Summary file for stable flow of downward-sloping stream

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Abstract

Under V2.0 and V3.0 we discuss some basic assumptions about granular flow, while some of them may be wrong. Thus Here we try to distinguish some mistakes and do a summary file for stable flow.

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Ref: Equ(From Shijie Z) AND V2.0 AND V3.0
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Basic equ.

Lame coefficients and derivations

First come back to the coordinates we have used in describing this system.

$$\begin{aligned} h_1 &= 1 \\ h_2 &= a\cos\alpha + b\cos\alpha \\ h_3 &= 1 \end{aligned}$$

For the $\text{time derivative of a function}$ in this system, we have

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\nabla$$

We seperate the second term by its components

$$\begin{aligned} \vec{u}\cdot\nabla &= (u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3)\cdot(\partial_1\vec{e}_1 + \partial_2\vec{e}_2 + \partial_3\vec{e}_3) \\ &= \frac{1}{h_1}u_1\partial_1 + \frac{1}{h_2}u_2\partial_2 + \frac{1}{h_3}u_3\partial_3 \\ &= u_a\partial_a + \frac{1}{h_2}u_\theta\partial_\theta + u_b\partial_b \end{aligned}$$

For the $\text{divergence of a function (velocity)}$ in this system, we have

$$\begin{aligned} \nabla\cdot\vec{u} &= (\partial_1\vec{e}_1 + \partial_2\vec{e}_2 + \partial_3\vec{e}_3)\cdot(u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3) \\ &= \frac{1}{h_1h_2h_3}\partial_1\left(\frac{h_1h_2h_3}{h_1}u_1\right) + \frac{1}{h_1h_2h_3}\partial_2\left(\frac{h_1h_2h_3}{h_2}u_2\right) + \frac{1}{h_1h_2h_3}\partial_3\left(\frac{h_1h_2h_3}{h_3}u_3\right) \\ &= \frac{1}{h_2}\partial_a\left((a\cos\alpha + b\sin\alpha)u_a\right) + \frac{1}{h_2}\partial_\theta u_\theta + \frac{1}{h_2}\partial_b\left((a\cos\alpha + b\sin\alpha)u_b\right) \\ &= \frac{1}{h_2}u_a\cos\alpha + \partial_a u_a + \frac{1}{h_2}\partial_\theta u_\theta + \frac{1}{h_2}\partial_b\left((a\cos\alpha + b\sin\alpha)u_b\right) \end{aligned}$$

For the stress term in momentum conservation equation, there may be different ways to describe its mathematical form. Here we focus on the simple form like:

$$\begin{aligned} T_{ij} &= -p\delta_{ij} + \tau_{ij} \\ \nabla\cdot\overset{\leftrightarrow}{T} &= -\nabla p + \mu\nabla^2\vec{u} \end{aligned}$$

For $\text{the gradient of pressure } p$, it is quite clear that

Here one thing needs to be noticed that the pressure term p is only related to b or $\color{red}{(b,h)}$ *when considering the stable flow. In other words, the disturbance case need to be reconsider the pressure term from the momentum conservation of b direction.*

$$\nabla p = \partial_a p\,\vec{e}_a + \frac{1}{h_2}\partial_\theta p\,\vec{e}_\theta + \partial_b p\,\vec{e}_b$$

For the laplacian of velocity, it may take some time to verify the concrete form.

