Correction of equs in granular flow system

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Abstract

This article recalculates equs in granular flow system based on Lame-coefficient expression. It includes coordinate transformation, strain-rate tensors and constitutive relations in this system. Also it points out some errors that occurred in senior Zhong's calulation.

Keywords: Lame-coefficient, coordinate transformation, strain-rate tensor, constitutive relation

1 General formula of calculation

In cartesian coordinate, we have definitions below

1. deviatoric strain-rate tensor

$$D_{ij} = -\frac{1}{2}(\mathrm{div}\vec{u})\delta_{ij} + \frac{1}{2}(\partial_j u_i + \partial_i u_j) \tag{1}$$

2. total strain-rate tensor

$$S_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j) \tag{2}$$

While in different coordinates, they will have different forms. Here I express these tensors based on Lame-coefficient expression.

Correction : in 3d space, the expression (1) is no longer the form as in 2d space. It should be

$$D_{ij} = -\frac{1}{3}(\mathrm{div}\vec{u})\delta_{ij} + \frac{1}{2}(\partial_j u_i + \partial_i u_j)$$

Provement: as we calculate in (10), the trace of 'old' $\overleftrightarrow{D}$ is not mathematically zero. Thus we need to rearrange the coefficients in (1). We assume two parameters A and B here as the arrangement.

$$D_{ij} = -A(\mathrm{div}\vec{u})\delta_{ij} + B(\partial_j u_i + \partial_i u_j)$$

Apply the trace calculation and let it be zero.

$$\begin{aligned}\mathrm{Tr}(\overleftrightarrow{D}) &= D_{11} + D_{22} + D_{33} \\ &= (-3A + 2B)(\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3) \\ &= 0\end{aligned}$$

Here from the physical definition we know that total strain-rate tensor maintains the same form in different dimensions. Thus parameter B is alwatys $\frac{1}{2}$

Thus $A = \frac{1}{3}$ Q.E.D.

In any curvlinear coordinate, we can rewrite total strain-rate tensor as below

$$\begin{aligned}
s_{11} &= \frac{1}{h_1}\frac{\partial u_1}{\partial q_1} + \frac{u_2}{h_1 h_2}\frac{\partial h_1}{\partial q_2} + \frac{u_3}{h_1 h_3}\frac{\partial h_1}{\partial q_3} \\
s_{22} &= \frac{1}{h_2}\frac{\partial u_2}{\partial q_2} + \frac{u_3}{h_2 h_3}\frac{\partial h_2}{\partial q_3} + \frac{u_1}{h_2 h_1}\frac{\partial h_2}{\partial q_1} \\
s_{33} &= \frac{1}{h_3}\frac{\partial u_3}{\partial q_3} + \frac{u_1}{h_3 h_1}\frac{\partial h_3}{\partial q_1} + \frac{u_2}{h_3 h_2}\frac{\partial h_3}{\partial q_2} \\
2s_{12} = 2s_{21} &= \frac{1}{h_2}\frac{\partial u_1}{\partial q_2} + \frac{1}{h_1}\frac{\partial u_2}{\partial q_1} - \frac{u_1}{h_1 h_2}\frac{\partial h_1}{\partial q_2} - \frac{u_2}{h_1 h_2}\frac{\partial h_2}{\partial q_1} \\
2s_{23} = 2s_{32} &= \frac{1}{h_3}\frac{\partial u_2}{\partial q_3} + \frac{1}{h_2}\frac{\partial u_3}{\partial q_2} - \frac{u_2}{h_2 h_3}\frac{\partial h_2}{\partial q_3} - \frac{u_3}{h_2 h_3}\frac{\partial h_3}{\partial q_2} \\
2s_{31} = 2s_{13} &= \frac{1}{h_1}\frac{\partial u_3}{\partial q_1} + \frac{1}{h_3}\frac{\partial u_1}{\partial q_3} - \frac{u_3}{h_3 h_1}\frac{\partial h_3}{\partial q_1} - \frac{u_1}{h_3 h_1}\frac{\partial h_1}{\partial q_3}
\end{aligned} \tag{3}$$

Also, we can write the divergence

$$\mathrm{div}\vec{u} = \frac{1}{h_1 h_2 h_3}\sum_{i=1}^{3}\frac{\partial}{\partial q_i}(u_i h_j h_k) \tag{4}$$

Thus from the definition of deviatoric strain-rate tensor (1), using (3) and (4), we can get the general form of deviatoric strain-rate tensor.

