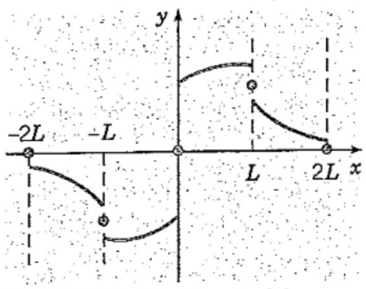

FIGURE 10.4.6 Graph of the function in Problem 38.

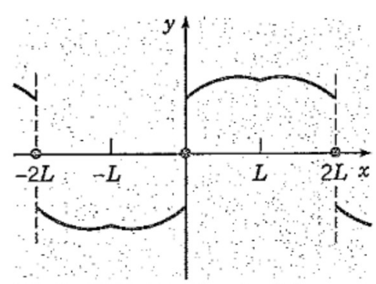

FIGURE 10.4.7 Graph of the function in Problem 39.

40. How should f, originally defined on $[0, L]$, be extended so as to obtain a Fourier series involving only the functions $\cos(\pi x/2L), \cos(3\pi x/2L), \cos(5\pi x/2L), \ldots$? Refer to Problems 38 and 39. If $f(x) = x$ for $0 \le x \le L$, sketch the function to which the Fourier series converges for $-4L \le x \le 4L$.

10.5 Separation of Variables; Heat Conduction in a Rod

The basic partial differential equations of heat conduction, wave propagation, and potential theory that we discuss in this chapter are associated with three distinct types of physical phenomena: diffusive processes, oscillatory processes, and time-independent or steady processes. Consequently, they are of fundamental importance in many branches of physics. They are also of considerable significance from a mathematical point of view. The partial differential equations whose theory is best developed and whose applications are most significant and varied are the linear equations of second order. All such equations can be classified into one of three categories: The heat conduction equation, the wave equation, and the potential equation, respectively, are prototypes of each of these categories. Thus a study of these three equations yields much information about more general second order linear partial differential equations.

During the last two centuries several methods have been developed for solving partial differential equations. The method of separation of variables is the oldest systematic method, having been used by D'Alembert, Daniel Bernoulli, and Euler about 1750 in their investigations of waves and vibrations. It has been considerably refined and generalized in the meantime, and it remains a method of great importance and frequent use today. To show how the method of separation of variables works we consider first a basic problem of heat conduction in a solid body. The mathematical study of heat conduction originated[9] about 1800, and it continues to command the

[9]The first important investigation of heat conduction was carried out by Joseph Fourier (1768–1830) while he was serving as prefect of the department of Isère (Grenoble) from 1801 to 1815. He presented basic papers on the subject to the Academy of Sciences of Paris in 1807 and 1811. However, these papers were criticized by the referees (principally Lagrange) for lack of rigor and so were not published. Fourier continued to develop his ideas and eventually wrote one of the classics of applied mathematics, *Théorie analytique de la chaleur*, published in 1822.

attention of modern scientists. For example, analysis of the dissipation and transfer of heat away from its sources in high-speed machinery is frequently an important technological problem.

Let us now consider a heat conduction problem for a straight bar of uniform cross section and homogeneous material. Let the x-axis be chosen to lie along the axis of the bar, and let $x = 0$ and $x = L$ denote the ends of the bar (see Figure 10.5.1). Suppose further that the sides of the bar are perfectly insulated so that no heat passes through them. We also assume that the cross-sectional dimensions are so small that the temperature u can be considered constant on any given cross section. Then u is a function only of the axial coordinate x and the time t.

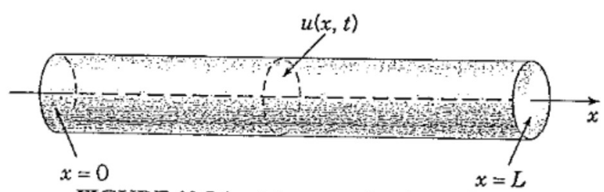

FIGURE 10.5.1 A heat-conducting solid bar.

