Equations for disturbance cases

Basic governing equations

The core idea is to set results of the depth-averaged stable case as the unperturbed quantities in expansion, like:

$$\boldsymbol{F} = \boldsymbol{F_s} + \hat{\boldsymbol{F}}$$

Here $\boldsymbol{F}$ can be any depth-averaged quantities we have discussed in the stable case, the s-marked term $\boldsymbol{F_s}$ means stable results, and the hat term $\hat{\boldsymbol{F}}$ means disturbance quantities, the actual quantity is consist of these two terms.

Notice that in our model the disturbance term is much smaller than the unperturbed term.

Recall our governing equations (Without applying integration along b direction):

$$\begin{aligned}
\text{Continum}: &\quad \frac{1}{h_2}u_a\cos\alpha + \partial_a u_a + \frac{1}{h_2}\partial_\theta u_\theta + \frac{1}{h_2}\partial_b\left(\left(a\cos\alpha + b\sin\alpha\right)u_b\right) = 0 \\
\hat{e}_a \ : &\quad \rho\phi\left(d_t u_a - \frac{1}{h_2}u_\theta^2\cos\alpha\right) = (\nabla\cdot\overleftrightarrow{T})_a + \rho\phi g\sin\alpha \\
\hat{e}_\theta \ : &\quad \rho\phi\left(d_t u_\theta + \frac{1}{h_2}u_\theta\left(u_a\cos\alpha + u_b\sin\alpha\right)\right) = (\nabla\cdot\overleftrightarrow{T})_\theta \\
\hat{e}_b \ : &\quad \rho\phi\left(d_t u_b - \frac{1}{h_2}u_\theta^2\sin\alpha\right) = (\nabla\cdot\overleftrightarrow{T})_b - \rho\phi g\cos\alpha
\end{aligned} \qquad (0.1 \sim 0.4)$$

In this thin-film model, the depth-averaged method should be applied to these four governing equations, just do integration along b diretion and get:

Some notations need to be claimed here before we move further:

$$\{\bar{u}_a, \bar{u}_\theta\} = \{u, v\} \quad \text{i.e.} \quad \{(\bar{u}_a)_s, (\bar{u}_\theta)_s\} = \{u_s, v_s\}$$

For $\hat{e}_b$ (0.4), it derives:

$$p = \rho\phi g\cos\alpha(h-b)$$

This pressure-depth relation will be applied to later calculations, like:

$$\hat{p} = \rho\phi g\cos\alpha\ \hat{h}$$

For the shear stress term, when we calculate its integration, it can be considered as two parts:

$$\frac{1}{\rho\phi}\frac{1}{h}\int_0^h \left(\nabla\cdot\overleftrightarrow{T}\right)_j db = -g\cos\alpha\ \partial_j h + \frac{1}{h}\int_0^h \frac{1}{\rho\phi}\tau_{ji}\ db$$

For the second term, recall what we have done in the stable case (Please see the (1.2*) in stable case) that $\text{its expression depends on what kind of velocity profile in this thin-film we choose}$, it can be denoted as:

$$\frac{1}{h}\int_0^h \frac{1}{\rho\phi}\tau_{ji}\ db \equiv \boldsymbol{Y}(h,u)\bar{u}_j$$

Here we define this $\boldsymbol{Y} = \boldsymbol{Y}(h,u)$ means that its form may relate to the depth of the thin-film h and the dominating flow velocity u.

Some terms need to discuss as well: the 1st term of (0.2) and the 1st term of (0.3)

$$\begin{aligned}
\frac{1}{h}\int_0^h d_t u_a\ db &= \frac{1}{h}\int_0^h\left(\partial_t u_a + u_a\partial_a u_a\right)db \\
&= \partial_t u + \frac{1}{2h}\int_0^h \partial_a {u_a}^2 db
\end{aligned}$$

Also, recall what we have done in the stable case (Also (1.2*) assumption in stable case), it becomes:

$$\begin{aligned}\frac{1}{h}\int_0^h d_t u_a\, db &= \partial_t u + \frac{1}{2}\boldsymbol{C}\partial_a u^2\\ &= \partial_t u + \boldsymbol{C}u\partial_a u\end{aligned}$$

You will find that here $\boldsymbol{C}$ and $\boldsymbol{Y}$ both depend on the velocity profile, it is just we have mentioned and discussed in stable case, for more calculations please see the file of stable case, later I will show corresponding relations before doing numerical calculations.

