5/4 Relative Velocity

The second approach to rigid-body kinematics is to use the principles of relative motion. In Art. 2/8 we developed these principles for motion relative to translating axes and applied the relative-velocity equation

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B} \qquad \textbf{[2/20]}$$

to the motions of two particles A and B.

Relative Velocity Due to Rotation

We now choose two points on the *same* rigid body for our two particles. The consequence of this choice is that the motion of one point as seen by an observer translating with the other point must be circular since the radial distance to the observed point from the reference point does not change. This observation is the *key* to the successful understanding of a large majority of problems in the plane motion of rigid bodies.

This concept is illustrated in Fig. 5/5*a*, which shows a rigid body moving in the plane of the figure from position AB to $A'B'$ during time Δt. This movement may be visualized as occurring in two parts. First, the body translates to the parallel position $A''B'$ with the displacement $\Delta\mathbf{r}_B$. Second, the body rotates about B' through the angle $\Delta\theta$. From the nonrotating reference axes x'-y' attached to the reference point B', you can see that this remaining motion of the body is one of simple rotation about B', giving rise to the displacement $\Delta\mathbf{r}_{A/B}$ of A with respect to B. To the nonrotating observer attached to B, the body appears to undergo fixed-axis rotation about B with A executing circular motion as emphasized in Fig. 5/5*b*. Therefore, the relationships developed for circular motion in Arts. 2/5 and 5/2 and cited as Eqs. 2/11 and 5/2 (or 5/3) describe the relative portion of the motion of point A.

Point B was arbitrarily chosen as the reference point for attachment of our nonrotating reference axes x-y. Point A could have been used just as well, in which case we would observe B to have circular motion about A considered fixed as shown in Fig. 5/5*c*. We see that the sense of the

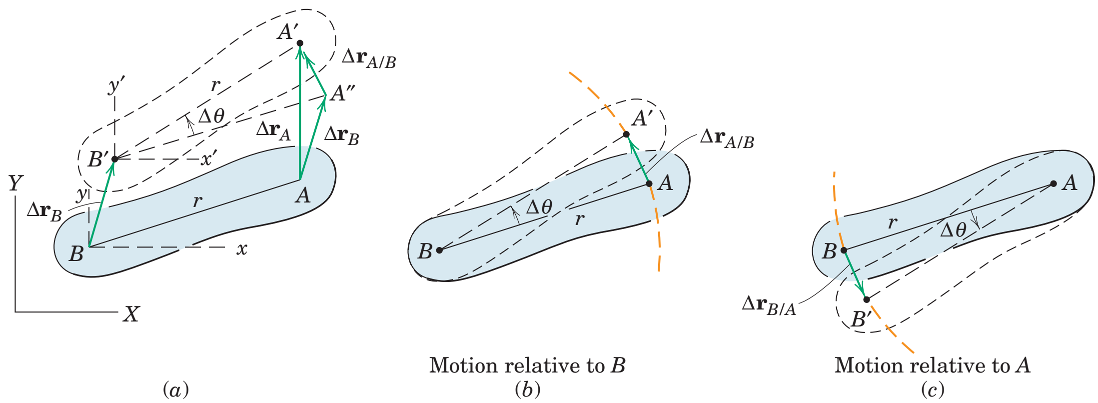


Figure 5/5

rotation, counterclockwise in this example, is the same whether we choose A or B as the reference, and we see that $\Delta\mathbf{r}_{B/A} = -\Delta\mathbf{r}_{A/B}$.

With B as the reference point, we see from Fig. 5/5*a* that the total displacement of A is

$$\Delta\mathbf{r}_A = \Delta\mathbf{r}_B + \Delta\mathbf{r}_{A/B}$$

where $\Delta\mathbf{r}_{A/B}$ has the magnitude $r\Delta\theta$ as $\Delta\theta$ approaches zero. We note that the *relative linear motion* $\Delta\mathbf{r}_{A/B}$ is accompanied by the *absolute angular motion* $\Delta\theta$, as seen from the translating axes x'-y'. Dividing the expression for $\Delta\mathbf{r}_A$ by the corresponding time interval Δt and passing to the limit, we obtain the relative-velocity equation

$$\boxed{\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}} \qquad \textbf{(5/4)}$$

This expression is the same as Eq. 2/20, with the one restriction that the distance r between A and B remains constant. The magnitude of the relative velocity is thus seen to be $v_{A/B} = \lim_{\Delta t \to 0} (|\Delta\mathbf{r}_{A/B}|/\Delta t) = \lim_{\Delta t \to 0} (r\Delta\theta/\Delta t)$ which, with $\omega = \dot{\theta}$, becomes

$$\boxed{v_{A/B} = r\omega} \qquad \textbf{(5/5)}$$

Using $\mathbf{r}$ to represent the vector $\mathbf{r}_{A/B}$ from the first of Eqs. 5/3, we may write the relative velocity as the vector

$$\boxed{\mathbf{v}_{A/B} = \boldsymbol{\omega} \times \mathbf{r}} \qquad \textbf{(5/6)}$$

where $\boldsymbol{\omega}$ is the angular-velocity vector normal to the plane of the motion in the sense determined by the right-hand rule. A critical observation seen from Figs. 5/5*b* and *c* is that the relative linear velocity is always perpendicular to the line joining the two points in question.

Interpretation of the Relative-Velocity Equation

We can better understand the application of Eq. 5/4 by visualizing the separate translation and rotation components of the equation. These components are emphasized in Fig. 5/6, which shows a rigid body

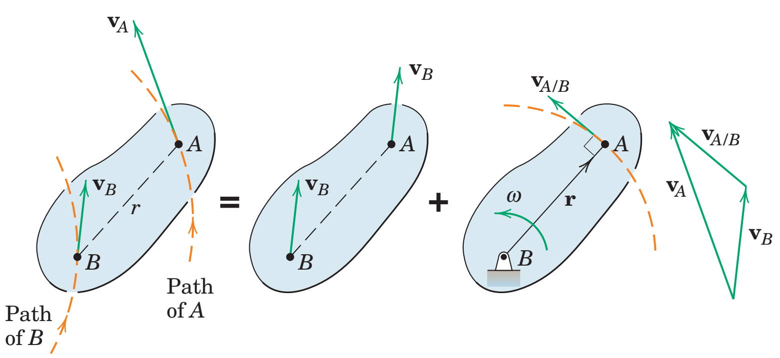


Figure 5/6

in plane motion. With B chosen as the reference point, the velocity of A is the vector sum of the translational portion $\mathbf{v}_B$, plus the rotational portion $\mathbf{v}_{A/B} = \boldsymbol{\omega} \times \mathbf{r}$, which has the magnitude $v_{A/B} = r\omega$, where $|\boldsymbol{\omega}| = \dot{\theta}$, the *absolute* angular velocity of AB. The fact that the *relative linear velocity* is *always perpendicular* to the line joining the two points in question is an important key to the solution of many problems. To reinforce your understanding of this concept, you should draw the equivalent diagram where point A is used as the reference point rather than B.