The variation of temperature in the bar is governed by a partial differential equation whose derivation appears in Appendix A at the end of this chapter. The equation is called the **heat conduction equation** and has the form

$$\alpha^2 u_{xx} = u_t, \qquad 0 < x < L, \qquad t > 0, \tag{1}$$

where α^2 is a constant known as the **thermal diffusivity**. The parameter α^2 depends only on the material from which the bar is made and is defined by

$$\alpha^2 = \kappa/\rho s, \tag{2}$$

where κ is the thermal conductivity, ρ is the density, and s is the specific heat of the material in the bar. The units of α^2 are (length)^2/time. Typical values of α^2 are given in Table 10.5.1.

TABLE 10.5.1 Values of the Thermal Diffusivity for Some Common Materials

Material	α^2 (cm^2/sec)
Silver	1.71
Copper	1.14
Aluminum	0.86
Cast iron	0.12
Granite	0.011
Brick	0.0038
Water	0.00144

In addition, we assume that the initial temperature distribution in the bar is given; thus

$$u(x,0) = f(x), \qquad 0 \le x \le L, \tag{3}$$

where f is a given function. Finally, we assume that the ends of the bar are held at fixed temperatures: the temperature T_1 at $x = 0$ and the temperature T_2 at $x = L$. However, it turns out that we need only consider the case where $T_1 = T_2 = 0$. We show in Section 10.6 how to reduce the more general problem to this special case. Thus in this section we will assume that u is always zero when $x = 0$ or $x = L$:

$$u(0,t) = 0, \qquad u(L,t) = 0, \qquad t > 0. \tag{4}$$

The fundamental problem of heat conduction is to find $u(x,t)$ that satisfies the differential equation (1) for $0 < x < L$ and for $t > 0$, the initial condition (3) when $t = 0$, and the boundary conditions (4) at $x = 0$ and $x = L$.

The problem described by Eqs. (1), (3), and (4) is an initial value problem in the time variable t; an initial condition is given and the differential equation governs what happens later. However, with respect to the space variable x, the problem is a boundary value problem; boundary conditions are imposed at each end of the bar and the differential equation describes the evolution of the temperature in the interval between them. Alternatively, we can consider the problem as a boundary value problem in the xt-plane (see Figure 10.5.2). The solution $u(x,t)$ of Eq. (1) is sought in the semi-infinite strip $0 < x < L$, $t > 0$, subject to the requirement that $u(x,t)$ must assume a prescribed value at each point on the boundary of this strip.

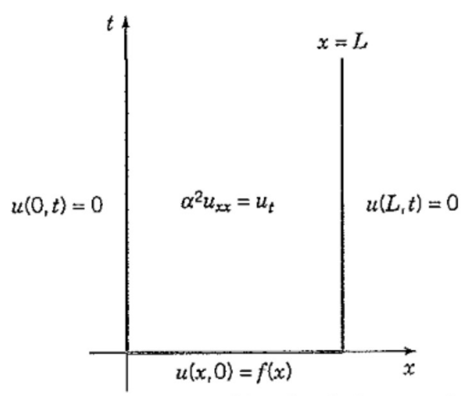

FIGURE 10.5.2 Boundary value problem for the heat conduction equation.

The heat conduction problem (1), (3), (4) is *linear* since u appears only to the first power throughout. The differential equation and boundary conditions are also *homogeneous*. This suggests that we might approach the problem by seeking solutions of the differential equation and boundary conditions, and then superposing them to satisfy the initial condition. The remainder of this section describes how this plan can be implemented.

One solution of the differential equation (1) that satisfies the boundary conditions (4) is the function $u(x,t) = 0$, but this solution does not satisfy the initial condition (3) except in the trivial case in which $f(x)$ is also zero. Thus our goal is to find other, nonzero solutions of the differential equation and boundary conditions. To find the needed solutions we start by making a basic assumption about the form of the solutions that has far-reaching, and perhaps unforeseen, consequences. The assumption is that $u(x,t)$ is a product of two other functions, one depending only on x and the other depending only on t; thus

$$u(x,t) = X(x)T(t). \tag{5}$$

Substituting from Eq. (5) for u in the differential equation (1) yields

$$\alpha^2 X''T = XT', \tag{6}$$

where primes refer to ordinary differentiation with respect to the independent variable, whether x or t. Equation (6) is equivalent to

