22. (a) Suppose that $\alpha^2 = 1$ and $s(x) = k$, a constant, in Eq. (i). Find $v(x)$.
(b) Assume that $T_1 = 0, T_2 = 0, L = 20, k = 1/5$, and that $f(x) = 0$ for $0 < x < L$. Determine $w(x,t)$. Then plot $u(x,t)$ versus x for several values of t; on the same axes also plot the steady-state part of the solution $v(x)$.

23. (a) Let $\alpha^2 = 1$ and $s(x) = kx/L$, where k is a constant, in Eq. (i). Find $v(x)$.
(b) Assume that $T_1 = 10, T_2 = 30, L = 20, k = 1/2$, and that $f(x) = 0$ for $0 < x < L$. Determine $w(x,t)$. Then plot $u(x,t)$ versus x for several values of t; on the same axes also plot the steady-state part of the solution $v(x)$.

10.7 The Wave Equation: Vibrations of an Elastic String

A second partial differential equation that occurs frequently in applied mathematics is the wave[10] equation. Some form of this equation, or a generalization of it, almost inevitably arises in any mathematical analysis of phenomena involving the propagation of waves in a continuous medium. For example, the studies of acoustic waves, water waves, electromagnetic waves, and seismic waves are all based on this equation.

Perhaps the easiest situation to visualize occurs in the investigation of mechanical vibrations. Suppose that an elastic string of length L is tightly stretched between two supports at the same horizontal level, so that the x-axis lies along the string (see Figure 10.7.1). The elastic string may be thought of as a violin string, a guy wire, or possibly an electric power line. Suppose that the string is set in motion (by plucking, for example) so that it vibrates in a vertical plane, and let $u(x,t)$ denote the vertical displacement experienced by the string at the point x at time t. If damping effects, such as air resistance, are neglected, and if the amplitude of the motion is not too large, then $u(x,t)$ satisfies the partial differential equation

$$a^2 u_{xx} = u_{tt} \tag{1}$$

in the domain $0 < x < L, t > 0$. Equation (1) is known as the one-dimensional **wave equation** and is derived in Appendix B at the end of the chapter. The constant coefficient a^2 appearing in Eq. (1) is given by

$$a^2 = T/\rho, \tag{2}$$

where T is the tension (force) in the string, and ρ is the mass per unit length of the string material. It follows that a has the units of length/time—that is, of velocity. In Problem 14 it is shown that a is the velocity of propagation of waves along the string.

To describe the motion of the string completely it is necessary also to specify suitable initial and boundary conditions for the displacement $u(x,t)$. The ends are

[10]The solution of the wave equation was one of the major mathematical problems of the mid-eighteenth century. The wave equation was first derived and studied by D'Alembert in 1746. It also attracted the attention of Euler (1748), Daniel Bernoulli (1753), and Lagrange (1759). Solutions were obtained in several different forms, and the merits of, and relations among, these solutions were argued, sometimes heatedly, in a series of papers extending over more than 25 years. The major points at issue concerned the nature of a function and the kinds of functions that can be represented by trigonometric series. These questions were not resolved until the nineteenth century.

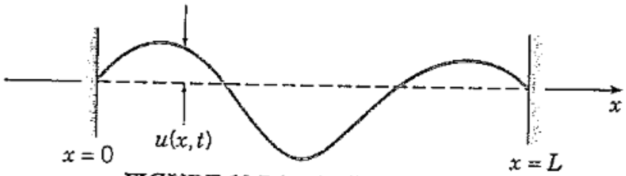

FIGURE 10.7.1 A vibrating string.

