Surface flows of granular materials: A short introduction to some recent models

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Surface flows of granular materials: A short introduction to some recent models

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Abstract. We present a short review of recent theoretical descriptions of flows occuring at the surface of granular piles, and focus mainly on two models: the phenomenological "BCRE" model and the hydrodynamic model, based on Saint-Venant equations. Both models distinguish a "static phase" and a "rolling" phase inside the granular packing and write coupled equations for the evolutions of the height of each of these phases, which prove similar in both approaches. The BCRE description provides a very intuitive picture of the flow, whereas the Saint-Venant hydrodynamic description establishes a general and rigorous framework for granular flow studies. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

granular materials/ avalanches/ surface flows

Modèles récents pour les coulements surfaciques d'empilements granulaires

Résumé. *Nous prsentons une rapide revue des modles rcents dcrivant les coulements de surface des empilements granulaires, en nous concentrant plus particulirement sur le modle phnom-nologique BCRE et le modle hydrodynamique, fond sur des quations de Saint-Venant. Ces deux approches font la distinction entre une phase statique et une phase roulante l'intrieur du tas, et crivent des quations d'volution couples pour l'paisseur de ces deux phases qui s'avrent similaires dans les deux cas. La description BCRE fournit une vision intuitive de l'coulement, tandis que la description hydrodynamique tablit un cadre d'tude rigoureux et gnral. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS*

matriaux granulaires/ avalanches/ coulements de surface

1. Introduction

Granular flows occur both naturally, for instance when debris roll down the side of a mountain or when sand avalanches on the slope of a desert dune, and in many industrial processes, which involve the conveyance of material in granular form. Describing and understanding this kind of flows thus appears as an issue of high importance from a practical point of view. Although some of the fundamentals laws at work inside these flows are still a matter of research at present (dissipation processes, existence of a stress-strain relation, etc.), significant progress has been made in the recent years with the introduction of several new theoretical models. This article is primarily intended to provide the non-specialist with a quick introduction to these recent models for granular flow, with a strong emphasis on two of them (the so-called "BCRE" and Saint-Venant models) which, in our mind, are the simplest currently available descriptions.

We restrict ourselves to the case of dry, cohesionless granular media, and to situations where the granular flow is confined to a layer at the surface of the granular system *(surface flows)*, i.e. situations where the typical depth of the flow is much smaller than its lateral dimensions (for instance, an avalanche at the surface of a dune). This stands in contrast with situations where the flow involves rearrangements deep inside the bulk of the system (e.g. when a heap of sand is bulldozed)—this latter situation is essentially the privilege of continuum mechanics description and requires extensive numerical computations.

Roughly speaking, one can distinguish three "generations" of models describing surface flows of granular materials. The first generation has originated in the large amount of work devoted to the description of granular dynamics by researchers in the fields of engineering sciences and applied mechanics, and has reached a very convincing form in the granular flow model of Savage and Hutter [1]. This very complete model writes classical hydrodynamic equations (incompressibility and momentum conservation) for the flowing material. After integration over the thickness of the rolling layer, depth-averaged (Saint-Venant) flow equations are obtained which allow the calculation of the flow thickness and mean velocity over the whole sample. In its principle, this model is restricted to the description of flows over *fixed* bottoms of given topography (e.g. the side of a mountain).

Others situations are though of interest, where the bottom is of the same nature and has the same properties as the flowing material. Consider for instance a sand avalanche on a dune: in this case, the position of the "bottom", i.e. the position of the border between flowing grains and material at rest, is not prescribed during the avalanche, since rolling grains colliding with initially immobile grains can either bring them into motion (*erosion* of the bottom by the flow), or get trapped into a hole between two of them (*deposition* of material on the bottom). Thus the static/mobile frontier (the "bottom" position) evolves as a dynamic variable of the problem. This distinction between a "flowing phase" and a "static phase", capable of exchanging grains through collisions processes, forms the central hypothesis of the second generation of models, and was developed by Mehta [2] and by Bouchaud, Cates, Ravi Prakash and Edwards (also known as BCRE) [3, 4]. The next section is devoted to the presentation of this BCRE model.

