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Kinetic energy sandpile model for conical sandpile development by revolving rivers

Xiang-Zhao Kong [a], Mao-Bin Hu [a], Qing-Song Wu [a,*], Yong-Hong Wu [b]

[a] *School of Engineering Science, University of Science and Technology of China, Hefei, Anhui 230026, PR China*
[b] *Department of Mathematics and Statistics, Curtin University of Technology, Perth WA6845, Australia*

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Abstract

In this Letter, a kinetic energy sandpile model, taking into account of grain inertia and the moving directions of the toppling grain, is developed and used to study the behaviour of sandpiles. In our model, the inertial effects are based on the toppling kinetic energy. The phenomenon of sandpile formation by revolving rivers is reproduced with the model, revolving velocity $\omega \sim t^{-2/3}$ and $\sim h^{-3/2}$, where t is the simulation time and h is the height of the sandpile.

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Keywords: Sandpile; Inertia; Energy; Revolving

1. Introduction

Since the concept of self-organized criticality (SOC) was proposed, intensive research, both experimental and theoretical, has been carried out worldwide to study the behaviour of sandpiles [1–4]. Various models have been developed to simulate the formation and the behaviour of sandpiles, including the cellular automata model [1], the Oslo rice pile model [5], the quasi-three-dimensional model [6], etc. But it is well known that inertia effects tend to destroy SOC behavior. This idea was proposed since SOC was applied to the real sandpile formation. The conventional understanding of sandpile formation is that as grains of sand are poured onto a horizontal surface, a conical pile develops and grows intermittently through avalanches that "adjust" the angle of repose of the pile about some critical value [7]. Recently, Altshuler et al. [7] discovered a new mechanism of the formation of conical sandpiles based on their experimental observation, namely, formation by revolving rivers.

* Corresponding author.
E-mail address: qswu@ustc.edu.cn (Q.-S. Wu).

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Existing sandpile models can satisfactorily simulate the sandpile behavior in certain aspects. For example, the models reported in Refs. [1,3,6] can simulate the behavior that, if the slope of the sandpile exceeds a threshold value, the sand will topple down to the neighboring sites. However, in these models, as Head and Rodgers [9] said that, the grain inertia is ignored and the dynamic friction between grains is assumed to be so great that a grain topples down from its original position and immediately comes to rest and then finds the next direction to topple down. But in Head's model, the inertial effects were charactered by a probability p_{diss}. The grain velocities are zero or infinite. This is not realistic and thus the models are not capable of reflecting certain behavior of a real sandpile. For example, the formation of conical sandpiles by revolving rivers cannot be simulated by these models.

In this Letter, we present a kinetic energy sandpile model taking into account of the grain inertia, the dynamic friction between grains and the moving directions of the toppling grain. The grain inertial effects are based on the toppling kinetic energy. The model is presented in Section 2 and the simulation of sandpile formation by revolving rivers is reported in Section 3. Section 4 gives a conclusion.

2. Model

The key elements of our model are the dynamics of grains for the grain inertia (represented by the kinetic energy) and toppling direction, and the nonlinear function for the height change of sandpile toppling reflecting the dynamic friction between grains. The nonlinear friction force between grains has been observed in experiments of sliding rocks [8] and the nonlinear function for the height change of sandpile toppling has also been applied in the construction of a quasi-3D sandpile model [6]. A 2D coupled map lattice (Fig. 1) is used in our model, and the system is characterized by the variable values at lattice at discrete time. To each site (i, j) in the lattice, there are three variables associated to it: H_{ij} representing the height of this site, E_{ij} and D_{ij} denoting respectively the magnitude of kinetic energy and the toppling direction of the toppling parts of this site. From $t \to t+1$, the rules for the updating of the variable values are given as the following different processes:

(1) Adding of sand.
If sand is added at a certain site (central or random), the variables H and E of this site is updated by $H^{t+1} = H^t + I$, $E^{t+1} = 0$, where I represents the input flux. Here we denote the kinetic energy of the inputted sand is zero, so that we can take $E^{t+1} = 0$ as the original kinetic energy of this site. The direction, D_{ij}, is set to 0 if no toppling occurred in the latest update step or set to the direction in which the toppling occurred in the latest update step.

