Solução da Lista 3 de Cálculo IV

1) $x^3 y' + x^2 y - y^2 = 2x^4$ (Eq. de Riccati) vejamos uma sol. do tipo $y_1 = ax^2$ (por inspeção).

$$x^3 2ax + x^2 a x^2 - a^2 x^4 = 2x^4 \quad \text{se } x \neq 0$$

$$2a + a - a^2 = 2$$

$$a^2 - 3a + 2 = 0$$

$$a = \frac{3 \pm \sqrt{9 - 4 \cdot 1 \cdot 2}}{2} = \frac{3 \pm 1}{2} = \begin{cases} 1 \\ 2 \end{cases}$$

Vamos fazer $y_1 = x^2$. e $y = y_1 + u$

$$y' = 2x + u'$$

$$x^3(2x + u') + x^2(x^2 + u) - (x^2 + u)^2 = 2x^4$$

$$\cancel{2x^4} + u'x^3 + \cancel{x^4} + x^2u - \cancel{x^4} - u^2 - 2x^2u = \cancel{2x^4}$$

$$u'x^3 - x^2u = u^2$$

$$u' - \frac{u}{x} = \frac{u^2}{x^3}$$

(Eq. de Bernoulli de ordem 2)

Fazendo $w = u^{-1} \to w' = -u^{-2}u' \to -\frac{w'}{w^2} = u'$

$$-\frac{w'}{w^2} - \frac{1}{wx} = \frac{1}{w^2x^3}$$

$$w' + \frac{w}{x} = -\frac{1}{x^3} \quad \text{(linear)}$$

$$w_1' + \frac{w_1}{x} = 0$$

$$w_1' = -\frac{w_1}{x}$$

$$\int \frac{dw_1}{w_1} = -\int \frac{1}{x}\,dx$$

$$\ln|w_1| = -\ln|x| + c$$

$$w_1 = \frac{1}{x}$$

$$w = \frac{v}{x}, \quad w' = \frac{v'}{x} - \frac{v}{x^2}$$

$$\frac{v'}{x} - \cancel{\frac{v}{x^2}} + \cancel{\frac{v}{x^2}} = -\frac{1}{x^3}$$

$$v' = -\frac{1}{x^2}$$

$$\int dv = -\int \frac{dx}{x^2}$$

$$v = \frac{1}{x} + c$$

$$w = \frac{1}{x^2} + \frac{c}{x}$$

$$u^{-1} = \frac{1}{x^2} + \frac{c}{x} = \frac{cx+1}{x^2}$$

$$u = \frac{x^2}{cx+1}$$

$$y = x^2 + u = x^2 + \frac{x^2}{cx+1}$$

$$\boxed{y = \frac{cx^3 + 2x^2}{cx+1}}$$

$$y' = \frac{3x^2c + 4x}{cx+1} - \frac{(cx^3+2x^2)c}{(cx+1)^2}$$

Na EDO

$$\frac{3x^5c + 4x^4}{cx+1} - \frac{(c^2x^6 + 2cx^5)}{(cx+1)^2}$$

$$+ \frac{cx^5 + 2x^4}{cx+1} - \frac{(cx^3+2x^2)^2}{(cx+1)^2}$$

$$=$$

$$\frac{(4cx^5 + 6x^4)(cx+1)}{(cx+1)^2} - \frac{(\cancel{c^2x^6} + 2c\cancel{x^5} + \cancel{c^2x^6} + 4x^4 + 4c\cancel{x^5})}{(cx+1)^2}$$

$$= (\cancel{4c^2x^6} + \cancel{6cx^5} + \cancel{4cx^5} + 6x^4 - \cancel{2c^2x^6} - \cancel{6cx^5} - 4x^4)/(cx+1)^2$$

$$= (2c^2x^6 + 4cx^5 + 2x^4)/(cx+1)^2$$

$$= 2x^4(c^2x^2 + 2cx + 1)/(cx+1)^2 = 2x^4$$

d) $y = xy' + \sqrt{(y')^2+1}$ (eq de Clairaut)

$$\cancel{y'} = \cancel{y'} + xy'' + \frac{1}{2}\left((y')^2+1\right)^{-1/2} \cdot 2y' \cdot y''$$

$$0 = xy'' + \frac{y'y''}{\sqrt{(y')^2+1}} = y''\left(x + \frac{y'}{\sqrt{(y')^2+1}}\right) \longrightarrow$$

$\to y'' = 0 \to y = \underline{Cx+d}$

$y' = C \to \boxed{y = xC + \sqrt{C^2+1}}$

A sol. singular paramétrica:

$$x + \frac{y'}{\sqrt{(y')^2+1}} = 0 \rightarrow x = \frac{c}{\sqrt{c^2+1}}$$

$$y = cx + \sqrt{c^2+1}$$

onde c é um parâmetro.

