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1\. a) $\left\{ \frac{5^n}{2^{n^2}} \right\}_{n=1}^{+\infty}$ (V)

Pelo teste da Razão:

$$\rho = \lim_{k \to +\infty} \frac{a_{k+1}}{a_k}$$

$a_k = \frac{5^k}{2^{k^2}}$; $a_{k+1} = \frac{5^{k+1}}{2^{(k+1)^2}}$; $\Rightarrow \frac{a_{k+1}}{a_k} = \frac{5^k \cdot 5^1}{2^{k^2} \cdot 2^{2k} \cdot 2} \cdot \frac{2^{k^2}}{5^k} = \frac{5}{2} \cdot \frac{1}{2^{2k}}$

$\Rightarrow \lim_{k \to +\infty} \frac{5}{2} \cdot \frac{1}{2^{2k}} \Rightarrow \frac{5}{2} \cdot \lim_{k \to +\infty} \frac{1}{2^{2k}} \Rightarrow \frac{5}{2} \lim_{k \to +\infty} \frac{1}{\infty}^{0} = \frac{5}{2} \cdot 0 = 0$

$\rho < 1 \to$ Converge.

Assim, para ser estritamente decrescente:

$$\frac{a_{n+1}}{a_n} < 1$$

$a_n = \frac{5^n}{2^{n^2}}$; a_{n+1} $\frac{5^n \cdot 5^1}{2^{n^2+2n+1}}$; $\Rightarrow \frac{\cancel{5^n} \cdot 5}{\cancel{2^{n^2}} \cdot 2^{2n} \cdot 2} \cdot \frac{\cancel{2^{n^2}}}{\cancel{5^n}} = \frac{5}{2^{2n+1}} < 1$

$$\frac{5}{2^{2n+1}} < 1$$

$$2n+1 = 0$$

$$2n = -1$$

$\boxed{n \neq -1/2}$ $\to$ Condição p/ $\frac{5}{2^{2n+1}} < 1$

b) Se $\lim_{k \to +\infty} (k^2 u_k) = 5$, $\sum_{k=1}^{\infty} u_k$ convergirá. (F)

. Pelo teorema do Teste da divergência:

i) Se $\lim_{k \to +\infty} u_k \neq 0 \to$ Diverge.

Assim, $\sum_{k=1}^{\infty} u_k \to$ Diverge, pois o $\lim_{k \to +\infty} (k^2 u_k) \neq 0$.

c) $\sum_{k=1}^{\infty} \frac{\alpha^k}{k^\alpha}$, $\alpha > 0$. (F)

p/ $\alpha = 1$:

$\sum_{k=1}^{\infty} 1/k \Rightarrow$ série harmônica, diverge

. Pelo teste da integral:

$\lim_{b \to +\infty} \int_1^b \frac{1}{x} dx \Rightarrow [\ln|x|]\Big|_1^b \Rightarrow [\ln|b| - \cancel{\ln|1|}] = +\infty$

. Portanto, não converge para todo $\alpha > 0$.

d) $\sum_{k=1}^{\infty} \left(\frac{-1}{\ln(k)} \right)^k$ (F)

. Pelo teste da Raiz:

$u_k = \left(\frac{-1}{\ln(k)} \right)^k \Rightarrow \sqrt[k]{|u_k|} \Rightarrow \sqrt[k]{\left| \left(\frac{-1}{\ln(k)} \right)^k \right|} \Rightarrow \frac{1}{\ln(k)}$

$\rho = \lim_{k \to +\infty} \frac{1}{\ln(k)} \Rightarrow \lim_{k \to +\infty} \frac{1}{\infty} = 0 \Rightarrow \rho = 0 < 1 \to$ converge, $\sum_{k=1}^{\infty} |u_k| \to$ converge.

