Alejandro Trabalho 14 de Calculo IV

$$\frac{\partial^2 u}{\partial t^2} = k^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = 0 = u(a,t)$$

$$u(x,0) = \frac{1}{6} x (a^2 - x^2)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

$$k = 1$$

Pora $u = XT$

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{T}\frac{d^2T}{dt^2} = -\lambda^2 < 0 \quad (\lambda^2 = 0 \text{ e } \lambda^2 > 0$$

lunom è sol. trivial)

$$X = A \operatorname{sen} \lambda x + B \cos \lambda x$$

$$T = C \operatorname{sen} \lambda t + D \cos \lambda t$$

Como

$X(0) = 0 = B$

$X(\ell) = 0 = A \operatorname{sen} \lambda \ell \rightarrow$ $\lambda \ell = n\pi$

$\lambda = \frac{n\pi}{\ell}$ com $n = 1, 2, 3 \ldots$

Como $\left.\dfrac{dT}{dt}\right|_{t=0} = 0$

$$\frac{dT}{dt} = C\lambda \cos \lambda t - D\lambda \,\mathrm{sen}\, \lambda t$$

$$C\lambda = 0$$

$$C = 0$$

$$T = D \cos \frac{n\pi}{a} t$$

$$\rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{a} t \,\mathrm{sen}\, \frac{n\pi}{a} x$$

Como

$$u(x,0) = \frac{1}{6}x(a^2-x^2) = \sum_{n=1}^{\infty} A_n \operatorname{sen}\frac{n\pi x}{a}$$

$$A_n = \frac{2}{a}\int_0^a \frac{1}{6}x(a^2-x^2)\operatorname{sen}\frac{n\pi x}{a}\,dx$$

$$\int_0^a x \operatorname{sen}\frac{n\pi x}{a}\,dx = -\frac{xa}{n\pi}\cos\frac{n\pi x}{a}\Big|_0^a + \frac{a}{n\pi}\int_0^a \cos\frac{n\pi x}{a}\,dx$$

$u = x \quad du = dx$

$dv = \operatorname{sen}\frac{n\pi x}{a}\,dx \quad v = -\frac{a}{n\pi}\cos\frac{n\pi x}{a}$

$$= -\frac{a^2}{n\pi}\cos n\pi + \cancelto{0}{\frac{a^2}{(n\pi)^2}\operatorname{sen}\frac{n\pi x}{a}\Big|_0^a}$$

$$= -\frac{a^2}{n\pi}(-1)^n$$

$$\int_0^a x^3 \sin\frac{n\pi x}{a}\,dx = -\frac{x^3 a}{n\pi}\cos\frac{n\pi x}{a}\Big|_0^a + \frac{3a}{n\pi}\int_0^a x^2\cos\frac{n\pi x}{a}\,dx$$

$u = x^3$

$du = 3x^2\,dx$

$dv = \sin\frac{n\pi x}{a}\,dx$

$v = -\frac{a}{n\pi}\cos\frac{n\pi x}{a}$

$$= -\frac{a^4}{n\pi}(-1)^n + \frac{3a}{n\pi}\int_0^a x^2\cos\frac{n\pi x}{a}\,dx$$

$$\int_0^a x^2\cos\frac{n\pi x}{a}\,dx = \overset{0}{\cancel{\frac{x^2 a}{n\pi}\sin\frac{n\pi x}{a}\Big|_0^a}} - \frac{2a}{n\pi}\int_0^a x\sin\frac{n\pi x}{a}\,dx$$

$u = x^2$

$du = 2x\,dx$

$dv = \cos\frac{n\pi x}{a}\,dx$

$v = \frac{a}{n\pi}\sin\frac{n\pi x}{a}$

$$= -\frac{2a}{n\pi}\int_0^a x\sin\frac{n\pi x}{a}\,dx = -\frac{2a}{n\pi}\left(-\frac{a^2}{n\pi}(-1)^n\right)$$

$$= \frac{2a^3}{n^2\pi^2}(-1)^n$$

$$A_n = \frac{2}{6a}\left[a^2\left(-\frac{a^2}{n\pi}(-1)^n\right) - \left\{-\frac{a^4}{n\pi}(-1)^n + \frac{3a}{n\pi}\left(\frac{2a^3}{n^2\pi^2}(-1)^n\right)\right\}\right]$$

$$= \frac{1}{3a}\left[\cancel{-\frac{a^4}{n\pi}(-1)^n} + \cancel{\frac{a^4}{n\pi}(-1)^n} - \frac{6a^4}{n^3\pi^3}(-1)^n\right]$$

$$= \frac{-1}{3a}\cdot\frac{6a^4}{n^3\pi^3}(-1)^n = -\frac{2a^3}{n^3\pi^3}(-1)^n$$

$$u(x,t) = \sum_{n=1}^{\infty} -\frac{2a^3}{n^3\pi^3}(-1)^n \cos\frac{n\pi t}{a}\,\mathrm{sen}\,\frac{n\pi x}{a}$$