The third generation of models is characterized by a return to a hydrodynamic description (Saint-Venant equations), as in the first generation's Savage and Hutter model, but with the integration of erosion/deposition mechanisms. This approach was initiated by Douady *et al.* [5], and recently reconsidered by Khakhar *et al.* [6] and Gray [7]. We devote section 3. to these hydrodynamic descriptions.

To this (non-exhaustive) list of models, we should also add a recent original proposal by Aranson and Tsimring [8, 9], based on an analogy of granular flows with the Landau theory of phase transitions: in this picture, the static phase is considered as "solid" and the mobile phase as "molten", and an order parameter continuously describes the system state from molten (order parameter equal to zero) to solid (order parameter equal to one). The granular flow is then governed by a Navier-Stokes equation, but with a hybrid stress tensor containing both fluid-like and static-like constraints whose relative importance is tuned by the value of the order parameter. This model has been shown [8, 9] to reproduce experimental observations in a variety of situations, especially when the system is only partially fluidized. A fair presentation of this approach would however go beyond the scope of the present article, and the interested reader is referred to
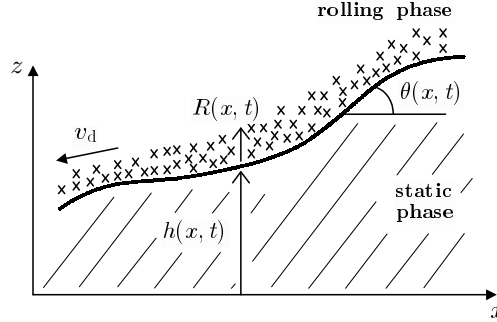
Figure 1: Avalanche description within the BCRE model.

the original articles [8, 9].

As announced, the present article mainly focuses on two of the above-mentioned models, because of their simplicity and their physical content. The paper is organized as follows. In section 2., we present the BCRE model. In section 3., we describe the Saint-Venant hydrodynamic approach. In the last section (sec 4.), we point out the relations and similarities that arise between the BCRE and the Saint-Venant descriptions, and conclude with a general discussion of our current understanding of avalanches and granular flows.

2.   The BCRE model

The BCRE model is essentially a phenomenological description of granular flows, based on the hypothesis that the granular pile can be separated in a rolling phase (made of mobile grains) atop a static phase (made of immobile grains), with a sharp, well-defined interface. The model appeared in its original form in [3, 4], and was later simplified and modified by Boutreux, Raphaël and de Gennes [10, 11, 12]. The basic quantities introduced in the BCRE description are depicted on Figure 1 (we restrict here to two-dimensional sandpiles): $h(x, t)$ is the local thickness of the static phase, at position x and time t, $R(x, t)$ the local thickness of the rolling layer, $\theta(x, t)$ the local slope of the flowing/static interface ($\theta \simeq \partial h/\partial x$ for small slopes), and v_d the downhill velocity of the rolling grains (which will be taken constant for a start). When attempting to describe a given situation of granular flow, one is interested in finding the evolution of the static and rolling height h and R. In the BCRE picture, the set of equations governing the temporal derivatives $\partial h/\partial t$ and $\partial R/\partial t$ of these quantities is very compact [10, 11]:

$$\frac{\partial h}{\partial t} = -\mathcal{E}(x, t)\,, \tag{1a}$$

$$\frac{\partial R}{\partial t} = v_d \frac{\partial R}{\partial x} + \mathcal{E}(x, t)\,. \tag{1b}$$

In these equations, $\mathcal{E}(x, t)$ is the *exchange term*, which represents the exchanges of grains that occur between static and mobile phases (dislodgement of immobile grains by rolling grains, or trapping of mobile grains by static grains). The physical meaning of the BCRE equations is very simple. Equation (1a) expresses that the static height locally increases when material is deposited from the rolling phase. Equation (1b) describes the evolution of the rolling thickness R: the first term on the right-hand side is a classical convection term describing the downhill motion of grains at velocity v_d, and the second term accounts for the local increase in R when grains are entrained out of the static part. Note that in their original model [3, 4], Bouchaud *et al.* included diffusion and curvature terms which are second-order (in most cases, but not all), and will here be omitted.

