(Edit1006) Corrigendum and discussion about equations and assumptions in (Equ.) files

This file aims to correct and explain some further derivations of (Equ.), at the end I will provide the the form that I will use it to calculate. Also these corrections need to be verified by the numerical calculation in the stable case. If something goes wrong, there may be another edit_equ file in the future.

1. Revise the explanation of $\frac{Dh}{Dt}$ and the integration of continum

Something insufficiently strict but leads to the correct answer, thus we need to point out mistakes to make the process more persuasive.

Recall what we have done when doing integration of continum:

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From (1.1) to (1.3) - (Equ.)Summary file for stable case
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The first mistake is about $\frac{1}{h}\int_0^h \partial_a u_a \, db = \partial_a \bar{u}$ term
When we use the linear velocity profile of u_a it may be correct, while when we use the parabolic velocity profile, something becomes a little bit different.

$$\begin{aligned}\frac{1}{h}\int_0^h \partial_a u_a \, db &= \frac{1}{h}\int_0^h \partial_a \left[\left(A\frac{b}{h}+B\frac{b^2}{h^2}\right)u_s\right] \, db \\ &= -\frac{1}{h}u_s\partial_a h\int_0^h \left(A\frac{b}{h^2}+2B\frac{b^2}{h^3}\right)db + \frac{1}{h}\partial_a u_s\int_0^h \left(A\frac{b}{h}+B\frac{b^2}{h^2}\right)db \\ &= -\frac{u_s}{h}\partial_a h\left(\frac{1}{2}A+\frac{2}{3}B\right)+\partial_a u_s\left(\frac{1}{2}A+\frac{1}{3}B\right)\end{aligned}$$

Before we move further, I want to point out another mistake first, it is about

$$\frac{D}{Dt}h = (\partial_t + u_s\cdot\nabla)h$$

Here when we consider the change of the height of the free surface, choosing the velocity on the surface is more reasonable instead of $\bar{u}$.

Thus we reconsider the continum, it becomes:

$$\begin{aligned}0 &= -\frac{u_s}{h}\partial_a h\left(\frac{1}{2}A+\frac{2}{3}B\right)+\partial_a u_s\left(\frac{1}{2}A+\frac{1}{3}B\right)+\frac{\bar{u}}{a}+\frac{1}{h}(\partial_t+u_s\partial_a)h \\ &= \frac{\bar{u}h}{a}+u_s\partial_a h\left(1-\frac{1}{2}A-\frac{2}{3}B\right)+h\partial_a u_s\left(\frac{1}{2}A+\frac{1}{3}B\right)+\partial_t h \\ &= \partial_t h+\frac{\bar{u}h}{a}+\bar{u}\partial_a h\frac{1-\frac{1}{2}A-\frac{2}{3}B}{\frac{1}{2}A+\frac{1}{3}B}+h\partial_a\bar{u}\end{aligned}$$

Here we know that $A=2, B=-1$, thus we have:

$$\lambda_{AB} = \frac{1-\frac{1}{2}A-\frac{2}{3}B}{\frac{1}{2}A+\frac{1}{3}B} = 1$$

So, results are the same, while two mistakes make for a correct coincidence.

2. Discussion about convection terms in the integral form of governing equations

Recall that we used to deal with this kind of terms like:

$$
\begin{aligned}
\frac{1}{h}\int_0^h d_t u_a\, db &= \frac{1}{h}\int_0^h \left(\partial_t u_a + u_a \partial_a u_a\right) db \\
&= \partial_t u + \frac{1}{2h}\int_0^h \partial_a {u_a}^2 db \\
&= \partial_t u + \boldsymbol{C} u \partial_a u
\end{aligned}
$$

While here something was ignored in $d_t = \partial_t + u\partial_a + \frac{1}{h_2} v \partial_\theta + w\partial_b$, in the stable case that $v = 0$, while tne contribution of $w\partial_b$ term was ignored when doing integration, here we focus on it.

