# Chapter 4
# Planar Kinematics

Kinematics is *Geometry of Motion*. It is one of the most fundamental disciplines in robotics, providing tools for describing the structure and behavior of robot mechanisms. In this chapter, we will discuss how the motion of a robot mechanism is described, how it responds to actuator movements, and how the individual actuators should be coordinated to obtain desired motion at the robot end-effecter. These are questions central to the design and control of robot mechanisms.

To begin with, we will restrict ourselves to a class of robot mechanisms that work within a plane, i.e. *Planar Kinematics*. Planar kinematics is much more tractable mathematically, compared to general three-dimensional kinematics. Nonetheless, most of the robot mechanisms of practical importance can be treated as planar mechanisms, or can be reduced to planar problems. General three-dimensional kinematics, on the other hand, needs special mathematical tools, which will be discussed in later chapters.

## 4.1 Planar Kinematics of Serial Link Mechanisms

***Example 4.1*** Consider the three degree-of-freedom planar robot arm shown in Figure 4.1.1. The arm consists of one fixed link and three movable links that move within the plane. All the links are connected by revolute joints whose joint axes are all perpendicular to the plane of the links. There is no closed-loop kinematic chain; hence, it is a serial link mechanism.

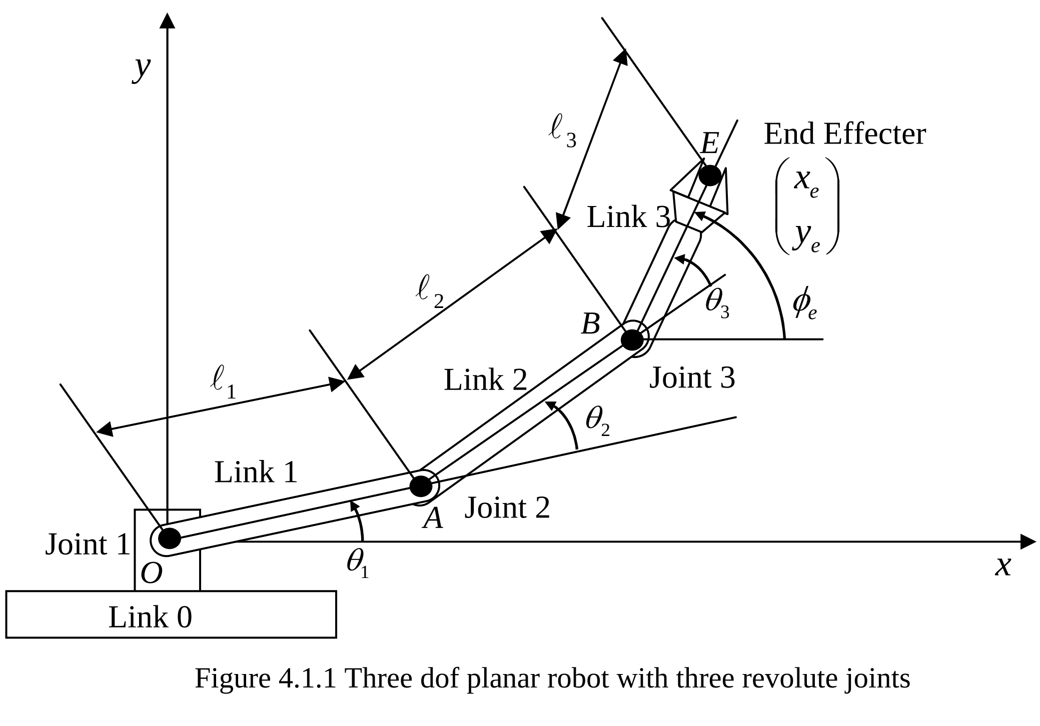


Figure 4.1.1 Three dof planar robot with three revolute joints

To describe this robot arm, a few geometric parameters are needed. First, the length of each link is defined to be the distance between adjacent joint axes. Let points *O, A,* and *B* be the locations of the three joint axes, respectively, and point E be a point fixed to the end-effecter. Then the link lengths are $\ell_1 = \overline{OA}$, $\ell_2 = \overline{AB}$, $\ell_3 = \overline{BE}$ . Let us assume that Actuator 1 driving link 1 is fixed to the base link (link 0), generating angle $\theta_1$, while Actuator 2 driving link 2 is fixed to the tip of Link 1, creating angle $\theta_2$ between the two links, and Actuator 3 driving Link 3 is fixed to the tip of Link 2, creating angle $\theta_3$, as shown in the figure. Since this robot arm performs tasks by moving its end-effecter at point E, we are concerned with the location of the end-effecter. To describe its location, we use a coordinate system, *O-xy*, fixed to the base link with the origin at the first joint, and describe the end-effecter position with coordinates $x_e$ and $y_e$. We can relate the end-effecter coordinates to the joint angles determined by the three actuators by using the link lengths and joint angles defined above:

$$x_e = \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1 + \theta_2) + \ell_3 \cos(\theta_1 + \theta_2 + \theta_3) \tag{4.1.1}$$

$$y_e = \ell_1 \sin\theta_1 + \ell_2 \sin(\theta_1 + \theta_2) + \ell_3 \sin(\theta_1 + \theta_2 + \theta_3) \tag{4.1.2}$$

This three dof robot arm can locate its end-effecter at a desired orientation as well as at a desired position. The orientation of the end-effecter can be described as the angle the centerline of the end-effecter measured from the positive *x* coordinate axis. This end-effecter orientation $\phi_e$ is related to the actuator displacements as

$$\phi_e = \theta_1 + \theta_2 + \theta_3 \tag{4.1.3}$$

□

The above three equations describe the position and orientation of the robot end-effecter viewed from the fixed coordinate system in relation to the actuator displacements. In general, a set of algebraic equations relating the position and orientation of a robot end-effecter, or any significant part of the robot, to actuator or active joint displacements, is called ***Kinematic Equations,*** or more specifically, ***Forward Kinematic Equations*** in the robotics literature.

***Exercise 4.1***

Shown below in Figure 4.1.2 is a planar robot arm with two revolute joints and one prismatic joint. Using the geometric parameters and joint displacements, obtain the kinematic equations relating the end-effecter position and orientation to the joint displacements.

