Equs of sandpile problem in granular flow system

Haoyu Lu
SPST Shanghaitech

Version: 3.0

Date: April 16, 2023

Abstract

This article recalculates equs of sandpile problem in granular flow system based on Lame-coefficient expression. It includes coordinate transformation, strain-rate tensors and constitutive relations in this system.

Keywords: Sandpile problem, Lame-coefficient, coordinate transformation, constitutive relation

1 Geometry settings

1.1 Coordinate transformation

To simplify this problem in calculation, we apply coordinate transformation as shown in figure 1. The transformation functions are defined as

$$\begin{aligned} a &= \sqrt{x^2+y^2}\cos\alpha - z\sin\alpha \\ \theta &= \arctan\frac{y}{x} \\ b &= \sqrt{x^2+y^2}\sin\alpha + z\cos\alpha \end{aligned} \tag{1}$$

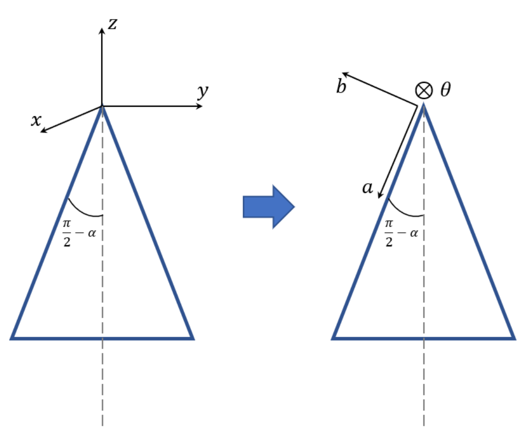

Figure 1: Coordinate transformation

1.2 Differential operations expressed by Lame coefficients

Lame coefficients:

$$\begin{aligned} h_1 &= 1 \\ h_2 &= a\cos\alpha + b\sin\alpha \\ h_3 &= 1 \end{aligned} \tag{2}$$

Differential operations:

Time D: here $\partial_t \equiv \frac{\partial}{\partial t}$, $d_t \equiv \frac{D}{Dt} = \partial_t + \vec{u}\cdot\nabla$

$$d_t = \partial_t + u_a\partial_a + \frac{1}{h_2}u_\theta\partial_\theta + u_b\partial_b \tag{3}$$

Div

$$\text{div}\,\vec{u} = \partial_a u_a + \frac{1}{h_2}\partial_\theta u_\theta + \partial_b u_b + \frac{1}{h_2}(u_a\cos\alpha + u_b\sin\alpha) \tag{4}$$

Directional derivative

First we calculate it for base vectors.

$$\left(\hat{e}_\tau\cdot\nabla\right)\hat{e}_\rho = -\frac{1}{h_\tau}\left(\nabla h_\rho\right)\delta_\tau^\rho + \frac{1}{h_\rho h_\tau}\frac{\partial h_\tau}{\partial x_\rho}\hat{e}_\tau \tag{5}$$

Thus in this (a,θ,b) coordinates, we have

$$\begin{array}{cccc} & \hat{e}_a & \hat{e}_\theta & \hat{e}_b \\ (\hat{e}_a\cdot\nabla) & 0 & 0 & 0 \\ (\hat{e}_\theta\cdot\nabla) & \frac{\cos\alpha}{h_2}\hat{e}_\theta & -\frac{\cos\alpha}{h_2}\hat{e}_a + \frac{\sin\alpha}{h_2}\hat{e}_b & \frac{\sin\alpha}{h_2}\hat{e}_\theta \\ (\hat{e}_b\cdot\nabla) & 0 & 0 & 0 \end{array} \tag{6}$$

Then we can obtain the convection terms by expanding vectors in coordinates

$$\begin{aligned} \partial_t + (\vec{u}\cdot\nabla)\vec{u} &= \partial_t + \left[(u_a\hat{e}_a + u_\theta\hat{e}_\theta + u_b\hat{e}_b)\cdot\nabla\right](u_a\hat{e}_a + u_\theta\hat{e}_\theta + u_b\hat{e}_b) \\ &= \partial_t + \left(u_a\mathbf{\hat{e}}_a\cdot\boldsymbol{\nabla}\right)\left(u_a\mathbf{\hat{e}}_a + u_\theta\mathbf{\hat{e}}_\theta + u_b\mathbf{\hat{e}}_b\right) \\ &\quad + \left(u_\theta\mathbf{\hat{e}}_\theta\cdot\boldsymbol{\nabla}\right)\left(u_a\mathbf{\hat{e}}_a + u_\theta\mathbf{\hat{e}}_\theta + u_b\mathbf{\hat{e}}_b\right) + \left(u_b\mathbf{\hat{e}}_b\cdot\boldsymbol{\nabla}\right)\left(u_a\mathbf{\hat{e}}_a + u_\theta\mathbf{\hat{e}}_\theta + u_b\mathbf{\hat{e}}_b\right) \end{aligned} \tag{7}$$

Thus the final form is

$$\begin{aligned} \left[\partial_t + (\vec{u}\cdot\nabla)\vec{u}\right]_a &= d_t u_a - \frac{1}{h_2}(u_\theta)^2\cos\alpha \\ \left[\partial_t + (\vec{u}\cdot\nabla)\vec{u}\right]_\theta &= d_t u_\theta + \frac{u_\theta}{h_2}(u_a\cos\alpha + u_b\sin\alpha) \\ \left[\partial_t + (\vec{u}\cdot\nabla)\vec{u}\right]_b &= d_t u_b - \frac{1}{h_2}(u_\theta)^2\sin\alpha \end{aligned} \tag{8}$$

1.3 Derivative strain-stress tensor D_{ij}

In 3d cartesian coordinate, we have definitions below

1. deviatoric strain-rate tensor

$$D_{ij} = -\frac{1}{3}(\text{div}\vec{u})\delta_{ij} + \frac{1}{2}(\partial_j u_i + \partial_i u_j) \tag{9}$$

2. total strain-rate tensor

$$S_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j) \tag{10}$$

While in different coordinates, they will have different forms. Here I express these tensors based on Lame-coefficient expression.

In any curvlinear coordinate, we can rewrite total strain-rate tensor as below

$$\begin{aligned}
s_{11} &= \frac{1}{h_1}\frac{\partial u_1}{\partial q_1} + \frac{u_2}{h_1 h_2}\frac{\partial h_1}{\partial q_2} + \frac{u_3}{h_1 h_3}\frac{\partial h_1}{\partial q_3} \\
s_{22} &= \frac{1}{h_2}\frac{\partial u_2}{\partial q_2} + \frac{u_3}{h_2 h_3}\frac{\partial h_2}{\partial q_3} + \frac{u_1}{h_2 h_1}\frac{\partial h_2}{\partial q_1} \\
s_{33} &= \frac{1}{h_3}\frac{\partial u_3}{\partial q_3} + \frac{u_1}{h_3 h_1}\frac{\partial h_3}{\partial q_1} + \frac{u_2}{h_3 h_2}\frac{\partial h_3}{\partial q_2} \\
2s_{12} = 2s_{21} &= \frac{1}{h_2}\frac{\partial u_1}{\partial q_2} + \frac{1}{h_1}\frac{\partial u_2}{\partial q_1} - \frac{u_1}{h_1 h_2}\frac{\partial h_1}{\partial q_2} - \frac{u_2}{h_1 h_2}\frac{\partial h_2}{\partial q_1} \\
2s_{23} = 2s_{32} &= \frac{1}{h_3}\frac{\partial u_2}{\partial q_3} + \frac{1}{h_2}\frac{\partial u_3}{\partial q_2} - \frac{u_2}{h_2 h_3}\frac{\partial h_2}{\partial q_3} - \frac{u_3}{h_2 h_3}\frac{\partial h_3}{\partial q_2} \\
2s_{31} = 2s_{13} &= \frac{1}{h_1}\frac{\partial u_3}{\partial q_1} + \frac{1}{h_3}\frac{\partial u_1}{\partial q_3} - \frac{u_3}{h_3 h_1}\frac{\partial h_3}{\partial q_1} - \frac{u_1}{h_3 h_1}\frac{\partial h_1}{\partial q_3}
\end{aligned} \tag{11}$$