(2) Toppling of sand.
The local slope S for each site is determined by $S = \max(H_{ij} - H_{nn})$, where nn denotes the three nearest neighbors corresponding to the intended direction D_{ij} as shown in Fig. 1. For example, if $D_{ij} = 3$, site 2, 3 and 4 will be the interested sites; if $D_{ij} = 4$, site 3, 4 and 5 will be the interested sites. It should also be addressed here that, if the site has no special intended direction ($D_{ij} = 0$), nn refers to eight neighbors (from site 1 to site 8) as shown in Fig. 1.
If S is smaller than the local critical slope $X_{th,ij}$ (its formula will be given below), the motion is stuck with $E_{ij} = 0$ and $D_{ij} = 0$, otherwise, the sandpile will topple to the lowest neighboring site in the preference direction as shown in Fig. 1. In the case that two or more neighboring sites have the same lower height, the direction is chosen stochastically in these sites. Let

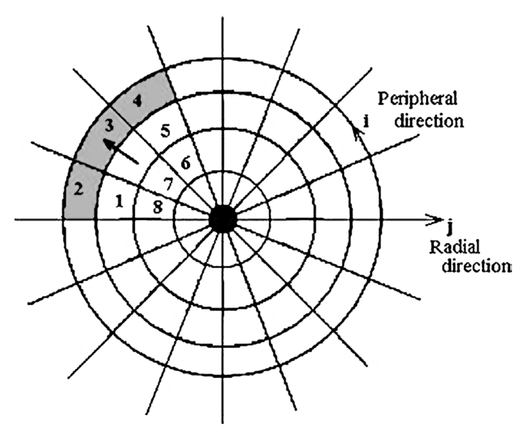

Fig. 1. Illustration of radial lattice map for our model (i for the peripheral direction and j for the radial direction). Direction preference (gray sites 2, 3 and 4) for local dropping, where the arrow indicates $D_{i,j}$ of the toppling site. The central dark disk represents the central site ($i = 0$, $j = 0$) of the map.

H_o^t, E_o^t and D_o^t be the origin site value at time t and H_d^t, E_d^t and D_d^t be the destination site value at time t, the toppling dynamics are as follows:

(i) Height change. $H_o^{t+1} = H_o^t - \delta H$, $H_d^{t+1} = H_d^t + \delta H$, where δH is determined by a given nonlinear function to be defined in formula (1). It should also be addressed here that, for generality, after the toppling process, $H_d^{t+1} \leqslant H_o^{t+1}$. As a result, before the toppling process, we get $\delta H \leqslant (H_o^t - H_d^t)/2$. So $\delta H \leqslant S/2$.

(ii) Kinetic energy change. The energy is conserved: there is $d \times \Delta H$ transformed into kinetic energy from potential energy, where d would be associated with the minimum drop in energy after an toppling event involving one single element (and also characterizes a Coulomb-type discontinuity) [6] and $\Delta H = S - \delta H$ is the height drop after an toppling event; because the collisions is being with the toppling, the kinetic energy of the origin site will have a loss when the sand moves to the destination site, which is taking into account of the grain inertia. Here we take $E_d^{t+1} = 0.5E_o^t + d \times \Delta H$, where $1/2$ kinetic energy of the origin site is lost.

(iii) Direction change. D_d^{t+1} changes to the direction in which the sand is toppled.

All sites are updated simultaneously under the rules.

Notice that the frictions between the grains are ineluctable. So we can defined a parameter a, which would be associated with the amount of dynamic friction between the grains. And we denote that the smaller the a is, the larger the friction is. After the toppling, the avalanching grains are accelerated. We make the critical slope $X_{th,ij}$ a decreasing function of the kinetic energy. That is larger E_{ij} will decrease the local critical slope $X_{th,ij}$ of each site, which is coincident with the experimental observations. In the simulation, we take $X_{th,ij} = \max(0, X_o - E_{ij})$, where X_o is the maximum critical slope. For generality, we can take $X_o = 1$. From (2)(i), $\delta H \leqslant S/2$, moreover the nonlinear height change function is taken to be:

$$\delta H = \big(1 - G(S - X_{th})\big)S/2, \tag{1}$$

where G is a given nonlinear function with two parameters a and d. The nonlinear function we use is [6]

$$G(x) = \begin{cases} 1 - d - ax & \text{when } x < \frac{1-d}{a}, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

3. Simulation of sandpile formation by revolving rivers

Various computer simulations have been carried out with different values of a and d under open boundary conditions (BC II in the experiment [7]), namely, H is set to 0 for the boundary sites. To simulate the sandpile formation, we use a special radial lattice map for the system as shown in Fig. 1. Note that if we take a special treatment for the central site ($i = 0$, $j = 0$), the radial lattice map can be splitted at the peripheral direction between the sites at $i = 1$ and the sites at $i = L$ with cyclic boundary condition for the i-direction (namely the $i = 1$ sites and the $i = L$ sites are neighboring sites) and open boundary condition for the sites at $j = L + 1$ (that is, $H = 0$ for the outer rim) where j is in the direction as shown in Fig. 1 and $j = 1$ refers to the sites nearest to the central site.