Equation 5/4 may also be used to analyze constrained sliding contact between two links in a mechanism. In this case, we choose points A and B as coincident points, one on each link, for the instant under consideration. In contrast to the previous example, in this case, the two points are on different bodies so they are not a fixed distance apart. This second use of the relative-velocity equation is illustrated in Sample Problem 5/10.

Solution of the Relative-Velocity Equation

Solution of the relative-velocity equation may be carried out by scalar or vector algebra, or a graphical analysis may be employed. A sketch of the vector polygon which represents the vector equation should always be made to reveal the physical relationships involved. From this sketch, you can write scalar component equations by projecting the vectors along convenient directions. You can usually avoid solving simultaneous equations by a careful choice of the projections. Alternatively, each term in the relative-motion equation may be written in terms of its **i**- and **j**-components, from which you will obtain two scalar equations when the equality is applied, separately, to the coefficients of the **i**- and **j**-terms.

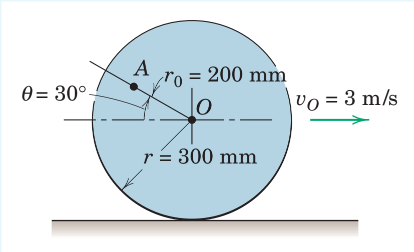
Many problems lend themselves to a graphical solution, particularly when the given geometry results in an awkward mathematical expression. In this case, we first construct the known vectors in their correct positions using a convenient scale. Then we construct the unknown vectors which complete the polygon and satisfy the vector equation. Finally, we measure the unknown vectors directly from the drawing.

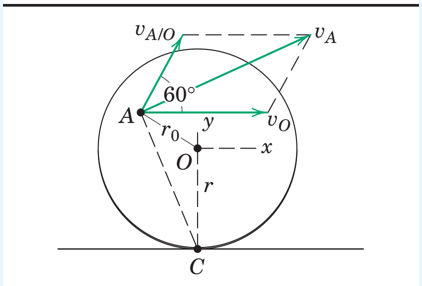
The choice of method to be used depends on the particular problem at hand, the accuracy required, and individual preference and experience. All three approaches are illustrated in the sample problems which follow.

Regardless of which method of solution we employ, we note that the single vector equation in two dimensions is equivalent to two scalar equations, so that at most two scalar unknowns can be determined. The unknowns, for instance, might be the magnitude of one vector and the direction of another. We should make a systematic identification of the knowns and unknowns before attempting a solution.

SAMPLE PROBLEM 5/7

The wheel of radius $r = 300$ mm rolls to the right without slipping and has a velocity $v_O = 3$ m/s of its center O. Calculate the velocity of point A on the wheel for the instant represented.





Solution I (Scalar-Geometric). The center O is chosen as the reference point for the relative-velocity equation since its motion is given. We therefore write

$$\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{A/O}$$

where the relative-velocity term is observed from the translating axes x-y attached to O. The angular velocity of AO is the same as that of the wheel which, from Sample Problem 5/4, is $\omega = v_O/r = 3/0.3 = 10$ rad/s. Thus, from Eq. 5/5 we have

$[v_{A/O} = r_0\dot{\theta}]$ $\qquad v_{A/O} = 0.2(10) = 2$ m/s

1 which is normal to AO as shown. The vector sum $\mathbf{v}_A$ is shown on the diagram and may be calculated from the law of cosines. Thus,

2 $$v_A{}^2 = 3^2 + 2^2 + 2(3)(2)\cos 60° = 19\ (\text{m/s})^2 \qquad v_A = 4.36 \text{ m/s} \qquad \textit{Ans.}$$

The contact point C momentarily has zero velocity and can be used alternatively as the reference point, in which case, the relative-velocity equation becomes $\mathbf{v}_A = \mathbf{v}_C + \mathbf{v}_{A/C} = \mathbf{v}_{A/C}$ where

$$v_{A/C} = \overline{AC}\omega = \frac{\overline{AC}}{\overline{OC}}v_O = \frac{0.436}{0.300}(3) = 4.36 \text{ m/s} \qquad v_A = v_{A/C} = 4.36 \text{ m/s}$$

The distance $\overline{AC} = 436$ mm is calculated separately. We see that $\mathbf{v}_A$ is normal to
3 AC since A is momentarily rotating about point C.

Solution II (Vector). We will now use Eq. 5/6 and write

$$\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{A/O} = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_0$$

where

4 $$\boldsymbol{\omega} = -10\mathbf{k} \text{ rad/s}$$

$$\mathbf{r}_0 = 0.2(-\mathbf{i}\cos 30° + \mathbf{j}\sin 30°) = -0.1732\mathbf{i} + 0.1\mathbf{j} \text{ m}$$

$$\mathbf{v}_O = 3\mathbf{i} \text{ m/s}$$

We now solve the vector equation

$$\mathbf{v}_A = 3\mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -10 \\ -0.1732 & 0.1 & 0 \end{vmatrix} = 3\mathbf{i} + 1.732\mathbf{j} + \mathbf{i}$$

$$= 4\mathbf{i} + 1.732\mathbf{j} \text{ m/s} \qquad \textit{Ans.}$$

The magnitude $v_A = \sqrt{4^2 + (1.732)^2} = \sqrt{19} = 4.36$ m/s and direction agree with the previous solution.

Helpful Hints

1 Be sure to visualize $v_{A/O}$ as the velocity which A appears to have in its circular motion relative to O.

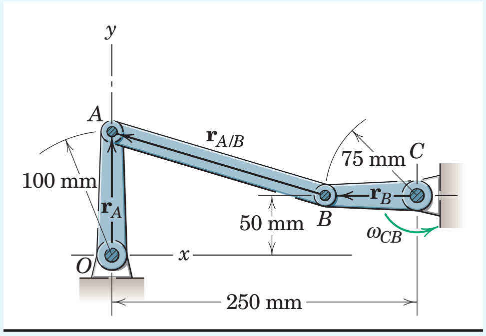
2 The vectors may also be laid off to scale graphically and the magnitude and direction of v_A measured directly from the diagram.

3 The velocity of any point on the wheel is easily determined by using the contact point C as the reference point. You should construct the velocity vectors for a number of points on the wheel for practice.

4 The vector $\boldsymbol{\omega}$ is directed into the paper by the right-hand rule, whereas the positive z-direction is out from the paper; hence, the minus sign.

SAMPLE PROBLEM 5/8

Crank CB oscillates about C through a limited arc, causing crank OA to oscillate about O. When the linkage passes the position shown with CB horizontal and OA vertical, the angular velocity of CB is 2 rad/s counterclockwise. For this instant, determine the angular velocities of OA and AB.