First we consider $\int_0^h u_b \partial_b u_a\, db$. This integration is hard to integrate by parts, thus we need to use the u_a velocity profile (we have already known) to find the form of u_b. The physical idea leads us to use the continuity to get this.

Recall the continuity in the stable case:

$$
\text{Continum}: \quad \frac{1}{h_2} u_a \cos\alpha + \partial_a u_a + \cancel{\frac{1}{h_2}\partial_\theta u_\theta} + \frac{1}{h_2}\partial_b\left(\left(a\cos\alpha + b\sin\alpha\right) u_b\right) = 0
$$

To find $u_b = u_b(b)$ relation, we can integrate along b direction from 0 to b, here the integral parameter is set to be ξ

$$
0 = \int_0^b \frac{1}{h_2}\cos\alpha u_a d\xi + \int_0^b \partial_a u_a d\xi + \int_0^b \frac{1}{h_2}\partial_\xi((a\cos\alpha + \xi\sin\alpha)u_\xi) d\xi
$$

For Term1, it is:

$$
\begin{aligned}
\int_0^b \frac{1}{h_2}\cos\alpha u_a d\xi &= \frac{u_s}{a}\int_0^b \left(A\frac{\xi}{h} + B\frac{\xi^2}{h^2}\right) d\xi \\
&= \frac{u_s}{a}\left(\frac{1}{2}\frac{A}{h} b^2 + \frac{1}{3}\frac{B}{h^2} b^3\right)
\end{aligned}
$$

For Term2, it is:

$$
\begin{aligned}
\int_0^b \partial_a u_a d\xi &= \int_0^b \partial_a \left[\left(A\frac{\xi}{h} + B\frac{\xi^2}{h^2}\right) u_s\right] d\xi \\
&= -u_s \partial_a h \int_0^b \left(A\frac{\xi}{h^2} + 2B\frac{\xi^2}{h^3}\right) d\xi + \partial_a u_s \int_0^b \left(A\frac{\xi}{h} + B\frac{\xi^2}{h^2}\right) d\xi \\
&= -u_s \partial_a h \left(\frac{1}{2}A\frac{b^2}{h^2} + \frac{2}{3}B\frac{b^3}{h^3}\right) + \partial_a u_s \left(\frac{1}{2}A\frac{b^2}{h} + \frac{1}{3}B\frac{b^3}{h^2}\right)
\end{aligned}
$$

For Term3, it is:

$$
\begin{aligned}
\int_0^b \frac{1}{h_2}\partial_\xi((a\cos\alpha + \xi\sin\alpha)u_\xi) d\xi &= \left(1 + \frac{b}{a}\tan\alpha\right) u_b \\
&\simeq u_b
\end{aligned}
$$

Thus we can obtain the velocity profile of u_b, it is

$$
u_b = b\left(\frac{\frac{1}{3}B\frac{b^2}{h^2}}{\frac{1}{2}A + \frac{1}{3}B}\frac{\bar{u}}{h}\partial_a h - \frac{\frac{1}{2}A\frac{b}{h} + \frac{1}{3}B\frac{b^2}{h^2}}{\frac{1}{2}A + \frac{1}{3}B}\left(\partial_a \bar{u} + \frac{\bar{u}}{a} - \frac{\bar{u}}{h}\partial_a h\right)\right)
$$

We can verify this form by applying this to boundary conditions:

1. When $b = 0$, it becomes $u_b|_{b=0} = 0$ clearly.
2. When $b = h$, it needs to be $\frac{D}{Dt}h$ form. (Using the continum we have discussed above with λ_{AB})

