Voronoi Graph of Point and Line Segments by Delaunay Mesh Refinement in 2-D

An algorithm for obtaining the Voronoi graph of point and line segment sites by the insertion of Steiner points along the lengths of the line segments in a Delaunay triangulation of points is presented. Initially, points are created along the lengths of segment sites such that successive points along the same segment are forced to form Delaunay edges. After an initial Delaunay triangulation, the resulting Voronoi graph is already a close approximation to the true Voronoi graph. Non-Delaunay triangles are then detected from the intersection of their circumcircles with adjacent segment sites, and more Steiner points are added to the triangulation along segments at locations that result in Delaunay triangles whose circumcircles are not intersected by any points or segments in a mechanism that is analogous to Delaunay edge flipping. From the data encoded in the Delaunay triangulation of original input points and added Steiner points, the Voronoi graph of input points and line segments can be created. The computational complexity of the algorithm depends on the geometry of the input line segments. The presented method can also be extended to create Voronoi graphs of any curves whose equation is known.

Introduction

The Voronoi graph of points and line segments is useful in many modern applications. The presented algorithm computes the Voronoi graph of points and line segments and is based on the implementation of the Delaunay triangulation of points, which has been thoroughly studied and refined by various groups. In the presented algorithm, Steiner points are added progressively to an initial Delaunay triangulation along line segment sites in different stages.

Initial Stage of Voronoi Graph Approximation

In the first stage, input segments are separated from their endpoints by creating new endpoint coordinates inside their corresponding segment some distance apart from the original endpoints. Then, segment sites are bisected until the lengths of the resulting sub-segments are smaller than the distance from their endpoints to their nearest neighbor site, and Steiner points are created at the endpoints of the resulting sub-segments. The process of segment bisection is illustrated in figure 1.

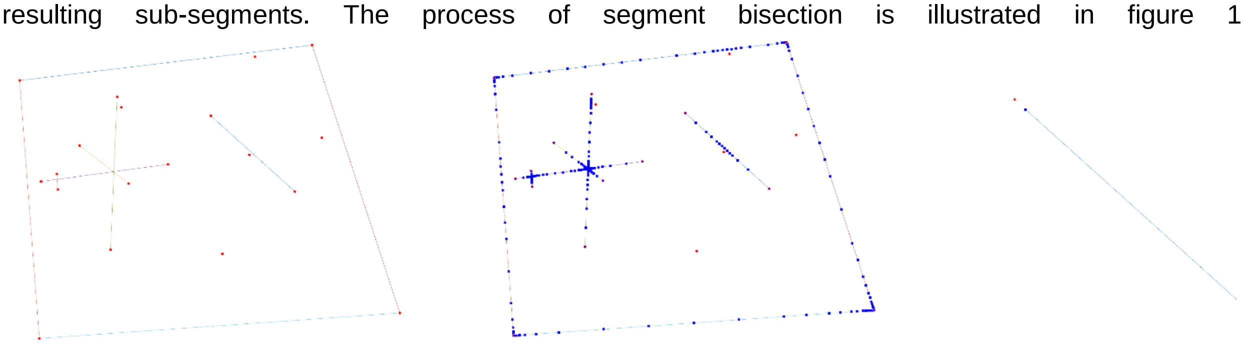

Figure 1. Left: arbitrarily chosen point and line segment sites are shown. All input points, including segment endpoints, are shown as red circles. Middle: The same sites are shown with new Steiner point sites drawn as blue squares inside line segment sites. The highest density of Steiner points is near segment intersections. Right: a close view of the separation of the interior of a segment from its endpoint.

The resulting Steiner points are labeled per the segment to which they belong. If input segments intersect, segment division will progress near their intersection until a predetermined minimum length is achieved by the resulting sub-segments. This pattern of segment division will result in a conforming triangulation with some triangle vertices inside input segments. Also, this pattern of segment division will result in higher concentrations of Steiner points near configurations of a higher geometric complexity, and lower concentrations of Steiner points near configurations of a lower geometric complexity. The concentration of Steiner points is also higher near regions where segments intersect weakly, and this is necessary to create a conforming triangulation where more than two segments intersect weakly. This pattern of segment division is similar to the computation of the Voronoi graph of line segments through a quadtree, but requires less memory. Image 2 illustrates the Delaunay triangulation and Voronoi graph at the initial stage of the Voronoi graph approximation.

