Solução do Trabalho 13 de Cálculo IV

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < 2 \qquad t > 0$$

$$u(0,t) = 0 \qquad \frac{\partial u}{\partial x}(2,t) = 0$$

$$f(x) = u(x,0) = \begin{cases} x, & 0 < x < 1 \\ 2 - x & 1 < x < 2 \end{cases}$$

$$\boxed{u = XT}$$

$$X\frac{\partial T}{\partial t} = T\frac{\partial^2 X}{\partial x^2}$$

$$\frac{1}{T}\frac{\partial T}{\partial t} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \lambda^2$$

1) $\lambda^2 > 0$

$$\frac{\partial^2 X}{\partial x^2} - \lambda^2 X = 0$$

$$r^2 - \lambda^2 = 0$$

$$r = \pm\lambda$$

$$X = Ae^{\lambda x} + Be^{-\lambda x}$$

$$X(0) = 0 = A + B$$

$$\left.\frac{dX}{dx}\right|_2 = A\lambda e^{\lambda 2} - B\lambda e^{-\lambda 2} = 0$$

$$\lambda A(e^{2\lambda} + e^{-2\lambda}) = 0$$

$$2\lambda A \cosh 2\lambda = 0 \quad X$$

2) $\lambda^2 = 0$

$T = cte$ $\quad X = Ax + B$

$X(0) = 0 = B$

$\left.\frac{dX}{dx}\right|_2 = A = 0$

X

3) $-\lambda^2 < 0$

$\frac{1}{T}\frac{dT}{dt} = -\lambda^2$

$\int_{T_0}^{T} \frac{dT^*}{T^*} = -\int_0^t \lambda^2 dt^*$

$\ln\left|\frac{T}{T_0}\right| = -\lambda^2 t$

$T = T_0 e^{-\lambda^2 t}$

$$X'' + \lambda^2 X = 0$$

$$r^2 + \lambda^2 = 0$$

$$r = \pm \lambda i$$

$$X = A \operatorname{sen} \lambda x + B \cos \lambda x$$

$$X(0) = 0 = B$$

$$\left.\frac{dX}{dx}\right|_2 = \lambda A \cos \lambda . 2 = 0$$

$$2\lambda = \frac{n\pi}{2} \qquad n = 1, 3, 5 \ldots$$

$$\lambda = \frac{(2k-1)\pi}{4} \qquad n = 2k - 1$$

$$k = 1, 2, 3 \ldots$$

$$X = A \operatorname{sen} \frac{(2k-1)\pi x}{4}$$

$$T = T_0 \, e^{-\frac{(2k-1)^2}{16}\pi^2 t}$$

$$u(x,t) = \sum_{k=1}^{\infty} A_k \, e^{-\frac{(2k-1)^2}{16}\pi^2 t} \operatorname{sen} \frac{(2k-1)\pi x}{4}$$

$$u(x,0) = f(x) = \sum_{k=1}^{\infty} A_k \operatorname{sen} \frac{(2k-1)\pi x}{4}$$

Pelo ortogonalidade:

$$\int_{-2}^{2} f(x)\,\text{sen}\left(\frac{2n-1}{4}\right)\pi x\,dx = \sum_{k=1}^{\infty}\int_{-2}^{2} A_n\,\text{sen}\left(\frac{2k-1}{4}\right)\pi x\,\text{sen}\left(\frac{2n-1}{4}\right)\pi x\,dx$$

usando a extensão ímpar de $f(x)$

se $k = n$

$$A_n\int_{-2}^{2}\text{sen}^2\left(\frac{2n-1}{4}\right)\pi x\,dx = \sum_{k=1}^{\infty} A_n\int_{-2}^{2}\frac{1-\cos\left(\frac{2n-1}{2}\right)\pi x}{2}\,dx$$

$$= A_n\left[\frac{1}{2}x\Big|_{-2}^{2} - \frac{1}{2}\,\frac{2}{(2n-1)\pi}\,\underbrace{\cancel{\text{sen}\,\frac{(2n-1)}{2}\pi x\Big|_{-2}^{2}}}_{0}\right] = A_n\left[\frac{1}{2}\cdot 4\right] = 2A_n$$

