Solutions for Fundamentals of Continuum Mechanics

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December 2, 2015

Contents

Part I

Mathematical Preliminaries

Chapter 1

Vectors

1. According to the right hand rule the direction of $\mathbf{u}\times\mathbf{v}$ is given by putting the fingers of the right hand in the direction of $\mathbf{u}$ and curling in the direction of $\mathbf{v}$. Then the thumb of the right hand gives the direction of $\mathbf{u}\times\mathbf{v}$. Using the same procedure with $\mathbf{v}\times\mathbf{u}$, the thumb of the right hand is oriented in the opposite direction corresponding to the insertion of a minus sign. Because $\mathbf{w}=\mathbf{u}\times\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, i.e., not in the plane of $\mathbf{u}$ and $\mathbf{v}$ the scalar product of $\mathbf{w}$ with $\mathbf{u}$ and $\mathbf{v}$ is zero.

2. If $\mathbf{u}\times\mathbf{v}$ has a positive scalar product with $\mathbf{w}$, then $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are right handed. In this case $\mathbf{u}\times\mathbf{v}\cdot\mathbf{w}=\mathbf{v}\times\mathbf{w}\cdot\mathbf{u}=\mathbf{w}\times\mathbf{u}\cdot\mathbf{v}$ are all equal to the volume of the parallelopiped with $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ as edges. Interchanging any two of the vectors in the vector product introduces a minus sign.

3. Letting $\mathbf{w}=\mathbf{u}$ in (1.5) gives $\mathbf{u}\times(\mathbf{v}\times\mathbf{u})=\alpha\mathbf{v}+\beta\mathbf{u}$, where α and β are scalars. Because $\mathbf{u}\times(\mathbf{v}\times\mathbf{u})$ is perpendicular to $\mathbf{u}$

$$\begin{aligned}\mathbf{u}\cdot\{\mathbf{u}\times(\mathbf{v}\times\mathbf{u})\} &= \mathbf{u}\cdot\{\alpha\mathbf{v}+\beta\mathbf{u}\}=0\\ &= \alpha\mathbf{u}\cdot\mathbf{v}+\beta\mathbf{u}\cdot\mathbf{u}=0\\ &= \alpha uv\cos\theta+\beta u^2=0\end{aligned}$$

   Since $u\neq 0$, $\alpha v\cos\theta+\beta u=0$.

4. If $\mathbf{u}\cdot\mathbf{v}=0$ for all $\mathbf{v}$, then the scalar product must be zero for $\mathbf{v}=\alpha\mathbf{u}$ where α is a nonzero scalar. Hence, $\mathbf{u}\cdot\alpha\mathbf{u}=\alpha u^2=0$. Therefore $u=0$ and $\mathbf{u}=0$.

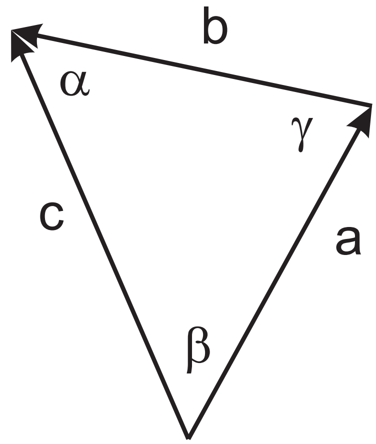
5. (a)

$$\begin{aligned}(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) &= \mathbf{u}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{v}\\ &= u^2+\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{v}-v^2\\ &= u^2-v^2\end{aligned}$$

(b)

$$\begin{aligned}(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v}) &= \mathbf{u}\times\mathbf{u}+\mathbf{v}\times\mathbf{u}-\mathbf{u}\times\mathbf{v}-\mathbf{v}\times\mathbf{v}\\ &= 0+(-\mathbf{u}\times\mathbf{v})-\mathbf{u}\times\mathbf{v}-0\\ &= -2\mathbf{u}\times\mathbf{v}\end{aligned}$$

6. Define $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ to coincide with the edges of the plane triangle with $\mathbf{c}=\mathbf{a}+\mathbf{b}$.



(a)

$$\begin{aligned}c^2 &= \mathbf{c}\cdot\mathbf{c}=(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}\\ &= a^2+2\mathbf{a}\cdot\mathbf{b}+b^2\\ &= a^2+2ab\cos(\pi-\gamma)+b^2\\ &= a^2-2ab\cos(\gamma)+b^2\end{aligned}$$

(b)

$$\begin{aligned}\mathbf{a}\times\mathbf{c} &= \mathbf{a}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}\\ &= \mathbf{a}\times\mathbf{b}\end{aligned}$$

Taking the magnitude of both sides gives $ab\sin(\pi-\gamma)=ac\sin\beta$ or

$$\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$$

Similarly, $\mathbf{b}\times\mathbf{a}=\mathbf{b}\times\mathbf{c}$. Taking the magnitude of both sides gives $ab\sin(\pi-\gamma)=bc\sin\alpha$ or

$$\frac{a}{\sin\alpha}=\frac{c}{\sin\gamma}=\frac{b}{\sin\beta}$$

7. Two edges of the fourth face are $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$. The oriented area of this face is

$$\begin{aligned} \mathbf{n}\Delta S &= \frac{1}{2}(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a}) \\ &= \frac{1}{2}(\mathbf{b}\times\mathbf{c}) - \frac{1}{2}(\mathbf{a}\times\mathbf{c}) - \frac{1}{2}(\mathbf{b}\times\mathbf{a}) \end{aligned}$$

Noting that

$$\frac{1}{2}(\mathbf{b}\times\mathbf{c}) = -S_1 n_1, \quad -\frac{1}{2}(\mathbf{a}\times\mathbf{c}) = -S_2 n_2, \quad -\frac{1}{2}(\mathbf{b}\times\mathbf{a}) = -S_3 n_3$$

gives the result.

8. Forming the vector product of $\mathbf{w}$ with both sides of (1.5) gives

$$\mathbf{w} \times \{\mathbf{u} \times (\mathbf{v} \times \mathbf{w})\} = \alpha \mathbf{w} \times \mathbf{v}$$

Forming the scalar product of both sides with $\mathbf{w} \times \mathbf{v}$ and solving for α gives

$$\alpha = \frac{(\mathbf{w}\times\mathbf{v}) \cdot [\mathbf{w} \times \{\mathbf{u} \times (\mathbf{v}\times\mathbf{w})\}]}{|\mathbf{w}\times\mathbf{v}|^2}$$

Similarly

$$\beta = \frac{(\mathbf{v}\times\mathbf{w}) \cdot [\mathbf{v} \times \{\mathbf{u} \times (\mathbf{v}\times\mathbf{w})\}]}{|\mathbf{v}\times\mathbf{w}|^2}$$

9. A line perpendicular to both of the given lines is $\mathbf{w} = \mathbf{c} + (\mathbf{l}\times\mathbf{m})\,s$. This line will intersect the line

$$\mathbf{u} = \mathbf{a} + \mathbf{l}s$$

if

$$\mathbf{c} + (\mathbf{l}\times\mathbf{m})\,s_{13} = \mathbf{a} + \mathbf{l}s_1$$

for some values of s_{13} and s_1. Forming the scalar product of both sides with $\mathbf{l}\times\mathbf{m}$ and rearranging gives

$$s_{13} = \frac{(\mathbf{l}\times\mathbf{m}) \cdot (\mathbf{a}-\mathbf{c})}{|\mathbf{l}\times\mathbf{m}|}$$

Similarly, $\mathbf{w} = \mathbf{c} + (\mathbf{l}\times\mathbf{m})\,s$ intersects $\mathbf{v} = \mathbf{b} + \mathbf{m}s$ for

$$s_{23} = \frac{(\mathbf{l}\times\mathbf{m}) \cdot (\mathbf{b}-\mathbf{c})}{|\mathbf{l}\times\mathbf{m}|}$$

Because $\mathbf{w} = \mathbf{c} + (\mathbf{l}\times\mathbf{m})\,s$ is perpendicular to both given lines, the minimum distance between them is

$$s_{13} - s_{23} = \frac{(\mathbf{l}\times\mathbf{m}) \cdot (\mathbf{a}-\mathbf{b})}{|\mathbf{l}\times\mathbf{m}|}$$

Consequently the two lines will intersect if

$$\mathbf{a} \cdot (\mathbf{l}\times\mathbf{m}) = \mathbf{b} \cdot (\mathbf{l}\times\mathbf{m})$$

10. Let P be a point on the line joining A and B and $\mathbf{w}$ be the vector joining point O to point P. Then
$$\mathbf{v} + \mathbf{AP} = \mathbf{w}$$
and
$$\mathbf{v} + \mathbf{AB} = \mathbf{u}$$
But since $\mathbf{AP}$ and $\mathbf{AB}$ are collinear, $\mathbf{AP} = t\mathbf{AB}$, where $0 \leq t \leq 1$. Consequently,
$$\mathbf{w} = t\mathbf{u} + (1-t)\,\mathbf{v}$$
is the equation of the line joining A and B. By letting $t = n/\,(m+n)$ gives
$$\mathbf{w} = \frac{n\mathbf{u} + m\mathbf{v}}{n+m}$$

11. Forming the scalar product of $\mathbf{z} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$ with $\mathbf{v} \times \mathbf{w}$ gives
$$\alpha = \frac{\mathbf{z} \cdot (\mathbf{v} \times \mathbf{w})}{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})}$$
and similarly for β and γ. If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar, then $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$, and it is not possible to express an arbitrary vector $\mathbf{z}$ in the given form.

12. Let $\mathbf{w}$ be a vector that lies in both planes. Therefore $\mathbf{w}$ can be expressed as
$$\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}$$
and as
$$\mathbf{w} = \gamma\mathbf{x} + \delta\mathbf{y}$$
Therefore $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{0}$ and $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{w} = \mathbf{0}$. Hence, $\mathbf{w}$ is perpendicular to the normal to the plane of $\mathbf{u}$ and $\mathbf{v}$ and to the normal to the plane of $\mathbf{x}$ and $\mathbf{y}$. Thus, $\mathbf{w}$ is proportional to $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{x} \times \mathbf{y})$. A unit vector in the direction of $\mathbf{w}$ is
$$\mathbf{n} = \frac{(\mathbf{u} \times \mathbf{v}) \times (\mathbf{x} \times \mathbf{y})}{|(\mathbf{u} \times \mathbf{v})|\,|(\mathbf{x} \times \mathbf{y})|}$$

Chapter 2

Tensors

1. Let $\mathbf{S} = \frac{1}{2}\left(\mathbf{F} + \mathbf{F}^T\right)$. If $\mathbf{S}\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{S}^T = \mathbf{u}\cdot\mathbf{S}$ for any $\mathbf{u}$ then $\mathbf{S}$ is symmetric.

$$\begin{aligned}
\mathbf{S}\cdot\mathbf{u} &= \frac{1}{2}\left(\mathbf{F} + \mathbf{F}^T\right)\cdot\mathbf{u} \\
&= \frac{1}{2}\left\{\mathbf{F}\cdot\mathbf{u} + \mathbf{F}^T\cdot\mathbf{u}\right\} \\
&= \frac{1}{2}\left\{\mathbf{u}\cdot\mathbf{F}^T + \mathbf{u}\cdot\mathbf{F}^{T^T}\right\} \\
&= \frac{1}{2}\mathbf{u}\cdot\left\{\mathbf{F}^T + \mathbf{F}\right\} = \mathbf{u}\cdot\mathbf{S}
\end{aligned}$$

Note:

$$\mathbf{F}\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{F}^T = \mathbf{F}^{T^T}\cdot\mathbf{u}$$

for all $\mathbf{u}$ implies the result $\mathbf{F}^{T^T} = \mathbf{F}$. Let $\mathbf{A} = \frac{1}{2}\left(\mathbf{F} - \mathbf{F}^T\right)$. If $\mathbf{A}\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{A}^T = -\mathbf{u}\cdot\mathbf{A}$ for any $\mathbf{u}$ then $\mathbf{A}$ is symmetric.

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{u} &= \frac{1}{2}\left(\mathbf{F} - \mathbf{F}^T\right)\cdot\mathbf{u} \\
&= \frac{1}{2}\left\{\mathbf{F}\cdot\mathbf{u} - \mathbf{F}^T\cdot\mathbf{u}\right\} \\
&= \frac{1}{2}\left\{\mathbf{u}\cdot\mathbf{F}^T - \mathbf{u}\cdot\mathbf{F}^{T^T}\right\} \\
&= \frac{1}{2}\mathbf{u}\cdot\left\{\mathbf{F}^T - \mathbf{F}\right\} = -\mathbf{u}\cdot\mathbf{A}
\end{aligned}$$

2. Substituting (2.10) into (2.5) and using (2.6) gives the result.

3.
$$\left(\mathbf{F}\cdot\mathbf{G}\right)^{-1}\cdot\mathbf{u} = \mathbf{v}$$

implies

$$\mathbf{u} = \mathbf{F}\cdot\mathbf{G}\cdot\mathbf{v}$$

Multiplying from the left by $\mathbf{F}^{-1}$and then $\mathbf{G}^{-1}$ gives

$$\mathbf{G}^{-1} \cdot \mathbf{F}^{-1} \cdot \mathbf{u} = \mathbf{v}$$

The result follows by noting that the two expressions for $\mathbf{v}$ are equal for all $\mathbf{u}$.