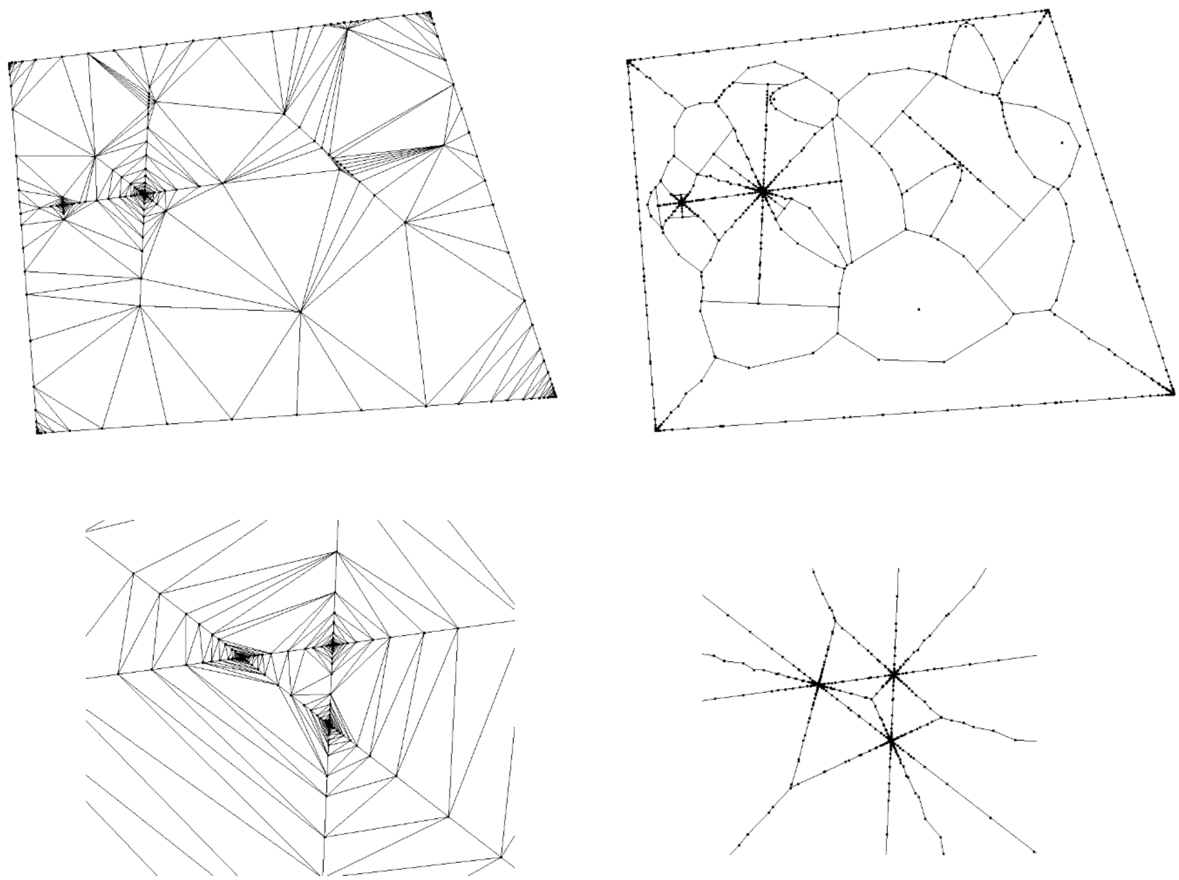

Figure 2. Top left: the Delaunay triangulation of input points and Steiner points from figure 1 is shown. Top right: the Voronoi graph of this initial triangulation is shown without the edges that correspond to pairs of triangles with two points on the same segment. Qualitatively, this Voronoi graph seems like a sufficient approximation that can easily be refined for applications like path planning. Bottom left: a close view of the triangulation where three input segments intersect each other. Bottom right: close view of the Voronoi graph that corresponds to the triangulation shown in the bottom-left section of this figure.

Second Stage of Voronoi Graph Approximation

In the second stage, the Delaunay triangulation of the input points and Steiner points from the first stage is used to detect triangles that have circumcircles intersected by input segment sites but do not have vertices inside the segments that intersect them. From these triangles, new Steiner points are created by projecting their circumcenters onto the segment that intersects them. The projection of their circumcenter onto their corresponding segments will ensure that all triangles have vertices labeled per the segments that intersect their circumcircle. Ensuring that triangles have points inside any segment that intersects their circumcircle will facilitate the detection of non-Delaunay triangles and further creation of Steiner points in the third stage. Figure 3 demonstrates an example of the second stage of the presented algorithm.

