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Constant Froude number in a circular hydraulic jump and its implication on the jump radius selection

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Abstract – The properties of a standard hydraulic jump depend critically on a Froude number Fr defined as the ratio of the flow velocity to the gravity waves speed. In the case of a horizontal circular jump, the question of the Froude number is not well documented. Our experiments show that Fr measured just after the jump is locked on a constant value that does not depend on the flow rate Q, the kinematic viscosity ν and the surface tension γ. Combining this result with a lubrication description of the outer flow leads, under appropriate conditions, to a new and simple law ruling the jump radius R_J: $R_J(\ln(\frac{R_\infty}{R_J}))^{3/8} \sim Q^{5/8}\nu^{-3/8}g^{-1/8}$, in excellent agreement with our experimental data. This unexpected result raises an unsolved question to all available models.

Structure formations in free surface flows remain a major source of complexity in hydrodynamics. Perhaps the most well-known example is the "hydraulic jump", in which one observes a sudden transition from a high-speed, supercritical, open channel flow to a subcritical one, with a sudden jump of the fluid depth [1,2]. This phenomenon is ubiquitous and can be observed at very different length scales: in dam release flows [1], in tidal bores on rivers [3] or in kitchen sinks when a vertical jet of liquid hits a horizontal surface. The circular liquid wall observed in the latter has even motivated recent model experiments for astrophysics [4], to mimic the competition between the speed of a wave and that of a radial flow (white hole equivalents) [5]. The natural dimensionless number used to describe this competition is the Froude number Fr defined [6] as the ratio of the flow velocity to the gravity waves speed $Fr = U/\sqrt{gh}$, with U the average flow velocity, h the fluid thickness and g the acceleration of gravity. In general, the properties of a standard hydraulic jump depends critically on the Fr value [1], that is always larger than one upstream and smaller than one downstream, the two values being dependent on each other via the mass and momentum conservation expressed at the jump location.

As is well known, several complex physical effects are mixed, even in the reasonably simple case of a circular hydraulic jump at moderate flow rate: on the one hand,

the jump can be understood as a shock front for surface gravity waves [2], but its formation can also be understood as something similar to the growth and detachment of a boundary layer close to the solid substrate [7], with the formation of toroidal recirculations all around the front. In turn, this association between a front and a toroidal vortex can become unstable leading to surprising faceting effects [8,9].

In the present letter, we show that even this simplest form of hydraulic jump (*i.e.* the circular hydraulic jump at moderate flow rate) involves an unsuspected selection mechanism that fixes the Froude number value at the jump exit. Up to now, this discovery has been missed despite numerous studies [7,8,10–21], the question of the precise value that this number can take remained unaddressed for a circular hydraulic jump. Most of available studies have focused rather on the question of the selection of the jump radius R_J, for which two main theories are available [12,14], among a rich literature trying to improve these ones or to propose alternatives [15,16,19,21–23].

We here investigate quantitatively the Froude number selection with experiments performed in the case of a liquid jet impacting a horizontal disk, with no confinement walls imposing the outer thickness. The Froude number Fr is accurately measured in a large range of flow rates Q and appears to be a constant, independent of Q,
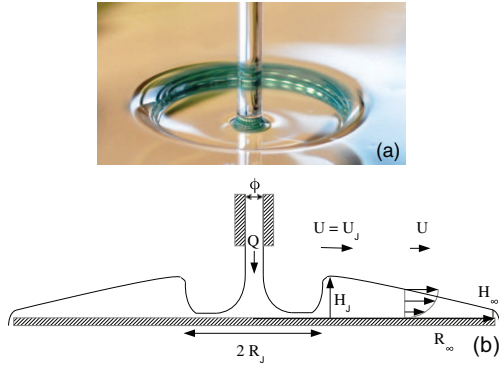
Fig. 1: (Colour on-line) (a) A jet of liquid (silicone oil (20 cS)) impacts vertically a horizontal surface. A circular hydraulic jump is observed. (b) Sketch of the experimental situation. A jet of liquid issued from a vertical tube of internal diameter $\varnothing = 3.2\,\mathrm{mm}$ hits the centre of a horizontal disk of radius $R_\infty = 15\,\mathrm{cm}$ placed $4\,\mathrm{cm}$ below the outlet. A circular hydraulic jump of radius R_J is observed. At the exit of the jump the height is H_J and the average speed U_J. There is no confinement wall on the disk perimeter, where H_∞ is the liquid thickness.

of kinematic viscosity ν and of surface tension γ. Surprisingly this result cannot be recovered from the two main theories of Watson [12] and Bohr *et al.* [14].