assumed to remain fixed, and therefore the boundary conditions are

$$u(0,t) = 0, \qquad u(L,t) = 0, \qquad t \geq 0. \tag{3}$$

Since the differential equation (1) is of second order with respect to t, it is plausible to prescribe two initial conditions. These are the initial position of the string,

$$u(x,0) = f(x), \qquad 0 \leq x \leq L, \tag{4}$$

and its initial velocity,

$$u_t(x,0) = g(x), \qquad 0 \leq x \leq L, \tag{5}$$

where f and g are given functions. In order for Eqs. (3), (4), and (5) to be consistent it is also necessary to require that

$$f(0) = f(L) = 0, \qquad g(0) = g(L) = 0. \tag{6}$$

The mathematical problem then is to determine the solution of the wave equation (1) that also satisfies the boundary conditions (3) and the initial conditions (4) and (5). Like the heat conduction problem of Sections 10.5 and 10.6, this problem is an initial value problem in the time variable t and a boundary value problem in the space variable x. Alternatively, it can be considered as a boundary value problem in the semi-infinite strip $0 < x < L$, $t > 0$ of the xt-plane (see Figure 10.7.2). One condition is imposed at each point on the semi-infinite sides, and two are imposed at each point on the finite base.

It is important to realize that Eq. (1) governs a large number of other wave problems besides the transverse vibrations of an elastic string. For example, it is only

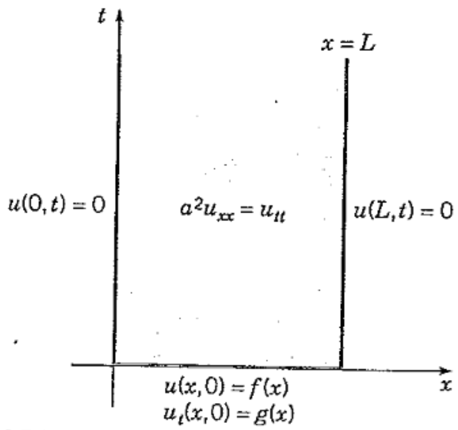

FIGURE 10.7.2 Boundary value problem for the wave equation.

necessary to interpret the function u and the constant a appropriately to have problems dealing with water waves in an ocean, acoustic or electromagnetic waves in the atmosphere, or elastic waves in a solid body. If more than one space dimension is significant, then Eq. (1) must be slightly generalized. The two-dimensional wave equation is

$$a^2(u_{xx} + u_{yy}) = u_{tt}. \tag{7}$$

This equation would arise, for example, if we considered the motion of a thin elastic sheet, such as a drumhead. Similarly, in three dimensions the wave equation is

$$a^2(u_{xx} + u_{yy} + u_{zz}) = u_{tt}. \tag{8}$$

In connection with the latter two equations, the boundary and initial conditions must also be suitably generalized.

We now solve some typical boundary value problems involving the one-dimensional wave equation.

Elastic String with Nonzero Initial Displacement. First suppose that the string is disturbed from its equilibrium position and then released at time $t = 0$ with zero velocity to vibrate freely. Then the vertical displacement $u(x,t)$ must satisfy the wave equation (1),

$$a^2 u_{xx} = u_{tt}, \qquad 0 < x < L, \quad t > 0;$$

the boundary conditions (3),

$$u(0,t) = 0, \qquad u(L,t) = 0, \qquad t \geq 0;$$

and the initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = 0, \qquad 0 \leq x \leq L, \tag{9}$$

where f is a given function describing the configuration of the string at $t = 0$.

The method of separation of variables can be used to obtain the solution of Eqs. (1), (3), and (9). Assuming that

$$u(x,t) = X(x)T(t) \tag{10}$$

and substituting for u in Eq. (1), we obtain

$$\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T} = -\lambda, \tag{11}$$

where λ is a separation constant. Thus we find that $X(x)$ and $T(t)$ satisfy the ordinary differential equations

$$X'' + \lambda X = 0, \tag{12}$$

$$T'' + a^2\lambda T = 0. \tag{13}$$

Further, by substituting from Eq. (10) for $u(x,t)$ in the boundary conditions (3), we find that $X(x)$ must satisfy the boundary conditions

$$X(0) = 0, \qquad X(L) = 0. \tag{14}$$

Finally, by substituting from Eq. (10) into the second of the initial conditions (9), we also find that $T(t)$ must satisfy the initial condition

$$T'(0) = 0. \tag{15}$$

Our next task is to determine $X(x)$, $T(t)$, and λ by solving Eq. (12) subject to the boundary conditions (14) and Eq. (13) subject to the initial condition (15).

