alpha\cdot u_\theta^2}{a\cos\alpha + h\sin\alpha}\right) - \rho_*\phi g\cos\alpha\cdot\partial_a h + \partial_a(\tau\cos 2\psi) + \frac{\partial_\theta(\tau\sin 2\psi)}{a\cos\alpha + h\sin\alpha} = \rho_*\phi g\sin\alpha \tag{34}$$

$$\rho_*\phi\left(\mathrm{d}_t u_\theta + \frac{\cos\alpha\cdot u_a u_\theta}{a\cos\alpha + h\sin\alpha}\right) + \partial_a(\tau\sin 2\psi) - \frac{\rho_*\phi g\cos\alpha\cdot\partial_\theta h}{a\cos\alpha + h\sin\alpha} - \frac{\partial_\theta\tau\cos 2\psi}{a\cos\alpha + h\sin\alpha} = 0 \tag{35}$$

$$(\partial_a u_a - \frac{\partial_\theta u_\theta}{a\cos\alpha + b\sin\alpha})\sin 2\psi - (\partial_a u_\theta + \frac{\partial_\theta u_a}{a\cos\alpha + b\sin\alpha})\cos 2\psi = 0 \tag{36}$$

$$\frac{\partial\tau}{\partial p} - \frac{I}{2p}\frac{\partial\tau}{\partial I} = f(p,\phi,I) + I\frac{\partial f}{\partial I} \equiv 0 \tag{37}$$

where

$$\mathrm{d}_t = \partial_t + u_a\partial_a + \frac{u_\theta\partial_\theta}{a\cos\alpha + h\sin\alpha}, \quad p = \rho_*\phi g\cos\alpha(b-h)$$

IV. Stability Analysis

The system has four scalar unknowns, $\boldsymbol{U} = (h, u_a, u_\theta, \psi)$. τ is an abbreviation for the yield function $Y(p,\phi,I)$. To linearize the equations, we substitute a perturbation of a base solution $\boldsymbol{U}^{(0)}(\boldsymbol{x},t)$ as:

$$\boldsymbol{U} = \boldsymbol{U}^{(0)} + \hat{\boldsymbol{U}} \tag{38}$$

Then, we plug it into the equations and retain only terms that are linear in the perturbation $\hat{\boldsymbol{U}}$ and freeze the coefficients at an arbitrary point $(\boldsymbol{x},t)$.

When expanding the fully nonlinear term $\|\mathbf{D}\|$, we take advantage of the rotational invariance of the equations to arrange that $\psi^* = 0$. Thus based on alignment criteria Eq (32) and Eq (29), we can get Eq (39).

$$\mathbf{D}^* = \frac{1}{2}\begin{bmatrix} \partial_a u_a^* - \frac{\partial_\theta u_\theta^*}{a^*\cos\alpha + b^*\sin\alpha} & 0 \\ 0 & -\partial_a u_a^* + \frac{\partial_\theta u_\theta^*}{a^*\cos\alpha+b^*\sin\alpha} \end{bmatrix} \tag{39}$$

$$\begin{aligned} \|\mathbf{D}^* + \hat{\mathbf{D}}\| &= \frac{1}{2}\sqrt{(\partial_a u_a^* - \frac{\partial_\theta u_\theta^*}{a^*\cos\alpha + b^*\sin\alpha} + \partial_a \hat{u}_a - \frac{\partial_\theta \hat{u}_\theta}{a^*\cos\alpha + b^*\sin\alpha})^2 + (\frac{\partial_\theta \hat{u}_a}{a^*\cos\alpha + b^*\sin\alpha} + \partial_a \hat{u}_\theta)^2} \\ &\approx \|\mathbf{D}^*\| + \frac{1}{2}\left|\partial_a \hat{u}_a - \frac{\partial_\theta \hat{u}_\theta}{a^*\cos\alpha + b^*\sin\alpha}\right| \end{aligned} \tag{40}$$

where the approximation follows from the expansion

$$\sqrt{(A \pm X)^2 + Y^2} = A + |X| + O\left(X^2 + Y^2\right)$$

if $A > 0$ and $|X|, |Y| \ll A$. Therefore, if we suppose that $\partial_a u_a^* > \partial_\theta u_\theta^*/(a^*\cos\alpha + b^*\sin\alpha)$, the formula becomes

$$\|\mathbf{D}^* + \hat{\mathbf{D}}\| \approx \|\mathbf{D}^*\| + \frac{1}{2}\left(\partial_a \hat{u}_a - \frac{\partial_\theta \hat{u}_\theta}{a^*\cos\alpha + b^*\sin\alpha}\right) \tag{41}$$

if we suppose that $\partial_a u_a^* < \partial_\theta u_\theta^*/(a^*\cos\alpha + b^*\sin\alpha)$, the formula becomes

$$\|\mathbf{D}^* + \hat{\mathbf{D}}\| \approx \|\mathbf{D}^*\| - \frac{1}{2}\left(\partial_a \hat{u}_a - \frac{\partial_\theta \hat{u}_\theta}{a^*\cos\alpha + b^*\sin\alpha}\right) \tag{42}$$

Here, I am not sure, since the original paper uses another expression

$$\sqrt{(-A + X)^2 + Y^2} = A - X + O(X^2 + Y^2)$$

I think it depends on the assumption of the sign of the second term.