Furthermore, by combining this result with a simple but accurate description of the outer flow (*i.e.* the thickness distribution of liquid), we obtain a new and simple law linking R_J and Q in an excellent agreement with our data (see eq. (4), below). This law is very close to the scaling proposed by Bohr *et al.* [14], but involves an additional logarithmic dependence upon R_J that cannot be neglected. These results are consistent with several developments of Bohr *et al.*'s theory [14,22], in which logarithmic effects were identified in the outer thickness distribution and in the numerical resolution of $R_J(Q)$. However, it is the first time that a so simple, analytical, law is proposed for $R_J(Q)$ and recovered experimentally.

Our experiment is depicted in fig. 1. A jet of liquid issued from a vertical tube of internal diameter $\varnothing = 3.2\,\mathrm{mm}$, hits the centre of a transparent glass disk of radius $R_\infty = 15\,\mathrm{cm}$, placed $4\,\mathrm{cm}$ below the outlet. The disk horizontality is tuned by using micrometer screws and by observing a spirit level. Boundary conditions are key parameters for such an experiment, as experimentally studied in [7]. Concerning this question, we chose the simplest experimental setup: no wall fixes the thickness of the liquid layer outside of the jump (see fig. 1(b)); the flow is free to adjust its own height. Fixing this boundary condition and watching the trajectories of tracers in the flow, we observed that the hydraulic jump was of type I [7] (*i.e.*, unidirectional surface flow with no vortex reversing the flow at the free surface). The accessible range of flow rates and the liquid viscosities in our experiments allowed us to always study the hydraulic jump in a steady laminar state [8] (no turbulence or instabilities of the jump).

The flow was imposed by a gear pump fabricated by Micropump, of ref. 75211-67, combined with a magnetically entrained head of ref. 81282, that uses helical gears to minimize the residual flow rate pulsation. Considering the fact that we are studying only steady flows, in condition of strong damping by the liquid viscosity, this possible residual pulsation can be neglected. We have checked this in the lowest range of flow rate (5–$10\,\mathrm{cm}^3 \cdot \mathrm{s}^{-1}$) by comparing the results with those obtained with a constant level tank, without finding any noticeable difference. The flow rates are measured by a flow meter calibrated by weighing the liquid. The accessible range of flow rates (dependent on the liquid viscosity) is typically 5–$60 \pm 0.25\,\mathrm{cm}^3 \cdot \mathrm{s}^{-1}$. We measured the hydraulic jump radii by visualizing from below, through the glass plate. These measurements were carried out with an accuracy of $\pm 0.2\,\mathrm{mm}$.

Experiments were conducted with two different types of liquids: silicone oils and water-glycerine mixtures. Silicone oil has a smaller surface tension ($\sim 20\,\mathrm{mN} \cdot \mathrm{m}^{-1}$), and a density close to that of water (between 0.95 and 0.965 at $25\,^\circ\mathrm{C}$ for the different silicone oils we used). We used three different kinematic viscosities: 20.4 ± 0.6, 44.9 ± 1.5 and $98.8 \pm 3\,\mathrm{cS}$. Water-glycerine mixtures had a larger surface tension (close to $65\,\mathrm{mN} \cdot \mathrm{m}^{-1}$) and a density around 1.2 (1.19 for the lower viscosity and 1.22 for the higher one). The kinematic viscosities chosen are: 18 ± 0.7 and $44 \pm 1.5\,\mathrm{cS}$. This range of parameter allowed us to investigate the influence on the critical Fr of the three main possible parameters: kinematic viscosity, surface tension and flow rate.

In order to determine the Froude number at the exit of the jump and the outer-liquid thickness distribution, we used a vernier height gauge having a needle pointer. A vertical needle is put into contact with the liquid free surface, and then with the disk surface, the difference of heights gives the liquid thickness. Examples of the obtained results are shown in fig. 2.

The observed thickness distribution allows one to propose two possible definitions of the liquid thickness at the jump exit H_J. The first one is the value measured just after the jump H_{J1}, *i.e.* the local maximum at the jump exit, after filtering qualitatively the possible high frequency fluctuations due to error bars. The second possible definition H_{J2}, indicated by the other arrow in the same figure, is the extrapolated value at the measured jump position R_J of a calculated liquid thickness profile: the profile obtained through the lubrication description of the outer flow. The calculation is detailed below (see eq. (3)). The value of H_{J2} is always higher than H_{J1}. Typically, in the example of fig. 2, the arrows indicate a value of order $3.2\,\mathrm{mm}$ for H_{J1}, and of order $3.6\,\mathrm{mm}$ for H_{J2}, while the inflection point of the thickness profile just at the jump location is somewhere between 1.5 and 2.5 mm of height.