$$\begin{aligned}
\nabla^2\vec{u} &= (\partial_1\vec{e}_1+\partial_2\vec{e}_2+\partial_3\vec{e}_3)\cdot(\partial_1\vec{e}_1+\partial_2\vec{e}_2+\partial_3\vec{e}_3)(u_1\vec{e}_1+u_2\vec{e}_2+u_3\vec{e}_3)\\
&= \frac{1}{h_1h_2h_3}\begin{pmatrix}\partial_1 h_2h_3\\ \partial_2 h_1h_3\\ \partial_3 h_1h_2\end{pmatrix}^T\begin{pmatrix}\frac{1}{h_1}\partial_1u_1 & \frac{1}{h_1}\partial_1u_2 & \frac{1}{h_1}\partial_1u_3\\ \frac{1}{h_2}\partial_2u_1 & \frac{1}{h_2}\partial_2u_2 & \frac{1}{h_2}\partial_2u_3\\ \frac{1}{h_3}\partial_3u_1 & \frac{1}{h_3}\partial_3u_2 & \frac{1}{h_3}\partial_3u_3\end{pmatrix}\\
&= \frac{1}{h_1h_2h_3}\left[\partial_1\left(\frac{h_2h_3}{h_1}\partial_1u_1\right)+\partial_2\left(\frac{h_1h_3}{h_2}\partial_2u_1\right)+\partial_3\left(\frac{h_1h_2}{h_3}\partial_3u_1\right)\right]\vec{e}_1\\
&+\frac{1}{h_1h_2h_3}\left[\partial_1\left(\frac{h_2h_3}{h_1}\partial_1u_2\right)+\partial_2\left(\frac{h_1h_3}{h_2}\partial_2u_2\right)+\partial_3\left(\frac{h_1h_2}{h_3}\partial_3u_2\right)\right]\vec{e}_2\\
&+\frac{1}{h_1h_2h_3}\left[\partial_1\left(\frac{h_2h_3}{h_1}\partial_1u_3\right)+\partial_2\left(\frac{h_1h_3}{h_2}\partial_2u_3\right)+\partial_3\left(\frac{h_1h_2}{h_3}\partial_3u_3\right)\right]\vec{e}_3
\end{aligned}$$

In our case, it becomes:

$$\begin{aligned}
(\nabla^2\vec{u})_i &= \frac{1}{h_2}\left[\partial_a(h_2\partial_a u_i)+\partial_\theta\left(\frac{1}{h_2}\partial_\theta u_i\right)+\partial_b(h_2\partial_b u_i)\right]\\
&= \frac{\cos\alpha}{h_2}\partial_a u_i+\partial_{aa}u_i+\frac{1}{(h_2)^2}\partial_{\theta\theta}u_i+\frac{\sin\alpha}{h_2}\partial_b u_i+\partial_{bb}u_i
\end{aligned}$$

Stable flow and integration

The core idea is to integrate the function along b direction, in this way we can use the similarity properties in thin-film flow analysis (relations about distributions between real and mean quantities)

Stable means every quantity is time-independent and only related to parameter a and b, which means:

$$\partial_t = 0 \quad ; \quad \partial_\theta = 0$$

In our case the main quantity is velocity along a direction when considering the stable flow. We define its mean as:

$$\bar{u} \equiv \frac{1}{h}\int_0^h u\,db \tag{1.1}$$

For the stable flow, the continuity becomes:

$$\begin{aligned}
0 &= \nabla\cdot\vec{u}\\
&= \frac{1}{h_2}u_a\cos\alpha+\partial_a u_a+\cancel{\frac{1}{h_2}\partial_\theta u_\theta}+\frac{1}{h_2}\partial_b\left((a\cos\alpha+b\sin\alpha)u_b\right)
\end{aligned}$$

Do the integration on both sides and we can get:

$$0=\partial_a\bar{u}+\frac{\cos\alpha}{h}\int_0^h\frac{1}{h_2}u\,db+\frac{1}{h}\int_0^h\frac{1}{h_2}\partial_b\left((a\cos\alpha+b\sin\alpha)u_b\right)db$$

For term III by using the boundary conditions we can obtain:(here use the approximation that $\frac{1}{h_2}\sim\frac{1}{a\cos\alpha}$, the provement and discussion see the **Term3** and **Appendix for $\frac{1}{h_2}$**)

$$\begin{aligned}
\text{Term3} &= \frac{1}{h}\int_0^h\frac{1}{h_2}\partial_b\left((a\cos\alpha+b\sin\alpha)u_b\right)db\\
&\simeq \frac{1}{h}\frac{1}{a\cos\alpha}(a\cos\alpha+b\sin\alpha)u_b\Big|_{b=0}^{b=h}\\
&= \frac{1}{h}\left[\left(1+\frac{h}{a}\tan\alpha\right)u_b|_h-(1+0)\cancel{u_b|_0}\right]\\
&= \left(1+\frac{h}{a}\tan\alpha\right)\frac{1}{h}\frac{D}{Dt}h
\end{aligned}$$