$\text{It is a pity that nearly all ariticles ignore the effect of coeff. } \boldsymbol{C}$
$\text{ and just think that after the integration this coeff. is still equal to 1}$
$\text{Although now I'm not sure about what difference and how serious the difference it will occur.}$

While for the 1st term of (0.3) there is something different:

$$\begin{aligned}\frac{1}{h}\int_0^h d_t u_\theta\, db &= \frac{1}{h}\int_0^h \left(\partial_t u_\theta + u_a\partial_a u_\theta\right) db\\ &= \partial_t v + \frac{1}{h}\int_0^h u_a\partial_a u_\theta\, db\end{aligned}$$

For the second term, we need to revisit that how we introduce the first assumption of (1.2*) in the stable flow, its core is to use the relation between u_a and $\bar{u}_a$ to obtain the relation between ${u_a}^2$ and ${\bar{u}_a}^2$. While here when we consider the u_θ we will get nothing at all! That is because in the stable flow: $(u_\theta)_s = (\bar{u}_\theta)_s = 0$. Thus the only term of v is its depth-averaged perturbation $\hat{v}$. Thus here we only need to consider the basic relation between u_a and $\bar{u}_a$, that is to say:

$$\frac{1}{h}\int_0^h u_a\partial_a u_\theta\, db \simeq u\partial_a v$$

For the 2nd term of (0.3), for the geometric relation $\frac{h}{a} \ll 1$, and the situation that the disturbance happens after a long time since the hydralic jump happens, we have:

$$\frac{1}{h}\int_0^h \frac{1}{h_2}u_\theta\left(u_a\cos\alpha + u_b\sin\alpha\right) db \simeq \frac{1}{h}\int_0^h \frac{1}{h_2}u_\theta u_a\cos\alpha\, db$$

And using the assumption what we have discussed, this term becomes:

$$\frac{1}{h}\int_0^h \frac{1}{h_2}u_\theta\left(u_a\cos\alpha + u_b\sin\alpha\right) db \simeq \frac{1}{h_2}uv\cos\alpha$$

After all these discussions above, we can obtain the integration form of these four governing equations:

$$\begin{aligned}\partial_a(uh) + \frac{h}{a}u + \frac{1}{h_2}\partial_\theta(vh) + \partial_t h = 0\\ \partial_t u + \boldsymbol{C}u\partial_a u = g\sin\alpha - g\cos\alpha\,\partial_a h + \boldsymbol{Y}u\\ \partial_t v + u\partial_a v + \frac{1}{h_2}uv\cos\alpha = -g\cos\alpha\frac{1}{h_2}\partial_\theta h + \boldsymbol{Y}v\end{aligned} \tag{1.1 \sim 1.3}$$

Now we can apply small perturbations to (1.1) (1.2) and (1.3), it becomes:

Wait wait wait, before I show the final form, we should first discuss perturbation forms of some special terms.

First we focus on the second term of (1.1):

$$\frac{h_s + \hat{h}}{a}(u_s + \hat{u}) = \frac{h_s}{a}u_s + \frac{h_s}{a}\hat{u} + \frac{\hat{h}}{a}u_s + \frac{\hat{h}}{a}\hat{u}$$

We will find that when we consider the geometric relation $\frac{h}{a} \ll 1$, except the stable flow term $\frac{h_s}{a}u_s$, other terms can be considered as small quantities compared with other terms in (1.1).