Portanto, a série infinita $\sum_{k=1}^{\infty} u_k$ é absolutamente convergente, pois a série $\sum_{k=1}^{\infty} |u_k|$ é convergente.

e) $1 + \frac{1 \cdot 3}{3!} + \frac{1 \cdot 3 \cdot 5}{5!} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{7!} + \ldots$

2\. a) $\displaystyle\sum_{k=1}^{\infty} \frac{k(k+3)}{(k+1)(k+2)(k+5)}$

$$\frac{k(k+3)}{(k+1)(k+2)(k+5)} = \frac{A}{(k+1)} + \frac{B}{(k+2)} + \frac{C}{(k+5)}$$

$$k(k+3) = A(k+2)(k+5) + B(k+1)(k+5) + C(k+1)(k+2)$$

p/ $x = -2$

$$-2(-2+3) = A(0) + B(-2+1)(-2+5) + C(0)$$

$$-2 = -3B$$

$$\boxed{B = 2/3}$$

p/ $k = -5$

$$-5(-5+3) = A(0) + B(0) + C(-5+1)(-5+2)$$

$$10 = 12C$$

$$\boxed{C = 5/6}$$

p/ $k = -1$

$$-1(-1+3) = A(-1+2)(-1+5) + B(0) + C(0)$$

$$-2 = 4A$$

$$\boxed{A = -1/2}$$

Assim:

$$\sum_{k=1}^{\infty} \frac{k(k+3)}{(k+1)(k+2)(k+5)} = \sum_{k=1}^{\infty}\left(\frac{-1}{2(k+1)} + \frac{2}{3(k+2)} + \frac{5}{6(k+5)}\right)$$

$$\sum_{k=1}^{\infty} \frac{k(k+3)}{(k+1)(k+2)(k+5)} = +\frac{1}{2}\sum_{k=1}^{\infty} \overset{\text{I}}{\frac{-1}{(k+1)}} + \frac{2}{3}\sum_{k=1}^{\infty} \overset{\text{II}}{\frac{1}{(k+2)}} + \frac{5}{6}\sum_{k=1}^{\infty} \overset{\text{III}}{\frac{1}{(k+5)}}$$

Pelo teste da integral

(I) $\displaystyle \frac{1}{2}\sum_{k=1}^{\infty}\left|\frac{-1}{(k+1)}\right| \Rightarrow \frac{1}{2}\lim_{b\to\infty}\int_1^b \frac{1}{(x+1)}\,dx \Rightarrow u = x+1 \Rightarrow du = dx \Rightarrow \int \frac{1}{u}\,du \Rightarrow \ln|u|$

$\displaystyle \Rightarrow \frac{1}{2}\lim_{b\to+\infty}\left[\ln|b+1| - \ln|2|\right] \Rightarrow \frac{1}{2}\lim_{b\to+\infty}\left[\ln|\infty|\right] = \infty$ // $\to$ Diverge

Assim $\displaystyle \frac{1}{2}\sum_{k=1}^{\infty}\left|-1/(k+1)\right|$ Diverge para valores absolutos, então $\displaystyle 1/2 \sum_{k=1}^{\infty}\left(-1/(x+1)\right) \to$ Diverge absolutamente //

(II) $\displaystyle \frac{2}{3}\sum_{k=1}^{\infty}\frac{1}{(k+2)} \Rightarrow \frac{2}{3}\lim_{b\to\infty}\int_1^b \frac{1}{(x+2)}\,dx \Rightarrow \frac{2}{3}\left[\ln|x+2|\right]\Big|_1^b \Rightarrow \frac{2}{3}\left[\ln|b+2| - \ln|3|\right] \Rightarrow$

$\displaystyle \Rightarrow \frac{2}{3}\lim_{b\to+\infty}\left[\ln|\infty|\right] = \infty \to$ Diverge //

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III) $\frac{5}{6} \sum_{k=1}^{\infty} \frac{1}{(k+5)} \Rightarrow \frac{5}{6} \lim_{b \to +\infty} \int_1^b \frac{1}{(x+5)} dx \Rightarrow \begin{matrix} u = x+5 \\ du = dx \end{matrix} \Rightarrow \int \frac{1}{u} du \Rightarrow \ln|u| \Rightarrow \left[\ln|x+5|\right]_1^b$

$\Rightarrow \frac{5}{6} \lim_{b \to +\infty} \left[\ln|b+5| - \ln|6|\right] = \infty \rightarrow$ Diverge