By chain rule,

$$\partial_j[\tau\cos(2\psi)] = \cos(2\psi)\left\{\partial_p \tau \partial_j p + \partial_\phi \tau \partial_j \phi + \partial_I \tau \left[\frac{2d}{\sqrt{p/\rho_*}}\partial_j\|\mathbf{D}\| - \frac{d\|\mathbf{D}\|}{\sqrt{p^3/\rho_*}}\partial_j p\right]\right\} - 2\tau\sin(2\psi)\partial_j\psi$$

Table 3 Linearization of terms

Original terms	Contributions
$\lVert\mathbf{D}\rVert$	$\hat{D}_{11}$
I	$\frac{I^*}{\lVert\mathbf{D}^*\rVert}\hat{D}_{11} - \frac{I^*}{2p^*}\hat{p}$
$\partial_j[\tau\cos(2\psi)]$	$\left(\partial_p\tau\right)^*\partial_j\hat{p} + \left(\partial_\phi\tau\right)^*\partial_j\hat{\phi} + (\partial_I\tau)^*\left\{\frac{I^*}{\lVert\mathbf{D}^*\rVert}\partial_j\hat{D}_{11} - \frac{I^*}{2p^*}\partial_j\hat{p}\right\}$
$\partial_j[\tau\sin(2\psi)]$	$2\tau^*\partial_j\hat{\psi}$
$f\lVert\mathbf{D}\rVert$	$f^*\hat{D}_{11} + \lVert D^*\rVert\left(\partial_p f\right)^*\hat{p} + \lVert D^*\rVert\left(\partial_\phi f\right)^*\hat{\phi} + \lVert D^*\rVert(\partial_I f)^*\left\{\frac{I^*}{\lVert D^*\rVert}\hat{D}_{11} - \frac{I^*}{2p^*}\hat{p}\right\}$

We can get (**upper sign for** $\partial_a u_a^* > \partial_\theta u_\theta^*/(a^*\cos\alpha + b^*\sin\alpha)$**; lower sign for** $\partial_a u_a^* < \partial_\theta u_\theta^*/(a^*\cos\alpha + b^*\sin\alpha)$):

$$\left[\mathrm{d}_t^* + \frac{\cos\alpha\cdot u_a^*}{a^*\cos\alpha + h^*\sin\alpha}\right]\hat{h} + \left[h^*\partial_a + \partial_a h^* + \frac{\cos\alpha\cdot h^*}{a^*\cos\alpha + h^*\sin\alpha}\right]\hat{u}_a + \frac{h^*\partial_\theta + \partial_\theta h^*}{a^*\cos\alpha + h^*\sin\alpha}\hat{u}_\theta = 0 \tag{43}$$

$$-\rho_*\phi g\cos\alpha\cdot\partial_a\hat{h} + \left[\rho_*\phi(\mathrm{d}_t^* + \partial_a u_a^*) \pm (\partial_I\tau)^*\frac{I^*}{2\lVert D^*\rVert}\partial_{aa}\right]\hat{u}_a + \left[\rho_*\phi\frac{\partial_\theta u_a^* - 2\cos\alpha\cdot u_\theta^*}{a^*\cos\alpha + h^*\sin\alpha} \pm (\partial_I\tau)^*\frac{I^*}{2\lVert D^*\rVert}\frac{-(a^*\cos\alpha + h^*\sin\alpha)\cdot\partial_{a\theta} + \cos\alpha\cdot\partial_\theta}{(a^*\cos\alpha + h^*\sin\alpha)^2}\right]\hat{u}_\theta + \frac{2\tau^*\partial_\theta\hat{\psi}}{a^*\cos\alpha + h^*\sin\alpha} = 0 \tag{44}$$

$$\frac{-\rho_*\phi g\cos\alpha\cdot\partial_\theta\hat{h}}{a^*\cos\alpha + h^*\sin\alpha} + \left[\rho_*\phi\left(\partial_a u_\theta^* + \frac{\cos\alpha\cdot u_\theta^*}{a^*\cos\alpha + h^*\sin\alpha}\right) \mp \frac{(\partial_I\tau)^* I^*/(2\lVert D^*\rVert)}{a^*\cos\alpha + h^*\sin\alpha}\partial_{a\theta}\right]\hat{u}_a + \left[\rho_*\phi\left(\mathrm{d}_t^* + \frac{\partial_\theta u_\theta^* + \cos\alpha\cdot u_a^*}{a^*\cos\alpha + h^*\sin\alpha}\right) \pm \frac{(\partial_I\tau)^* I^*/(2\lVert D^*\rVert)}{(a^*\cos\alpha + h^*\sin\alpha)^2}\partial_{\theta\theta}\right]\hat{u}_\theta + (2\tau^*)\partial_a\hat{\psi} = 0 \tag{45}$$

$$\frac{\partial_\theta\hat{u}_a}{a^*\cos\alpha + h^*\sin\alpha} - \partial_a\hat{u}_\theta \pm 4\left\lVert D^*\right\rVert\hat{\psi} = 0 \tag{46}$$

where

$$\mathrm{d}_t^* = \partial_t + u_a^*\partial_a + \frac{u_\theta^*\partial_\theta}{a\cos\alpha + h\sin\alpha}$$

V. The Eigenvalue Problem

We now look for exponential solutions of the system,

$$\hat{\boldsymbol{U}}(\boldsymbol{x}, t) = e^{i(k_a a + k_\theta\theta) + \lambda t}\tilde{\boldsymbol{U}} \tag{47}$$

where $\tilde{\boldsymbol{U}} = (\tilde{h}, \tilde{u}_a, \tilde{u}_\theta, \tilde{\psi})$, $k_b, k_\theta \in \mathbb{R}$, $\lambda \in \mathbb{C}$.