Their concrete form is

$$\begin{aligned}
S_{11} &= \partial_a u_a \\
S_{22} &= \frac{1}{h_2}\partial_\theta u_\theta + \frac{u_b \sin\alpha + u_a \cos\alpha}{h_2} \\
S_{33} &= \partial_b u_b \\
2S_{12} = 2S_{21} &= \partial_a u_\theta + \frac{1}{h_2}\partial_\theta u_a + -\frac{u_\theta \cos\alpha}{h_2} \\
2S_{23} = 2S_{32} &= \partial_b u_\theta + \frac{1}{h_2}\partial_\theta u_b - \frac{u_\theta \sin\alpha}{h_2} \\
2S_{31} = 2S_{13} &= \partial_a u_b + \partial_b u_a
\end{aligned} \tag{12}$$

Plug (4) and (12) in (9), we can obtain

$$
\begin{aligned}
D_{11} &= \frac{2}{3}\partial_a u_a - \frac{1}{3}\frac{1}{h_2}\partial_\theta u_\theta - \frac{1}{3}\partial_b u_b - \frac{1}{3}\frac{u_a\cos\alpha + u_b\sin\alpha}{h_2} \\
D_{22} &= -\frac{1}{3}\partial_a u_a + \frac{2}{3}\frac{1}{h_\theta}\partial_\theta u_\theta - \frac{1}{3}\partial_b u_b + \frac{2}{3}\frac{u_b\sin\alpha + u_a\cos\alpha}{h_2} \\
D_{33} &= -\frac{1}{3}\partial_a u_a - \frac{1}{3}\frac{1}{h_\theta}\partial_\theta u_\theta + \frac{2}{3}\partial_b u_b - \frac{1}{3}\frac{u_b\sin\alpha + u_a\cos\alpha}{h_2} \\
2D_{12} = 2D_{21} &= \partial_a u_\theta + \frac{1}{h_2}\partial_\theta u_a + -\frac{u_\theta\cos\alpha}{h_2} \\
2D_{23} = 2D_{32} &= \partial_b u_\theta + \frac{1}{h_2}\partial_\theta u_b - \frac{u_\theta\sin\alpha}{h_2} \\
2D_{31} = 2D_{13} &= \partial_a u_b + \partial_b u_a
\end{aligned}
\tag{13}
$$

2 Kinetic equations and constitutive relations

Mass conversation:

$$
(\partial_t + \vec{u}\cdot\nabla)\phi + \phi(\nabla\cdot\vec{u}) = 0 \tag{14}
$$

Momentum conversation: (Note: $T_{ij} = -p\delta_{ij} + \tau_{ij}$)

$$
\hat{a}: \quad \rho\phi\left(d_t u_a - \frac{(u_\theta)^2}{h_2}\cos\alpha\right) = (\nabla\cdot\overleftrightarrow{T})_a + \rho\phi g\sin\alpha \tag{15}
$$

$$
\hat{\theta}: \quad \rho\phi\left(d_t u_\theta + \frac{u_\theta}{h_2}(u_a\cos\alpha + u_b\sin\alpha)\right) = (\nabla\cdot\overleftrightarrow{T})_\theta \tag{16}
$$

$$
\hat{b}: \quad \rho\phi\left(d_t u_b - \frac{(u_\theta)^2}{h_2}\sin\alpha\right) = (\nabla\cdot\overleftrightarrow{T})_b - \rho\phi g\cos\alpha \tag{17}
$$

Constitutive relations:(Coulomb constitutive model)

$$
\begin{gathered}
\text{Flow rule: div } \vec{u} = 0 \\
\text{Yield condition: } \|\tau\| = \mu p \\
\text{Alignment: } \frac{D_{ij}}{\|D\|} = \frac{\tau_{ij}}{\|\tau\|}
\end{gathered}
\tag{18}
$$

Here functions (14) to (18) represent all granular flow governing equations to describe this problem in our model.

Notice that It is well-posed or ill-posed depends on the restriction functions that are mentioned in different approximations.

3 Pertubation criterions[1]

Before we simplify every restriction function under this approximation, we should announce pertubation criterions here.

$$\begin{aligned} &M = (u_a, u_\theta, u_b, \phi, p) \\ &M = M^{(0)} + \hat{M} \end{aligned} \tag{19}$$

Here (0) means the base state.

Notice that it is convenient to temporarily drop most terms not of maximal order and estimate their effect in a calculation at the end of the argument.[1] This method can be summed up as 2 principles

1.Retain only terms that are linear in the pertubation

2.Freeze coefficients at an arbitrary point(x^*, t^*)

Here it is convenient to set this 'arbitrary point' to the initial point(base-state). We use eq(14) as an example.

Its linear form is

$$\begin{aligned} &u_j^* = u_j^{(0)}(x^*, t^*) \\ &\phi^* = \phi^{(0)}(x^*, t^*) \\ &(\partial_t + u_1^*\partial_1 + u_2^*\partial_2)\,\hat{\phi} + \phi^*\,(\partial_1\hat{u}_1 + \partial_2\hat{u}_2) = 0 \end{aligned} \tag{20}$$

Notice that lower order terms like $\partial_j\phi^*\hat{u}_j$ and $\partial_j u_j^*\hat{\phi}$ have been dropped in eq(20).

For these pertubation terms, our goal is to using them to find relationships in these modes. We can expand these terms in exponential way.

$$\hat{M}(\mathbf{x}, t) = e^{i\langle\mathbf{k},\mathbf{x}\rangle + \lambda t}\tilde{M} \tag{21}$$

In this notation, $\tilde{M}$ means the amplitude of every parameter, λ means the growth rate.

Our goal is to find Re$[\lambda] = f(k_a, k_\theta, k_b)$, and it needs to satisfy properties as below.

$$\max_{i,j=1,2,3}\sup_{k\in\mathbb{R}^2}\text{Re}[\lambda_i(k_j)] < \infty \tag{22}$$

4 Case 1: If without friction what happens?

Here if we ignore $\overleftrightarrow{\tau} \longrightarrow$ no friction. There are $(u_a, u_\theta, u_b, \phi, p)$ 5 parameters with 5 restriction functions. Thus this case can be seemed as well-posed.

