Deriving Circle Formulas from Square Formulas: Relating Spheres, Cubes, and their Cross-sections

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Let us assume that we do not know anything about circles besides the following equation:

$$S = C \cdot \frac{\pi}{2D} \mid s = 2r \tag{1}$$

Where S stands for the formula of a sphere or circle, C stands for the formula of a cube or square, D stands for the number of dimensions in the geometric shape, s stands for the side length of the square, and r stands for the radius of the circle or sphere. If the circle is inscribed in the square, then the length of the square's side will be $s = 2r$ as shown in Figure 1.

The logic for designing formula (1) is simple: as long as the dimensions of a square or a cube are given, the dimensions of its incribed circle or sphere can be calculated. Analyzing 2-dimensionally, squares have a larger area than their inscribed circles because circles are missing four corners; however, the missing area of those four corners is accounted for in equation (1).

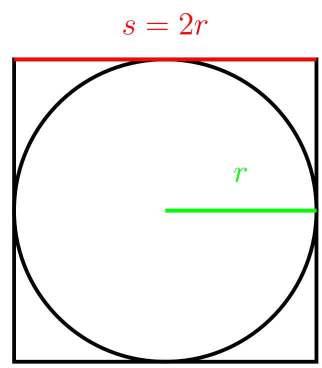

Figure 1: Illustration of a circle inscribed in a square.

Applying the Formula

Equation (1) can be used to find the volume, surface area, area, perimeter (circumference), and lenght (arc length) of spheres and circles.

Volume

To see how the equation works, consider the formula for the volume of a cube and the number of dimensions a cube has:

$$C_{volume} = s^3 \qquad D = 3 \tag{2}$$

In order to find the corresponding volume of a sphere inscribed in such cube, we must plug D, the number of dimensions, into equation (1), as shown below:

$$\begin{aligned} S_{volume} &= C_{volume} \cdot \frac{\pi}{2D} \mid s = 2r \\ &= s^3 \cdot \frac{\pi}{2 \cdot 3} \mid s = 2r \\ &= (2r)^3 \cdot \frac{\pi}{6} \\ &= \frac{8\pi r^3}{6} \\ &= \frac{4\pi r^3}{3} \end{aligned}$$

Surface Area

Consider the formula for the surface area of a cube and the number of dimensions a cube has:

$$C_{surfaceArea} = 6s^2 \qquad D = 3 \tag{3}$$

Then the corresponding surface area of a sphere inscribed in such cube can be found using equation (1), as shown below:

$$\begin{aligned} S_{surfaceArea} &= C_{surfaceArea} \cdot \frac{\pi}{2D} \mid s = 2r \\ &= 6s^2 \cdot \frac{\pi}{2 \cdot 3} \mid s = 2r \\ &= 6(2r)^2 \cdot \frac{\pi}{6} \\ &= 4\pi r^2 \end{aligned}$$

Area

Consider the formula for the area of a square (the cross-section of a cube) and the number of dimensions a square has:

$$C_{area} = s^2 \qquad D = 2 \tag{4}$$

Then the corresponding area of a circle (the cross-section of a sphere) inscribed in such square can be found using equation (1), as shown below:

$$\begin{aligned} S_{area} &= C_{area} \cdot \frac{\pi}{2D} \mid s = 2r \\ &= s^2 \cdot \frac{\pi}{2 \cdot 2} \mid s = 2r \\ &= (2r)^2 \cdot \frac{\pi}{4} \\ &= \frac{4\pi r^2}{4} \\ &= \pi r^2 \end{aligned}$$

Perimeter

Consider the formula for the perimeter of a square and the number of dimensions a square has:

$$C_{perimeter} = 4s \qquad D = 2 \tag{5}$$

Then the corresponding perimeter (circumference) of a circle inscribed in such square can be found using equation (1), as shown below:

$$\begin{aligned} S_{circumference} &= C_{perimeter} \cdot \frac{\pi}{2D} \mid s = 2r \\ &= 4s \cdot \frac{\pi}{2 \cdot 2} \mid s = 2r \\ &= 4(2r) \cdot \frac{\pi}{4} \\ &= 2\pi r \end{aligned}$$

Figure (2) clarifies the formula's meaning:

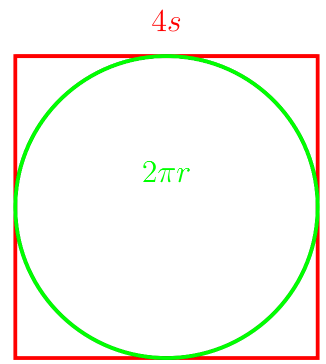

Figure 2: Illustration showing that a square's perimeter corresponds to its incribed circle's circumference.