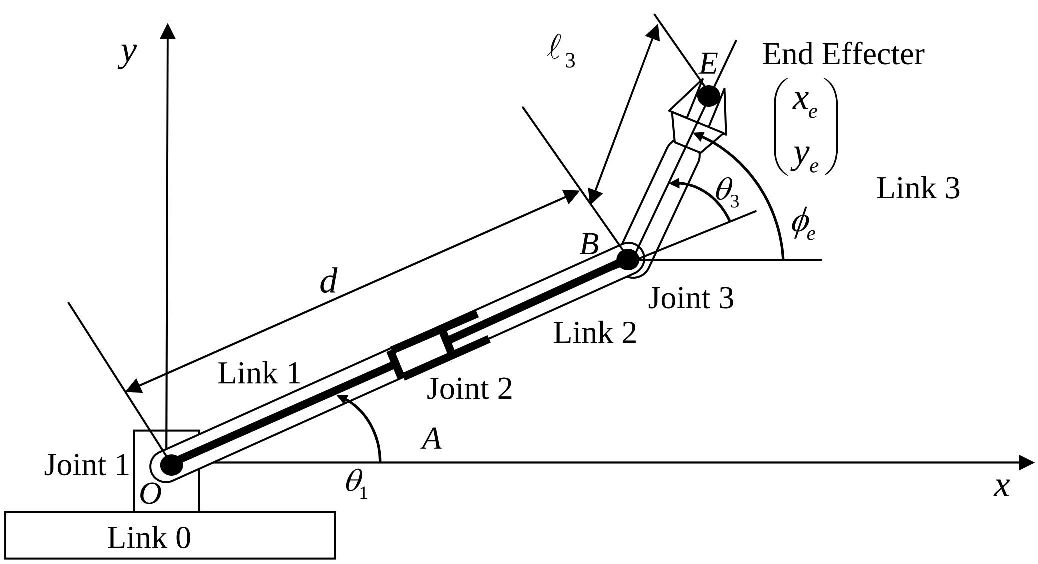


Figure 4.1.2 Three dof robot with two revolute joints and one prismatic joint

Now that the above Example and Exercise problems have illustrated kinematic equations, let us obtain a formal expression for kinematic equations. As mentioned in the previous chapter, two types of joints, prismatic and revolute joints, constitute robot mechanisms in most cases. The displacement of the *i-th* joint is described by distance $d_i$ if it is a prismatic joint, and by angle $\theta_i$ for a revolute joint. For formal expression, let us use a generic notation: $q_i$. Namely, joint displacement $q_i$ represents either distance $d_i$ or angle $\theta_i$ depending on the type of joint.

$$q_i = \begin{cases} d_i & \text{Prismatic joint} \\ \theta_i & \text{Revolute joint} \end{cases} \qquad (4.1.4)$$

We collectively represent all the joint displacements involved in a robot mechanism with a column vector: $q = [q_1 \quad q_2 \quad \cdots \quad q_n]^T$, where *n* is the number of joints. Kinematic equations relate these joint displacements to the position and orientation of the end-effecter. Let us collectively denote the end-effecter position and orientation by vector ***p.*** For planar mechanisms, the end-effecter location is described by three variables:

$$p = \begin{bmatrix} x_e \\ y_e \\ \phi_e \end{bmatrix} \qquad (4.1.5)$$

Using these notations, we represent kinematic equations as a vector function relating ***p*** to ***q***:

$$p = f(q), \quad p \in \Re^{3x1}, q \in \Re^{nx1} \qquad (4.1.6)$$

For a serial link mechanism, all the joints are usually active joints driven by individual actuators. Except for some special cases, these actuators uniquely determine the end-effecter position and orientation as well as the configuration of the entire robot mechanism. If there is a link whose location is not fully determined by the actuator displacements, such a robot mechanism is said to be ***under-actuated***. Unless a robot mechanism is under-actuated, the collection of the joint displacements, i.e. the vector ***q,*** uniquely determines the entire robot configuration. For a serial link mechanism, these joints are independent, having no geometric constraint other than their stroke limits. Therefore, these joint displacements are ***generalized coordinates*** that locate the robot mechanism uniquely and completely. Formally, the number of generalized coordinates is called ***degrees of freedom.*** Vector ***q*** is called joint coordinates, when they form a complete and independent set of generalized coordinates.

## 4.2 Inverse Kinematics of Planar Mechanisms

The vector kinematic equation derived in the previous section provides the functional relationship between the joint displacements and the resultant end-effecter position and orientation. By substituting values of joint displacements into the right-hand side of the kinematic equation, one can immediately find the corresponding end-effecter position and orientation. The problem of finding the end-effecter position and orientation for a given set of joint displacements is referred to as the *direct kinematics problem.* This is simply to evaluate the right-hand side of the kinematic equation for known joint displacements. In this section, we discuss the problem of moving the end-effecter of a manipulator arm to a specified position and orientation. We need to find the joint displacements that lead the end-effecter to the specified position and orientation. This is the inverse of the previous problem, and is thus referred to as the *inverse kinematics problem.* The kinematic equation must be solved for joint displacements, given the end-effecter position and orientation. Once the kinematic equation is solved, the desired end-effecter motion can be achieved by moving each joint to the determined value.

In the direct kinematics problem, the end-effecter location is determined uniquely for any given set of joint displacements. On the other hand, the inverse kinematics is more complex in the sense that multiple solutions may exist for the same end-effecter location. Also, solutions may not always exist for a particular range of end-effecter locations and arm structures. Furthermore, since the kinematic equation is comprised of nonlinear simultaneous equations with many trigonometric functions, it is not always possible to derive a closed-form solution, which is the explicit inverse function of the kinematic equation. When the kinematic equation cannot be solved analytically, numerical methods are used in order to derive the desired joint displacements.

***Example 4.2*** Consider the three dof planar arm shown in Figure 4.1.1 again. To solve its inverse kinematics problem, the kinematic structure is redrawn in Figure 4.2.1. The problem is to find three joint angles $\theta_1, \theta_2, \theta_3$ that lead the end effecter to a desired position and orientation, $x_e, y_e, \phi_e$. We take a two-step approach. First, we find the position of the wrist, point B, from $x_e, y_e, \phi_e$. Then we find $\theta_1, \theta_2$ from the wrist position. Angle $\theta_3$ can be determined immediately from the wrist position.

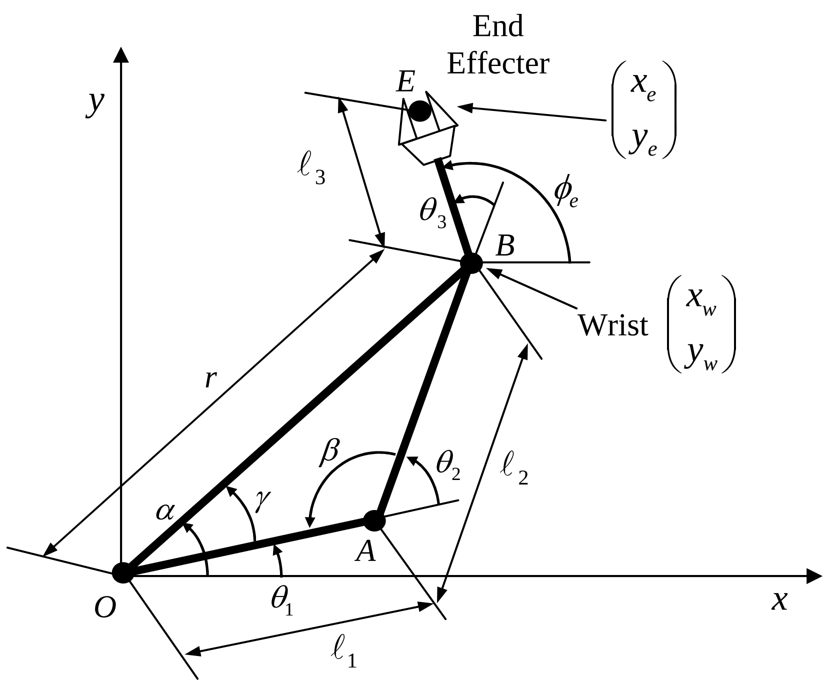


Figure 4.2.1 Skeleton structure of the robot arm of Example 4.1

Let $x_w$ and $y_w$ be the coordinates of the wrist. As shown in Figure 4.2.1, point B is at distance $\ell_3$ from the given end-effecter position E. Moving in the opposite direction to the end effecter orientation $\phi_e$, the wrist coordinates are given by

$$
\begin{aligned}
x_w &= x_e - \ell_3 \cos\phi_e \\
y_w &= y_e - \ell_3 \sin\phi_e
\end{aligned}
\tag{4.2.1}
$$