Case 1.1 Here we set the base-state as

$$(a^{(0)}, \theta^{(0)}, b^{(0)}, u_a^{(0)}, u_\theta^{(0)}, u_b^{(0)}, \vec{p}^{(0)}, t^{(0)}) = (a^*, 0, h, u_a^*, 0, 0, \vec{p}^*, t^*) \tag{23}$$

Assume initial conditions are

$$a^*(t=0) = 0 \quad b^*(t=0) = h \quad u_a^*(t=0) = u_0 \quad \vec{p}^*(t=0) = (0, 0, p_0)$$

Reconsider all governing equations in base-state:

(14):

$$(\partial_t + \vec{u} \cdot \nabla)\phi = 0 \longrightarrow \phi = \text{Const} \tag{24}$$

(3):

$$d_t = \partial_t + u_a^* \partial_a \tag{25}$$

(18):

$$\text{div } \vec{u} = \partial_a u_a^* + \frac{1}{h_2} u_a^* \cos\alpha = 0 \tag{26}$$

(15):

$$d_t u_a^* = -\frac{1}{\rho\phi} \partial_a p^* + g \sin\alpha \tag{27}$$

(16):

$$0 = -\frac{1}{\rho\phi} \partial_\theta p^* \tag{28}$$

(17):

$$0 = -\frac{1}{\rho\phi} \partial_b p^* - g\cos\alpha \tag{29}$$

From (28):

$$p_\theta^* = C = 0 \tag{30}$$

From (29):

$$p_b^* = p_0 + \rho\phi g \cos\alpha (h - b) \tag{31}$$

From (26):

$$u_a^* = \frac{1}{h_2} u_0 h \sin\alpha \tag{32}$$

From (27) use the result(32):

$$p_a^* = \rho\phi \left(g \sin\alpha a + \frac{1}{2}(u_0)^2 \left(1 - \left(\frac{h \sin\alpha}{h_2} \right)^2 \right) \right) \tag{33}$$

Reconsider all governing equations in real-state (No small quantities have been abandoned at this time):

(14) and (3) remain the form as (24) and (25) give.

(18):

$$\begin{aligned}&\partial_a(\cancel{u_a^*} + \hat{u}_a) + \tfrac{1}{h_2}\partial_\theta \hat{u}_\theta + \partial_b \hat{u}_b + \tfrac{1}{h_2}[(\cancel{u_a^*} + \hat{u}_a)\cos\alpha + \hat{u}_b \sin\alpha] = 0 \\ &\partial_a \hat{u}_a + \tfrac{1}{h_2}\partial_\theta \hat{u}_\theta + \partial_b \hat{u}_b + \tfrac{\cos\alpha}{h_2}\hat{u}_a + \tfrac{\sin\alpha}{h_2}\hat{u}_b = 0\end{aligned} \tag{34}$$

(15):

$$\begin{aligned}&d_t(\cancel{u_a^*} + \hat{u}_a) - \tfrac{(\hat{u}_\theta)^2}{h_2}\cos\alpha = -\tfrac{1}{\rho\phi}\partial_a(\cancel{p_a^*} + \hat{p}_a) + \cancel{g\sin\alpha} \\ &(\partial_t + u_a^*\partial_a)\hat{u}_a - \tfrac{\cos\alpha}{h_2}(\hat{u}_\theta)^2 + \tfrac{1}{\rho\phi}\partial_a \hat{p}_a = 0\end{aligned} \tag{35}$$

(16):

$$\begin{aligned}&d_t \hat{u}_\theta + \tfrac{\hat{u}_\theta}{h_2}[(u_a^* + \hat{u}_a)\cos\alpha + \hat{u}_b\sin\alpha] = -\tfrac{1}{\rho\phi}\tfrac{1}{h_2}\partial_\theta(\cancel{p_\theta^*} + \hat{p}_\theta) \\ &(\partial_t + u_a^*\partial_a)\hat{u}_\theta + \tfrac{1}{\rho\phi}\tfrac{1}{h_2}\partial_\theta \hat{p}_\theta + \tfrac{\hat{u}_\theta}{h_2}[(u_a^* + \hat{u}_a)\cos\alpha + \hat{u}_b\sin\alpha] = 0\end{aligned} \tag{36}$$

(17):

$$\begin{aligned}&d_t \hat{u}_b - \tfrac{(\hat{u}_\theta)^2}{h_2}\sin\alpha = -\tfrac{1}{\rho\phi}\partial_b(\cancel{p_b^*} + \hat{p}_b) - \cancel{g\cos\alpha} \\ &(\partial_t + u_a^*\partial_a)\hat{u}_b + \tfrac{1}{\rho\phi}\partial_b \hat{p}_b - \tfrac{\sin\alpha}{h_2}(\hat{u}_\theta)^2 = 0\end{aligned} \tag{37}$$

Now apply (21) to (34),(35),(36),(37) and use the pertubation principle 1, we can obtain

(34):

$$\left(ik_a + \frac{\cos\alpha}{h_2}\right)\hat{u}_a + \frac{1}{h_2}(ik_\theta)\hat{u}_\theta + \left(ik_b + \frac{\sin\alpha}{h_2}\right)\hat{u}_b = 0 \tag{38}$$

(35):

$$(\lambda + ik_a u_a^*)\hat{u}_a + \frac{1}{\rho\phi}ik_a\hat{p}_a = 0 \tag{39}$$

(36):

$$\left(\lambda + ik_a u_a^* + \frac{\cos\alpha}{h_2}u_a^*\right)\hat{u}_\theta + \frac{1}{\rho\phi}\frac{1}{h_2}ik_\theta\hat{p}_\theta = 0 \tag{40}$$

(37):

$$(\lambda + ik_a u_a^*)\hat{u}_b + \frac{1}{\rho\phi}ik_b\hat{p}_b = 0 \tag{41}$$

Rewrite (38),(39),(40),(41) as $Ax = 0$ form

$$\begin{pmatrix} \lambda + ik_a u_a^* & 0 & 0 & \frac{1}{\rho\phi}ik_a \\ 0 & \lambda + ik_a u_a^* + \frac{\cos\alpha}{h_2}u_a^* & 0 & \frac{1}{\rho\phi}\frac{1}{h_2}ik_\theta \\ 0 & 0 & \lambda + ik_a u_a^* & \frac{1}{\rho\phi}ik_b \\ ik_a + \frac{\cos\alpha}{h_2} & \frac{1}{h_2}ik_\theta & ik_b + \frac{\sin\alpha}{h_2} & 0 \end{pmatrix} \begin{pmatrix} \hat{u}_a \\ \hat{u}_\theta \\ \hat{u}_b \\ \hat{p} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{42}$$

Directly calculate det(A)=0. Here we use symbolic notation to apply calculation in **Sympy**

$$A = u_a^*; \quad B = \frac{1}{\rho\phi}; \quad C = \alpha; \quad D = k_a; \quad E = k_\theta; \quad F = k_b; \quad H = \frac{1}{h_2}; \quad X = \lambda;$$

Results group is

$$[x_1, x_2] = \left[-iAD, \; \frac{A\left(-2iD^3 - 4D^2H\cos(C) - 2iDE^2H^2 - 2iDF^2 - 2DFH\sin(C) + 2iDH^2\cos^2(C) - 2F^2H\cos(C) + iFH^2\sin(2C)\right)}{2\left(D^2 - iDH\cos(C) + E^2H^2 + F^2 - iFH\sin(C)\right)} \right] \tag{43}$$

