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ABSTRACT

We consider the stability problem for wide, uniform stationary open flows down a slope with constant inclination under gravity. Depth-averaged equations are used with arbitrary bottom friction as a function of the flow depth and depth-averaged velocity. The stability conditions for perturbations propagating along the flow are widely known. In this paper, we focus on the effect of oblique perturbations that propagate at an arbitrary angle to the velocity of the undisturbed flow. We show that under certain conditions, oblique perturbations can grow even when the perturbations propagating along the flow are damped. This means that if oblique perturbations exist, the stability conditions found in the investigation of the one-dimensional problem are insufficient for the stability of the flow. New stability criteria are formulated as explicit relations between the slope and the flow parameters. The ranges of the growing disturbances propagation angles are indicated for unstable flows.

I. INTRODUCTION

Over the last century, flows down an inclined channel or surface have been widely studied experimentally, theoretically, and numerically. It is well known that a flow down an incline can lose stability, and progressing bores (roll waves) are formed, which are shocklike disturbances. The first data about these phenomena were obtained by Cornish[1] in 1905. There are a lot of observations of roll waves in nature and in laboratories, such as those by Qian,[2] McArdell *et al.*,[3] Sharp and Nobles,[4] Pierson,[5] Jeffreys,[6] Brock,[7] and Coussot.[8] An extensive review of theoretical and experimental results is given, e.g., by Zanuttigh and Lamberti.[9] Most of the authors investigate the stability of downslope flow using hydraulic approximation, i.e., depth-averaged equations.

As a rule, authors investigate disturbances propagating along the flow velocity vector (hereunder, we will call this statement the 1D problem). Jeffreys[6] conducted linear temporal stability analysis, for the bottom friction described by the Chézy formula, and obtained the critical Froude number $\mathrm{Fr}_{cr} = 2$; when $\mathrm{Fr} > 2$, the flow is unstable. Dressler and Pohle[10] found the critical Froude number $\mathrm{Fr}_{cr} = n/m$ for the flow, in which the bottom friction, divided by depth and density, had the form $-\lambda u^n/y^m$ (u is the flow velocity, y is the hydraulic radius, and λ describes the channel roughness and is independent on u and y, $m, n > 0$). This result coincides with those of Vedernikov[11] and Craya.[12] Values $m = 4/3$, $n = 2$, and $\mathrm{Fr}_{\mathrm{cr}} = 1.5$ correspond to the Manning formula (Dressler and Pohle[10]).

Trowbridge[13] analyzed the linear temporal stability of the 1D problem for the flow at the slope of the small constant angle. Shear stress at the bottom τ was supposed to be an arbitrary single-valued, differentiable function of local depth and velocity: $\tau = \tau(u, h)$, and the derivative $\partial\tau/\partial u$ was assumed to be positive. An instability condition in this case is

$$\sqrt{gh} < \frac{\tau - h\frac{\partial \tau}{\partial h}}{\frac{\partial \tau}{\partial u}},$$

where h is the depth of the steady, uniform basic flow. Trowbridge[13] applied this criterion to several special cases of the bottom shear stress, expressed in terms of the Darcy-Weisbach friction factor. Turbulent Newtonian fluid flows in smooth and rough channels; laminar and turbulent Bingham flows have been considered. Critical Froude number $\mathrm{Fr}_{\mathrm{cr}} = 0.5$ was obtained for the laminar flow of Newtonian fluid.

Ng and Mei[14] investigated the linear temporal stability of the 1D problem of the flow of power law fluid with the power-law index $0 < n \leq 1$.

Later, Thual *et al.*[15] considered 1D slope flows, modeled by Saint-Venant equations, in which drag coefficient C_f was an arbitrary differentiable and positive function. They used two dimensionless functions

$$\mu(h,u) = -\frac{h}{C_f}\frac{\partial C_f}{\partial h}, \quad \chi = \frac{1}{2}\frac{u}{C_f}\frac{\partial C_f}{\partial u}.$$

The study was conducted for both temporal and spatial stability, and the critical Froude number for the onset of both types of instabilities was found. It was also shown that roll-wave instability is of a convective nature.

Coussot[8] used Trowbridge's approach to obtain the stability criterion for the uniform flow of the Herchel-Bulkley fluid on an infinitely wide inclined plane. Di Cristo *et al.*[16] analyzed the 1D stability of the Herschel-Bulkley fluid wide channel flow, described by depth-averaged momentum and mass conservation equations, using the near-front expansion technique of Whitham.[17] The streamwise nonuniformity of the basic flow was taken into account. Campomaggiore *et al.*[18] conducted a near-front expansion analysis of the gradually varied flow with the bottom stress $\tau = \lambda \rho u^2/8$, where λ is determined through the Colebrook-White formula. For the flow of the Bingham fluid, linear temporal stability analysis was done by Liu and Mei.[19]

Dressler[20] first gave the theory of roll waves. He constructed the solution for nonlinear shallow water equations of the flow in the inclined channel, joining together sections of continuous solutions through hydraulic jumps. The Chézy formula was used for the friction at the bottom. It was shown that roll waves do not occur if the friction is zero or larger than a certain critical value. Ng and Mei[14] studied nonlinear roll waves in the flow of power law fluid; Liu and Mei[19] studied them in the flow of the Bingham fluid using Dressler's approach. Dressler[20] also obtained the solution for roll waves using another method and showed that they can be approximated by continuous "cnoidal" waves.

In papers mentioned above, the depth-averaged approach was used. There are also a number of papers that study the stability of Newtonian and non-Newtonian flows on an inclined plane, using equations that are not averaged over the depth of the flow. In some papers, the Navier-Stokes equations are used; in others, equations simplified due to the thinness of the layer, e.g., Benjamin,[21] Yih,[22] Gupta and Rai,[23] Mogilevskii and Shkadov,[24] Allouche *et al.*,[25] Ghosh and Usha,[26] Ellaban *et al.*,[27] Chakraborty *et al.*,[28] Mogilevskiy and Vakhitova,[30] and others.

In particular, Benjamin[21] and Yih[22] investigated the linear stability of the Newtonian fluid downslope flow with free surface, solving the Orr-Sommerfeld type equation. It was determined that for low Reynolds numbers and long waves for which Re > (5/6) $\cot\alpha$, some disturbances become amplified (it also can be formulated as $\mathrm{Fr}_{\mathrm{cr}} = 0.572$, which is close to Trowbridge's result $\mathrm{Fr}_{\mathrm{cr}} = 0.5$). We emphasize that all mentioned papers consider two-dimensional flows and two-dimensional small perturbations propagating in the plane of the flow. We are aware of only two papers[23,25] in which the influence of three-dimensional (oblique) perturbations was investigated.

Gupta and Rai[23] found that Squire's theorem, which states that two-dimensional instabilities are more dangerous than three-dimensional, is not satisfied for the flow of viscoelastic liquid film on an inclined plane.

Allouche *et al.*[25] investigated the solution stability of the system of Navier-Stokes equations and the continuity equation for flows of Newtonian and generalized Newtonian fluid down an inclined plane with respect to two-dimensional and tree-dimensional perturbations. Effective viscosity $\mu_{eff} = \tau/(\gamma)$ (γ is the shear rate in simple shear flow) for the generalized Newtonian fluid was taken in the form of the four-parameter Carreau model

$$\mu_{eff} = \mu_\infty + (\mu_0 - \mu_\infty)\left[1 + \left(\delta\frac{\partial u}{\partial z}\right)^2\right]^{\frac{n-1}{2}},$$

where δ is a characteristic time, $0 < n < 1$ corresponds to shear-thinning fluids, and $n > 1$ corresponds to shear-thickening fluids. A system of two Orr-Sommerfeld type equations for velocity perturbations v', w' is obtained. Stability maps and neutral curves are numerically calculated in terms of the Reynolds number vs the streamwise wave number, slope inclination, or dimensionless parameter $L = \delta Q((\rho g \sin\alpha)/(\mu_0 Q))^{2/3}$ (Q is the flow rate). Series of numerical calculations were done for surface tension $\sigma = 0$, different fixed values of the bottom inclination, parameters n, L, and the obliquity angles of perturbations. In particular, neutral curves on the plane Reynolds number—streamwise wave number were calculated for the angle of plane inclination $\alpha = 2°$ and different obliquity angles of three-dimensional waves for Newtonian fluid and generalized Newtonian fluids with several different values of n. It was shown that Squire's theorem is valid for the open flow of the Newtonian fluid down an incline. For the generalized Newtonian fluids with strong shear-thinning properties, cases were found for which thresholds for three-dimensional waves are smaller than for two-dimensional ones.

This literature review shows that criteria of stability to longitudinal waves have been formulated in many papers for the 1D problems for the flows modeled by the depth-averaged equations. Studies of stability to oblique perturbations were made for the flows described by not-averaged equations, but only particular cases of the flow rheologies were considered, and calculations for particular sets of flow and medium properties were made.

The aim of this study is to investigate the effect of oblique perturbations analytically. We consider the stability problem for wide uniform stationary open flows down an inclined plane under the gravity using the depth-averaged equations

$$\frac{\partial h}{\partial t} + \frac{\partial h u_x}{\partial x} + \frac{\partial h u_y}{\partial y} = 0, \tag{1}$$

$$\frac{d\vec{u}}{dt} = g\sin\alpha\vec{e} - g\cos\alpha \,\mathrm{grad}\, h - \frac{\vec{\tau}}{\rho h}. \tag{2}$$

In (1) and (2), α is a constant inclination angle of the slope, $\vec{e}$ is the unit vector along a steepest descent line at the bottom, h is the depth of the flow along the z axis that is normal to the bottom, ρ is the fluid density, g is the gravity acceleration, u_x and u_y are the depth-averaged velocity components, and $\vec{\tau} = \vec{\tau}(h, \vec{u})$ is the friction force at the bottom per unit area. For the sake of simplicity, we do not take into account the correction factor connected with nonuniformity of the velocity profile. Here, we investigate the stability of solutions of

Eqs. (1) and (2) to oblique perturbations, propagating at an angle θ to the unperturbed flow velocity (we will call this the 2D stability problem).