In our case, $u_\theta^* = 0$. Then the original system can be simplified as

$$\left(\lambda + u_a^*\partial_a + \frac{\cos\alpha\cdot u_a^*}{a^*\cos\alpha + h^*\sin\alpha}\right)\tilde{h} + \left(\partial_a h^* + \frac{\cos\alpha\cdot h^*}{a^*\cos\alpha + h^*\sin\alpha} + h^*\partial_a\right)\tilde{u}_a + \frac{\partial_\theta h^* + ik_\theta h^*}{a^*\cos\alpha + h^*\sin\alpha}\tilde{u}_\theta = 0 \tag{48}$$

$$-\rho_*\phi g\cos\alpha\cdot\partial_a\tilde{h} + \left(\rho_*\phi(\lambda + u_a^*\partial_a + \partial_a u_a^*) \pm \frac{(\partial_I\tau)^* I^*}{2\|D^*\|}\partial_{aa}\right)\tilde{u}_a + \left[\frac{\rho_*\phi\partial_\theta u_a^*}{a^*\cos\alpha + h^*\sin\alpha} \pm \frac{ik_\theta(\partial_I\tau)^* I^*}{2\|D^*\|}\frac{\cos\alpha - (a^*\cos\alpha + h^*\sin\alpha)\partial_a}{(a^*\cos\alpha + h^*\sin\alpha)^2}\right]\tilde{u}_\theta + \frac{ik_\theta(2\tau^*)\tilde{\psi}}{a^*\cos\alpha + h^*\sin\alpha} = 0 \tag{49}$$

$$-\frac{ik_\theta\rho_*\phi g\cos\alpha}{a^*\cos\alpha + h^*\sin\alpha}\tilde{h} \mp \frac{ik_\theta(\partial_I\tau)^* I^*/(2\|D^*\|)}{a^*\cos\alpha + h^*\sin\alpha}\partial_a\tilde{u}_a + \rho_*\phi\left(\lambda + u_a^*\partial_a + \frac{\cos\alpha\cdot u_a^*}{a^*\cos\alpha + h^*\sin\alpha}\right)\tilde{u}_\theta \pm \frac{-k_\theta^2(\partial_I\tau)^* I^*/(2\|D^*\|)}{(a^*\cos\alpha + h^*\sin\alpha)^2}\tilde{u}_\theta + 2\tau^*\partial_a\tilde{\psi} = 0 \tag{50}$$

$$\frac{ik_\theta\tilde{u}_a}{a^*\cos\alpha + h^*\sin\alpha} - \partial_a\tilde{u}_\theta \pm 4\left\|D^*\right\|\tilde{\psi} = 0 \tag{51}$$

To simplify the equations, we introduce **new notations** as

$$A = \frac{1}{a^*\cos\alpha + h^*\sin\alpha}, \quad B = \rho_*\phi, \quad C = \pm\frac{(\partial_I\tau)^* I^*}{2\|D^*\|}, \quad D = \pm 4\|D^*\| \tag{52}$$

And we can get:

$$\left(\lambda + u_a^*\partial_a + A\cos\alpha\cdot u_a^*\right)\tilde{h} + (\partial_a h^* + A\cos\alpha\cdot h^* + h^*\partial_a)\tilde{u}_a + (A\partial_\theta h^* + ik_\theta A h^*)\tilde{u}_\theta = 0 \tag{53}$$

$$-Bg\cos\alpha\cdot\partial_a\tilde{h} + \left[B(\lambda + u_a^*\partial_a + \partial_a u_a^*) + C\partial_{aa}\right]\tilde{u}_a + \left[AB\partial_\theta u_a^* + ik_\theta(A^2C\cos\alpha - AC\partial_a)\right]\tilde{u}_\theta + ik_\theta A(2\tau^*)\tilde{\psi} = 0 \tag{54}$$

$$-ik_\theta ABg\cos\alpha\cdot\tilde{h} - ik_\theta AC\partial_a\tilde{u}_a + \left[B(\lambda + u_a^*\partial_a + A\cos\alpha u_a^*) - k_\theta^2 A^2 C\right]\tilde{u}_\theta + 2\tau^*\partial_a\tilde{\psi} = 0 \tag{55}$$

$$ik_\theta A\tilde{u}_a - \partial_a\tilde{u}_\theta + D\tilde{\psi} = 0, \quad \text{i.e.} \quad \tilde{u}_a = \frac{\partial_a\tilde{u}_\theta - D\tilde{\psi}}{ik_\theta A} \tag{56}$$

A. Non-dimensional version (2022.3.21 Updated)

Denote

$$a = L\tilde{a}, \quad \partial_a = \frac{1}{L}\partial_{\tilde{a}}, \quad t = T\tau, \quad \partial_t = \frac{1}{T}\partial_\tau, \quad h = L\tilde{h}, \quad u_a = U\tilde{u}_a, \quad u_\theta = U\tilde{u}_\theta. \tag{57}$$

