Solução da Lista 13 de Cálculo IV

1) $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$

$u(0,t)=0 \qquad u(L,t)=0$

$u(x,0) = x(L-x)$

$u = XT$

$\dfrac{X''}{X} = \dfrac{T'}{kT} = -\lambda^2 < 0$

$X = A\cos\lambda x + B\,\text{sen}\,\lambda x$

$X(0) = 0 = A$

$X(L) = 0 = B\,\text{sen}\,\lambda L \rightarrow \lambda L = n\pi$

$\lambda = \dfrac{n\pi}{L}$ n inteiro

$X = B\,\text{sen}\,\dfrac{n\pi}{L}x$

$$T = c\, e^{-\kappa \lambda^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty} D_n\, e^{-\kappa \frac{n^2\pi^2}{L^2} t} \operatorname{sen} \frac{n\pi x}{L}$$

$$u(x,0) = x(L-x) = \sum_{n=1}^{\infty} D_n \operatorname{sen} \frac{n\pi x}{L}$$

$$D_n = \frac{2}{L} \int_0^L x(L-x) \operatorname{sen} \frac{n\pi x}{L}\, dx$$

$$= \frac{2}{L} \left[\int_0^L Lx \operatorname{sen} \frac{n\pi x}{L}\, dx - \int_0^L x^2 \operatorname{sen} \frac{n\pi x}{L}\, dx \right]$$

$$I = \int_0^L x \sin\frac{n\pi x}{L}\,dx = -\frac{xL}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L + \overset{0}{\cancel{\frac{L}{n\pi}\int_0^L \cos\frac{n\pi x}{L}\,dx}}$$

$$u = x \qquad dv = \sin\frac{n\pi x}{L}\,dx$$

$$du = dx$$

$$v = -\frac{L}{n\pi}\cos\frac{n\pi x}{L}$$

$$I_1 = -\frac{L^2}{n\pi}\cos n\pi = \frac{L^2}{n\pi}(-1)^{n+1}$$

$$I_2 = \int_0^L x^2 \operatorname{sen}\frac{n\pi x}{L}\,dx = -\frac{x^2 L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L + \frac{2L}{n\pi}\int_0^L x\cos\frac{n\pi x}{L}\,dx$$

$u = x^2 \qquad dv = \operatorname{sen}\frac{n\pi x}{L}\,dx$

$du = 2x\,dx \qquad v = \frac{-L}{n\pi}\cos\frac{n\pi x}{L}$

$$\underbrace{I_2 = -\frac{L^3}{n\pi}(-1)^n}+\frac{2L}{n\pi}I_3$$

$$I_3 = \int_0^L x\cos\frac{n\pi x}{L}\,dx = \frac{xL}{n\pi}\cancel{\operatorname{sen}\frac{n\pi x}{L}}\Big|_0^L - \frac{L}{n\pi}\int_0^L \operatorname{sen}\frac{n\pi x}{L}\,dx$$

0

$u = x \quad du = dx \qquad dv = \cos\frac{n\pi x}{L}\,dx$

$v = \frac{L}{n\pi}\operatorname{sen}\frac{n\pi x}{L}$

$$I_3 = +\frac{L^2}{n^2\pi^2}\cos\frac{n\pi x}{L}\Big|_0^L = \frac{L^2}{n^2\pi^2}\left((-1)^n-1\right)$$

$$D_n = \frac{2}{L}\left[L I_1 - I_2\right] = \frac{2}{L}\left\{L\cdot\frac{L^2}{n\pi}(-1)^{n+1}\right.$$

$$\left. - \left[-\frac{L^3}{n\pi}(-1)^n + \frac{2L}{n\pi}\frac{L^2}{n^2\pi^2}\left((-1)^n-1\right)\right]\right\}$$

$$= \cancel{\frac{2L^2}{n\pi}(-1)^{n+1}} + \cancel{\frac{2L^2}{n\pi}(-1)^n} - \frac{4L^2}{n^3\pi^3}\left((-1)^n-1\right)$$

$$D_n = \frac{4L^2}{n^3\pi^3}\left(1-(-1)^n\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4L^2}{n^3\pi^3}\left(1-(-1)^n\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \operatorname{sen} \frac{n\pi x}{L}$$