4.

$$\mathbf{u} = \mathbf{F} \cdot \mathbf{v}$$

implies

$$\mathbf{v} = \mathbf{F}^{-1} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{F}^{-1^T}$$

But $\mathbf{u} = \mathbf{F} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{F}^T$ gives

$$\mathbf{v} = \mathbf{u} \cdot \mathbf{F}^{T^{-1}}$$

Noting that the two expressions for $\mathbf{v}$ are equal for all $\mathbf{u}$ gives the result.

5. Consider

$$\mathbf{v} = (\mathbf{R} \cdot \mathbf{S}) \cdot \mathbf{u}$$

Forming the scalar product of $\mathbf{v}$ with itself gives

$$\mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot (\mathbf{R} \cdot \mathbf{S})^T \cdot (\mathbf{R} \cdot \mathbf{S}) \cdot \mathbf{u}$$

where the definition of the transpose has been used in the first expression for $\mathbf{v}$ on the right hand side. Using the rule for the transpose of a product (Example 2.5.2) gives

$$\mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \left(\mathbf{S}^T \cdot \mathbf{R}^T\right) \cdot (\mathbf{R} \cdot \mathbf{S}) \cdot \mathbf{u}$$

Rearranging the parentheses gives

$$\mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{S}^T \cdot \left(\mathbf{R}^T \cdot \mathbf{R}\right) \cdot \mathbf{S} \cdot \mathbf{u}$$

But if $\mathbf{R}$ is orthogonal, $\mathbf{R}^T \cdot \mathbf{R} = \mathbf{I}$ yielding

$$\mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{S}^T \cdot \mathbf{S} \cdot \mathbf{u}$$

Similarly, if $\mathbf{S}$ is orthogonal, $\mathbf{S}^T \cdot \mathbf{S} = \mathbf{I}$ and $\mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{u}$. Hence, $\mathbf{R} \cdot \mathbf{S}$ is orthogonal.

6. The angle θ between vectors $\mathbf{u}$ and $\mathbf{v}$ is given in terms of the scalar product by (1.2). Consider the scalar product of $\mathbf{A} \cdot \mathbf{u}$ and $\mathbf{A} \cdot \mathbf{v}$ where $\mathbf{A}$ is orthogonal:

$$(\mathbf{A} \cdot \mathbf{u}) \cdot (\mathbf{A} \cdot \mathbf{v}) = \left(\mathbf{u} \cdot \mathbf{A}^T\right) \cdot (\mathbf{A} \cdot \mathbf{v})$$

where definition of the transpose has been used in the first term on the right. Rearranging the parentheses gives

$$\begin{aligned} (\mathbf{A} \cdot \mathbf{u}) \cdot (\mathbf{A} \cdot \mathbf{v}) &= \mathbf{u} \cdot \left(\mathbf{A}^T \cdot \mathbf{A}\right) \cdot \mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{v} \end{aligned}$$

and the second line follows because $\mathbf{A}$ is orthogonal. Hence, the angle between $\mathbf{u}$ and $\mathbf{v}$ is the same as between $\mathbf{A} \cdot \mathbf{u}$ and $\mathbf{A} \cdot \mathbf{v}$.

Chapter 3

Cartesian Coordinates

1. (a)

$$\begin{aligned} v_1 &= H_{11}u_1 + H_{21}u_2 + H_{31}u_3 \\ v_2 &= H_{12}u_1 + H_{22}u_2 + H_{32}u_3 \\ v_3 &= H_{13}u_1 + H_{23}u_2 + H_{33}u_3 \end{aligned}$$

(b)

$$e = e_{11} + e_{22} + e_{33}$$

(c)

$$\begin{aligned} W &= F_{11}H_{11} + F_{12}H_{12} + F_{13}H_{13} \\ &\quad + F_{21}H_{21} + F_{22}H_{22} + F_{23}H_{23} \\ &\quad + F_{31}H_{31} + F_{32}H_{32} + F_{33}H_{33} \end{aligned}$$

(d)

$$\begin{array}{lll} F_{11} = \alpha G_{11} & F_{12} = \alpha G_{12} & F_{13} = \alpha G_{13} \\ F_{21} = \alpha G_{21} & F_{22} = \alpha G_{22} & F_{23} = \alpha G_{23} \\ F_{31} = \alpha G_{31} & F_{32} = \alpha G_{32} & F_{33} = \alpha G_{33} \end{array}$$

2. (a)

$$\begin{aligned} \delta_{mm} &= \delta_{11} + \delta_{22} + \delta_{33} \\ &= 1 + 1 + 1 = 3 \end{aligned}$$

(b)

$$\begin{aligned}
\delta_{mn}\delta_{mn} &= \delta_{11}\delta_{11} + \delta_{12}\delta_{12} + \delta_{13}\delta_{13} \\
&\quad + \delta_{21}\delta_{21} + \delta_{22}\delta_{22} + \delta_{23}\delta_{23} \\
&\quad + \delta_{31}\delta_{31} + \delta_{32}\delta_{32} + \delta_{33}\delta_{33} \\
&= 1 + 0 + 0 \\
&\quad + 0 + 1 + 0 \\
&\quad + 0 + 0 + 1 \\
&= 3
\end{aligned}$$

3.

$$\begin{aligned}
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} H_{11} & H_{21} & H_{31} \\ H_{12} & H_{22} & H_{32} \\ H_{13} & H_{23} & H_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \\
&= \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix}
\end{aligned}$$

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} = \alpha \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix}$$

4. (a)

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{I} &= (v_i \mathbf{e}_i) \cdot (\delta_{kl} \mathbf{e}_k \mathbf{e}_l) \\
&= v_i \delta_{kl} (\mathbf{e}_i \cdot \mathbf{e}_k) \mathbf{e}_l \\
&= v_i \delta_{kl} \delta_{ik} \mathbf{e}_l \\
&= v_i \delta_{il} \mathbf{e}_l = v_l \mathbf{e}_l = \mathbf{v}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{F} \cdot \mathbf{I} &= (F_{ij} \mathbf{e}_i \mathbf{e}_j) \cdot (\delta_{kl} \mathbf{e}_k \mathbf{e}_l) \\
&= F_{ij} \mathbf{e}_i \delta_{kl} (\mathbf{e}_j \cdot \mathbf{e}_k) \mathbf{e}_l \\
&= F_{ij} \mathbf{e}_i \delta_{kl} \delta_{jk} \mathbf{e}_l \\
&= F_{ij} \mathbf{e}_i \delta_{lj} \mathbf{e}_l = F_{ij} \mathbf{e}_i \mathbf{e}_j = \mathbf{F}
\end{aligned}$$

5. If $\mathbf{u} \cdot \mathbf{v} = u_i v_i = 0$ for any $\mathbf{v}$, then choose

$$v_1 = 1,\ v_2 = 0,\ v_3 = 0$$

$\Rightarrow u_1 = 0$. Choose

$$v_1 = 0,\ v_2 = 1,\ v_3 = 0$$

$\Rightarrow u_2 = 0$. Choose

$$v_1 = 0,\ v_2 = 0,\ v_3 = 1$$

$\Rightarrow u_2 = 0$. Hence, $\mathbf{u} = 0$.

6.

$$\begin{aligned}
tr\mathbf{F} &= \mathbf{F}\cdot\cdot\mathbf{I} \\
&= (F_{ij}\mathbf{e}_i\mathbf{e}_j)\cdot\cdot(\delta_{kl}\mathbf{e}_k\mathbf{e}_l) \\
&= F_{ij}\delta_{kl}\left(\mathbf{e}_j\cdot\mathbf{e}_k\right)\left(\mathbf{e}_i\cdot\mathbf{e}_l\right) \\
&= F_{ij}\delta_{kl}\delta_{jk}\delta_{il} = F_{ij}\delta_{ij} \\
&= F_{jj} = F_{11}+F_{22}+F_{33}
\end{aligned}$$

7.

$$\begin{aligned}
\mathbf{F}\cdot\cdot\mathbf{G} &= F_{ij}G_{ji} \\
&= G_{ji}F_{ij} \\
&= G_{ji}F_{kl}\delta_{ik}\delta_{lj} \\
&= G_{ji}F_{kl}\left(\mathbf{e}_i\cdot\mathbf{e}_k\right)\left(\mathbf{e}_j\cdot\mathbf{e}_l\right) \\
&= (G_{ji}\mathbf{e}_j\mathbf{e}_i)\cdot\cdot(F_{kl}\mathbf{e}_k\mathbf{e}_l) \\
&= \mathbf{G}\cdot\cdot\mathbf{F}
\end{aligned}$$

8.

$$\begin{aligned}
\mathbf{F}\cdot\cdot\mathbf{G} &= F_{ij}G_{ji} \\
&= F_{ij}G^T_{ij} \\
&= \mathbf{F}:\mathbf{G}^T
\end{aligned}$$

$$\begin{aligned}
\mathbf{F}\cdot\cdot\mathbf{G} &= F_{ij}G_{ji} \\
&= F^T_{ji}G_{ji} \\
&= \mathbf{F}^T:\mathbf{G}
\end{aligned}$$

9.

$$\begin{aligned}
\mathbf{F}\cdot\mathbf{G}\cdot\cdot\mathbf{H} &= (F_{ij}\mathbf{e}_i\mathbf{e}_j)\cdot(G_{kl}\mathbf{e}_k\mathbf{e}_l)\cdot\cdot(H_{mn}\mathbf{e}_m\mathbf{e}_n) \\
&= (F_{ij}\mathbf{e}_i\delta_{jk}G_{kl}\mathbf{e}_l)\cdot\cdot(H_{mn}\mathbf{e}_m\mathbf{e}_n) \\
&= (F_{ij}G_{jl}\mathbf{e}_i\mathbf{e}_l)\cdot\cdot(H_{mn}\mathbf{e}_m\mathbf{e}_n) \\
&= F_{ij}G_{jl}\delta_{in}\delta_{lm}H_{mn} = F_{ij}G_{jl}H_{li}
\end{aligned}$$

$$\begin{aligned}
\mathbf{H}\cdot\mathbf{F}\cdot\cdot\mathbf{G} &= (H_{mn}\mathbf{e}_m\mathbf{e}_n)\cdot(F_{ij}\mathbf{e}_i\mathbf{e}_j)\cdot\cdot(G_{kl}\mathbf{e}_k\mathbf{e}_l) \\
&= (H_{mn}\mathbf{e}_m\delta_{ni}F_{ij}\mathbf{e}_j)\cdot\cdot(G_{kl}\mathbf{e}_k\mathbf{e}_l) \\
&= (H_{mi}\mathbf{e}_mF_{ij}\mathbf{e}_j)\cdot\cdot(G_{kl}\mathbf{e}_k\mathbf{e}_l) \\
&= H_{mi}F_{ij}\delta_{jk}\delta_{ml}G_{kl} = H_{li}F_{ij}G_{jl}
\end{aligned}$$

$$\begin{aligned}
\mathbf{G}\cdot\mathbf{H}\cdot\cdot\mathbf{F} &= (G_{kl}\mathbf{e}_k\mathbf{e}_l)\cdot(H_{mn}\mathbf{e}_m\mathbf{e}_n)\cdot\cdot(F_{ij}\mathbf{e}_i\mathbf{e}_j) \\
&= (G_{kl}\mathbf{e}_k\delta_{lm}H_{mn}\mathbf{e}_n)\cdot\cdot(F_{ij}\mathbf{e}_i\mathbf{e}_j) \\
&= (G_{kl}\mathbf{e}_kH_{ln}\mathbf{e}_n)\cdot\cdot(F_{ij}\mathbf{e}_i\mathbf{e}_j) \\
&= G_{kl}\delta_{kj}H_{ln}\delta_{ni}F_{ij} = G_{jl}H_{li}F_{ij}
\end{aligned}$$

10.