Here using the lame derivation I have shown above we can get

$$\begin{aligned}
\frac{D}{Dt}h &= (\partial_t+\bar{u}\cdot\nabla)\,h\\
&= \bar{u}\partial_a h
\end{aligned}$$

Here one thing needs to mention that the orange part means we use the b-mean(which means u_a takes average along b direction) velocity $\bar{u}$ to 'observe' the change of h. The reason of choosing this value is because it is the mean velocity of u_a along b direction, which will be shown in ***velocity similarity and velocity profile part*** *below.*

When we consider the flow in the thin film, we can assume that the velocity u_a (now denoted by u) has the similarity with its free-surface value ($u|_{b=h} \equiv u_s$). The form can be expressed as:

$$\frac{u}{u_s} = f(\eta) \quad \text{where } \eta = \frac{b}{h} \tag{1.2}$$

And by researches of earlier authors, we can express the distribution function as:

$$f(\eta) = A\eta + B\eta^2 \tag{1.2}$$

For term II it can be written as:

$$\begin{aligned}
\text{Term2} &= \frac{\cos\alpha}{h}\int_0^h \frac{1}{h_2} u db \\
&= \frac{\cos\alpha}{h}\int_0^h \frac{\bar{u}}{a\cos\alpha + b\sin\alpha}\left(A\frac{b}{h} + B\frac{b^2}{h^2}\right) db \\
&= \frac{\bar{u}\cos\alpha}{h}\int_0^h \frac{1}{a\cos\alpha + b\sin\alpha}\left(A\frac{b}{h} + B\frac{b^2}{h^2}\right) db
\end{aligned}$$

The integration is quite complex, we can only focus on $o\left(\frac{h}{a}\right) \sim o(1)$ terms (in the final form)

$$\begin{aligned}
\text{RHS} &= \frac{u_s\cos\alpha}{h}\frac{1}{a\cos\alpha}\int_0^h \left(1 + \frac{b}{a}\tan\alpha\right)^{-1}\left(A\frac{b}{h} + B\frac{b^2}{h^2}\right) db \\
&= \frac{u_s}{ah}\int_0^h \left(1 - \frac{b}{a}\tan\alpha + o\left(\frac{b^2}{a^2}\tan^2\alpha\right)\right)\left(A\frac{b}{h} + B\frac{b^2}{h^2}\right) db \\
&= \frac{u_s}{a}\left(\frac{1}{2}A - \frac{1}{3}\frac{h\tan\alpha}{a}A + \frac{1}{3}B - \frac{1}{4}\frac{h\tan\alpha}{a}B\right)
\end{aligned}$$

Now we come back to see our velocity profile assumption in (1.1) and get

$$\begin{aligned}
\bar{u} &\equiv \frac{1}{h}\int_0^h u db \\
&= u_s\int_0^1 (A\eta + B\eta^2) d\eta \\
&= \left(\frac{1}{2}A + \frac{1}{3}B\right) u_s
\end{aligned}$$

Thus the u_s can be rewritten by the form of $\bar{u}$ as below:

$$\begin{aligned}
\text{Term2} &= \left(\frac{1}{2}A + \frac{1}{3}B\right)\frac{u_s}{a}\left(1 - \frac{h}{a}\tan\alpha\frac{\frac{1}{3}A + \frac{1}{4}B}{\frac{1}{2}A + \frac{1}{3}B}\right) \\
&= \frac{\bar{u}}{a}\left(1 - \frac{h}{a}\frac{4A + 3B}{6A + 4B}\tan\alpha\right)
\end{aligned}$$

Thus the final form of continuum is:

$$\partial_a\bar{u} + \frac{\bar{u}}{a}\left(1 - \frac{h}{a}\frac{4A + 3B}{6A + 4B}\tan\alpha\right) + \left(1 + \frac{h}{a}\tan\alpha\right)\frac{1}{h}\bar{u}\partial_a h + \cancel{\frac{1}{h}\int_0^h \frac{1}{h_2}\partial_\theta u_\theta db} = 0 \tag{1.3}$$

Now we correct it to the non-zero first order form, the continuum of stable form becomes:

$$\partial_a(\bar{u}h) + \frac{\bar{u}h}{a} = 0 \tag{1.3*}$$

Appendix for $\frac{1}{h_2}$: from (1.3) to (1.3^*) we know that in most integration in our model, $\left(\frac{1}{h_2}, u\right) \sim \left(\frac{1}{a\cos\alpha}, \bar{u}\right)$ is a well-done zeroth-ordered approximation.