Note that the right hand sides of the above equations are functions of $x_e, y_e, \phi_e$ alone. From these wrist coordinates, we can determine the angle $\alpha$ shown in the figure.[1]

$$\alpha = \tan^{-1}\frac{y_w}{x_w} \tag{4.2.2}$$

Next, let us consider the triangle *OAB* and define angles $\beta, \gamma$ , as shown in the figure. This triangle is formed by the wrist *B*, the elbow *A*, and the shoulder *O*. Applying the law of cosines to the elbow angle $\beta$ yields

$$\ell_1^2 + \ell_2^2 - 2\ell_1\ell_2\cos\beta = r^2 \tag{4.2.3}$$

where $r^2 = x_w^2 + y_w^2$ , the squared distance between O and B. Solving this for angle $\beta$ yields

$$\theta_2 = \pi - \beta = \pi - \cos^{-1}\frac{\ell_1^2 + \ell_2^2 - x_w^2 - y_w^2}{2\ell_1\ell_2} \tag{4.2.4}$$

Similarly,

$$r^2 + \ell_1^2 - 2r\ell_1\cos\gamma = \ell_2^2 \tag{4.2.5}$$

Solving this for $\gamma$ yields

$$\theta_1 = \alpha - \gamma = \tan^{-1}\frac{y_w}{x_w} - \cos^{-1}\frac{x_w^2 + y_w^2 + \ell_1^2 - \ell_2^2}{2\ell_1\sqrt{x_w^2 + y_w^2}} \tag{4.2.6}$$

From the above $\theta_1, \theta_2$ we can obtain

$$\theta_3 = \phi_e - \theta_1 - \theta_2 \tag{4.2.7}$$

Eqs. (4), (6), and (7) provide a set of joint angles that locates the end-effecter at the desired position and orientation. It is interesting to note that there is another way of reaching the same end-effecter position and orientation, i.e. another solution to the inverse kinematics problem. Figure 4.2.2 shows two configurations of the arm leading to the same end-effecter location: the elbow down configuration and the elbow up configuration. The former corresponds to the solution obtained above. The latter, having the elbow position at point *A'*, is symmetric to the former configuration with respect to line *OB*, as shown in the figure. Therefore, the two solutions are related as

$$\begin{aligned} \theta_1' &= \theta_1 + 2\gamma \\ \theta_2' &= -\theta_2 \\ \theta_3' &= \phi_e - \theta_1' - \theta_2' = \theta_3 + 2\theta_2 - 2\gamma \end{aligned} \tag{4.2.8}$$

Inverse kinematics problems often possess multiple solutions, like the above example, since they are nonlinear. Specifying end-effecter position and orientation does not uniquely determine the whole configuration of the system. This implies that vector ***p***, the collective position and orientation of the end-effecter, cannot be used as generalized coordinates.

The existence of multiple solutions, however, provides the robot with an extra degree of flexibility. Consider a robot working in a crowded environment. If multiple configurations exist for the same end-effecter location, the robot can take a configuration having no interference with

[1] Unless noted specifically we assume that the arc tangent function takes an angle in a proper quadrant consistent with the signs of the two operands.

the environment. Due to physical limitations, however, the solutions to the inverse kinematics problem do not necessarily provide feasible configurations. We must check whether each solution satisfies the constraint of movable range, i.e. stroke limit of each joint.

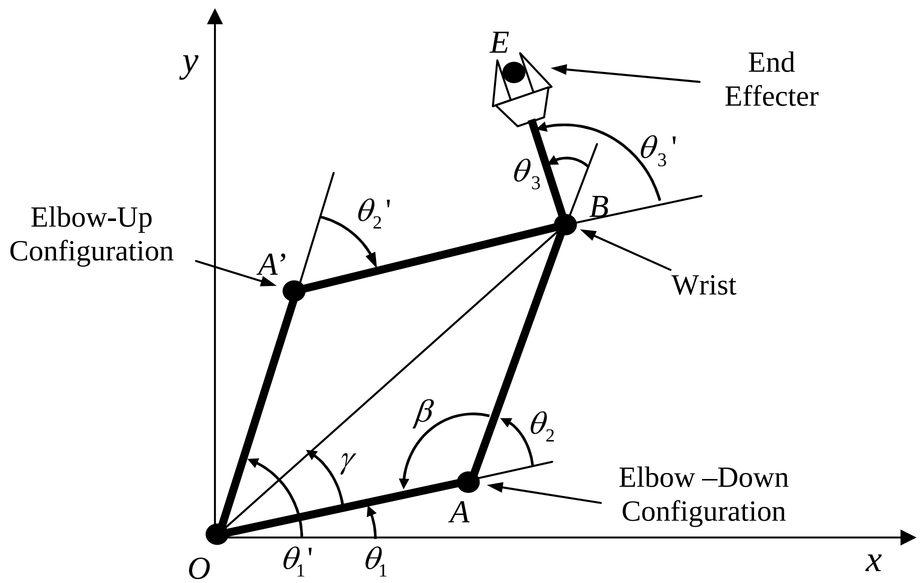


Figure 4.2.2 Multiple solutions to the inverse kinematics problem of Example 4.2

## 4.3 Kinematics of Parallel Link Mechanisms

***Example 4.3*** Consider the five-bar-link planar robot arm shown in Figure 4.3.1.

$$\begin{aligned} x_e &= \ell_1 \cos\theta_1 + \ell_2 \cos\theta_2 \\ y_e &= \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2 \end{aligned} \tag{4.3.1}$$

Note that Joint 2 is a passive joint. Hence, angle $\theta_2$ is a dependent variable. Using $\theta_2$, however, we can obtain the coordinates of point A:

$$\begin{aligned} x_A &= \ell_1 \cos\theta_1 + \ell_5 \cos\theta_2 \\ y_A &= \ell_1 \sin\theta_1 + \ell_5 \sin\theta_2 \end{aligned} \tag{4.3.2}$$

Point A must be reached via the branch comprising Links 3 and 4. Therefore,

$$\begin{aligned} x_A &= \ell_3 \cos\theta_3 + \ell_4 \cos\theta_4 \\ y_A &= \ell_3 \sin\theta_3 + \ell_4 \sin\theta_4 \end{aligned} \tag{4.3.3}$$

Equating these two sets of equations yields two constraint equations:

$$\begin{aligned} \ell_1 \cos\theta_1 + \ell_5 \cos\theta_2 &= \ell_3 \cos\theta_3 + \ell_4 \cos\theta_4 \\ \ell_1 \sin\theta_1 + \ell_5 \sin\theta_2 &= \ell_3 \sin\theta_3 + \ell_4 \sin\theta_4 \end{aligned} \tag{4.3.4}$$

Note that there are four variables and two constraint equations. Therefore, two of the variables, such as $\theta_1, \theta_3$, are independent. It should also be noted that multiple solutions exist for these constraint equations.