Both definitions allow to propose, in turn, two different definitions of the Froude number at the jump exit that reads, respectively, $Fr = U_J/\sqrt{gH_J}$, with $H_J = H_{J1}$ and $H_J = H_{J2}$, and where $U_J = Q/(2\pi R_J H_J)$ is the averaged
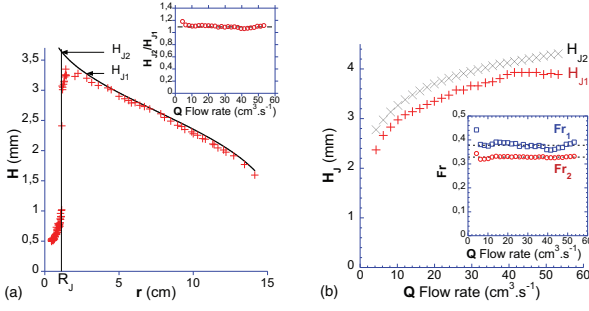
Fig. 2: (Colour on-line) (a) Liquid height profile. The liquid used is a silicone oil (20 cS). The measured flow rate Q is $17\,\text{cm}^3 \cdot \text{s}^{-1}$ and the measured thickness at the disk position $H_\infty = 1.4\,\text{mm}$. Equation (3) is plotted with a black solid line. Two values for H_J can be extracted: H_{J1} is the measured height just at the jump exit and H_{J2} the extrapolated value of the liquid thickness calculated from eq. (3) at the jump radius R_J. Insert: ratio of the two possible definition of H_J. This ratio appears to be constant and close to 1.1. (b) Outer height H_J as a function of the flow rate Q for silicone oil (20 cS). Insert: critical Froude number observed at the jump exit with the two possible definitions in panel (a).

velocity of the liquid on the liquid thickness considered, deduced from the flow rate conservation. So the two expressions for Fr are $Fr = Q/(2\pi R_J g^{1/2} H_J^{3/2})$ with $H_J = H_{J1}$ or $H_J = H_{J2}$. Both definitions have the advantage to avoid the precise knowledge of the 3D flow structure at the jump exit. The first definition, involving the experimentally measured value H_{J1}, is more natural, but the second (with $H_J = H_{J2}$) will allow us to develop analytical calculations more easily.

Surprisingly a constant Froude number Fr independent of the flow rate Q is found (see fig. 2(b), insert):

$$Fr_2 \sim 0.33 \pm 0.01, \tag{1}$$

with the second definition, while a constant value is also obtained with the first one ($Fr_1 \sim 0.38 \pm 0.02$). This second result is consistent with the fact that, experimentally, H_{J1} and H_{J2} were found to be proportional (see fig. 2(a), insert), i.e. $H_{J2} = \alpha H_{J1}$ with $\alpha \approx 1.09 \pm 0.03$. Note that the constancy of the Froude number and the momentum conservation impose the previous relation between the two heights with α depending only on the Froude numbers.

For both definitions, these values are smaller than one, as expected from the theory for a Froude number in the subcritical zone. But a Froude number fixed to a constant value appears to be unexpected and is not predicted by the two main hydraulic jump theories:

i) In its simplest form, Bohr et al.'s theory leads to a scaling $U_J \sim Q^{1/8}\nu^{1/8}g^{3/8}[4\ln(R_S/R_J)]^{1/4}$, where R_S is the radius where the singularity happened (typically the end of the plate R_∞), and a outer height $H_J \sim Q^{1/4}\nu^{1/4}g^{-1/4}[4\ln(R_S/R_J)]^{-1/4}$, which would lead to a Froude number: $Fr_{Bohr} \sim [4\ln(R_S/R_J)]^{3/8}$. In this approximation, Fr would depend on the flow rate through

R_J, which is in contradiction with our study. A more complete version of this theory is available in refs. [14,22] but is solved numerically with input constants (r_0, h_0), which have to be measured. These authors did not investigate the behaviour and possible selection of Fr upon the imposed conditions.

ii) In the case of the Bush-Watson theory the outer height is not fixed and has to be experimentally measured to determine the jump radius. So this theory lets a free parameter available and cannot predict our outer Froude number.