To simplify the simulation, the area of each site is taken to be equal so that it is not necessary to recalculate the height change in the updating process (2). Through simple geometrical calculation, if $r_{j+1}^2 - r_j^2 = \text{const.}$ (which means that the r_j^2 is an arithmetical progression), the area of every site can be taken to be equal, where r_j is radius of the sandpile. Initially the site variables are all set to 0. The sand is then added to the central site and then the sand topples to the sites around.

Fig. 2 shows the snapshots of the sandpile formation by revolving rivers. The typical process of sandpile formation can be described as follows. The sand topples from the central site and finds its way down the slope. The successive toppling sand will follow its trace because of the inertia effect. When the sandpile is small, no river is formed and also there is no pre-

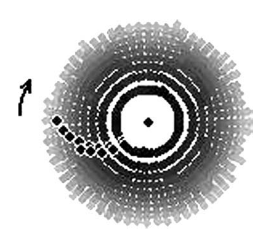

Fig. 2. Snapshot of sandpile's planform with $I = 0.1$, $a = 0.8$ and $d = 0.6$ in the open boundary condition. $H_{i,j}$ is displayed with the gray scale of the site, and the dark solid circles show the toppling sites. The right arrow represents the revolving direction.

ferred direction of rotation. When the sandpile grows bigger, a river of sand develops that flows down the slope from the apex of the pile and leaves some sand on the way to the base of the pile. Once one side of the river is higher than the other side, the river rotates to the lower side. The river rotation is not stable at the beginning; it rotates back and forth with a slightly preferred direction. But when the sandpile grows to a sufficient size, the revolution becomes steady. In our simulation, the rivers, once formed, may revolve for dozens of rounds. But when the sandpile grows bigger and bigger (about 13 in the j coordinate), the sand toppling becomes unstable and chaotic and consequently no river can be formed. Altshuler et al. [7] claimed that the reason for the curved shape of a revolving river is that there is a growing inverted V shaped "delta" bending the river. In our simulation, we also found such a "delta" showed as Fig. 3. It is clear to see that sand how to choose the one side of the "delta" to flow down that side of the delta. From the apex of the "delta", the toppling sand could find two paths to flow down, illustrated in Fig. 3. Once it chooses a path, it continues to flow down that path of the "delta", then, it gradually developes to rotate about the sandpile.

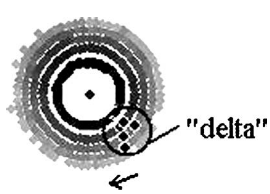

Fig. 3. A growing inverted V shaped "delta" bending the river for open boundary condition with $I = 0.1$, $a = 0.8$ and $d = 0.6$. $H_{i,j}$ is displayed with the gray scale of the site, and the dark solid circles show the toppling sites. The right arrow represents the revolving direction. The pile is viewed in a planform.

We calculated the time evolution of the angular velocity of river rotation and the dependence of the angular velocity and the sandpile height (Fig. 4) for open boundary condition with $I = 0.1$. In the Altshuler's experiment [7], revolving velocity $\omega \sim t^{-2/3}$ and $\sim r^{-2}$, where t is the revolution time and r is the sandpile radius. Notice that, the sandpile height h is associated with r in this way: $h = r \tan\theta_c$, where θ_c is the angle of repose of the piles. As a result, we can have $\omega \sim h^{-2}$. We take 0.1 s as our simulation time step. Our simulation results are qualitatively in agreement with their experiment results (Fig. 4). If we take $I = 1.0$, $a = 0.8$ and $d = 0.6$, the pile grows through randomly distributed topplings. If we take $I = 0.01$, $a = 0.8$ and $d = 0.6$, also, the pile grows through randomly distributed topplings. That is to say, the revolving river can only exist in a regime of inputted amount. If we keep $I = 0.1$, the revolving river can only exist in a few combination of a and d, which means that this phenomenon is quite sensitive to the type of sand.