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T}, \tag{7}$$

in which the variables are separated; that is, the left side depends only on x and the right side only on t. For Eq. (7) to be valid for $0 < x < L, t > 0$, it is necessary that both sides of Eq. (7) be equal to the same constant. Otherwise, if one independent variable (say x) were kept fixed and the other were allowed to vary, one side (the left in this case) of Eq. (7) would remain unchanged while the other varied, thus violating the equality. If we call this separation constant $-\lambda$, then Eq. (7) becomes

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda. \tag{8}$$

Hence we obtain the following two ordinary differential equations for $X(x)$ and $T(t)$:

$$X'' + \lambda X = 0, \tag{9}$$

$$T' + \alpha^2\lambda T = 0. \tag{10}$$

We denote the separation constant by $-\lambda$ (rather than λ) because it turns out that it must be negative, and it is convenient to exhibit the minus sign explicitly.

The assumption (5) has led to the replacement of the partial differential equation (1) by the two ordinary differential equations (9) and (10). Each of these equations can be readily solved for *any* value of λ. The product of two solutions of Eq. (9) and (10), respectively, provides a solution of the partial differential equation (1). However, we are interested only in those solutions of Eq. (1) that also satisfy the boundary conditions (4). As we now show, this severely restricts the possible values of λ.

Substituting for $u(x,t)$ from Eq. (5) in the boundary condition at $x = 0$, we obtain

$$u(0,t) = X(0)T(t) = 0. \tag{11}$$

If Eq. (11) is satisfied by choosing $T(t)$ to be zero for all t, then $u(x, t)$ is zero for all x and t, and we have already rejected this possibility. Therefore Eq. (11) must be satisfied by requiring that

$$X(0) = 0. \tag{12}$$

Similarly, the boundary condition at $x = L$ requires that

$$X(L) = 0. \tag{13}$$

We now want to consider Eq. (9) subject to the boundary conditions (12) and (13). This is an eigenvalue problem and, in fact, is the same problem that we discussed in detail at the end of Section 10.1; see especially the paragraph following Eq. (29) in that section. The only difference is that the dependent variable there was called y rather than X. If we refer to the results obtained earlier [Eq. (31) of Section 10.1], the only nontrivial solutions of Eqs. (9), (12), and (13) are the eigenfunctions

$$X_n(x) = \sin(n\pi x/L), \qquad n = 1, 2, 3, \dots \tag{14}$$

associated with the eigenvalues

$$\lambda_n = n^2\pi^2/L^2, \qquad n = 1, 2, 3, \dots. \tag{15}$$

Turning now to Eq. (10) for $T(t)$ and substituting $n^2\pi^2/L^2$ for λ, we have

$$T' + (n^2\pi^2\alpha^2/L^2)T = 0. \tag{16}$$

Thus $T(t)$ is proportional to $\exp(-n^2\pi^2\alpha^2 t/L^2)$. Hence, multiplying solutions of Eqs. (9) and (10) together, and neglecting arbitrary constants of proportionality, we conclude that the functions

$$u_n(x, t) = e^{-n^2\pi^2\alpha^2 t/L^2}\sin(n\pi x/L), \qquad n = 1, 2, 3, \dots \tag{17}$$

satisfy the partial differential equation (1) and the boundary conditions (4) for each positive integer value of n. The functions u_n are sometimes called fundamental solutions of the heat conduction problem (1), (3), and (4).

It remains only to satisfy the initial condition (3),

$$u(x, 0) = f(x), \qquad 0 \le x \le L. \tag{18}$$

Recall that we have often solved initial value problems by forming linear combinations of a set of fundamental solutions and then choosing the coefficients to satisfy the initial conditions. The analogous step in the present problem is to form a linear combination of the functions (17) and then to choose the coefficients to satisfy Eq. (18). The main difference from earlier problems is that there are infinitely many functions (17), so a general linear combination of them is an infinite series. Thus we assume that

$$u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2}\sin\frac{n\pi x}{L}, \tag{19}$$

where the coefficients c_n are as yet undetermined. The individual terms in the series (19) satisfy the differential equation (1) and boundary conditions (4). We will assume that the infinite series of Eq. (19) converges and also satisfies Eqs. (1) and (4). To satisfy the initial condition (3) we must have

