A07

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$$\vec{F} = \left(\frac{2xy}{(x^2+y^2)^2} , \frac{(y^2-x^2)}{(x^2+y^2)^2} \right)$$

. Antes de tudo, precisamos saber se $\vec{F}(x,y)$ é conservativo, para isso:

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$

$g(x,y) = (y^2-x^2).(x^2+y^2)^{-2}$

$$\frac{\partial g}{\partial x} = (-2x)(x^2+y^2)^{-2} + (y^2-x^2)(-2)(x^2+y^2)^{-3}(2x) = \frac{-2x}{(x^2+y^2)^2} + \frac{-4x(y^2-x^2)}{(x^2+y^2)^3}$$

$$\frac{\partial g}{\partial x} = \frac{-2x(x^2+y^2) - 4x(y^2-x^2)}{(x^2+y^2)^3} = \frac{-2x^3-2xy^2-4xy^2+4x^3}{(x^2+y^2)^3} = \frac{2x^3-6xy^2}{(x^2+y^2)^3}$$

Por outro lado,

$f(x,y) = (2xy)(x^2+y^2)^{-2}$:

$$\frac{\partial f}{\partial y} = (2x)(x^2+y^2)^{-2} + (2xy)(-2)(x^2+y^2)^{-3}(2y) = \frac{2x}{(x^2+y^2)^2} + \frac{-8xy^2}{(x^2+y^2)^3}$$

$$\frac{\partial f}{\partial y} = \frac{2x(x^2+y^2) - 8xy^2}{(x^2+y^2)^3} = \frac{2x^3+2xy^2-8xy^2}{(x^2+y^2)^3} = \frac{2x^3-6xy^2}{(x^2+y^2)^3}$$

Como

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$

Logo, o $\vec{F}(x,y)$ é conservativo, sendo possível determinar uma função potencial associada à ele;

$$\vec{F} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j}$$

$$\begin{cases} \text{(I)} \quad \dfrac{\partial \phi}{\partial x} = \dfrac{2xy}{(x^2+y^2)^2} \\[2ex] \text{(II)} \quad \dfrac{\partial \phi}{\partial y} = \dfrac{(y^2-x^2)}{(x^2+y^2)^2} \end{cases}$$

Ⓘ: $\int \partial\phi = \int \frac{2xy}{(x^2+y^2)^2}\,\partial x \Rightarrow \phi = \int y u^{-2}\,\partial u \Rightarrow y\left[-\frac{1}{u}\right] \Rightarrow y\left[\frac{-1}{(x^2+y^2)}\right]$

$u = x^2+y^2$
$\partial u = 2x\,\partial x$

$$\phi = \frac{-y}{(x^2+y^2)} + C(y)$$

Assim em Ⓘ:

$$\phi = -y(x^2+y^2)^{-1} + C(y)$$

$$\frac{\partial\phi}{\partial y} = (-1)(x^2+y^2)^{-1} + (-y)(-1)(x^2+y^2)^{-2}(2y) + \frac{\partial C}{\partial y} \Rightarrow \frac{-1}{(x^2+y^2)} + \frac{2y^2}{(x^2+y^2)^2} + \frac{\partial C}{\partial y}$$

$$\frac{\partial\phi}{\partial y} = \frac{(-x^2-y^2)+2y^2}{(x^2+y^2)^2} \Rightarrow \frac{(y^2-x^2)}{(x^2+y^2)^2} + \frac{\partial C}{\partial y}$$

Assim

$$\frac{\partial\phi}{\partial y} = \frac{(y^2-x^2)}{(x^2+y^2)^2} \Rightarrow \cancel{\frac{(y^2-x^2)}{(x^2+y^2)^2}} + \frac{\partial C}{\partial y} = \cancel{\frac{(y^2-x^2)}{(x^2+y^2)^2}} \Rightarrow \partial C = 0\,\partial y$$

$$\int \partial C = \int 0\,\partial y \Rightarrow C = 0 + K$$

Logo:

$$\phi = \frac{-y}{(x^2+y^2)} + C(y)$$

$$C(y) = 0 + K$$

$$\therefore \phi = \frac{-y}{(x^2+y^2)} + K \quad //$$

. Verificando se $\vec{\nabla}\phi = \vec{F}(x,y)$

$$\vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j}$$

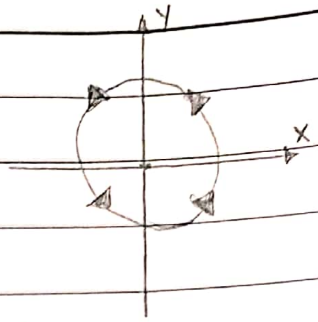
$$\frac{\partial\phi}{\partial x} = -y(-1)(x^2+y^2)^{-2}(2x) \Rightarrow \frac{2xy}{(x^2+y^2)^2}\hat{i} \quad //$$

$$\frac{\partial\phi}{\partial y} = (-1)(x^2+y^2)^{-1} + (-y)(-1)(x^2+y^2)^{-2}(2y) = \frac{-1}{(x^2+y^2)} + \frac{2y^2}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{(y^2-x^2)}{(x^2+y^2)^2}\hat{j} \quad //$$

$$\therefore \vec{\nabla}\phi = \vec{F}(x,y)$$

Logo, $f(x,y)$ e $g(x,y)$ são contínuas e suas derivadas de 1ª ordem também, numa região D simplesmente conexa. Assim, $\vec{F}(x,y)$ é conservativo, então existe uma função potencial associada. Portanto, $\oint_C \vec{F}\cdot d\vec{r} = 0$ para qualquer caminho que envolva a origem.

Escolhendo um caminho C envolvendo a origem para demonstração



$$x^2 + y^2 = R^2$$

$$\begin{cases} x = R\cos t \\ y = R\,\text{sen}\,t \end{cases} ; t \in [0, 2\pi]$$

$$\vec{r}(t) = (R\cos t, R\,\text{sen}\,t)$$

$$\vec{r}\,'(t) = (-R\,\text{sen}\,t, R\cos t)$$

$$\vec{F}(x,y) = \frac{1}{(x^2+y^2)^2}\left[2xy, (y^2-x^2)\right] \Rightarrow \vec{F}(\vec{r}(t)) = \frac{1}{[R^2(\cos^2 t + \text{sen}^2 t)]^2}\left[2R^2\cos t\,\text{sen}\,t, R^2(\text{sen}^2 t - \cos^2 t)\right]$$

$$\vec{F}(\vec{r}(t)) = \frac{1}{R^4}\left[2R^2\cos t\,\text{sen}\,t, R^2(\text{sen}^2 t - \cos^2 t)\right] \Rightarrow \frac{1}{R^2}\left[2\cos t\,\text{sen}\,t, \text{sen}^2 t - \cos^2 t\right]$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \left(\frac{1}{R^2}\left[2\cos t\,\text{sen}\,t, \text{sen}^2 t - \cos^2 t\right]\right) \cdot (-R\,\text{sen}\,t, R\cos t)$$

$$= \frac{1}{R^2}\left[-2R\cos t\,\text{sen}^2 t + R\cos t(\text{sen}^2 t - \cos^2 t)\right] = \frac{1}{R}\left[-2\cos t\,\text{sen}^2 t + \cos t\,\text{sen}^2 t - \cos^3 t\right]$$

$$= \frac{1}{R}\left[-\cos t\,\text{sen}^2 t - \cos^3 t\right] \Rightarrow \frac{1}{R}\left[-\cos t(\text{sen}^2 t + \cos^2 t)\right] \Rightarrow \frac{1}{R}\left[-\cos t\right]$$

Assim

$$\oint_C \vec{F}\cdot d\vec{r} = \frac{1}{R}\int_0^{2\pi} -\cos t\,dt \to \frac{-1}{R}\left[\text{sen}\,t\right]_0^{2\pi} \Rightarrow \frac{-1}{R}\left[\text{sen}(2\pi) - \text{sen}(0)\right] \Rightarrow \frac{-1}{R}\left[0 - 0\right]$$

$$\oint_C \vec{F}\cdot d\vec{r} = 0$$

Portanto, $\vec{F}(x,y)$ é conservativo, existe uma função potencial $\vec{\nabla}\phi$, de modo que, $\vec{F}(x,y) = \vec{\nabla}\phi$. Assim, é possível afirmar que $\oint_C \vec{F}\cdot d\vec{r} = 0$ para qualquer caminho C que envolva a origem.