The next step in the BCRE description is to suggest a structure for the exchange term $\mathcal{E}$. Bouchaud *et*

3

al. [3, 4, 11] proposed the following expression:

$$\mathcal{E}(x,t) = \gamma R(\theta - \theta_{\rm n}) . \tag{2}$$

This structure can be interpreted in simple physical terms as follows. Let γ denote the typical collision frequency of a mobile grain with the static phase. Then the number of collisions per unit time experienced by the static phase when surmounted by a rolling layer of thickness R is proportional to γR. To evaluate the exchanges between phases, one further has to estimate the probability of each of these collisions of mobile grains with the static phase to result either in the dislodgement of a static grain or the trapping of the mobile grain. BCRE assume this probability to be mainly governed by the local slope $\theta \simeq \partial h/\partial x$ of the static phase. When θ equals a certain *neutral angle* $\theta_{\rm n}$, the probability of dislodgement is equal to that of trapping, and there is no net erosion nor deposition of grains (in other words, there is no amplification nor damping of the avalanche). But when $\theta > \theta_{\rm n}$ (resp. $\theta < \theta_{\rm n}$), there is a net erosion of grains (resp. deposition). For slopes that remain close to $\theta_{\rm n}$ (usually around $30°$ for dry sand), one can take a linear form $\theta - \theta_{\rm n}$ for this angular dependence of the erosion/deposition processes, and finally obtain the structure given in equation (2) for the global exchanges between static and mobile phase.

With the above form (2) of $\mathcal{E}$, the general equations (1) then become

$$\begin{aligned}\frac{\partial h}{\partial t} &= -\gamma R(\theta - \theta_{\rm n}) , \\ \frac{\partial R}{\partial t} &= v_{\rm d}\frac{\partial R}{\partial x} + \gamma R(\theta - \theta_{\rm n}) .\end{aligned} \tag{3}$$

The evolutions of R and h thus appear as intimately coupled, through the expression of the exchange term (via its R-dependence and the local slope $\theta \simeq \partial h/\partial x$). Solving this set of equations (analytically or numerically) with appropriate boundary and initial conditions allows to describe the granular flow through the knowledge of h and R.

The expression (2) of the exchange term has however been criticized by Boutreux *et al.* for flows thicker than a few grain diameters: in that case (assuming a a more or less layered flow structure), grains from the upper layers of the flow have no opportunity to interact and collide with the underlying static phase, because they are "screened" by lower flowing layers. Thus the number of collisions, and therefore the exchange term, cannot remain proportional to the rolling thickness R when R becomes larger than a screening length λ. Boutreux *et al.* suggest that the exchange term must thus saturate for thicker flows, i.e. must be written as

$$\mathcal{E}(x,t) = v_{\rm up}(\theta - \theta_{\rm n}) \qquad (R > \lambda) , \tag{4}$$

where $v_{\rm up} = \gamma\lambda$ has the dimensions of a velocity (one can indeed show that this corresponds to the velocity of uphill waves running at the surface of the static phase in avalanches, see [12]).

If we use this saturated form (4) of the exchange term into the set of governing equations (1), we obtain the *saturated* BCRE equations:

$$\begin{aligned}\frac{\partial h}{\partial t} &= -v_{\rm up}(\theta - \theta_{\rm n}) , \\ \frac{\partial R}{\partial t} &= v_{\rm d}\frac{\partial R}{\partial x} + v_{\rm up}(\theta - \theta_{\rm n}) .\end{aligned} \tag{5}$$

One pleasant feature of this saturated set of equations is that it is now *linear* in both h and R, and that a general solution can easily be formulated, in terms of uphill and downhill wave motions [12].

Due to its simplicity and to the clear physical picture that it conveys, the BCRE model has since its creation and until recently, played a central role for the theoretical study and discussion of various situations

of granular flow, either in its form (3) or (5), or in other variants for some complex cases. Examples of problems that have been tackled with the help of the BCRE description are the surface dynamics of a pile in a rotating drum [10], the segregation and stratification phenomena in mixed granular media [13], the formation of a sandpile from a point source [14], or the formation of ripples on a sand bed submitted to the action of wind [15], etc.