$$
\begin{aligned}
\mathbf{F}\cdot\mathbf{G}:\mathbf{H} &= (F_{ij}\mathbf{e}_i\mathbf{e}_j)\cdot(G_{kl}\mathbf{e}_k\mathbf{e}_l):(H_{mn}\mathbf{e}_m\mathbf{e}_n) \\
&= (F_{ij}\mathbf{e}_i\delta_{jk}G_{kl}\mathbf{e}_l):(H_{mn}\mathbf{e}_m\mathbf{e}_n) \\
&= (F_{ij}\mathbf{e}_iG_{jl}\mathbf{e}_l):(H_{mn}\mathbf{e}_m\mathbf{e}_n) \\
&= F_{ij}G_{jl}\delta_{im}\delta_{ln}H_{mn} = F_{ij}G_{jn}H_{in} \\
&= (F_{ij}\mathbf{e}_j\mathbf{e}_i):(G_{km}H_{lm}\mathbf{e}_k\mathbf{e}_l) \\
&= (F_{ij}\mathbf{e}_j\mathbf{e}_i):(G_{kn}\mathbf{e}_k\mathbf{e}_n)\cdot(H_{lm}\mathbf{e}_m\mathbf{e}_l) \\
&= \mathbf{F}^T:\mathbf{G}\cdot\mathbf{H}^T
\end{aligned}
$$

11.

$$
\begin{aligned}
\mathrm{tr}\ (\mathbf{F}\cdot\mathbf{G}) &= \mathrm{tr}\ (F_{ij}G_{jk}\mathbf{e}_i\mathbf{e}_k) \\
&= \mathbf{I}\cdot\cdot(F_{ij}G_{jk}\mathbf{e}_i\mathbf{e}_k) \\
&= \delta_{mn}(\mathbf{e}_m\cdot\mathbf{e}_k)(\mathbf{e}_n\cdot\mathbf{e}_i)(F_{ij}G_{jk}) \\
&= \delta_{mn}\delta_{mk}\delta_{ni}(F_{ij}G_{jk}) \\
&= F_{ij}G_{ji} = \mathbf{F}\cdot\cdot\mathbf{G}
\end{aligned}
$$

Chapter 4

Vector (Cross) Product

1.

$$\begin{aligned}
\mathbf{e}_p \cdot (\mathbf{e}_i \times \mathbf{e}_j) &= \mathbf{e}_p \cdot (\epsilon_{ijk}\mathbf{e}_k) \\
&= \epsilon_{ijk}(\mathbf{e}_p \cdot \mathbf{e}_k) \\
&= \epsilon_{ijk}\delta_{pk} = \epsilon_{ijp}
\end{aligned}$$

2. If j, k, m and n are the free indices, then the equality only needs to be verified for the six cases: (i) $j = k$; (ii) $m = n$; (iii) $j = m$; (iv) $k = n$; (v) $j = n$; (vi) $k = m$.

3. (a)

$$\begin{aligned}
\epsilon_{pqi}\epsilon_{pqj} &= \delta_{qq}\delta_{ij} - \delta_{qi}\delta_{qj} \\
&= 3\delta_{ij} - \delta_{ij} = 2\delta_{ij}
\end{aligned}$$

(b)

$$\epsilon_{pqr}\epsilon_{pqr} = 2\delta_{rr} = 6$$

4.

$$\begin{aligned}
\mathbf{e}_i \times \mathbf{e}_j &= \epsilon_{ijk}\mathbf{e}_k \\
\epsilon_{ijp}(\mathbf{e}_i \times \mathbf{e}_j) &= \epsilon_{ijp}\epsilon_{ijk}\mathbf{e}_k \\
&= 2\delta_{pk}\mathbf{e}_k \\
&\Rightarrow \quad e_p = \frac{1}{2}\epsilon_{ijp}\mathbf{e}_i \times \mathbf{e}_j
\end{aligned}$$

5. (a)

$$\begin{aligned}\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) &= u_i\mathbf{e}_i\times(\epsilon_{pqr}\mathbf{e}_p v_q w_r)\\ &= \epsilon_{psi}\epsilon_{pqr}u_i v_q w_r\mathbf{e}_s\\ &= (\delta_{sq}\delta_{ir}-\delta_{iq}\delta_{sr})\,u_i v_q w_r\mathbf{e}_s\\ &= u_i w_i v_s\mathbf{e}_s - u_i v_i w_r\mathbf{e}_r\\ &= (\mathbf{u}\cdot\mathbf{w})\,\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}\\ &= \mathbf{u}\cdot(\mathbf{wv}-\mathbf{vw})\end{aligned}$$

(b)

$$\begin{aligned}&\mathbf{u}\times(\mathbf{v}\times\mathbf{w})+\mathbf{v}\times(\mathbf{w}\times\mathbf{u})+\mathbf{w}\times(\mathbf{u}\times\mathbf{v})\\ =\;&(\mathbf{u}\cdot\mathbf{w})\,\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}+(\mathbf{v}\cdot\mathbf{u})\,\mathbf{w}-(\mathbf{v}\cdot\mathbf{w})\,\mathbf{u}\\ &+(\mathbf{w}\cdot\mathbf{v})\,\mathbf{u}-(\mathbf{w}\cdot\mathbf{u})\,\mathbf{v}\\ =\;&0\end{aligned}$$

6.

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\times\mathbf{w} &= (\mathbf{e}_i\epsilon_{ijk}u_j v_k)\times w_m\mathbf{e}_m\\ &= \epsilon_{imn}\mathbf{e}_n\epsilon_{ijk}u_j v_k w_m\\ &= (\delta_{mj}\delta_{nk}-\delta_{mk}\delta_{nj})\,\mathbf{e}_n u_j v_k w_m\\ &= u_j w_j v_k\mathbf{e}_k - v_k w_k u_j\mathbf{e}_j\\ &= (\mathbf{u}\cdot\mathbf{w})\,\mathbf{v}-(\mathbf{v}\cdot\mathbf{w})\,\mathbf{u}\end{aligned}$$

Therefore, $\alpha=-\mathbf{v}\cdot\mathbf{w}$ and $\beta=\mathbf{u}\cdot\mathbf{w}$

7. Let $\mathbf{u}=\mathbf{n}$ and $\mathbf{w}=\mathbf{n}$ in (4.14) to get

$$\mathbf{n}\times(\mathbf{v}\times\mathbf{n})=(\mathbf{n}\cdot\mathbf{n})\,\mathbf{v}-(\mathbf{n}\cdot\mathbf{v})\,\mathbf{n}$$

Since $\mathbf{n}$ is a unit vector $\mathbf{n}\cdot\mathbf{n}=1$. Rearranging give

$$\mathbf{v}=(\mathbf{n}\cdot\mathbf{v})\,\mathbf{n}+\mathbf{n}\times(\mathbf{v}\times\mathbf{n})$$

8. (a)

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= \epsilon_{ijk}a_j b_k\epsilon_{imn}c_m d_n\\ &= (\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{mk})\,(a_j b_k c_m d_n)\\ &= (a_m c_m)\,(b_k d_k)-(a_j d_j)\,(b_m c_m)\\ &= (\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})\,(\mathbf{b}\cdot\mathbf{c})\end{aligned}$$

(b)

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) &= \mathbf{e}_i\epsilon_{ijk}\epsilon_{jmn}a_m b_n\epsilon_{kpq}c_p d_q\\ &= \mathbf{e}_i\,(\delta_{mk}\delta_{in}-\delta_{mi}\delta_{kn})\,a_m b_n\epsilon_{kpq}c_p d_q\\ &= (\mathbf{e}_n b_n)\,\epsilon_{kpq}a_k c_p d_q-(\mathbf{e}_m a_m)\,\epsilon_{kpq}b_k c_p d_q\\ &= [\mathbf{c}\cdot(\mathbf{d}\times\mathbf{a})]\,\mathbf{b}-[\mathbf{c}\cdot(\mathbf{d}\times\mathbf{b})]\,\mathbf{a}\end{aligned}$$

9. Writing in index notation

$$W_{ik}u_k = \epsilon_{ijk}w_j u_k$$

Because this applies for all u_k

$$W_{ik} = \epsilon_{ijk}w_j$$

Multiplying the preceding equation by ϵ_{ikl} gives

$$\begin{aligned}
\epsilon_{ikl}W_{ik} &= \epsilon_{ikl}\epsilon_{ijk}w_j \\
&= -\epsilon_{ikl}\epsilon_{ikj}w_j \\
&= -2\delta_{lj}w_j \\
&= -2w_l
\end{aligned}$$

Therefore,

$$w_l = -\frac{1}{2}\epsilon_{ikl}W_{ik}$$

If $\mathbf{W}$ is not anti-symmetric, then $\mathbf{w} = \mathbf{0}$.

10. (a)

$$\begin{aligned}
\mathbf{n} \times \dot{\mathbf{n}} &= \mathbf{n} \times (\mathbf{w} \times \mathbf{n}) \\
&= \mathbf{n} \cdot (\mathbf{n}\mathbf{w} - \mathbf{w}\mathbf{n})
\end{aligned}$$

where the second line follows from problem 5. Noting that $\mathbf{n} \cdot \mathbf{n} = 1$ because $\mathbf{n}$ is a unit vector and $\mathbf{n} \cdot \mathbf{w} = 0$ because $\mathbf{n}$ is orthogonal to $\mathbf{w}$ gives the result.

(b)

$$\begin{aligned}
W_{ik} &= \epsilon_{ijk}w_j \\
&= \epsilon_{ijk}\left(\epsilon_{jlm}n_l\dot{n}_m\right) \\
&= \left(\delta_{kl}\delta_{im} - \delta_{il}\delta_{km}\right)n_l\dot{n}_m \\
&= \dot{n}_i n_k - n_i\dot{n}_k
\end{aligned}$$

or

$$\mathbf{W} = \dot{\mathbf{n}}\mathbf{n} - \mathbf{n}\dot{\mathbf{n}}$$

in dyadic notation.

11.

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{F}) &= u_i\mathbf{e}_i \cdot (v_j\mathbf{e}_j \times F_{kl}\mathbf{e}_k\mathbf{e}_l) \\
&= u_i\mathbf{e}_i \cdot (v_j F_{kl}\epsilon_{jkm}\mathbf{e}_m\mathbf{e}_l) \\
&= u_i v_j F_{kl}\epsilon_{jki}\mathbf{e}_l \\
&= \epsilon_{ijk}u_i v_j\mathbf{e}_k \cdot F_{ml}\mathbf{e}_m\mathbf{e}_l \\
&= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{F}
\end{aligned}$$

12. Writing both sides in index form gives

$$\epsilon_{ijr} u_j v_r F^*_{it} \mathbf{e}_t = \epsilon_{iqt} \mathbf{e}_t F_{ij} u_j F_{qr} v_r$$

Because this applies for all $\mathbf{u}$ and $\mathbf{v}$,

$$\epsilon_{ijr} F^*_{it} = \epsilon_{iqt} F_{ij} F_{qr}$$

Multiplying both sides by ϵ_{jrk} and summing gives

$$\begin{aligned} \epsilon_{jrk}\epsilon_{ijr} F^*_{it} &= \epsilon_{jrk}\epsilon_{iqt} F_{ij} F_{qr} \\ 2\delta_{ki} F^*_{it} &= \epsilon_{jrk}\epsilon_{iqt} F_{ij} F_{qr} \end{aligned}$$

or

$$F^*_{kt} = \frac{1}{2}\epsilon_{jrk}\epsilon_{iqt} F_{ij} F_{qr}$$

and re-labelling indices gives the result.

