Plane Kinematics of Rigid Bodies

5

CHAPTER OUTLINE

5/1 Introduction

In Chapter 2 on particle kinematics, we developed the relationships governing the displacement, velocity, and acceleration of points as they moved along straight or curved paths. In rigid-body kinematics we use these same relationships but must also account for the rotational motion of the body. Thus rigid-body kinematics involves both linear and angular displacements, velocities, and accelerations.

We need to describe the motion of rigid bodies for two important reasons. First, we frequently need to generate, transmit, or control certain motions by the use of cams, gears, and linkages of various types. Here we must analyze the displacement, velocity, and acceleration of the motion to determine the design geometry of the mechanical parts. Furthermore, as a result of the motion generated, forces may be developed which must be accounted for in the design of the parts.

Second, we must often determine the motion of a rigid body caused by the forces applied to it. Calculation of the motion of a rocket under the influence of its thrust and gravitational attraction is an example of such a problem.

We need to apply the principles of rigid-body kinematics in both situations. This chapter covers the kinematics of rigid-body motion which may be analyzed as occurring in a single plane. In Chapter 7 we will present an introduction to the kinematics of motion in three dimensions.

Rigid-Body Assumption

In the previous chapter we defined a *rigid body* as a system of particles for which the distances between the particles remain unchanged. Thus, if each particle of such a body is located by a position vector from reference axes attached to and rotating with the body, there will be no change in any position vector as measured from these axes. This is, of course, an ideal case since all solid materials change shape to some extent when forces are applied to them.

Nevertheless, if the movements associated with the changes in shape are very small compared with the movements of the body as a whole, then the assumption of rigidity is usually acceptable. The displacements due to the flutter of an aircraft wing, for instance, do not affect the description of the flight path of the aircraft as a whole, and thus the rigid-body assumption is clearly acceptable. On the other hand, if the problem is one of describing, as a function of time, the internal wing stress due to wing flutter, then the relative motions of portions of the wing cannot be neglected, and the wing may not be considered a rigid body. In this and the next two chapters, almost all of the material is based on the assumption of rigidity.

Plane Motion

A rigid body executes plane motion when all parts of the body move in parallel planes. For convenience, we generally consider the *plane of motion* to be the plane which contains the mass center, and we treat the body as a thin slab whose motion is confined to the plane of the slab. This idealization adequately describes a very large category of rigid-body motions encountered in engineering.

The plane motion of a rigid body may be divided into several categories, as represented in Fig. 5/1.

Translation is defined as any motion in which every line in the body remains parallel to its original position at all times. In translation there is *no rotation of any line in the body*. In *rectilinear translation*, part *a* of Fig. 5/1, all points in the body move in parallel straight lines. In *curvilinear translation*, part *b*, all points move on congruent curves. We note that in each of the two cases of translation, the motion of the body is completely specified by the motion of any point in the body, since all points have the same motion. Thus, our earlier study of the motion of a point (particle) in Chapter 2 enables us to describe completely the translation of a rigid body.

Rotation about a fixed axis, part *c* of Fig. 5/1, is the angular motion about the axis. It follows that all particles in a rigid body move in circular paths about the axis of rotation, and all lines in the body which are perpendicular to the axis of rotation (including those which do not pass through the axis) rotate through the same angle in the same time. Again, our discussion in Chapter 2 on the circular motion of a point enables us to describe the motion of a rotating rigid body, which is treated in the next article.

General plane motion of a rigid body, part *d* of Fig. 5/1, is a combination of translation and rotation. We will utilize the principles of relative motion covered in Art. 2/8 to describe general plane motion.

David Parker/Photo Researchers, Inc.

These nickel microgears are only 150 microns ($150(10^{-6})$ m) thick and have potential application in microscopic robots.