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} = f(x). \tag{20}$$

In other words, we need to choose the coefficients c_n so that the series of sine functions in Eq. (20) converges to the initial temperature distribution $f(x)$ for $0 \le x \le L$. The series in Eq. (20) is just the Fourier sine series for f; according to Eq. (8) of Section 10.4 its coefficients are given by

$$c_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \tag{21}$$

Hence the solution of the heat conduction problem of Eqs. (1), (3), and (4) is given by the series in Eq. (19) with the coefficients computed from Eq. (21).

EXAMPLE 1

Find the temperature $u(x,t)$ at any time in a metal rod 50 cm long, insulated on the sides, which initially has a uniform temperature of 20°C throughout and whose ends are maintained at 0°C for all $t > 0$.

The temperature in the rod satisfies the heat conduction problem (1), (3), (4) with $L = 50$ and $f(x) = 20$ for $0 < x < 50$. Thus, from Eq. (19), the solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/2500} \sin \frac{n\pi x}{50}, \tag{22}$$

where, from Eq. (21),

$$\begin{aligned} c_n &= \frac{4}{5}\int_0^{50} \sin \frac{n\pi x}{50}\, dx \\ &= \frac{40}{n\pi}(1 - \cos n\pi) = \begin{cases} 80/n\pi, & n \text{ odd;} \\ 0, & n \text{ even.} \end{cases} \end{aligned} \tag{23}$$

Finally, by substituting for c_n in Eq. (22), we obtain

$$u(x,t) = \frac{80}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n} e^{-n^2\pi^2\alpha^2 t/2500} \sin \frac{n\pi x}{50}. \tag{24}$$

The expression (24) for the temperature is moderately complicated, but the negative exponential factor in each term of the series causes the series to converge quite rapidly, except for small values of t or α^2. Therefore accurate results can usually be obtained by using only a few terms of the series.

In order to display quantitative results, let us measure t in seconds; then α^2 has the units of cm²/sec. If we choose $\alpha^2 = 1$ for convenience, this corresponds to a rod of a material whose thermal properties are somewhere between copper and aluminum. The behavior of the solution can be seen from the graphs in Figures 10.5.3 through 10.5.5. In Figure 10.5.3 we show the temperature distribution in the bar at several different times. Observe that the temperature diminishes steadily as heat in the bar is lost through the end points. The way in which the temperature decays at a given point in the bar is indicated in Figure 10.5.4, where temperature is plotted against time for a few selected points in the bar. Finally, Figure 10.5.5 is a three-dimensional plot of u versus both x and t. Observe that we obtain the graphs in Figures 10.5.3 and 10.5.4 by intersecting the surface in Figure 10.5.5 by planes on which either t or x is constant. The slight waviness in Figure 10.5.5 at $t = 0$ results from using only a finite number of terms in the series for $u(x, t)$ and from the slow convergence of the series for $t = 0$.

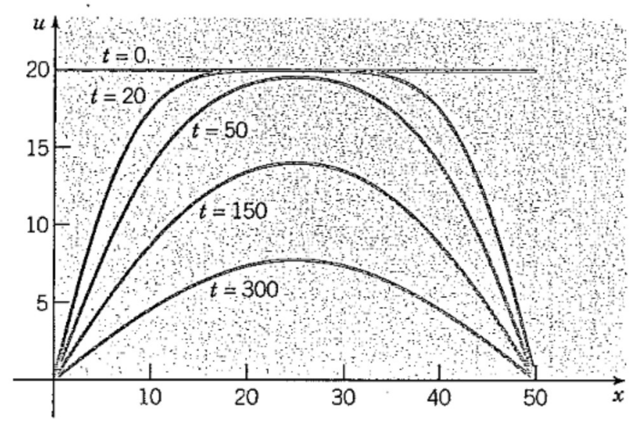

FIGURE 10.5.3 Temperature distributions at several times for the heat conduction problem of Example 1.