13.

$$\begin{aligned} g &= \mathbf{g}_1 \cdot (\mathbf{g}_2 \times \mathbf{g}_3) \\ &= (\mathbf{e}_2 + \mathbf{e}_3) \cdot [(\mathbf{e}_1 + \mathbf{e}_3) \times (\mathbf{e}_1 + \mathbf{e}_2)] \\ &= (\mathbf{e}_2 + \mathbf{e}_3) \cdot [\mathbf{e}_3 + \mathbf{e}_2 - \mathbf{e}_1] = 2 \end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{g}^1 &= \frac{1}{2}(\mathbf{g}_2 \times \mathbf{g}_3) \\ &= \frac{1}{2}(\mathbf{e}_3 + \mathbf{e}_2 - \mathbf{e}_1) \end{aligned}$$

$$\begin{aligned} \mathbf{g}^2 &= \frac{1}{2}(\mathbf{g}_3 \times \mathbf{g}_1) \\ &= \frac{1}{2}(\mathbf{e}_3 - \mathbf{e}_2 + \mathbf{e}_1) \end{aligned}$$

$$\begin{aligned} \mathbf{g}^3 &= \frac{1}{2}(\mathbf{g}_1 \times \mathbf{g}_2) \\ &= \frac{1}{2}(-\mathbf{e}_3 + \mathbf{e}_2 + \mathbf{e}_1) \end{aligned}$$

14. (a) Since $\mathbf{g}^1$ is perpendicular to $\mathbf{g}_2$, $\mathbf{g}^1 \cdot \mathbf{g}_2 = 0$ and $\mathbf{g}^1$ also lies in the 12 plane. Therefore

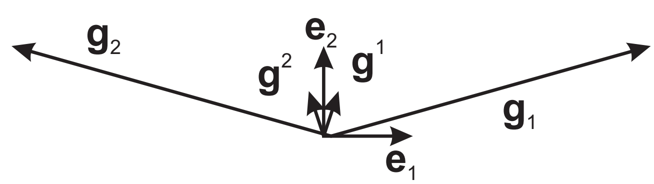
$$\begin{aligned} \mathbf{g}^1 &= \alpha \mathbf{g}_2 \times \mathbf{e}_3 \\ &= \alpha (3\mathbf{e}_2 + \mathbf{e}_1) \end{aligned}$$

But $\mathbf{g}^1 \cdot \mathbf{g}_1 = 1$ gives $\alpha = 1/6$. Therefore

$$\mathbf{g}^1 = \frac{1}{6}\left(\mathbf{e}_1 + 3\mathbf{e}_2\right)$$

Similarly

$$\mathbf{g}^2 = \frac{1}{6}\left(-\mathbf{e}_1 + 3\mathbf{e}_2\right)$$



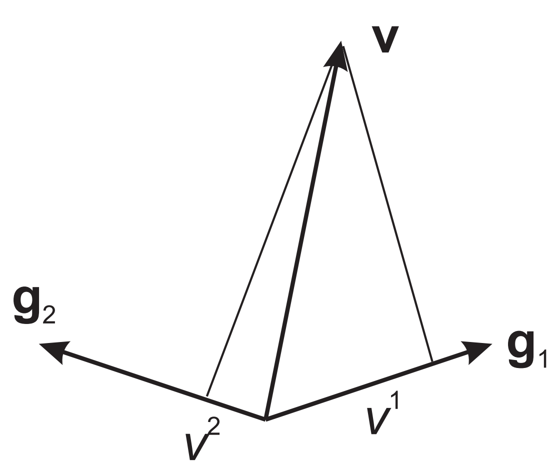
Original and dual base vectors.

(b) $\mathbf{v} = \mathbf{e}_1 + 5\mathbf{e}_2 = v^i \mathbf{g}_i = v_i \mathbf{g}^i$.

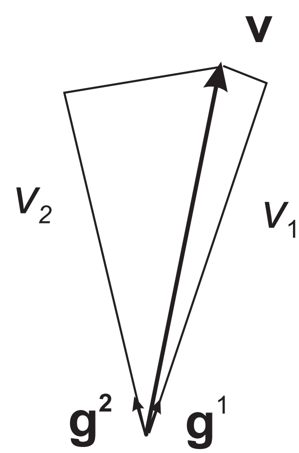
$$\begin{aligned} v^1 &= \mathbf{g}^1 \cdot \mathbf{v} = \frac{8}{3} \\ v^2 &= \mathbf{g}^2 \cdot \mathbf{v} = \frac{7}{3} \\ v_1 &= \mathbf{g}_1 \cdot \mathbf{v} = 8 \\ v_2 &= \mathbf{g}_2 \cdot \mathbf{v} = 2 \end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{v} &= 8\mathbf{g}^1 + 2\mathbf{g}^2 \\ &= \frac{8}{3}\mathbf{g}_1 + \frac{7}{3}\mathbf{g}_2 \end{aligned}$$



Components of $\mathbf{v}$ relative to original base vectors.



Components of **v** relative to dual base vectors.

Chapter 5

Determinants

1.

$$\begin{aligned}
\det M &= \epsilon_{ijk} M_{i1} M_{j2} M_{k3} \\
&= M_{11} \{\epsilon_{123} M_{22} M_{33} + \epsilon_{132} M_{32} M_{23}\} \\
&\quad + M_{21} \{\epsilon_{213} M_{32} M_{13} + \epsilon_{213} M_{12} M_{33}\} \\
&\quad + M_{31} \{\epsilon_{312} M_{12} M_{23} + \epsilon_{321} M_{22} M_{13}\} \\
&= M_{11} \{(+1)\, M_{22} M_{33} + (-1)\, M_{32} M_{23}\} \\
&\quad + M_{21} \{(+1)\, M_{32} M_{13} + (-1)\, M_{12} M_{33}\} \\
&\quad + M_{31} \{(+1)\, M_{12} M_{23} + (-1)\, M_{22} M_{13}\} \\
&= M_{11} \{M_{22} M_{33} - M_{32} M_{23}\} \\
&\quad - M_{21} \{M_{12} M_{33} - M_{32} M_{13}\} \\
&\quad + M_{31} \{M_{12} M_{23} - M_{22} M_{13}\}
\end{aligned}$$

Therefore

$$\begin{aligned}
\det M &= M_{11} \begin{vmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{vmatrix} - M_{21} \begin{vmatrix} M_{12} & M_{13} \\ M_{32} & M_{33} \end{vmatrix} \\
&\quad + M_{31} \begin{vmatrix} M_{12} & M_{13} \\ M_{22} & M_{23} \end{vmatrix}
\end{aligned}$$

2.

$$\det M = \epsilon_{ijk} M_{1i} M_{2j} M_{3k}$$

Interchanging row 1 and row 2 gives

$$\begin{aligned}
& \epsilon_{ijk} M_{2i} M_{1j} M_{3k} \\
&= -\epsilon_{jik} M_{2i} M_{1j} M_{3k} \\
&= -\epsilon_{ijk} M_{1i} M_{2j} M_{3k} \\
&= -\det M
\end{aligned}$$

3. Consider

$$h_{pqr} = \epsilon_{ijk} M_{ip} M_{jq} M_{kr}$$

Then $h_{123} = \epsilon_{ijk} M_{i1} M_{j2} M_{k3} = \det M$. If any two indices are the same, the result is zero:

$$\begin{aligned}
h_{pqq} &= \epsilon_{ijk} M_{ip} M_{jq} M_{kq}, \text{ (no sum on } q) \\
&= -\epsilon_{ikj} M_{ip} M_{jq} M_{kq}, \text{ (no sum on } q) \\
&= -\epsilon_{ijk} M_{ip} M_{kq} M_{jq}, \text{ (no sum on } q) \\
&= -\epsilon_{ijk} M_{ip} M_{jq} M_{kq}, \text{ (no sum on } q) \\
&= 0
\end{aligned}$$

The result is zero because the top and bottom lines are the negative of each other. Cyclically rotating the indices does not change the value; e.g.,

$$\begin{aligned}
h_{rpq} &= \epsilon_{ijk} M_{ir} M_{jp} M_{kq} \\
&= \epsilon_{jki} M_{ir} M_{jp} M_{kq} \\
&= \epsilon_{ijk} M_{kr} M_{ip} M_{jq} \\
&= \epsilon_{ijk} M_{ip} M_{jq} M_{kr} \\
&= h_{pqr}
\end{aligned}$$

But reversing any two indices introduces a minus sign. E.g.,

$$\begin{aligned}
h_{mln} &= \epsilon_{ijk} M_{im} M_{jl} M_{kn} \\
&= -\epsilon_{jik} M_{im} M_{jl} M_{kn} \\
&= -\epsilon_{ijk} M_{jm} M_{il} M_{kn} \\
&= -\epsilon_{ijk} M_{il} M_{jm} M_{kn} \\
&= -h_{lmn}
\end{aligned}$$

With these results it is possible to easy enumerate results for all numeric values. Consequently,

$$h_{lmn} = \epsilon_{lmn} \det M$$

4.

$$\begin{aligned}
c_{i1} &= \epsilon_{ijk} M_{j2} M_{k3} \\
&= \frac{1}{2} \left(\epsilon_{123} \epsilon_{ijk} M_{j2} M_{k3} + \epsilon_{132} \epsilon_{ikj} M_{j2} M_{k3} \right) \\
&= \frac{1}{2} \epsilon_{ijk} \left(\epsilon_{123} M_{j2} M_{k3} + \epsilon_{132} M_{j3} M_{k2} \right) \\
&= \frac{1}{2} \epsilon_{ijk} \epsilon_{1mn} M_{jm} M_{kn}
\end{aligned}$$

Because the second index is arbitrary, it can be changed to "p". Relabelling indices then gives Equation (5.6).

5.

$$c_{li} = \frac{1}{2}\epsilon_{lmn}\epsilon_{ijk}M_{mj}M_{nk}$$

$$\begin{aligned}
c_{21} &= \frac{1}{2}\epsilon_{2mn}\epsilon_{1jk}M_{mj}M_{nk} \\
&= \frac{1}{2}\epsilon_{2mn}\left\{\epsilon_{123}M_{m2}M_{n3} + \epsilon_{132}M_{m3}M_{n2}\right\} \\
&= \frac{1}{2}\left\{\epsilon_{2mn}M_{m2}M_{n3} - \epsilon_{2mn}M_{m3}M_{n2}\right\} \\
&= \frac{1}{2}\left\{\epsilon_{2mn}M_{m2}M_{n3} - \epsilon_{2nm}M_{n3}M_{m2}\right\} \\
&= \frac{1}{2}\left\{\epsilon_{2mn}M_{m2}M_{n3} - \epsilon_{2mn}M_{m3}M_{n2}\right\} \\
&= \epsilon_{2mn}M_{m2}M_{n3} \\
&= M_{32}M_{13} - M_{12}M_{33} \\
&= (-1)^{2+1}\begin{vmatrix} M_{12} & M_{13} \\ M_{32} & M_{33} \end{vmatrix}
\end{aligned}$$

$$\begin{aligned}
c_{13} &= \frac{1}{2}\epsilon_{1nm}\epsilon_{3jk}M_{mj}M_{nk} \\
&= \frac{1}{2}\epsilon_{3jk}\left(\epsilon_{123}M_{2j}M_{3k} + \epsilon_{312}M_{3j}M_{2k}\right) \\
&= \frac{1}{2}\left(\epsilon_{3jk}M_{2j}M_{3k} - \epsilon_{3jk}M_{3j}M_{2k}\right) \\
&= \epsilon_{3jk}M_{2j}M_{3k} \\
&= \epsilon_{312}M_{21}M_{32} + \epsilon_{321}M_{22}M_{31} \\
&= M_{21}M_{32} - M_{22}M_{31} \\
&= (-1)^{3+1}\begin{vmatrix} M_{21} & M_{22} \\ M_{31} & M_{32} \end{vmatrix}
\end{aligned}$$

6. From Equation (4.10)

$$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\mathbf{d}\cdot\mathbf{e}\times\mathbf{f} = \begin{vmatrix} d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{vmatrix}$$

where the second equality results because the determinant of matrix is equal to the determinant of its transpose. The product of the two matrices is

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{bmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{bmatrix} = \begin{bmatrix} \mathbf{a}\cdot\mathbf{d} & \mathbf{a}\cdot\mathbf{e} & \mathbf{a}\cdot\mathbf{f} \\ \mathbf{b}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{e} & \mathbf{b}\cdot\mathbf{f} \\ \mathbf{c}\cdot\mathbf{d} & \mathbf{c}\cdot\mathbf{e} & \mathbf{c}\cdot\mathbf{f} \end{bmatrix}$$

Noting that the determinant of a product is the product of the determinants establishes the result.