. Portanto

$$\sum_{k=1}^{\infty} \frac{k(k+3)}{(k+1)(k+2)(k+5)} = \underbrace{\frac{1}{2} \sum_{k=1}^{\infty} \frac{-1}{(k+1)}}_{\text{Diverge}} + \underbrace{\frac{2}{3} \sum_{k=1}^{\infty} \frac{1}{(k+2)}}_{\text{Diverge}} + \underbrace{\frac{5}{6} \sum_{k=1}^{\infty} \frac{1}{(k+5)}}_{\text{Diverge}}$$

Assim

$$\sum_{k=1}^{\infty} \frac{k(k+3)}{(k+1)(k+2)(k+5)} \rightarrow \text{Diverge}$$

b) $\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!} = \sum_{k=0}^{\infty} \frac{(k)!(k)!}{(2k)!}$

Pelo teste da Razão

$$\rho = \lim_{k \to +\infty} \frac{a_{k+1}}{a_k}$$

$a_k = \frac{(k!)(k!)}{(2k)!}$; $a_{k+1} = \frac{(k+1)!(k+1)!}{(2(k+1))!}$

$\frac{a_{k+1}}{a_k} = \frac{(k+1)!(k+1)!}{(2k+2)!} \cdot \frac{(2k)!}{(k)!(k)!} \Rightarrow \frac{[(k+1)\cancel{(k)\dots}][(k+1)\cancel{(k)\dots}]}{[(2k+2)(2k+1)\cancel{(2k)\dots}]} \cdot \frac{[\cancel{(2k)\dots}]}{[\cancel{(k)(k-1)}][\cancel{(k)(k-1)\dots}]}$

$\Rightarrow \frac{(k+1)(k+1)}{(2k+2)(2k+1)} \Rightarrow \frac{k^2+2k+1}{4k^2+6k+2} \Rightarrow \rho = \lim_{k \to +\infty} \frac{k^2+2k+1}{4k^2+6k+2} \begin{matrix} \div k^2 \\ \div k^2 \end{matrix} \Rightarrow \frac{\frac{k^2}{k^2} + \frac{2k}{k^2}^{\to 0} + \frac{1}{k^2}^{\to 0}}{\frac{4k^2}{k^2} + \frac{6k}{k^2} + \frac{2}{k^2}}$

$\Rightarrow \lim_{k \to +\infty} \frac{1}{4} = \frac{1}{4} \Rightarrow \rho = \frac{1}{4} < 1 \rightarrow$ Converge!

Portanto $\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)}$ Converge.

3. $\sum_{k=0}^{\infty} \frac{(2x-3)^k}{4^{2k}}$

. Pelo teste da Razão para convergência absoluta

$$\rho = \lim_{k \to +\infty} \frac{|a_{k+1}|}{|a_k|}$$

$a_k = \frac{(2x-3)^k}{4^{2k}}$; $a_{k+1} = \frac{(2x-3)^{k+1}}{4^{2(k+1)}}$

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(2x-3)^k\cdot(2x-3)^1}{4^{2k}\cdot 4^2}\cdot\frac{4^{2k}}{(2x-3)^k}\right| = \left|\frac{(2x-3)}{16}\right| \Rightarrow \lim_{k\to\infty}\left|\frac{(2x-3)}{16}\right|$$

$$\Rightarrow = \left|\frac{(2x-3)}{16}\right| \Rightarrow \rho < 1 \Rightarrow |(2x-3)| < 16 \Rightarrow -16 < 2x-3 < 16 \Rightarrow$$

$$\Rightarrow -16+3 < 2x-3+3 < 16+3 \Rightarrow -13 < 2x < 19 \Rightarrow -\tfrac{13}{2} < x < \tfrac{19}{2}$$

Assim:

$(2x-3)=0$

$x_0 = 3/2$

$$\frac{19}{2} - \frac{3}{2} = \frac{16}{2} = 8$$

$$I = \left(-\tfrac{13}{2}, \tfrac{19}{2}\right)$$

p/ $x = 19/2$

$$\sum_{k=0}^{\infty}\frac{(2(19/2)-3)^k}{4^{2k}} \Rightarrow \sum_{k=0}^{\infty}(1)^k = 1^0 + 1^1 + 1^2 + 1^3 \rightarrow \text{série geométrica}$$

$a_1 = 1$; $r = 1/1 \Rightarrow |r| \geq 1 \rightarrow$ Diverge. Assim p/ $x = 19/2$ a série $\sum_{k=0}^{\infty}\frac{(2x-3)^k}{4^{2k}} \rightarrow$ Diverge.

