Chapter 2
Derivation of the Shallow Water Equations

2.1 Introduction

This chapter presents a derivation of the shallow water equations, which describe flows where horizontal length-scales are much larger than the vertical. This occurs when hydrodynamic processes are small in comparison to gravity effects, and is therefore appropriate for tidal flows in channels where the tidal wavelength λ_w is greater than the channel depth h, such that $\lambda_w/h \ll 2\pi$.

The shallow water equations are commonly used in tidal hydrodynamic modelling [2] because they provide a greatly simplified set of governing equations, thereby improving ease of computation. This however comes at the cost of accuracy as to the resolution of vertical flow features. Depth-averaged models are unable to simulate secondary flows and shear stresses in the water column as they do not model vertical momentum transfer and thus do not account for phenomena resulting from these such as separation and horizontal mixing of flow due to complex bathymetries or the presence of tidal turbines [4, 10, 12]. However, in the absence of extreme stratification, bulk flow phenomena are well modelled and hence the shallow water equations provide a good estimate for tidal power resource assessments of large turbine arrays, for example, spanning the entire width of a channel because the power is dependent on bulk flow rather than local spatial gradients [2].

2.2 Shallow Water Equations

The shallow water equations may be derived by depth integrating the continuity and Navier-Stokes equations (see for example [5]) or from mass and momentum conservation analysis applied to a control volume of infinitesimal plan area extending through the whole depth of the fluid column (see Abbott [1]). The following is a brief presentation of the derivation following the former approach.

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For an incompressible and Newtonian fluid, the continuity equation $\partial\rho/\partial t + \nabla \cdot (\rho\mathbf{u}_*) = 0$ may be reduced to

$$\frac{\partial u_*}{\partial x} + \frac{\partial v_*}{\partial y} + \frac{\partial w_*}{\partial z} = 0, \tag{2.1}$$

where $\mathbf{u}_* = (u_*, v_*, w_*)^T$ is the three-dimensional instantaneous velocity vector with components in the $\mathbf{x} = (x, y, z)^T$ direction, and ρ is fluid density. This equation describes the conservation of mass in a fluid of constant and uniform density.

Conservation of momentum is commonly expressed by the Navier-Stokes equations. In non-conservative form, these may be expressed as

$$\frac{\partial u_*}{\partial t} + u_*\frac{\partial u_*}{\partial x} + v_*\frac{\partial u_*}{\partial y} + w_*\frac{\partial u_*}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\nabla^2 u_*, \tag{2.2a}$$

$$\frac{\partial v_*}{\partial t} + u_*\frac{\partial v_*}{\partial x} + v_*\frac{\partial v_*}{\partial y} + w_*\frac{\partial v_*}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\nabla^2 v_*, \tag{2.2b}$$

$$\frac{\partial w_*}{\partial t} + u_*\frac{\partial w_*}{\partial x} + v_*\frac{\partial w_*}{\partial y} + w_*\frac{\partial w_*}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\nabla^2 w_*, \tag{2.2c}$$

where t is time, p is pressure, μ the dynamic viscosity, and $\mathbf{X} = (X, Y, Z)^T$ represents the body force vector. The equations express the balance between local and convective acceleration of a fluid parcel, body forces, pressure gradient, and viscous momentum diffusion. For examples of a detailed derivation of the Navier-Stokes equations, see Hughes and Brighton [6] and White [13].

By multiplying the incompressible continuity equation (2.1) with u_* and adding the resulting expression to the x-momentum equation (2.2a), the conservative form of the momentum balance in the x-direction can be obtained after some rearranging:

$$\frac{\partial u_*}{\partial t} + \frac{\partial u_*^2}{\partial x} + \frac{\partial u_* v_*}{\partial y} + \frac{\partial u_* w_*}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\nabla^2 u_*. \tag{2.3}$$

Equivalent expressions may be obtained in the y- and z-directions.

The non-conservative momentum equations (2.2) are given in terms of primitive variables (velocity components) which are not conserved across discontinuities and do not hold in scenarios with discontinuous features. In contrast, while mathematically equivalent, the conservative x-momentum equation (2.3) (and equivalent expressions in the y- and z-directions) strictly adhere to the conservation law of momentum and hence hold across discontinuities such as hydraulic jumps. This makes the conservative formulation useful in numerical schemes.