However, the phenomenological nature of the BCRE description inevitably raises a series of important questions. (i) The first of these questions is to determine whether this model respects the fundamental laws of mass and momentum conservation. It is easy to show that mass (i.e. the total number of particles $R + h$) is conserved, but conservation of momentum is open to doubt. (ii) It is known from experiments that the velocity profile within the flowing layer is not constant, implying that the hypothesis $v_{\mathrm{d}} = \mathrm{const.}$ is not acceptable. How are then the BCRE equations to be modified? Is it enough to replace v_{d} by the average velocity, or is there more to it? (iii) A last shortcoming is due to the introduction of several phenomenological parameters like θ_{n} or v_{up}, which would have to be fitted to the data. One would naturally prefer to derive directly the expressions of these parameters from the fundamental quantities describing the system.

All these questions have found a satisfactory answer with the recent (re-)introduction of Saint-Venant hydrodynamic descriptions that we are presenting in the next section.

3.   Saint-Venant hydrodynamic description

We now present the recent hydrodynamic descriptions based on depth-averaged (Saint-Venant) equations, which represent an essential step: as they start from first principle equations, they provide a rigourous and general framework for the study and understanding of surface flows of granular materials.

The first hydrodynamic model incorporating the exchanges of grains between flowing and static part of the granular system is due to Douady *et al.* [5]. This work was revisited very recently by Khakhar *et al.* [6] and also by Gray [7]. For conciseness, in this presentation, we will not enter into all details [5, 6, 16] and only give the general lines.

The idea of these hydrodynamic models is to write conservation equations for mass and momentum in the flowing layer, but averaging all quantities across the depth of the layer (Saint-Venant equations). It is important to realize that, as a consequence, these models *do not* provide any information on the inner structure of the flow, and can only give information on the flowing layer thickness and the static height.

The BCRE equations given in the previous section were written in the fixed gravity frame ((x, z) axis, see figure 1). Here, in order to write the hydrodynamic conservation equations in a compact way, it will be easier to work temporarily in the frame (X, Z) locally tangent to the flow at each point, as shown on figure 2-a. The static height h and the local slope θ are defined as before, but the thickness of the rolling layer is now measured in the direction *perpendicular to the flow* (along Z) and will be noted $\mathcal{R}$ (notice the difference with R which is reserved to the rolling thickness measured in the vertical direction). We now present the different conservation equations, and explain them one by one.

The first equation is mass conservation. Following the notations of Khakhar *et al.* [6], we have

$$\frac{\partial}{\partial t}(\overline{\rho}\,\mathcal{R}) + \frac{\partial}{\partial X}(\overline{\rho v_X}\,\mathcal{R}) = (\rho v_Z)|_{Z=0}\,, \tag{6}$$

where ρ is the density of the rolling phase, and (v_X, v_Z) are the components of the velocity field in the rolling phase. The notation $\overline{A}$ represents the average of the quantity A over the rolling depth: $\overline{A} = (1/\mathcal{R})\int_0^{\mathcal{R}} A(Z)\,\mathrm{d}Z$. Equation (6) can be easily interpreted by considering a small slice of fluid of width $\mathrm{d}X$ and height $\mathcal{R}$ as represented on figure 2-b. The first term on the l.h.s. gives the local variation of the mass of the slice. The second term on the l.h.s. gives the difference between input of mass at one side of the slice and output of mass at the other side due to the flow along the X-direction: this difference is equal to the spatial derivative of the mass flow, which writes $Q_X^{\mathrm{mass}} = \int_0^{\mathcal{R}} \rho v_X\,\mathrm{d}Z = (\overline{\rho v_X})\,\mathcal{R}$. The last term (on the r.h.s.) reflects that mass can also enter the slice from the bottom border of the slice (at $Z = 0$), due to
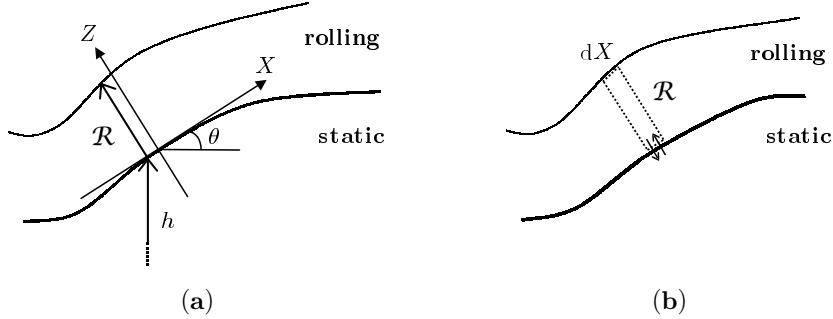
Figure 2: (a) Locally tangent frame (X, Z). The origin $Z = 0$ is chosen at the rolling/static interface. (b) Slice of fluid for which mass and momentum conservation are written. The arrows represent the exchanges between rolling and static phase.