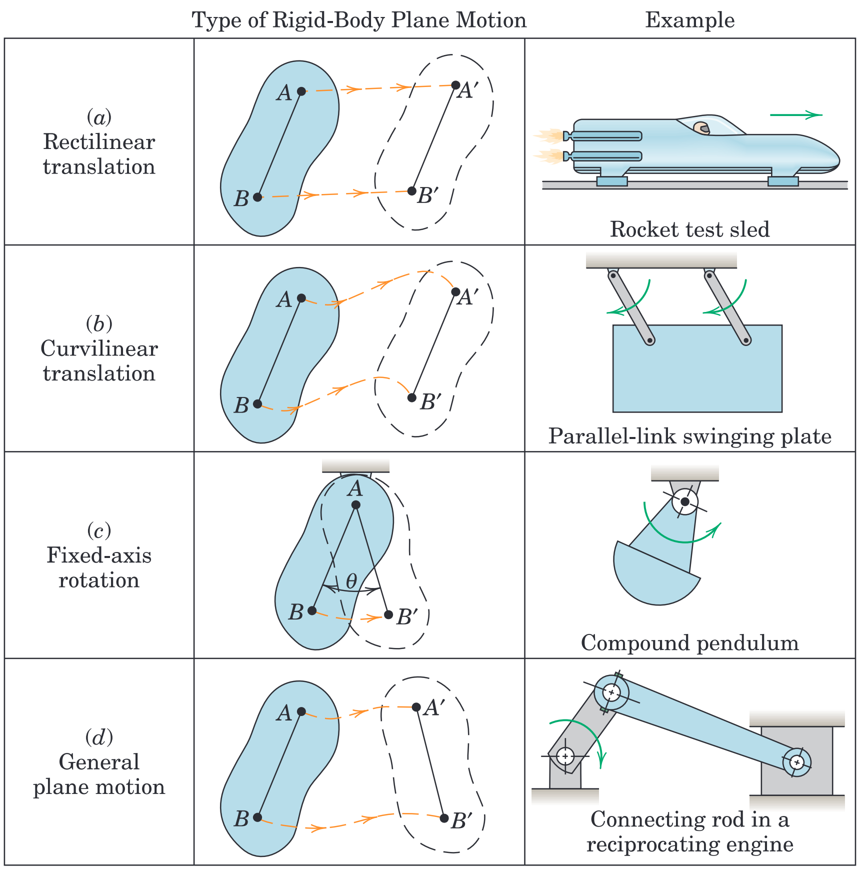


Figure 5/1

Note that in each of the examples cited, the actual paths of all particles in the body are projected onto the single plane of motion as represented in each figure.

Analysis of the plane motion of rigid bodies is accomplished either by directly calculating the absolute displacements and their time derivatives from the geometry involved or by utilizing the principles of relative motion. Each method is important and useful and will be covered in turn in the articles which follow.

5/2 Rotation

The rotation of a rigid body is described by its angular motion. Figure 5/2 shows a rigid body which is rotating as it undergoes plane motion in the plane of the figure. The angular positions of any two lines 1 and 2 attached to the body are specified by θ_1 and θ_2 measured from any convenient fixed reference direction. Because the angle β is invariant, the relation $\theta_2 = \theta_1 + \beta$ upon differentiation with respect to time gives $\dot{\theta}_2 = \dot{\theta}_1$ and $\ddot{\theta}_2 = \ddot{\theta}_1$ or, during a finite interval, $\Delta\theta_2 = \Delta\theta_1$. Thus, *all lines on a rigid body in its plane of motion have the same angular displacement, the same angular velocity, and the same angular acceleration*.

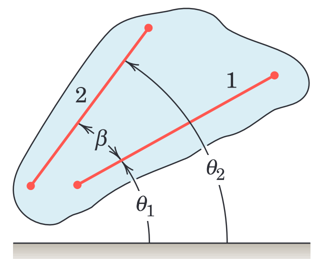


Figure 5/2

Note that the angular motion of a line depends only on its angular position with respect to any arbitrary fixed reference and on the time derivatives of the displacement. Angular motion does not require the presence of a fixed axis, normal to the plane of motion, about which the line and the body rotate.

KEY CONCEPTS

Angular-Motion Relations

The angular velocity ω and angular acceleration α of a rigid body in plane rotation are, respectively, the first and second time derivatives of the angular position coordinate θ of any line in the plane of motion of the body. These definitions give

$$\omega = \frac{d\theta}{dt} = \dot{\theta}$$

$$\alpha = \frac{d\omega}{dt} = \dot{\omega} \qquad \text{or} \qquad \alpha = \frac{d^2\theta}{dt^2} = \ddot{\theta} \tag{5/1}$$

$$\omega\, d\omega = \alpha\, d\theta \qquad \text{or} \qquad \dot{\theta}\, d\dot{\theta} = \ddot{\theta}\, d\theta$$

The third relation is obtained by eliminating dt from the first two. In each of these relations, the positive direction for ω and α, clockwise or counterclockwise, is the same as that chosen for θ. Equations 5/1 should be recognized as analogous to the defining equations for the rectilinear motion of a particle, expressed by Eqs. 2/1, 2/2, and 2/3. In fact, all relations which were described for rectilinear motion in Art. 2/2 apply to the case of rotation in a plane if the linear quantities s, v, and a are replaced by their respective equivalent angular quantities θ, ω, and α. As we proceed further with rigid-body dynamics, we will find that the analogies between the relationships for linear and angular motion are almost complete throughout kinematics and kinetics. These relations are important to recognize, as they help to demonstrate the symmetry and unity found throughout mechanics.

For rotation with *constant* angular acceleration, the integrals of Eqs. 5/1 becomes

$$\omega = \omega_0 + \alpha t$$

$$\omega^2 = {\omega_0}^2 + 2\alpha(\theta - \theta_0)$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

Here θ_0 and ω_0 are the values of the angular position coordinate and angular velocity, respectively, at $t = 0$, and t is the duration of the motion considered. You should be able to carry out these integrations easily, as they are completely analogous to the corresponding equations for rectilinear motion with constant acceleration covered in Art. 2/2.