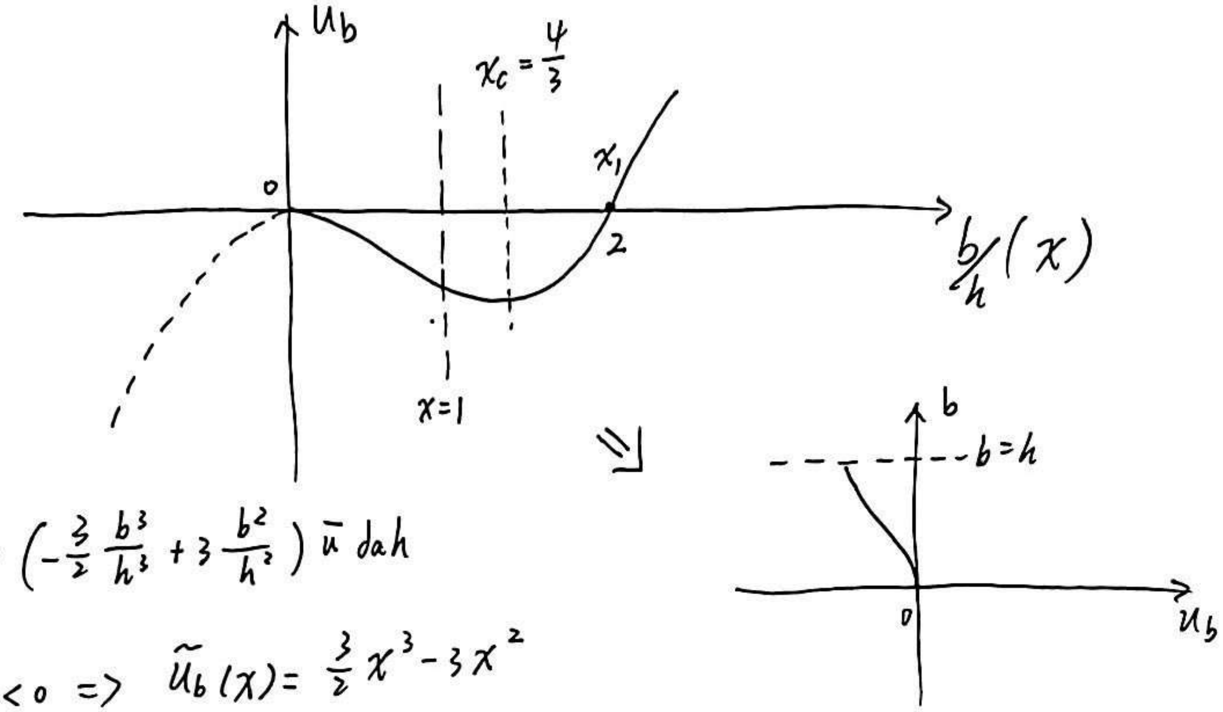
$$
\begin{aligned}
u_b|_{b=h} &= \bar{u}\partial_a h\left(\frac{\frac{1}{2}A+\frac{2}{3}B}{\frac{1}{2}A+\frac{1}{3}B}\right) - h\partial_a\bar{u} - \frac{\bar{u}h}{a} \\
&= \bar{u}\partial_a h\left(\frac{\frac{1}{2}A+\frac{2}{3}B}{\frac{1}{2}A+\frac{1}{3}B}\right) - \left(\bar{u}\partial_a h\frac{\frac{1}{2}A+\frac{2}{3}B-1}{\frac{1}{2}A+\frac{1}{3}B} - \partial_t h\right) \\
&= \partial_t h + \frac{\bar{u}}{\frac{1}{2}A+\frac{1}{3}B}\partial_a h \\
&= \partial_t h + u_s\partial_a h \\
&\equiv \frac{D}{Dt}h
\end{aligned}
$$