Pe $k \neq n$

$$\int_{-2}^{2} \sin\frac{(2k-1)\pi x}{4} \sin\frac{(2n-1)\pi x}{4}\,dx$$

$$=\frac{1}{2}\int_{-2}^{2}\cos\left[\left(\frac{2k-1}{4}\right)-\left(\frac{2n-1}{4}\right)\right]\pi x - \cos\left[\left(\frac{2k-1}{4}\right)+\left(\frac{2n-1}{4}\right)\right]\pi x\,dx$$

$$=\frac{1}{2}\int_{-2}^{2}\cos\frac{(k-n)}{2}\pi x - \cos\left(\frac{k+n-1}{2}\right)\pi x\,dx$$

$$=\frac{1}{2}\left[\frac{2}{(k-n)\pi}\sin\frac{(k-n)}{2}\pi x - \frac{2}{(k+n-1)\pi}\sin\left(\frac{k+n-1}{2}\right)\pi x\right]\Big|_{-2}^{2} =$$

$$= \frac{1}{2}\left[\frac{2}{(k-n)\pi}\sin\frac{(k-n)\pi x}{2} - \frac{2}{(k+n-1)\pi}\sin\frac{(k+n-1)\pi x}{2}\right]\Bigg|_{-2}^{2} =$$

$$= \frac{1}{2}\left[\frac{2}{\pi(k-n)}\cdot 2\underbrace{\cancel{\sin(k-n)\pi}}_{0} - \frac{2}{(k+n-1)\pi}\cdot 2\underbrace{\cancel{\sin(k+n-1)\pi}}_{0}\right] = 0$$

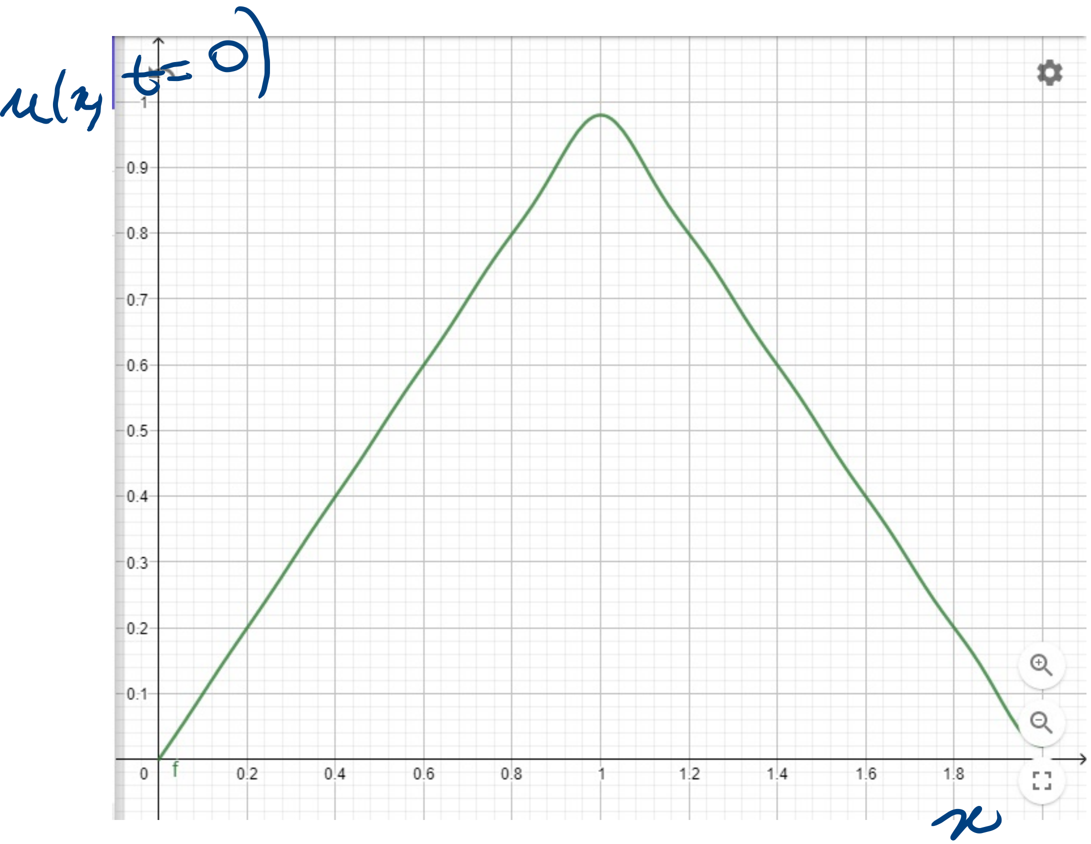
$$\to \quad A_n = \int_0^2 f(x)\sin\frac{(2n-1)}{4}\pi x\,dx$$