Next is about the perturbation of $\boldsymbol{Y}\bar{u}_j$ terms, it can be expressed as:

$$\delta(\boldsymbol{Y}\bar{u}_j) = \left(\frac{\partial \boldsymbol{Y}}{\partial h} + \frac{\partial \boldsymbol{Y}}{\partial u}\frac{\partial u}{\partial h}\right)\bar{u}_j(\delta h) + \frac{\partial \boldsymbol{Y}}{\partial u}\bar{u}_j(\delta u) + \boldsymbol{Y_s}\,\delta u$$

Here we denote that:

$$\begin{aligned}\boldsymbol{P} &= \frac{\partial \boldsymbol{Y}}{\partial h} + \frac{\partial \boldsymbol{Y}}{\partial u}\frac{\partial u}{\partial h}\\ \boldsymbol{Q} &= \frac{\partial \boldsymbol{Y}}{\partial u}\end{aligned}$$

Thus we have:

$$\delta(\boldsymbol{Y}\bar{u}_j) = \boldsymbol{P}\bar{u}_j\hat{h} + \boldsymbol{Q}\bar{u}_j(\hat{\bar{u}}_j) + \boldsymbol{Y_s}(\hat{\bar{u}}_j)$$

Thus the perturbation form of these four integration governing equations is:

$$\begin{aligned}\partial_t\hat{h} + u_s\partial_a\hat{h} + h_s\partial_a\hat{u} + \frac{1}{h_2}h_s\partial_\theta\hat{v} &= 0\\ g\cos\alpha\,\partial_a\hat{h} - \boldsymbol{Q}u_s\hat{h} + \partial_t\hat{u} + \boldsymbol{C}u_s\partial_a\hat{u} - \boldsymbol{P}u_s\hat{u} - \boldsymbol{Y_s}\hat{u} &= 0\\ g\cos\alpha\frac{1}{h_2}\partial_\theta\hat{h} + \partial_t\hat{v} + u_s\partial_a\hat{v} + \frac{1}{h_2}u_s\cos\alpha\,\hat{v} - \boldsymbol{Y_s}\hat{v} &= 0\end{aligned} \tag{2.1 \sim 2.3}$$

Here we notice that in (2.1) (2.2) and (2.3), all equations are linear and homogeneous, thus their solution form must be expoential form. Thus using the normal mode method to express the perturbation quantities are well applicable.

$$\begin{aligned}\hat{h} &= A_h\,e^{i(k_1a + k_2\theta - \omega t)}\\ \hat{u} &= A_u\,e^{i(k_1a + k_2\theta - \omega t)}\\ \hat{v} &= A_v\,e^{i(k_1a + k_2\theta - \omega t)}\end{aligned}$$

Apply these perturbation expressions to (2.1) (2.2) (2.3) and obtain:

$$\begin{aligned}(-i\omega + ik_1u_s)A_h + (ik_1h_s)A_u + \left(ik_2\frac{1}{h_2}h_s\right)A_v &= 0\\ (ik_1g\cos\alpha - \boldsymbol{Q}u_s)A_h + (-i\omega + ik_1u_s\boldsymbol{C} - \boldsymbol{P}u_s - \boldsymbol{Y_s})A_u &= 0\\ \left(ik_2\frac{1}{h_2}g\cos\alpha\right)A_h + \left(-i\omega + ik_1u_s + \frac{1}{h_2}u_s\cos\alpha - \boldsymbol{Y_s}\right)A_v &= 0\end{aligned} \tag{3.1 \sim 3.3}$$

It is very clear that these perturbations are strongly connected, we can rewrite (3.1) (3.2) (3.3) as the matrix form:

$$\begin{pmatrix} -i\omega + ik_1u_s & ik_1h_s & ik_2\frac{1}{h_2}h_s\\ ik_1g\cos\alpha - \boldsymbol{Q}u_s & -i\omega + ik_1u_s\boldsymbol{C} - \boldsymbol{P}u_s - \boldsymbol{Y_s} & 0\\ ik_2\frac{1}{h_2}g\cos\alpha & 0 & -i\omega + ik_1u_s + \frac{1}{h_2}u_s\cos\alpha - \boldsymbol{Y_s}\end{pmatrix}\begin{pmatrix}A_h\\ A_u\\ A_v\end{pmatrix} = 0$$

Denote as:

$$\xi A = 0$$

To let the solution of the amplitude matrix A won't be trivial, the condition is

$$\det(\xi) = 0$$

This step is quite complicated, please see the symcal file and numerical file for more discussion.