The problem of solving the differential equation (12) subject to the boundary conditions (14) is *precisely the same problem* that arose in Section 10.5 in connection with the heat conduction equation. Thus we can use the results obtained there and at the end of Section 10.1: The problem (12), (14) has nontrivial solutions if and only if λ is an eigenvalue,

$$\lambda = n^2\pi^2/L^2, \qquad n = 1, 2, \ldots, \tag{16}$$

and $X(x)$ is proportional to the corresponding eigenfunction $\sin(n\pi x/L)$.

Using the values of λ given by Eq. (16) in Eq. (13), we obtain

$$T'' + \frac{n^2\pi^2a^2}{L^2}T = 0. \tag{17}$$

Therefore

$$T(t) = k_1 \cos\frac{n\pi at}{L} + k_2 \sin\frac{n\pi at}{L}, \tag{18}$$

where k_1 and k_2 are arbitrary constants. The initial condition (15) requires that $k_2 = 0$, so $T(t)$ must be proportional to $\cos(n\pi at/L)$.

Thus the functions

$$u_n(x,t) = \sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}, \qquad n = 1, 2, \ldots, \tag{19}$$

satisfy the partial differential equation (1), the boundary conditions (3), and the second initial condition (9). These functions are the fundamental solutions for the given problem.

To satisfy the remaining (nonhomogeneous) initial condition (9) we will consider a superposition of the fundamental solutions (19) with properly chosen coefficients. Thus we assume that $u(x,t)$ has the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}, \tag{20}$$

where the constants c_n remain to be chosen. The initial condition $u(x,0) = f(x)$ requires that

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} = f(x). \tag{21}$$

Consequently, the coefficients c_n must be the coefficients in the Fourier sine series of period $2L$ for f; hence

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \ldots. \tag{22}$$

Thus the formal solution of the problem of Eqs. (1), (3), and (9) is given by Eq. (20) with the coefficients calculated from Eq. (22).

For a fixed value of n the expression $\sin(n\pi x/L)\cos(n\pi at/L)$ in Eq. (19) is periodic in time t with the period $2L/na$; it therefore represents a vibratory motion of the string having this period, or having the frequency $n\pi a/L$. The quantities $\lambda a = n\pi a/L$

for $n = 1, 2, \ldots$ are the **natural frequencies** of the string—that is, the frequencies at which the string will freely vibrate. The factor $\sin(n\pi x/L)$ represents the displacement pattern occurring in the string when it is executing vibrations of the given frequency. Each displacement pattern is called a **natural mode** of vibration and is periodic in the space variable x; the spatial period $2L/n$ is called the **wavelength** of the mode of frequency $n\pi a/L$. Thus the eigenvalues $n^2\pi^2/L^2$ of the problem (12), (14) are proportional to the squares of the natural frequencies, and the eigenfunctions $\sin(n\pi x/L)$ give the natural modes. The first three natural modes are sketched in Figure 10.7.3. The total motion of the string, given by the function $u(x,t)$ of Eq. (20), is thus a combination of the natural modes of vibration and is also a periodic function of time with period $2L/a$.

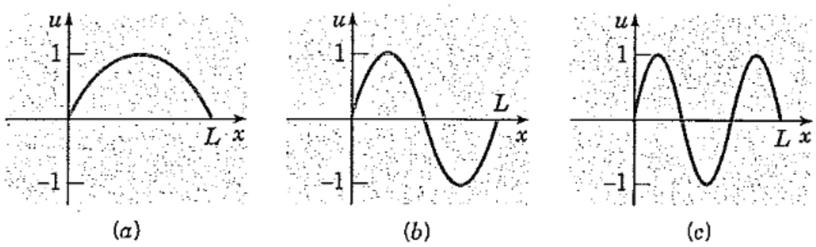

FIGURE 10.7.3 First three fundamental modes of vibration of an elastic string. (*a*) Frequency $= \pi a/L$, wavelength $= 2L$; (*b*) frequency $= 2\pi a/L$, wavelength $= L$; (*c*) frequency $= 3\pi a/L$, wavelength $= 2L/3$.