Notice that the result x_1 is the trivial solution of this system that just the base-state we have mentioned above.

$$x_1 = -iAD = -ik_a u_a^* \longrightarrow e^{\lambda t} = e^{-ik_a u_a^* t} \tag{44}$$

The mode characteristic frequency is

$$\omega^{(0)} \equiv \omega^* = k_a u_a^* \tag{45}$$

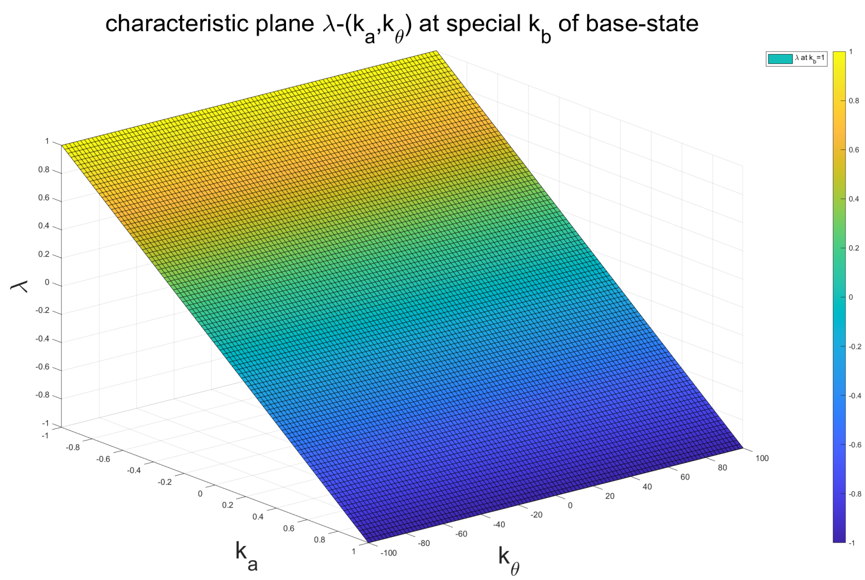

Figure 2: characteristic plane of base-state

For result x_2 is too messy and we simplify it and divide it into real-part and im-part

$$\mathrm{Re}[x_2] = \frac{AH\left(-D^4 - D^2E^2H^2 - 2D^2F^2 - D^2H^2\cos^2(C) - 2DFH^2\sin(C)\cos(C) - E^2F^2H^2 - F^4 + F^2H^2\cos^2(C) - F^2H^2\right)\cos(C)}{H^2\left(D\cos(C) + F\sin(C)\right)^2 + \left(D^2 + E^2H^2 + F^2\right)^2} \tag{46}$$

$$\begin{aligned} \mathrm{Im}[x_2] = & \frac{A\left(-2D^5 - 4D^3E^2H^2 - 4D^3F^2 - D^3H^2\cos(2C) - D^3H^2 - 2D^2FH^2\sin(2C)\right)}{2\left(H^2\left(D\cos(C) + F\sin(C)\right)^2 + \left(D^2 + E^2H^2 + F^2\right)^2\right)} \\ & + \frac{-2DE^4H^4 - 4DE^2F^2H^2 + DE^2H^4\cos(2C) + DE^2H^4 - 2DF^4 + DF^2H^2\cos(2C) - DF^2H^2 + E^2FH^4\sin(2C)}{2\left(H^2\left(D\cos(C) + F\sin(C)\right)^2 + \left(D^2 + E^2H^2 + F^2\right)^2\right)} \end{aligned} \tag{47}$$

Here we can get the $\lambda - (k_a, k_\theta)$ distribution by choosing different k_b values.

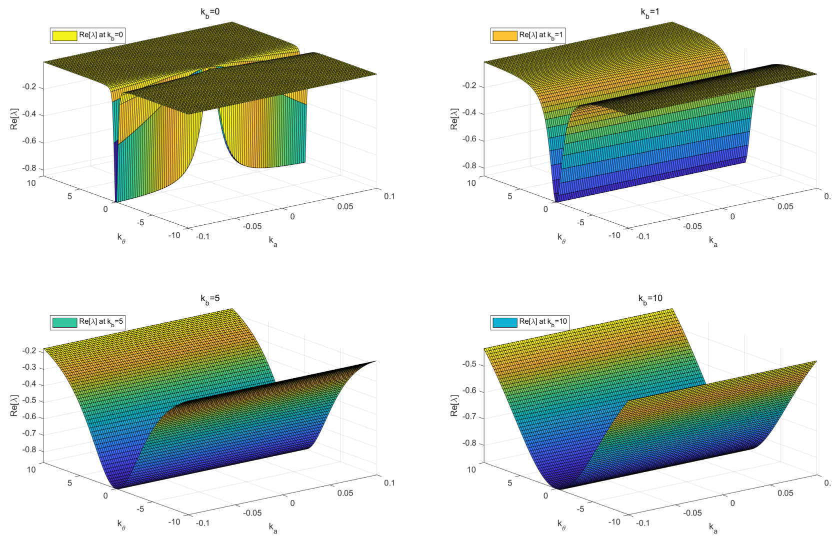

Figure 3: characteristic plane $\text{Re}[\lambda] - (k_a, k_\theta)$ at different k_b of real-state

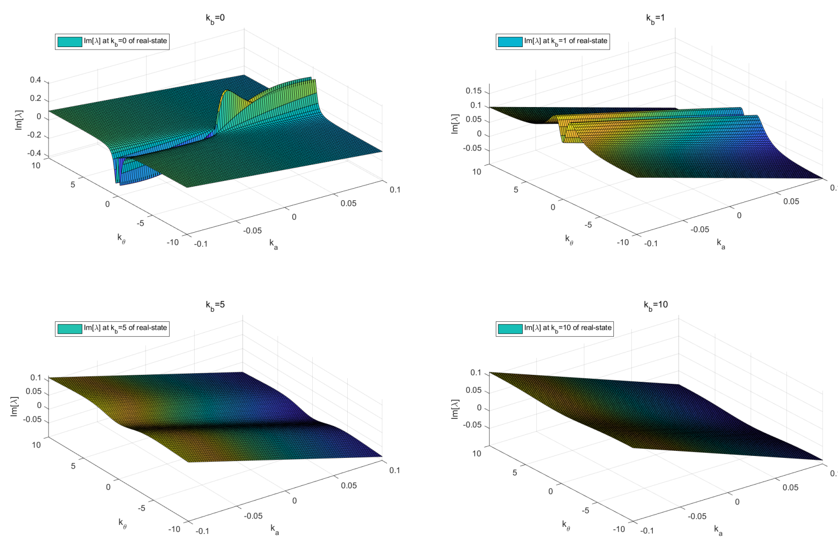

Figure 4: characteristic plane $\text{Im}[\lambda] - (k_a, k_\theta)$ at different k_b of real-state

From figure 3 and figure 4 we can conclude two points:

1. k_b plays an important role in this flow, and its presence or absence will have a relatively large impact on the existing model.

2. When k_b raises, the 'real-state mode-plane' becomes flattening and close to the 'base-state mode-plane'. For more details of comparison, see the image below.