Then, the original system becomes

$$\frac{L}{UT}\partial_t \tilde{h} + \tilde{h}\partial_{\tilde{a}}\tilde{u}_a + \tilde{u}_a\partial_{\tilde{a}}\tilde{h} + \frac{\partial_\theta(\tilde{u}_\theta\tilde{h}) + \cos\alpha(\tilde{u}_a\tilde{h})}{\tilde{a}\cos\alpha + \tilde{h}\sin\alpha} = 0 \tag{58}$$

$$\frac{U}{gT}\partial_t\tilde{u}_a + \frac{U^2}{gL}\tilde{u}_a\partial_{\tilde{a}}\tilde{u}_a + \frac{U^2}{gL}\frac{\tilde{u}_\theta\partial_\theta\tilde{u}_a - \cos\alpha(\tilde{u}_\theta^2)}{\tilde{a}\cos\alpha + \tilde{h}\sin\alpha} - \cos\alpha(\partial_{\tilde{a}}\tilde{h}) + \frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\partial_{\tilde{a}}(\tilde{\tau}\cos(2\psi)) + \frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\frac{\partial_\theta(\tilde{\tau}\sin(2\psi))}{\tilde{a}\cos\alpha + \tilde{h}\sin\alpha} - \sin\alpha = 0 \tag{59}$$

$$\frac{U}{gT}\partial_t\tilde{u}_\theta + \frac{U^2}{gL}\tilde{u}_a\partial_{\tilde{a}}\tilde{u}_\theta + \frac{U^2}{gL}\frac{\tilde{u}_\theta\partial_\theta\tilde{u}_\theta - \cos\alpha(\tilde{u}_a\tilde{u}_\theta)}{\tilde{a}\cos\alpha + \tilde{h}\sin\alpha} - \frac{\cos\alpha(\partial_\theta\tilde{h})}{\tilde{a}\cos\alpha + \tilde{h}\sin\alpha} + \frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\partial_{\tilde{a}}(\tilde{\tau}\sin(2\psi)) + \frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\frac{\partial_\theta(\tilde{\tau}\cos(2\psi))}{\tilde{a}\cos\alpha + \tilde{h}\sin\alpha} = 0 \tag{60}$$

$$\partial_{\tilde{a}}\tilde{u}_a\sin(2\psi) - \frac{\partial_\theta\tilde{u}_a\cos(2\psi)}{\tilde{a}\cos\alpha + \tilde{h}\sin\alpha} = 0 \tag{61}$$

$$\tau = 4k_\tau\rho_* d^2\,\|D\|^2 = \rho_* U^2\frac{k_\tau d^2}{L^2}\tilde{\tau} \tag{62}$$

By grouping the non-dimensional group, we can get:

$$\frac{L}{UT}\partial_t \tilde{h} + \tilde{h}\partial_{\tilde{a}}\tilde{u}_a + \tilde{u}_a\partial_{\tilde{a}}\tilde{h} + \frac{\partial_\theta(\tilde{u}_\theta\tilde{h}) + \cos\alpha(\tilde{u}_a\tilde{h})}{\tilde{a}\cos\alpha + \tilde{h}\sin\alpha} = 0 \tag{63}$$

$$\frac{U}{gT}\partial_t\tilde{u}_a + \frac{U^2}{gL}\left(\tilde{u}_a\partial_{\tilde{a}}\tilde{u}_a + \frac{\tilde{u}_\theta\partial_\theta\tilde{u}_a - \cos\alpha(\tilde{u}_\theta^2)}{\tilde{a}\cos\alpha + \tilde{h}\sin\alpha}\right) + \frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\left(\partial_{\tilde{a}}(\tilde{\tau}\cos(2\psi)) + \frac{\partial_\theta(\tilde{\tau}\sin(2\psi))}{\tilde{a}\cos\alpha + \tilde{h}\sin\alpha}\right) - \cos\alpha(\partial_{\tilde{a}}\tilde{h}) - \sin\alpha = 0 \tag{64}$$

$$\frac{U}{gT}\partial_t\tilde{u}_\theta + \frac{U^2}{gL}\left(\tilde{u}_a\partial_{\tilde{a}}\tilde{u}_\theta + \frac{\tilde{u}_\theta\partial_\theta\tilde{u}_\theta - \cos\alpha(\tilde{u}_a\tilde{u}_\theta)}{\tilde{a}\cos\alpha + \tilde{h}\sin\alpha}\right) + \frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\left(\partial_{\tilde{a}}(\tilde{\tau}\sin(2\psi)) + \frac{\partial_\theta(\tilde{\tau}\cos(2\psi))}{\tilde{a}\cos\alpha + \tilde{h}\sin\alpha}\right) - \frac{\cos\alpha(\partial_\theta\tilde{h})}{\tilde{a}\cos\alpha + \tilde{h}\sin\alpha} = 0 \tag{65}$$

$$\partial_{\tilde{a}}\tilde{u}_a\sin(2\psi) - \frac{\partial_\theta\tilde{u}_a\cos(2\psi)}{\tilde{a}\cos\alpha + \tilde{h}\sin\alpha} = 0 \tag{66}$$

$$\tau = 4k_\tau\rho_* d^2\,\|D\|^2 = \rho_* U^2\frac{k_\tau d^2}{L^2}\tilde{\tau} \tag{67}$$

Later on for simplification ,we still use $a, h, u_a, u_\theta, \tau$ to denote the non-dimensional variables.