EXAMPLE 1

Consider a vibrating string of length $L = 30$ that satisfies the wave equation

$$4u_{xx} = u_{tt}, \qquad 0 < x < 30, \quad t > 0. \tag{23}$$

Assume that the ends of the string are fixed and that the string is set in motion with no initial velocity from the initial position

$$u(x,0) = f(x) = \begin{cases} x/10, & 0 \le x \le 10, \\ (30 - x)/20, & 10 < x \le 30. \end{cases} \tag{24}$$

Find the displacement $u(x,t)$ of the string and describe its motion through one period.

The solution is given by Eq. (20) with $a = 2$ and $L = 30$; that is,

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{30} \cos\frac{2n\pi t}{30}, \tag{25}$$

where c_n is calculated from Eq. (22). Substituting from Eq. (24) into Eq. (22), we obtain

$$c_n = \frac{2}{30}\int_0^{10} \frac{x}{10}\sin\frac{n\pi x}{30}\,dx + \frac{2}{30}\int_{10}^{30} \frac{30 - x}{20}\sin\frac{n\pi x}{30}\,dx. \tag{26}$$

By evaluating the integrals in Eq. (26), we find that

$$c_n = \frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}, \qquad n = 1, 2, \ldots. \tag{27}$$

The solution (25), (27) gives the displacement of the string at any point x at any time t. The motion is periodic in time with period 30, so it is sufficient to analyze the solution for $0 \le t \le 30$.

The best way to visualize the solution is by a computer animation showing the dynamic behavior of the vibrating string. Here we indicate the motion of the string in Figures 10.7.4, 10.7.5, and 10.7.6. Plots of u versus x for $t = 0, 4, 7.5, 11$, and 15 are shown in Figure 10.7.4. Observe that the maximum initial displacement is positive and occurs at $x = 10$, while at $t = 15$, a half-period later, the maximum displacement is negative and occurs at $x = 20$. The string then retraces its motion and returns to its original configuration at $t = 30$. Figure 10.7.5 shows the behavior of the points $x = 10, 15$, and 20 by plots of u versus t for these fixed values of x. The plots confirm that the motion is indeed periodic with period 30. Observe also that each interior point on the string is motionless for one-third of each period. Figure 10.7.6 shows a three-dimensional plot of u versus both x and t, from which the overall nature of the solution is apparent. Of course, the curves in Figures 10.7.4 and 10.7.5 lie on the surface shown in Figure 10.7.6.

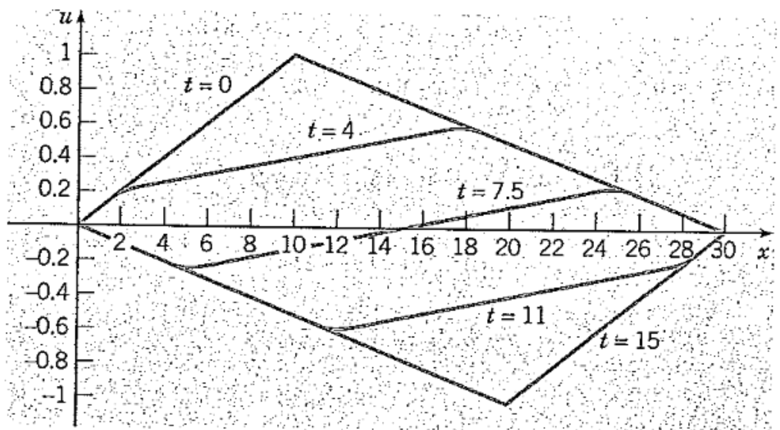

FIGURE 10.7.4 Plots of u versus x for fixed values of t for the string in Example 1.

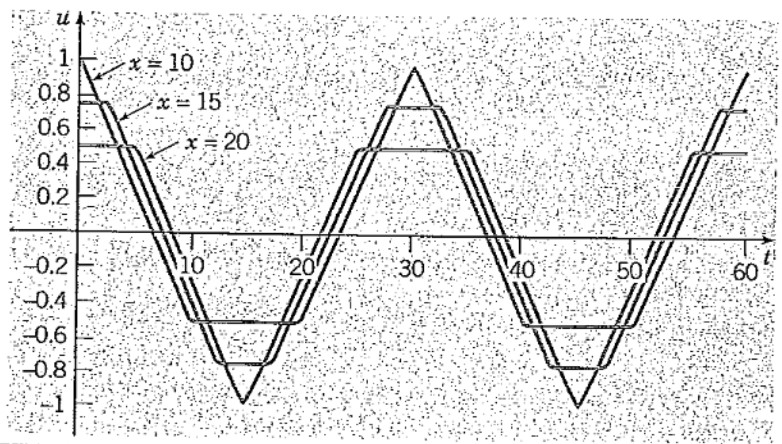

FIGURE 10.7.5 Plots of u versus t for fixed values of x for the string in Example 1.