We show that under certain conditions, oblique perturbations can grow even when the perturbations propagating along the flow are damped. This means that if oblique perturbations exist, the stability conditions found in the investigation of the one-dimensional problem are insufficient for the stability of the flow. New stability criteria are formulated as explicit relations between the slope and the flow parameters. The ranges of the growing disturbances' propagation angles are indicated for unstable flows, and the wave numbers of growing perturbations are found.

The structure of the paper is as follows. In Sec. II, we give the statement of the problem and basic equations. In Sec. III, the dispersion equation for small perturbations with arbitrary propagation angle is derived. In Sec. IV, the dispersion equation is investigated and the stability criterion of the flow to oblique perturbations is obtained in general terms (Statements I and II). Relations, including the slope and flow parameters, as well as obliquity angle θ and wave numbers for unstable modes that provide fulfillment of Statements I and II, are derived in Sec. V. Results obtained for oblique perturbations are summarized and compared with known results for longitudinal perturbations in Sec. VI. Section VII is devoted to the application of the stability criterion to several models of downslope flows. Conclusions are given in Sec. VIII.

II. STATEMENT OF THE PROBLEM

Consider the wide uniform stationary open downslope flow under gravity, using the depth-averaged Eqs. (1) and (2). The coordinate system is shown in Fig. 1. Define the function $F(h, u)$ by the relation

$$\frac{\vec{\tau}}{\rho h} = F(h,u)\vec{u}, \quad \text{i.e.,} \quad \vec{\tau} = \tau\frac{\vec{u}}{u}, \quad F = \frac{\tau}{\rho h u}, \tag{3}$$

where $\vec{u} = \{u_x, \ u_y\} = \{u\cos\theta, \ u\sin\theta\}$ is the depth-averaged velocity vector, $u = \sqrt{u_x^2 + u_y^2}$.

A. Disturbances propagation speeds in large-scale and small-scale approximations

Equations (1) and (2) can be simplified for the flows with large time and length scales (large-scale approximation or kinematic waves' theory). In this case, the terms with derivatives over time and space coordinates in (2) are small in comparison with the terms without derivatives and can be neglected to obtain

$$g\sin\alpha\vec{e} = F(h,u)\vec{u}, \qquad \text{i.e.,} \quad \frac{\vec{u}}{u} = \vec{e}, \ \ g\sin\alpha = uF(h,u). \tag{4}$$

This means that in large-scale approximation, $\vec{u}$ is parallel to $\vec{e}$, and the fluid moves along the steepest lines. Besides, it follows from (4) that $u = u(h, \alpha)$. If the incline of the bottom is constant, then we can direct the x axis along $\vec{e}$ to write the continuity equation in the form

$$\frac{\partial h}{\partial t} + a\frac{\partial h}{\partial x} = 0, \quad \text{where} \qquad a = \frac{\partial hu(h)}{\partial h} = u + h\frac{\partial u}{\partial h}. \tag{5}$$

If h' is the small disturbance of the uniform flow with $h = h_0$, then

$$\frac{\partial h'}{\partial t} + a_0\frac{\partial h'}{\partial x} = 0, \ \ \text{i.e.,} \ \ \frac{dh'}{dt} = 0 \ \ \text{for} \ \ \frac{dx}{dt} = a_0.$$

Here, $a_0 = a(h_0)$. So, small disturbances propagate along the flow velocity with the speed a_0. The value of a_0 can be found from (4). Differentiation by h of both parts of the third equality of (4) leads to the expression

$$\frac{\partial F}{\partial h}u + \frac{\partial F}{\partial u}\frac{\partial u}{\partial h}u + F\frac{\partial u}{\partial h} = 0.$$

From here, the relation for the partial derivative $\partial u/\partial h$ can be obtained and substituted to the formula (5) that gives

$$a = u + N, \quad N = -u\frac{h\dfrac{\partial F}{\partial h}}{F + u\dfrac{\partial F}{\partial u}}. \tag{6}$$

Here, N is the disturbances propagation speed relative to the moving medium.

Another approach can be used for small-scale motion. The momentum equation then reads

$$\frac{d\vec{u}}{dt} = -g\cos\alpha\,\mathrm{grad}\,h$$

for the flow at the bottom of constant inclination. For small disturbances of the uniform flow in the coordinate system, moving with the flow, we have

$$\frac{\partial \vec{u}'}{\partial t} = -g\cos\alpha\,\mathrm{grad}\,h',$$
$$\frac{\partial h'}{\partial t} = -h_0\,\mathrm{div}\,\vec{u}'.$$

The equation for the small disturbances of the depth in the coordinate system moving with the fluid can be derived from these relations

$$\frac{\partial^2 h'}{\partial t^2} = c_0^2\Delta h',$$

where the perturbations velocity relative to the moving medium is

$$c_0 = \sqrt{gh_0\cos\alpha}.$$

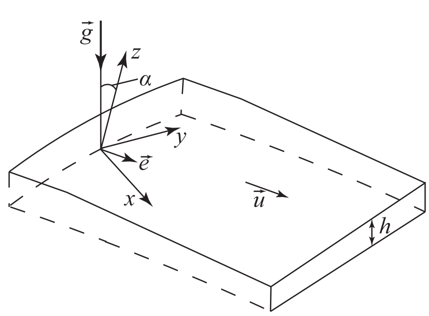

FIG. 1. The scheme of the flow and coordinate system.

Note that small plane waves both in large- and small-scale approximations propagate without change in shape and amplitude.

In the general case, we have to use a full momentum equation (2) to describe the behavior of small perturbations.

B. Parameters, which play a crucial role in the flow stability problem

We will mark by 0 all quantities related to the unperturbed flow. In particular, $F_0 = F(h_0, u_0)$,

$$\frac{\partial F}{\partial u_0} = \frac{\partial F}{\partial u}\bigg|_{h=h_0,u=u_0}, \quad \frac{\partial F}{\partial h_0} = \frac{\partial F}{\partial h}\bigg|_{h=h_0,u=u_0}.$$

Introduce the following parameters:

$$p = u_0\frac{\partial F}{\partial u_0}, \quad q = -h_0\frac{\partial F}{\partial h_0}, \quad A = F_0 + p, \qquad \text{then} \quad N_0 = u_0\frac{q}{A}. \tag{7}$$

These parameters can be expressed in terms of the friction τ,

$$F = \frac{\tau}{\rho h u}; \quad A = \frac{1}{\rho h}\frac{\partial \tau}{\partial u}; \quad p = \frac{1}{\rho h u}\left(u\frac{\partial \tau}{\partial u} - \tau\right); \qquad q = \frac{1}{\rho h u}\left(\tau - h\frac{\partial \tau}{\partial h}\right); \quad N = \frac{\tau - h\dfrac{\partial \tau}{\partial h}}{\dfrac{\partial \tau}{\partial u}}. \tag{8}$$

Flow stability depends on the values and signs of these parameters. Furthermore, we always take into account that $F_0 > 0$ since the direction of the friction force is always opposite to the flow velocity. Besides, commonly $A > 0$, the friction increases as the velocity increases. However, some flows can move in a Coulombic frictional regime, i.e., the friction does not depend on the flow velocity (Ancey and Meunier[30]), then $A = 0$. Moreover, Kytomaa and Prasad[31] and Coussot and Piau[32] found the flows with very high solid concentrations, for which a flow curve had a shear-stress minimum, that is, the flow regime exists, where $A < 0$. As to the signs of q and N, usually $q > 0$, i.e., the friction decreases as h increases. There can be situations, when the friction increases with h, for example, due to the friction at side walls of the channel[33] or in the problem of traffic flows, when the cars velocity decreases with the increase of the density of cars at the road.[17] Concerning the sign of the parameter p, it can be different.

Consider a model of the power-law fluid as an example. In laminar flow of this fluid $\tau = K_1 u_0^n/h_0^n$, $K_1, n > 0$ are constants (see details in Sec. VII). Therefore, $F_0 = (K_1/\rho)(u_0^{n-1}/h_0^{n+1})$, $A = nF_0 > 0$, $q = (n+1)F_0 > 0$, $p = (n-1)F_0$; $p < 0$ for $n < 1$ (pseudoplastic or shear-thinning fluid) and $p > 0$ for $n > 1$ (dilatant or shear-thickening fluid); $p = 0$ if $n = 1$ (the Newtonian fluid).

C. Stability criterion without accounting oblique perturbations

The stability criterion for the 1D stability problem for flows down constant slopes can be written in the general form as follows:[11,17]

$$|N_0| \le c_0, \quad \text{i.e.,} \quad |\xi| \le 1, \tag{9}$$

$$\xi \equiv \frac{N_0}{c_0} = \mathrm{Fr}\frac{q}{A} = \mathrm{Fr}\frac{\tau - h\dfrac{\partial \tau}{\partial h}}{u\dfrac{\partial \tau}{\partial u}},$$

$$\mathrm{Fr} = \frac{u_0}{\sqrt{gh_0\cos\alpha}}.$$

The criterion (9) can be rewritten as a condition on the value of the Froude number of the unperturbed flow

$$\mathrm{Fr} \le \left|\frac{A}{q}\right|. \tag{10}$$

Below, we investigate the 2D stability problem of oblique perturbations with nonzero propagation angle θ to the unperturbed flow velocity $\vec{u}_0$ (further θ will be referred as an obliquity angle). Our study shows that in the presence of oblique perturbations, the fulfillment of condition (9) does not always ensure stability. Stability is sometimes provided by the condition $|\xi| \le \xi_* < 1$. This criterion is derived below, and the value ξ_* is calculated.