Let me show that waht kind of shape does this velocity profile look like. In our case $A=2, B=1$, it becomes:

$$
\begin{aligned}
u_b &= \frac{A\frac{b^2}{h^2}+B\frac{b^3}{h^3}}{\frac{1}{2}A+\frac{1}{3}B}\bar{u}\partial_a h \\
&= \left(\frac{1}{2}\frac{b^3}{h^2}-\frac{3}{2}\frac{b^2}{h}\right)\left(\partial_a h+\frac{\bar{u}}{a}-\frac{\bar{u}}{h}\partial_a h\right) - \left(\frac{1}{2}\frac{b^2}{h^2}\frac{\bar{u}}{h}\partial_a h\right)b \\
&= \left(-\frac{3}{2}\frac{b^3}{h^3}+3\frac{b^2}{h^2}\right)\bar{u}\partial_a h
\end{aligned}
$$

Using the continuity when $A=2, B=-1$,
(and also choosing the stable condition when calculating other integral $\partial_t h = 0$)

$$
\partial_a\bar{u}+\frac{\bar{u}}{a} = -\frac{\bar{u}}{h}\partial_a h - \cancel{\partial_t h}
$$



$u_b = \left(-\frac{3}{2}\frac{b^3}{h^3}+3\frac{b^2}{h^2}\right)\bar{u}\,\partial_a h$

for $\partial_a h<0 \Rightarrow \tilde{u}_b(x) = \frac{3}{2}x^3-3x^2$

Now we can calculate $\int_0^h u_b\partial_b u_a\,db$ directly:

$$
\begin{aligned}
\frac{1}{h}\int_0^h u_b\partial_b u_a\,db &= \frac{1}{h}\int_0^h\left(-\frac{3}{2}\frac{b^3}{h^3}+3\frac{b^2}{h^2}\right)\bar{u}\partial_a h\left(1-\frac{b}{h}\right)3\frac{\bar{u}}{h}\,db \\
&= 9\frac{\bar{u}^2}{h}\partial_a h\int_0^1\left(-\frac{1}{2}x^3+x^2\right)(1-x)dx \\
&= 9\frac{\bar{u}^2}{h}\partial_a h\cdot\frac{7}{120} \\
&= \frac{21}{40}\frac{\bar{u}^2}{h}\partial_a h
\end{aligned}
$$

3. Discussion about approximation (1.2*-1) in (Equ.)

Here we try to calculate $\frac{1}{h}\int_0^h u_a\partial_a u_a$ directly and we will see that something was ignored when setting approximation (1.2*-1).

$$\begin{aligned}\frac{1}{h}\int_0^h u_a\partial_a u_a\,db &= \int_0^1 u_s(Ax+Bx^2)\partial_a[(Ax+Bx^2)u_s]dx\\ &= -\frac{{u_s}^2}{h}\partial_a h\int_0^1(Ax+Bx^2)(Ax+2Bx^2)dx + u_s\partial_a u_s\int_0^1(Ax+Bx^2)^2\,dx\\ &= -\frac{7}{30}\frac{{u_s}^2}{h}\partial_a h + \frac{8}{15}u_s\partial_a u_s\\ &= -\frac{21}{40}\frac{\bar{u}^2}{h}\partial_a h + \frac{6}{5}\bar{u}\partial_a\bar{u}\\ &= -\frac{21}{40}\frac{\bar{u}^2}{h}\partial_a h + c_1\partial_a\bar{u}^2\end{aligned}$$

Now we can see that we lost the first term (blue one) when setting assumption.

4. Somthing needs to be re-discussed in $f(u,h,a)=0$ relation

When we take these terms into account, (1.7) in (Equ.) becomes:

$$-\frac{21}{40}\frac{\bar{u}^2}{h}\partial_a h + c_1\partial_a\bar{u}^2 + \frac{21}{40}\frac{\bar{u}^2}{h}\partial_a h + g\cos\alpha\partial_a h = g\sin\alpha + \frac{\mu}{\rho\phi}\frac{c_2}{ah}\bar{u}\tan\alpha - \frac{\mu}{\rho\phi}\frac{1}{h^2}c_2\bar{u}$$

So nothing happens while the process is totally insufficiently strict in the past.

Thus in (Equ.)Disturbance, the form of (1.2) is still correct.