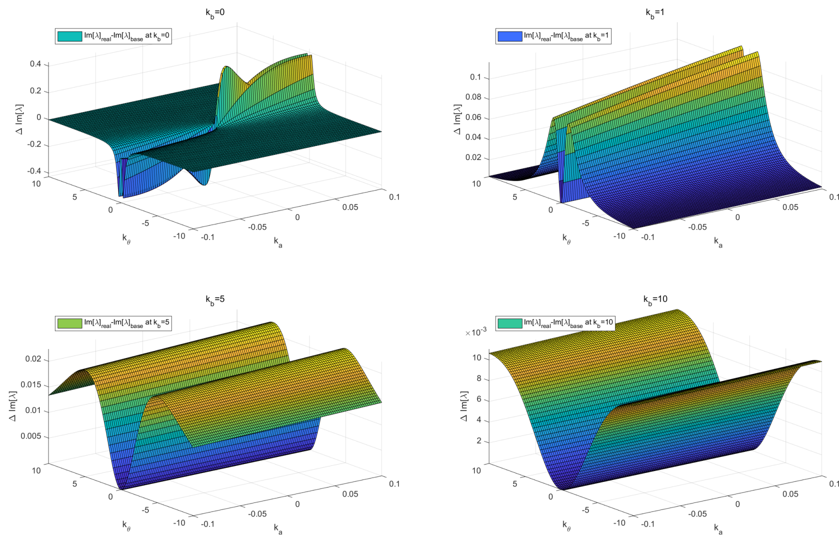

Figure 5: Comparison of characteristic planes between real-state and base-state

case 1.2 Here we set the base-state as

$$(a^{(0)}, \theta^{(0)}, b^{(0)}, u_a^{(0)}, u_\theta^{(0)}, u_b^{(0)}, \vec{p}^{(0)}, t^{(0)}) = (a^*, 0, h, u_a^*, u_\theta^*, 0, \vec{p}^*, t^*) \tag{48}$$

Assume initial conditions are

$$a^*(t=0) = 0 \quad b^*(t=0) = h$$
$$u_a^*(t=0) = u_1 \quad u_\theta^*(t=0) = u_2$$
$$\vec{p}^*(t=0) = (0, 0, p_0)$$

Reconsider all governing equations in base-state:

(14):

$$(\partial_t + \vec{u} \cdot \nabla)\phi = 0 \longrightarrow \phi = \text{Const} \tag{49}$$

(3):

$$\begin{aligned} d_t &= \partial_t + u_a^* \partial_a + \frac{1}{h_2} u_\theta^* \partial_\theta \\ &= \partial_t + u_1 \partial_a + \frac{1}{h_2} u_2 \partial_\theta \end{aligned} \tag{50}$$

(18):

$$\text{div } \vec{u} = \partial_a u_1 + \frac{1}{h_2} \partial_\theta u_2 + \frac{1}{h_2} u_1 \cos\alpha = 0 \tag{51}$$

(15):

$$d_t u_1 - \frac{(u_2)^2}{h_2} \cos\alpha = -\frac{1}{\rho\phi} \partial_a p^* + g \sin\alpha \tag{52}$$

(16):

$$d_t u_2 + \frac{\cos\alpha}{h_2} u_2 u_1 = -\frac{1}{\rho\phi}\frac{1}{h_2}\partial_\theta p^* \tag{53}$$

(17):

$$-\frac{\sin\alpha}{h_2}(u_2)^2 = -\frac{1}{\rho\phi}\partial_b p^* - g\cos\alpha \tag{54}$$

(Here we can also calculate all terms of u^* and p^* like (30),(31),(32),(33), we won't go into details here.)

Reconsider all governing equations in real-state (larger than 2-order small quantities have been abandoned at this time):

(14) and (3) remain the form as (49) and (50) give.

(18):

$$\begin{aligned}&\partial_a(\cancel{u_1} + \hat{u}_a) + \tfrac{1}{h_2}\partial_\theta(\cancel{u_2} + \hat{u}_\theta) + \partial_b\hat{u}_b + \tfrac{1}{h_2}(\cancel{u_1} + \hat{u}_a)\cos\alpha + \tfrac{1}{h_2}\hat{u}_b\sin\alpha = 0\\ &\left(\partial_a + \tfrac{\cos\alpha}{h_2}\right)\hat{u}_a + \tfrac{1}{h_2}\partial_\theta\hat{u}_\theta + \left(\partial_b + \tfrac{\sin\alpha}{h_2}\right)\hat{u}_b = 0\end{aligned} \tag{55}$$

(15):

$$\begin{aligned}&d_t(\cancel{u_1} + \hat{u}_a) - \tfrac{\cancel{(u_2)^2} + 2u_2\hat{u}_\theta + \cancel{(u_\theta)^2}}{h_2}\cos\alpha = -\tfrac{1}{\rho\phi}\partial_a(\cancel{p^*_a} + \hat{p}_a) + \cancel{g\sin\alpha}\\ &d_t\hat{u}_a - \tfrac{2u_2\cos\alpha}{h_2}\hat{u}_\theta + \tfrac{1}{\rho\phi}\partial_a\hat{p}_a = 0\end{aligned} \tag{56}$$

(16):

$$\begin{aligned}&d_t(u_2 + \hat{u}_\theta) + \tfrac{(u_2+\hat{u}_\theta)}{h_2}\left[(u_1 + \hat{u}_a)\cos\alpha + \hat{u}_b\sin\alpha\right] = -\tfrac{1}{\rho\phi}\tfrac{1}{h_2}(p^*_\theta + \hat{p}_\theta)\\ &d_t(\cancel{u_2} + \hat{u}_\theta) + \tfrac{\cos\alpha}{h_2}(\cancel{u_1u_2} + u_2\hat{u}_a + u_1\hat{u}_\theta + \cancel{\hat{u}_\theta\hat{u}_a}) + \tfrac{\sin\alpha}{h_2}(u_2\hat{u}_b + \cancel{\hat{u}_\theta\hat{u}_b}) + \tfrac{1}{\rho\phi}\tfrac{1}{h_2}\partial_\theta(\cancel{p^*_\theta} + \hat{p}_\theta) = 0\\ &\tfrac{u_2\cos\alpha}{h_2}\hat{u}_a + \left(d_t + \tfrac{u_1\cos\alpha}{h_2}\right)\hat{u}_\theta + \tfrac{u_2\sin\alpha}{h_2}\hat{u}_b + \tfrac{1}{\rho\phi}\tfrac{1}{h_2}\partial_\theta\hat{p}_\theta = 0\end{aligned} \tag{57}$$

(17):

$$\begin{aligned}&d_t\hat{u}_b - \tfrac{\cancel{(u_2)^2} + 2u_2\hat{u}_\theta + \cancel{(u_\theta)^2}}{h_2}\cos\alpha = -\tfrac{1}{\rho\phi}\partial_b(\cancel{p^*_b} + \hat{p}_b) - \cancel{g\cos\alpha}\\ &d_t\hat{u}_b - \tfrac{2u_2\cos\alpha}{h_2}\hat{u}_\theta + \tfrac{1}{\rho\phi}\partial_b\hat{p}_b = 0\end{aligned} \tag{58}$$