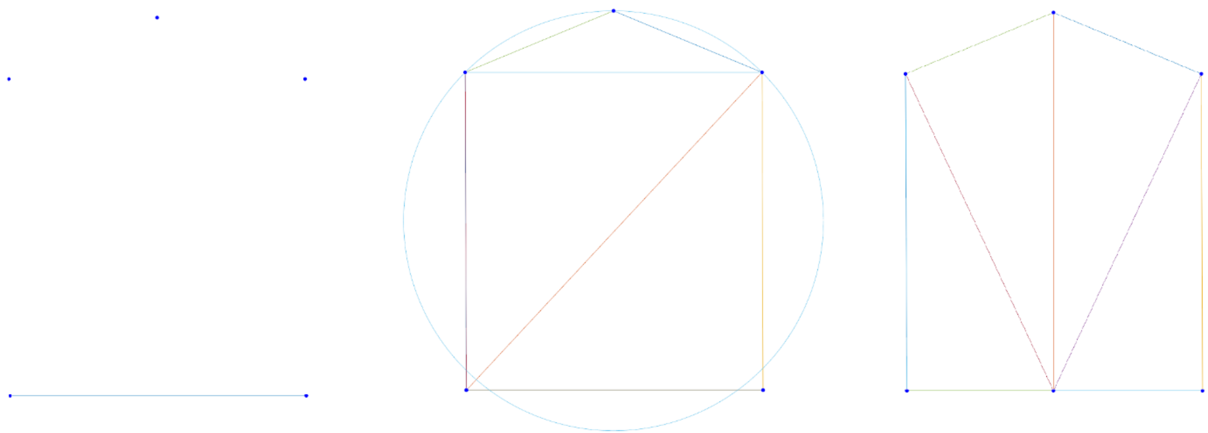

Figure 3. Left: An arbitrary set of one segment site and five point sites, including segment endpoints, is shown. Middle: The Delaunay triangulation of all input points is shown after the first stage in the presented algorithm. Note that no Steiner points were created in the first stage because the input segment already forms a Delaunay edge. There is a triangle whose circumcircle is intersected by the input segment, although this triangle does not have any points inside of the segment that intersects its circumcircle. Right: A Steiner point was created inside the input segment from the projection of the circumcenter of the triangle whose circumcircle was intersected. All triangles now have vertices inside the segments that potentially intersect their circumcircles.

Third Stage of the Voronoi Graph Approximation

In the third stage, the Delaunay triangulation of all input point and Steiner point sites created thus far is referenced to detect triangles with one vertex inside a segment and two points not inside segments, two vertices inside two different segments and one vertex not inside a segment, or three vertices inside three different segments whose circumcircles are intersected by their corresponding segments. Steiner points are created from these triangles. Before creating the new Steiner points, the circumcenter of intersected circumcircles is projected and recalculated repeatedly while repeatedly substituting the vertices of the triangles whose circumcenter is intersected with the projection of their circumcenter onto the intersecting segments. This process is analogous to Delaunay edge flipping for infinitely many infinitesimally thin adjacent triangles, but the recalculation of the circumcenter of triangles after substituting their vertices with the projection of their circumcenter onto segments that intersect their circumcircles enables the flipping of Delaunay edges by large gaps instead of by infinitesimally small steps.

For triangles with one vertex inside a segment, if their circumcenter is intersected by their corresponding segment, then their circumcenter is projected onto the segment by which they are intersected and recalculated repeatedly by substituting the vertex inside the intersecting segment with the projection of its corresponding circumcenter onto the segment until the intersection of the circumcircle cannot be detected at a predetermined precision or until the circumcenter has been projected a predetermined number of times.

Generally, a sequence of less than five circumcenter projections and recalculations yields a highly accurate approximation. After this cycle of circumcenter projections, Steiner points are created at the projections of the new circumcenters of the triangles with previously intersected circumcircles onto the intersecting segments and labeled accordingly. For triangles with two or three vertices inside a segment, the vertices that are inside segments are substituted with the projection of their circumcenter onto any segment that intersects their circumcircle and their circumcenter is recalculated from the updated points while an intersection can be detected or while their circumcenter has been projected less than a predetermined number of times. When no circumcircle intersections can be detected, or after circumcenters have been projected a certain number of times, Steiner points are created inside the segments that previously intersected the circumcircles of these triangles by projecting the recalculated circumcenter onto the intersecting segments. Figure 4 illustrates the process of Steiner point creation by circumcenter projection for each case. The Voronoi graph and Delaunay triangulation of the input sites from figure 1 after the third stage of the proposed algorithm are illustrated in figure 5.