We now examine the implications of these results on the jump radius selection. If most of the available theories mainly target the internal flow possibly coupled to an outer flow, we propose here to calculate the thickness distribution of the liquid imposed by the lubrication theory at large radius distance r compared to the jump radius $R_J(Q)$. We match it to the constant Froude number Fr_{J2} found experimentally and obtain $R_J(Q)$.

First, the boundary conditions (no-slip condition on the substrate and free slip at the free interface) allow us to identify the outer flow to a parabolic profile flow governed by a balance between hydrostatic pressure and viscous friction. More precisely the equation governing the flow reads as follows:

$$U(r) \simeq -\frac{H(r)^2}{3\nu}g\frac{\mathrm{d}H(r)}{\mathrm{d}r}, \tag{2}$$

combining this equation with the mass conservation $Q = 2\pi r U H$ leads to the following solution:

$$H(r) = \left(H_\infty^4 + \frac{6}{\pi}\frac{\nu Q}{g}\ln\left(\frac{R_\infty}{r}\right)\right)^{1/4}, \tag{3}$$

where H_∞ is the liquid thickness at the disk perimeter ("end" of the flow for $r = R_\infty$). We have to notice that Bohr et al. [14,22] and Rojas et al. [21] have already found a similar solution but not combined, as here, with a constant Froude number.

This result can be experimentally tested: for a given flow rate we accurately measured the outer-liquid thickness profile $H(r)$ with our needle pointer. A typical example is reproduced in fig. 2(a), where the black solid line represents eq. (3) without any adjustable parameter, the H_∞ value being extracted from our measurements. An excellent agreement with our data can be observed, which is not a surprise as the lubrication equation holds rigorously far enough from the jump, the local slope of the interface and the reduced Reynolds number being very small (typically $Re' \sim QH_J/(2\pi R_\infty^2 \nu) \sim 5 \cdot 10^{-2}$ and the local slope is of the order of $2.5 \cdot 10^{-2}$).

With the silicon oils we were in the situation of total wetting on the disk. We observed that the liquid, after leaving the top surface of the disk was also wetting the lateral edges with a flowing film that presumably fixed the value of H_∞. We did not try to calculate theoretically H_∞ in these conditions, but it appeared experimentally that it was following a non-trivial but weak power law

upon the flow rate while remaining at the same order of magnitude as the silicone capillary length $l_c = 1.5\,\text{mm}$.

In these conditions, the comparison between the two terms of eq. (3) shows that H_∞^4 is negligible compared to the other term, and so we can consider that $H_J \simeq \frac{6}{\pi}\frac{\nu Q}{g}\ln(\frac{R_\infty}{R_J})$. This results holds for our range of flow rate $(5\text{--}60\,\text{cm}^3\cdot\text{s}^{-1})$ and for every viscosities we used, in this total wetting case.

Using this expression in the one given at the beginning of this paper for the Froude number (*i.e.* $Fr = Q/(2\pi R_J g^{1/2} H_J^{3/2})$) and assuming that Fr is a constant as experimentally shown, leads after some calculations to the final expression for the jump radius:

$$R_J\left(\ln\left(\frac{R_\infty}{R_J}\right)\right)^{3/8} = \beta Fr^{-1}Q^{5/8}\nu^{-3/8}g^{-1/8}, \quad (4)$$

where $\beta = (\frac{6}{\pi})^{-3/8}\frac{1}{2\pi}$. It can be noticed here that we recover Bohr scaling but with a non-negligible logarithmic dependence and the exact value of the pre factor, when one knows the experimental value of the constant Froude number Fr_2. This result is compared with the experimental data we collected using the silicone oils on fig. 3(a). All the points line and fall on the same straight line, which slope is defined by the Froude number value ($Fr_2 = 0.33$ here). To be more precise, we have tried to fit independently each set of data with (4), each of the set giving the same value of the Froude number required $Fr_2 = 0.33$.

As mentioned at the beginning of the paper, we also performed some complementary measurements of the jump radius upon the flow rate in the case of water-glycerol mixtures. The situation is less simple in this case, as these liquids are flowing on the glass under partial wetting conditions. However, provided that one works with a clean enough glass disk, all the top surface of the disk remains wetted, dewetting only occurring on the lateral edge of the disk, with formation of rivulets, these rivulets breaking a bit the axial symmetry of the flow at the disk perimeter. The total number of rivulets N depends on the total flow rate and on the kinematic viscosity used, but is always rather large (typically around $N = 20$), which limits a lot the symmetry breaking. Qualitatively, we observed that the axial symmetry was preserved everywhere, except in a very narrow region at the disk periphery, where H_∞ had to be measured.