exchanges with the static phase: these exchanges occur along the Z-direction and the corresponding mass input is $Q_Z^{\text{mass}} = (\rho v_Z)|_{Z=0}$.

The second equation [6] comes from momentum conservation in the X-direction:

$$\frac{\partial}{\partial t}\big(\overline{\rho v_X}\,\mathcal{R}\big) + \frac{\partial}{\partial X}\big(\overline{\rho v_X{}^2}\,\mathcal{R}\big) = \sigma_{XZ}|_{Z=0} - \overline{\rho}g\mathcal{R}\sin\theta\,, \tag{7}$$

where σ_{XZ} is one of the components of the stress tensor $\boldsymbol{\sigma}$ within the rolling layer and g is the gravity field. The most general form of this conservation law normally includes a couple of additional terms (see [6]), but for simplicity and to make our point, we here restrict to the simplest cases where these terms are negligible. Here again, we can understand the meaning of each term of equation (7) by considering the slice of figure 2-b. The first term on the l.h.s. accounts for the local variation of momentum of the slice. The second term of the l.h.s. computes the difference between the momentum transferred by the flow into the slice at one side and out of it at the other side, as the derivative of the momentum flux $Q_X^{\text{moment.}} = \int_0^{\mathcal{R}}(\rho v_X)\cdot v_X\,\mathrm{d}Z = \big(\overline{\rho v_X{}^2}\big)\mathcal{R}$. On the r.h.s. of the equation are gathered the sources of momentum due to the forces applied on the slice. The first term on the r.h.s. takes into account the friction force $\sigma_{XZ}|_{Z=0}$ exerted by the static phase on the bottom of the slice, and the second term is simply the X-component of the slice weight.

Both equations (6) and (7) are concerned with the rolling thickness R. We now need an equation governing the evolution of the static height h. This is easily obtained by considering again our small slice of fluid: the mass of grains settling from the mobile phase to the static phase in a time $\mathrm{d}t$ is $-(\rho v_Z)|_{Z=0}\cdot\mathrm{d}t\,\mathrm{d}X$. Assuming a continuous density ρ across the flowing/static interface, this mass input induces a rise of the static height $\mathrm{d}h$, so that the mass increase can also be rewritten $\rho|_{Z=0}\cos\theta\cdot\mathrm{d}h\,\mathrm{d}X$ (the angular factor originates in the fact that h and Z are at an angle θ). Equating these two expressions of the mass, we obtain

$$\frac{\partial h}{\partial t} = -\frac{1}{\cos\theta}\,v_Z|_{Z=0}\,. \tag{8}$$

The three equations (6), (7) and (8) form the set governing the granular flow in the hydrodynamic description. Let us enumerate the unknown quantities in these three equations: of course, $\mathcal{R}$ and h are unknown, as is $v_Z|_{Z=0}$. But we also ignore the exact expressions of the density profile *within* the flow $\rho(Z)$ and of the velocity profile $v_X(Z)$, which would allow the calculation of averages like $\overline{\rho v_X}$ or $\overline{\rho v_X{}^2}$. Thus, the set of three equations (6)–(8) is incomplete. Furthermore, there is no hope within this kind of depth-averaged description to gain any further knowledge on the internal profile of ρ and v_X. Ideally, one would need an

internal relation equivalent to the Navier-Stokes equation of ordinary fluids (and a state equation) to reach these internal profiles, but such a relation (if it exists!) is unknown at present for granular materials.