D. Dimensionless variables and equations

In the dimensionless form, Eqs. (1) and (2) at the constant bottom inclination angle are

$$\frac{\partial \bar{h}}{\partial \bar{t}} + \mathrm{div}\bar{h}\bar{\vec{u}} = 0, \tag{11}$$

$$\frac{d\bar{\vec{u}}}{d\bar{t}} = \frac{1}{\mathrm{Fr}^2}\tan\alpha\vec{e} - \frac{1}{\mathrm{Fr}^2}\mathrm{grad}\,\bar{h} - \bar{F}\bar{\vec{u}}, \tag{12}$$

where the bar marks dimensionless values. Velocity vector $\vec{u}$ makes an angle θ with the x axis, so $u_x = u\cos\theta$ and $u_y = u\sin\theta$. The depth h_0 and the velocity modulus u_0 of the unperturbed flow are taken as the depth and the velocity scales, respectively. The dimensionless characteristics of the flow are defined as follows:

$$\bar{h} = \frac{h}{h_0}, \quad \bar{h}_0 = 1, \quad \bar{u}_x = \frac{u\cos\theta}{u_0}, \quad \bar{u}_{0x} = \cos\theta,$$
$$\bar{u}_y = \frac{u\sin\theta}{u_0}, \quad \bar{u}_{0y} = \sin\theta, \quad \bar{t} = tu_0/h_0,$$
$$\bar{F} = Fh_0/u_0, \quad \bar{A} = \bar{F}_0 + \bar{p},$$
$$\bar{p} = \partial\bar{F}/\partial\overline{u_0}, \quad \bar{q} = -\partial\bar{F}/\partial\overline{h_0},$$
$$\bar{N}_0 = \bar{q}/\bar{A}, \quad \bar{a}_0 = 1 + \bar{N}_0, \quad \bar{c}_0 = \frac{c_0}{u_0} = 1/\mathrm{Fr}.$$

From this point forward, we will write all equations in the dimensionless form, omitting the bar above the letters.

III. EQUATIONS FOR SMALL PERTURBATIONS AND DISPERSION EQUATION

A. Equations for small perturbations

Consider the small perturbations of arbitrary space and time scales. Let

$$h = 1 + h',$$
$$u_x = \cos\theta + u_x',$$
$$u_y = \sin\theta + u_y',$$

where h', u'_x, u'_y are small disturbances. Equations (11) and (12) linearized near the undisturbed flow with $h_0 = 1$, $\vec{u}_0 = \{\cos\theta, \sin\theta\}$ are

$$\frac{\partial h'}{\partial t} + \cos\theta\frac{\partial h'}{\partial x} + \frac{\partial u'_x}{\partial x} + \sin\theta\frac{\partial h'}{\partial y} + \frac{\partial u'_y}{\partial y} = 0, \tag{13}$$

$$\frac{\partial u'_x}{\partial t} + \cos\theta\frac{\partial u'_x}{\partial x} + \sin\theta\frac{\partial u'_x}{\partial y} + \frac{1}{\mathrm{Fr}^2}\frac{\partial h'}{\partial x} = -p\cos^2\theta u'_x - p\cos\theta\sin\theta u'_y + q\cos\theta h' - F_0 u'_x, \tag{14}$$

$$\frac{\partial u'_y}{\partial t} + \cos\theta\frac{\partial u'_y}{\partial x} + \sin\theta\frac{\partial u'_y}{\partial y} + \frac{1}{\mathrm{Fr}^2}\frac{\partial h'}{\partial y} = -p\cos\theta\sin\theta u'_x - p\sin^2\theta u'_y + q\sin\theta h' - F_0 u'_y. \tag{15}$$

B. Dispersion equation

Consider small perturbations in the form of plane waves

$$h' = He^{i(kx-\omega t)},\quad u'_x = U_x e^{i(kx-\omega t)},\quad u'_y = U_y e^{i(kx-\omega t)}. \tag{16}$$

Here, it is assumed that the x axis is directed along the wave vector $\vec{k}$ so that $\vec{k} = \{k, 0\}$ and $\vec{k}$ has an arbitrary angle θ with $\vec{u}_0$ (Fig. 2).

Let us insert (16) into (13)–(15) to obtain the system of linear algebraic equations for H, U_x, and U_y,

$$(-i\omega + i\cos\theta k)H + (ik)U_x = 0, \tag{17}$$

$$\left(-ik\frac{1}{\mathrm{Fr}^2} - q\cos\theta\right)H + (-i\omega + ik\cos\theta + p\cos^2\theta + F_0)U_x + (p\cos\theta\sin\theta)U_y = 0, \tag{18}$$

$$(-q\sin\theta)H + (p\cos\theta\sin\theta)U_x + (-i\omega + ik\cos\theta + p\sin^2\theta + F_0)U_y = 0. \tag{19}$$

The condition of existence for the nonzero solution of systems (17)–(19) is its discriminant being equal to zero. This condition yields a dispersion equation that can be written in the following form, taking into account relations (6) and (7):

$$D(U, k, \theta) = k^2(U-\cos\theta)(U-a_+)(U-a_-) - F_0 A(U-a_{0x}) + ikF_0(U-a_+)(U-a_-) + ikA(U-\cos\theta)(U-a_{0x}) - ikc_0^2 p\sin^2\theta = 0. \tag{20}$$

In (20), $U = \omega/k$, $a_+ = \cos\theta + c_0$, $a_- = \cos\theta - c_0$, $a_{0x} = a_0\cos\theta = \cos\theta + N_{0x}$, and $N_{0x} = N_0\cos\theta$. Recall that the projection of the dimensionless basic flow velocity on the direction of the wave propagation (on the x axis) equals $\cos\theta$.

For all real k and all angles θ, we can find U, and thus ω, by Eq. (20). All disturbances damp if $\mathrm{Im}\,\omega < 0$, i.e., if $\mathrm{Im}\,U < 0$ for all $k > 0$ and $\mathrm{Im}\,U > 0$ for all $k < 0$ and all θ. They are neutral (neither growing no damping) if the imaginary parts of all dispersion equation roots are zero for all k and all θ. Growing disturbances exist if at least for one wave number $k > 0$ and for one θ there is a root U with a positive imaginary part (for one $k < 0$ and one θ, there is a root with a negative imaginary part).

Let us note that if for a certain $k = k_p$, the dispersion equation (20) has the root $\omega_p = \lambda_1 + i\lambda_2$, then the root of the Eq. (20), in which $k = k_n = -k_p$, is $\omega_n = -\lambda_1 + i\lambda_2$. It is therefore sufficient to investigate the roots of the dispersion equation (20) for all real $k > 0$. Furthermore, $k > 0$ everywhere.

IV. STABILITY CRITERION WITH ACCOUNT OF OBLIQUE PERTURBATIONS. FORMULATION IN GENERAL TERMS

A. Argument principle

The existence of roots of (20) with positive imaginary parts for different given values of k and θ is investigated using the argument principle of the complex variables theory. The function $D(U)$ is the third-order polynomial. Consider the contour C in the U-plane, supposing that there are no zeros of $D(U)$ at this contour. Then, according to the argument principle, the number of $D(U)$ zeros inside the contour C is equal to the argument change of the function $D(U)$, when U goes around the contour C, divided by 2π (a winding number). In our case, we have to find conditions, under which there are no roots U of the Eq. (20) in the upper half-plane $\mathrm{Im}\,U > 0$. The contour C, consisting of a semicircle of infinitely large radius and the real axis, is chosen for this reason [Fig. 3(a)]. The flow is stable if and only if the argument of $D(U)$ does not change in the result of U going around this contour.

Let us follow the change of the argument D as U moves along the contour C. Let U start to move along the semicircle from the

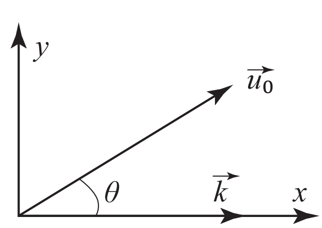

FIG. 2. The x axis is directed along the wave vector $\vec{k}$, and the velocity vector $\vec{u}_0$ makes an angle θ to the x axis.

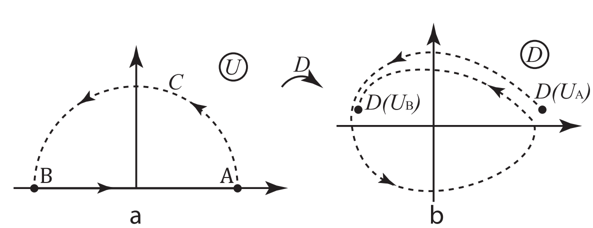

FIG. 3. Contour C at the complex plane U (a); image of the semicircle in D-plane under the function D for the condition $F_0 + A > 0$ (b).

point A to the point B. The behavior of the function $D(U)$ on this part of the circuit C is determined by the principal term k^2U^3, so the argument D increases approximately by 3π, and the image of the semicircle in plane D is a curve making approximately 1.5 turns around the origin in the counterclockwise direction [Fig. 3(b)]. Furthermore, U moves along the real axis from the point B to the point A. For flow stability [the absence of roots of $D(U) = 0$ in the upper half-plane U, that is, inside contour C], the total change in the argument D must be zero. This means that the image of the real U axis on the D plane should be a curve that also makes approximately 1.5 turns around the origin, but in the opposite direction, that is, clockwise (Fig. 4).

So, in order to derive the stability criterion, we should study the behavior of $D(U)$ at the real U axis. Denote by D_R the function D at real U. Figures 3(b) and 4(a) correspond to the case $F_0 + A > 0$ (i.e., for $p > -2F_0$). We see from Eq. (20) that $\operatorname{Re} D_R(U_A) > 0$, $\operatorname{Re} D_R(U_B) < 0$, $\operatorname{Im} D_R(U_A) > 0$, and $\operatorname{Im} D_R(U_B) > 0$. So, for $F_0 + A > 0$, the general stability criterion can be formulated as the following statement.