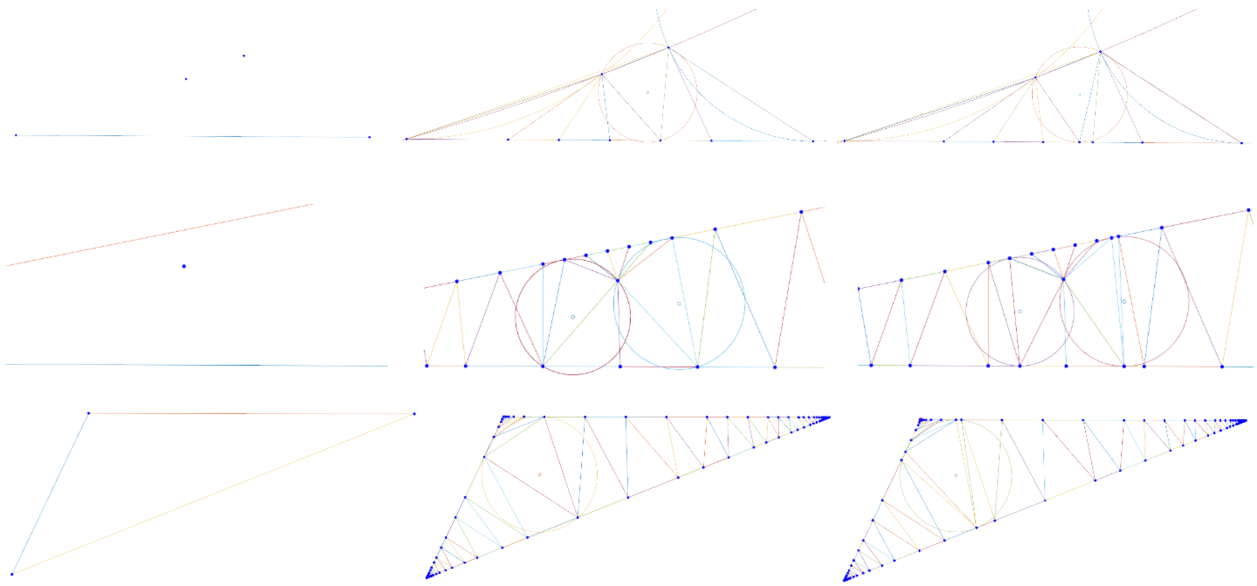

Figure 4. The creation of Steiner points to enforce the Delaunay criterion is shown for the case of two points and one segment (top), one point and two segments (middle), and three segments (bottom.)

Voronoi Graph Extraction from the Delaunay Triangulation of Points

After the Delaunay triangulation of all input and Steiner points is calculated, the Voronoi graph of input points and line segments can be calculated. The Voronoi graph of points and line segments is composed of straight line segments and parabolic arcs. Straight line Voronoi edges are formed in two ways. Adjacent triangles with three vertices inside of the same two segments can be combined into quadrilateral polygons that are analogous with groups of infinitely many infinitesimally thin adjacent Delaunay triangles whose long edges are virtually parallel and whose Voronoi vertices are aligned along straight line segments. Adjacent triangles with no vertices on any segment, one vertex on a segment, two vertices on two different segments, or all vertices on three different segments also form straight Voronoi edges. The parabolic arcs of the Voronoi graph are formed by combining adjacent triangles with one point that is not inside any segment, and two points inside of the same segment. This type of triangle is analogous with groups of infinitely many infinitesimally thin adjacent triangles with one point not inside a segment and two points inside of the same segment. The circumcenters of these triangles trace a parabolic arc whose focus is the vertex of the triangles that is not inside of any segment and whose directrix is a line that includes the triangle vertices that are inside of the segment. Figure 6 illustrates the extraction of the Voronoi graph of points and line segments and its corresponding quasi-triangulation from the Delaunay triangulation of all points from the third stage of the presented process.

Generalization to Arbitrary Curves

The presented algorithm can also be used for finding the Voronoi graph of points and curves of any shape if the equations of the input curves are known. In the case of curves with any shape, Steiner points should be created along the curves during the same three stages described in this paper. In the first stage, curves should be subdivided such that successive points along a curve form Delaunay edges. It should be noted that arc-length should be considered during the subdivision stage instead of only the chord length between successive points along the same arc. Because free-form curves can self-intersect, curves should be subdivided enough times to prevent Delaunay edges that intersect the input curves. In the second and third stages, the same considerations as in the case of straight line segments should be applied, but instead of projecting the circumcenters of triangles onto straight line segments, circumcenters will be projected onto free-form curves and visibility should be considered. A possible step to eliminate the need to consider the visibility of the curves from each circumcenter is to add an additional stage of curve sub-division before the other stages. In this suggested stage, free-form curves should be partitioned into piece-wise curves whose curvatures do not change sign and whose endpoint normal vectors are less than ninety degrees apart.