Formulas as functions of θ

It is possible to express all the formulas as functions of θ by using a slightly different conversion fomula; however, the formula is only relevant for finding the arc lenght and area of a sector:

$$S = C \cdot \frac{\pi}{2D} \mid s = 2r$$
$$S(\theta) = C \cdot \frac{\pi}{2D} \cdot \frac{\theta}{2\pi} \mid s = 2r$$

This yields equation number (6):

$$S(\theta) = C \cdot \frac{\theta}{4D} \mid s = 2r \tag{6}$$

Arc Lenght

The formula for the arc length of a circle can be obtained from the perimeter formula of a square:

$$C_{perimeter} = 4s \qquad D = 2 \tag{7}$$

The conversion formula is applied as normal, but the final result is multiplied by $\frac{\theta}{2\pi}$ as shown below:

$$\begin{aligned} S_{arcLength} &= C_{perimeter} \cdot \frac{\pi}{2D} \cdot \frac{\theta}{2\pi} \mid s = 2r \\ &= 4s \cdot \frac{\pi}{2 \cdot 2} \cdot \frac{\theta}{2\pi} \mid s = 2r \\ &= (2r) \cdot \frac{\theta}{2} \\ &= \theta r \end{aligned}$$

Sector Area

The formula for the sector area of a circle can be obtained from the area formula of a square:

$$C_{area} = s^2 \qquad D = 2 \tag{8}$$

The conversion formula is applied as normal, but the final result is multiplied by $\frac{\theta}{2\pi}$ as shown below:

$$\begin{aligned} S_{sectorArea} &= C_{area} \cdot \frac{\pi}{2D} \cdot \frac{\theta}{2\pi} \mid s = 2r \\ &= s^2 \cdot \frac{\pi}{2 \cdot 2} \cdot \frac{\theta}{2\pi} \mid s = 2r \\ &= (2r)^2 \cdot \frac{\theta}{8} \\ &= 4r^2 \cdot \frac{\theta}{8} \\ &= \frac{1}{2}\theta r^2 \end{aligned}$$

Formulas

3D Shapes

$$\begin{aligned}
S_{volume} &= C_{volume} \cdot \frac{\pi}{2D} \mid s = 2r \\
&= s^3 \cdot \frac{\pi}{2 \cdot 3} \mid s = 2r \\
&= (2r)^3 \cdot \frac{\pi}{6} \\
&= \frac{8\pi r^3}{6} \\
&= \frac{4\pi r^3}{3}
\end{aligned}$$

$$\begin{aligned}
S_{surfaceArea} &= C_{surfaceArea} \cdot \frac{\pi}{2D} \mid s = 2r \\
&= 6s^2 \cdot \frac{\pi}{2 \cdot 3} \mid s = 2r \\
&= 6(2r)^2 \cdot \frac{\pi}{6} \\
&= 4\pi r^2
\end{aligned}$$

2D Shapes

$$\begin{aligned}
S_{area} &= C_{area} \cdot \frac{\pi}{2D} \mid s = 2r \\
&= s^2 \cdot \frac{\pi}{2 \cdot 2} \mid s = 2r \\
&= (2r)^2 \cdot \frac{\pi}{4} \\
&= \frac{4\pi r^2}{4} \\
&= \pi r^2 \\
S(\theta)_{sectorArea} &= \pi r^2 \cdot \frac{\theta}{2\pi} \\
&= \frac{1}{2}\theta r^2
\end{aligned}$$

$$\begin{aligned}
S_{circumference} &= C_{perimeter} \cdot \frac{\pi}{2D} \mid s = 2r \\
&= 4s \cdot \frac{\pi}{2 \cdot 2} \mid s = 2r \\
&= 4(2r) \cdot \frac{\pi}{4} \\
&= 2\pi r \\
S(\theta)_{arcLength} &= 2\pi r \cdot \frac{\theta}{2\pi} \\
&= \theta r
\end{aligned}$$