The graphical relationships described for s, v, a, and t in Figs. 2/3 and 2/4 may be used for θ, ω, and α merely by substituting the corresponding symbols. You should sketch these graphical relations for plane rotation. The mathematical procedures for obtaining rectilinear velocity and displacement from rectilinear acceleration may be applied to rotation by merely replacing the linear quantities by their corresponding angular quantities.

Rotation about a Fixed Axis

When a rigid body rotates about a fixed axis, all points other than those on the axis move in concentric circles about the fixed axis. Thus, for the rigid body in Fig. 5/3 rotating about a fixed axis normal to the plane of the figure through O, any point such as A moves in a circle of radius r. From the previous discussion in Art. 2/5, you should already be familiar with the relationships between the linear motion of A and the angular motion of the line normal to its path, which is also the angular motion of the rigid body. With the notation $\omega = \dot{\theta}$ and $\alpha = \dot{\omega} = \ddot{\theta}$ for the angular velocity and angular acceleration, respectively, of the body we have Eqs. 2/11, rewritten as

$$\begin{aligned} v &= r\omega \\ a_n &= r\omega^2 = v^2/r = v\omega \\ a_t &= r\alpha \end{aligned} \tag{5/2}$$

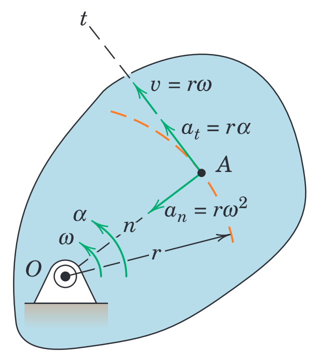


Figure 5/3

These quantities may be expressed alternatively using the cross-product relationship of vector notation. The vector formulation is especially important in the analysis of three-dimensional motion. The angular velocity of the rotating body may be expressed by the vector $\boldsymbol{\omega}$ normal to the plane of rotation and having a sense governed by the right-hand rule, as shown in Fig. 5/4*a*. From the definition of the vector cross product, we see that the vector $\mathbf{v}$ is obtained by crossing $\boldsymbol{\omega}$ into $\mathbf{r}$. This cross product gives the correct magnitude and direction for $\mathbf{v}$ and we write

$$\mathbf{v} = \dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$$

The order of the vectors to be crossed must be retained. The reverse order gives $\mathbf{r} \times \boldsymbol{\omega} = -\mathbf{v}$.

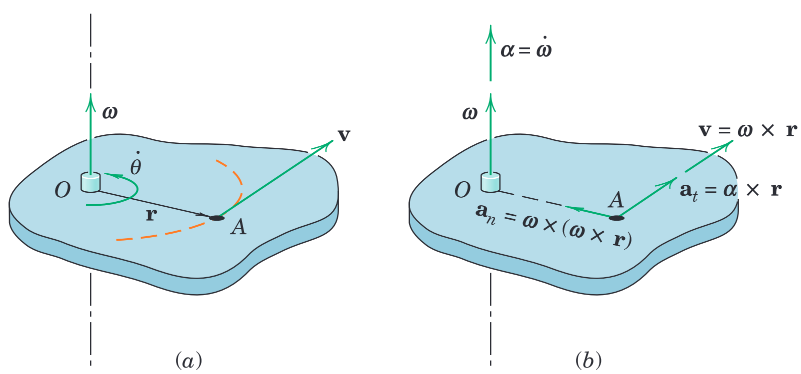
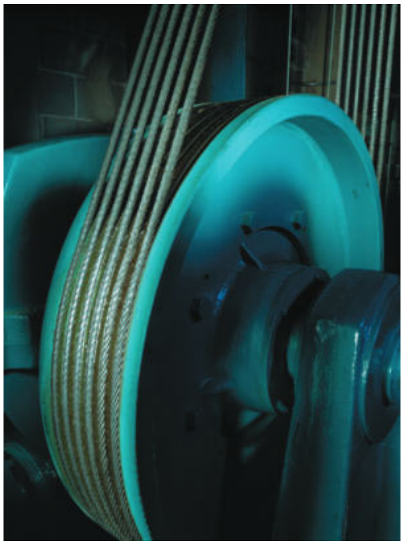


Figure 5/4



This pulley-cable system is part of an elevator mechanism.