2 Calculation 1: D_{ij} complete form

In this part, I will calculate D_{ij} in coordinates that mentioned by senior Zhong.

In this coordinate, we first list parameters that will be used

$$h_a = 1 \quad h_\theta = a\cos\alpha + b\sin\alpha \quad h_b = 1 \tag{5}$$

$$\mathrm{div}\vec{u} = \partial_a u_a + \frac{1}{h_\theta}\partial_\theta u_\theta + \partial_b u_b + \frac{u_a\cos\alpha + u_b\sin\alpha}{h_\theta} \tag{6}$$

In order to simplify the expression, I decide to maintain $h_\theta = h_2$ form without expanding it until when it needs further calculation.

From (3), we can get

$$\begin{aligned}
S_{11} &= \partial_a u_a \\
S_{22} &= \frac{1}{h_2}\partial_\theta u_\theta + \frac{u_b\sin\alpha + u_a\cos\alpha}{h_2} \\
S_{33} &= \partial_b u_b \\
2S_{12} = 2S_{21} &= \partial_a u_\theta + \frac{1}{h_2}\partial_\theta u_a + -\frac{u_\theta\cos\alpha}{h_2} \\
2S_{23} = 2S_{32} &= \partial_b u_\theta + \frac{1}{h_2}\partial_\theta u_b - \frac{u_\theta\sin\alpha}{h_2} \\
2S_{31} = 2S_{13} &= \partial_a u_b + \partial_b u_a
\end{aligned} \tag{7}$$

Thus we can get $\overleftrightarrow{D}$ from (6) and (7)

$$\overleftrightarrow{D} = \frac{1}{2}\begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \tag{8}$$

See below for the expansion of each item.

$$\begin{aligned}
D_{11} &= \partial_a u_a - \frac{1}{h_\theta}\partial_\theta u_\theta + \partial_b u_b - \frac{u_a\cos\alpha + u_b\sin\alpha}{h_\theta} \\
D_{22} &= -\partial_a u_a + \frac{1}{h_\theta}\partial_\theta u_\theta - \partial_b u_b + \frac{u_b\sin\alpha + u_a\cos\alpha}{h_2} \\
D_{33} &= -\partial_a u_a - \frac{1}{h_\theta}\partial_\theta u_\theta + \partial_b u_b - \frac{u_b\sin\alpha + u_a\cos\alpha}{h_2} \\
2D_{12} = 2D_{21} &= \partial_a u_\theta + \frac{1}{h_2}\partial_\theta u_a + -\frac{u_\theta\cos\alpha}{h_2} \\
2D_{23} = 2D_{32} &= \partial_b u_\theta + \frac{1}{h_2}\partial_\theta u_b - \frac{u_\theta\sin\alpha}{h_2} \\
2D_{31} = 2D_{13} &= \partial_a u_b + \partial_b u_a
\end{aligned} \tag{9}$$

Correction : Due to the (1) -> (1*), each term of $\overleftrightarrow{D}$ should be rewritten as

$$D_{11} = \frac{2}{3}\partial_a u_a - \frac{1}{3}\frac{1}{h_2}\partial_\theta u_\theta - \frac{1}{3}\partial_b u_b - \frac{1}{3}\frac{u_a \cos\alpha + u_b \sin\alpha}{h_2}$$

$$D_{22} = -\frac{1}{3}\partial_a u_a + \frac{2}{3}\frac{1}{h_\theta}\partial_\theta u_\theta - \frac{1}{3}\partial_b u_b + \frac{2}{3}\frac{u_b \sin\alpha + u_a \cos\alpha}{h_2}$$

$$D_{33} = -\frac{1}{3}\partial_a u_a - \frac{1}{3}\frac{1}{h_\theta}\partial_\theta u_\theta + \frac{2}{3}\partial_b u_b - \frac{1}{3}\frac{u_b \sin\alpha + u_a \cos\alpha}{h_2}$$

$$2D_{12} = 2D_{21} = \partial_a u_\theta + \frac{1}{h_2}\partial_\theta u_a + -\frac{u_\theta \cos\alpha}{h_2}$$

$$2D_{23} = 2D_{32} = \partial_b u_\theta + \frac{1}{h_2}\partial_\theta u_b - \frac{u_\theta \sin\alpha}{h_2}$$

$$2D_{31} = 2D_{13} = \partial_a u_b + \partial_b u_a$$

Here we focus on the trace of this tensor, that is because, for a deviatoric tensor its trace needs to be 0.