For the stable flow, we also need to determine the momentum relations in different directions. Due to Shijie Zhong's form, we can get:

Here we introduce the time derivative as

$$\begin{aligned} d_t &= \partial_t + \vec{u} \cdot \nabla \\ &= \partial_t + u_a \partial_a + \frac{1}{h_2} u_\theta \partial_\theta + u_b \partial_b \end{aligned}$$

and the gravity as $\vec{g} = g \sin\alpha\, \hat{e}_a - g \cos\alpha\, \hat{e}_b$

$$\begin{aligned} a-\text{direction:} \quad & \rho\phi \left(d_t u_a - \frac{1}{h_2} {u_\theta}^2 \cos\alpha \right) = (\nabla \cdot \overleftrightarrow{T})_a + \rho\phi g \sin\alpha \\ \theta-\text{direction:} \quad & \rho\phi \left(d_t u_\theta + \frac{1}{h_2} u_\theta (u_a \cos\alpha + u_b \sin\alpha) \right) = (\nabla \cdot \overleftrightarrow{T})_\theta \\ b-\text{direction:} \quad & \rho\phi \left(d_t u_b - \frac{1}{h_2} {u_\theta}^2 \sin\alpha \right) = (\nabla \cdot \overleftrightarrow{T})_b - \rho\phi g \cos\alpha \end{aligned}$$

Something needs to be discussed that here we can use different models to describe the stress term $(\nabla \cdot \overleftrightarrow{T})$. To verify our model, we choose the basic constitutive relation applied here:

$$T_{ij} = -p\delta_{ij} + \tau_{ij} \qquad \text{where } \tau_i = (\mu \nabla^2 \vec{u})_i$$

In stable flow case, things become quite easy that everything relates to θ and ∂_t will be zero. We first apply the Lame form we have calculated above to show the form of the shear stress term:

$$(\mu\nabla^2 \vec{u})_i = \mu \left[\frac{\cos\alpha}{h_2} \partial_a u_i + \partial_{aa} u_i + \frac{1}{(h_2)^2} \partial_{\theta\theta} u_i + \frac{\sin\alpha}{h_2} \partial_b u_i + \partial_{bb} u_i \right]$$

While in the thin-film, terms along the wall direction can be considered as small quantities compared with those terms perpendicular to the wall, thus the actual relations of momentum are:

$$\begin{aligned} a-\text{direction:} \quad & \rho\phi \left(u_a \partial_a u_a + u_b \partial_b u_a \right) = -\partial_a p + \mu \frac{\sin\alpha}{h_2} \partial_b u_a + \mu \partial_{bb} u_a + \rho\phi g \sin\alpha \\ b-\text{direction:} \quad & \rho\phi \left(u_a \partial_a u_b + u_b \partial_b u_b \right) = -\partial_b p + (\mu\nabla^2 u_b) - \rho\phi g \cos\alpha \end{aligned}$$

Here we use the assumption that the velocity along the b direction is much smaller than the velocity along the a & θ direction (and other partial-derivative terms), thus we can simplify those functions above as:

$$\begin{aligned} a-\text{direction:} \quad & \rho\phi \left(u_a \partial_a u_a + \cancel{u_b} \partial_b u_a \right) = -\partial_a p + \mu \frac{\sin\alpha}{h_2} \partial_b u_a + \mu \partial_{bb} u_a + \rho\phi g \sin\alpha & (1.4) \\ b-\text{direction:} \quad & \rho\phi \left(u_a \cancel{\partial_a u_b} + \cancel{u_b} \partial_b u_b \right) = -\partial_b p + \cancel{(\mu\nabla^2 u_b)} - \rho\phi g \cos\alpha & (1.5) \end{aligned}$$