Now apply (21) to (55),(56),(57),(58) and rewrite them as $Ax = 0$ form

$$\begin{pmatrix} \lambda + ik_a u_1 + \frac{1}{h_2}ik_\theta u_2 & -\frac{2\cos\alpha}{h_2}u_2 & 0 & \frac{1}{\rho\phi}ik_a \\ \frac{\cos\alpha}{h_2}u_2 & \lambda + ik_a u_1 + \frac{1}{h_2}ik_\theta u_2 + \frac{\cos\alpha}{h_2}u_1 & \frac{\sin\alpha}{h_2}u_2 & \frac{1}{\rho\phi}\frac{1}{h_2}ik_\theta \\ 0 & -\frac{2\cos\alpha}{h_2}u_2 & \lambda + ik_a u_1 + \frac{1}{h_2}ik_\theta u_2 & \frac{1}{\rho\phi}ik_b \\ ik_a + \frac{\cos\alpha}{h_2} & \frac{1}{h_2}ik_\theta & ik_b + \frac{\sin\alpha}{h_2} & 0 \end{pmatrix}\begin{pmatrix}\hat{u}_a\\ \hat{u}_\theta\\ \hat{u}_b\\ \hat{p}\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \tag{59}$$

Directly calculate det(A)=0. Here we use symbolic notation to apply calculation in **Sympy**

$$A = u_a^* = u_1; \quad B = \frac{1}{\rho\phi}; \quad C = \alpha; \quad D = k_a; \quad E = k_\theta; \quad F = k_b; \quad G = u_\theta^* = u_2; \quad H = \frac{1}{h_2}; \quad X = \lambda;$$

$$\begin{pmatrix} iAD + iEGH + X & -2GH\cos(C) & 0 & iBD \\ GH\cos(C) & iAD + AH\cos(C) + iEGH + X & GH\sin(C) & iBEH \\ 0 & -2GH\cos(C) & iAD + iEGH + X & iBF \\ iD + H\cos(C) & iEH & iF + H\sin(C) & 0 \end{pmatrix} \begin{pmatrix} \hat{u}_a \\ \hat{u}_\theta \\ \hat{u}_b \\ \hat{p} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{60}$$

Results group is (It is too complicated thus I separate results into parts)

$$[x_1, x_2] = \left[-\frac{N_1 - N_2}{D_e}, -\frac{N_1 + N_2}{D_e} \right] \tag{61}$$

Here each term is

$$N_1 = -\sqrt{\begin{array}{l} -8\,\left(D^2 - 1\mathrm{i}\,\cos(C)\,D\,H + E^2\,H^2 + F^2 - 1\mathrm{i}\,\sin(C)\,F\,H\right) \\ * \left(\begin{array}{l} -2\,A^2\,D^4 + A^2\,D^3\,H\,\cos(C)\,4\mathrm{i} - 2\,A^2\,D^2\,E^2\,H^2 - 2\,A^2\,D^2\,F^2 \\ +2\mathrm{i}\,\sin(C)\,A^2\,D^2\,F\,H + 2\,A^2\,D^2\,H^2\,\cos(C)^2 + A^2\,D\,F^2\,H\,\cos(C)\,2\mathrm{i} \\ +\sin(2\,C)\,A^2\,D\,F\,H^2 - 4\,A\,D^3\,E\,G\,H \\ +A\,D^2\,E\,G\,H^2\,\cos(C)\,8\mathrm{i} - 4\,A\,D\,E^3\,G\,H^3 - 4\,A\,D\,E\,F^2\,G\,H \\ +A\,D\,E\,F\,G\,H^2\,\cos(C)\,4\mathrm{i} + 2\mathrm{i}\,\sin(C)\,A\,D\,E\,F\,G\,H^2 \\ +6\,A\,D\,E\,G\,H^3\,\cos(C)^2 + 2\,\sin(2\,C)\,A\,D\,E\,G\,H^3 \\ +A\,E\,F^2\,G\,H^2\,\cos(C)\,2\mathrm{i} + \sin(2\,C)\,A\,E\,F\,G\,H^3 - 2\,D^2\,E^2\,G^2\,H^2 \\ +2\,\sin(2\,C)\,D^2\,G^2\,H^2 + D\,E^2\,G^2\,H^3\,\cos(C)\,4\mathrm{i} \\ -4\,D\,F\,G^2\,H^2\,\cos(C)^2 - 2\,\sin(2\,C)\,D\,F\,G^2\,H^2 - 2\,E^4\,G^2\,H^4 - 2\,E^2\,F^2\,G^2\,H^2 \\ +E^2\,F\,G^2\,H^3\,\cos(C)\,4\mathrm{i} + 4\,E^2\,G^2\,H^4\,\cos(C)^2 \\ +2\,\sin(2\,C)\,E^2\,G^2\,H^4 + 4\,F^2\,G^2\,H^2\,\cos(C)^2 \end{array} \right) \\ + \left(\begin{array}{l} 4\mathrm{i}\,A\,D^3 + 4\mathrm{i}\,G\,D^2\,E\,H + 6\,A\,D^2\,H\,\cos(C) \\ +4\mathrm{i}\,A\,D\,E^2\,H^2 + 6\,G\,D\,E\,H^2\,\cos(C) + 4\mathrm{i}\,A\,D\,F^2 + 4\,A\,\sin(C)\,D\,F\,H \\ -2\mathrm{i}\,A\,D\,H^2\,\cos(C)^2 + 4\mathrm{i}\,G\,E^3\,H^3 + 4\mathrm{i}\,G\,E\,F^2\,H + 4\,G\,E\,F\,H^2\,\cos(C) \\ +2\,G\,\sin(C)\,E\,F\,H^2 - 4\mathrm{i}\,G\,E\,H^3\,\cos(C)^2 - 2\mathrm{i}\,G\,\sin(2\,C)\,E\,H^3 \\ +2\,A\,F^2\,H\,\cos(C) - 1\mathrm{i}\,A\,\sin(2\,C)\,F\,H^2 \end{array} \right)^2 \end{array}} \tag{62}$$

$$
\begin{aligned}
N_2 = {} & 3\,A\,D^2\,H\,\cos(C) \\
& + A\,F^2\,H\,\cos(C) + 2\,A\,D\,F\,H\,\sin(C) + 3\,D\,E\,G\,H^2\,\cos(C) + 2\,E\,F\,G\,H^2\,\cos(C) + E\,F\,G\,H^2\,\sin(C) \\
& + A\,D^3\,2\mathrm{i} + A\,D\,F^2\,2\mathrm{i} + E^3\,G\,H^3\,2\mathrm{i} + A\,D\,E^2\,H^2\,2\mathrm{i} - A\,D\,H^2\cos(C)^2\,1\mathrm{i} \\
& - E\,G\,H^3\cos(C)^2\,2\mathrm{i} - \frac{A\,F\,H^2\,\sin(2\,C)\,1\mathrm{i}}{2} - E\,G\,H^3\,\sin(2\,C)\,1\mathrm{i} + D^2\,E\,G\,H\,2\mathrm{i} + E\,F^2\,G\,H\,2\mathrm{i}
\end{aligned} \tag{63}
$$

$$
D_e = 2\,\left(D^2 - 1\mathrm{i}\,\cos(C)\,D\,H + E^2\,H^2 + F^2 - 1\mathrm{i}\,\sin(C)\,F\,H\right) \tag{64}
$$

