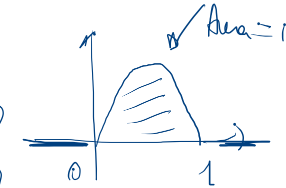
ESERCIZIO

Sia X una v.a. con densità continua $f_X(x) = 6x(1-x)\,\mathbb{1}_{(0;1)}(x)$

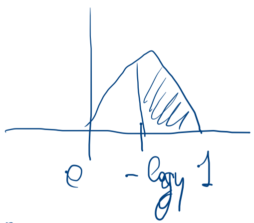


Trovare la densità continua di $Y = e^{-X}$

RISPOSTA

La funzione $f(x) = e^{-x}$ è decrescente; quindi Y assume valori in $(f(1), f(0)) = (e^{-1}, e^{0}) = (e^{-1}, 1)$

$$F_Y(y) = \begin{cases} 0 & \text{per } y \le e^{-1} \\ \ast & \text{per } y \in (e^{-1}, 1). \\ 1 & \text{per } y \ge 1. \end{cases}$$



$$\ast = P(e^{-X} \le y) = P(-X \le \log y) = P(X \ge -\log y) = \int_{-\log y}^{+\infty} f_X(x)\,dx =$$

$$= \int_{-\log y}^{1} 6x(1-x)\,dx = 6\int_{-\log y}^{1} x - x^2\,dx = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{y=-\log y}^{y=1} = 6\left[\frac{1}{2} - \frac{1}{3} - \left(\frac{(-\log y)^2}{2} - \frac{(-\log y)^3}{3}\right)\right] =$$

$$= 6\left(\underbrace{\frac{1}{2}-\frac{1}{3}}_{=\frac{3-2}{6}} - \frac{(\log y)^2}{2} - \frac{(\log y)^3}{3}\right) = 6\left(\frac{1}{6} - \cdots\cdots\right) = 1 - 3(\log y)^2 - 2(\log y)^3$$

$$F_y(y) = \begin{cases} 0 & \text{per } y \leq e^{-1} \\ 1 - 3(\log y)^2 - 2(\log y)^3 & \text{per } y \in (e^{-1}, 1) \\ 1 & \text{per } y \geq 1 \end{cases}$$

Verifica per $y = e^{-1}$:

$$1 - 3\cdot(-1)^2 - 2(-1)^3$$

$$= 1 - 3 + 2 = 0$$ ok

$$f_Y(y) = \left\{-3\cdot\frac{2}{y}\log y - 2\cdot\frac{3}{y}(\log y)^2\right\} \mathbb{1}_{(e^{-1},1)}(y)$$

$$= \frac{6}{y}\log y\left(-1 + \log y\right) \mathbb{1}_{(e^{-1},1)}(y)$$