7. Setting $i = m$ gives

$$\begin{aligned}
\epsilon_{mjk}\epsilon_{mnp} &= \begin{vmatrix} \delta_{mm} & \delta_{mn} & \delta_{mp} \\ \delta_{jm} & \delta_{jn} & \delta_{jp} \\ \delta_{km} & \delta_{kn} & \delta_{kp} \end{vmatrix} \\
&= \delta_{mm} \begin{vmatrix} \delta_{jn} & \delta_{jp} \\ \delta_{kn} & \delta_{kp} \end{vmatrix} - \delta_{mn} \begin{vmatrix} \delta_{jm} & \delta_{jp} \\ \delta_{km} & \delta_{kp} \end{vmatrix} \\
&\quad + \delta_{mp} \begin{vmatrix} \delta_{jm} & \delta_{jn} \\ \delta_{km} & \delta_{kn} \end{vmatrix} \\
&= 3\left(\delta_{jn}\delta_{kp} - \delta_{jp}\delta_{kn}\right) - \delta_{mn}\left(\delta_{jm}\delta_{kp} - \delta_{jp}\delta_{km}\right) \\
&\quad + \delta_{mp}\left(\delta_{jm}\delta_{kn} - \delta_{km}\delta_{jn}\right) \\
&= 3\left(\delta_{jn}\delta_{kp} - \delta_{jp}\delta_{kn}\right) - \left(\delta_{nj}\delta_{kp} - \delta_{kn}\delta_{jp}\right) \\
&\quad - \left(\delta_{nj}\delta_{kp} - \delta_{kn}\delta_{jp}\right) \\
&= \left(\delta_{jn}\delta_{kp} - \delta_{jp}\delta_{kn}\right)
\end{aligned}$$

8.

$$\epsilon_{lmn} \det(M) = \epsilon_{ijk} M_{il} M_{jm} M_{kn}$$

Multiplying each side by ϵ_{lmn} and summing gives

$$\begin{aligned}
\epsilon_{lmn}\epsilon_{lmn} \det(M) &= \epsilon_{lmn}\epsilon_{ijk} M_{il} M_{jm} M_{kn} \\
6 \det(M) &= \epsilon_{lmn}\epsilon_{ijk} M_{il} M_{jm} M_{kn}
\end{aligned}$$

or

$$\det(M) = \frac{1}{6}\epsilon_{lmn}\epsilon_{ijk} M_{il} M_{jm} M_{kn}$$

9.

$$\begin{aligned}
(\mathbf{M}\cdot\mathbf{a}\times\mathbf{M}\cdot\mathbf{b})\cdot\mathbf{M} &= (M_{ik}a_k\mathbf{e}_i \times M_{jl}b_l\mathbf{e}_j)\cdot M_{pq}\mathbf{e}_p\mathbf{e}_q \\
&= (M_{ik}a_k M_{jl}b_l\epsilon_{ijn}\mathbf{e}_n)\cdot M_{pq}\mathbf{e}_p\mathbf{e}_q \\
&= M_{ik}a_k M_{jl}b_l\epsilon_{ijn}M_{nq}\mathbf{e}_q \\
&= \left(\epsilon_{ijn}M_{ik}M_{jl}M_{nq}\right) a_k b_l \mathbf{e}_q \\
&= \epsilon_{klq}\det(M) a_k b_l \mathbf{e}_q \\
&= \det(M)\,(\mathbf{a}\times\mathbf{b})
\end{aligned}$$

10.

$$\begin{aligned}
(\mathbf{M}\cdot\mathbf{a})\cdot(\mathbf{M}\cdot\mathbf{b})\times(\mathbf{M}\cdot\mathbf{c}) &= (M_{il}a_l\mathbf{e}_i)\cdot(M_{jm}b_m\mathbf{e}_j)\times(M_{kn}c_n\mathbf{e}_k) \\
&= (M_{il}a_l\mathbf{e}_i)\cdot(M_{jm}b_m M_{kn}c_n\epsilon_{jkp}\mathbf{e}_p) \\
&= \left(\epsilon_{jkp}M_{il}M_{jm}M_{kn}\right) a_l b_m c_n \\
&= \epsilon_{lmn}\det(M) a_l b_m c_n \\
&= \det(M)\,(\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})
\end{aligned}$$

11.

$$\begin{aligned}
\mathbf{A}^T \cdot \mathbf{A} &= \mathbf{I} \\
\det\left(\mathbf{A}^T \cdot \mathbf{A}\right) &= \det(\mathbf{I}) = 1 \\
\det\left(\mathbf{A}^T\right)\det(\mathbf{A}) &= (\det(\mathbf{A}))^2 = 1
\end{aligned}$$

Therefore

$$\det(\mathbf{A}) = \pm 1$$

From Problem 10

$$(\mathbf{A} \cdot \mathbf{a}) \cdot (\mathbf{A} \cdot \mathbf{b}) \times (\mathbf{A} \cdot \mathbf{c}) = \det(A)\,(\mathbf{a} \cdot \mathbf{b} \times \mathbf{c})$$

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are right-handed, $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} > \mathbf{0}$. If $\mathbf{A}$ transforms a right-handed coordinate system to a right-handed coordinate system, then the left side is also positive and $\det(A) = +1$. If $\mathbf{A}$ transforms a right-handed coordinate system to a left-handed coordinate system, then the left side is negative and $\det(A) = -1$.

Chapter 6

Change of Orthonormal Basis

1.

$$\mathbf{e}'_k = \mathbf{A} \cdot \mathbf{e}_k = \mathbf{e}_k \cdot \mathbf{A}^T$$

Therefore, the components of a vector in the primed system are given by

$$\begin{aligned} v'_m &= \mathbf{e}'_m \cdot \mathbf{v} = \left(\mathbf{e}_k \cdot \mathbf{A}^T\right) \cdot \mathbf{v} \\ &= \mathbf{e}_k \cdot \mathbf{A}^T \cdot \mathbf{v} \end{aligned}$$

2.

$$\begin{aligned} v_i &= \mathbf{e}_i \cdot \mathbf{v} \\ &= \mathbf{e}_i \cdot (v'_k \mathbf{e}'_k) \\ &= (\mathbf{e}_i \cdot \mathbf{e}'_k)\, v'_k \\ &= A_{ik} v'_k \end{aligned}$$

3.

$$\begin{aligned} F_{ij} &= \mathbf{e}_i \cdot \mathbf{F} \cdot \mathbf{e}_j \\ &= \mathbf{e}_i \cdot (F'_{kl} \mathbf{e}'_k \mathbf{e}'_l) \cdot \mathbf{e}_j \\ &= (\mathbf{e}_i \cdot \mathbf{e}'_k)\, F'_{kl}\, (\mathbf{e}'_l \cdot \mathbf{e}_j) \\ &= A_{ik} F'_{kl} A_{jl} \end{aligned}$$

4. $\mathbf{A} = A_{ij}\mathbf{e}_i\mathbf{e}_j$, $\mathbf{A}^T = A_{lk}\mathbf{e}_k\mathbf{e}_l$ and $\mathbf{I} = \delta_{mn}\mathbf{e}_m\mathbf{e}_n$

$$\begin{aligned} \mathbf{e}_p \cdot (A_{lk}\mathbf{e}_k\mathbf{e}_l \cdot A_{ij}\mathbf{e}_i\mathbf{e}_j) \cdot \mathbf{e}_q &= \mathbf{e}_p \cdot (\delta_{mn}\mathbf{e}_m\mathbf{e}_n) \cdot \mathbf{e}_q \\ \mathbf{e}_p \cdot (A_{lk}\mathbf{e}_k\delta_{li}A_{ij}\mathbf{e}_j) \cdot \mathbf{e}_q &= \delta_{pm}\delta_{mn}\delta_{nq} \\ A_{lk}\boldsymbol{\delta}_{pk}A_{lj}\delta_{jq} &= \delta_{pm}\delta_{mq} \\ A_{lp}A_{lq} &= \delta_{pq} \end{aligned}$$

Beginning with $\mathbf{A} \cdot \mathbf{A}^T = \mathbf{I}$ establishes the first equality of (6.4).

5.
$$\mathbf{e}'_3 = \boldsymbol{\lambda}$$

Because $\mathbf{e}'_2$ is perpendicular to $\mathbf{e}'_3$ and $\mathbf{e}_1$

$$\begin{aligned} \mathbf{e}'_2 &= \alpha \mathbf{e}'_3 \times \mathbf{e}_1 \\ &= \alpha \left(\lambda_3 \mathbf{e}_2 - \lambda_2 \mathbf{e}_3 \right) \end{aligned}$$

where α is a scalar. Choose α so that $\mathbf{e}'_2$ is a unit vector:

$$\alpha = \pm \sqrt{\lambda_2^2 + \lambda_3^2}$$

and take the + sign so that $\mathbf{e}'_2$ has a positive projection on $\mathbf{e}_2$. Choose $\mathbf{e}'_1$ so that the primed system is right-handed:

$$\begin{aligned} \mathbf{e}'_1 &= \mathbf{e}'_2 \times \mathbf{e}'_3 \\ &= \sqrt{\lambda_2^2 + \lambda_3^2}\, \mathbf{e}_1 - \frac{\lambda_1 \lambda_2}{\sqrt{\lambda_2^2 + \lambda_3^2}} \mathbf{e}_2 - \frac{\lambda_1 \lambda_3}{\sqrt{\lambda_2^2 + \lambda_3^2}} \mathbf{e}_3 \end{aligned}$$

Noting that $\mathbf{e}'_k = A_{ki} \mathbf{e}_i$ and $\mathbf{e}'_k \cdot \mathbf{e}_l = A_{lk}$ establishes the result.

6. Because $u_k = A_{kl} u'_l$,

$$\begin{aligned} u_k v_k &= A_{kl} u'_l v_k \\ &= u'_l \left(A_{kl} v_k \right) = \alpha \end{aligned}$$

for a scalar α. Defining
$$v'_l = A_{kl} v_k$$

establishes the v_k as components of a vector because they transform like one under a rotation of the coordinate system.

7. Because $F_{pq} = A_{pr} F'_{rs} A_{qs}$,

$$\begin{aligned} A_{pr} F'_{rs} A_{qs} G_{qp} &= \alpha \\ F'_{rs} \left(A_{qs} G_{qp} A_{pr} \right) &= \alpha \end{aligned}$$

for a scalar α. Defining

$$G'_{sr} = A_{qs} G_{qp} A_{pr}$$

establishes the G_{qp} as components of a tensor because they transform like one under a rotation of axes.

8.

$$\begin{aligned} w_k &= \epsilon_{ijk} u_i v_j \\ &= \epsilon_{ijk} \left(A_{il} u'_l \right) \left(A_{jm} v'_m \right) \end{aligned}$$

Multiplying both sides by A_{kr} and summing gives

$$\begin{aligned} A_{kr}w_k &= \epsilon_{ijk}A_{il}A_{jm}A_{kr}u'_l v'_m \\ &= \det(A)\epsilon_{lmr}u'_l v'_m \end{aligned}$$

For transformation from a right (left) - handed system to a right (left) - handed system $\det(A) = 1$. In this case $w'_r = A_{kr}w_k$ and the components of **w** transform like components of a vector. If, however, the A_{kr} transform from a right (left) - handed system to a left (right) - handed system $\det(A) = -1$ and the components of the vector product are not tensor components as defined here.

9. Because

$$F'_{mn} = A_{im}A_{jn}F_{ij}$$

and

$$G_{kl} = A_{kp}A_{lr}G'_{pr}$$

$$\begin{aligned} F'_{mn} &= A_{im}A_{jn}H_{ijkl}A_{kp}A_{lq}G'_{pr} \\ &= H'_{mnpr}G'_{pr} \end{aligned}$$

Therefore

$$H'_{mnpr} = A_{im}A_{jn}H_{ijkl}A_{kp}A_{lq}$$

10.

$$\begin{aligned} F'_{pq} &= A_{ip}A_{jq}F_{ij} \\ &= A_{ip}A_{jq}F_{ji} \\ &= A_{jq}A_{ip}F_{ji} \\ &= F'_{qp} \end{aligned}$$

Hence the components are symmetric in any rectangular cartesian coordinate system.

11.

$$\begin{aligned} F'_{pq} &= A_{ip}A_{jq}F_{ij} \\ &= -A_{ip}A_{jq}F_{ji} \\ &= -A_{jq}A_{ip}F_{ji} \\ &= -F'_{qp} \end{aligned}$$

Hence the components are anti-symmetric in any rectangular cartesian coordinate system.

12.

$$F'_{pq} = A_{ip}A_{jq}F_{ij}$$

$$\begin{aligned}
\mathrm{tr}\mathbf{F} &= F'_{pp} \\
&= A_{ip}A_{jp}F_{ij} \\
&= \delta_{ij}F_{ij} \\
&= F_{jj}
\end{aligned}$$

13.

$$\begin{aligned}
F_{kl}F_{lk} &= \left(A_{kp}A_{lq}F'_{pq}\right)\left(A_{lr}A_{ks}F'_{rs}\right) \\
&= \left(A_{lq}A_{lr}\right)\left(A_{kp}A_{ks}\right)F'_{pq}F'_{rs} \\
&= \delta_{qr}\delta_{ps}F'_{pq}F'_{rs} \\
&= F'_{pq}F'_{qp}
\end{aligned}$$

14.

$$\begin{aligned}
\mathbf{e}'_k &= \mathbf{A}\cdot\mathbf{e}_k \\
&= \left(A_{ij}\mathbf{e}_i\mathbf{e}_j\right)\cdot\mathbf{e}_k \\
&= A_{ij}\mathbf{e}_i\delta_{jk} \\
&= A_{ik}\mathbf{e}_i
\end{aligned}$$

From Figure 6.2

$$\begin{aligned}
\mathbf{e}'_1 &= \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2 \\
\mathbf{e}'_2 &= -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2 \\
\mathbf{e}'_3 &= \mathbf{e}_3
\end{aligned}$$

Comparing yields the given result.