$$\begin{aligned} \mathrm{Tr}(\overleftrightarrow{D}) &= D_{11} + D_{22} + D_{33} \\ &= -\partial_a u_a - \frac{1}{h_\theta}\partial_\theta u_\theta - \partial_b u_b - \frac{u_a \cos\alpha + u_b \sin\alpha}{h_\theta} \\ &= -(\mathrm{div}\vec{u}) \end{aligned} \tag{10}$$

Correction : When calculating the trace of $\overleftrightarrow{D}$, the coefficient $\frac{1}{2}$ is ingored! Thus the trace of 'old' $\overleftrightarrow{D}$ needs to be

$$\begin{aligned} \mathrm{Tr}(\overleftrightarrow{D}) &= \frac{1}{2}(D_{11} + D_{22} + D_{33}) \\ &= \frac{1}{2}(-\partial_a u_a - \frac{1}{h_\theta}\partial_\theta u_\theta - \partial_b u_b - \frac{u_a \cos\alpha + u_b \sin\alpha}{h_\theta}) \\ &= -\frac{1}{2}(\mathrm{div}\vec{u}) \end{aligned}$$

The trace of 'new' $\overleftrightarrow{D}$ is 0 mathematically without approximation of 'incompressible'.

3 Calculation 2: Approximation and continuity equation

Here we apply three kinds of conditions.

1. Assume that the sand maintains incompressible when flowing.

$$\mathrm{div}\vec{u} = 0 \tag{11}$$

It implies that equation (10) is always zero and it satisfies the zero-trace property of deviatoric tensor.

2. Apply shallow water conditions

$$\partial_b u_a = 0 \quad \partial_b u_\theta = 0 \tag{12}$$

$$\partial_b u_a = C_a \quad \partial_b u_\theta = C_\theta \quad b \ll a$$

It means that along the b direction, the a and θ components of the velocity are nolonger equally distributed.

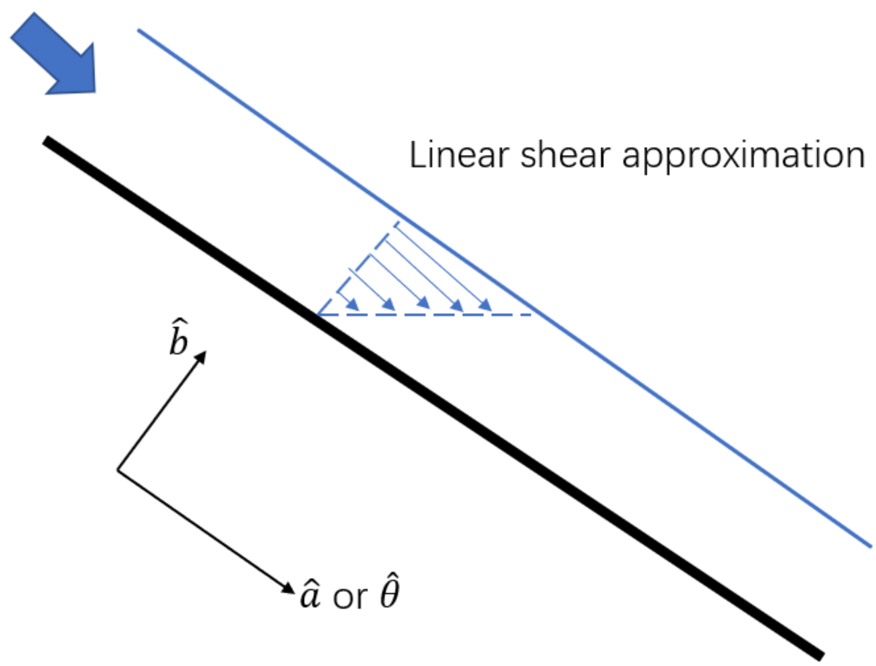

Figure 1: Linear shear approximation to the shallow water condition

3. Considered in steady state

$$h = h(a, \theta) \quad u_b = 0 \tag{13}$$

It means that the thickness of sand h is a const along b direction, while it is still $a - \theta$ distributed. (Also the second term shows that it is independent of time t.)