$$\frac{L}{UT}\partial_t h + h\partial_a u_a + u_a\partial_a h + \frac{\partial_\theta(u_\theta h) + \cos\alpha(u_a h)}{a\cos\alpha + h\sin\alpha} = 0 \tag{68}$$

$$\frac{U}{gT}\partial_t u_a + \frac{U^2}{gL}\left(u_a\partial_a u_a + \frac{u_\theta\partial_\theta u_a - \cos\alpha(u_\theta^2)}{a\cos\alpha + h\sin\alpha}\right) + \frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\left(\partial_a(\tau\cos(2\psi)) + \frac{\partial_\theta(\tau\sin(2\psi))}{a\cos\alpha + h\sin\alpha}\right) - \cos\alpha(\partial_a h) - \sin\alpha = 0 \tag{69}$$

$$\frac{U}{gT}\partial_t u_\theta + \frac{U^2}{gL}\left(u_a\partial_a u_\theta + \frac{u_\theta\partial_\theta u_\theta - \cos\alpha(u_a u_\theta)}{a\cos\alpha + h\sin\alpha}\right) + \frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\left(\partial_a(\tau\sin(2\psi)) + \frac{\partial_\theta(\tau\cos(2\psi))}{a\cos\alpha + h\sin\alpha}\right) - \frac{\cos\alpha(\partial_\theta h)}{a\cos\alpha + h\sin\alpha} = 0 \tag{70}$$

$$\partial_a u_a \sin(2\psi) - \frac{\partial_\theta u_a\cos(2\psi)}{a\cos\alpha + h\sin\alpha} = 0 \tag{71}$$

$$\tau = 4\,\|D\|^2 \quad (\|D\|^2 \text{ is also non-dimensional}) \tag{72}$$

And the linearized version is:

$$\left(\frac{L}{UT}\partial_t + u_a^*\partial_a + \frac{u_\theta^*\partial_\theta}{a^*\cos\alpha + h^*\sin\alpha} + \partial_a u_a^* + \frac{\partial_\theta u_\theta^* + \cos\alpha u_a^*}{a^*\cos\alpha + h^*\sin\alpha}\right)\hat{h} + \left(h^*\partial_a + \partial_a h^* + \frac{\cos\alpha h^*}{a^*\cos\alpha + h^*\sin\alpha}\right)\hat{u}_a + \frac{h^*\partial_\theta + \partial_\theta h^*}{a^*\cos\alpha + h^*\sin\alpha}\hat{u}_\theta = 0 \tag{73}$$

$$\begin{aligned}
&\left[\frac{U}{gT}\partial_t + \frac{U^2}{gL}\left(u_a^*\partial_a + \frac{u_\theta^*\partial_\theta}{a^*\cos\alpha + h^*\sin\alpha} + \partial_a u_a^*\right)\right]\hat{u}_a + \frac{U^2}{gL}\frac{\partial_\theta u_a^* - 2\cos\alpha u_\theta^*}{a^*\cos\alpha + h^*\sin\alpha}\hat{u}_\theta \\
&\quad + 2\frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\left(\tau^*\sin(2\psi^*)\partial_a + \frac{\tau^*\cos(2\psi^*)\partial_\theta}{a^*\cos\alpha + h^*\sin\alpha} + \partial_a(\tau^*\sin(2\psi^*)) + \partial_\theta(\frac{\tau^*\cos(2\psi^*)}{a^*\cos\alpha + h^*\sin\alpha})\right)\hat{\psi} \\
&\quad + \frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\left(\cos(2\psi^*)\partial_a + \frac{\sin(2\psi^*)\partial_\theta}{a^*\cos\alpha + h^*\sin\alpha} + \partial_a(\cos(2\psi^*)) + \frac{\partial_\theta(\sin(2\psi^*))}{a^*\cos\alpha + h^*\sin\alpha}\right)\hat{\tau} - \cos\alpha\partial_a\hat{h} = 0
\end{aligned} \tag{74}$$

$$\begin{aligned}
&\left[\frac{U}{gT}\partial_t + \frac{U^2}{gL}\left(u_a^*\partial_a + \frac{u_\theta^*\partial_\theta}{a^*\cos\alpha + h^*\sin\alpha} + \frac{\partial_\theta u_\theta^* - \cos\alpha u_a^*}{a^*\cos\alpha + h^*\sin\alpha}\right)\right]\hat{u}_\theta + \frac{U^2}{gL}\left(\partial_a u_\theta^* - \frac{\cos\alpha u_\theta^*}{a^*\cos\alpha + h^*\sin\alpha}\right)\hat{u}_a \\
&\quad + 2\frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\left(\tau^*\cos(2\psi^*)\partial_a + \frac{\tau^*\sin(2\psi^*)\partial_\theta}{a^*\cos\alpha + h^*\sin\alpha} + \partial_a(\tau^*\cos(2\psi^*)) + \partial_\theta(\frac{\tau^*\sin(2\psi^*)}{a^*\cos\alpha + h^*\sin\alpha})\right)\hat{\psi} \\
&\quad + \frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\left(\sin(2\psi^*)\partial_a + \frac{\cos(2\psi^*)\partial_\theta}{a^*\cos\alpha + h^*\sin\alpha} + \partial_a(\sin(2\psi^*)) + \frac{\partial_\theta(\cos(2\psi^*))}{a^*\cos\alpha + h^*\sin\alpha}\right)\hat{\tau} - \frac{\sin\alpha\partial_a\hat{h}}{a^*\cos\alpha + h^*\sin\alpha} = 0
\end{aligned} \tag{75}$$