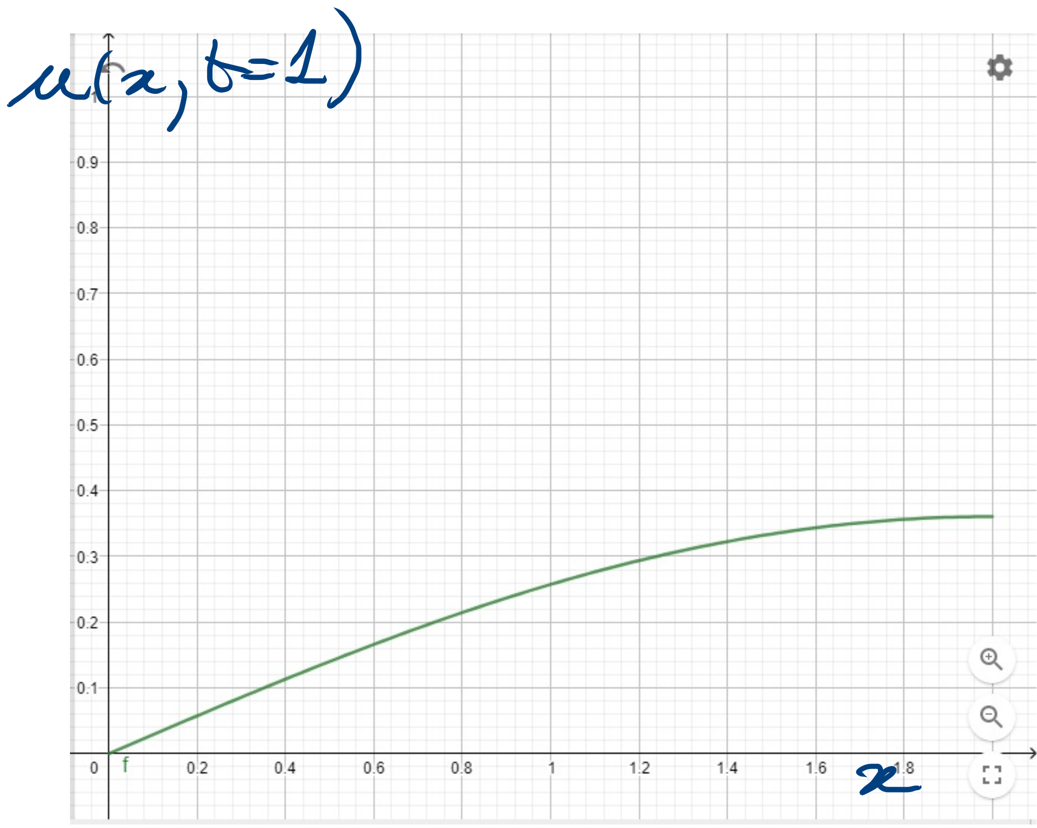
$$= \int_0^1 x\sin\frac{(2n-1)}{4}\pi x\,dx + \int_1^2 (2-x)\sin\frac{(2n-1)}{4}\pi x\,dx$$

$$A_n = \frac{32}{(2n-1)^2\pi^2}\left(\sin\frac{(2n-1)\pi}{4} + \frac{\cos n\pi}{2}\right)$$

$$\Rightarrow$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{32}{(2n-1)^2\pi^2}\left(\sin\frac{(2n-1)\pi}{4} + \frac{\cos n\pi}{2}\right) e^{-\frac{(2n-1)^2}{16}\pi^2 t} \sin\left(\frac{2n-1}{4}\right)\pi x$$


u(x, t=0)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
f
x


u(x, t=1)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
f
x