Computing the S-Bandwidth of a Quantum Network's Graph Representation

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Quantum Computers/Networks

While classical computers use **bits** (0 or 1), quantum computers use **qubits** (a "superposition" of both). Quantum computers can solve certain problems far faster but are very sensitive to external "noise."

- A **superposition** is a probability distribution of distinct states (made possible when **local realism** fails at the quantum scale)
- Recall the famous example of **Schrödinger's cat**:

Figure 1: My cat, **Ash**. (She may be both **awake** and **asleep**. . .)

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When constructing quantum systems, we use undirected graphs to represent groups of linked processors, or **quantum networks**.

- **Vertices** represent separated quantum processors, while **edges** indicate the ability for qubits to move between these processors
- **Edge weights** (typically positive) represent voltages in cases where the processors are part of the same machine
- A system with **quantum channels** between all but two nodes:

Figure 2: The complete multipartite graph $K_{1,1,1,2}$ on 5 vertices.

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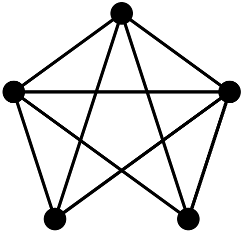

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Quantum Computers/Networks

Definition (Perfect State Transfer)

*A quantum network with localities $u \perp v$ exhibits **perfect state transfer** over a period T of unitary evolution if and only if a qubit p's **wave function** satisfies* $|\Psi(p \in u, t_0)| = |\Psi(p \in v, t_0 + T)| = 1.$

- i.e., when a particle holding quantum information completely "redistributes" from node u at time t_0 to node v at time $t_0 + T$
- Perfect state transfer (PST) thus facilitates high-fidelity information transmission in quantum communication systems
- A qubit's position is a probability distribution ("wave function") rather than a precise location, so PST is not easy to achieve...
- ...but a certain property of network graph representations called S-bandwidth serves as an indicator in some cases

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S-Bandwidth and PST

A small number of **Laplacian integral** graphs are $\{-1, 1\}$- and $\{-1, 0, 1\}$**-diagonalizable**. This can be an indicator of PST in quantum networks.

Definition (Matrix Bandwidth)

A matrix X has a ***matrix bandwidth*** *of $\beta(X) = k$ if and only if $x_{ij} = 0$ whenever $|i - j| \geq k$.*

- For instance, the following matrix has a bandwidth of 2:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 4 & 0 & 5 & 0 & 0 & 0 \\ 0 & 4 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 3 & 0 \\ 0 & 0 & 0 & 3 & 3 & 7 \\ 0 & 0 & 0 & 0 & 5 & 0 \end{bmatrix}$$

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*A graph G has an **S-bandwidth** of $\beta_S(G) = k$ if and only if $L(G) = PDP^{-1}$ for some $P \in S^{n \times n} \subseteq \mathbb{R}^{n \times n}$ with $\beta(P^T P) = k$.*

- i.e., when G's **Laplacian matrix** has a full set of eigenvectors $\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n \in S^n$ such that $\mathrm{v}_i \cdot \mathrm{v}_j = 0$ whenever $|i - j| \geq k$
- We say that a graph is **S-diagonalizable** if its S-bandwidth is finite—when $S = \{-1, 1\}$ or $\{-1, 0, 1\}$, this may indicate PST
- To test a graph for $\{-1, 1\}$- and $\{-1, 0, 1\}$-diagonalizability and determine its exact bandwidths, we first need to find the eigenvectors of its Laplacian matrix...