$$\left(\sin(2\psi^*)\partial_a - \frac{\cos(2\psi^*)\partial_\theta}{a^*\cos\alpha + h^*\sin\alpha}\right)\hat{u}_a + 2\left(\cos(2\psi^*)\partial_a u_a^* - \frac{\sin(2\psi^*)\partial_\theta u_a^*}{a^*\cos\alpha + h^*\sin\alpha}\right)\hat{\psi} = 0 \tag{76}$$

$$\hat{\tau} = 4\left(\left\|D^* + \hat{D}\right\|^2 - \|D^*\|^2\right) \tag{77}$$

Steady state solution ($\partial_t = 0, \partial_\theta = 0, u_\theta = 0$):

$$h^* \partial_a u_a^* + u_a^* \partial_a h^* + \frac{\cos\alpha (u_a^* h^*)}{a\cos\alpha + h\sin\alpha} = 0 \tag{78}$$

$$\frac{U^2}{gL} u_a^* \partial_a u_a^* + \frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\partial_a(\tau^* \cos(2\psi^*)) - \cos\alpha(\partial_a h^*) - \sin\alpha = 0 \tag{79}$$

$$\frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\partial_a(\tau^* \sin(2\psi^*)) = 0 \tag{80}$$

$$\partial_a u_a^* \sin(2\psi^*) = 0 \tag{81}$$

$$\tau^* = 4\,\|D^*\|^2 \quad (\|D^*\|^2 \text{ is also non-dimensional}) \tag{82}$$

It is easy to find that $\psi^* = 0$, and the system can be reduced to

$$\tau^* = (\partial_a u_a^*)^2 \tag{83}$$

$$h^* \partial_a u_a^* + u_a^* \partial_a h^* + \frac{\cos\alpha (u_a^* h^*)}{a^*\cos\alpha + h^*\sin\alpha} = 0 \tag{84}$$

$$\frac{U^2}{gL}\left(u_a^* + \frac{2k_\tau d^2}{\phi L^2}\partial_{aa} u_a^*\right)\partial_a u_a^* - \cos\alpha(\partial_a h^*) - \sin\alpha = 0 \tag{85}$$

When $h^* \ll a^*$ and $\partial_a h^* \ll 1$, the above system can be reduced to

$$\partial_a(h^* \cdot u_a^* \cdot a^*) = 0 \tag{86}$$

$$\frac{U^2}{gL}\left(u_a^* + \frac{2k_\tau d^2}{\phi L^2}\partial_{aa} u_a^*\right)\partial_a u_a^* - \sin\alpha = 0 \tag{87}$$

CASE 1: Suppose: $u_a^* = C_0 + C_1 a^*$, where $C_1^2 a^* \ll 1$, then it should satisfy:

$$C_0 C_1 = \frac{\sin\alpha}{U^2/gL} \tag{88}$$

CASE 2: Suppose: $\frac{k_\tau d^2}{\phi L^2} \ll 1$, then the have

$$u_a^* = \sqrt{\frac{2(\sin\alpha \cdot a^* + C_0)}{U^2/(gL)}}, \quad \partial_a u_a^* = \frac{2\sin\alpha}{\sqrt{2\frac{U^2}{gL}(\sin\alpha \cdot a^* + C_0)}} \tag{89}$$

where $C_0 > 0$.

By plugging Eq (47) into the linearized equation system, we can get:

$$\left(\frac{L}{UT}\lambda + iu_a^* k_a + \partial_a u_a^* + \frac{\cos\alpha u_a^*}{a^*\cos\alpha + h^*\sin\alpha}\right)|\hat{h}| + \left(ih^* k_a + \partial_a h^* + \frac{\cos\alpha h^*}{a^*\cos\alpha + h^*\sin\alpha}\right)|\hat{u}_a| + \frac{ih^* k_\theta}{a^*\cos\alpha + h^*\sin\alpha}|\hat{u}_\theta| = 0 \tag{90}$$

$$\begin{aligned}
&\left[\frac{U}{gT}\lambda + \frac{U^2}{gL}(iu_a^* k_a + \partial_a u_a^*)\right]|\hat{u}_a| + 2\frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\left(\frac{i\tau^* k_\theta}{a^*\cos\alpha + h^*\sin\alpha}\right)|\hat{\psi}| + \frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\partial_a\hat{\tau} - i\cos\alpha k_a|\hat{h}| = 0\\
&\left[\frac{U}{gT}\lambda + \frac{U^2}{gL}\left(iu_a^* k_a - \frac{\cos\alpha u_a^*}{a^*\cos\alpha + h^*\sin\alpha}\right)\right]|\hat{u}_\theta| + 2\frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}(i\tau^* k_a + \partial_a(\tau^*))|\hat{\psi}| + \frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\left(\frac{\partial_\theta}{a^*\cos\alpha + h^*\sin\alpha}\right)\hat{\tau} - \frac{i\sin\alpha k_a|\hat{h}|}{a^*\cos\alpha + h^*\sin\alpha} = 0\\
&\frac{-ik_\theta}{a^*\cos\alpha + h^*\sin\alpha}|\hat{u}_a| + 2\partial_a u_a^*|\hat{\psi}| = 0
\end{aligned} \tag{91}$$