Statement I. The flow is stable (when $F_0 + A > 0$) if and only if the following two conditions are respected for all θ and all real k:

$$\begin{aligned}&1)\ \text{the equation}\quad \operatorname{Im} D_R(U) = 0 \quad \text{has two real roots:}\\ &\qquad U_{\mathrm{Im}\,1} < U_{\mathrm{Im}\,2},\\ &2)\quad \operatorname{Re} D_R(U_{\mathrm{Im}\,1}) \ge 0 \quad \text{and} \quad \operatorname{Re} D_R(U_{\mathrm{Im}\,2}) \le 0.\end{aligned} \tag{21}$$

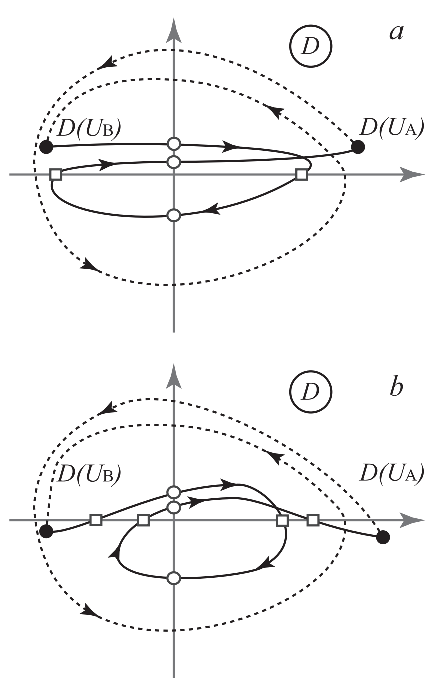

FIG. 4. The images of the contour C under the function $D(U, k, \theta)$ [see Fig. 3(a)]. Circles mark the roots of $\operatorname{Re} D_R = 0$, and squares mark the roots of $\operatorname{Im} D_R = 0$. $\operatorname{Im} D_R = 0$ has two roots for real U, the number of $D(U, k, \theta) = 0$ roots with $\operatorname{Im} U > 0$ is zero; $F_0 + A > 0$ (a). The configuration, which is necessary to decrease the argument of $D(U)$ by 3π, while moving along the real axis, at the condition $F_0 + A < 0$ (b).

Looking at Fig. 4(a), we can give the second formulation of the criterion of the flow stability. We see that if the flow is stable, then when moving along the real axis from point B ($U = -\infty$) to A ($U = \infty$), we first meet the point where $\operatorname{Re} D_R = 0$, then the point where $\operatorname{Im} D_R = 0$, then the new point where $\operatorname{Re} D_R = 0$, and so on. Therefore, the following statement is true.

Statement II (the principle of alternation of roots). When $F_0 + A > 0$, the flow is stable if and only if for all θ and all real k, there are 3 real roots $U_{\mathrm{Re}1} < U_{\mathrm{Re}2} < U_{\mathrm{Re}3}$ of the equation $\operatorname{Re} D_R(U) = 0$ and 2 real roots $U_{\mathrm{Im}1} < U_{\mathrm{Im}2}$ of the equation $\operatorname{Im} D_R(U) = 0$, and the relative position of these roots on the real axis is as follows:

$$U_{\mathrm{Re}1} \le U_{\mathrm{Im}1} \le U_{\mathrm{Re}2} \le U_{\mathrm{Im}2} \le U_{\mathrm{Re}3}. \tag{22}$$

Furthermore, we will use both Statement I and Statement II.

Remark I. Condition (22) in Statement II is reduced to a hierarchy of waves,[17] if only longitudinal perturbations propagating along the flow velocity are considered. In this case, the roots of $\operatorname{Re} D_R$ and $\operatorname{Im} D_R$ are equal to the propagation speeds of small perturbations in the small-scale and large-scale approximations, respectively. We see that when oblique perturbations are taken into account and the additional parameter θ appears, the principle of the hierarchy of waves is generalized as "the principle of alternation of roots" of $\operatorname{Re} D_R(U) = 0$ and $\operatorname{Im} D_R(U) = 0$.

Remark II. If the conditions (21) and (22) are true not for all k and θ but only for the particular disturbance with certain k and certain θ, then this disturbance will not grow.

Remark III. The mutual arrangement of $\operatorname{Re} D_R$ and $\operatorname{Im} D_R$ roots may change as k and θ vary, then the stability with respect to the disturbance under consideration will change to instability. The transition from one relative position of roots to another occurs when the $\operatorname{Re} D_R$ root coincides with the root of $\operatorname{Im} D_R$.

Remark IV. If $\operatorname{Re} D_R(U_{\mathrm{Im}1}) = 0$ or $\operatorname{Re} D_R(U_{\mathrm{Im}2}) = 0$, the argument principle is not applicable to the contour C since the root of $D(U)$ lies on C. However, the part of the contour BA [Fig. 3(a)] can be moved upward by a small distance ε. There will be no $D(U)$ zeros at this new contour C_0, so the argument principle can be applied to C_0. The result for C can be obtained by $\varepsilon \to 0$.

In conclusion, we have to note that for $F_0 + A < 0$ (i.e., for $p < -2F_0$) $\operatorname{Re} D(U_A) > 0$, $\operatorname{Re} D(U_B) < 0$, $\operatorname{Im} D(U_A) < 0$, and $\operatorname{Im} D(U_B) < 0$. Such location of the points $D(U_A)$, $D(U_B)$ at the complex plane D yields no possibility to decrease the argument of the function D by 3π while moving along the real axis, taking into account that maximum number of the roots of $\operatorname{Im} D_R = 0$ can be two. It is shown in Fig. 4(b) that if the argument of D decreases by 3π, while moving from B to A, there will be 4 roots of $\operatorname{Im} D_R = 0$ at the real axis U. Consequently, the flow is unstable for $F_0 + A < 0$. Moreover, we proved that the flow with $A < 0$ is unstable.

V. ANALYSIS OF THE DISPERSION EQUATION

A. Re D_R, Im D_R in reference points

Here, we look for the relations for the slope angle α, rheological function F, and obliquity angle θ, for which Statements I and II are respected. The real and imaginary parts of the function $D_R(U)$ are

$$\operatorname{Re} D_R = k^2(U - \cos\theta)(U - a_+)(U - a_-) - F_0 A(U - a_{0x}),$$

$$\operatorname{Im} D_R = k\big[F_0(U-a_+)(U-a_-) + A(U-\cos\theta)(U-a_{0x}) - c_0^2 p \sin^2\theta\big].$$

To investigate the locations of the roots of $\operatorname{Re} D_R(U)$ and $\operatorname{Im} D_R(U)$, we consider the signs of $\operatorname{Re} D_R$ and $\operatorname{Im} D_R$ at the reference points a_-, $\cos\theta$, a_{0x}, a_+, while U goes along the real axis from point B to point A. It is useful to take into account that $a_- - a_{0x} = -(c_0 + N_{0x})$, $a_+ - a_{0x} = c_0 - N_{0x}$, and $\xi = N_0/c_0$.

Let us write expressions for $\operatorname{Re} D_R$ and $\operatorname{Im} D_R$ at the points a_-, $\cos\theta$, a_{0x}, a_+,

$$\begin{aligned}
\operatorname{Re} D_R(a_-) &= F_0 A c_0 (1+\xi\cos\theta),\\
\operatorname{Re} D_R(\cos\theta) &= F_0 A c_0 \xi\cos\theta,\\
\operatorname{Re} D_R(a_{0x}) &= k^2 c_0^3 \xi \cos\theta(\xi^2\cos^2\theta - 1),\\
\operatorname{Re} D_R(a_+) &= F_0 c_0 A(\xi\cos\theta - 1),
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
\operatorname{Im} D_R(a_-) &= k c_0^2 A(\xi\cos\theta + n_1),\\
\operatorname{Im} D_R(\cos\theta) &= -k c_0^2 \big(F_0 + p\sin^2\theta\big),\\
\operatorname{Im} D_R(a_{0x}) &= k F_0 c_0^2 \left(\xi^2\cos^2\theta - 1 - \frac{p}{F_0}\sin^2\theta\right),\\
\operatorname{Im} D_R(a_+) &= k c_0^2 A(n_1 - \xi\cos\theta).
\end{aligned} \tag{24}$$

Here,

$$n_1 \equiv \left(1 - \frac{p}{A}\sin^2\theta\right).$$

B. Flows with A > 0, p > 0

Consider the flows,[33] for which

$$p \ge 0, \quad A > 0. \tag{25}$$

See examples in Sec. VII. Under the conditions (25), $0 < n_1 \le 1$.

Consider first the case

$$\frac{N_{0x}}{c_0} = \xi\cos\theta \ge 0. \tag{26}$$

It means that either $\partial F/\partial h_0 \le 0$ and $\cos\theta \ge 0$ or $\partial F/\partial h_0 \ge 0$ and $\cos\theta \le 0$. Besides, $a_{0x} \ge \cos\theta$. At the conditions (25) and (26),

$$a_- < \cos\theta \le a_{0x} \le a_+.$$

The first step is to investigate the behavior of the function D_R, when

$$0 < \xi\cos\theta \le n_1 \le 1. \tag{27}$$

It follows from the formulas (23) and (24) that $\operatorname{Re} D_R(U_B) < 0$, $\operatorname{Re} D_R(a_-) > 0$, $\operatorname{Re} D_R(\cos\theta) > 0$, $\operatorname{Re} D_R(a_{0x}) < 0$, $\operatorname{Re} D_R(a_+) < 0$, and $\operatorname{Re} D_R(U_A) > 0$. Therefore,

$$U_{\mathrm{Re}\,1} < a_-, \quad \cos\theta < U_{\mathrm{Re}\,2} < a_{0x}, \quad U_{\mathrm{Re}\,3} > a_+.$$

Besides $\operatorname{Im} D_R(U_B) > 0$, $\operatorname{Im} D_R(a_-) > 0$, $\operatorname{Im} D_R(\cos\theta) < 0$, $\operatorname{Im} D_R(a_{0x}) < 0$, $\operatorname{Im} D_R(a_+) \ge 0$, and $\operatorname{Im} D_R(U_A) > 0$, so

$$a_- < U_{\mathrm{Im}\,1} < \cos\theta, \quad a_{0x} < U_{\mathrm{Im}\,2} < a_+$$

[Figure 5(a)]. The mutual arrangement of the roots obeys the principle of alternation of roots (22), and perturbations with all k at the conditions (27) do not grow. The same conclusion can be obtained using Statement I. The image of the real axis under $D(U)$ for (27) is shown in Fig. 5(b).