The acceleration of point A is obtained by differentiating the cross-product expression for $\mathbf{v}$, which gives

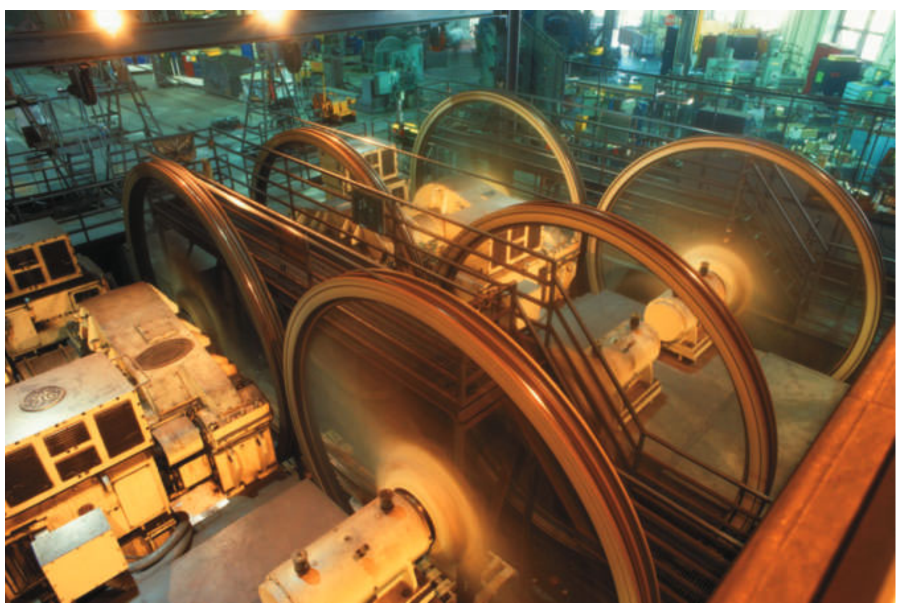
$$\begin{aligned}\mathbf{a} = \dot{\mathbf{v}} &= \boldsymbol{\omega} \times \dot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} \\ &= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r} \\ &= \boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\alpha} \times \mathbf{r}\end{aligned}$$

Here $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$ stands for the angular acceleration of the body. Thus, the vector equivalents to Eqs. 5/2 are

$$\begin{aligned}\mathbf{v} &= \boldsymbol{\omega} \times \mathbf{r} \\ \mathbf{a}_n &= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \\ \mathbf{a}_t &= \boldsymbol{\alpha} \times \mathbf{r}\end{aligned} \qquad \textbf{(5/3)}$$

and are shown in Fig. 5/4*b*.

For three-dimensional motion of a rigid body, the angular-velocity vector $\boldsymbol{\omega}$ may change direction as well as magnitude, and in this case, the angular acceleration, which is the time derivative of angular velocity, $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$, will no longer be in the same direction as $\boldsymbol{\omega}$.



These pulleys and cables are part of the San Francisco cable-car system.

5/3 Absolute Motion

We now develop the approach of absolute-motion analysis to describe the plane kinematics of rigid bodies. In this approach, we make use of the geometric relations which define the configuration of the body involved and then proceed to take the time derivatives of the defining geometric relations to obtain velocities and accelerations.

In Art. 2/9 of Chapter 2 on particle kinematics, we introduced the application of absolute-motion analysis for the constrained motion of connected particles. For the pulley configurations treated, the relevant velocities and accelerations were determined by successive differentiation of the lengths of the connecting cables. In this earlier treatment, the geometric relations were quite simple, and no angular quantities had to be considered. Now that we will be dealing with rigid-body motion, however, we find that our defining geometric relations include both linear and angular variables and, therefore, the time derivatives of these quantities will involve both linear and angular velocities and linear and angular accelerations.

Ski-lift pulley tower near the Matterhorn in Switzerland.

In absolute-motion analysis, it is essential that we be consistent with the mathematics of the description. For example, if the angular position of a moving line in the plane of motion is specified by its counterclockwise angle θ measured from some convenient fixed reference axis, then the positive sense for both angular velocity $\dot{\theta}$ and angular acceleration $\ddot{\theta}$ will also be counterclockwise. A negative sign for either quantity will, of course, indicate a clockwise angular motion. The defining relations for linear motion, Eqs. 2/1, 2/2, and 2/3, and the relations involving angular motion, Eqs. 5/1 and 5/2 or 5/3, will find repeated use in the motion analysis and should be mastered.

The absolute-motion approach to rigid-body kinematics is quite straightforward, provided the configuration lends itself to a geometric description which is not overly complex. If the geometric configuration is awkward or complex, analysis by the principles of relative motion may be preferable. Relative-motion analysis is treated in this chapter beginning with Art. 5/4. The choice between absolute- and relative-motion analyses is best made after experience has been gained with both approaches.

The next three sample problems illustrate the application of absolute-motion analysis to three commonly encountered situations. The kinematics of a rolling wheel, treated in Sample Problem 5/4, is especially important and will be useful in much of the problem work because the rolling wheel in various forms is such a common element in mechanical systems.