15. (a)

$$\begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\begin{aligned}
u'_1 &= u_1\cos\theta + u_2\sin\theta \\
u'_2 &= -u_1\sin\theta + u_2\cos\theta \\
u'_3 &= u_3
\end{aligned}$$

(b)

$$[F'] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$
\begin{aligned}
F'_{11} &= F_{11}\cos^2\theta + (F_{12}+F_{21})\cos\theta\sin\theta + F_{22}\sin^2\theta \\
F'_{12} &= (F_{22}-F_{11})\cos\theta\sin\theta + F_{12}\cos^2\theta - F_{21}\sin^2\theta \\
F'_{13} &= F_{13}\cos\theta + F_{23}\sin\theta \\
F'_{21} &= (F_{22}-F_{11})\cos\theta\sin\theta + F_{21}\cos^2\theta - F_{12}\sin^2\theta \\
F'_{22} &= F_{11}\sin^2\theta - (F_{12}+F_{21})\cos\theta\sin\theta + F_{22}\cos^2\theta \\
F'_{23} &= -F_{13}\sin\theta + F_{23}\cos\theta \\
F'_{31} &= F_{13}\cos\theta + F_{23}\sin\theta \\
F'_{32} &= -F_{13}\sin\theta + F_{23}\cos\theta \\
F'_{33} &= F_{33}
\end{aligned}
$$

(c)

$$
\begin{aligned}
u'_1 &= u_1 + u_2 d\theta \\
u'_2 &= -u_1 d\theta + u_2 \\
u'_3 &= u_3
\end{aligned}
$$

$$
\begin{aligned}
F'_{11} &= F_{11} + (F_{12}+F_{21})\,d\theta \\
F'_{12} &= (F_{22}-F_{11})\,d\theta + F_{12} \\
F'_{13} &= F_{13} + F_{23}d\theta \\
F'_{21} &= (F_{22}-F_{11})\,d\theta + F_{21} \\
F'_{22} &= -(F_{12}+F_{21})\,d\theta + F_{22} \\
F'_{23} &= -F_{13}d\theta + F_{23} \\
F'_{31} &= F_{13} + F_{23}d\theta \\
F'_{32} &= -F_{13}d\theta + F_{23} \\
F'_{33} &= F_{33}
\end{aligned}
$$

16.

$$\dot{\mathbf{A}}(\theta) = \dot\theta\left\{-\sin(\theta)\left(\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2\right) + \cos(\theta)\left(\mathbf{e}_2\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_2\right)\right\}$$

$$\dot{\mathbf{A}}(0) = \dot\theta\left(\mathbf{e}_2\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_2\right)$$

$$\mathbf{A}(\theta)\cdot\dot{\mathbf{A}}(0) = \dot{\mathbf{A}}(\theta)$$

17.

$$\mathbf{e}'_p = \mathbf{A}(t)\cdot\mathbf{e}_p$$

$$\frac{d}{dt}\mathbf{e}'_p = \dot{\mathbf{A}}(t)\cdot\mathbf{e}_p$$

but substituting $\mathbf{e}_p = \mathbf{A}^T(t)\cdot\mathbf{e}'_p$ gives the result. Because $\mathbf{A}\cdot\mathbf{A}^T = \mathbf{I}$

$$
\begin{aligned}
\dot{\mathbf{A}}\cdot\mathbf{A}^T + \mathbf{A}\cdot\dot{\mathbf{A}}^T &= 0 \\
\dot{\mathbf{A}}\cdot\mathbf{A}^T &= -\left(\dot{\mathbf{A}}\cdot\mathbf{A}^T\right)^T
\end{aligned}
$$

18. (a) A rotation of base vectors 90° about the x_2 is

$$\begin{bmatrix}\mathbf{e}_1'\\\mathbf{e}_2'\\\mathbf{e}_3'\end{bmatrix}=\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}\mathbf{e}_1\\\mathbf{e}_2\\\mathbf{e}_3\end{bmatrix}$$

or in dyadic form

$$\begin{aligned}\mathbf{A}' &= \mathbf{e}_1'\mathbf{e}_1+\mathbf{e}_2'\mathbf{e}_2+\mathbf{e}_3'\mathbf{e}_3\\ \mathbf{A}' &= -\mathbf{e}_3\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2+\mathbf{e}_1\mathbf{e}_3\end{aligned}$$

(b) A rotation of base vectors 90° about the x_3 is

$$\begin{bmatrix}\mathbf{e}_1''\\\mathbf{e}_2''\\\mathbf{e}_3''\end{bmatrix}=\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}\mathbf{e}_1'\\\mathbf{e}_2'\\\mathbf{e}_3'\end{bmatrix}$$

or in dyadic form

$$\begin{aligned}\mathbf{A}'' &= \mathbf{e}_1''\mathbf{e}_1'+\mathbf{e}_2''\mathbf{e}_2'+\mathbf{e}_3''\mathbf{e}_3'\\ &= \mathbf{e}_2'\mathbf{e}_1'-\mathbf{e}_1'\mathbf{e}_2'+\mathbf{e}_3'\mathbf{e}_3'\\ &= -\mathbf{e}_2\mathbf{e}_3+\mathbf{e}_3\mathbf{e}_2+\mathbf{e}_1\mathbf{e}_1\end{aligned}$$

(c) Successive 90° about the x_2 and x_3 axes are given by

$$\begin{aligned}\begin{bmatrix}\mathbf{e}_1''\\\mathbf{e}_2''\\\mathbf{e}_3''\end{bmatrix} &= \begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}\mathbf{e}_1\\\mathbf{e}_2\\\mathbf{e}_3\end{bmatrix}\\ &= \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}\mathbf{e}_1\\\mathbf{e}_2\\\mathbf{e}_3\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{e}_2\\\mathbf{e}_3\\\mathbf{e}_1\end{bmatrix}\end{aligned}$$

or in dyadic form

$$\mathbf{A}''\cdot\mathbf{A}'=\mathbf{e}_2\mathbf{e}_1+\mathbf{e}_3\mathbf{e}_2+\mathbf{e}_1\mathbf{e}_3$$

19. (a) The matrix rotating the x_3 direction into a direction $\boldsymbol{\lambda}$ is given by the transpose of the matrix from Problem 5. Multiplying this matrix by the matrix from Problem 14 rotates the axes through an angle θ. Then multiplying by the matrix from Problem 5 rotates the back into the original axes. Hence, the result is given by the matrix product

$$\begin{bmatrix}\sqrt{\lambda_2^2+\lambda_3^2}&0&\lambda_1\\ \frac{\lambda_1\lambda_2}{\sqrt{\lambda_2^2+\lambda_3^2}}&\frac{\lambda_3}{\sqrt{\lambda_2^2+\lambda_3^2}}&\lambda_2\\ -\frac{\lambda_1\lambda_3}{\sqrt{\lambda_2^2+\lambda_3^2}}&-\frac{\lambda_2}{\sqrt{\lambda_2^2+\lambda_3^2}}&\lambda_3\end{bmatrix}\begin{bmatrix}\cos\theta&\sin\theta&0\\-\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}\begin{bmatrix}\sqrt{\lambda_2^2+\lambda_3^2}&\frac{\lambda_1\lambda_2}{\sqrt{\lambda_2^2+\lambda_3^2}}&-\frac{\lambda_1\lambda_3}{\sqrt{\lambda_2^2+\lambda_3^2}}\\0&\frac{\lambda_3}{\sqrt{\lambda_2^2+\lambda_3^2}}&-\frac{\lambda_2}{\sqrt{\lambda_2^2+\lambda_3^2}}\\\lambda_1&\lambda_2&\lambda_3\end{bmatrix}$$

(b) The dyadic form follows from setting $\boldsymbol{\lambda} = \mathbf{e}_3$, using $\mathbf{I} = \mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3$ and noting that $W_{12} = -W_{21} = -1$ are components of the antisymmetric matrix with axial vector $\mathbf{e}_3$.

20. Setting $\boldsymbol{\lambda} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$ and $\theta = 120^\circ$ in the answer to Problem 19a gives the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

which the same as the result of Problem 18c.

21.

$$\begin{aligned} \epsilon'_{ijk} &= A_{pi}A_{qj}A_{rk}\epsilon_{pqr} \\ &= \epsilon_{ijk}\det(A) \\ &= \epsilon_{ijk} \end{aligned}$$

Hence ϵ_{ijk} has the same components in any rectangular cartesian coordinate system if $\det(A) = 1$. This will be the case for transformations from right (left) - handed systems to right (left) - handed systems. But for reflections, i.e., transformations from a right (left) - handed systems to a left (right) - handed system, $\det(A) = -1$ and ϵ_{ijk} is not isotropic.

Chapter 7

Principal Values and Principal Directions

1. Substituting $\lambda_{II} = 5$ back into equation (7.1) gives

$$\begin{aligned}
2\left(\boldsymbol{\mu}^{II}\right)_1 + 0\left(\boldsymbol{\mu}^{II}\right)_2 - 2\left(\boldsymbol{\mu}^{II}\right)_3 &= 0 \\
0 + (5-5)\left(\boldsymbol{\mu}^{II}\right)_2 - 0 &= 0 \\
-2\left(\boldsymbol{\mu}^{II}\right)_1 + 0 + (-1)\left(\boldsymbol{\mu}^{II}\right)_3 &= 0
\end{aligned}$$

The first and third equations are linearly independent and have only the solution $\left(\boldsymbol{\mu}^{II}\right)_1 = \left(\boldsymbol{\mu}^{II}\right)_3 = 0$. Because the coefficient of $\left(\boldsymbol{\mu}^{II}\right)_2$ in the second equation is zero, it can be arbitrary and is taken as $\left(\boldsymbol{\mu}^{II}\right)_2 = 1$ to make $\boldsymbol{\mu}^{II}$ a unit vector.

2.

$$\begin{aligned}
F'_{jj} &= A_{kj}A_{lj}F_{kl} \\
&= \delta_{kl}F_{kl} = F_{kk}
\end{aligned}$$

$$\begin{aligned}
F'_{ij}F'_{ji} &= \left(A_{ki}A_{lj}F_{kl}\right)\left(A_{mj}A_{ni}F_{mn}\right) \\
&= \left(A_{ki}A_{ni}\right)\left(A_{lj}A_{mj}\right)F_{kl}F_{mn} \\
&= \delta_{kn}\delta_{lm}F_{kl}F_{mn} = F_{kl}F_{lk}
\end{aligned}$$

$$\begin{aligned}
\det\left(F'\right) &= \epsilon_{ijk}F'_{1i}F'_{2j}F'_{3k} \\
&= \epsilon_{ijk}\left(A_{l1}A_{mi}F_{lm}\right)\left(A_{n2}A_{pj}F_{np}\right)\left(A_{r3}A_{sk}F_{rs}\right) \\
&= \left(\epsilon_{ijk}A_{mi}A_{pj}A_{sk}\right)\left(A_{l1}A_{n2}A_{r3}\right)F_{lm}F_{np}F_{rs} \\
&= \epsilon_{mps}\det(A)\left(A_{l1}A_{n2}A_{r3}\right)F_{lm}F_{np}F_{rs} \\
&= \det(A)\left(\epsilon_{mps}F_{lm}F_{np}F_{rs}\right)\left(A_{l1}A_{n2}A_{r3}\right) \\
&= \epsilon_{lnr}\det(F)\left(A_{l1}A_{n2}A_{r3}\right) \\
&= \left(\det(A)\right)^2\det(F) = \det(F)
\end{aligned}$$

because $|\det(A)| = 1$.

3. Taking the trace of both sides of the Cayley Hamilton theorem (7.13) yields

$$\mathrm{tr}\left(\mathbf{F}\cdot\mathbf{F}\cdot\mathbf{F}\right) = I_1 \mathrm{tr}\left(\mathbf{F}\cdot\mathbf{F}\right) + I_2 \mathrm{tr}\mathbf{F} + I_3 \mathrm{tr}\mathbf{I}$$

But $\mathrm{tr}\mathbf{F} = I_1$, $\mathrm{tr}\mathbf{I} = 3$ and

$$\mathrm{tr}\left(\mathbf{F}\cdot\mathbf{F}\right) = 2I_2 + I_1^2$$

Substituting and rearranging gives the result.

4. (a) Multiplying both sides of (7.13) by $\mathbf{F}^{-1}$ gives the result.

   (b) Take the trace of both sides, use

$$\mathrm{tr}\left(\mathbf{F}\cdot\mathbf{F}\right) = 2I_2 + I_1^2$$

   and rearrange.