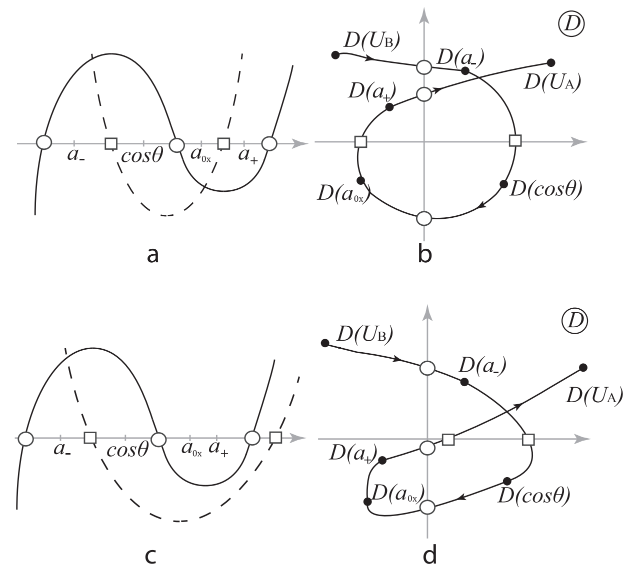

FIG. 5. Functions $\operatorname{Re} D_R(U) = 0$ (solid line) and $\operatorname{Im} D_R(U) = 0$ (dashed line) [(a) and (c)]. Images of the real axis under the function $D(U)$ [(b) and (d)]. Circles mark the roots of $\operatorname{Re} D_R = 0$, squares mark the roots of $\operatorname{Im} D_R = 0$. (a) and (b) correspond to conditions $0 < \xi\cos\theta < n_1$; (c) and (d) correspond to conditions $n_1 < \xi\cos\theta < 1$.

Now, let us consider the second interval of the values of $\xi\cos\theta$,

$$n_1 < \xi\cos\theta < 1. \tag{28}$$

In this case, the signs of $\operatorname{Re} D_R$ and $\operatorname{Im} D_R$ in all reference points are the same as in the previous case, except for $U = a_+$, where now $\operatorname{Im} D_R(a_+) < 0$. It means that $U_{\mathrm{Im}2} > a_+$ [Fig. 5(c)]. But $U_{\mathrm{Re}3} > a_+$ as well, so both roots are located in the same interval. To respect the alternation of roots principle, it should be $U_{\mathrm{Re}3} > U_{\mathrm{Im}2}$. We explain below that the latter inequality is not true at large values of the wave number k. Note that the roots of $\operatorname{Re} D_R$ depend on k, while the roots of $\operatorname{Im} D_R$ do not depend on it. Namely, $U_{\mathrm{Re}3}$ decreases with increasing k, approaching a_+ as k tends to infinity and increases, tending to infinity with decreasing k. Let $U_{\mathrm{Re}3} > U_{\mathrm{Im}2}$ for some value of k. With an increase in k, $U_{\mathrm{Re}3}$ decreases, approaches $U_{\mathrm{Im}2}$, and at a certain $k = k^*$, these roots become equal to each other, and at $k > k^*$, we get $U_{\mathrm{Re}3} < U_{\mathrm{Im}2}$. This violates the order of alternation of roots. Therefore, the corresponding perturbations grow. The value of k^* is calculated by the relation $\operatorname{Re} D_R(U_{\mathrm{Im}2}) = 0$,

$$k^* = \sqrt{\frac{F_0 A(U_{\mathrm{Im}\,2} - a_{0x})}{(U_{\mathrm{Im}\,2} - \cos\theta)(U_{\mathrm{Im}\,2} - a_+)(U_{\mathrm{Im}\,2} - a_-)}}.$$

The image of the real axis under $D(U)$ for (28) and $k > k^*$ is shown in Fig. 5(d).

At the condition,

$$\xi\cos\theta = 1, \tag{29}$$

$a_{0x} = a_+$, $U_{\mathrm{Re}1} < a_-$, and $U_{\mathrm{Re}2} = U_{\mathrm{Re}3} = a_{0x}$. If $\cos\theta \ne 1$, then $U_{\mathrm{Im}1} < a_-$ and $U_{\mathrm{Im}2} > a_{0x}$. So the principle of alternation of roots is not respected, disturbances grow. If $\cos\theta = 1$ (longitudinal disturbances), then $U_{\mathrm{Im}\,2} = U_{\mathrm{Re}\,2} = a_{0x}$; there is coincidence of roots that mean stability. We got known result: with $\xi = 1$, the longitudinal perturbations do not grow.

The next case is

$$\xi\cos\theta > 1, \tag{30}$$

then $a_+ < a_{0x}$ and the real parts of $D(a_+)$ and $D(a_{0x})$ become greater than zero. Statement I is not respected, because $\mathrm{Re}\, D_R\,(U_{\mathrm{Im}\,2}) > 0$ for $k > k^*$. Hence, there are growing perturbations.

The main obtained result here is that at the condition (26) there are no growing perturbations for $0 < \xi\cos\theta \le n_1$.

If we consider the case

$$\frac{N_{0x}}{c_0} = \xi\cos\theta \le 0,$$

then by reasoning similar to that used previously, we find that there are no growing perturbation for $-n_1 \le \xi\cos\theta < 0$.

Several particular cases also have been analyzed. The equality

$$\xi\cos\theta = 0 \tag{31}$$

can be either for $\cos\theta = 0$ (i.e., $\theta = \pm\pi/2$) or for

$$N_0 = \frac{q}{A} = -\frac{1}{A}\frac{\partial F}{\partial h_0} = 0,$$

which can occur if $\partial F/\partial h_0 = 0$, i.e., τ is the linear function of h (as in Coulombic friction) with the coefficient probably dependent on u.

The signs of $\mathrm{Re}\, D$ and $\mathrm{Im}\, D$ at the points a_-, $a_{0x} = \cos\theta$, a_+ are under the condition (31) as follows:

$$\begin{aligned}
&\mathrm{Re}\, D_R(a_-) > 0;\ \mathrm{Im}\, D_R(a_-) > 0;\\
&\mathrm{Re}\, D_R(\cos\theta) = \mathrm{Re}\, D_R(a_{0x}) = 0;\\
&\mathrm{Im}\, D_R(\cos\theta) = \mathrm{Im}\, D_R(a_{0x}) < 0;\\
&\mathrm{Re}\, D_R(a_+) < 0;\ \mathrm{Im}\, D_R(a_+) > 0.
\end{aligned}$$

The image of C under D is shown in Fig. 6. The winding number of the contour C under the function D is zero; thus, Statement I is satisfied. So, transversal perturbations

$$\cos\theta = 0 \Leftrightarrow \theta = \pm\frac{\pi}{2} \tag{32}$$

do not grow.

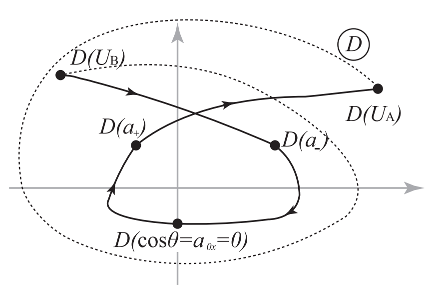

FIG. 6. The images of the contour C under the function D [see Fig. 3(a)] for cases (32). Dashed line corresponds to U running along the semicircle of C; full line shows the image of the real axis (part BA of C) under D.

For $\theta = 0$ and $\theta = \pi$, dispersion equation (20) splits into two equations

$$kU \pm k + iF_0 = 0, \tag{33}$$

$$k(U \pm 1)^2 + iA(U \pm 1) - (kc_0^2 \pm iAN_0) = 0. \tag{34}$$

(The signs "+" and "−" correspond to $\theta = 0$ and $\theta = \pi$, respectively.) Equation (33) gives the roots $U = \pm 1 - iF_0/k$ with the negative imaginary part, Eq. (34) coincides with the dispersion equation for longitudinal perturbations, for which the stability analysis with the use of the argument principle was conducted by Eglit,[35] and stability criterion (9) is derived. Consequently, the flow is stable with respect to perturbations propagating with the angles $\theta = 0$ and $\theta = \pi$, if (9) is satisfied.

Finally, we have shown that at the conditions (25), i.e., $F_0 > 0$, $p > 0$, and $A > 0$, the perturbations do not grow for

$$|\xi\cos\theta| \le n_1,$$

i.e.,

$$|\xi\cos\theta| \le \left(1 - \frac{p}{A}\sin^2\theta\right), \tag{35}$$

where $\xi = N_0/c_0$. Condition (35) includes θ. So, in order to complete the study of flows with $A > 0$ and $p > 0$, we must find θ, for which (35) is satisfied, and then obtain relations between the flow parameters that ensure fulfillment (35) for all θ, i.e., flow stability.

C. Obliquity angles of nongrowing perturbations

Condition (35) can be rewritten in the forms

$$\begin{cases} \xi\cos\theta \ge 0,\\ \dfrac{p}{A}\cos^2\theta - \xi\cos\theta + 1 - \dfrac{p}{A} \ge 0, \end{cases} \tag{36}$$

and

$$\begin{cases} \xi\cos\theta \le 0,\\ \dfrac{p}{A}\cos^2\theta + \xi\cos\theta + 1 - \dfrac{p}{A} \ge 0. \end{cases} \tag{37}$$

Since $p > 0$ and $A > 0$,

$$0 < \frac{p}{A} < 1,$$

so (36) and (37) are always valid for $\xi = 0$ and $\theta = \pm\pi/2$. In the following, we will investigate (36) and (37) for $\cos\theta \ne 0$.

For the sake of brevity, consider the case $\cos\theta > 0$ and $\xi > 0$, and we will study inequality (36).