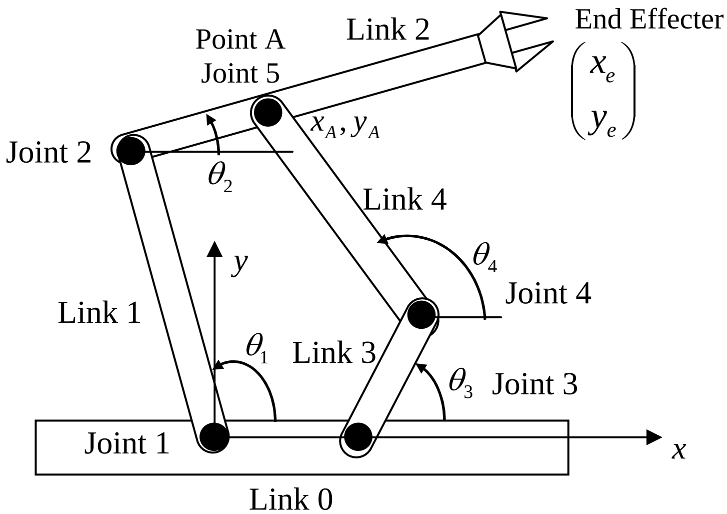


Figure 4.3.1 Five-bar-link mechanism

Although the forward kinematic equations are difficult to write out explicitly, the inverse kinematic equations can be obtained for this parallel link mechanism. The problem is to find $\theta_1, \theta_3$ that lead the endpoint to a desired position: $x_e, y_e$. We will take the following procedure:

Step 1 Given $x_e, y_e$, find $\theta_1, \theta_2$ by solving the two-link inverse kinematics problem.

Step 2 Given $\theta_1, \theta_2$, obtain $x_A, y_A$. This is a forward kinematics problem.

Step 3 Given $x_A, y_A$, find $\theta_3, \theta_4$ by solving another two-link inverse kinematics problem.

***Example 4.4*** Obtain the joint angles of the dog's legs, given the body position and orientation.

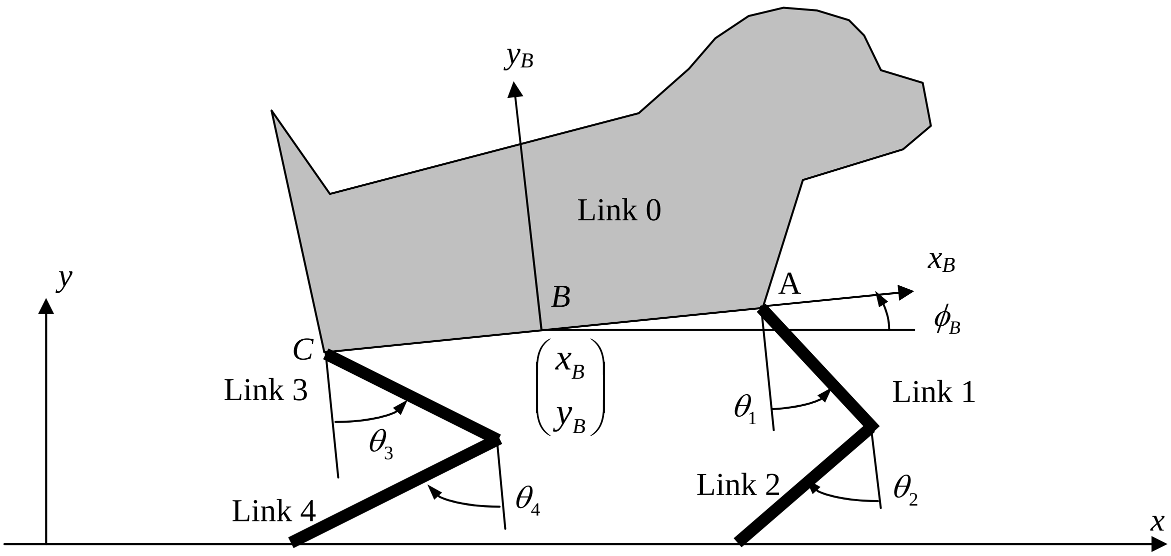


Figure 4.3.2 A doggy robot with two legs on the ground

The inverse kinematics problem:

Step 1 Given $x_B$, $y_B$, $\phi_B$, find $x_A$, $y_A$ and $x_C$, $y_C$

Step 2 Given $x_A$, $y_A$, find $\theta_1$, $\theta_2$

Step 3 Given $x_C$, $y_C$, find $\theta_3$, $\theta_4$

**4.4 Redundant mechanisms**

A manipulator arm must have at least six degrees of freedom in order to locate its end-effecter at an arbitrary point with an arbitrary orientation in space. Manipulator arms with less than 6 degrees of freedom are not able to perform such arbitrary positioning. On the other hand, if a manipulator arm has more than 6 degrees of freedom, there exist an infinite number of solutions to the kinematic equation. Consider for example the human arm, which has seven degrees of freedom, excluding the joints at the fingers. Even if the hand is fixed on a table, one can change the elbow position continuously without changing the hand location. This implies that there exist an infinite set of joint displacements that lead the hand to the same location. Manipulator arms with more than six degrees of freedom are referred to as *redundant manipulators.* We will discuss redundant manipulators in detail in the following chapter.

# Center of mass of a system of particles

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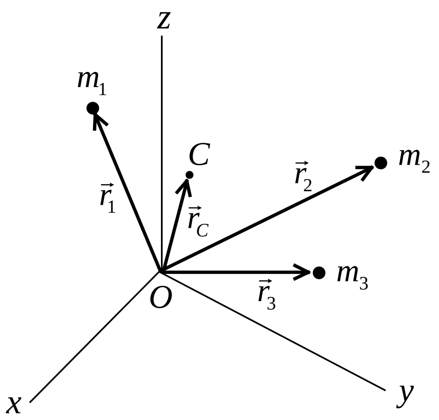
## 1. Definition of the center of mass

Consider a system of particles of masses $m_1$ , $m_2$ , $m_3$ ,... Assume that at some particular moment the particles are located at the points of space with corresponding position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, \cdots,$ relative to a reference point $O$ which is typically chosen to be the origin of an inertial[1] frame of reference (see figure).