Direct measurements of this limit thickness, performed at nearly 5 mm of the disk perimeter give a nearly constant value of 3 mm with relative fluctuations due to the vicinity of the rivulet formation that were not exceeding a few per cent. This value is consistent with the constant value H_∞ obtained by Dressaire *et al.* in [18] with a very similar setup. This constant value being very close to that of the capillary length $\sqrt{\frac{\gamma}{\rho g}} \sim 3\,\text{mm}$, we suspect that it is imposed by some balance of forces between hydrostatic pressure and surface tension at the disk perimeter, between each pair of rivulet.
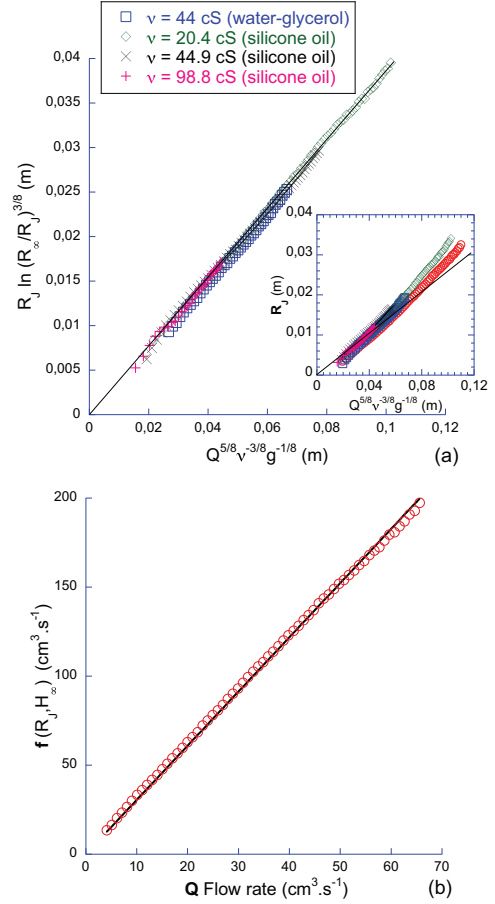

Fig. 3: (Colour on-line) (a) Equation (4) is tested experimentally for two different kind of liquids: silicone oil and water-glycerine mixture and for 4 different kinematic viscosities (20.4 cS, 44 cS, 44.9 cS and 98.8 cS). These curves, collapsing on a single master curve, are fitted by a linear function and a Froude number can be extracted ($Fr_2 \approx 0.33$). Insert: the radius of the jump R_J is plotted *vs.* the Bohr *et al.* scaling $Q^{5/8}\nu^{-3/8}g^{-1/8}$ for all the liquids presented in this figure. Such a scaling law clearly failed to describe accurately our data. (b) For lower viscosities of the water-glycerine mixture H_∞ has to be taken into account: eq. (5) is plotted for the water-glycerine mixture with a kinematic viscosity of 18 cS. We observed that $H_\infty \simeq l_c \simeq 3\,\text{mm}$. A linear function is also observed and the pre-factor leads to a $Fr_2 \approx 0.33$ which is consistent with the other results reported in panel (a).

Anyway, in these imperfect partial wetting conditions two cases have to be distinguished:

i) If the viscosity is large enough we will have $H_\infty^4 \ll \frac{6}{\pi}\frac{\nu Q}{g}\ln(\frac{R_\infty}{R_J})$ and the data should fall again on the straight line of fig. 3(a), following (4). This can be checked also in fig. 3(a), where we have also added the data obtained with a water-glycerol mixture of 45 cS. Again, the collapse of the data is excellent, with again the same Froude number deduced from the slope, *i.e.* $Fr_2 = 0.33$.

ii) If the viscosity is too low, as H_∞ has been doubled between silicon oils and glycerol-water solution, the above approximation may break as observed with a $18\,$cS mixture. In this case, (4) must be generalized, without neglecting the H_∞ term. After a straightforward calculation, we got the following equation that allows one to calculate R_J:

$$Q = Fr\, f(R_J, H_\infty), \qquad (5)$$

with $f(R_J, H_\infty) = \sqrt{g}\, 2\pi R_J (H_\infty^4 + \frac{6}{\pi}\frac{\nu Q}{g}\ln(\frac{R_\infty}{R_J}))^{3/8}$. As it was explained before, the value of H_∞ is experimentally a bit ambiguous, because of the small variations of the liquid thickness observed near the disk perimeter where rivulet formation occurs. However, to check (5), we chose to use the approximate mean value found above, *i.e.* $H_\infty = 3\,$mm. The predicted law is checked experimentally in fig. 3(b), where the lining of the points is very impressive. We found again the same outer Froude number deduced from the slope of the obtained line, *i.e.* $Fr_2 = 0.33$. We also consider the influence of the error on H_∞ for the evaluated value of the Froude number: if we consider $H_\infty = 2.5\,$mm, we obtained $Fr_2 = 0.31$ and if we choose $H_\infty = 3.5\,$mm, $Fr_2 = 0.35$ (note that in this limit case the lining of the points becomes questionable). So our experimental estimation of H_∞ appears to be also the best fit to recover the constant Froude number obtained in all the other situations.

From this detailed comparison between data and calculations, two significant lessons may be learned:

i) At first we can conclude that our description provides an analytical law in excellent agreement with experiments in all the tested cases and with the same jump exit Froude number, as expected from the matching condition. Particularly, we can observe that the effect of the viscosity is well described by the $-3/8$ exponent. More precisely, the value of the Froude number that can be extracted from all the set of data (fig. 3(a) and fig. 3(b)) is the one expected: $Fr_2 = 0.33$. The logarithmic term and the influence of H_∞ in (5) appear to be important: the scaling proposed by Bohr *et al.* is tested in the insert of fig. 3(a) for all our data set and shows clearly that it failed to predict accurately the dependence of R_J on the different physical parameters.

ii) As one can observe, the surface tension, which is the main difference between silicone oils and water-glycerol mixtures, is absent in eqs. (4) and (5). So one can conclude that the role of surface tension is negligible for large enough radius. Indeed, according to Bush *et al.*, surface tension plays a minor role in the momentum conservation at the jump level [16,23] that reads:

$$\frac{1}{2}g(H_J^2 - h_J^2) + \frac{\gamma}{\rho}\frac{H_J - h_J}{R_J} = \int_0^{h_J} u^2 \mathrm{d}z - \int_0^{H_J} U^2 \mathrm{d}z, \quad (6)$$

where h_J, u, H_J and U are, respectively, the heights and the liquid velocities at the either side of the jump and ρ the liquid density.

A scaling analysis gives an estimate for a jump radius where the surface tension term has the same weight as the others: $R_J \sim \frac{2\gamma}{\rho g(H_J + h_J)} \sim 10^{-3}\,$m (calculated for silicone oil). So we recover that for large enough radii the surface tension becomes negligible.

Our result of a fixed Froude number has numerous consequences, particularly about the inner zone of the hydraulic jump. Using the momentum conservation (cf. eq. (6)) one can predict that the Froude number just before the jump is also constant (except for low radii when surface tension has to be taken into account). In this way, the average speed before the jump u can also be estimated. This average speed u appears to be weakly dependent on the flow rate ($u \sim Q^{0.05}$). This result is consistent with a previous observation reported by the authors of ref. [24] when observing the rotation of levitating drops trapped in a circular jump, that the surface velocity at the jump entry seemed to be nearly independent of the flow rate Q.

To sum up our result, at the jump position a constant Froude number Fr independent of the flow rate, viscosity and surface tension is observed in the case of a circular hydraulic jump without any confinement wall. A lubrication description of the outer-liquid thickness profile has been proposed and is well confirmed by experimental data. This profile combined with the locking condition on the Froude number leads to a law linking the radius of the jump and the experimental conditions: $R_J(\ln(\frac{R_\infty}{R_J}))^{3/8} \sim Q^{5/8}\nu^{-3/8}g^{-1/8}$, valid for a large range of experimental conditions. This law was experimentally tested and our data show a good agreement with our theoretical investigation. This law recovers the scaling expected by Bohr *et al.* but with a non-negligible logarithmic correction.

The origin of a constant Froude number is still unknown. We have explored its possible dependence upon the geometrical parameters (nozzle diameters, disk diameters, nozzle-to-disk distance). It is independent of the nozzle-to-disk distance (1% of variation for a change between 4 and $40\,$mm) and slightly dependent on the nozzle diameter (10% for a variation from 1.1 to $7\,$mm) and on the disk radius (5% for a variation from $R_\infty = 10\,$cm to $R_\infty = 15\,$cm). However we emphasize here that in all the tested cases we found a constant Froude number and an excellent agreement between the jump radius and the laws (5) and (4) observed above.

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