For this form is too complicated and we want to verify that the result is structured correctly. Thus we need a benchmark $G = 0$ and see the result x_1

Recall that (44) has mentioned that

$$
x_1(u_b^* = 0) = -iAD
$$

Thus we can just verify the real-part of result x_1 is zero or not in this case (calculate the residual graph).

$$
\begin{aligned}
\mathrm{Re}[x_1]|_{G=0} = {} & \mathrm{Re}\left[-\frac{N_1 - N_2}{D_e}\right]|_{G=0} \\
= {} & \frac{\left(\begin{array}{l} 2\,A\,D^3 + 2\,A\,D\,F^2 + 2\,A\,D\,E^2\,H^2 \\ -|\cos(C)|\ \mathrm{imag}\left(\sqrt{\begin{array}{l} D^4 - D^3\,H\,\cos(C)\,2\mathrm{i} + 2\,D^2\,F^2 - D^2\,F\,H\,\sin(C)\,2\mathrm{i} \\ -D^2\,H^2\,\cos(C)^2 - D\,F^2\,H\,\cos(C)\,2\mathrm{i} - \sin(2\,C)\,D\,F\,H^2 \\ +F^4 - F^3\,H\,\sin(C)\,2\mathrm{i} - F^2\,H^2\,\sin(C)^2 \end{array}}\right)|A|\,|H| \\ -A\,D\,H^2\,\cos(C)^2 - \frac{A\,F\,H^2\,\sin(2\,C)}{2} \end{array}\right)}{2\left((D^2+E^2\,H^2+F^2)^2 + (D\,H\,\cos(C)+F\,H\,\sin(C))^2\right)/(D\,H\,\cos(C)+F\,H\,\sin(C))} \\
& - \frac{\left(\begin{array}{l} 3\,A\,D^2\,H\,\cos(C) + A\,F^2\,H\,\cos(C) \\ -|\cos(C)|\ \mathrm{real}\left(\sqrt{\begin{array}{l} D^4 - D^3\,H\,\cos(C)\,2\mathrm{i} + 2\,D^2\,F^2 - D^2\,F\,H\,\sin(C)\,2\mathrm{i} \\ -D^2\,H^2\,\cos(C)^2 - D\,F^2\,H\,\cos(C)\,2\mathrm{i} - \sin(2\,C)\,D\,F\,H^2 \\ +F^4 - F^3\,H\,\sin(C)\,2\mathrm{i} - F^2\,H^2\,\sin(C)^2 \end{array}}\right)|A|\,|H| \\ +2\,A\,D\,F\,H\,\sin(C) \end{array}\right)}{2\left((D^2+E^2\,H^2+F^2)^2 + (D\,H\,\cos(C)+F\,H\,\sin(C))^2\right)/(D^2+E^2\,H^2+F^2)}
\end{aligned} \tag{65}
$$

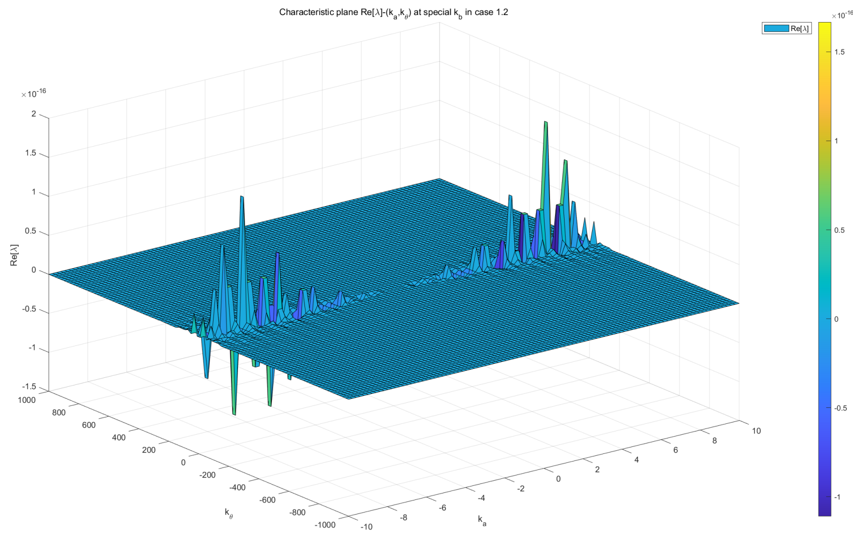

Figure 6: residual graph of real-part of result x_1

From this residual graph we can see that the difference between these two results is smaller than 10^{-16}, some peaks are occured by MATLAB floating point accuracy. Thus this result maintains a great calculation form, and we can calculate by using this form below.

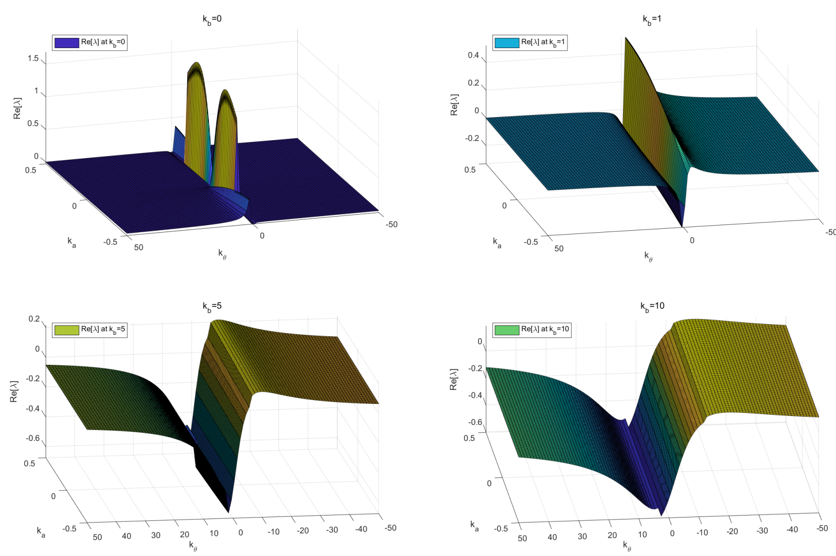

Figure 7: characteristic plane $\mathrm{Re}[\lambda] - (k_a, k_\theta)$ at different k_b of base-state (x_1) graph

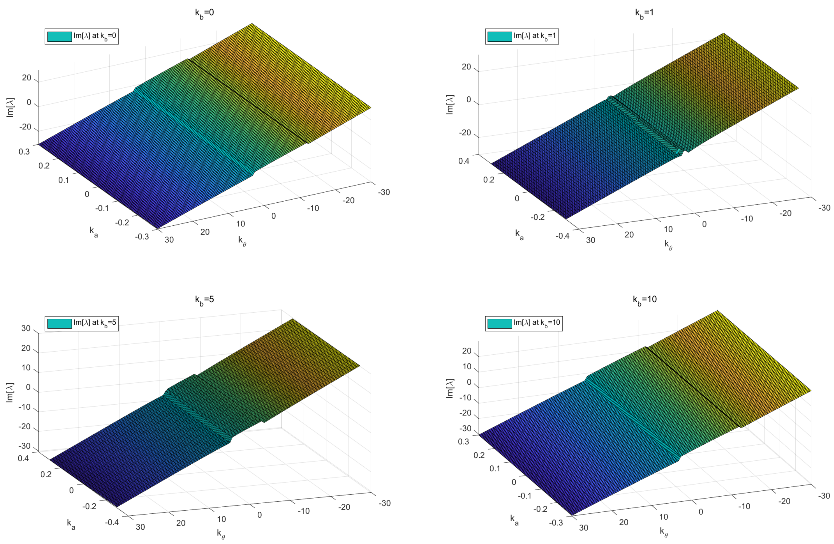

Figure 8: characteristic plane $\mathrm{Im}[\lambda] - (k_a, k_\theta)$ at different k_b of base-state (x_1) graph