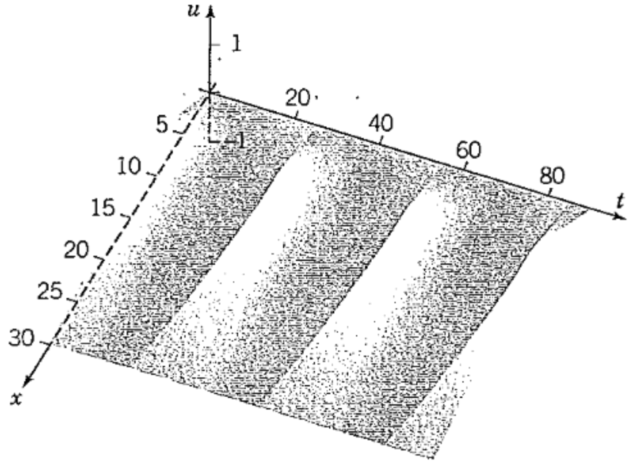

FIGURE 10.7.6 Plot of u versus x and t for the string in Example 1.

Justification of the Solution. As in the heat conduction problem considered earlier, Eq. (20) with the coefficients c_n given by Eq. (22) is only a *formal* solution of Eqs. (1), (3), and (9). To ascertain whether Eq. (20) *actually* represents the solution of the given problem requires some further investigation. As in the heat conduction problem, it is tempting to try to show this directly by substituting Eq. (20) for $u(x,t)$ in Eqs. (1), (3), and (9). However, upon formally computing u_{xx}, for example, we obtain

$$u_{xx}(x,t) = -\sum_{n=1}^{\infty} c_n \left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L} \cos\frac{n\pi at}{L};$$

due to the presence of the n^2 factor in the numerator, this series may not converge. This would not necessarily mean that the series (20) for $u(x,t)$ is incorrect, but only that the series (20) cannot be used to calculate u_{xx} and u_{tt}. A basic difference between solutions of the wave equation and those of the heat conduction equation is that the latter contain negative exponential terms that approach zero very rapidly with increasing n, which ensures the convergence of the series solution and its derivatives. In contrast, series solutions of the wave equation contain only oscillatory terms that do not decay with increasing n.

However, there is an alternative way to establish the validity of Eq. (20) indirectly. At the same time, we will gain additional information about the structure of the solution. First we will show that Eq. (20) is equivalent to

$$u(x,t) = \tfrac{1}{2}\left[h(x-at) + h(x+at)\right], \tag{28}$$

where h is the function obtained by extending the initial data f into $(-L,0)$ as an odd function, and to other values of x as a periodic function of period $2L$. That is,

$$h(x) = \begin{cases} f(x), & 0 \le x \le L, \\ -f(-x), & -L < x < 0; \end{cases} \tag{29}$$

$$h(x+2L) = h(x).$$

To establish Eq. (28) note that h has the Fourier series

$$h(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}, \tag{30}$$

where c_n is given by Eq. (22). Then, using the trigonometric identities for the sine of a sum or difference, we obtain

$$h(x - at) = \sum_{n=1}^{\infty} c_n \left(\sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L} - \cos \frac{n\pi x}{L} \sin \frac{n\pi at}{L} \right),$$

$$h(x + at) = \sum_{n=1}^{\infty} c_n \left(\sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L} + \cos \frac{n\pi x}{L} \sin \frac{n\pi at}{L} \right),$$