Solution I (Vector). The relative-velocity equation $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$ is rewritten as

① $$\boldsymbol{\omega}_{OA} \times \mathbf{r}_A = \boldsymbol{\omega}_{CB} \times \mathbf{r}_B + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B}$$

where

$$\boldsymbol{\omega}_{OA} = \omega_{OA}\mathbf{k} \qquad \boldsymbol{\omega}_{CB} = 2\mathbf{k} \text{ rad/s} \qquad \boldsymbol{\omega}_{AB} = \omega_{AB}\mathbf{k}$$

$$\mathbf{r}_A = 100\mathbf{j} \text{ mm} \qquad \mathbf{r}_B = -75\mathbf{i} \text{ mm} \qquad \mathbf{r}_{A/B} = -175\mathbf{i} + 50\mathbf{j} \text{ mm}$$

Substitution gives

$$\omega_{OA}\mathbf{k} \times 100\mathbf{j} = 2\mathbf{k} \times (-75\mathbf{i}) + \omega_{AB}\mathbf{k} \times (-175\mathbf{i} + 50\mathbf{j})$$

$$-100\omega_{OA}\mathbf{i} = -150\mathbf{j} - 175\omega_{AB}\mathbf{j} - 50\omega_{AB}\mathbf{i}$$

Matching coefficients of the respective **i**- and **j**-terms gives

$$-100\omega_{OA} + 50\omega_{AB} = 0 \qquad 25(6 + 7\omega_{AB}) = 0$$

the solutions of which are

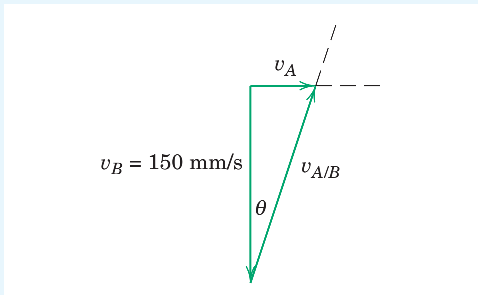
② $$\omega_{AB} = -6/7 \text{ rad/s} \quad \text{and} \quad \omega_{OA} = -3/7 \text{ rad/s}$$ *Ans.*

Solution II (Scalar-Geometric). Solution by the scalar geometry of the vector triangle is particularly simple here since $\mathbf{v}_A$ and $\mathbf{v}_B$ are at right angles for this special position of the linkages. First, we compute v_B, which is

$[v = r\omega]$ $$v_B = 0.075(2) = 0.150 \text{ m/s}$$

and represent it in its correct direction as shown. The vector $\mathbf{v}_{A/B}$ must be perpendicular to AB, and the angle θ between $\mathbf{v}_{A/B}$ and $\mathbf{v}_B$ is also the angle made by AB with the horizontal direction. This angle is given by

$$\tan\theta = \frac{100 - 50}{250 - 75} = \frac{2}{7}$$



③ The horizontal vector $\mathbf{v}_A$ completes the triangle for which we have

$$v_{A/B} = v_B/\cos\theta = 0.150/\cos\theta$$

$$v_A = v_B \tan\theta = 0.150(2/7) = 0.30/7 \text{ m/s}$$

The angular velocities become

$[\omega = v/r]$ $$\omega_{AB} = \frac{v_{A/B}}{AB} = \frac{0.150}{\cos\theta}\,\frac{\cos\theta}{0.250 - 0.075}$$

$$= 6/7 \text{ rad/s CW}$$ *Ans.*

$[\omega = v/r]$ $$\omega_{OA} = \frac{v_A}{OA} = \frac{0.30}{7}\,\frac{1}{0.100} = 3/7 \text{ rad/s CW}$$ *Ans.*

Helpful Hints

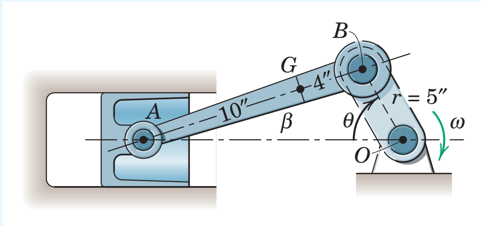
① We are using here the first of Eqs. 5/3 and Eq. 5/6.

② The minus signs in the answers indicate that the vectors $\boldsymbol{\omega}_{AB}$ and $\boldsymbol{\omega}_{OA}$ are in the negative **k**-direction. Hence, the angular velocities are clockwise.

③ Always make certain that the sequence of vectors in the vector polygon agrees with the equality of vectors specified by the vector equation.

SAMPLE PROBLEM 5/9

The common configuration of a reciprocating engine is that of the slider-crank mechanism shown. If the crank OB has a clockwise rotational speed of 1500 rev/min, determine for the position where $\theta = 60°$ the velocity of the piston A, the velocity of point G on the connecting rod, and the angular velocity of the connecting rod.



Solution. The velocity of the crank pin B as a point on AB is easily found, so that B will be used as the reference point for determining the velocity of A. The relative-velocity equation may now be written

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

The crank-pin velocity is

1 $[v = r\omega]$ $$v_B = \frac{5}{12}\frac{1500\,(2\pi)}{60} = 65.4 \text{ ft/sec}$$

and is normal to OB. The direction of $\mathbf{v}_A$ is, of course, along the horizontal cylinder axis. The direction of $\mathbf{v}_{A/B}$ must be perpendicular to the line AB as explained in the present article and as indicated in the lower diagram, where the reference point B is shown as fixed. We obtain this direction by computing angle β from the law of sines, which gives

$$\frac{5}{\sin\beta} = \frac{14}{\sin 60°} \qquad \beta = \sin^{-1} 0.309 = 18.02°$$

We now complete the sketch of the velocity triangle, where the angle between $\mathbf{v}_{A/B}$ and $\mathbf{v}_A$ is $90° - 18.02° = 72.0°$ and the third angle is $180° - 30° - 72.0° = 78.0°$. Vectors $\mathbf{v}_A$ and $\mathbf{v}_{A/B}$ are shown with their proper sense such that the head-to-tail sum of $\mathbf{v}_B$ and $\mathbf{v}_{A/B}$ equals $\mathbf{v}_A$. The magnitudes of the unknowns are now calculated from the trigonometry of the vector triangle or are scaled from the diagram if a graphical solution is used. Solving for v_A and $v_{A/B}$ by the law of sines gives

2 $$\frac{v_A}{\sin 78.0°} = \frac{65.4}{\sin 72.0°} \qquad v_A = 67.3 \text{ ft/sec} \qquad \textit{Ans.}$$

$$\frac{v_{A/B}}{\sin 30°} = \frac{65.4}{\sin 72.0°} \qquad v_{A/B} = 34.4 \text{ ft/sec}$$

The angular velocity of AB is counterclockwise, as revealed by the sense of $\mathbf{v}_{A/B}$, and is

$[\omega = v/r]$ $$\omega_{AB} = \frac{v_{A/B}}{AB} = \frac{34.4}{14/12} = 29.5 \text{ rad/sec} \qquad \textit{Ans.}$$

We now determine the velocity of G by writing

$$\mathbf{v}_G = \mathbf{v}_B + \mathbf{v}_{G/B}$$

where $$v_{G/B} = \overline{GB}\omega_{AB} = \frac{\overline{GB}}{AB}v_{A/B} = \frac{4}{14}(34.4) = 9.83 \text{ ft/sec}$$