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Identifying $\{-1, 0, 1\}$-Eigenvectors

- The linear conditions we have identified give the following bound on $\{-1, 0, 1\}$-eigenvectors for an order n Laplacian:

$$\frac{1}{2}\sum_{k=1}^{n}\binom{n}{k}\binom{n-k}{k} \leq 3^{n-2} \quad \forall n \in \mathbb{N}$$

- Therefore, eigenvector identification has complexity $O(3^{n-2})$
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Finding k-Orthogonal Bases

Now, we have all $\{-1,0,1\}$-eigenvectors **stored as columns in matrices** (one for each eigenspace):

$$\begin{array}{rl}
\textbf{Eigenspace 1:} & \left[v_{\lambda_1,1} \mid v_{\lambda_1,2} \mid \dots \mid v_{\lambda_1,r_1}\right] \\
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- If any eigenspace contains less $\{-1,0,1\}$-eigenvectors than its multiplicity, the graph is not $\{-1,0,1\}$-diagonalizable
- Otherwise, convert each matrix to RREF (saving the pivots) to verify that we have enough **linearly independent** eigenvectors
- Similarly, test for $\{-1,1\}$-diagonalizability

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We must find not only linearly independent eigenbases, but those with minimum **k-orthogonality** (i.e., with a Gram matrix $P^T P$ of bandwidth k). Since matrix bandwidth is **permutation-variant**:

- For $k = 1$ (pairwise orthogonality): Check whether the Gram matrix of the basis set is **diagonal**
- For $k = 2$ (quasi-orthogonality): Check whether the Gram matrix of the basis is the adjacency matrix of a **path subgraph**
- For $2 < k < n$: Here we must resort to some more complex **subgraph isomorphism** tests, which can still be improved...

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Now to actually find a basis. . .

- We construct a search tree of eigenvector indices, starting from roots with one index each
- Create children by adding column indices corresponding to eigenvectors, only keeping options that preserve linear independence and maintain k-orthogonality
- Conduct a depth-first search on this lazily constructed tree until a k-orthogonal basis is found
- If no basis is found, repeat for $(k+1)$-orthogonality . . .
- Perform a search for $\{-1, 1\}$-bandwidth as well if appropriate
- Unfortunately, this search tree grows in factorial time, making it another bottleneck. . .

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Minimizing S-Bandwidth

We can now use these foundations to construct the first algorithm to compute a graph's S-bandwidth!

- First, we check each eigenspace for a **k-orthogonal basis** up to $k = \mu - 1$ (where μ is the dimension of the eigenspace)
- If no $(\mu - 1)$-orthogonal basis is found, we use the pivots from our original row reduction to get a μ-orthogonal basis
- With every test, save the value of k and only start checking subsequent eigenspaces against that **orthogonality parameter**
- Complete this process for all eigenspaces to minimize the overall S-bandwidth of the graph, and we are done!

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Results / Findings / Future Work

We have tabulated, with exact $\{-1,1\}$- and $\{-1,0,1\}$-bandwidths,

- all $\{-1,0,1\}$-diagonalizable simple connected graphs on $n \leq 11$ vertices, and
- all $\{-1,0,1\}$-diagonalizable simple connected regular graphs (a superset of the bipartite ones) on $n \leq 14$ vertices

We have also proven several results about the Laplacian spectra of S-diagonalizable graphs, the effects of different edge weights on S-bandwidths, and S-diagonalizable graph composition.

Next, we hope to improve methods of matrix bandwidth reduction, reduce the complexity of our DFS search tree, investigate remaining conjectures on S-diagonalizable graphs.

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We have also proven several results about the Laplacian spectra of S-diagonalizable graphs, the effects of different edge weights on S-bandwidths, and S-diagonalizable graph composition.

Next, we hope to improve methods of matrix bandwidth reduction, reduce the complexity of our DFS search tree, investigate remaining conjectures on S-diagonalizable graphs.