The total mass of the system is

$$M = m_1 + m_2 + m_3 + \cdots = \sum_i m_i \quad (1)$$

The *center of mass* of the system is defined as the point $C$ of space having the position vector

$$\vec{r}_C = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots) = \frac{1}{M}\sum_i m_i\vec{r}_i \quad (2)$$

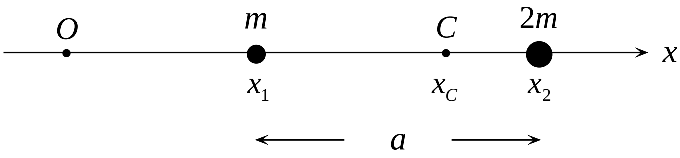
In relation (2) the position vectors of the particles and of the center of mass are defined with respect to the fixed origin $O$ of our coordinate system. If we choose a different reference point $O'$, these position vectors will, of course, change. However, as will be shown below, the position of the center of mass $C$ *relative to the system of particles* will remain the same, regardless of the choice of reference point.

[1] At least, insofar as Newton's laws are to be used.

If $(x_i, y_i, z_i)$ and $(x_C, y_C, z_C)$ are the coordinates of $m_i$ and $C$, respectively, we can replace the vector relation (2) with three scalar equations:

$$x_C = \frac{1}{M}\sum_i m_i x_i \ , \quad y_C = \frac{1}{M}\sum_i m_i y_i \ , \quad z_C = \frac{1}{M}\sum_i m_i z_i \tag{3}$$

As an example, consider two particles of masses $m_1=m$ and $m_2=2m$, located at points $x_1$ and $x_2$ of the $x$-axis. Call $a = x_2 - x_1$ the distance between these particles:




The total mass of the system is $M=m_1+m_2=3m$ . From relations (3) it follows that the center of mass $C$ of the system is located on the $x$-axis. Indeed, $y_i=z_i=0$ ($i$=1,2) so that $y_C=z_C=0$ (the $y$ and $z$-axes have not been drawn). Furthermore,

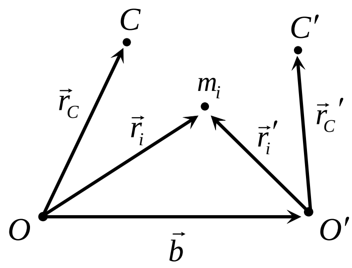
$$x_C = \frac{1}{M}(m_1 x_1 + m_2 x_2) = \frac{1}{3}(x_1 + 2x_2) = x_1 + \frac{2}{3}a$$

where we have used the fact that $x_2 = x_1 + a$ . Thus, the center of mass $C$ is located at a distance $2a/3$ from $m$. Note that the position of $C$ *relative to the system of particles* does not depend on the choice of the reference point $O$ with respect to which the coordinates of the particles are determined.

As the above example demonstrates, the position of the center of mass does not necessarily coincide with the position of a particle of the system. (Give examples of systems in which a particle is located at $C$, as well as of systems where no such coincidence occurs.)

## 2. Independence from the point of reference

We must now show that the location of $C$ in space does not depend on the choice of the reference point $O$. Let us assume for the moment, however, that the position of $C$ *does* depend on the choice of reference point. So, let $C$ and $C'$ be two different positions of the center of mass, corresponding to the reference points $O$ and $O'$. We call $\vec{r}_C$ and $\vec{r}_C\,'$ the position vectors of $C$ and $C'$ with respect to $O$ and $O'$, respectively, and we let $\vec{r}_i$ and $\vec{r}_i\,'$ be the position vectors of the particle $m_i$ relative to $O$ and $O'$. For convenience, we denote by $\vec{b}$ the vector $\overline{OO'}$ (see figure).




The defining equation (2), expressed successively for $O$ and $O'$, yields

$$\vec{r}_C = \frac{1}{M}\sum_i m_i \vec{r}_i \ , \qquad \vec{r}_C{}' = \frac{1}{M}\sum_i m_i \vec{r}_i'$$

where $\vec{r}_i' = \vec{r}_i - \vec{b}$ . Now,

$$\begin{aligned}
\overrightarrow{CC'} &= \overrightarrow{CO} + \overrightarrow{OO'} + \overrightarrow{O'C'} = -\vec{r}_C + \vec{b} + \vec{r}_C{}' \quad \Rightarrow \\
\overrightarrow{CC'} &= -\frac{1}{M}\sum_i m_i \vec{r}_i + \vec{b} + \frac{1}{M}\sum_i m_i \vec{r}_i' = \vec{b} - \frac{1}{M}\sum_i m_i (\vec{r}_i - \vec{r}_i') \\
&= \vec{b} - \frac{1}{M}\sum_i m_i \vec{b} = \vec{b} - \frac{1}{M}\left(\sum_i m_i\right)\vec{b} = \vec{b} - \frac{1}{M} M \vec{b} = 0
\end{aligned}$$

which suggests that the points $C$ and $C'$ coincide. Hence, the center of mass of the system is a uniquely determined point of space, independent of the origin of our coordinate system.

## 3. Center of mass and Newton's laws

We define the *total (linear) momentum* of the system at time $t$ , relative to an inertial reference frame, as the vector sum

$$\vec{P} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i \tag{4}$$

Let $\vec{F}_i$ be the *external* force acting on $m_i$ at this instant. The *total external force* acting on the system at time $t$ is $\vec{F}_{\text{ext}} = \sum_i \vec{F}_i$ . By Newton's 2nd and 3rd laws we find that

$$\frac{d\vec{P}}{dt} = \vec{F}_{\text{ext}} \tag{5}$$

[see, e.g., Papachristou (2020)]. We now prove the following:

1. *The total momentum of the system is equal to the momentum of a hypothetical particle having mass equal to the total mass M of the system and moving with the velocity of the center of mass of the system.*
2. *The equation of motion of the center of mass of the system is that of a hypothetical particle of mass equal to the total mass M of the system, subject to the total external force* $\vec{F}_{\text{ext}}$ *acting on the system.*

*Proof:*

1. Differentiating (2) with respect to time, we find the velocity of the center of mass of the system:

$$\vec{v}_C = \frac{d\vec{r}_C}{dt} = \frac{d}{dt}\left(\frac{1}{M}\sum_i m_i \vec{r}_i\right) = \frac{1}{M}\sum_i m_i \frac{d\vec{r}_i}{dt} \quad \Rightarrow$$

$$\vec{v}_C = \frac{1}{M}\sum_i m_i \vec{v}_i = \frac{1}{M}\sum_i \vec{p}_i \tag{6}$$

Combining this with (4), we have:

$$\vec{P} = M\,\vec{v}_C \tag{7}$$

2. Differentiating (7), we have:

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}(M\,\vec{v}_C) = M\,\frac{d\vec{v}_C}{dt} = M\,\vec{a}_C$$