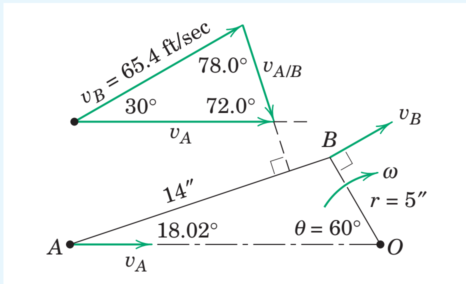
As seen from the diagram, $\mathbf{v}_{G/B}$ has the same direction as $\mathbf{v}_{A/B}$. The vector sum is shown on the last diagram. We can calculate v_G with some geometric labor or simply measure its magnitude and direction from the velocity diagram drawn to scale. For simplicity we adopt the latter procedure here and obtain

$$v_G = 64.1 \text{ ft/sec} \qquad \textit{Ans.}$$

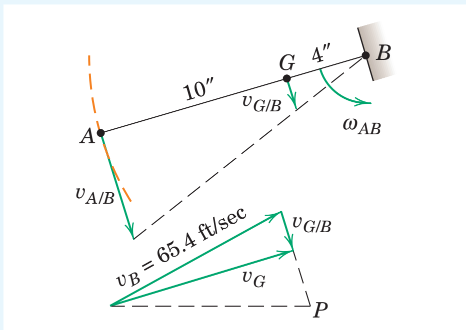
As seen, the diagram may be superposed directly on the first velocity diagram.

Helpful Hints

1 Remember always to convert ω to radians per unit time when using $v = r\omega$.

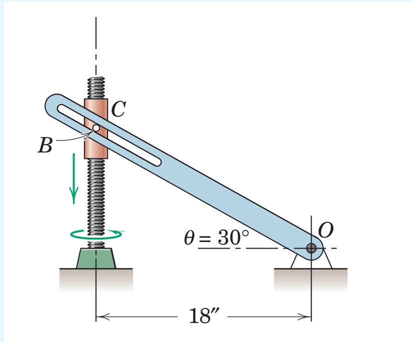


2 A graphical solution to this problem is the quickest to achieve, although its accuracy is limited. Solution by vector algebra can, of course, be used but would involve somewhat more labor in this problem.



SAMPLE PROBLEM 5/10

The power screw turns at a speed which gives the threaded collar C a velocity of 0.8 ft/sec vertically down. Determine the angular velocity of the slotted arm when $\theta = 30°$.



Solution. The angular velocity of the arm can be found if the velocity of a point on the arm is known. We choose a point A on the arm coincident with the
1 pin B of the collar for this purpose. If we use B as our reference point and write $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$, we see from the diagram, which shows the arm and points A and B an instant before and an instant after coincidence, that $\mathbf{v}_{A/B}$ has a direction along the slot away from O.

2 The magnitudes of $\mathbf{v}_A$ and $\mathbf{v}_{A/B}$ are the only unknowns in the vector equation, so that it may now be solved. We draw the known vector $\mathbf{v}_B$ and then obtain the intersection P of the known directions of $\mathbf{v}_{A/B}$ and $\mathbf{v}_A$. The solution gives

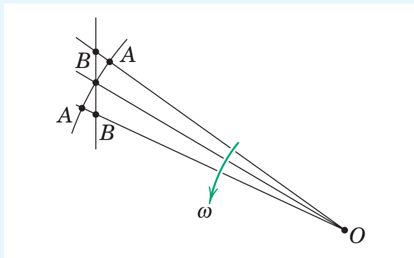
$$v_A = v_B \cos\theta = 0.8 \cos 30° = 0.693 \text{ ft/sec}$$

$$[\omega = v/r] \qquad \omega = \frac{v_A}{OA} = \frac{0.693}{(\frac{18}{12})/\cos 30°}$$
$$= 0.400 \text{ rad/sec CCW} \qquad \textit{Ans.}$$

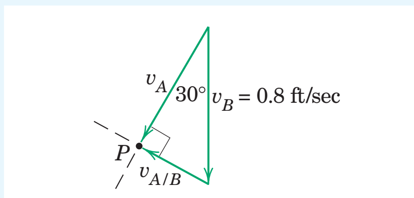
We note the difference between this problem of constrained sliding contact between two links and the three preceding sample problems of relative velocity, where no sliding contact occurred and where the points A and B were located on the same rigid body in each case.

Helpful Hints

1 Physically, of course, this point does not exist, but we can imagine such a point in the middle of the slot and attached to the arm.



2 Always identify the knowns and unknowns before attempting the solution of a vector equation.



5/6 Relative Acceleration

Consider the equation $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$, which describes the relative velocities of two points A and B in plane motion in terms of nonrotating reference axes. By differentiating the equation with respect to time, we may obtain the relative-acceleration equation, which is $\dot{\mathbf{v}}_A = \dot{\mathbf{v}}_B + \dot{\mathbf{v}}_{A/B}$ or

$$\boxed{\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}} \qquad \textbf{(5/7)}$$

In words, Eq. 5/7 states that the acceleration of point A equals the vector sum of the acceleration of point B and the acceleration which A appears to have to a nonrotating observer moving with B.

Relative Acceleration Due to Rotation

If points A and B are located on the same rigid body and in the plane of motion, the distance r between them remains constant so that the observer moving with B perceives A to have circular motion about B, as we saw in Art. 5/4 with the relative-velocity relationship. Because the relative motion is circular, it follows that the relative-acceleration term will have both a normal component directed from A toward B due to the change of direction of $\mathbf{v}_{A/B}$ and a tangential component perpendicular to AB due to the change in magnitude of $\mathbf{v}_{A/B}$. These acceleration components for circular motion, cited in Eqs. 5/2, were covered earlier in Art. 2/5 and should be thoroughly familiar by now.

Thus we may write

$$\boxed{\mathbf{a}_A = \mathbf{a}_B + (\mathbf{a}_{A/B})_n + (\mathbf{a}_{A/B})_t} \qquad \textbf{(5/8)}$$

where the magnitudes of the relative-acceleration components are

$$\boxed{\begin{aligned}(a_{A/B})_n &= v_{A/B}{}^2/r = r\omega^2\\ (a_{A/B})_t &= \dot{v}_{A/B} = r\alpha\end{aligned}} \qquad \textbf{(5/9)}$$

In vector notation the acceleration components are

$$\boxed{\begin{aligned}(\mathbf{a}_{A/B})_n &= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\\ (\mathbf{a}_{A/B})_t &= \boldsymbol{\alpha} \times \mathbf{r}\end{aligned}} \qquad \textbf{(5/9}\boldsymbol{a}\textbf{)}$$

In these relationships, $\boldsymbol{\omega}$ is the angular velocity and $\boldsymbol{\alpha}$ is the angular acceleration of the body. The vector locating A from B is $\mathbf{r}$. It is important to observe that the *relative* acceleration terms depend on the respective *absolute* angular velocity and *absolute* angular acceleration.