Let us find the discriminant d of the equation

$$\frac{p}{A}\cos^2\theta - \xi\cos\theta + 1 - \frac{p}{A} = 0. \tag{38}$$

It is

$$d = \xi^2 - \xi_*^2,$$

where

$$\xi_* \equiv 2\sqrt{\frac{p}{A} - \frac{p^2}{A^2}} = \frac{2\sqrt{F_0 p}}{F_0 + p}.$$

Note that $0 < \xi_* \le 1$; $\xi_* = 1$ only for $p = F_0$.

Let $d \le 0$, i.e.,

$$\xi \le \xi_*.$$

In this case, the inequality (35) is satisfied for all θ. All perturbations do not grow; the flow is stable.

If

$$\xi > \xi_*,$$

then $d > 0$ and there are two roots of Eq. (38). These roots are denoted by x_1 and x_2,

$$x_1 = \cos\theta_1 = \frac{\xi - \sqrt{\xi^2 - \xi_*^2}}{2p/A}, \tag{39}$$

$$x_2 = \cos\theta_2 = \frac{\xi + \sqrt{\xi^2 - \xi_*^2}}{2p/A}. \tag{40}$$

Solutions for the other alternatives ($\cos\theta < 0$, $\xi > 0$; $\cos\theta > 0$, $\xi < 0$; and $\cos\theta < 0$, $\xi < 0$) can be obtained using similar arguments.

Note that $x_2 > x_1 > 0$.

There are three possibilities for the location of the roots of Eq. (38)

$$x_2 > x_1 \geq 1, \tag{41}$$

$$x_1 > 0, \quad x_2 \leq 1, \tag{42}$$

$$x_1 < 1, \quad x_2 > 1. \tag{43}$$

For (41), the inequality (36) is satisfied. The roots locations (41) takes place for

$$\xi_* < \xi \leq 1 \quad \text{at} \quad 0 < \frac{p}{A} \leq 0.5. \tag{44}$$

Under condition (44), perturbations do not grow for all propagation angles.

Relation (42) means that

$$\frac{\xi + \sqrt{\xi^2 - \xi_*^2}}{2p/A} \leq 1.$$

The solution of this inequality, taking into account condition $d > 0$, yields

$$\xi_* < \xi \leq 1 \quad \text{at} \quad 0.5 < \frac{p}{A} < 1. \tag{45}$$

This means that for conditions (45), there are two roots, x_1 and x_2, of Eq. (38), $0 < x_1 < x_2 \leq 1$. There are two ranges of values of $\cos\theta$ in the interval from 0 to 1 in which the perturbations do not grow

$$0 \leq \cos\theta \leq x_1 \quad \text{and} \quad x_2 \leq \cos\theta \leq 1,$$

and there is one range where they grow

$$x_1 < \cos\theta < x_2.$$

The next case, described by (43), takes place for

$$\xi > 1 \quad \text{at} \quad 0 < \frac{p}{A} < 1.$$

It means that perturbations do not grow for

$$0 \leq \cos\theta \leq x_1,$$

but grow for

$$x_1 < \cos\theta \leq 1.$$

Thus, at conditions (25), the stability criteria are

$$|\xi| \leq 1 \;\text{ for }\; \frac{p}{A} \leq \frac{1}{2}; \qquad |\xi| \leq \xi_* \;\text{ for }\; \frac{1}{2} < \frac{p}{A}.$$

Study of the problem at conditions (25) is finished.

D. Flows with different p, q, A

We will move to several cases when not all the conditions in (25) are satisfied.

The first case is

$$p = 0, \; A > 0.$$

For instance, the linear-viscous fluid satisfies this relation (see Sec. VII). The left-hand part of the dispersion equation for this case is the product of two factors

$$\big(k(U - \cos\theta) + iF_0\big)\big(k^2(U - a_+)(U - a_-) + ikA(U - a_{0x})\big) = 0. \tag{46}$$

The root of the first factor is $U = \cos\theta - iF_0/k$ and has the negative imaginary part. The second factor of Eq. (46) is a second-order polynomial and can be studied by the argument method. The stability condition coincides with (9), and waves with obliquity angles

$$-\arccos\frac{1}{\xi} + \pi m < \theta < \arccos\frac{1}{\xi} + \pi m, \quad m = 0, 1, \tag{47}$$

are growing for $|\xi| > 1$. The waves propagating at other angles are damped or neutral.

The next case is

$$A = 0.$$

The kinematic approximation can not be considered under this condition, so this case demands special reasoning. The analysis of the dispersion equation for $A = 0$ shows that there always exist the dispersion equation roots with positive imaginary parts, except the case, when $\partial F/\partial h_0 = 0$ simultaneously with $A = 0$. Under these conditions, there are no roots with positive imaginary parts for all real k and all obliquity angles θ.

Conditions for the flow stability can also be obtained using Statement I if the flow parameters satisfy inequalities

$$p < 0, A > 0 \quad (\text{i.e.,} \quad F_0 > |p|).$$

Applying the argument method, we obtain the condition

$$|\xi\cos\theta| \leq \sqrt{1 + \frac{p}{F_0}\sin^2\theta}.$$

The solution of this inequality for ξ shows that the flow is stable in relation to the perturbations with all angles of propagation for $|\xi| \leq 1$. For $|\xi| > 1$, the perturbations with propagation angles

$$-\arccos\sqrt{\frac{F_0 + p}{F_0\xi^2 + p}} + \pi m < \theta < \arccos\sqrt{\frac{F_0 + p}{F_0\xi^2 + p}} + \pi m, \quad m = 0, 1, \tag{48}$$

are growing.

The flow is always unstable if $A < 0$ (this agrees with Coussot[36] results). Note that $A < 0$ means $\partial\tau/\partial u < 0$. This type of the $\tau(u)$ dependence was observed in flows with very high solid concentrations for which a flow curve had a shear-stress minimum.[32,31]

Note that the waves propagating normally to the basic flow velocity never grow.

VI. SUMMARY OF RESULTS

Now, let us summarize the results of our analysis and compare them with criterion (9) for longitudinal perturbations. In Sec. V, we have analyzed the stability of downslope flows subjected to different conditions

$$p > 0, \quad A > 0 \quad \text{conditions (25)},$$

$$p = 0,\ A > 0, \tag{49}$$

$$A = 0, \tag{50}$$

$$p < 0, A > 0 \quad (\text{i.e.,} \quad F_0 > |p|), \tag{51}$$

$$A < 0. \tag{52}$$

For the cases (49) and (51), the stability criterion coincides with (9) for both longitudinal and oblique perturbations. For $|\xi| > 1$, propagation angles of growing perturbations are found [relations (47) and (48), respectively].

For (50), it was obtained that the flow is unstable for all ξ and all obliquity angles θ, when $\partial F/\partial h_0 \neq 0$, and the flow is stable, when both parameters A and $\partial F/\partial h_0$ are zero.

For negative A [condition (52)], the flow is always unstable, which agrees with the result obtained by Coussot.[36]

At conditions (25), the stability criterion with account of oblique perturbations is again (9) and (10), i.e., the flow is stable for all θ, if $p/A \leq 0.5$. However, if $0.5 < p/A$, the flow is stable for all θ, if and only if $|\xi| \leq \xi_* < 1$ (region 1 in Fig. 7). For $0.5 < p/A$ and $\xi_* < |\xi| \leq 1$, there exist growing oblique perturbations, while the flow is stable to longitudinal waves. Propagation angles of growing oblique perturbations are

$$\pm\arccos\left(\frac{\xi + \sqrt{\xi^2 - \xi_*^2}}{2p/A}\right) + \pi m < \theta < \pm\arccos\left(\frac{\xi - \sqrt{\xi^2 - \xi_*^2}}{2p/A}\right) + \pi m, \quad m = 0,\ 1 \tag{53}$$

(regions 2 in Fig. 7 and gray sectors in Fig. 8(a)].

For $|\xi| > 1$, the flow is unstable in the whole range of parameter p/A satisfying (25); see regions 3 in Fig. 7. But there are oblique waves, which do not grow, their propagation angles are

$$\arccos\left(\frac{\xi - \sqrt{\xi^2 - \xi_*^2}}{2p/A}\right) + \pi m \leq \theta \leq \pi - \arccos\left(\frac{\xi - \sqrt{\xi^2 - \xi_*^2}}{2p/A}\right) + \pi m, \quad m = 0,\ 1 \tag{54}$$

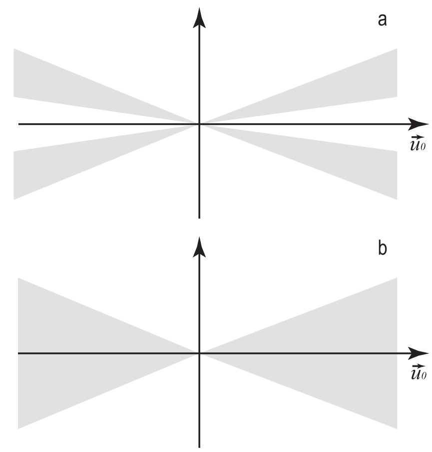

FIG. 8. Sectors of propagation angles for growing (gray sectors) and nongrowing (white sectors) perturbations. (a) corresponds to the regions 2 from Fig. 7; (b) corresponds to the regions 3 from Fig. 7.

[white sectors in Fig. 8(b)].