Thus from the (1.5) we can directly get the pressure distribution:

$$p = -\rho\phi g \cos\alpha\, b + C$$

Apply the B.C. on the free-surface: $p|_{b=h} = p_0$, it becomes:

$$C = p_0 + \rho\phi g \cos\alpha\, h$$

Here we use the additional pressure $p - p_0$ to express p, it becomes:

$$p = \rho\phi g \cos\alpha (h - b) \tag{1.6}$$

For (1.4) we integrate it along b direction and apply (1.6) in it and obtain:

$$\frac{1}{h} \int_0^h u \partial_a u db + g \cos\alpha \partial_a h = g \sin\alpha + \frac{\mu}{\rho\phi} \frac{\sin\alpha}{h} \int_0^h \frac{1}{h_2} \partial_b u db + \frac{\mu}{\rho\phi} \frac{1}{h} \int_0^h \partial_{bb} u db$$

Also use the appendix for $\frac{1}{h2}$, the approach of $\mathrm{Term2}$ and B.C., we can find that

$$\frac{1}{2h} \int_0^h \partial_a u^2\, db + g \cos\alpha \partial_a h = g \sin\alpha + \frac{\mu}{\rho\phi} \frac{c_2}{ah} \tan\alpha\, \bar{u} + \frac{\mu}{\rho\phi} \frac{1}{h} (0 - \partial_b u|_{b=0})$$

Here we need two approximations to determine the first term and the stress caused by the wall, and I will prove that these two assumptions can be obtained by the velocity profile later.

$$\begin{aligned} \frac{1}{2h}\int_0^h \partial_a u^2\, db &= c_1 \partial_a \bar{u}^2 \\ \tau_\omega = \partial_b u|_{b=0} &= c_2 \frac{\bar{u}}{h} \end{aligned} \tag{1.2*}$$

Thus the integration becomes:

$$c_1 \partial_a \bar{u}^2 + g\cos\alpha\, \partial_a h = g\sin\alpha + \frac{\mu}{\rho\phi}\frac{c_2}{ah}\bar{u}\tan\alpha - \frac{\mu}{\rho\phi}\frac{1}{h^2}c_2\bar{u} \tag{1.7}$$

Here we calculate the function $(1.3*)$ and get its solution is

$$\bar{u}ha = c_0 \tag{1.8}$$

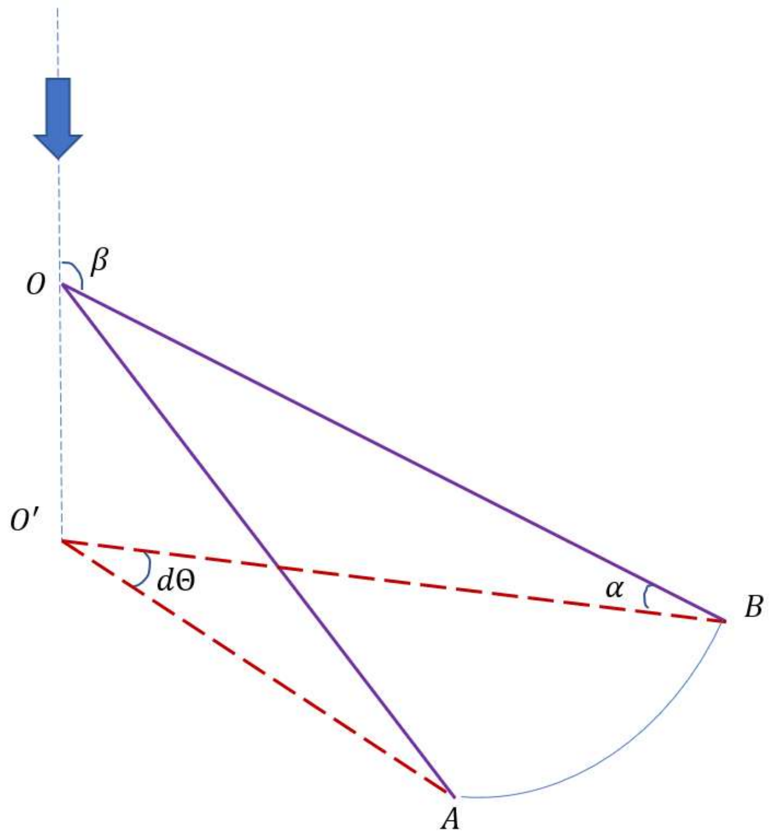
(Notice that here we can determine the const c_0 by the definition of $c_0 = q^* \equiv \frac{Q^*}{2\pi}$)