Interpretation of the Relative-Acceleration Equation

The meaning of Eqs. 5/8 and 5/9 is illustrated in Fig. 5/9, which shows a rigid body in plane motion with points A and B moving along separate curved paths with absolute accelerations $\mathbf{a}_A$ and $\mathbf{a}_B$. Contrary to the case with velocities, the accelerations $\mathbf{a}_A$ and $\mathbf{a}_B$ are, in general, not tangent to the paths described by A and B when these

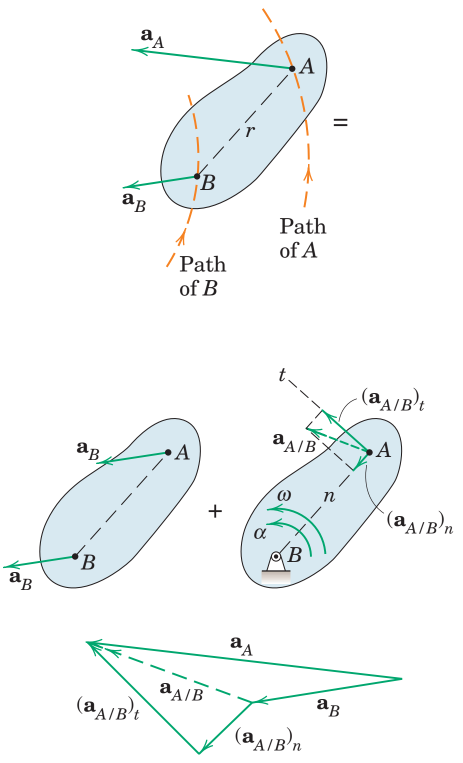


Figure 5/9

paths are curvilinear. The figure shows the acceleration of A to be composed of two parts: the acceleration of B and the acceleration of A with respect to B. A sketch showing the reference point as fixed is useful in disclosing the correct sense of each of the two components of the relative-acceleration term.

Alternatively, we may express the acceleration of B in terms of the acceleration of A, which puts the nonrotating reference axes on A rather than B. This order gives

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

Here $\mathbf{a}_{B/A}$ and its n- and t-components are the negatives of $\mathbf{a}_{A/B}$ and its n- and t-components. To understand this analysis better, you should make a sketch corresponding to Fig. 5/9 for this choice of terms.

Solution of the Relative-Acceleration Equation

As in the case of the relative-velocity equation, we can handle the solution to Eq. 5/8 in three different ways, namely, by scalar algebra and geometry, by vector algebra, or by graphical construction. It is helpful to be familiar with all three techniques. You should make a sketch of the vector polygon representing the vector equation and pay close attention to the head-to-tail combination of vectors so that it agrees with the equation. Known vectors should be added first, and the unknown vectors will become the closing legs of the vector polygon. It is vital that you visualize the vectors in their geometrical sense, as only then can you understand the full significance of the acceleration equation.

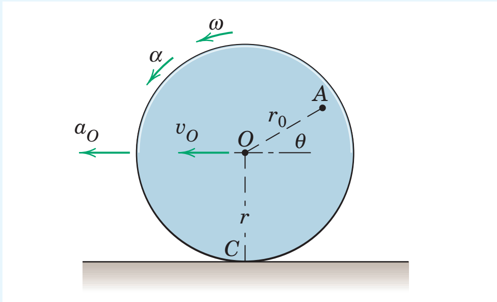
Before attempting a solution, identify the knowns and unknowns, keeping in mind that a solution to a vector equation in two dimensions can be carried out when the unknowns have been reduced to two scalar quantities. These quantities may be the magnitude or direction of any of the terms of the equation. When both points move on curved paths, there will, in general, be six scalar quantities to account for in Eq. 5/8.

Because the normal acceleration components depend on velocities, it is generally necessary to solve for the velocities before the acceleration calculations can be made. Choose the reference point in the relative-acceleration equation as some point on the body in question whose acceleration is either known or can be easily found. Be careful *not* to use the instantaneous center of zero velocity as the reference point unless its acceleration is known and accounted for.

An instantaneous center of zero acceleration exists for a rigid body in general plane motion, but will not be discussed here since its use is somewhat specialized.

SAMPLE PROBLEM 5/13

The wheel of radius r rolls to the left without slipping and, at the instant considered, the center O has a velocity $\mathbf{v}_O$ and an acceleration $\mathbf{a}_O$ to the left. Determine the acceleration of points A and C on the wheel for the instant considered.



Solution. From our previous analysis of Sample Problem 5/4, we know that the angular velocity and angular acceleration of the wheel are

$$\omega = v_O/r \qquad \text{and} \qquad \alpha = a_O/r$$

The acceleration of A is written in terms of the given acceleration of O. Thus,

$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{A/O} = \mathbf{a}_O + (\mathbf{a}_{A/O})_n + (\mathbf{a}_{A/O})_t$$

The relative-acceleration terms are viewed as though O were fixed, and for this relative circular motion they have the magnitudes

$$(a_{A/O})_n = r_0\omega^2 = r_0\left(\frac{v_O}{r}\right)^2$$

$$(a_{A/O})_t = r_0\alpha = r_0\left(\frac{a_O}{r}\right)$$

1 and the directions shown.

Adding the vectors head-to-tail gives $\mathbf{a}_A$ as shown. In a numerical problem, we may obtain the combination algebraically or graphically. The algebraic expression for the magnitude of $\mathbf{a}_A$ is found from the square root of the sum of the squares of its components. If we use n- and t-directions, we have

2
$$\begin{aligned} a_A &= \sqrt{(a_A)_n{}^2 + (a_A)_t{}^2} \\ &= \sqrt{[a_O \cos\theta + (a_{A/O})_n]^2 + [a_O \sin\theta + (a_{A/O})_t]^2} \\ &= \sqrt{(r\alpha \cos\theta + r_0\omega^2)^2 + (r\alpha \sin\theta + r_0\alpha)^2} \qquad \textit{Ans.} \end{aligned}$$

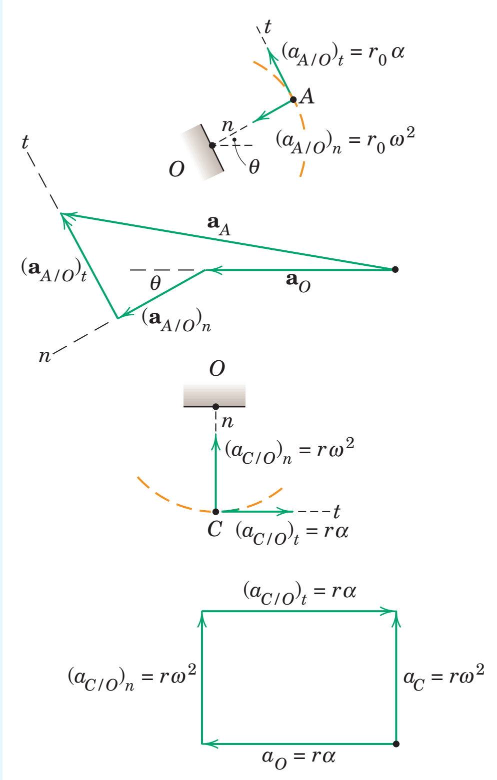
The direction of $\mathbf{a}_A$ can be computed if desired.