To make progress, it is therefore necessary to add "manually" extra information on the physics of the system. The first usual step is to neglect density variations in the flow [5, 6], i.e. impose that $\rho = \overline{\rho} = $ const. (although this hypothesis might be a matter of debate if one looks for refined equations, it is useful for a start).

Next, we need information on the velocity profile $v_X(Z)$. Here, experiments provide precious data: Rajchenbach *et al.* [17, 18] and Bonamy *et al.* [19] have shown that flows occurring at the surface of bidimensional piles (made of a monolayer of metal beads confined between two vertical walls) present a *linear velocity profile*, with a vanishing velocity at the flowing/static interface (no slippage):

$$v_X(Z) = -\Gamma_0 Z \,, \tag{9}$$

where the velocity gradient Γ_0 is independent of the rolling thickness $\mathcal{R}$. ($\Gamma_0 \simeq \sqrt{g/d}$, with d the grain diameter.) It is important to note, however, that if linear velocity profiles appear in 2D piles, the case of 3D piles is unknown at present (to our knowledge). We note also that, in the different situation when the granular flow occurs at the surface of a fixed, inclined plane which cannot be eroded, the profile is non-linear with a mean velocity $\overline{v_X} \sim R^{3/2}$ [20, 21] (for recent theoretical proposals concerning this fact, see [22, 23]), whereas $\overline{v_X} \sim R$ for the linear profile of eq. (9).

Finally, we need to find the expression for the friction force $\sigma_{XZ}|_{Z=0}$ exerted by the static phase on the rolling phase (eq. (7)). We choose the simplest form, i.e. a classical Coulomb force: $\sigma_{XZ}|_{Z=0}$ is taken equal to the Z-component of the weight times a (constant) dynamic friction coefficient μ_{dyn}, and thus writes

$$\sigma_{XZ}|_{Z=0} = \mu_{\text{dyn}}\, \rho g \mathcal{R} \cos \theta \,. \tag{10}$$

We shall now incorporate all this new information into the hydrodynamic equations (6)–(8): after using the assumption of constant density $\rho = $ const. in the flow, substituting the expressions of $\overline{v_X}$ and $\overline{v_X{}^2}$ as deduced from (9), and inserting the expression (10) of the friction force, the only remaining unknowns in the three equations are h, $\mathcal{R}$ and $v_Z|_{Z=0}$. One can then combine these equations together, and after some straightforward algebra, eventually obtain the two following differential equations for the static height h and the rolling thickness R:

$$\frac{\partial h}{\partial t} = -\frac{g}{\Gamma_0 \cos \theta}(\sin \theta - \mu_{\text{dyn}} \cos \theta) \tag{11}$$

$$\frac{\partial R}{\partial t} = (\Gamma_0 \cos^2 \theta)\, R\frac{\partial R}{\partial x} + \frac{g}{\Gamma_0 \cos \theta}(\sin \theta - \mu_{\text{dyn}} \cos \theta) \,. \tag{12}$$

Note that, in these expressions, we made the geometric transformation ($\mathcal{R} \to R, X \to x$) in order to come back from the locally tangent frame, where the hydrodynamic equations were initially written, to the fixed gravity frame (x, z) of figure 1, where the rolling R thickness is measured vertically. We skip the details of this transformation from locally tangent to fixed frame, which can be tedious in the general case [5, 6]. Here, we simply assumed, as happens in most practical cases, that the slope θ of the static phase remains roughly constant over the whole system [16]: $\theta \simeq \theta_{\text{dyn}}$, where the angle θ_{dyn} is the "dynamic friction angle" defined by $\tan \theta_{\text{dyn}} = \mu_{\text{dyn}}$.