and Eq. (28) follows immediately upon adding the last two equations. From Eq. (28) we see that $u(x,t)$ is continuous for $0 < x < L, t > 0$, provided that h is continuous on the interval $(-\infty, \infty)$. This requires that f be continuous on the original interval $[0, L]$. Similarly, u is twice continuously differentiable with respect to either variable in $0 < x < L, t > 0$, provided that h is twice continuously differentiable on $(-\infty, \infty)$. This requires that f' and f'' be continuous on $[0, L]$. Furthermore, since h'' is the odd extension of f'', we must also have $f''(0) = f''(L) = 0$. However, since h' is the even extension of f', no further conditions are required on f'. Provided that these conditions are met, u_{xx} and u_{tt} can be computed from Eq. (28), and it is an elementary exercise to show that these derivatives satisfy the wave equation. Some of the details of the argument just indicated are given in Problems 19 and 20.

If some of the continuity requirements stated in the last paragraph are not met, then u is not differentiable at some points in the semi-infinite strip $0 < x < L, t > 0$, and thus is a solution of the wave equation only in a somewhat restricted sense. An important physical consequence of this observation is that if there are any discontinuities present in the initial data f, then they will be preserved in the solution $u(x,t)$ for all time. In contrast, in heat conduction problems, initial discontinuities are instantly smoothed out (Section 10.6). Suppose that the initial displacement f has a jump discontinuity at $x = x_0$, $0 \leq x_0 \leq L$. Since h is a periodic extension of f, the same discontinuity is present in $h(\xi)$ at $\xi = x_0 + 2nL$ and at $\xi = -x_0 + 2nL$, where n is any integer. Thus $h(x - at)$ is discontinuous when $x - at = x_0 + 2nL$, or when $x - at = -x_0 + 2nL$. For a fixed x in $[0, L]$ the discontinuity that was originally at x_0 will reappear in $h(x - at)$ at the times $t = (x \pm x_0 - 2nL)/a$. Similarly, $h(x + at)$ is discontinuous at the point x at the times $t = (-x \pm x_0 + 2mL)/a$, where m is any integer. If we refer to Eq. (28), it then follows that the solution $u(x,t)$ is also discontinuous at the given point x at these times. Since the physical problem is posed for $t > 0$, only those values of m and n that yield positive values of t are of interest.

General Problem for the Elastic String. Let us modify the problem considered previously by supposing that the string is set in motion from its equilibrium position with a given velocity. Then the vertical displacement $u(x,t)$ must satisfy the wave equation (1),

$$a^2 u_{xx} = u_{tt}, \qquad 0 < x < L, \quad t > 0;$$

the boundary conditions (3),

$$u(0,t) = 0, \qquad u(L,t) = 0, \qquad t \geq 0;$$

and the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = g(x), \qquad 0 \le x \le L, \tag{31}$$

where $g(x)$ is the initial velocity at the point x of the string.

The solution of this new problem can be obtained by following the procedure described above for the problem (1), (3), and (9). Upon separating variables, we find that the problem for $X(x)$ is exactly the same as before. Thus, once again, $\lambda = n^2\pi^2/L^2$ and $X(x)$ is proportional to $\sin(n\pi x/L)$. The differential equation for $T(t)$ is again Eq. (17), but the associated initial condition is now

$$T(0) = 0, \tag{32}$$

corresponding to the first of the initial conditions (31). The general solution of Eq. (17) is given by Eq. (18), but now the initial condition (32) requires that $k_1 = 0$. Therefore $T(t)$ is now proportional to $\sin(n\pi at/L)$ and the fundamental solutions for the problem (1), (3), and (31) are

$$u_n(x,t) = \sin\frac{n\pi x}{L}\sin\frac{n\pi at}{L}, \qquad n = 1, 2, 3, \ldots. \tag{33}$$

Each of the functions $u_n(x,t)$ satisfies the wave equation (1), the boundary conditions (3), and the first of the initial conditions (31). The main consequence of using the initial conditions (31) rather than (9) is that the time-dependent factor in $u_n(x,t)$ involves a sine rather than a cosine.