The acceleration of the instantaneous center C of zero velocity, considered a point on the wheel, is obtained from the expression

$$\mathbf{a}_C = \mathbf{a}_O + \mathbf{a}_{C/O}$$

where the components of the relative-acceleration term are $(a_{C/O})_n = r\omega^2$ directed from C to O and $(a_{C/O})_t = r\alpha$ directed to the right because of the counterclockwise angular acceleration of line CO about O. The terms are added together in the lower diagram and it is seen that

3
$$a_C = r\omega^2 \qquad \textit{Ans.}$$



Helpful Hints

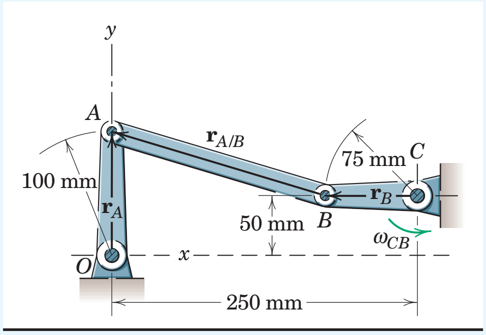
1 The counterclockwise angular acceleration α of OA determines the positive direction of $(a_{A/O})_t$. The normal component $(a_{A/O})_n$ is, of course, directed toward the reference center O.

2 If the wheel were rolling to the right with the same velocity v_O but still had an acceleration a_O to the left, note that the solution for a_A would be unchanged.

3 We note that the acceleration of the instantaneous center of zero velocity is independent of α and is directed toward the center of the wheel. This conclusion is a useful result to remember.

SAMPLE PROBLEM 5/14

The linkage of Sample Problem 5/8 is repeated here. Crank CB has a constant counterclockwise angular velocity of 2 rad/s in the position shown during a short interval of its motion. Determine the angular acceleration of links AB and OA for this position. Solve by using vector algebra.



Solution. We first solve for the velocities which were obtained in Sample Problem 5/8. They are

$$\omega_{AB} = -6/7 \text{ rad/s} \qquad \text{and} \qquad \omega_{OA} = -3/7 \text{ rad/s}$$

where the counterclockwise direction (+**k**-direction) is taken as positive. The acceleration equation is

$$\mathbf{a}_A = \mathbf{a}_B + (\mathbf{a}_{A/B})_n + (\mathbf{a}_{A/B})_t$$

where, from Eqs. 5/3 and 5/9a, we may write

1
$$\begin{aligned}
\mathbf{a}_A &= \boldsymbol{\alpha}_{OA} \times \mathbf{r}_A + \boldsymbol{\omega}_{OA} \times (\boldsymbol{\omega}_{OA} \times \mathbf{r}_A) \\
&= \alpha_{OA}\mathbf{k} \times 100\mathbf{j} + (-\tfrac{3}{7}\mathbf{k}) \times (-\tfrac{3}{7}\mathbf{k} \times 100\mathbf{j}) \\
&= -100\alpha_{OA}\mathbf{i} - 100(\tfrac{3}{7})^2\mathbf{j} \text{ mm/s}^2
\end{aligned}$$

$$\begin{aligned}
\mathbf{a}_B &= \boldsymbol{\alpha}_{CB} \times \mathbf{r}_B + \boldsymbol{\omega}_{CB} \times (\boldsymbol{\omega}_{CB} \times \mathbf{r}_B) \\
&= \mathbf{0} + 2\mathbf{k} \times (2\mathbf{k} \times [-75\mathbf{i}]) \\
&= 300\mathbf{i} \text{ mm/s}^2
\end{aligned}$$

2
$$\begin{aligned}
(\mathbf{a}_{A/B})_n &= \boldsymbol{\omega}_{AB} \times (\boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B}) \\
&= -\tfrac{6}{7}\mathbf{k} \times [(-\tfrac{6}{7}\mathbf{k}) \times (-175\mathbf{i} + 50\mathbf{j})] \\
&= (\tfrac{6}{7})^2(175\mathbf{i} - 50\mathbf{j}) \text{ mm/s}^2
\end{aligned}$$

$$\begin{aligned}
(\mathbf{a}_{A/B})_t &= \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{A/B} \\
&= \alpha_{AB}\mathbf{k} \times (-175\mathbf{i} + 50\mathbf{j}) \\
&= -50\alpha_{AB}\mathbf{i} - 175\alpha_{AB}\mathbf{j} \text{ mm/s}^2
\end{aligned}$$

We now substitute these results into the relative-acceleration equation and equate separately the coefficients of the **i**-terms and the coefficients of the **j**-terms to give

$$-100\alpha_{OA} = 429 - 50\alpha_{AB}$$
$$-18.37 = -36.7 - 175\alpha_{AB}$$

The solutions are

$$\alpha_{AB} = -0.1050 \text{ rad/s}^2 \qquad \text{and} \qquad \alpha_{OA} = -4.34 \text{ rad/s}^2 \qquad \textit{Ans.}$$

Since the unit vector **k** points out from the paper in the positive z-direction, we see that the angular accelerations of AB and OA are both clockwise (negative).

It is recommended that the student sketch each of the acceleration vectors in its proper geometric relationship according to the relative-acceleration equation to help clarify the meaning of the solution.

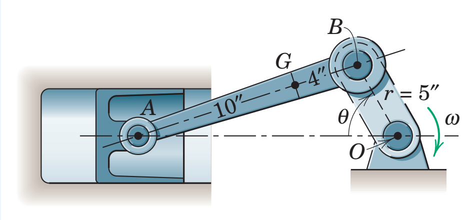
Helpful Hints

1 Remember to preserve the order of the factors in the cross products.

2 In expressing the term $\mathbf{a}_{A/B}$ be certain that $\mathbf{r}_{A/B}$ is written as the vector from B to A and not the reverse.

SAMPLE PROBLEM 5/15

The slider-crank mechanism of Sample Problem 5/9 is repeated here. The crank OB has a constant clockwise angular speed of 1500 rev/min. For the instant when the crank angle θ is 60°, determine the acceleration of the piston A and the angular acceleration of the connecting rod AB.



Solution. The acceleration of A may be expressed in terms of the acceleration of the crank pin B. Thus,

$$\mathbf{a}_A = \mathbf{a}_B + (\mathbf{a}_{A/B})_n + (\mathbf{a}_{A/B})_t$$

1 Point B moves in a circle of 5-in. radius with a constant speed so that it has only a normal component of acceleration directed from B to O.

$$[a_n = r\omega^2] \qquad a_B = \frac{5}{12}\left(\frac{1500[2\pi]}{60}\right)^2 = 10{,}280 \text{ ft/sec}^2$$

The relative-acceleration terms are visualized with A rotating in a circle relative to B, which is considered fixed, as shown. From Sample Problem 5/9, the angular velocity of AB for these same conditions is $\omega_{AB} = 29.5$ rad/sec so that

2 $$[a_n = r\omega^2] \qquad (a_{A/B})_n = \frac{14}{12}(29.5)^2 = 1015 \text{ ft/sec}^2$$

directed from A to B. The tangential component $(\mathbf{a}_{A/B})_t$ is known in direction only since its magnitude depends on the unknown angular acceleration of AB. We also know the direction of $\mathbf{a}_A$ since the piston is confined to move along the horizontal axis of the cylinder. There are now only two scalar unknowns left in the equation, namely, the magnitudes of $\mathbf{a}_A$ and $(\mathbf{a}_{A/B})_t$, so the solution can be carried out.