2.2.1 Reynolds-Averaged Navier-Stokes Equations (RANS)

Because the flow in coastal basins is often turbulent (see Taylor [11]), the dependent flow variables, u_*, v_*, w_*, and p, may be separated into a time-averaged component and a randomly fluctuating component. This permits the equations to be manipulated into a form that is more tractable, following [9]. For example, the velocity component in the x-direction u, may be written as $u_* = \overline{u_*} + u'_*$, where the time-averaged component $\overline{u_*} = \frac{1}{\Delta t}\int_t^{t+\Delta t} u_*\,\mathrm{d}t$, and the randomly fluctuating component is u'_*, with $\overline{u'_*} = \frac{1}{\Delta t}\int_t^{t+\Delta t} u'_*\,\mathrm{d}t = 0$. The time period Δt is long compared to the timescale of the fluctuations but short compared to the tidal period. Inserting this decomposition into the continuity equation (2.1) and time-averaging gives

$$\frac{\partial \overline{u_*}}{\partial x} + \frac{\partial \overline{v_*}}{\partial y} + \frac{\partial \overline{w_*}}{\partial z} = 0. \tag{2.4}$$

Applying the same procedure to the conservative x-momentum equation (2.3), noting that $\overline{\overline{u_*} + u'_*} = \overline{u_*}$ and $\overline{(\overline{u_*} + u'_*)(\overline{u_*} + u'_*)} = \overline{u_*}\,\overline{u_*} + \overline{u'_* u'_*}$, gives, after some rearranging, the Reynolds-averaged Navier-Stokes (RANS) equation for incompressible turbulent flow in the x-direction:

$$\frac{\partial \overline{u_*}}{\partial t} + \frac{\partial \overline{u_*}^2}{\partial x} + \frac{\partial \overline{u_*}\,\overline{v_*}}{\partial y} + \frac{\partial \overline{u_*}\,\overline{w_*}}{\partial z} = \overline{X} - \frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \frac{1}{\rho}\left[\frac{\partial \sigma'_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right], \tag{2.5}$$

where

$$\sigma'_{xx} = \mu\frac{\partial \overline{u_*}}{\partial x} - \rho\overline{u'_* u'_*}, \quad \tau_{xy} = \mu\frac{\partial \overline{u_*}}{\partial y} - \rho\overline{u'_* v'_*}, \text{ and } \tau_{xz} = \mu\frac{\partial \overline{u_*}}{\partial z} - \rho\overline{u'_* w'_*}. \tag{2.6}$$

Here we observe Reynolds stress terms (second terms in (2.6)) that are quadratic in the fluctuating velocity components which represent interactions between the fluctuating turbulent quantities, in addition to the viscous stress terms (first terms in (2.6)). In general, the dispersive momentum transport due to turbulent eddies is much larger than due to laminar diffusion, such that the molecular viscosity contributions may be neglected.

The time-averaged body force components per unit mass, $\overline{X}$, $\overline{Y}$, and $\overline{Z}$ are given by combination of the Coriolis force acting in the xy-plane, and the effect of gravity in the downwards z-direction, i.e. $\overline{\mathbf{X}} = \overline{\mathbf{f}} \wedge \overline{\mathbf{u}_*} + \overline{\mathbf{g}}$ or (longhand)

$$\begin{pmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} \wedge \begin{pmatrix} \overline{u_*} \\ \overline{v_*} \\ \overline{w_*} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} = \begin{pmatrix} f\overline{v_*} \\ -f\overline{u_*} \\ -g \end{pmatrix}, \tag{2.7}$$

where $\wedge$ is the cross product and $f = 2\Omega_E \sin\phi$ is the Coriolis parameter resulting from the Earth's rotation, with Ω_E being the angular frequency of rotation of the Earth and ϕ the angle of latitude.

The overbar notation is subsequently dropped to avoid clutter.

2.2.2 Depth-Integration

For shallow basins where the length scales are predominantly horizontal, and particularly for tidal flows where the vertical velocity component w_* is much smaller than its horizontal counterparts u_* and v_*, the Reynolds-averaged continuity and Navier-Stokes momentum equations may be integrated over the flow depth to give a simpler system of equations whose solution yields the depth-averaged velocity field and local water level in time and space.