Since $\theta \simeq \theta_{\text{dyn}}$, we can further simplify equations (11) and (12) by letting $\cos \theta \simeq \cos \theta_{\text{dyn}}$ and $\sin(\theta - \theta_{\text{dyn}}) \simeq \theta - \theta_{\text{dyn}}$. After some rearrangements, we finally obtain the set of equations, which govern the evolution of the static height h and the rolling thickness R in the Saint-Venant hydrodynamic model:

$$\frac{\partial h}{\partial t} = -\frac{g}{\Gamma}(\theta - \theta_{\text{dyn}}) \tag{13a}$$

$$\frac{\partial R}{\partial t} = \Gamma\, R\frac{\partial R}{\partial x} + \frac{g}{\Gamma}(\theta - \theta_{\text{dyn}}) \,, \tag{13b}$$

where we used the shorthand notation $\Gamma = \Gamma_0 \cos^2 \theta_{\mathrm{dyn}}$.

In this last form, it appears that the equations (13) describing the flow within the Saint-Venant hydrodynamic description bear a striking similarity with the saturated version of the BCRE model, as given by equations (5). We will discuss this similarity in the next section, but we should first give a word of caution concerning the outcome of the Saint-Venant model: the final structure shown by equations (13) is directly dependent on the various physical assumptions that were introduced along the presentation (constant density, linear velocity profile, Coulomb friction force with a constant friction coefficient, near-constant static slope). However, except for special cases like, for instance, flows on a fixed bottom (non-linear velocity profile), or, possibly, very thin flowing layers (where it has been proposed [5] that the friction coefficient μ_{dyn} may show a dependence on R), we believe that these physical assumptions are rather robust and thus it is plausible that eqs. (13) may hold in a number of situations.

4.   Relation between models and perspectives

4.1.   Relation between BCRE and Saint-Venant descriptions

After having presented the BCRE and the saint-Venant hydrodynamic approach in turn, we concluded that the final sets of equations obtained within both models (saturated eqs. (5) and eqs. (13), respectively) have *exactly* the same structure. This outcome is especially interesting since these two approaches result from rather opposed points of view, the BCRE model being essentially phenomenological whereas the Saint-Venant equations originates in the application of hydrodynamic first principles.

We are now in a position to answer the questions raised about the BCRE model at the end of section 2.. First, the fact that a first-principle derivation leads to the same equations as the BCRE equations ensures that BCRE does indeed verify conservation of momentum. Second, the term-to-term comparison of the saturated equations (5) with the saturated equations (13) allows to give the expressions of the phenomenological parameters v_{up} and θ_{n} with respect to characteristic quantities in the problem:

$$v_{\mathrm{up}} = \frac{g}{\Gamma_0 \cos^2 \theta_{\mathrm{dyn}}}\,, \qquad \theta_{\mathrm{n}} = \theta_{\mathrm{dyn}} = \arctan \mu_{\mathrm{dyn}}\,. \tag{14}$$

Finally, we see how the BCRE model would have to be modified to take into account that the actual velocity profile in the flow is linear (instead of the model's original assumption of a constant downhill velocity v_{d}): one simply has to take the following expression $v_{\mathrm{d}} = v_{\mathrm{d}}(R) = \Gamma_0 \cos^2 \theta_{\mathrm{dyn}} R$, which is equal (within angular factors) to the depth-averaged velocity of the flow. However, though this process seems rather intuitive, one should not infer that the BCRE model can be extended to *any* type of velocity profile by simply introducing the mean velocity in place of the downhill velocity v_{d}: this deceptively simple conclusion proves erroneous in the general case, as first pointed out by Douady *et al.* [5]. In fact, if the velocity profile in the flowing layer is not linear (nor constant), the hydrodynamic approach proves that the h-equation (13a) includes several additional terms and becomes significantly more complex. (And the simple result for linear, or constant, velocity can actually easily be seen to come as the result of a favorable cancellation of terms ...).

One of the great merits of the BCRE model is to give a very intuitive picture of the granular flow through the idea of grain "exchanges" between the static and rolling phases occurring by dislodgement and trapping mechanisms, and it is indeed very satisfiying to see this picture confirmed by the more rigorous hydrodynamic approach. In our mind, the BCRE description remains therefore useful as a way of thinking and "visualizing" granular surface flows. On the other hand, the hydrodynamic description is both more general and more rigourous, and thus constitutes, as of today, a reference tool for the study of granular flows.