5.

$$\begin{aligned}
\mathbf{F}\cdot\boldsymbol{\mu}^I &= \lambda_I \boldsymbol{\mu}^I \\
\mathbf{F}\cdot\boldsymbol{\mu}^{II} &= \lambda_{II} \boldsymbol{\mu}^{II} \\
\mathbf{F}\cdot\boldsymbol{\mu}^{III} &= \lambda_{III} \boldsymbol{\mu}^{III}
\end{aligned}$$

Form the scalar product of the first equation with $\boldsymbol{\mu}^{II}$ and the second with $\boldsymbol{\mu}^I$. Subtracting then gives

$$\left(\lambda_I - \lambda_{II}\right)\boldsymbol{\mu}^I\cdot\boldsymbol{\mu}^{II} = 0$$

Because $\lambda_I \neq \lambda_{II}$, $\boldsymbol{\mu}^{II}$ must be orthogonal to $\boldsymbol{\mu}^I$. By the same argument, $\boldsymbol{\mu}^{III}$ must be orthogonal to $\boldsymbol{\mu}^I$. Forming the scalar product of the second equation with $\boldsymbol{\mu}^{III}$ and the third with $\boldsymbol{\mu}^{II}$ and subtracting yields

$$\left(\lambda_{II} - \lambda_{III}\right)\boldsymbol{\mu}^{III}\cdot\boldsymbol{\mu}^{II} = 0$$

Because $\lambda_{II} = \lambda_{III}$, the equation is automatically satisfied and the only requirement on $\boldsymbol{\mu}^{II}$ and $\boldsymbol{\mu}^{III}$ is that they be orthogonal to $\boldsymbol{\mu}^I$.

6. (a) Principal values are $\lambda_I = 13$, $\lambda_{II} = 7$ and $\lambda_{III} = 3$. Principal directions are

$$\begin{aligned}
\boldsymbol{\mu}^I &= \pm\frac{2}{\sqrt{5}}\mathbf{e}_1 \pm \frac{1}{\sqrt{5}}\mathbf{e}_2 \\
\boldsymbol{\mu}^{II} &= \mathbf{e}_3 \\
\boldsymbol{\mu}^{III} &= \pm\frac{1}{\sqrt{5}}\mathbf{e}_1 \mp \frac{2}{\sqrt{5}}\mathbf{e}_2
\end{aligned}$$

(b)

$$\begin{bmatrix} \pm\frac{2}{\sqrt{5}} & \pm\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \pm\frac{1}{\sqrt{5}} & \mp\frac{2}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 11 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} \pm\frac{2}{\sqrt{5}} & 0 & \pm\frac{1}{\sqrt{5}} \\ \pm\frac{1}{\sqrt{5}} & 0 & \mp\frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

7.

$$\begin{aligned} \mathrm{tr}\mathbf{F} &= \mathbf{a}\cdot\mathbf{b} \\ \frac{1}{2}\left(F_{kk}F_{ll} - F_{kl}F_{lk}\right) &= 0 \\ \det(F) &= 0 \end{aligned}$$

8.

$$\begin{aligned} \det \begin{bmatrix} \cos\theta - \lambda & \sin\theta & 0 \\ -\sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} &= (1-\lambda)\left(\left(\cos\theta - \lambda\right)^2 + \sin^2\theta\right) \\ &= 0 \end{aligned}$$

Therefore $\lambda = 1$ is one solution and the remaining two satisfy

$$\left(\cos\theta - \lambda\right)^2 + \sin^2\theta = 0$$

which has the solutions

$$\lambda = \cos\theta \pm \iota\sqrt{1-\cos^2\theta}$$

Because $\cos^2\theta \leq 1$, the only real solution is $\lambda = 1$ and corresponding principal direction is $\mathbf{e}_3$.

9.

$$\begin{aligned} \det\left(F_{ij} - \lambda\delta_{ij}\right) &= \frac{1}{6}\epsilon_{pqr}\epsilon_{ijk}\left(F_{ip} - \lambda\delta_{ip}\right)\left(F_{jq} - \lambda\delta_{jq}\right)\left(F_{kr} - \lambda\delta_{kr}\right) \\ &= \frac{1}{6}\epsilon_{pqr}\epsilon_{ijk}F_{ip}F_{jq}F_{kr} \\ &\quad -\frac{1}{6}\lambda\epsilon_{pqr}\left(\epsilon_{iqk}F_{ip}F_{rk} + \epsilon_{ijr}F_{ip}F_{qj} + \epsilon_{pjk}F_{qj}F_{rk}\right) \\ &\quad +\frac{1}{6}\lambda^2\epsilon_{pqr}\left(F_{ip}\epsilon_{pqr}\epsilon_{iqr} + F_{rk}\epsilon_{pqr}\epsilon_{pqk} + F_{qj}\epsilon_{pqr}\epsilon_{pqj}\right) \\ &\quad -\frac{1}{6}\epsilon_{pqr}\epsilon_{pqr}\lambda^3 \end{aligned}$$

Using the $\epsilon - \delta$ identities gives

$$\det\left(F_{ij} - \lambda\delta_{ij}\right) = \det\left(F\right) + \frac{1}{2}\lambda\left(F_{ik}F_{ik} - F_{kk}F_{pp}\right) + F_{kk}\lambda^2 - \lambda^3$$

Chapter 8

Gradient

1.
$$\mathbf{v} = \mathbf{e}_j \frac{\partial \phi}{\partial x_j}$$

Substituting in $\boldsymbol{\nabla} \times \mathbf{v}$ gives

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{v} &= \mathbf{e}_i \frac{\partial}{\partial x_i} \times \mathbf{e}_j \\ &= \mathbf{e}_k \epsilon_{kij} \frac{\partial^2 \phi}{\partial x_j \partial x_i} \end{aligned}$$

Because ϵ_{kij} is anti-symmetric with respect to interchange of i and j and $\partial^2 \phi / \partial x_j \partial x_i$ is symmetric, their summed product vanishes.

2. A vector in the direction of the normal to a surface is proportional to the gradient:

$$\begin{aligned} \mathbf{y} &= \mathbf{e}_i \frac{\partial}{\partial x_i} \left(x_j F_{jk} x_k \right) \\ &= \mathbf{e}_i \left(\delta_{ij} F_{jk} x_k + x_j F_{jk} \delta_{ik} \right) \\ &= \mathbf{e}_i \left(F_{ik} x_k + x_j F_{ji} \right) \\ &= 2\mathbf{e}_i \left(F_{ik} x_k \right) \end{aligned}$$

where the last line follows because $F_{ij} = F_{ji}$. Dividing by the magnitude of $\mathbf{y}$ to get a unit vector gives

$$\mathbf{n} = \mathbf{e}_k \frac{F_{kl} x_l}{\sqrt{F_{pq} F_{pr} x_q x_r}}$$

3.
$$\begin{aligned} \mathbf{v} \cdot (\boldsymbol{\nabla} \mathbf{u}) &= v_i \mathbf{e}_i \cdot \left(\partial_j u_k \mathbf{e}_j \mathbf{e}_k \right) \\ &= v_i \mathbf{e}_i \cdot \mathbf{e}_j \partial_j \left(u_k \mathbf{e}_k \right) \\ &= \left(\mathbf{v} \cdot \boldsymbol{\nabla} \right) \mathbf{u} \end{aligned}$$

4. (a)

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\phi\mathbf{v}) &= \mathbf{e}_i\frac{\partial}{\partial x_i}\cdot(\phi v_j\mathbf{e}_j)\\
&= \mathbf{e}_i\frac{\partial\phi}{\partial x_i}\cdot v_j\mathbf{e}_j+\phi\mathbf{e}_i\cdot\frac{\partial v_j}{\partial x_i}\mathbf{e}_j\\
&= \mathbf{v}\cdot\boldsymbol{\nabla}\phi+\phi\boldsymbol{\nabla}\cdot\mathbf{v}
\end{aligned}$$

(b)

$$\begin{aligned}
\boldsymbol{\nabla}\times(\phi\mathbf{v}) &= \mathbf{e}_i\frac{\partial}{\partial x_i}\times(\phi v_j\mathbf{e}_j)\\
&= \mathbf{e}_i\frac{\partial\phi}{\partial x_i}\times(v_j\mathbf{e}_j)+\phi\mathbf{e}_i\frac{\partial}{\partial x_i}\times(v_j\mathbf{e}_j)\\
&= \boldsymbol{\nabla}\phi\times\mathbf{v}+\phi\boldsymbol{\nabla}\times\mathbf{v}
\end{aligned}$$

5. (a)

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\mathbf{u}\times\mathbf{v}) &= \mathbf{e}_i\frac{\partial}{\partial x_i}\cdot(u_j\mathbf{e}_j\times v_k\mathbf{e}_k)\\
&= \mathbf{e}_i\frac{\partial}{\partial x_i}\cdot(\epsilon_{jkl}u_jv_k\mathbf{e}_l)\\
&= \epsilon_{jkl}\delta_{il}\left(\frac{\partial u_j}{\partial x_i}v_k+u_j\frac{\partial v_k}{\partial x_i}\right)\\
&= (\epsilon_{jki}\partial_i u_j)\,v_k+u_j\,(\epsilon_{jki}\partial_i v_k)\\
&= (\epsilon_{ijk}\partial_i u_j)\,v_k-u_j\,(\epsilon_{jik}\partial_i v_k)\\
&= (\boldsymbol{\nabla}\times\mathbf{u})\cdot\mathbf{v}-\mathbf{u}\cdot(\boldsymbol{\nabla}\times\mathbf{v})
\end{aligned}$$

(b)

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\mathbf{F}\cdot\mathbf{u}) &= \mathbf{e}_i\frac{\partial}{\partial x_i}\cdot(F_{kl}\mathbf{e}_k\mathbf{e}_l\cdot u_m\mathbf{e}_m)\\
&= \mathbf{e}_i\frac{\partial}{\partial x_i}\cdot(F_{km}\mathbf{e}_k u_m)\\
&= \frac{\partial F_{km}}{\partial x_k}u_m+F_{km}\frac{\partial u_m}{\partial x_k}\\
&= u_m\mathbf{e}_m\cdot\left(\mathbf{e}_n\frac{\partial}{\partial x_n}\cdot F_{kl}\mathbf{e}_k\mathbf{e}_l\right)+F_{kl}\left(\mathbf{e}_k\cdot\mathbf{e}_n\right)\left(\mathbf{e}_l\cdot\mathbf{e}_m\right)\partial_n u_m\\
&= \mathbf{u}\cdot(\boldsymbol{\nabla}\cdot\mathbf{F})+\mathbf{F}^T\cdot\cdot\boldsymbol{\nabla}\mathbf{u}
\end{aligned}$$

6. (a)

$$\begin{aligned}
\boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{v}) &= \mathbf{e}_i \frac{\partial}{\partial x_i} \times (a_j e_j \times b_k e_k) \\
&= \mathbf{e}_i \frac{\partial}{\partial x_i} \times (\mathbf{e}_m \epsilon_{mjk} a_j b_k) \\
&= \mathbf{e}_n \left(\epsilon_{nim} \epsilon_{mjk}\right) \frac{\partial}{\partial x_i} \left(a_j b_k\right) \\
&= \mathbf{e}_n \left(\delta_{nj} \delta_{ik} - \delta_{nk} \delta_{ij}\right) \frac{\partial}{\partial x_i} \left(a_j b_k\right) \\
&= \mathbf{e}_n \frac{\partial}{\partial x_k} \left(a_n b_k\right) - \mathbf{e}_n \frac{\partial}{\partial x_j} \left(a_j b_n\right) \\
&= \mathbf{e}_n a_n \frac{\partial b_k}{\partial x_k} + b_k \frac{\partial}{\partial x_k} \left(a_n \mathbf{e}_n\right) - \mathbf{e}_n b_n \frac{\partial a_j}{\partial x_j} - a_j \frac{\partial}{\partial x_j} \left(b_n \mathbf{e}_n\right) \\
&= \mathbf{a} \boldsymbol{\nabla} \cdot \mathbf{b} + \mathbf{b} \cdot \boldsymbol{\nabla} \mathbf{a} - \mathbf{b} \boldsymbol{\nabla} \cdot \mathbf{a} - \mathbf{a} \cdot \boldsymbol{\nabla} \mathbf{b}
\end{aligned}$$

(b)

$$\begin{aligned}
\boldsymbol{\nabla} \times (\bigtriangledown \times \mathbf{v}) &= \mathbf{e}_i \frac{\partial}{\partial x_i} \times \left(\mathbf{e}_j \frac{\partial}{\partial x_j} \times v_k \mathbf{e}_k \right) \\
&= \mathbf{e}_i \frac{\partial}{\partial x_i} \times \left(\mathbf{e}_m \epsilon_{mjk} \frac{\partial}{\partial x_j} v_k \right) \\
&= \epsilon_{nim} \epsilon_{mjk} \mathbf{e}_n \frac{\partial^2 v_k}{\partial x_i \partial x_j} \\
&= \left(\delta_{nj} \delta_{ik} - \delta_{nk} \delta_{ij}\right) \mathbf{e}_n \frac{\partial^2 v_k}{\partial x_i \partial x_j} \\
&= \mathbf{e}_n \frac{\partial}{\partial x_n} \frac{\partial v_k}{\partial x_k} - \frac{\partial^2}{\partial x_j \partial x_j} v_n \mathbf{e}_n \\
&= \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \mathbf{v}\right) - \boldsymbol{\nabla}^2 \mathbf{v}
\end{aligned}$$

