a) $\vec{f}(t_0) = P = \left(\frac{1}{2}e^{\pi/3}, \frac{\sqrt{3}}{2}e^{\pi/3}, 2\right)$

$(e^{t_0}\cos(t_0), e^{t_0}\operatorname{sen}(t_0), 2) = \left(\frac{1}{2}e^{\pi/3}, \frac{\sqrt{3}}{2}e^{\pi/3}, 2\right)$

$e^{t_0}\cos t_0 = \frac{1}{2}e^{\pi/3}$

$\boxed{t_0 = \pi/3}$

Como o vetor tangente unitário é:

$$\vec{T}(t) = \frac{\vec{f}'(t)}{|\vec{f}'(t)|}$$

$\vec{f}'(t) = e^t(\cos t - \operatorname{sen} t)\hat{i} + e^t(\cos t + \operatorname{sen} t)\hat{j}$

$|\vec{f}'(t)| = \sqrt{(e^t(\cos t - \operatorname{sen} t))^2 + (e^t(\cos t + \operatorname{sen} t))^2}$

$|\vec{f}'(t)| = \sqrt{e^{2t}(\cos^2 t - \cancel{2\cos t \operatorname{sen} t} + \operatorname{sen}^2 + \cos^2 t + \cancel{2\cos t \operatorname{sen} t} + \operatorname{sen}^2 t)}$

$|\vec{f}'(t)| = \sqrt{2e^{2t}} \to \sqrt{2}e^t$

$$\vec{T}(t) = \frac{[e^t(\cos t - \operatorname{sen} t)\hat{i} + e^t(\cos t + \operatorname{sen} t)\hat{j}]}{\sqrt{2}e^t}$$

Assim:

$$\vec{T}(t) = \frac{\sqrt{2}}{2}[(\cos t - \operatorname{sen} t)\hat{i} + (\cos t + \operatorname{sen} t)\hat{j}]$$

Já o vetor normal unitário:

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

$\vec{T}'(t) = \frac{\sqrt{2}}{2}[(-\operatorname{sen} t - \cos t)\hat{i} + (-\operatorname{sen} t + \cos t)\hat{j}]$

$|\vec{T}'(t)| = \sqrt{2}/2\left[\sqrt{(-\operatorname{sen} t - \cos t)^2\hat{i} + (-\operatorname{sen} t + \cos t)^2\hat{j}}\right.$

$|\vec{T}'(t)| = \sqrt{2}/2\left[\sqrt{(\operatorname{sen}^2 t + 2\operatorname{sen} t\cos t + \cos^2 t) + (\operatorname{sen}^2 t - \cancel{2\cos t \operatorname{sen} t} + \cos^2 t)}\right.$

$|\vec{T}'(t)| = \sqrt{2}/2\sqrt{2(\cos^2 t + \operatorname{sen}^2 t)}$

$|\vec{T}'(t)| = 2/2 = 1$

$$\vec{N}(t) = \frac{\sqrt{2}}{2}\left[(-\operatorname{sen} t - \cos t)\hat{i} + (\cos t - \operatorname{sen} t)\hat{j}\right]$$

$t_0 = \pi/3$:

$$\vec{N}(t_0) = \frac{\sqrt{2}}{2}\left[\left(-\operatorname{sen}(\pi/3) - \cos(\pi/3)\right)\hat{i} + \left(\cos(\pi/3) - \operatorname{sen}(\pi/3)\right)\hat{j}\right]$$

$$\vec{N}(t_0) = \sqrt{2}/2\left[\left(-\sqrt{3}/2 - 1/2\right)\hat{i} + \left(1/2 - \sqrt{3}/2\right)\hat{j}\right]$$

$$\therefore \vec{N}(t_0) = \left[\frac{-\sqrt{6} - \sqrt{2}}{4}\hat{i} + \frac{\sqrt{2} - \sqrt{6}}{4}\hat{j}\right]$$

FORONI

Assim a equação do plano normal da curva no ponto P será:

$P = (\frac{1}{2}e^{\pi/3}, \frac{\sqrt{3}}{2}e^{\pi/3}, 2)$

$\vec{N}(t_0) = \frac{-\sqrt{6}-\sqrt{2}}{4}\hat{i} + \frac{\sqrt{2}-\sqrt{6}}{4}\hat{j}$

$$\vec{PR} \cdot \vec{N}(t_0) = 0$$

$$\left(x - \frac{1}{2}e^{\pi/3}\right)\left(\frac{-\sqrt{6}-\sqrt{2}}{4}\right) + \left(y - \frac{\sqrt{3}}{2}e^{\pi/3}\right)\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right) + (z-2)(0) = 0$$

$$\left(x - \frac{1}{2}e^{\pi/3}\right)\frac{-\sqrt{6}-\sqrt{2}}{4} + \left(y - \frac{\sqrt{3}}{2}e^{\pi/3}\right)\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right) = 0$$

Logo o plano Normal será:

$$\therefore \left(x - \frac{1}{2}e^{\pi/3}\right)\left(\frac{-\sqrt{6}-\sqrt{2}}{4}\right) + \left(y - \frac{\sqrt{3}}{2}e^{\pi/3}\right)\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right) = 0$$

b) $t \geq 0$, reparametrize a curva C pelo comprimento de arco.

$$\vec{f}(t) = e^t \cos t\,\hat{i} + e^t \operatorname{sen} t\,\hat{j} + 2\hat{k}, \quad t \in \mathbb{R}$$

Como:

$$S(t) = \int_0^t \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

$x'(t) = e^t \cos t + e^t(-\operatorname{sen} t)$

$y'(t) = e^t \operatorname{sen} t + e^t \cos t$

$z'(t) = 0$

Assim:

$$S(t) = \int_0^t \sqrt{(e^t(\cos t - \operatorname{sen} t))^2 + (e^t(\operatorname{sen} t + \cos t))^2 + 0^2}\, dt$$

$$S(t) = \int_0^t \sqrt{e^{2t}(\cos^2 t - 2\cos t \operatorname{sen} t + \operatorname{sen}^2 t) + e^{2t}(\operatorname{sen}^2 t + 2\operatorname{sen} t \cos t + \cos^2 t)}\, dt$$

$$S(t) = \int_0^t \sqrt{2e^{2t}(\cos^2 t + \operatorname{sen}^2 t)}\, dt$$

$$S(t) = \int_0^T e^t \sqrt{2}\, dt$$

$$S(t) = \sqrt{2} \int_0^t e^{t'}\, dt'$$

$$S(t) = \sqrt{2}\,[e^{t'}]\Big|_0^t$$

$$S(t) = \sqrt{2}\,[e^t - e^0]$$

$$\therefore S(t) = \sqrt{2}\,[e^t - 1]$$

Isolando a t:

$$S + \sqrt{2} = \sqrt{2}\, e^t$$

$$\frac{S}{\sqrt{2}} + 1 = e^t$$

$$t = \ln\left|\frac{S}{\sqrt{2}} + 1\right|$$

Logo:

$$t \geq 0$$

$$S \geq 0$$

$$\therefore \vec{f}(S) = \left(\frac{S}{\sqrt{2}} + 1\right)\cos\left(\ln\left|\frac{S}{\sqrt{2}} + 1\right|\right)\hat{i} + \left(\frac{S}{\sqrt{2}} + 1\right)\operatorname{sen}\left(\ln\left|\frac{S}{\sqrt{2}} + 1\right|\right)\hat{j} + 2\hat{k}$$