If we adopt an algebraic solution using the geometry of the acceleration polygon, we first compute the angle between AB and the horizontal. With the law of sines, this angle becomes 18.02°. Equating separately the horizontal components and the vertical components of the terms in the acceleration equation, as seen from the acceleration polygon, gives

$$a_A = 10{,}280 \cos 60^\circ + 1015 \cos 18.02^\circ - (a_{A/B})_t \sin 18.02^\circ$$

$$0 = 10{,}280 \sin 60^\circ - 1015 \sin 18.02^\circ - (a_{A/B})_t \cos 18.02^\circ$$

The solution to these equations gives the magnitudes

$$(a_{A/B})_t = 9030 \text{ ft/sec}^2 \qquad \text{and} \qquad a_A = 3310 \text{ ft/sec}^2 \qquad \textit{Ans.}$$

With the sense of $(\mathbf{a}_{A/B})_t$ also determined from the diagram, the angular acceleration of AB is seen from the figure representing rotation relative to B to be

$$[\alpha = a_t/r] \qquad \alpha_{AB} = 9030/(14/12) = 7740 \text{ rad/sec}^2 \text{ clockwise} \qquad \textit{Ans.}$$

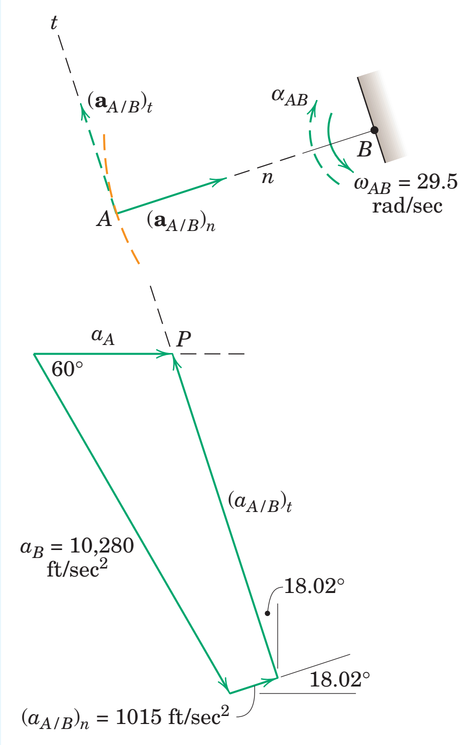
3 If we adopt a graphical solution, we begin with the known vectors $\mathbf{a}_B$ and $(\mathbf{a}_{A/B})_n$ and add them head-to-tail using a convenient scale. Next we construct the direction of $(\mathbf{a}_{A/B})_t$ through the head of the last vector. The solution of the equation is obtained by the intersection P of this last line with a horizontal line through the starting point representing the known direction of the vector sum $\mathbf{a}_A$. Scaling the magnitudes from the diagram gives values which agree with the calculated results.

$$a_A = 3310 \text{ ft/sec}^2 \qquad \text{and} \qquad (a_{A/B})_t = 9030 \text{ ft/sec}^2 \qquad \textit{Ans.}$$

Helpful Hints

1 If the crank OB had an angular acceleration, $\mathbf{a}_B$ would also have a tangential component of acceleration.

2 Alternatively, the relation $a_n = v^2/r$ may be used for calculating $(a_{A/B})_n$, provided the relative velocity $v_{A/B}$ is used for v. The equivalence is easily seen when it is recalled that $v_{A/B} = r\omega$.



3 Except where extreme accuracy is required, do not hesitate to use a graphical solution, as it is quick and reveals the physical relationships among the vectors. The known vectors, of course, may be added in any order as long as the governing equation is satisfied.

5/5 Instantaneous Center of Zero Velocity

In the previous article, we determined the velocity of a point on a rigid body in plane motion by adding the relative velocity due to rotation about a convenient reference point to the velocity of the reference point. We now solve the problem by choosing a unique reference point which momentarily has zero velocity. As far as velocities are concerned, the body may be considered to be in pure rotation about an axis, normal to the plane of motion, passing through this point. This axis is called the *instantaneous axis of zero velocity*, and the intersection of this axis with the plane of motion is known as the *instantaneous center of zero velocity*. This approach provides us with a valuable means for visualizing and analyzing velocities in plane motion.

Locating the Instantaneous Center

The existence of the instantaneous center is easily shown. For the body in Fig. 5/7, assume that the directions of the absolute velocities of any two points A and B on the body are known and are not parallel. If there is a point about which A has absolute circular motion at the instant considered, this point must lie on the normal to $\mathbf{v}_A$ through A. Similar reasoning applies to B, and the intersection of the two perpendiculars fulfills the requirement for an absolute center of rotation *at the instant considered*. Point C is the instantaneous center of zero velocity and may lie on or off the body. If it lies off the body, it may be visualized as lying on an imaginary extension of the body. The instantaneous center need not be a fixed point in the body or a fixed point in the plane.

If we also know the magnitude of the velocity of one of the points, say, v_A, we may easily obtain the angular velocity ω of the body and the linear velocity of every point in the body. Thus, the angular velocity of the body, Fig. 5/7*a*, is

$$\omega = \frac{v_A}{r_A}$$

which, of course, is also the angular velocity of *every* line in the body. Therefore, the velocity of B is $v_B = r_B\omega = (r_B/r_A)v_A$. Once the instantaneous center is located, the direction of the instantaneous velocity of

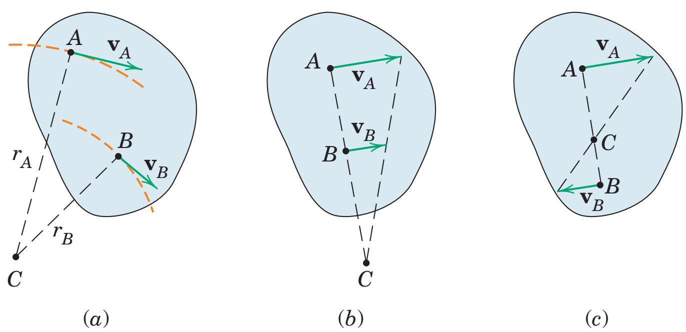


Figure 5/7

every point in the body is readily found since it must be perpendicular to the radial line joining the point in question with C.

If the velocities of two points in a body having plane motion are parallel, Fig. 5/7*b* or 5/7*c*, and the line joining the points is perpendicular to the direction of the velocities, the instantaneous center is located by direct proportion as shown. We can readily see from Fig. 5/7*b* that as the parallel velocities become equal in magnitude, the instantaneous center moves farther away from the body and approaches infinity in the limit as the body stops rotating and translates only.