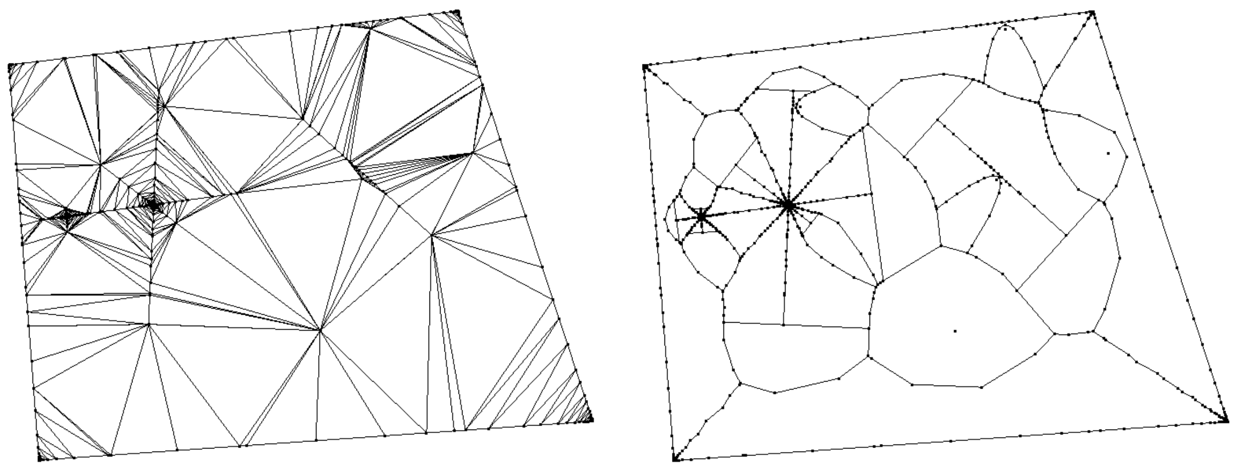

Figure 5. The triangulation of all input and Steiner points is shown on the left, and its corresponding Voronoi graph is shown on the right. Compared with figure 1, the Delaunay triangulation has several more points after the third stage, but the differences between the corresponding Voronoi graphs are minimal.

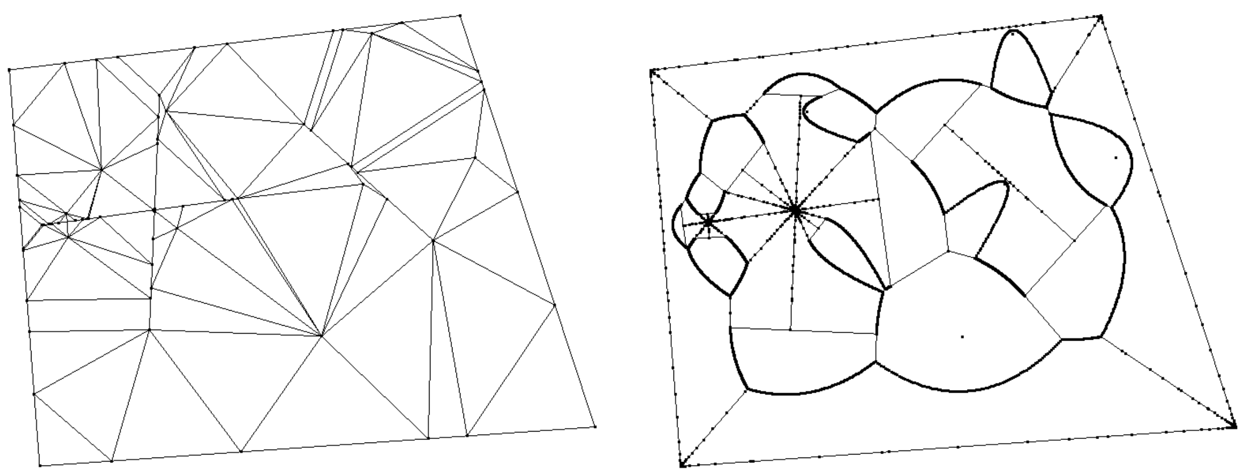

Figure 6. The quasi-triangulation obtained by simplifying the Delaunay triangulation of all the points from the third stage of the presented algorithm is shown on the left, and the Voronoi graph of the input sites, obtained from the quasi-triangulation, is shown on the right.