Taking the bed to be at $z = -z_b$ and the free surface at $z = \zeta$, where the still water level is at $z = 0$ (see Fig. 2.1), integration of the Reynolds-averaged continuity equation (2.4) over the flow depth and application of Leibniz's rule gives

$$\frac{\partial}{\partial x}\int_{-z_b}^{\zeta} u_* \,\mathrm{d}z + \frac{\partial}{\partial y}\int_{-z_b}^{\zeta} v_* \,\mathrm{d}z - \left(u_*|_\zeta \frac{\partial \zeta}{\partial x} + u_*|_{-z_b} \frac{\partial(-z_b)}{\partial x} \right) - \left(v_*|_\zeta \frac{\partial \zeta}{\partial y} + v_*|_{-z_b} \frac{\partial(-z_b)}{\partial y} \right) + \left(w_*|_\zeta - w_*|_{-z_b} \right) = 0. \quad (2.8)$$

The following boundary conditions are applied. At $z = -z_b$ the bed is assumed solid, and a no-flow kinematic boundary condition applies i.e.

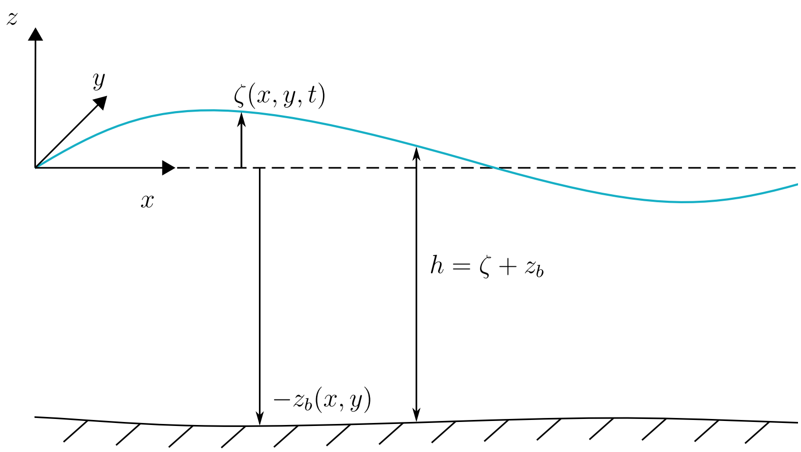

Fig. 2.1 Definition sketch for the shallow water equations

$$w_*|_{-z_b} = \frac{\mathrm{d}(-z_b)}{\mathrm{d}t} = \frac{\partial(-z_b)}{\partial t} + u_*|_{-z_b}\frac{\partial(-z_b)}{\partial x} + v_*|_{-z_b}\frac{\partial(-z_b)}{\partial y}. \tag{2.9}$$

That is, there is no flow through the solid bed. Similarly, at $z = \zeta$, the kinematic free surface boundary condition states that particles at the free surface remain there. Hence,

$$w|_\zeta = \frac{\mathrm{d}(\zeta)}{\mathrm{d}t} = \frac{\partial\zeta}{\partial t} + u_*|_\zeta\frac{\partial\zeta}{\partial x} + v|_\zeta\frac{\partial\zeta}{\partial y}. \tag{2.10}$$

Substituting the above conditions into the depth-averaged continuity equation (2.8) then gives

$$\frac{\partial(h)}{\partial t} + \frac{\partial}{\partial x}\int_{-z_b}^{\zeta} u_*\,\mathrm{d}z + \frac{\partial}{\partial y}\int_{-z_b}^{\zeta} v_*\,\mathrm{d}z = 0, \tag{2.11}$$

where $h = \zeta + z_b$. Defining the depth-averaged velocity components using the first mean value theorem as

$$u = \frac{1}{h}\int_{-z_b}^{\zeta} u_*\,\mathrm{d}z \quad \text{and} \quad v = \frac{1}{h}\int_{-z_b}^{\zeta} v_*\,\mathrm{d}z, \tag{2.12}$$

the depth-averaged continuity equation becomes

$$\frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} + \frac{\partial(vh)}{\partial y} = 0. \tag{2.13}$$

For a non-erodible bed, the position of the bed is constant with respect to time, $\partial(-z_b)/\partial t = 0$ such that $\partial h/\partial t = \partial\zeta/\partial t$ and (2.13) further reduces to

$$\frac{\partial\zeta}{\partial t} + \frac{\partial(uh)}{\partial x} + \frac{\partial(vh)}{\partial y} = 0. \tag{2.14}$$

Next consider the pressure acting on the fluid column. Neglecting vertical acceleration and shear stress terms, the z-component of the momentum conservation equation reduces to a balance between gravitational acceleration and the pressure gradient:

$$\frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0. \tag{2.15}$$

Integration over the fluid depth gives the hydrostatic pressure distribution

$$p(z) = p_a + \rho g(\zeta - z), \tag{2.16}$$

where p_a is the atmospheric pressure at the free surface $z = \zeta$. Neglecting atmospheric pressure gradients, the spatial derivatives of pressure may then be expressed in terms of water elevations, i.e.