4.2.   Perspectives on the surface flows of granular materials

We hope to have shown in the present short review that the theoretical description of surface flows of granular materials has made significant advances in the last fifteen years, and that we presently are in
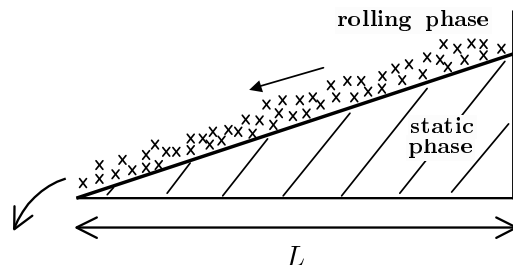
Figure 3: An "open" pile of grains with an avalanche taking place at its surface.

possession of reasonably efficient and reliable models.

It is worth mentioning that recent experimental results by Khakhar *et al.* [6] have confirmed some the-oretical predictions made with the help of the governing equations (13): in ref. [24], it had been predicted that when an avalanche occurs at the surface of an "open" pile of grains (i.e. a pile where the grains are free to fall at the bottom end), as shown on figure 3, part of the rolling profile $R(x, t)$ has a parabolic shape (and therefore, the maximum thickness of the avalanche scales as the square root of the system size, i.e. $R_{\max} \sim \sqrt{L}$). This is the same type of parabolic profile that has indeed been observed experimentally by Khakhar *et al.* [6].

Yet, at present, our understanding of the surface flows of granular flows remains fragmented and many elements lack to form a coherent and global picture. We would like to conclude this presentation by pointing out two of the least understood points.

Perhaps one of the most important issue is now to understand how an avalanche *starts*. The initiation process still remains obscure, and one can think of several mechanisms. For instance, the avalanche may start by a delocalized mechanism: above a certain maximum stability angle, the top layer of the pile is destabilized, starts to slide as a whole and is rapidly fluidized by collisions, hence forming a thin initial layer of rolling grains over the surface. On the contrary, one may rather favor a localized mechanism: at some moment, the most unstable grain(s) start to roll, and progressively disturb their neighborhood, thereby creating avalanching regions which spread around. Recent experiments by Rajchenbach [25] seem to sup-port this second scenario. We can even think that the delocalized and localized pictures are not completely incompatible: the delocalized scenario might be considered as a "coarse-grained" view of the avalanche initiation process, not valid at the grain scale where localized nucleation mechanisms predominate, but sen-sible after a short time for which the whole surface has finally been disturbed and put into motion. It would be interesting to have an estimate of this "complete destabilization" time of the surface originating from a few, localized, triggering points. We may also note that, in describing such early processes of initiation, the granular flow model by Aranson and Tsimring [8, 9] (mentioned in the Introduction) might be better suited, as its very principle is to allow for *partially* fluidized states.

Another significant (and very difficult) issue is to get a better understanding of the internal rheology of these surface flows: beyond the depth-averaged descriptions presented here, one would for instance like to better understand why linear velocity profiles emerge in flows taking place at the surface of piles, and why non-linear profiles arise in flows over fixed bottoms. We may mention here two arguments that have been proposed in the literature: the first is due to Komatsu *et al.* [26] who noticed that when a flow takes place at the surface of a pile, the static phase in fact undergoes a slow creeping motion; these authors suggest that the supression of this creeping motion when the flow occurs on a *rigid* bottom may be responsible for the change in the nature of the velocity profile between these two types of experiments. The second proposal, by Bonamy *et al.* [27] is very recent: these authors have observed experimentally that the texture of granular

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flows is strongly inhomogeneous, with "solid" clusters of grains embedded within the flowing layer. For flows at the surface of a pile, it was found that the size-distribution of these clusters follows a power-law with sizes ranging from the grain size to the flowing layer thickness. Bonamy *et al.* [27] then suggest that flows over fixed bottoms should display a very different cluster size distribution, which may be at the origin of the very different rheological behaviour observed in these systems as compared to piles.

Much remains to be done before we reach a comprehensive theory of surface flows of granular materials. However, the rapid pace sustained within this field on both the experimental and theoretical sides, over the last few years, may be interpreted as a good sign in the exploration of one of the (many) intriguing aspects of granular matter.

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