In conclusion of this section, note that the obtained criteria can be presented as the relations for the friction τ, taking into account (8). In particular, the instability criterion $|\xi| > 1$ in dimensional form is

$$\mathrm{Fr} > \left|\frac{u_0 \dfrac{\partial \tau}{\partial u_0}}{\tau_0 - h_0 \dfrac{\partial \tau}{\partial h_0}}\right| \quad \text{or} \quad \left|\frac{\tau_0 - h_0 \dfrac{\partial \tau}{\partial h_0}}{\dfrac{\partial \tau}{\partial u_0}}\right| > c_0.$$

This form of the instability criterion coincides with Trowbridge.[13]

VII. EXAMPLES OF STABILITY CONDITIONS FOR DIFFERENT BOTTOM FRICTION LAWS

This section is devoted to applying the obtained results to the formulation of stability criteria for flows of various physical nature, in particular, for laminar flows of Newtonian fluid, power-law fluids with different values of the power-law index, and fluids with the Herschel-Bulkley rheological equation. Here, we use the basic assumption adopted in the hydraulics of unsteady flows (Assumption 1): the friction at the bottom in the unsteady flow is a function of the velocity u and depth h corresponding to the friction in the

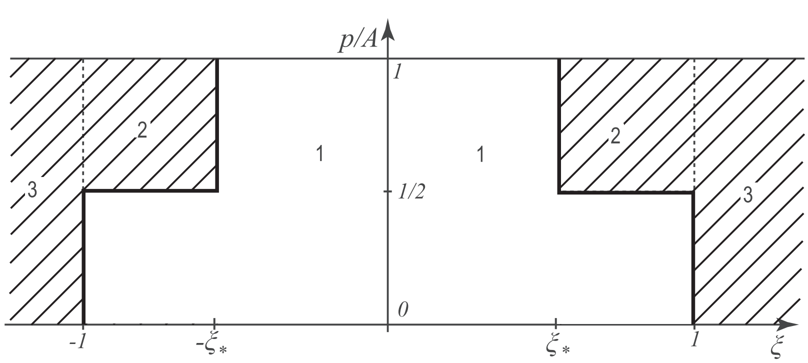

FIG. 7. Stability regions in the plane of parameters $\xi - p/A$. Nonshadowed region 1 including thick lines corresponds to the stability of downslope flow; instability regions 2 and 3 are shaded.

steady flow with the same values of u and h. This is justified by the fact that the hydraulic approach is applicable only to the description of long waves.

As elsewhere in this paper, we consider flows on the wide slopes of a constant incline. The x axis is directed along the velocity of the basic uniform steady flow, the z axis (where it is used) is directed along the normal to the bottom. We use dimensional values in this section.

A. Laminar downslope flow of Newtonian fluid

First, let us consider the laminar flow of Newtonian (linear viscous) fluid under the gravity at the slope of the constant inclination to the horizon. Velocity averaged over the depth in the steady basic flow is

$$u_0 = \frac{1}{3}\frac{\rho g \sin\alpha}{\mu} h_0^2.$$

The shear stress at the bottom, the function F_0, and other parameters are

$$\tau = \frac{3}{\mathrm{Re}}\rho u_0^2, \quad F_0 = \frac{3\mu}{\rho h_0^2}, \quad p = 0, \quad q = \frac{6\mu}{\rho h_0^2}, \quad \xi = 2\mathrm{Fr},$$

and $\mathrm{Re} = \rho u h/\mu$. The conditions (49) are satisfied. Therefore, the stability condition is $|\xi| \le 1$, which results in $\mathrm{Fr} \le 0.5$. This agrees with the result of Trowbridge.[13]

For $\mathrm{Fr} > 0.5$, the waves propagating at angles

$$-\arccos\frac{1}{2\mathrm{Fr}} + \pi m < \theta < \arccos\frac{1}{2\mathrm{Fr}} + \pi m, \quad m = 0, 1,$$

grow [see (47)].

B. Chézy and Manning formulas for the bottom friction in the turbulent downslope flows

There are different approaches for turbulent flows, in which the friction at the bottom is supposed to be governed by specific empirical relations. For instance, the Chézy formula is

$$\tau = \frac{g}{C^2}\rho u_0^2, \; C = const.$$

In this case,

$$F_0 = \frac{g}{C^2}\frac{u_0}{h_0}, \quad p = q = F_0, \quad \xi = \frac{\mathrm{Fr}}{2}, \quad \xi_* = 1,$$

thus, the conditions (25) are satisfied, and $p/A = 0.5$. The stability condition is

$$|\xi| \le 1, \text{ i.e., } \mathrm{Fr} \le 2,$$

which corresponds to Jeffreys[6] result.

For $\mathrm{Fr} > 2$, only the waves with obliquity angles

$$-\arccos\frac{\mathrm{Fr} - \sqrt{\mathrm{Fr}^2 - 4}}{2} + \pi m < \theta < \arccos\frac{\mathrm{Fr} - \sqrt{\mathrm{Fr}^2 - 4}}{2} + \pi m, \quad m = 0, 1,$$

grow [compare with (54)].

The friction at the bottom can be governed by the Manning formula

$$\tau = \frac{gl^2}{h_0^{1/3}}\rho u_0^2,$$

where l is related with the bottom roughness. In this case,

$$F_0 = \frac{gl^2}{h_0^{4/3}}\rho u_0, \quad p = F_0, \quad q = \frac{4}{3}F_0, \quad \xi = \frac{2}{3}\mathrm{Fr}, \quad \xi_* = 1,$$

the conditions (25) are satisfied, and $p/A = 0.5$. The stability condition is

$$\mathrm{Fr} \le 1.5,$$

which corresponds to Dressler and Pohle[10] result.

For $\mathrm{Fr} > 1.5$, the waves with angles

$$-\arccos\left(\frac{2}{3}\mathrm{Fr} - \frac{2}{3}\sqrt{\mathrm{Fr}^2 - \frac{9}{4}}\right) + \pi m < \theta < \arccos\left(\frac{2}{3}\mathrm{Fr} - \frac{2}{3}\sqrt{\mathrm{Fr}^2 - \frac{9}{4}}\right) + \pi m, \quad m = 0, 1,$$

grow [compare with (54)].

In examples considered so far general stability criterion coincides with the criterion for the 1D problem. The next example demonstrates the opposite situation.

C. Laminar downslope flow of power-law fluid

The rheological relation for the power-law fluid in the simplest shear flow is

$$|\tau_{xz}| = K\left|\frac{\partial u_x}{\partial z}\right|^n,$$

where K is the consistency index and $n > 0$ is the power-law index. The friction at the bottom τ for the nonperturbed flow is

$$\tau = \rho h_0 u_0 F_0, \quad \text{where} \quad F_0 = \left(\frac{2n+1}{n}\right)^n \frac{K}{\rho}\frac{u_0^{n-1}}{h_0^{n+1}}.$$

In this case, $p = (n-1)F_0$, $A = nF_0$, and $p/A = (n-1)/n$ so that

$$\frac{p}{A} < 0 \text{ for } n < 1,$$

$$\frac{p}{A} = 0 \text{ for } n = 1 \text{ (the linear viscous fluid)},$$

$$\frac{p}{A} > 0 \text{ for } n > 1.$$

The propagation velocity of the large-scale perturbations with respect to the nonperturbed flow is

$$N_0 = \frac{n+1}{n}u_0$$

so that

$$|\xi| = \frac{n+1}{n}\mathrm{Fr}.$$

For pseudoplastic (shear-thinning) fluids, $0 < n < 1$, and the stability condition is

$$|\xi| \le 1 \quad \text{i.e.,} \quad \mathrm{Fr} \le \frac{n}{n+1}. \tag{55}$$

If $\mathrm{Fr} > n/(n+1)$, the waves propagating with the angles

$$-\arccos\sqrt{\frac{n^3}{(n+1)^2\mathrm{Fr}^2+n^2(n-1)}}+\pi m<\theta<\arccos\sqrt{\frac{n^3}{(n+1)^2\mathrm{Fr}^2+n^2(n-1)}}+\pi m,\quad m=0,\ 1,$$

are growing [see (48)].

For the dilatant (shear-thickening) fluids with $1 < n \le 2$, which corresponds to $0 < p/A \le 0.5$, the stability condition coincides with (55); perturbations are growing, if they propagate at the following angles:

$$-\arccos\left(\frac{\mathrm{Fr}}{2}\frac{n+1}{n-1}-\sqrt{\frac{(n+1)^2}{(n-1)^2}\frac{\mathrm{Fr}^2}{4}-\frac{1}{n-1}}\right)+\pi m<\theta<\arccos\left(\frac{\mathrm{Fr}}{2}\frac{n+1}{n-1}-\sqrt{\frac{(n+1)^2}{(n-1)^2}\frac{\mathrm{Fr}^2}{4}-\frac{1}{n-1}}\right)+\pi m,$$
$$m=0,\ 1.$$

For dilatant fluids with $n > 2$, we have $p/A > 0.5$ and $\xi_* = 2\sqrt{(n-1)}/n$. The stability condition is

$$\mathrm{Fr}\le 2\frac{\sqrt{n-1}}{n+1}.$$

For

$$2\frac{\sqrt{n-1}}{n+1}<\mathrm{Fr}\le\frac{n}{n+1},$$

there are growing oblique perturbations, while the longitudinal perturbations are damped or neutral [the situation is like that in Fig. 8(a)]. Perturbations propagating with angles

$$\pm\arccos\left(\frac{\mathrm{Fr}}{2}\frac{n+1}{n-1}+\sqrt{\frac{(n+1)^2}{(n-1)^2}\frac{\mathrm{Fr}^2}{4}-\frac{1}{n-1}}\right)+\pi m<\theta<\pm\arccos\left(\frac{\mathrm{Fr}}{2}\frac{n+1}{n-1}-\sqrt{\frac{(n+1)^2}{(n-1)^2}\frac{\mathrm{Fr}^2}{4}-\frac{1}{n-1}}\right)+\pi m,$$
$$m=0,\ 1,$$

grow.

Let us compare results obtained by Allouche *et al.*[25] with ours. We obtained that for dilatant fluids with $1 < n \le 2$, oblique perturbations can not cause the loss of stability if longitudinal perturbations do not grow. The same result was obtained by Allouche *et al.*[25] by numerical calculations for $n = 1.2$, 1.5, and 1.65. However, our study shows that for $n > 2$ (not considered by Allouche *et al.*[25]) the situation differs. As for shear-thinning fluids, our results are not in agreement with those by Allouche *et al.*[25] Note that rheological models used in the paper of Allouche *et al.*[25] and in our investigation are different.