$$\hat{\tau} = 4\left(\left\|D^* + \hat{D}\right\|^2 - \|D^*\|^2\right) = 2\left(\partial_a u_a^* - \frac{\partial_\theta u_\theta^*}{a^*\cos\alpha + h^*\sin\alpha}\right)\left(\partial_a\hat{u}_a - \frac{\partial_\theta\hat{u}_\theta}{a^*\cos\alpha + h^*\sin\alpha}\right) = 2\partial_a u_a^*\left(ik_a|\hat{u}_a| - \frac{ik_\theta|\hat{u}_\theta|}{a^*\cos\alpha + h^*\sin\alpha}\right) \tag{92}$$

it can be simplified as **(adding viscous effects)**

$$\left(\frac{L}{UT}\lambda + iu_a^* k_a + \partial_a u_a^* + \frac{\cos\alpha u_a^*}{a^*\cos\alpha + h^*\sin\alpha}\right)|\hat{h}| + \left(ih^* k_a + \partial_a h^* + \frac{\cos\alpha h^*}{a^*\cos\alpha + h^*\sin\alpha}\right)|\hat{u}_a| + \frac{ih^* k_\theta}{a^*\cos\alpha + h^*\sin\alpha}|\hat{u}_\theta| = 0 \tag{93}$$

$$-i\cos\alpha k_a|\hat{h}| + \left[\frac{U}{gT}\lambda + \frac{U^2}{gL}\left(iu_a^* k_a + \partial_a u_a^* + 2i\frac{k_\tau d^2}{\phi L^2}\partial_{aa}u_a^* k_a + \frac{\partial_{aa}}{Re} + \frac{\partial_{\theta\theta}}{Re\cdot a^2}\right)\right]|\hat{u}_a| - 2i\frac{U}{gT}\frac{k_\tau d^2}{\phi L^2}\partial_{aa}u_a^* k_\theta|\hat{u}_\theta| + 2\frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\left(\frac{i\tau^* k_\theta}{a^*\cos\alpha + h^*\sin\alpha}\right)|\hat{\psi}| = 0 \tag{94}$$

$$-\frac{i\sin\alpha k_a|\hat{h}|}{a^*\cos\alpha + h^*\sin\alpha} + \left[\frac{U}{gT}\lambda + \frac{U^2}{gL}\left(iu_a^* k_a - \frac{\cos\alpha u_a^*}{a^*\cos\alpha + h^*\sin\alpha} + \frac{\partial_{aa}}{Re} + \frac{\partial_{\theta\theta}}{Re\cdot a^2}\right)\right]|\hat{u}_\theta| + 2\frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}(i\tau^* k_a + 2(\partial_a u_a^*)(\partial_{aa}u_a^*))|\hat{\psi}| = 0 \tag{95}$$

$$\frac{-ik_\theta}{a^*\cos\alpha + h^*\sin\alpha}|\hat{u}_a| + 2\partial_a u_a^*|\hat{\psi}| = 0 \tag{96}$$