Thus we apply these approximations to (6) and integrate it along b direction

$$\int_0^h (\mathrm{div}\vec{u})db = 0 \tag{14}$$

$$\begin{aligned}\mathrm{LHS} &= \partial_a(u_a h) + (\partial_\theta u_\theta + u_a \cos\alpha)\frac{1}{\sin\alpha}\ln\left(1 + \frac{h}{a}\tan\alpha\right) \\ &= u_a\partial_a h + h\partial_a u_a + (\partial_\theta u_\theta + u_a\cos\alpha)\frac{1}{\sin\alpha}\left(\frac{h}{a}\tan\alpha + O\left(-\frac{1}{2}\frac{h^2}{a^2}\tan^2\alpha\right)\right)\end{aligned} \tag{15}$$

Consider $h\partial_a u_a$ is also $O(1) \equiv O\left(\frac{h}{a}\right)$, thus we correct to first order and obtain the continuity equation as below

$$\partial_a(u_a h) + \frac{1}{a\cos\alpha}\partial_\theta(u_\theta h) + \frac{h}{a}u_a = 0 \tag{16}$$

Correction :Due to modification of the shallow water conditions, the equ (16) needs to be corrected as

$$\partial_a(u_a h) + \frac{1}{a\cos\alpha}\partial_\theta(u_\theta h) + \frac{h}{a}u_a(a,\theta) = 0$$

While in this 'new' relation, when we apply differiate operations, these terms which contain b will be 0, thus (16) is actually not changed.

Provement :Due to shallow water condition, we can expand each velocity component as

$$\begin{aligned} u_a \equiv u_a(a,\theta,b) \quad &= C_a \cdot b + u_a(a,\theta) \\ u_\theta \equiv u_\theta(a,\theta,b) \quad &= C_\theta \cdot b + u_\theta(a,\theta)\end{aligned}$$

Here we rewrite (15) as

$$\begin{aligned}\mathrm{LHS} &= \int_0^h \partial_a u_a + \frac{1}{h_2}\partial_\theta u_\theta + \cancel{\partial_b u_b} + \frac{u_a\cos\alpha + \cancel{u_b\sin\alpha}}{h_2}db \\ &= \partial_a(u_a h) + \mathrm{term2} + \mathrm{term3}\end{aligned}$$

In term2, $\partial_\theta u_\theta$ is independent of b, thus it can be written as

$$\begin{aligned}\mathrm{term2} &= \partial_\theta u_\theta \int_0^h \frac{1}{a\cos\alpha + b\sin\alpha}db \\ &= \frac{1}{\sin\alpha}\partial_\theta u_\theta \ln\left(1 + \frac{h}{a}\tan\alpha\right) \\ &\approx \frac{1}{\sin\alpha}\partial_\theta u_\theta \frac{h}{a}\tan\alpha \\ &= \frac{1}{a\cos\alpha}\partial_\theta(u_\theta h)\end{aligned}$$

In term3, we just plug the expansion of u_a in it and obtain

$$\begin{aligned}\mathrm{term3} &= \int_0^h \frac{u_a(a,\theta) + C_a\cdot b}{a\cos\alpha + b\sin\alpha}db \\ &= \frac{h}{a}u_a(a,\theta) + C_a\cos\alpha\int_0^h \frac{b}{a\cos\alpha + b\sin\alpha}db\end{aligned}$$

For the second integrate term, we have integration by parts

$$\begin{aligned}
\text{term3}_2 &= \frac{1}{\sin\alpha}\left(b - a\frac{\cos\alpha}{\sin\alpha}\ln(a\cos\alpha + b\sin\alpha)\right)\Bigg|_{b=0}^{b=h} \\
&= \frac{h}{\sin\alpha} - a\frac{\cos\alpha}{\sin^2\alpha}\ln\left(1 + \frac{h}{a}\tan\alpha\right) \\
&\approx \frac{h}{\sin\alpha} - a\frac{\cos\alpha}{\sin^2\alpha}\frac{h}{a}\frac{\sin\alpha}{\cos\alpha} \\
&= 0
\end{aligned}$$

Q.E.D.