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial \zeta}{\partial x}. \tag{2.17}$$

Repeating the depth-averaging process for the Reynolds-averaged Navier-Stokes x-momentum equation (2.5), and substituting (2.17), yields

$$\int_{-z_b}^{\zeta} \frac{\partial u_*}{\partial t}\, \mathrm{d}z + \int_{-z_b}^{\zeta} \frac{\partial {u_*}^2}{\partial x}\, \mathrm{d}z + \int_{-z_b}^{\zeta} \frac{\partial u_* v_*}{\partial y}\, \mathrm{d}z + \int_{-z_b}^{\zeta} \frac{\partial u_* w_*}{\partial z}\, \mathrm{d}z = \int_{-z_b}^{\zeta} f v\, \mathrm{d}z$$
$$-g \int_{-z_b}^{\zeta} \frac{\partial \zeta}{\partial x}\, \mathrm{d}z + \frac{1}{\rho}\left[\int_{-z_b}^{\zeta} \frac{\partial \sigma'_{xx}}{\partial x}\, \mathrm{d}z + \int_{-z_b}^{\zeta} \frac{\partial \tau_{xy}}{\partial y}\, \mathrm{d}z + \int_{-z_b}^{\zeta} \frac{\partial \tau_{xz}}{\partial z}\, \mathrm{d}z\right]. \tag{2.18}$$

Following a similar procedure as above, the application of Leibniz's rule and boundary conditions (2.9) and (2.10), gives

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\int_{-z_b}^{\zeta} u_*^2\, \mathrm{d}z + \frac{\partial}{\partial y}\int_{-z_b}^{\zeta} u_* v_*\, \mathrm{d}z = f v h - g h \frac{\partial \zeta}{\partial x}$$
$$+ \frac{\tau_{wx} - \tau_{bx}}{\rho} + \frac{1}{\rho}\left[\frac{\partial}{\partial x}\int_{-z_b}^{\zeta} \sigma'_{xx}\, \mathrm{d}z + \frac{\partial}{\partial y}\int_{-z_b}^{\zeta} \tau_{xy}\, \mathrm{d}z\right], \tag{2.19}$$

where the wind and the bottom friction stress components are defined as

$$\tau_{wx} = \sigma'_{xx}\big|_{z=\zeta} \frac{\partial \zeta}{\partial x} - \tau_{xy}\big|_{z=\zeta} \frac{\partial \zeta}{\partial y} + \tau_{xz}|_{z=\zeta}\,, \tag{2.20}$$

and

$$\tau_{bx} = \sigma'_{xx}\big|_{z=-z_b} \frac{\partial (-z_b)}{\partial x} - \tau_{xy}\big|_{z=-z_b} \frac{\partial (-z_b)}{\partial y} + \tau_{xz}|_{z=-z_b}\,, \tag{2.21}$$

It should be noted that this derivation neglects form drag at the sea bed. A more formal derivation of this may be found in Papadopoulos et al. [7] and Pokrajac [8] Defining

$$\beta_{xx} = \frac{1}{u^2 h}\int_{-z_b}^{\zeta} u_*^2\, \mathrm{d}z \quad \text{and} \quad \beta_{xy} = \frac{1}{u v h}\int_{-z_b}^{\zeta} u_* v_*\, \mathrm{d}z, \tag{2.22}$$

Equation (2.19) may be recast as

$$\frac{\partial(uh)}{\partial t} + \frac{\partial(\beta_{xx}u^2h)}{\partial x} + \frac{\partial(\beta_{xy}uvh)}{\partial y} = fvh - gh\frac{\partial\zeta}{\partial x} + \frac{\tau_{wx} - \tau_{bx}}{\rho} + \frac{1}{\rho}\left[\frac{\partial}{\partial x}\int_{-z_b}^{\zeta}\sigma_{xx}\,\mathrm{d}z + \frac{\partial}{\partial y}\int_{-z_b}^{\zeta}\tau_{xy}\,\mathrm{d}z\right], \tag{2.23}$$

The multipliers β_{xx} and β_{xy} represent correction factors resulting from dispersive horizontal momentum exchanges due to the non-uniformity of the velocity profiles over the depth of the fluid (see Falconer [5] for details). In most practical applications these multipliers are very close to unity and are hence approximated as $\beta_{xx} = \beta_{xy} = 1$ in this thesis.