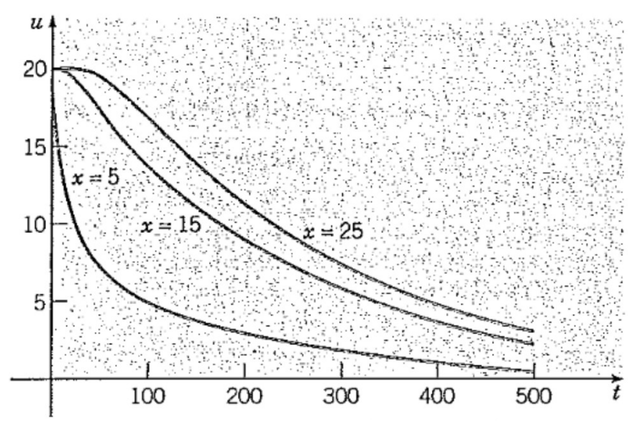

FIGURE 10.5.4 Dependence of temperature on time at several locations for the heat conduction problem of Example 1.

A problem with possible practical implications is to determine the time τ at which the entire bar has cooled to a specified temperature. For example, when is the temperature in the entire bar no greater than 1°C? Because of the symmetry of the initial temperature distribution and

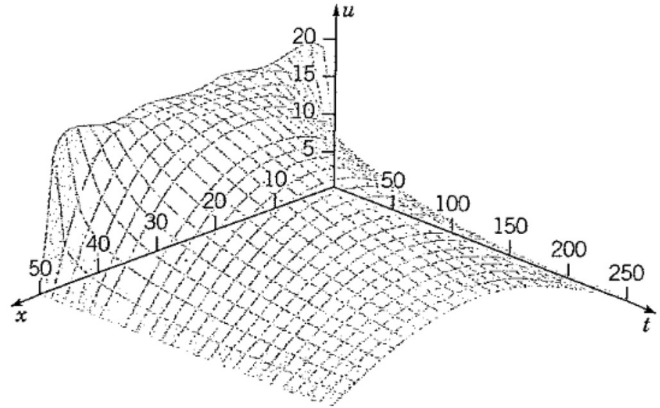

FIGURE 10.5.5 Plot of temperature u versus x and t for the heat conduction problem of Example 1.

the boundary conditions, the warmest point in the bar is always the center. Thus τ is found by solving $u(25, t) = 1$ for t. Using one term in the series expansion (24), we obtain

$$\tau = \frac{2500}{\pi^2} \ln(80/\pi) \cong 820 \text{ sec.}$$

PROBLEMS

In each of Problems 1 through 6 determine whether the method of separation of variables can be used to replace the given partial differential equation by a pair of ordinary differential equations. If so, find the equations.

1. $xu_{xx} + u_t = 0$
2. $tu_{xx} + xu_t = 0$
3. $u_{xx} + u_{xt} + u_t = 0$
4. $[p(x)u_x]_x - r(x)u_{tt} = 0$
5. $u_{xx} + (x + y)u_{yy} = 0$
6. $u_{xx} + u_{yy} + xu = 0$
7. Find the solution of the heat conduction problem

$$100u_{xx} = u_t, \quad 0 < x < 1, \quad t > 0;$$
$$u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0;$$
$$u(x, 0) = \sin 2\pi x - \sin 5\pi x, \quad 0 \le x \le 1.$$

8. Find the solution of the heat conduction problem

$$u_{xx} = 4u_t, \quad 0 < x < 2, \quad t > 0;$$
$$u(0, t) = 0, \quad u(2, t) = 0, \quad t > 0;$$
$$u(x, 0) = 2\sin(\pi x/2) - \sin \pi x + 4 \sin 2\pi x, \quad 0 \le x \le 2.$$

Consider the conduction of heat in a rod 40 cm in length whose ends are maintained at 0°C for all $t > 0$. In each of Problems 9 through 12 find an expression for the temperature $u(x, t)$ if the initial temperature distribution in the rod is the given function. Suppose that $\alpha^2 = 1$.

9. $u(x, 0) = 50, \quad 0 < x < 40$