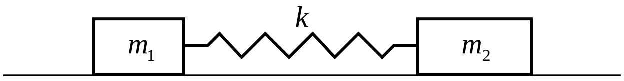
where $\vec{a}_C$ is the acceleration of the center of mass. Hence, by (5),

$$\vec{F}_{\text{ext}} = M\,\vec{a}_C \tag{8}$$

A system of particles is said to be *isolated* if (*a*) it is not subject to any external interactions (a situation that is only theoretically possible) or (*b*) the total external force on the system is zero: $\vec{F}_{\text{ext}} = 0$. In this case, relations (5) and (7) lead to the following conclusions:

1. *The total momentum of an isolated system of particles retains a constant value relative to an inertial frame of reference (principle of conservation of momentum).*
2. *The center of mass C of an isolated system of particles moves with constant velocity relative to an inertial reference frame.*

As an example, consider two masses $m_1$ and $m_2$ connected to each other with a spring. The masses can move on a frictionless horizontal plane, as shown in the figure:




The system may be considered isolated since the total external force on it is zero (explain this!). Thus, the total momentum of the system and the velocity of the center of mass $C$ remain constant while the two masses move on the plane. Note that the *internal* force $F_{\text{int}}=k\Delta l$, where $\Delta l$ is the deformation of the spring relative to its natural length, *cannot* produce any change to the total momentum and the velocity of $C$.

## 4. Center of mass and angular momentum

The *total angular momentum* of the system at time $t$, relative to an arbitrary point $O$, is defined as

$$\vec{L} = \sum_i \vec{L}_i = \sum_i m_i \, (\vec{r}_i \times \vec{v}_i) \tag{9}$$

In particular, the total angular momentum relative to the center of mass $C$ of the system is

$$\vec{L}' = \sum_i m_i \, (\vec{r}_i\,' \times \vec{v}_i\,') \tag{10}$$

where primed quantities are measured with respect to $C$. We have:

$$\vec{r}_i = \vec{r}_i\,' + \vec{r}_C \quad , \quad \vec{v}_i = \vec{v}_i\,' + \vec{v}_C \ .$$

Substituting these into (9) and using (1) and (10), we get:

$$\vec{L} = \vec{L}' + M\,(\vec{r}_C \times \vec{v}_C) + \left[ \left( \sum_i m_i \, \vec{r}_i\,' \right) \times \vec{v}_C \right] + \left[ \vec{r}_C \times \sum_i m_i \vec{v}_i\,' \right] \ .$$

But, $\Sigma m_i \, \vec{r}_i\,' = 0$ and $\Sigma m_i \vec{v}_i\,' = 0$, since these quantities are proportional to the position vector and the velocity, respectively, of the center of mass $C$ relative to $C$ itself. Thus, finally,

$$\vec{L} = \vec{L}' + M\,(\vec{r}_C \times \vec{v}_C) \tag{11}$$

We may interpret this result as follows:

> *The total angular momentum of the system, with respect to a point O, is the sum of the angular momentum relative to the center of mass (“spin angular momentum”) and the angular momentum relative to O, of a hypothetical particle of mass equal to the total mass of the system, moving with the center of mass (“orbital angular momentum”).*

Now, suppose $O$ is the origin of an *inertial* reference frame. Let $\vec{F}_i$ be the external force acting on $m_i$ at time $t$. The *total external torque* acting on the system at this time, relative to $O$, is given by

$$\vec{T}_{\text{ext}} = \sum_i \vec{r}_i \times \vec{F}_i \tag{12}$$

If we make the assumption that all *internal* forces in the system are *central* (as the case is in most physical situations of interest), then the following relation exists between the total angular momentum and the total external torque, both quantities measured relative to $O$ [see, e.g., Papachristou (2020)]:

$$\frac{d\vec{L}}{dt} = \vec{T}_{\text{ext}} \tag{13}$$

Equation (13) is strictly valid relative to the origin $O$ of an *inertial* frame. If the system of particles is *isolated*, the center of mass $C$ moves with constant velocity (relative to $O$) thus is a proper choice of point of reference for the vector relation (13). That is, (13) is valid with respect to the center of mass of an isolated system. But, what if the system of particles is *not* isolated? Then $C$ is *accelerating* (relative to $O$) and it would appear that (13) is not valid relative to $C$ in this case. This is not so, however:

> *Equation (13) is always valid with respect to the center of mass C, even when C is accelerating (i.e., even if the system of particles is not isolated)!*

Indeed, by differentiating (11) with respect to time and by using (13), (12) and (8), we have:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{L}'}{dt} + M(\vec{r}_C \times \vec{a}_C) \, \big(+M(\vec{v}_C \times \vec{v}_C) \text{ which vanishes}\big) \;\Rightarrow$$

$$\vec{T}_{\text{ext}} \equiv \sum_i \vec{r}_i \times \vec{F}_i = \frac{d\vec{L}'}{dt} + (\vec{r}_C \times \vec{F}_{\text{ext}}) \;\Rightarrow$$

$$\frac{d\vec{L}'}{dt} = \sum_i \vec{r}_i \times \vec{F}_i - \left( \vec{r}_C \times \sum_i \vec{F}_i \right) = \sum_i (\vec{r}_i - \vec{r}_C) \times \vec{F}_i$$

$$= \sum_i \vec{r}_i{}' \times \vec{F}_i = \vec{T}_{\text{ext}}{}'$$

where $\vec{T}_{\text{ext}}{}'$ is the total external torque relative to the center of mass.

This observation justifies using (13) to analyze, e.g., the motion of a rolling body on an inclined plane by choosing an axis of rotation that passes through the *accelerating* center of mass of the body.

## 5. Center of mass and kinetic energy

The *total kinetic energy* of the system relative to an external observer $O$ is

$$E_k = \sum_i \frac{1}{2} m_i v_i^2 \tag{14}$$

The total kinetic energy with respect to the center of mass $C$ is

$$E_k' = \sum_i \frac{1}{2} m_i v_i'^2 \tag{15}$$

(as before, primed quantities are measured with respect to $C$). We have:

$$\vec{v}_i = \vec{v}_i' + \vec{v}_C \;\Rightarrow\; v_i^2 = \vec{v}_i \cdot \vec{v}_i = v_i'^2 + v_C^2 + 2\vec{v}_i' \cdot \vec{v}_C \;.$$

Substituting this into (14) and using (1) and (15), we get:

$$E_k = E_k' + \frac{1}{2} M v_C^2 + \left( \sum_i m_i \vec{v}_i' \right) \cdot \vec{v}_C \;.$$

But, as noted previously, the sum in the last term vanishes, being proportional to the velocity of the center of mass $C$ relative to $C$. Thus, finally,

$$E_k = E_k' + \frac{1}{2} M v_C^2 \tag{16}$$

This may be interpreted as follows:

> *The total kinetic energy of the system, relative to an observer O, is the sum of the kinetic energy relative to the center of mass and the kinetic energy relative to O, of a hypothetical particle of mass equal to the total mass of the system, moving with the center of mass.*

## 6. Adding a particle at – or removing a particle from – the center of mass

We now prove the following:

(*a*) Consider a system of $N$ particles of masses $m_1$ , $m_2$ , ... , $m_N$ . Let $C$ be the center of mass of the system. If a new particle, of mass $m$ , is placed at $C$, the center of mass of the enlarged system of ($N$+1) particles will still be at $C$.