Using the Boussinesq eddy viscosity hypothesis, presented by Boussinesq [3], the Reynolds stresses from (2.6) may be expressed in terms of mean velocity components:

$$\sigma'_{xx} = -\rho\overline{u'_*u'_*} = \rho\nu_t 2\frac{\partial u_*}{\partial x} \quad \text{and} \quad \tau_{xy} = -\rho\overline{u'_*v'_*} = \rho\nu_t\left(\frac{\partial u_*}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{2.24}$$

where ν_t is the turbulent eddy viscosity. The integrals of these terms may be simplified by introducing a depth-scaled turbulent eddy viscosity ν_T which allows rewriting the integrals as

$$\int_{-z_b}^{\zeta}\nu_t\frac{\partial u_*}{\partial x}\,\mathrm{d}z = \nu_T h\frac{\partial u}{\partial x}, \quad \int_{-z_b}^{\zeta}\nu_t\frac{\partial u_*}{\partial y}\,\mathrm{d}z = \nu_T h\frac{\partial u}{\partial y}, \quad \text{and} \quad \int_{-z_b}^{\zeta}\nu_t\frac{\partial v_*}{\partial x}\,\mathrm{d}z = \nu_T h\frac{\partial v}{\partial x}. \tag{2.25}$$

Hence (2.23) may be expressed as

$$\frac{\partial(uh)}{\partial t} + \frac{\partial(\beta_{xx}u^2h)}{\partial x} + \frac{\partial(\beta_{xy}uvh)}{\partial y} = fvh - gh\frac{\partial\zeta}{\partial x} + \frac{\tau_{wx} - \tau_{bx}}{\rho} + \frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(2\nu_T h\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu_T h\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right)\right]. \tag{2.26}$$

For large tidal sites where currents vary smoothly over the spatial domain, viscous and turbulence terms may be neglected as they involve second order derivatives in space of depth-averaged velocity. The two-dimensional shallow water equations may then be simplified to

$$\frac{\partial(uh)}{\partial t} + \frac{\partial u^2h}{\partial x} + \frac{\partial uvh}{\partial y} = fvh - gh\frac{\partial\zeta}{\partial x} + \frac{\tau_{wx} - \tau_{bx}}{\rho} \tag{2.27a}$$

$$\frac{\partial(vh)}{\partial t} + \frac{\partial uvh}{\partial x} + \frac{\partial v^2h}{\partial y} = -fuh - gh\frac{\partial\zeta}{\partial y} + \frac{\tau_{wy} - \tau_{by}}{\rho}. \tag{2.27b}$$

Here the left hand side expresses the local and convective acceleration of the flow and the right hand side expresses Coriolis force acceleration, the pressure head driving the flow, and the surface (wind) and bed friction stresses.

For the tidal flow applications considered in Chap. 3, we will consider a reduced form of the above equations, written in one-spatial dimension, and ignoring the terms related to turbulence and wind stress, given by

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (uh)}{\partial x} = 0, \tag{2.28}$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial (u^2 h)}{\partial x} = -gh\frac{\partial \zeta}{\partial x} - \frac{\tau_{bx}}{\rho}. \tag{2.29}$$

The bed stress is calculated using the following (commonly employed) empirical quadratic approach,

$$\tau_{bx} = \rho C_d \left| u \right| u, \tag{2.30}$$

where C_d is either prescribed, or given by the Chézy or Manning formula by

$$C_d = \sqrt{\frac{g}{C^2}} = \frac{gn^2}{h^{1/3}}, \tag{2.31}$$

where C is Chézy coefficient, and n is the Manning coefficient.

2.3 Conclusions

This chapter describes the derivation of the one-dimensional shallow water equations, in the form of (2.28) and (2.29). These equations constitute the governing equations solved numerically, as described in Chap. 3, for flow in a strait containing a fence of turbines. Further, the shallow water equations are used in Chap. 4 to derive analytic power models for flow through turbine farms modelled as enhanced bed roughness.

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