p/ $x = -13/2$

$$\sum_{k=0}^{\infty}\frac{(2(-13/2)-3)^k}{4^{2k}} \Rightarrow \sum_{k=0}^{\infty}(-1)^k\left(\frac{16}{16}\right)^k = \sum_{k=0}^{\infty}(-1)^k\cdot(1)^k = 1^0 - 1^1 + 1^2 - 1^3 \ldots$$

Assim,

$a_1 = 1$; $r = -1 \Rightarrow |r| \geq 1 \rightarrow$ Diverge. Logo, p/ $x = -13/2$ a série $\sum_{k=0}^{\infty}\frac{(2x-3)^k}{4^{2k}}$ Diverge.

Portanto:

Intervalo de convergência: $\left(-\tfrac{13}{2}, \tfrac{19}{2}\right)$

Raio de convergência: 8

Assim, a soma será:

$$\sum_{k=0}^{\infty}\frac{(2(8)-3)^k}{4^{2k}} = \sum_{k=0}^{\infty}\frac{(13)^k}{4^{2k}} = 1 + \frac{13}{4^2} + \frac{13^2}{4^4} + \ldots \rightarrow \text{série geométrica}$$

$a_1 = 1$; $r = 13/16 \rightarrow |r| < 1 \rightarrow$ Converge

Soma:

$$\frac{a_1}{1-r} = \frac{1}{1-13/16} = \frac{1}{3/16} = \frac{16}{3}$$

$\boxed{\therefore\ 16/3}$ → Soma de acordo com o raio de convergência.

4. a) $f(x) = e^{x^4}$

$f(u) = e^u$

. Pela série de Maclaurin $(x_0 = 0)$:

$$f(u) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots$$

$f(0) = e^0 \rightarrow 1$

$f'(u) = e^u \rightarrow f'(0) = e^0 \rightarrow 1$

$f''(u) = e^u \rightarrow f''(0) = e^0 \rightarrow 1$

$f'''(u) = e^u \rightarrow f'''(0) = e^0 \rightarrow 1$

$$f(u) = 1 + 1.u + \frac{1.u^2}{2!} + \frac{1.u^3}{3!} + \dots$$

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{u^k}{k!}$$

Assim p/ e^{x^4}:

$$e^{x^4} = 1 + x^4 + \frac{x^8}{2!} + \frac{x^{12}}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^{4k}}{k!}$$ //

b) $\int e^{x^4} dx$

$$e^{x^4} = \sum_{k=0}^{\infty} \frac{x^{4k}}{k!}$$

Assim: $$\int e^{x^4} dx = \int \sum_{k=0}^{\infty} \frac{x^{4k}}{k!} dx \Rightarrow \sum_{k=0}^{\infty} \int \frac{x^{4k}}{k!} dx \Rightarrow \sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)k!} + C$$

Pelo Teste da Razão p/ convergência absoluta:

$$\rho = \lim_{k \to +\infty} \frac{|a_{k+1}|}{|a_k|} \; ; \quad a_k = \frac{x^{4k+1}}{(4k+1)k!} \; ; \quad a_{k+1} = \frac{x^{4(k+1)+1}}{(4(k+1)+1)(k+1)!} \; ;$$

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\cancel{x^{4k}}.x^5}{\cancel{x^{4k}}.x} \cdot \frac{(4k+1)k!}{(4k+5)(k+1)!}\right| = |x^4| . \left|\frac{(4k+1)(k)(k-1)(k-2)\dots}{(4k+5)(k+1)(k)(k-1)(k-2)\dots}\right|$$

$$\Rightarrow |x^4| . \left|\frac{(4k+1)}{(4k^2+5k+4k+5}\right| \Rightarrow \lim_{k \to +\infty} |x^4| . \left|\frac{4k}{4k^2}\right| \Rightarrow |x^4| . \lim_{k \to +\infty} \frac{1}{k} \to 0 \Rightarrow |x^4| . 0 = 0$$

$\rho = 0$. Portanto converge para todo x, o intervalo de convergência é: $(-\infty, +\infty)$, o Raio de convergência é: ∞.