To satisfy the remaining (nonhomogeneous) initial condition we assume that $u(x,t)$ can be expressed as a linear combination of the fundamental solutions (33); that is,

$$u(x,t) = \sum_{n=1}^{\infty} k_n u_n(x,t) = \sum_{n=1}^{\infty} k_n \sin\frac{n\pi x}{L}\sin\frac{n\pi at}{L}. \tag{34}$$

To determine the values of the coefficients k_n we differentiate Eq. (34) with respect to t, set $t = 0$, and use the second initial condition (31); this gives the equation

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi a}{L} k_n \sin\frac{n\pi x}{L} = g(x). \tag{35}$$

Hence the quantities $(n\pi a/L)k_n$ are the coefficients in the Fourier sine series of period $2L$ for g. Therefore

$$\frac{n\pi a}{L} k_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \ldots. \tag{36}$$

Thus Eq. (34), with the coefficients given by Eq. (36), constitutes a formal solution to the problem of Eqs. (1), (3), and (31). The validity of this formal solution can be established by arguments similar to those previously outlined for the solution of Eqs. (1), (3), and (9).

Finally, we turn to the problem consisting of the wave equation (1), the boundary conditions (3), and the general initial conditions (4), (5):

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad 0 < x < L, \tag{37}$$

where $f(x)$ and $g(x)$ are the given initial position and velocity, respectively, of the string. Although this problem can be solved by separating variables, as in the cases discussed previously, it is important to note that it can also be solved simply by adding together the two solutions that we obtained above. To show that this is true, let $v(x,t)$ be the solution of the problem (1), (3), and (9), and let $w(x,t)$ be the solution of the problem (1), (3), and (31). Thus $v(x,t)$ is given by Eqs. (20) and (22), and $w(x,t)$ is given by Eqs. (34) and (36). Now let $u(x,t) = v(x,t) + w(x,t)$; what problem does $u(x,t)$ satisfy? First, observe that

$$a^2 u_{xx} - u_{tt} = (a^2 v_{xx} - v_{tt}) + (a^2 w_{xx} - w_{tt}) = 0 + 0 = 0, \tag{38}$$

so $u(x,t)$ satisfies the wave equation (1). Next, we have

$$u(0,t) = v(0,t) + w(0,t) = 0 + 0 = 0, \qquad u(L,t) = v(L,t) + w(L,t) = 0 + 0 = 0, \tag{39}$$

so $u(x,t)$ also satisfies the boundary conditions (3). Finally, we have

$$u(x,0) = v(x,0) + w(x,0) = f(x) + 0 = f(x) \tag{40}$$

and

$$u_t(x,0) = v_t(x,0) + w_t(x,0) = 0 + g(x) = g(x). \tag{41}$$

Thus $u(x,t)$ satisfies the general initial conditions (37).

We can restate the result we have just obtained in the following way. To solve the wave equation with the general initial conditions (37) you can solve instead the somewhat simpler problems with the initial conditions (9) and (31), respectively, and then add together the two solutions. This is another use of the principle of superposition.

PROBLEMS

Consider an elastic string of length L whose ends are held fixed. The string is set in motion with no initial velocity from an initial position $u(x,0) = f(x)$. In each of Problems 1 through 4 carry out the following steps. Let $L = 10$ and $a = 1$ in parts (b) through (d).

(a) Find the displacement $u(x,t)$ for the given initial position $f(x)$.

(b) Plot $u(x,t)$ versus x for $0 \le x \le 10$ and for several values of t between $t = 0$ and $t = 20$.

(c) Plot $u(x,t)$ versus t for $0 \le t \le 20$ and for several values of x.

(d) Construct an animation of the solution in time for at least one period.

(e) Describe the motion of the string in a few sentences.

1. $f(x) = \begin{cases} 2x/L, & 0 \le x \le L/2, \\ 2(L-x)/L, & L/2 < x \le L \end{cases}$

2. $f(x) = \begin{cases} 4x/L, & 0 \le x \le L/4, \\ 1, & L/4 < x < 3L/4, \\ 4(L-x)/L, & 3L/4 \le x \le L \end{cases}$

3. $f(x) = 8x(L-x)^2/L^3$.

4. $f(x) = \begin{cases} 1, & L/2 - 1 < x < L/2 + 1 \quad (L > 2), \\ 0, & \text{otherwise} \end{cases}$