Now we can plug (1.8) in the (1.7) to eliminate other variable $\bar{u}$, let it be $f(h,a) = 0$

$$\begin{aligned} c_1 \partial_a \left(\frac{c_0}{ha}\right)^2 + g\cos\alpha\, \partial_a h &= g\sin\alpha + \frac{\mu}{\rho\phi}\frac{c_0 c_2}{h^2 a^2}\tan\alpha - \frac{\mu}{\rho\phi}\frac{1}{ah^3}c_2 c_0 \\ -c_1 {c_0}^2 \frac{2}{h^3 a^3}(a\partial_a h + h) + g\cos\alpha\, \partial_a h &= g\sin\alpha + \frac{\mu}{\rho\phi}\frac{c_0 c_2}{h^2 a^2}\tan\alpha - \frac{\mu}{\rho\phi}\frac{1}{ah^3}c_2 c_0 \\ \left((-2c_1 {c_0}^2)\frac{1}{h^3 a^2} + g\cos\alpha\right)\partial_a h &= c_1 {c_0}^2 \frac{2}{h^2 a^3} + g\sin\alpha + \frac{\mu}{\rho\phi}\frac{c_0 c_2}{h^2 a^2}\tan\alpha - \frac{\mu}{\rho\phi}\frac{1}{ah^3}c_2 c_0 \end{aligned}$$

Try to simplify the form, the final result is:

$$\left(2c_1 \frac{{c_0}^2}{a^2} - gh^3\cos\alpha\right)\partial_a h = -gh^3\sin\alpha - 2c_1\frac{h{c_0}^2}{a^3} + c_2\frac{\mu}{\rho\phi}\frac{c_0}{a}\left(1 - \tan\alpha\frac{h}{a}\right) \tag{1.9}$$



Compare our result with Guangzhao Zhou's result[1], we can find that

$$\begin{aligned} &\left(\frac{54}{35}\frac{q^2}{r^2\sin^2\beta} - gh^3\sin\beta\right)\frac{\mathrm{d}h}{\mathrm{d}r} \\ &= h^3 g\cos\beta - \frac{54}{35}\frac{hq^2}{r^3\sin^2\beta} + \frac{3vq}{r\sin\beta}\left(1 + \cot\beta\frac{h}{r}\right) \end{aligned} \tag{G1}$$

In spherical coordinates, the q-flux can be expressed as:

$$\begin{aligned}
Q &\equiv \iint_S \vec{u}\cdot\vec{n} dS \\
&= -2\pi r^2 \int_0^{\Psi} \sin\theta \; u \; d\psi \\
&= -2\pi r^2 \int_0^{\Psi} \sin(\beta-\psi) \; u \; d\psi \\
&= -2\pi r^2 \int_0^{\Psi} (\sin\beta\cos\psi - \sin\psi\cos\beta) \; u \; d\psi \\
&\simeq -2\pi r^2 \sin\beta \int_0^{\Psi} u \; d\psi
\end{aligned}$$

recall the mean velocity definition, it is:

$$\bar{u} = \frac{\int_0^{\Psi} ru \; d\psi}{\int_0^{\Psi} r \; d\psi}$$

So it becomes:

$$\begin{aligned}
q \equiv \frac{Q}{2\pi} &= -r^2 \sin\beta \Psi \bar{u} \\
&= rh\bar{u}\sin\beta \\
&= q^* \sin\beta
\end{aligned}$$

Thus our result can correspond formally to the result of Guangzhao Zhou's result here. And it also shows that, when we use the velocity profile like below:

$$\frac{u}{u_s} = \frac{\psi}{\Psi}\left(2-\frac{\psi}{\Psi}\right)$$

Thus we can find the corresponding coefficients:

$$(A=2, B=-1) \longrightarrow \left(c_1 = \frac{27}{35}, c_2 = 3\right)$$

To verify this relation, we need to calculate the (c_1, c_2) from (A, B) directly! The core calculation is to use (1.2) to obtain the $(1.2*)$, see below:

Here $(A=2, B=-1)$ means that $\bar{u} = \frac{2}{3}u_s$

$$\begin{aligned}
\partial_b u|_{b=0} &= \frac{A}{h}u_s \\
&= \frac{2}{h}\frac{3}{2}\bar{u} \longrightarrow c_2 = 3 \\
&= 3\frac{\bar{u}}{h}
\end{aligned}$$

To verify the relation 1, we first assume that $\partial_a h \sim o(1)$

$$\begin{aligned}
\frac{1}{2h}\int_0^h \partial_a u^2 \; db &= \frac{1}{2}\partial_a\left(\frac{1}{h}\int_0^h u^2 \; db\right) \\
&= \frac{1}{2}\partial_a\left(\int_0^1 \left(A\eta+B\eta^2\right)^2 u_s \; d\eta\right) \longrightarrow c_1 = \frac{3}{5} \\
&= \frac{1}{2}\left(\frac{4}{3}-1+\frac{1}{5}\right)\frac{9}{4}\partial_a\bar{u}^2 \\
&= \frac{3}{5}\partial_a\bar{u}^2
\end{aligned}$$

This result can correspond to Kasimov's result. While this assumption may cause some error, if we consider the $h=h(a)$ and take $\partial_a h$ terms into consideration, Guangzhao Zhou's result is more reasonable.

Supplementary instruction

The green part intergration between (1.6) and (1.7)

$$
\begin{aligned}
\frac{\sin\alpha}{h}\int_0^h \frac{1}{h_2}\partial_b u db &= \frac{\sin\alpha}{h}\int_0^h \frac{1}{a\cos\alpha + b\sin\alpha}\left(\frac{A}{h}+2B\frac{b}{h^2}\right)u_s\, db \\
&= \frac{\sin\alpha}{h}u_s\left(\frac{A}{h}\int_0^1 \frac{1}{\frac{a}{h}\cos\alpha+\eta\sin\alpha}d\eta\right. \\
&\left. + \frac{2B}{h}\int_0^1 \frac{\eta}{\frac{a}{h}\cos\alpha+\eta\sin\alpha}d\eta\right)
\end{aligned}
$$

For term 1, there is:

$$
\begin{aligned}
\frac{A}{h}\int_0^1 \frac{1}{\frac{a}{h}\cos\alpha+\eta\sin\alpha}d\eta &= \frac{A}{h}\frac{1}{\sin\alpha}\ln\left(1+\frac{h}{a}\frac{\sin\alpha}{\cos\alpha}\right) \\
&\simeq \frac{A}{h}\frac{1}{\sin\alpha}\frac{h}{a}\frac{\sin\alpha}{\cos\alpha} \\
&= \frac{A}{a\cos\alpha}
\end{aligned}
$$

For term 2, there is:

$$
\begin{aligned}
\frac{2B}{h}\int_0^1 \frac{\eta}{\frac{a}{h}\cos\alpha+\eta\sin\alpha}d\eta &= \frac{2B}{h}\frac{1}{\sin^2\alpha}\left(\sin\alpha - \frac{a\cos\alpha}{h}\ln\left(1+\frac{h}{a}\frac{\sin\alpha}{\cos\alpha}\right)\right) \\
&\simeq \frac{2B}{h}\frac{1}{\sin\alpha}\left(1-\frac{a}{h}\frac{\cos\alpha}{\sin\alpha}\frac{h}{a}\frac{\sin\alpha}{\cos\alpha}\right) \\
&= 0
\end{aligned}
$$

Due to

$$
\begin{aligned}
\bar{u} &= \left(\frac{1}{2}A+\frac{1}{3}B\right)u_s \\
&= \frac{2}{3}u_s
\end{aligned}
$$

Thus this green integration becomes:

$$
\begin{aligned}
\frac{\sin\alpha}{h}\int_0^h \frac{1}{h_2}\partial_b u db &= \frac{A}{ah}u_s\tan\alpha \\
&= \frac{1}{ah}A\frac{3}{2}\tan\alpha\,\bar{u} \\
&= \frac{3}{ah}\tan\alpha\,\bar{u} \\
&= \frac{c_2}{ah}\tan\alpha\,\bar{u}
\end{aligned}
$$