4 D_{ij} approx form

Here we first apply approximation conditions to divergence and see what happends to it

$$\begin{aligned}
\mathrm{div}\vec{u} &= \partial_a u_a + \frac{1}{h_2}\partial_\theta u_\theta + \cancel{\partial_b u_b} + \frac{u_a\cos\alpha + \cancel{u_b\sin\alpha}}{h_2} \\
&= \partial_a u_a + \frac{1}{h_2}\partial_\theta u_\theta + \frac{u_a\cos\alpha}{h_2} \\
&= 0
\end{aligned} \tag{17}$$

It is noted that $h_2 = \frac{1}{a\cos\alpha + b\sin\alpha}$ is not $O\left(\frac{h}{a}\right)$, while we expand it later and use $\frac{1}{h_2} \equiv \lambda$ form to simplify the tensor $\overleftrightarrow{D}$ right now.

The deviatoric tensor becomes

$$\overleftrightarrow{D} =$$

$$\frac{1}{2}\begin{bmatrix} \partial_a u_a - \lambda(\partial_\theta u_\theta + u_a\cos\alpha) & \partial_a u_\theta + \lambda(\partial_\theta u_a - u_\theta\cos\alpha) & \partial_b u_a \\ \partial_a u_\theta + \lambda(\partial_\theta u_a - u_\theta\cos\alpha) & -\partial_a u_a + \lambda(\partial_\theta u_\theta + u_a\cos\alpha) & \partial_b u_\theta - \lambda(u_\theta\sin\alpha) \\ \partial_b u_a & \partial_b u_\theta - \lambda(u_\theta\sin\alpha) & -\partial_a u_a - \lambda(\partial_\theta u_\theta + u_a\cos\alpha) \end{bmatrix} \tag{18}$$

Correction :

Due to new terms we have mentioned in part II, apply (12*) , (13) and (17) to them and obtain the correction

$$\begin{aligned}
D_{11} &= \frac{2}{3}\partial_a u_a - \frac{1}{3}\frac{1}{h_2}\partial_\theta u_\theta - \frac{1}{3}\partial_b u_b - \frac{1}{3}\frac{u_a\cos\alpha + u_b\sin\alpha}{h_2} \\
&= \frac{2}{3}\partial_a u_a - 0 - \frac{1}{3}(-\partial_a u_a) \\
&= \partial_a u_a \\
D_{22} &= -\frac{1}{3}\partial_a u_a + \frac{2}{3}\frac{1}{h_2}\partial_\theta u_\theta - \frac{1}{3}\partial_b u_b + \frac{2}{3}\frac{u_b\sin\alpha + u_a\cos\alpha}{h_2} \\
&= \frac{1}{h_2}(\partial_\theta u_\theta + u_a\cos\alpha) \\
D_{33} &= -\frac{1}{3}\partial_a u_a - \frac{1}{3}\frac{1}{h_\theta}\partial_\theta u_\theta + \frac{2}{3}\partial_b u_b - \frac{1}{3}\frac{u_b\sin\alpha + u_a\cos\alpha}{h_2} \\
&= -\frac{1}{3}(\mathrm{div}\vec{u}) \\
&= 0 \\
2D_{12} &= \partial_a u_\theta + \frac{1}{h_2}\partial_\theta u_a - \frac{u_\theta\cos\alpha}{h_2} \\
2D_{23} &= C_\theta - \frac{u_\theta\sin\alpha}{h_2} \\
2D_{31} &= C_a
\end{aligned}$$

Thus the 'new' deviatoric tensor is

$$\overleftrightarrow{D} = \frac{1}{2}\begin{bmatrix} 2\partial_a u_a & \partial_a u_\theta + \frac{1}{h_2}\partial_\theta u_a - \frac{u_\theta \cos\alpha}{h_2} & C_a \\ \partial_a u_\theta + \frac{1}{h_2}\partial_\theta u_a - \frac{u_\theta \cos\alpha}{h_2} & 2\cdot\frac{1}{h_2}(\partial_\theta u_\theta + u_a\cos\alpha) & C_\theta - \frac{u_\theta \sin\alpha}{h_2} \\ C_a & C_\theta - \frac{u_\theta \sin\alpha}{h_2} & 0 \end{bmatrix}$$

And we can see two kinds of colored terms, the red terms mean they are applied the shallow water conditions and the blue terms mean they are applied incompressible condition.