The matrix is

$$
\begin{aligned}
M &= \begin{bmatrix}
\frac{L}{UT}\lambda + iu_a^* k_a + \partial_a u_a^* + \frac{\cos\alpha u_a^*}{a^*\cos\alpha + h^*\sin\alpha} & ih^* k_a + \partial_a h^* + \frac{\cos\alpha h^*}{a^*\cos\alpha + h^*\sin\alpha} & \frac{ih^* k_\theta}{a^*\cos\alpha + h^*\sin\alpha} & 0 \\
-i\cos\alpha k_a & \frac{U}{gT}\lambda + \frac{U^2}{gL}\left(iu_a^* k_a + \partial_a u_a^* + 2i\frac{k_\tau d^2}{\phi L^2}\partial_{aa}u_a^* k_a\right) & -2i\frac{U}{gT}\frac{k_\tau d^2}{\phi L^2}\partial_{aa}u_a^* k_\theta & +2\frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\left(\frac{i\tau^* k_\theta}{a^*\cos\alpha + h^*\sin\alpha}\right) \\
-\frac{i\sin\alpha k_a}{a^*\cos\alpha + h^*\sin\alpha} & 0 & \frac{U}{gT}\lambda + \frac{U^2}{gL}\left(iu_a^* k_a - \frac{\cos\alpha u_a^*}{a^*\cos\alpha + h^*\sin\alpha}\right) & 2\frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\left(i\tau^* k_a + 2(\partial_a u_a^*)(\partial_{aa}u_a^*)\right) \\
0 & \frac{-ik_\theta}{a^*\cos\alpha + h^*\sin\alpha} & 0 & 2\partial_a u_a^*
\end{bmatrix} \\
&= \begin{bmatrix}
\frac{L}{UT}\lambda + iu_a^* k_a + \partial_a u_a^* + \frac{\cos\alpha u_a^*}{a^*\cos\alpha + h^*\sin\alpha} & ih^* k_a + \partial_a h^* + \frac{\cos\alpha h^*}{a^*\cos\alpha + h^*\sin\alpha} & \frac{ih^* k_\theta}{a^*\cos\alpha + h^*\sin\alpha} \\
-i\cos\alpha k_a & \frac{U}{gT}\lambda + \frac{U^2}{gL}\left(iu_a^* k_a + \partial_a u_a^* + 2i\frac{k_\tau d^2}{\phi L^2}\partial_{aa}u_a^* k_a - \frac{k_\tau d^2}{\phi L^2}\frac{\partial_a u_a^* k_\theta^2}{a^*\cos\alpha + h^*\sin\alpha}\right) & -2i\frac{U}{gT}\frac{k_\tau d^2}{\phi L^2}\partial_{aa}u_a^* k_\theta \\
-\frac{i\sin\alpha k_a}{a^*\cos\alpha + h^*\sin\alpha} & 2\frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\frac{-(\partial_a u_a^*)k_a k_\theta + 2(\partial_{aa}u_a^*)}{2(a^*\cos\alpha + h^*\sin\alpha)} & \frac{U}{gT}\lambda + \frac{U^2}{gL}\left(iu_a^* k_a - \frac{\cos\alpha u_a^*}{a^*\cos\alpha + h^*\sin\alpha}\right)
\end{bmatrix} \\
&= \begin{bmatrix}
\frac{L}{UT}\lambda + iu_a^* k_a + \frac{\cos\alpha u_a^*}{a^*\cos\alpha + h^*\sin\alpha} & ih^* k_a + \frac{\cos\alpha h^*}{a^*\cos\alpha + h^*\sin\alpha} & \frac{ih^* k_\theta}{a^*\cos\alpha + h^*\sin\alpha} \\
-i\cos\alpha k_a & \frac{U}{gT}\lambda + \frac{U^2}{gL}\left(iu_a^* k_a + 2i\frac{k_\tau d^2}{\phi L^2}\partial_{aa}u_a^* k_a - \frac{k_\tau d^2}{\phi L^2}\frac{\partial_a u_a^* k_\theta^2}{a^*\cos\alpha + h^*\sin\alpha}\right) & -2i\frac{U}{gT}\frac{k_\tau d^2}{\phi L^2}\partial_{aa}u_a^* k_\theta \\
-\frac{i\sin\alpha k_a}{a^*\cos\alpha + h^*\sin\alpha} & 2\frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\frac{-(\partial_a u_a^*)k_a k_\theta}{2(a^*\cos\alpha + h^*\sin\alpha)} & \frac{U}{gT}\lambda + \frac{U^2}{gL}\left(iu_a^* k_a - \frac{\cos\alpha u_a^*}{a^*\cos\alpha + h^*\sin\alpha}\right)
\end{bmatrix} \\
&= \begin{bmatrix}
\lambda + iu_a^* k_a + \frac{u_a^*}{a^*} & ih^* k_a & \frac{ih^* k_\theta}{a^*\cos\alpha} \\
-i\cos\alpha k_a & \frac{U^2}{gL}\left(\lambda + iu_a^* k_a + 2i\frac{k_\tau d^2}{\phi L^2}\partial_{aa}u_a^* k_a - \frac{k_\tau d^2}{\phi L^2}\frac{\partial_a u_a^* k_\theta^2}{a^*\cos\alpha}\right) & -2i\frac{k_\tau d^2}{\phi L^2}\partial_{aa}u_a^* k_\theta \\
-\frac{i\sin\alpha k_a}{a^*\cos\alpha} & 2\frac{U^2}{gL}\frac{k_\tau d^2}{\phi L^2}\frac{-(\partial_a u_a^*)k_a k_\theta}{2a^*\cos\alpha} & \frac{U^2}{gL}\left(\lambda + iu_a^* k_a - \frac{u_a^*}{a^*}\right)
\end{bmatrix} \\
&\equiv \begin{bmatrix}
\lambda + iu_a^* k_a + \frac{u_a^*}{a^*} & ih^* k_a & \frac{ih^* k_\theta}{a^*\cos\alpha} \\
-i\frac{gL}{U^2}\cos\alpha k_a & \lambda + iu_a^* k_a + 2i\frac{k_\tau d^2}{\phi L^2}\partial_{aa}u_a^* k_a - \frac{k_\tau d^2}{\phi L^2}\frac{\partial_a u_a^* k_\theta^2}{a^*\cos\alpha} & -2i\frac{gL}{U^2}\frac{k_\tau d^2}{\phi L^2}\partial_{aa}u_a^* k_\theta \\
-\frac{gL}{U^2}\frac{i\sin\alpha k_a}{a^*\cos\alpha} & 2\frac{k_\tau d^2}{\phi L^2}\frac{-(\partial_a u_a^*)k_a k_\theta}{2a^*\cos\alpha} & \lambda + iu_a^* k_a - \frac{u_a^*}{a^*}
\end{bmatrix}
\end{aligned}
\tag{97}
$$

Therefore, the solution for $\det M = 0$ is:

Reference scale:

$$U = 3.5 \times 10^{-1}\,\mathrm{m/s}, \quad L = 6.558 \times 10^{-2}\,\mathrm{m}, \quad g = 9.8\,\mathrm{m/s}^2 \tag{98}$$

then, we have

$$\frac{L}{U} = 1.87 \times 10^{-1}\,\mathrm{s}, \quad \frac{U}{g} = 3.57 \times 10^{-2}\,\mathrm{s}, \quad \frac{U^2}{gL} = 1.91 \times 10^{-1}\,\mathrm{s} \tag{99}$$