As to the closed cylindrical container condition, the boundary condition at the outer rim (cylinder wall) is taked as

$$\left.\frac{\partial H}{\partial j}\right|_{j=j_{\max}} = 0. \tag{3}$$

This means that $H(i, j_{\max}) = H(i, j_{\max} + 1)$ with i from 1 to L. Under the conditions of $I = 0.1$, $a = 0.8$ and $d = 0.6$, for small j (less than 4), no stable rotating rivers were formed. For j between 4 and

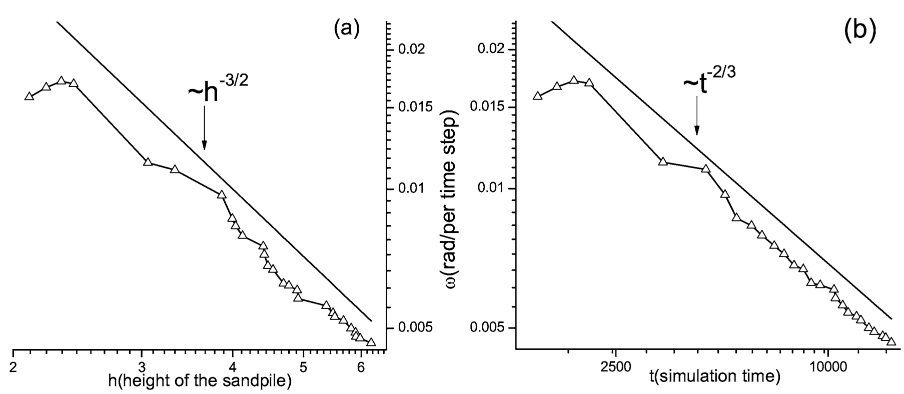

Fig. 4. Height (a) and simulation time (b) dependence of the angular speed of revolving rivers for open boundary condition with $I = 0.1$, $a = 0.8$ and $d = 0.6$.

12, steady "continuous" rotating rivers can be easily formed. When j is larger than 12, the rivers become instable and intermittent, sometimes even rollback in a small angle and then move on again. We also calculated the relation between the simulation time and the angular velocity of the rivers which is approximately constant after the sandpile reaches the cylinder wall. Fig. 5 shows their relation.

As for the boundary condition (BC III), on a cylindrical base of finite size with no walls, in our simulation, we make $j_{\max} = 6$ and $H(i, j_{\max}) = 0$ ($i = 1, 2, \ldots, L$). In this case, once the sandpile grows to reach the edge of the base, the river does not revolve again. The sand topples down in a straight line (see Fig. 6). The reason is that, claimed by Altshuler et al. [7], for BC III there is impossible to produce a "delta", so no bending nor revolving of the river is presented.

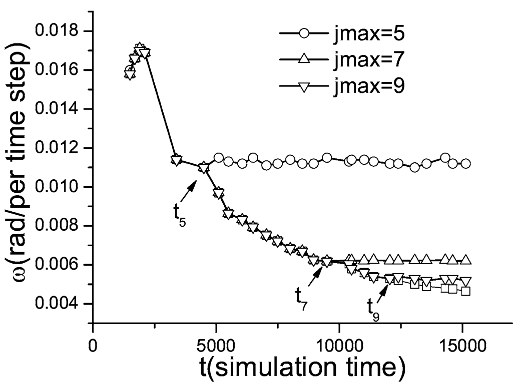

Fig. 5. Simulation time dependence of the angular speed of revolving rivers for the closed cylindrical container boundary condition with $I = 0.1$, $a = 0.8$ and $d = 0.6$. Time t_k ($k = 5, 7, 9$) is the time when sand reaches the cylinder wall whose outer time is $j_{\max} = k$. It is clear that the angular velocity of the rivers is approximately constant after the sandpile reaches the cylinder wall.

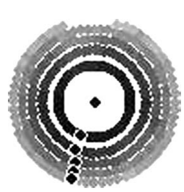

Fig. 6. Snapshot of sandpile's planform with $I = 0.1$, $a = 0.8$ and $d = 0.6$ on a cylindrical base of finite size with no walls, where $j_{\max} = 6$. $H_{i,j}$ is displayed with the gray scale of the site, and the dark solid circles show the toppling sites.

4. Conclusion

We have proposed a new sandpile model taking into account of the effect of grain inertia by association with the kinetic energy of the toppling sand. In our model, the revolving river are based on the grain inertia and the rule of how to choose the toppling direction, which is the most important dynamic parameter of the smooth and heavy sand. The phenomenon of sandpile formation by revolving rivers is reproduced by our model. The revolving velocity $\omega \sim t^{-2/3}$ and $\sim h^{-3/2}$. The scaling relations give some detailed information of the sandpile formation. The phenomenon of revolving river is quite sensitive to the type of sand and the inputted amount of sand. The toppling sand revolves about the pile only under the condition of its radius $r \in [r_l, r_u]$, where r_l is its radius lower-limit in the phenomenon and r_u is its radius upper-limit, which means that the revolving river is also sensitive to the sandpile radius. Our simulation results are qualitatively in agreement with experiment observations [7].

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