(*b*) Consider a system of $N$ particles of masses $m_1$ , $m_2$ , ... , $m_N$ . It is assumed that the location of one of the particles, say of $m_N$ , coincides with the center of mass $C$ of the system. If we now remove this particle from the system, the center of mass of the remaining system of ($N$–1) particles will still be at $C$.

*Proof:*

(*a*) The total mass of the original system of $N$ particles is $M=m_1+m_2+...+m_N$ . The center of mass of this system is located at the point $C$ with position vector

$$\vec{r}_C = \frac{1}{M}\,(m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N)$$

relative to some fixed reference point $O$. For the additional particle, which we name $m_{N+1}$ , we are given that $m_{N+1}=m$ and $\vec{r}_{N+1} = \vec{r}_C$. The total mass of the enlarged system of $(N+1)$ particles $m_1$ , $m_2$ , ... , $m_N$ , $m_{N+1}$ is $M'= M+m$, and the center of mass of this system, relative to $O$, is located at

$$\vec{r}_C{}' = \frac{1}{M'}\,(m_1\vec{r}_1 + \cdots + m_N\vec{r}_N + m\vec{r}_C) \ .$$

Now, $m_1\vec{r}_1 + \cdots + m_N\vec{r}_N = M\vec{r}_C$, so that

$$\vec{r}_C{}' = \frac{1}{M+m}\,(M\vec{r}_C + m\vec{r}_C) = \vec{r}_C \ .$$

(*b*) Although this statement is obviously a corollary of part (*a*), we will prove this independently. Here we are given that $\vec{r}_N = \vec{r}_C$. Thus,

$$\frac{1}{M}\,(m_1\vec{r}_1 + \cdots + m_N\vec{r}_N) = \vec{r}_N \ .$$

The mass of the reduced system of $(N–1)$ particles $m_1$ , $m_2$ , ... , $m_{N–1}$ is $M'= M–m_N$ , while the center of mass of this system is located at

$$\vec{r}_C{}' = \frac{1}{M'}\,(m_1\vec{r}_1 + \cdots + m_{N-1}\vec{r}_{N-1}) \ .$$

But, $m_1\vec{r}_1 + \cdots + m_{N-1}\vec{r}_{N-1} + m_N\vec{r}_N = M\vec{r}_N \quad \Rightarrow$

$$m_1\vec{r}_1 + \cdots + m_{N-1}\vec{r}_{N-1} = (M - m_N)\,\vec{r}_N = M'\vec{r}_N \ .$$

Thus, finally,

$$\vec{r}_C{}' = \frac{1}{M'}M'\vec{r}_N = \vec{r}_N = \vec{r}_C \ .$$

## 7. Center of mass of a continuous mass distribution

A *rigid body* is a physical object the structure of which exhibits a *continuous* mass distribution. Such an object can be considered as a system consisting of an enormous (practically infinite) number of particles of infinitesimal masses $dm_i$ , placed in such a way that the distance between any two neighboring particles is zero. The total mass of the body is

$$M=\sum_i dm_i=\int dm$$

where the sum has been replaced by an integral due to the fact that the $dm_i$ are infinitesimal and the distribution of mass is continuous.

A point in a rigid body can be specified by its position vector $\vec{r}$ , or its coordinates $(x, y, z)$, relative to the origin $O$ of some frame of reference. Let $dV$ be an infinitesimal volume centered at $\vec{r}\equiv(x,y,z)$, and let $dm$ be the infinitesimal mass contained in this volume element. The *density* $\rho$ of the body at point $\vec{r}$ is defined by

$$\rho(\vec{r})=\rho(x,y,z)=\frac{dm}{dV} \ .$$

Then,

$$dm=\rho(\vec{r})\,dV$$

and the total mass of the body is written

$$M=\int\rho(\vec{r})\,dV$$

where the integration takes place over the entire volume of the body. (The integral is in fact a *triple* one since, in Cartesian coordinates, $dV=dxdydz$.) The center of mass $C$ of the body is found by using (2):

$$\vec{r}_C=\frac{1}{M}\sum_i(dm_i)\vec{r}_i=\frac{1}{M}\int\vec{r}\,dm \quad\Rightarrow$$

$$\vec{r}_C=\frac{1}{M}\int\vec{r}\,\rho(\vec{r})\,dV \tag{17}$$

where the $\vec{r}$ and $\vec{r}_C$ are measured relative to the origin $O$ of our coordinate system. (Remember, however, that the location of $C$ *with respect to the body* is uniquely determined and is independent of the choice of the reference point $O$.)

In a *homogeneous* body the density has a constant value $\rho$, independent of $\vec{r}$ . Then,

$$M=\int\rho\,dV=\rho\int dV=\rho V$$

where $V$ is the total volume of the body. Also, from (17) we have:

$$\vec{r}_C = \frac{\rho}{M}\int \vec{r}\, dV = \frac{1}{V}\int \vec{r}\, dV \tag{18}$$

Imagine now that, instead of a mass distribution in space, we have a *linear* distribution of mass (e.g., a very thin rod) along the $x$-axis. We define the *linear density* of the distribution by

$$\rho(x) = \frac{dm}{dx} \ .$$

The total mass of the distribution is

$$M = \int dm = \int \rho(x)\, dx \ .$$

The position of the center of mass of the distribution is given by

$$x_C = \frac{1}{M}\int x\, dm = \frac{1}{M}\int x\,\rho(x)\, dx \tag{19}$$

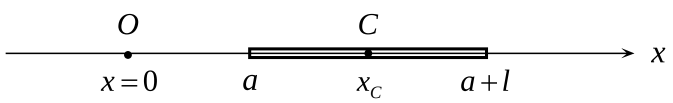
If the density $\rho$ is constant, independent of $x$, then

$$M = \int \rho\, dx = \rho \int dx = \rho\, l$$

where $l$ is the total length of the distribution. Furthermore,

$$x_C = \frac{\rho}{M}\int x\, dx = \frac{1}{l}\int x\, dx \tag{20}$$

As an example, consider a thin, homogeneous rod of length $l$, placed along the $x$-axis from $x=a$ to $x=a+l$, as shown in the figure:




By equation (20),

$$x_C = \frac{1}{l}\int_a^{a+l} x\, dx = \frac{1}{2l}\left[(a+l)^2 - a^2\right] = a + \frac{l}{2} \ .$$

That is, the center of mass $C$ of the rod is located at the center of the rod. Note that the location of $C$ on the rod is uniquely determined, independently of the choice of the origin $O$ of the $x$-axis (although the value of the coordinate $x_C$ does, of course, depend on this choice).