D. Laminar downslope flow of Herschel-Bulkley fluid

The Herschel-Bulkley fluid is the medium with the yield stress τ_y and the power-law dependence of the shear stresses on shear-strain rates; rheological relation for simple shear flow of this fluid is

$$\begin{cases}\dfrac{\partial u}{\partial z}=0 & \text{for}\quad |\tau_{xz}|\le\tau_y,\\ |\tau_{xz}|=\tau_y+K\left|\dfrac{\partial u}{\partial z}\right|^n & \text{for}\quad |\tau_{xz}|\ge\tau_y.\end{cases}$$

In open flows of medium with a yield stress, a layer adjacent to the upper surface exists that moves without deformation ("quasisolid" layer). The small value of the tangential stress on the upper surface (usually it is assumed to be zero) is the reason. Denote the thickness of the quasisolid layer by h_b. In a stationary shear flow, $h_b = \tau_y/(\rho g\sin\alpha)$, i.e., the thickness of the quasisolid layer depends on neither the velocity nor the depth of the flow. We can express the friction at the bottom τ in a stationary basic flow in terms of its depth, the depth-averaged velocity, and the value of h_b. The result is

$$\tau=\rho h_0u_0F_0,$$
$$F_0=\frac{K}{\rho}\left(\frac{(n+1)(2n+1)}{n(nh_b+(n+1)h_0)}\right)^n\frac{h_0^n}{(h_0-h_b)^{1+n}}u_0^{n-1}.$$

In accordance with Assumption 1, while using the hydraulic approach, we can assume that in nonstationary flows, these formulas are still valid and h_b does not depend on h_0 and u_0. Then, $p/A = (n-1)/n$ and the criteria of stability with account of oblique perturbations are different for fluids with $n \le 1$ and $n > 1$. The large-scale perturbation velocity is

$$N_0=-u_0\frac{1}{n}\left(\frac{n^2h_b}{nh_b+(1+n)h_0}-\frac{(1+n)h_0}{h_0-h_b}\right)\ge 0$$

for the laminar flow of the Herschel-Bulkley fluid. It follows from here that for the power-law index $n \le 2$, the stability condition for the flow of the Herschel-Bulkley fluid is

$$|\xi|\le 1,\ \text{i.e., }\mathrm{Fr}\le n\left(\frac{(1+n)h_0}{h_0-h_b}-\frac{n^2h_b}{nh_b+(1+n)h_0}\right)^{-1},$$

and the angles of growing and nongrowing waves for $|\xi| > 1$ can be found using (54). The critical Froude number

$$\mathrm{Fr}_{cr}=n\left(\frac{(1+n)h_0}{h_0-h_b}-\frac{n^2h_b}{nh_b+(1+n)h_0}\right)^{-1}$$

coincides with the result obtained by Coussot[8] for the downslope flow of the Herschel-Bulkley fluid under longitudinal perturbations.

The stability condition for the flow of Herschel-Bulkley fluid with $n > 2$ is $|\xi| \le \xi_*$ $[\xi_* = 2\sqrt{(n-1)}/n]$ and can be written through the Froude number as

$$\mathrm{Fr}\le 2\sqrt{n-1}\left(\frac{(1+n)h_0}{h_0-h_b}-\frac{n^2h_b}{nh_b+(1+n)h_0}\right)^{-1}.$$

Thus, for the flow of the Herschel-Bulkley fluid with $n > 2$, there is the range of Froude numbers

$$2\sqrt{n-1}\left(\frac{(1+n)h_0}{h_0-h_b}-\frac{n^2h_b}{nh_b+(1+n)h_0}\right)^{-1}<\mathrm{Fr}\le n\left(\frac{(1+n)h_0}{h_0-h_b}-\frac{n^2h_b}{nh_b+(1+n)h_0}\right)^{-1},$$

at which the longitudinal perturbations are damped or neutral, but there are growing oblique waves [the propagation angles of these waves are given by the formula (53)].

Remark. In fact, if the depth of the flow changes as a result of the propagation of disturbances, then the upper quasisolid layer undergoes deformation (bending, stretching, and shortening). In a more accurate description, it would be necessary to take into account the effect of these deformations, considering this layer as a highly viscous or, probably, as viscoelastic. We assumed that for long waves of small amplitude, these deformations can be neglected. However, this problem requires more complete analysis.[37]

E. Numerical example

Let us consider the set of parameters for the flow of the Herschel-Bulkley fluid close to the data given by Kern *et al.*,[38] which can be used for dense avalanches. Namely, let us take

$$n = 2.5,\ \tau_y = 1000\ \frac{\mathrm{kg}}{\mathrm{m\,s^2}},\ K = 44.5\ \mathrm{m}^2,$$
$$\rho = 500\ \frac{\mathrm{kg}}{\mathrm{m}^3},\ \sin\alpha = 0.3. \tag{56}$$

In this case, $p/A = 0.6 > 0.5$. Let us take Fr = 0.367. In this case, we have $u_0 = 1.27$ m/s and $h_0 = 1.285$ m. We have obtained the roots of the dispersion equation (20) for different perturbation propagation angles numerically. Imaginary parts of the roots of the dispersion equation are negative for longitudinal (damped) wave, and $\mathrm{Im}\,U_1$ is shown in Fig. 9(b) by the dashed line (U_1 is one of the roots of $D = 0$). Full lines in Figs. 9(a) and 9(b) correspond to the root U_1 of the dispersion equation for the wave propagating at angle, lying at the center of the unstable sector; $\mathrm{Im}\,U_1$ changes the sign [Fig. 9(b)]. In Fig. 9(d), we show the behavior of the root U_1 with the angle increasing. Lines 1 and 6 lie in the stable region [line 1 corresponds to the lightest gray color in Fig. 9(c), line 6 corresponds to the black lines]. Lines 2 and 5 lie at the boundary of that region [at the boundaries of the gray sectors in Fig. 9(c)]; here, perturbations are also damped, but $\mathrm{Im}\,U_1 \to 0$ as $k \to \infty$. Lines 3 and 4 lie in the region of instability [inside the gray sectors in Fig. 9(c)]: there are the roots with positive imaginary parts. Here, we also have to give one more clarification. The boundary of the angle sector, for which perturbations grow, is not the line of neutral stability for finite k. It can be shown analytically that for these angles, the dispersion equation linearized as $k \to \infty$ has the roots with negative imaginary parts, but $\mathrm{Im}\,U \to 0$ as $k \to \infty$.

VIII. CONCLUSIONS

The stability of the downslope flows to oblique perturbations is studied. The flow is described by the continuity and momentum equations in the hydraulic approximation. The dispersion equation is derived and analyzed using the argument method for complex analytical functions. It is shown that the absence of growth of longitudinal perturbations does not guarantee the stability of the flow. The ranges of the bottom and flow parameters are obtained, for which oblique perturbations grow, while longitudinal perturbations are damped or neutral. Angles for the growing perturbations are indicated. Several cases of flow parameters are considered, and it is shown that for the laminar downslope flow of the Newtonian fluid and for the flows with the bottom friction given by the Chézy and Manning formulas, the results obtained in this study agree with

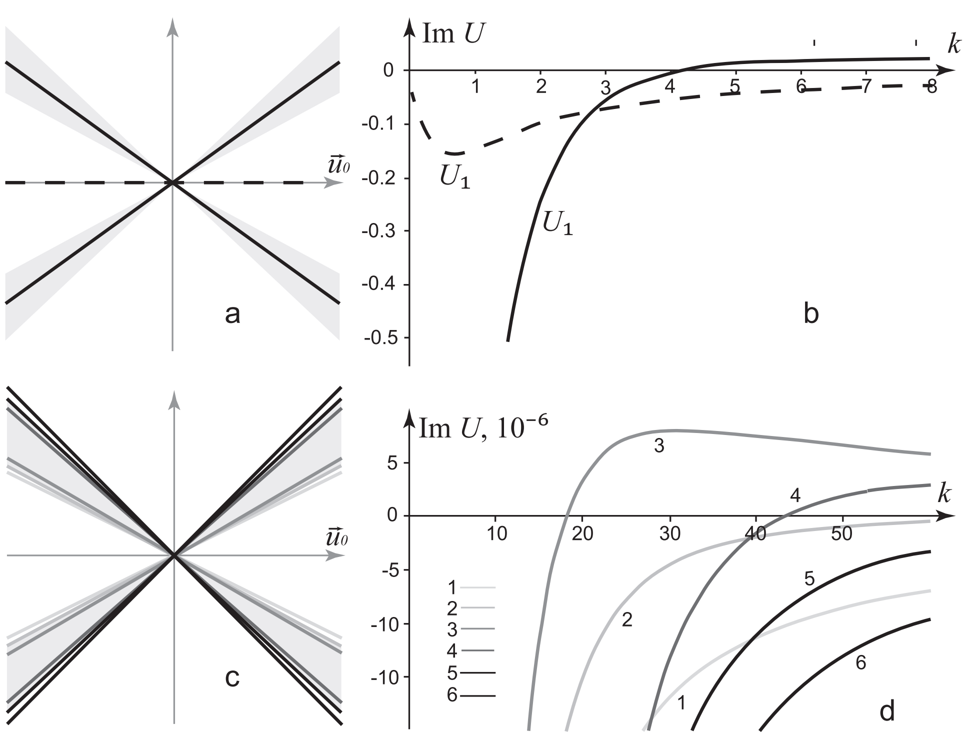

FIG. 9. Sectors of angles for growing (gray) and nongrowing (white) perturbations [(a) and (c)]; lines in (a) and (c) show different obliquity angles of the disturbances considered for the calculation with the parameters (56). Numerically obtained dependence $\mathrm{Im}\,U_1$ on k for longitudinal perturbation [dashed line, (b)] and oblique perturbation [full line, (b)]. (d) shows the behavior of $\mathrm{Im}\,U_1$ for different obliquity angles θ, while θ increases from the values in the stability region (lines 1 and 2) to the region of instability (lines 3 and 4), and again to the stability region for larger obliquity angles (lines 5 and 6).

known results from the literature. It is found that for the flows of the power-law fluid and the Herschel-Bulkley fluid, there is the range of parameter n, for which the instability of the flow is governed by the behavior of oblique perturbations.

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