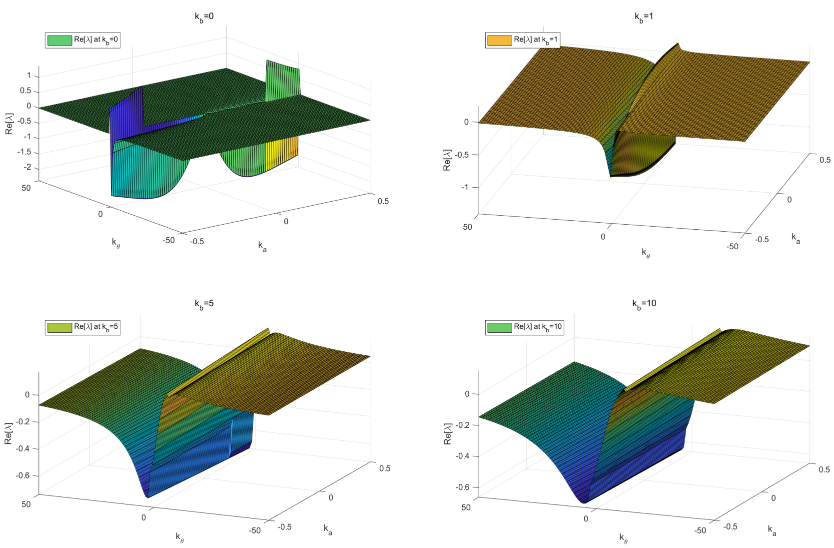

Figure 9: characteristic plane $\mathrm{Re}[\lambda] - (k_a, k_\theta)$ at different k_b of real-state (x_2) graph

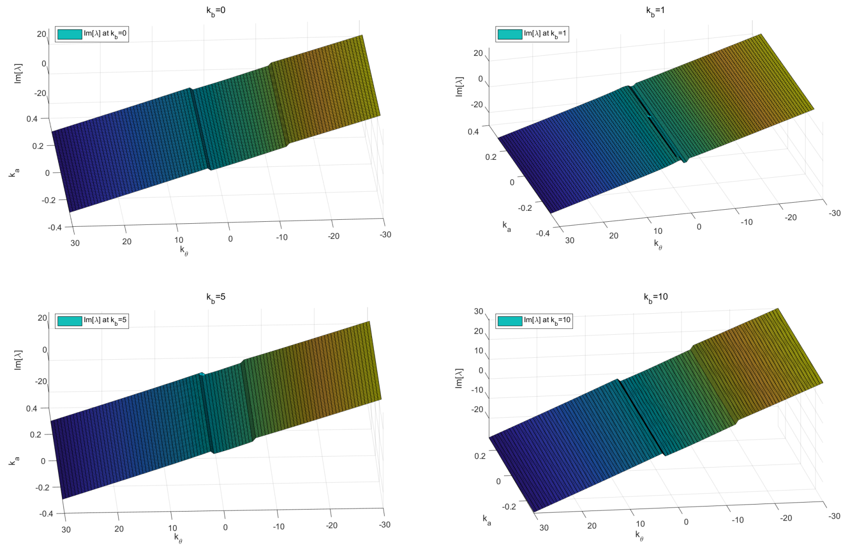

Figure 10: characteristic plane $\mathrm{Im}[\lambda] - (k_a, k_\theta)$ at different k_b of real-state (x_2) graph

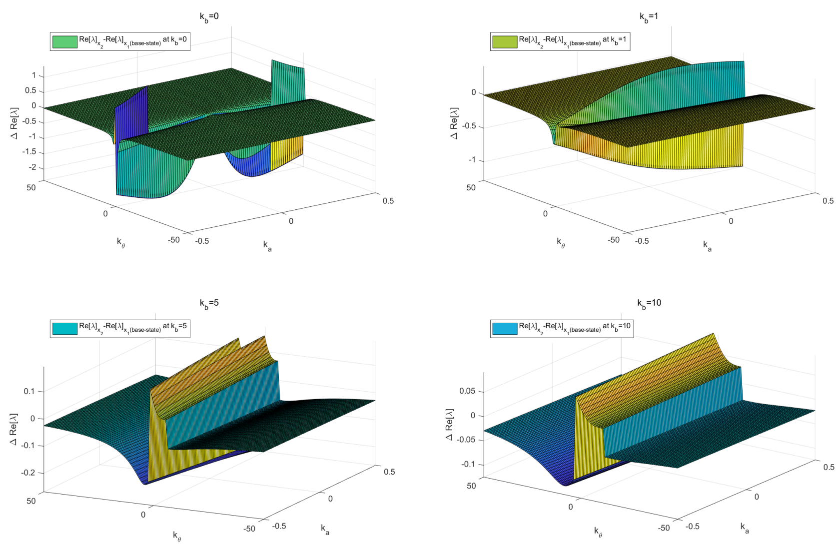

Figure 11: Comparison of characteristic Re-plane between real-state (x_2) and base-state (x_1) graph

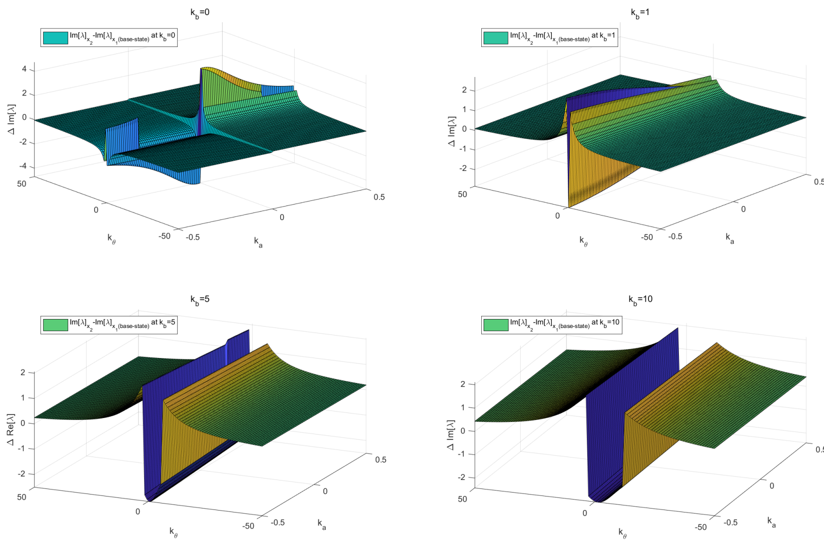

Figure 12: Comparison of characteristic Im-plane between real-state (x_2) and base-state (x_1) graph