n阶方程的初边值问题

$$\begin{cases} y^{(n)} = f(x, y, y', \cdots, y^{(n-1)}) \\ y(x_0)=y_0;\ y'(x_0)=y_0';\ \cdots;\ y^{(n-1)}(x_0)=y_0^{(n-1)} \end{cases}$$

引入新变量 $y_1 = y';\ y_2 = y'';\ \cdots;\ y_{n-1} = y^{(n-1)}$

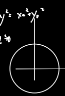
$$\Rightarrow \begin{cases} y_{n-1}' = f(x, y, y_1, \cdots, y_{n-1}) \\ y' = y_1 \\ y_1' = y_2 \\ \vdots \\ y_{n-2}' = y_{n-1} \end{cases} \qquad Y = \begin{pmatrix} y \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

写作： $\dfrac{dY}{dx} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ f(x, y, y_1, \cdots, y_{n-1}) \end{pmatrix} = F(x, Y)$

$\Rightarrow \begin{cases} \dfrac{dY}{dx} = F(x, Y) \\ \vec{Y}(x_0) = \vec{Y}_0 \end{cases}$ 化为一阶方程初值问题

例 b.2 $\begin{cases} y' = -x \\ x' = y \\ x(t_0) = x_0;\ y(t_0) = y_0 \end{cases}$ 自变量为 t　在相平面中轨道为 $(xx' + yy' = 0)$

$x^2 + y^2 = x_0^2 + y_0^2$



引入 $Y = \begin{pmatrix} x \\ y \end{pmatrix}$

反对称阵：辛结构

$\dfrac{d}{dt} Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y = AY$

梯形公式 $\vec{Y}_{n+1} = \vec{Y}_n + \frac{h}{2} [A\vec{Y}_n + A\vec{Y}_{n+1}]$

$\Rightarrow \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \frac{h}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_n + x_{n+1} \\ y_n + y_{n+1} \end{pmatrix}$

$\Rightarrow \begin{cases} x_{n+1} = x_n + \frac{h}{2}(y_n + y_{n+1}) \\ y_{n+1} = y_n - \frac{h}{2}(x_n + x_{n+1}) \end{cases}$

改写为： $\begin{cases} x_{n+1} - x_n = \frac{h}{2}(y_n + y_{n+1}) & \text{—— ①} \\ y_{n+1} - y_n = -\frac{h}{2}(x_n + x_{n+1}) & \text{—— ②} \end{cases}$

① $\cdot (x_n + x_{n+1}) +$ ② $\cdot (y_n + y_{n+1})$

$= x_{n+1}^2 - x_n^2 + y_{n+1}^2 - y_n^2 = 0$

$\Rightarrow x_n^2 + y_n^2 = x_{n+1}^2 + y_{n+1}^2$ Conservative!

例 b.3

$y'' = -xy' - x^2 y + x + 1,\ y(0) = 1;\ y'(0) = -1$

记 $z = y'$

$\Rightarrow \begin{cases} y' = z \\ z' = -xz - x^2 y + x + 1 \end{cases}$

① 显式 Euler： $h = 0.01$，计算 $y(0.02)$ 近似值

$\begin{cases} y_{n+1} = y_n + h z_n \\ z_{n+1} = z_n + h(-x_n z_n - x_n^2 y_n + x_n + 1) \end{cases}$

$\left.\begin{matrix} y_0 = 1,\ z_0 = -1 \\ y_1 = 0.99,\ z_1 = -0.99 \end{matrix}\right\} \Rightarrow y_2 = \cdots$ ✓

② 梯形 Euler

$\begin{cases} y_{n+1} = y_n + \frac{h}{2}(z_n + z_{n+1}) \\ z_{n+1} = z_n + \frac{h}{2}(z_n' + z_{n+1}') \end{cases}$ ← 不展开写

消去 z_{n+1} 可得 y_{n+1} 表达式

上述方法解决初值问题

边值问题

$\begin{cases} y'' + ay' + by = f(x, y) \\ y(0) = A,\ y(1) = B \end{cases}$

↓

边界条件齐次化，$y^*(0) = 0,\ y^*(1) = 0$，在 $[0,1]$ 上寻找近似解

$h = \frac{1}{n},\ x_i = ih;\ i = 0, 1, \cdots, n$

$y_0 = 0$　　$y_n = 0$

需要构造格式计算 $y_1, y_2, \cdots, y_{n-1}$

↓

离散导数： $\dfrac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = \dfrac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2}$

Taylor expansion

$y''(x_i) + o(h^2)$

$\dfrac{y_{i+1} - y_{i-1}}{2h} \approx y'(x_i)$：两步 Euler 方法.

$\hookrightarrow = y'(x_i) + o(h^2)$

$\dfrac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \dfrac{a}{2h}(y_{i+1} - y_{i-1}) + b y_i = f(x_i, y_i)$

$i = 1, 2, \cdots, n-1$；为 $o(h^2)$ 误差

除去 $1, n-1$ 项有 2 项，其余方程均为 3 个未知量

⇓

三对角方程