Verifichiamo $\quad y = cx + \sqrt{c^2+1}$

$$y' = c$$

no EDO

$$y = xc + \sqrt{c^2+1}$$

ok

3) $xy'' = y' + x(y')^2$

$u = y' \qquad u' = y''$

$$xu' = u + xu^2$$

$u' - \frac{u}{x} = u^2$ (Bernoulli order 2)

$v = u^{-1}$

$v' = -u^{-2}u' = -v^2u'$

$$-\frac{v'}{v^2} - \frac{1}{vx} = \frac{1}{v^2}$$

$$v' + \frac{v}{x} = -1$$

$$v_1' + \frac{v_1}{x} = 0$$

$$\int \frac{dv_1}{v_1} = \int -\frac{1}{x}\,dx$$

$$\ln|v_1| = -\ln|x| + c$$

$$v_1 = \frac{1}{x}$$

$$v = \frac{w}{x}, \quad v' = \frac{w'}{x} - \frac{w}{x^2}$$

$$\frac{w'}{x} - \cancel{\frac{w}{x^2}} + \cancel{\frac{w}{x^2}} = -1$$

$$w' = -x$$

$$w = -\frac{x^2}{2} + C = vx$$

$$v = -\frac{x}{2} + \frac{C}{x} = \frac{1}{u}$$

$$\frac{-x^2 + 2C}{2x} = \frac{1}{u} \quad \Big| \quad u = \frac{2x}{-x^2 + 2C} = y'$$

$$\int dy = \int \frac{2x}{-x^2+2c}\,dx$$

$$a = -x^2+2c$$
$$da = -2x\,dx$$

$$y = -\int \frac{da}{a}$$

$$\boxed{y = -\ln|-x^2+2c| + d}$$

Verificação

$$y' = \frac{1}{-x^2+2c} \cdot 2x$$

$$y'' = \frac{2}{-x^2+2c} + \frac{2x}{(-x^2+2c)^2} \cdot 2x$$

Não ~~é~~ EDO

$$x\left(\frac{2}{-x^2+2c}+\frac{4x^2}{(-x^2+2c)^2}\right)=\frac{2x}{-x^2+2c}+x\left(\frac{2x}{(-x^2+2c)}\right)^2$$

$$\frac{2x}{-x^2+2c}+\frac{4x^3}{(-x^2+2c)^2}=\frac{2x}{-x^2+2c}+\frac{4x^3}{(-x^2+2c)^2}$$

<u>OK</u>

4) $\left(2+e^{-x/y}\right)dx + 2\left(1-\frac{x}{y}\right)dy = 0$

$x' = \frac{dx}{dy}$

$u = -\frac{x}{y}$ $\qquad u' = -\frac{x'}{y} + \frac{x}{y^2}$

$xu' = -\frac{x}{y}x' + \frac{x^2}{y^2} = ux' + u^2$

$x = -uy$

$-\cancel{u}y\,u' = \cancel{u}x' + \cancel{u^2}$

$x' = -yu' - u$

no EDO

$$(2+e^u)x' + 2(1+u) = 0$$

$$(2+e^u)(-y u' - u) + 2(1+u) = 0$$

$$y u' + u = \frac{2(1+u)}{2+e^u}$$

$$y u' = \frac{2(1+u)}{2+e^u} - u$$

$$\frac{u'(2+e^u)}{2(1+u) - u(2+e^u)} = -\frac{1}{y}$$

$$\int du \left(\frac{2+e^u}{2-ue^u}\right) = \int \frac{dy}{y}$$

$$\frac{2+e^u}{2-ue^u} = \frac{2+e^u-ue^u+ue^u}{2-ue^u}$$

$$\frac{2-ue^u}{2-ue^u} + \frac{e^u+ue^u}{2-ue^u}$$

$$= \frac{e^u+ue^u}{2-ue^u} + 1$$

$$\int \frac{e^u + ue^u}{2 - ue^u} + 1 \, du = \int \frac{e^u + ue^u}{2 - ue^u} \, du + u$$

$$v = 2 - ue^u$$

$$dv = -e^u - ue^u \, du$$

$$\int \frac{e^u + ue^u}{2 - ue^u} \, du = \int -\frac{dv}{v} = -\ln|2 - ue^u| + c$$

$$\ln|y| = -\ln|2 - ue^u| + u + c$$

$$\ln|y| = -\ln\left|2+\frac{x}{y}e^{-x/y}\right| - \frac{x}{y} + c$$

Verificação Derivando em relação a y.

$$\frac{1}{y} = -\frac{1}{2+\frac{x}{y}e^{-x/y}} \cdot \left(\frac{x'}{y}e^{-x/y} - \frac{x}{y^2}e^{-x/y} - \frac{x}{y}e^{-x/y}\left(\frac{x'}{y} - \frac{x}{y^2}\right)\right)$$

$$- \frac{x'}{y} + \frac{x}{y^2}$$

$$\frac{1}{y} = \frac{-e^{-x/y}}{2+\frac{x}{y}e^{-x/y}} \left(\frac{x'}{y} - \frac{x}{y^2} - \frac{x}{y} - \frac{xx'}{y^2} + \frac{x^2}{y^3} \right) - \frac{x'}{y} + \frac{x}{y^2}$$

$$1 = \frac{-e^{-x/y}}{2+\frac{x}{y}e^{-x/y}} \left(x' - \frac{x}{y} - x - \frac{xx'}{y} + \frac{x^2}{y^2} \right) + \frac{\left(\frac{x}{y} - x'\right)\left(2+\frac{x}{y}e^{-x/y}\right)}{2+\frac{x}{y}e^{-x/y}}$$

$$\left(2+\cancel{\frac{x}{y}e^{-x/y}}\right) = -x'e^{-x/y} + \cancel{\frac{x}{y}e^{-x/y}} + \cancel{\frac{x}{y}x'e^{-x/y}} - \cancel{\frac{x^2}{y^2}e^{-x/y}} + \frac{2x}{y} - 2x'$$

$$+ \cancel{\frac{x^2}{y^2}e^{-x/y}} - \cancel{\frac{xx'}{y}e^{-x/y}}$$

$$\mathcal{L} = -x' e^{-x/y} + \frac{2x}{y} - 2x'$$

$$\boxed{x'\left(2 + e^{-x/y}\right) + 2\left(1 - \frac{x}{y}\right) = 0}$$

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