We now can recall the tensor expression (19) and the constitutive relation (20)

$$(\boldsymbol{\nabla}\cdot\tau)_i = \frac{1}{h_j}\partial_j(\tau_{ij}) - \frac{1}{h_i}\partial_i\mathbb{P}(\boldsymbol{\nabla}\boldsymbol{u},\phi) \tag{19}$$

$$\frac{D_{ij}}{\|\mathbf{D}\|} = \frac{\tau_{ij}}{\|\tau\|} \tag{20}$$

Thus we actually need to calculate the $\partial_j(\tau_{ij}) \longrightarrow \nabla\cdot\overleftrightarrow{D}$, its $\hat{b}$ components are as belows

$$\begin{aligned} 2(\nabla\cdot\overleftrightarrow{D})_b &= \begin{pmatrix}\partial_a & \frac{1}{h_2}\partial_\theta & \partial_b\end{pmatrix}\begin{pmatrix} C_a \\ C_\theta - \frac{u_\theta\sin\alpha}{h_2} \\ 0\end{pmatrix} \\ &= \partial_a C_a + \frac{1}{h_2}\partial_\theta C_\theta - \partial_\theta u_\theta \frac{\sin\alpha}{(h_2)^2} \\ &= -\frac{\sin\alpha}{(h_2)^2}\partial_\theta u_\theta \end{aligned} \tag{21}$$

Recall (19), we expand it in $\hat{b}$ direction $S_{ij} = -p\delta_{ij} + \tau_{ij}$

$$\begin{aligned} (\boldsymbol{\nabla}\cdot S)_b &= \frac{1}{h_j}\partial_j(\tau_{bj}) - \partial_b p \\ &= \left(\nabla\cdot\left(\frac{\|\tau\|}{\|\mathbf{D}\|}\overleftrightarrow{D}\right)\right)_b - \partial_b p \\ &= \left(\nabla\cdot\left(\frac{\mu(I)p}{\|\mathbf{D}\|}\overleftrightarrow{D}\right)\right)_b - \partial_b p \\ &= \frac{\mu(I)}{\|\mathbf{D}\|}\partial_b p\cdot 0 + \frac{\mu(I)p}{\|\mathbf{D}\|}(\nabla\cdot\overleftrightarrow{D})_b - \partial_b p \\ &= -\frac{1}{2}\frac{\mu(I)\sin\alpha}{(h_2)^2\|\mathbf{D}\|}(\partial_\theta u_\theta)p - \partial_b p \end{aligned} \tag{22}$$

Recall the momentum conservation function

$$\rho\phi\left(d_t u_b + \frac{-\sin\alpha\cdot u_\theta^2}{h_2}\right) = (\boldsymbol{\nabla}\cdot S)_b - \rho\phi g\cos\alpha \tag{23}$$

It can be simplified as

$$\partial_b p = \rho\phi\left((u_\theta)^2\frac{\sin\alpha}{h_2} - g\cos\alpha\right) - \frac{1}{2}\frac{\mu(I)\sin\alpha}{(h_2)^2\|\mathbf{D}\|}(\partial_\theta u_\theta)p \tag{24}$$

Here to make the formula clear, we can gather messy coefficients like

$$\frac{1}{2}\frac{\mu(I)\sin\alpha}{\|\mathbf{D}\|} = \zeta \tag{25}$$

Thus (24) becomes

$$\partial_b p + \frac{\zeta \partial_\theta u_\theta}{(h_2)^2} p = \rho\phi \left(\frac{(u_\theta)^2 \sin\alpha}{h_2} - g\cos\alpha \right) \tag{26}$$

The equ (26) can be solved by the constant variation method, after applying the approximation to it we can get

$$p(b) = p_0 + \rho\phi \left(g\cos\alpha - \frac{(u_\theta)^2}{a} \tan\alpha \right) (b - h) \tag{27}$$