Motion of the Instantaneous Center

As the body changes its position, the instantaneous center C also changes its position both in space and on the body. The locus of the instantaneous centers in space is known as the *space centrode*, and the locus of the positions of the instantaneous centers on the body is known as the *body centrode*. At the instant considered, the two curves are tangent at the position of point C. It can be shown that the body-centrode curve rolls on the space-centrode curve during the motion of the body, as indicated schematically in Fig. 5/8.

Although the instantaneous center of zero velocity is momentarily at rest, its acceleration generally is *not* zero. Thus, this point may *not* be used as an instantaneous center of zero acceleration in a manner analogous to its use for finding velocity. An instantaneous center of zero acceleration does exist for bodies in general plane motion, but its location and use represent a specialized topic in mechanism kinematics and will not be treated here.

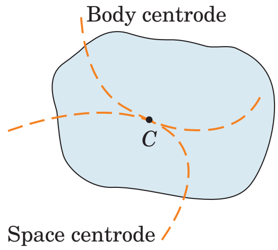
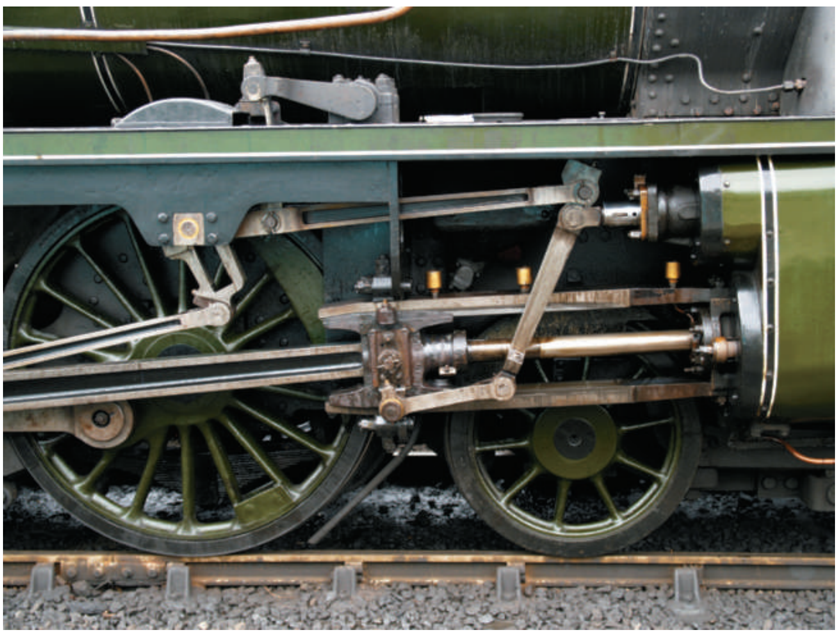


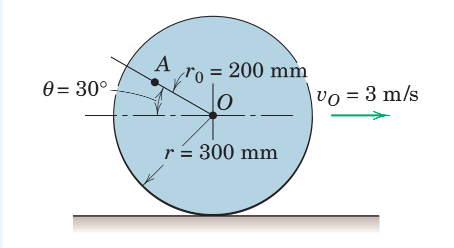
Figure 5/8

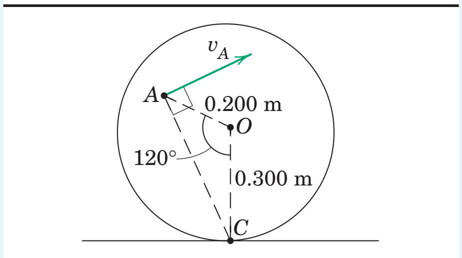


This valve gear of a steam locomotive provides an interesting (albeit not cutting-edge) study in rigid-body kinematics.

SAMPLE PROBLEM 5/11

The wheel of Sample Problem 5/7, shown again here, rolls to the right without slipping, with its center O having a velocity $v_O = 3$ m/s. Locate the instantaneous center of zero velocity and use it to find the velocity of point A for the position indicated.





Solution. The point on the rim of the wheel in contact with the ground has no velocity if the wheel is not slipping; it is, therefore, the instantaneous center C of zero velocity. The angular velocity of the wheel becomes

$[\omega = v/r]$ $\qquad \omega = v_O / \overline{OC} = 3/0.300 = 10$ rad/s

The distance from A to C is

① $$\overline{AC} = \sqrt{(0.300)^2 + (0.200)^2 - 2(0.300)(0.200)\cos 120°} = 0.436 \text{ m}$$

The velocity of A becomes

② $[v = r\omega]$ $\qquad v_A = \overline{AC}\omega = 0.436(10) = 4.36$ m/s $\qquad$ *Ans.*

The direction of $\mathbf{v}_A$ is perpendicular to AC as shown.

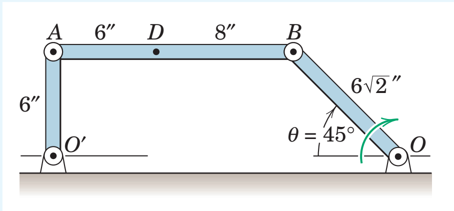
Helpful Hints

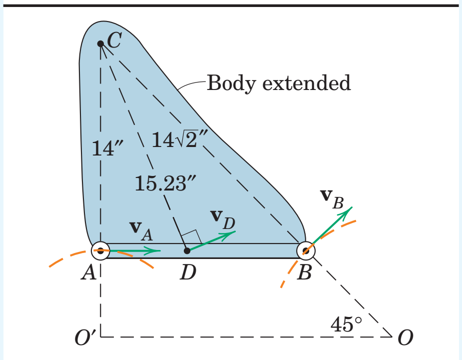
① Be sure to recognize that the cosine of 120° is itself negative.

② From the results of this problem, you should be able to visualize and sketch the velocities of all points on the wheel.

SAMPLE PROBLEM 5/12

Arm OB of the linkage has a clockwise angular velocity of 10 rad/sec in the position shown where $\theta = 45°$. Determine the velocity of A, the velocity of D, and the angular velocity of link AB for the position shown.





Solution. The directions of the velocities of A and B are tangent to their circular paths about the fixed centers O' and O as shown. The intersection of the two ① perpendiculars to the velocities from A and B locates the instantaneous center C for the link AB. The distances $\overline{AC}$, $\overline{BC}$, and $\overline{DC}$ shown on the diagram are computed or scaled from the drawing. The angular velocity of BC, considered a line on the body extended, is equal to the angular velocity of AC, DC, and AB and is

$[\omega = v/r]$ $$\omega_{BC} = \frac{v_B}{\overline{BC}} = \frac{\overline{OB}\omega_{OB}}{\overline{BC}} = \frac{6\sqrt{2}(10)}{14\sqrt{2}}$$

$$= 4.29 \text{ rad/sec CCW} \qquad \textit{Ans.}$$

Thus, the velocities of A and D are

$[v = r\omega]$ $$v_A = \frac{14}{12}(4.29) = 5.00 \text{ ft/sec} \qquad \textit{Ans.}$$

$$v_D = \frac{15.23}{12}(4.29) = 5.44 \text{ ft/sec} \qquad \textit{Ans.}$$

in the directions shown.

Helpful Hint

① For the instant depicted, we should visualize link AB and its body extended to be rotating as a single unit about point C.