Results / Findings / Future Work

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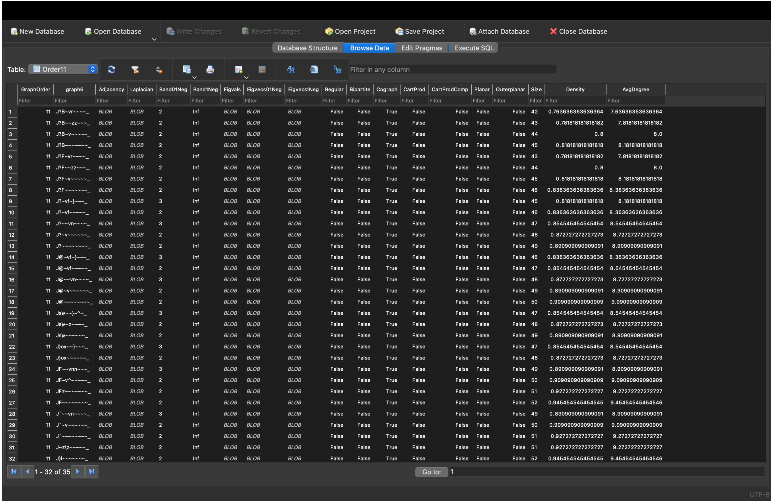

	GraphOrder	graph6	Adjacency	Laplacian	Band01Neg	Band1Neg	Eigvals	Eigvecs01Neg	Eigvecs1Neg	Regular	Bipartite	Cograph	CartProd	CartProdComp	Planar	Outerplanar	Size	Density	AvgDegree
1	11	J?B~vr~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	42	0.763636363636364	7.63636363636364
2	11	J?B~~zz~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	43	0.781818181818182	7.81818181818182
3	11	J?B~v~~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	44	0.8	8.0
4	11	J?B~~~~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	45	0.818181818181818	8.18181818181818
5	11	J?F~vr~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	43	0.781818181818182	7.81818181818182
6	11	J?F~~zz~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	44	0.8	8.0
7	11	J?F~v~~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	45	0.818181818181818	8.18181818181818
8	11	J?F~~~~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	46	0.836363636363636	8.36363636363636
9	11	J?~vf~}~~~_	BLOB	BLOB	3	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	45	0.818181818181818	8.18181818181818
10	11	J?~vf~~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	46	0.836363636363636	8.36363636363636
11	11	J?~~vn~~~~_	BLOB	BLOB	3	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	47	0.854545454545454	8.54545454545454
12	11	J?~v~~~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	48	0.872727272727273	8.72727272727273
13	11	J?~~~~~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	49	0.890909090909091	8.90909090909091
14	11	J@~vf~}~~~_	BLOB	BLOB	3	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	46	0.836363636363636	8.36363636363636
15	11	J@~vf~~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	47	0.854545454545454	8.54545454545454
16	11	J@~~vn~~~~_	BLOB	BLOB	3	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	48	0.872727272727273	8.72727272727273
17	11	J@~v~~~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	49	0.890909090909091	8.90909090909091
18	11	J@~~~~~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	50	0.909090909090909	9.09090909090909
19	11	Js\v~~}~^~_	BLOB	BLOB	3	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	47	0.854545454545454	8.54545454545454
20	11	Js\v~z~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	48	0.872727272727273	8.72727272727273
21	11	Js\v~~~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	49	0.890909090909091	8.90909090909091
22	11	J}ox~~}~~~_	BLOB	BLOB	3	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	47	0.854545454545454	8.54545454545454
23	11	J}ox~~~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	48	0.872727272727273	8.72727272727273
24	11	JF~~vnn~~~_	BLOB	BLOB	3	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	49	0.890909090909091	8.90909090909091
25	11	JF~v^~~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	50	0.909090909090909	9.09090909090909
26	11	JFz~~~~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	51	0.927272727272727	9.27272727272727
27	11	JF~~~~~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	52	0.945454545454545	9.45454545454546
28	11	J`~~vn~~~~_	BLOB	BLOB	3	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	49	0.890909090909091	8.90909090909091
29	11	J`~v~~~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	50	0.909090909090909	9.09090909090909
30	11	J`~~~~~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	51	0.927272727272727	9.27272727272727
31	11	J~z\z~~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	51	0.927272727272727	9.27272727272727
32	11	J}l~~~~~~~_	BLOB	BLOB	2	Inf	BLOB	BLOB	BLOB	False	False	True	False	False	False	False	52	0.945454545454545	9.45454545454546

Figure 3: Some tabulated data on $\{-1, 0, 1\}$-diagonalizable graphs.

Thank you!

Figure 4: Pusheen the Cat <3