## 8. Center of mass and center of gravity

We have seen that the center of mass $C$ of a system of particles moves in space as if it were a particle of mass equal to the total mass $M$ of the system, subject to the total external force acting on the system. The same is true for a rigid body. Let us assume that the only external forces that act on the system (or the rigid body) are those due to gravity. The total external force is then equal to the *total weight* of the system:

$$\vec{w} = \sum_i \vec{w}_i = \sum_i (m_i \vec{g}) = \left( \sum_i m_i \right) \vec{g} \quad \Rightarrow$$

$$\vec{w} = M \vec{g} \quad \text{where} \quad M = \sum_i m_i \ .$$

The acceleration of gravity $\vec{g}$ is constant in a region of space where the gravitational field may be considered uniform.

Note that $\vec{w}$ is a sum of forces that act on separate particles (or elementary masses $dm_i$ in the case of a rigid body) located at various points of space. The question now is whether there exists some specific point of application of the total weight $\vec{w}$ of the system and, in particular, of a rigid body. A reasonable assumption is that this point could be the center of mass $C$ of the body, given that, as mentioned above, the point $C$ behaves as if it concentrates the entire mass $M$ of the body and the total external force acting on it. And, in our case, $\vec{w}$ is indeed the total external force due to gravity.

There is a subtle point here, however: In contrast to a point particle (such as the hypothetical "particle" of mass $M$ moving with the center of mass $C$ ) that simply changes its location in space, a rigid body may execute a more complex motion, specifically, a combination of translation and rotation. The *translational* motion of the body under the action of gravity is indeed represented by the motion of the center of mass $C$, if this point is regarded as a "particle" of mass $M$ on which the total force $\vec{w}$ is applied. For the *rotational* motion of the body, however, it is the *torques* of the external forces, rather than the forces themselves, that are responsible. Where should we place the total force $\vec{w}$ in order that the rotational motion it produces on the body be the same as that caused by the simultaneous action of the elementary gravitational forces $d\vec{w}_i = (dm_i)\vec{g}$ ? Equivalently, where should we place $\vec{w}$ in order that its torque *with respect to any point O* be equal to the total torque of the $d\vec{w}_i$ with respect to $O$ ?

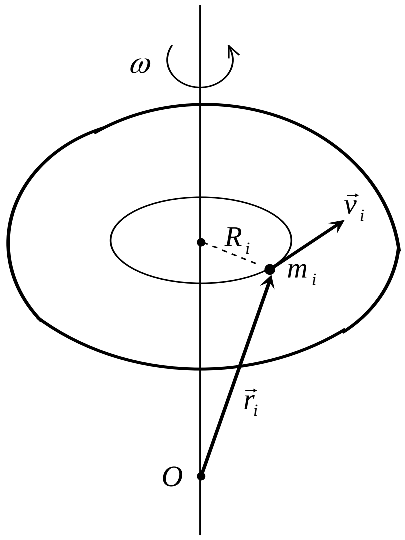
You may have guessed the answer already: at the center of mass $C$ ! [See, e.g., Papachristou (2020).] In conclusion:

> *By placing the total weight $\vec{w}$ of the body at the center of mass C we manage to describe not only the translational but also the rotational motion of the body under the action of gravity.*

It is for this reason that $C$ is frequently called the *center of gravity* of the body. Note that this point does *not necessarily* belong to the body (consider, for example, the cases of a ring and a spherical shell).

## 9. Mechanical energy of a rigid body

Consider a rigid body rotating with angular velocity $\omega$ about an axis passing from a fixed point $O$ of space:




During rotation, every elementary mass $m_i$ in the body moves circularly about the axis of rotation, with the common angular velocity $\omega$. If $R_i$ is the perpendicular distance of $m_i$ from the axis (thus, the radius of the circular path of $m_i$), the speed of this mass element is $v_i = R_i\,\omega$ . The total *kinetic energy of rotation* is the sum of the kinetic energies of all elementary masses $m_i$ contained in the body:

$$E_{k,rot} = \sum_i \left( \frac{1}{2} m_i v_i^{\,2} \right) = \sum_i \left( \frac{1}{2} m_i R_i^{\,2} \omega^2 \right) = \frac{1}{2}\omega^2 \sum_i m_i R_i^{\,2} \quad \Rightarrow$$

$$E_{k,rot} = \frac{1}{2} I\,\omega^2 \tag{21}$$

where

$$I = \sum_i m_i R_i^{\,2}$$

is the *moment of inertia* of the body relative to the axis of rotation.

Relation (21) represents the total kinetic energy of the body when the latter executes *pure rotation* about a fixed axis. A more general kind of motion is a rotation about an axis that is moving in space. Specifically, assume that the axis of rotation passes from the center of mass $C$ of the body, while $C$ itself moves in space with velocity $\vec{v}_C$ . The body thus executes a composite motion consisting of a *translation* of the center of mass $C$ and a *rotation* about $C$. According to equation (16), the total kinetic energy of the body is the sum of two quantities: a *kinetic energy of translation*,

$$E_{k,trans} = \frac{1}{2} M v_C^{\,2}$$

(where $M$ is the mass of the body and $v_C$ is the speed of the center of mass $C$) and a *kinetic energy of rotation about C*,

$$E_{k,rot} = \frac{1}{2} I_C\, \omega^2$$

(where $\omega$ is the angular velocity of rotation about an axis passing from $C$, while $I_C$ is the moment of inertia of the body relative to this axis[2]). Hence, the total kinetic energy of the body is

$$E_k = E_{k,trans} + E_{k,rot} = \frac{1}{2} M v_C{}^2 + \frac{1}{2} I_C\, \omega^2 \tag{22}$$

If the body is subject to external forces that are conservative, we can define an *external potential energy $E_p$* as well as a *total mechanical energy E*, the latter assuming a constant value during the motion of the body:

$$E = E_k + E_p = \frac{1}{2} M v_C{}^2 + \frac{1}{2} I_C\, \omega^2 + E_p = const. \tag{23}$$

For example, if the body moves under the sole action of gravity, its potential energy is

$$E_p = M g\, y_C$$

where $y_C$ is the vertical distance (the height) of the center of mass $C$ with respect to an arbitrary horizontal plane of reference. Indeed, by relation (3),

$$y_C = \frac{1}{M} \sum_i m_i\, y_i$$

where $y_i$ is the height of the point of location of the elementary mass $m_i$ in the body. The total gravitational potential energy of the body, equal to the sum of the potential energies of all elementary masses $m_i$, is then

$$E_p = \sum_i (m_i\, g\, y_i) = g \sum_i m_i\, y_i = M g\, y_C \ .$$

The total mechanical energy of the body is constant and equal to

$$E = \frac{1}{2} M v_C{}^2 + \frac{1}{2} I_C\, \omega^2 + M g\, y_C \tag{24}$$

[2] The moment of inertia $I$ relative to an axis parallel to this axis is given by the *parallel-axis theorem* [see, e.g., Papachristou (2020)]. Specifically, $I=I_C+Ma^2$, where $a$ is the perpendicular distance between the two axes.

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