7. (a)

$$\begin{aligned}
\boldsymbol{\nabla} r &= \mathbf{e}_i \frac{\partial}{\partial x_i} \sqrt{x_k x_k} \\
&= \mathbf{e}_i \frac{x_k}{\sqrt{x_l x_l}} \frac{\partial x_k}{\partial x_i} \\
&= \mathbf{e}_i \frac{x_k}{\sqrt{x_l x_l}} \delta_{ki} \\
&= \frac{x_i \mathbf{e}_i}{\sqrt{x_l x_l}} \\
&= \frac{\mathbf{x}}{r}
\end{aligned}$$

(b)

$$\begin{aligned}
\boldsymbol{\nabla}\frac{\mathbf{x}}{r} &= \mathbf{e}_i\frac{\partial}{\partial x_i}\left(\frac{x_j\mathbf{e}_j}{\sqrt{x_kx_k}}\right) \\
&= \mathbf{e}_i\mathbf{e}_j\frac{1}{\sqrt{x_kx_k}}\frac{\partial x_j}{\partial x_i} + \mathbf{e}_i\mathbf{e}_jx_j\frac{\partial}{\partial x_i}\frac{1}{\sqrt{x_kx_k}} \\
&= \frac{1}{\sqrt{x_kx_k}}\delta_{ij} + \mathbf{e}_i\mathbf{e}_jx_j\left(\frac{-x_i}{(x_lx_l)^{3/2}}\right) \\
&= \frac{\mathbf{e}_i\mathbf{e}_j}{r}\left(\delta_{ij} - \frac{x_ix_j}{r^2}\right)
\end{aligned}$$

(c)

$$\begin{aligned}
\nabla^2 r &= \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}r \\
&= \mathbf{e}_k\frac{\partial}{\partial x_k}\cdot\left(\frac{x_i\mathbf{e}_i}{\sqrt{x_lx_l}}\right) \\
&= \delta_{ki}\left(\frac{\delta_{ki}}{r} - \frac{x_ix_k}{r^3}\right) \\
&= \frac{2}{r}
\end{aligned}$$

8.

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{u} &= \mathbf{e}_i\epsilon_{ijk}\partial_ju_k \\
&= -\mathbf{e}_i\epsilon_{ikj}u_{k,j} \\
&= -\mathbf{u}\times\boldsymbol{\nabla}
\end{aligned}$$

9.

$$\begin{aligned}
\mathbf{F}\times\boldsymbol{\nabla} &= \left(\frac{\partial}{\partial x_k}F_{ij}\right)\mathbf{e}_i\mathbf{e}_j\times\mathbf{e}_k \\
&= \left(\frac{\partial}{\partial x_k}F_{ij}\right)\epsilon_{jkl}\mathbf{e}_i\mathbf{e}_l \\
&= \left(\frac{\partial F_{ij}}{\partial x_k}\epsilon_{jkl}\mathbf{e}_l\mathbf{e}_i\right)^T \\
&= -\left(\frac{\partial F_{ij}}{\partial x_k}\epsilon_{jlk}\mathbf{e}_l\mathbf{e}_i\right)^T \\
&= \left(\frac{\partial F^T_{ji}}{\partial x_k}\epsilon_{kjl}\mathbf{e}_l\mathbf{e}_i\right)^T \\
&= -\left(\boldsymbol{\nabla}\times\mathbf{F}^T\right)^T
\end{aligned}$$

10.

$$\begin{aligned}
\mathbf{M} &= \boldsymbol{\nabla} \times \mathbf{E} \\
&= \mathbf{e}_k \frac{\partial}{\partial x_k} \times E_{mn} \mathbf{e}_m \mathbf{e}_n \\
&= E_{mn},_k \, \epsilon_{kml} \mathbf{e}_l \mathbf{e}_n
\end{aligned}$$

Therefore,

$$M_{ij} = \epsilon_{ikm} E_{mj},_k$$

$$\begin{aligned}
M_{11} &= \epsilon_{1km} E_{m1},_k \\
&= E_{31},_2 - E_{21},_3
\end{aligned}$$

$$\begin{aligned}
M_{22} &= \epsilon_{2km} E_{m2},_k \\
&= E_{12},_3 - E_{32},_1
\end{aligned}$$

$$\begin{aligned}
M_{12} &= \epsilon_{1km} E_{m2},_k \\
&= E_{32},_2 - E_{22},_3
\end{aligned}$$

$$\begin{aligned}
M_{21} &= \epsilon_{2km} E_{m1},_k \\
&= E_{11},_3 - E_{31},_1
\end{aligned}$$

11. To determine the Laplacian in cylindrical coordinates:

$$\begin{aligned}
\nabla^2 \phi &= \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \phi) \\
&= \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial \theta} \right) \cdot \left(\mathbf{e}_r \frac{\partial \phi}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial \phi}{\partial \theta} + \mathbf{e}_z \frac{\partial \phi}{\partial z} \right) \\
&= \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\mathbf{e}_\theta}{r} \cdot \frac{d\mathbf{e}_r}{d\theta} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} \\
&= \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} \\
&= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}
\end{aligned}$$

12. To determine the curl in cylindrical coordinates

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{v} &= \left(\mathbf{e}_r\frac{\partial}{\partial r}+\frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial\theta}+\mathbf{e}_z\frac{\partial}{\partial z}\right)\times(v_r\mathbf{e}_r+v_\theta\mathbf{e}_\theta+v_z\mathbf{e}_z)\\
&= \mathbf{e}_r\times\mathbf{e}_\theta\frac{\partial v_\theta}{\partial r}+\mathbf{e}_r\times\mathbf{e}_z\frac{\partial v_z}{\partial r}+\mathbf{e}_z\times\mathbf{e}_r\frac{\partial v_r}{\partial z}+\mathbf{e}_z\times\mathbf{e}_\theta\frac{\partial v_\theta}{\partial z}\\
&\quad \ldots+\frac{\mathbf{e}_\theta}{r}\times\mathbf{e}_r\frac{\partial v_r}{\partial\theta}+\frac{\mathbf{e}_\theta}{r}\times\frac{d\mathbf{e}_\theta}{d\theta}v_\theta+\frac{\mathbf{e}_\theta}{r}\times\mathbf{e}_z\frac{\partial v_z}{\partial\theta}\\
&= \mathbf{e}_z\frac{\partial v_\theta}{\partial r}-\mathbf{e}_\theta\frac{\partial v_z}{\partial r}+\mathbf{e}_\theta\frac{\partial v_r}{\partial z}-\mathbf{e}_r\frac{\partial v_\theta}{\partial z}\\
&\quad \ldots-\mathbf{e}_z\frac{\partial v_r}{r\partial\theta}+\mathbf{e}_z\frac{v_\theta}{r}+\mathbf{e}_r\frac{\partial v_z}{r\partial\theta}\\
&= \mathbf{e}_r\left(\frac{\partial v_z}{r\partial\theta}-\frac{\partial v_\theta}{\partial z}\right)+\mathbf{e}_\theta\left(\frac{\partial v_r}{\partial z}-\frac{\partial v_z}{\partial r}\right)+\mathbf{e}_z\left(\frac{\partial v_\theta}{\partial r}-\frac{\partial v_r}{r\partial\theta}+\frac{v_\theta}{r}\right)
\end{aligned}$$

13. To determine $\boldsymbol{\nabla}\mathbf{v}$ in cylindrical coordinates

$$\begin{aligned}
\boldsymbol{\nabla}\mathbf{v} &= \left(\mathbf{e}_r\frac{\partial}{\partial r}+\frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial\theta}+\mathbf{e}_z\frac{\partial}{\partial z}\right)(v_r\mathbf{e}_r+v_\theta\mathbf{e}_\theta+v_z\mathbf{e}_z)\\
&= \mathbf{e}_r\mathbf{e}_r\frac{\partial v_r}{\partial r}+\mathbf{e}_r\mathbf{e}_\theta\frac{\partial v_\theta}{\partial r}+\mathbf{e}_r\mathbf{e}_z\frac{\partial v_z}{\partial r}\\
&\quad +\mathbf{e}_\theta\mathbf{e}_r\frac{\partial v_r}{r\partial\theta}+\mathbf{e}_\theta\frac{d\mathbf{e}_r}{d\theta}\frac{v_r}{r}+\mathbf{e}_\theta\mathbf{e}_\theta\frac{\partial v_\theta}{r\partial\theta}+\mathbf{e}_\theta\frac{d\mathbf{e}_\theta}{d\theta}\frac{v_\theta}{r}+\mathbf{e}_\theta\mathbf{e}_z\frac{\partial v_z}{r\partial\theta}\\
&\quad +\mathbf{e}_z\mathbf{e}_r\frac{\partial v_r}{\partial z}+\mathbf{e}_z\mathbf{e}_\theta\frac{\partial v_\theta}{\partial z}+\mathbf{e}_z\mathbf{e}_z\frac{\partial v_z}{\partial z}\\
&= \mathbf{e}_r\mathbf{e}_r\frac{\partial v_r}{\partial r}+\mathbf{e}_r\mathbf{e}_\theta\frac{\partial v_\theta}{\partial r}+\mathbf{e}_r\mathbf{e}_z\frac{\partial v_z}{\partial r}\\
&\quad +\mathbf{e}_\theta\mathbf{e}_r\left(\frac{\partial v_r}{r\partial\theta}-\frac{v_\theta}{r}\right)+\mathbf{e}_\theta\mathbf{e}_\theta\left(\frac{\partial v_\theta}{r\partial\theta}+\frac{v_r}{r}\right)+\mathbf{e}_\theta\mathbf{e}_z\frac{\partial v_z}{r\partial\theta}\\
&\quad +\mathbf{e}_z\mathbf{e}_r\frac{\partial v_r}{\partial z}+\mathbf{e}_z\mathbf{e}_\theta\frac{\partial v_\theta}{\partial z}+\mathbf{e}_z\mathbf{e}_z\frac{\partial v_z}{\partial z}
\end{aligned}$$

14. To calculate the divergence of a tensor $\nabla \cdot \mathbf{F}$ write

$$
\begin{aligned}
\nabla \cdot \mathbf{F} &= \left(\mathbf{e}_r \frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}\right) \cdot \mathbf{F} \\
&= \left(\mathbf{e}_r \frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}\right) \cdot \left(F_{rr}\mathbf{e}_r\mathbf{e}_r + F_{r\theta}\mathbf{e}_r\mathbf{e}_\theta + F_{rz}\mathbf{e}_r\mathbf{e}_z\right) \\
&\quad + \left(\mathbf{e}_r \frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}\right) \cdot \left(F_{\theta r}\mathbf{e}_\theta\mathbf{e}_r + F_{\theta\theta}\mathbf{e}_\theta\mathbf{e}_\theta + F_{\theta z}\mathbf{e}_\theta\mathbf{e}_z\right) \\
&\quad + \left(\mathbf{e}_r \frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}\right) \cdot \left(F_{zr}\mathbf{e}_z\mathbf{e}_r + F_{z\theta}\mathbf{e}_z\mathbf{e}_\theta + F_{zz}\mathbf{e}_z\mathbf{e}_z\right) \\
&= \frac{\partial F_{rr}}{\partial r}\mathbf{e}_r + \frac{\partial F_{r\theta}}{\partial r}\mathbf{e}_\theta + \frac{\partial F_{rz}}{\partial r}\mathbf{e}_z + \frac{\mathbf{e}_\theta}{r} \cdot \frac{d\mathbf{e}_r}{d\theta}\left(F_{rr}\mathbf{e}_r + F_{r\theta}\mathbf{e}_\theta + F_{rz}\mathbf{e}_z\right) \\
&\quad + \frac{1}{r}\frac{\partial F_{\theta r}}{\partial \theta}\mathbf{e}_r + \frac{1}{r}\frac{\partial F_{\theta\theta}}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r}\frac{\partial F_{\theta z}}{\partial \theta}\mathbf{e}_z \\
&\quad + \left(\frac{F_{\theta r}}{r}\frac{d\mathbf{e}_r}{d\theta} + \frac{F_{\theta\theta}}{r}\frac{d\mathbf{e}_\theta}{d\theta}\right) + \frac{\partial F_{zr}}{\partial z}\mathbf{e}_r + \frac{\partial F_{z\theta}}{\partial z}\mathbf{e}_\theta + \frac{\partial F_{zz}}{\partial z}\mathbf{e}_z \\
&= \frac{\partial F_{rr}}{\partial r}\mathbf{e}_r + \frac{\partial F_{r\theta}}{\partial r}\mathbf{e}_\theta + \frac{\partial F_{rz}}{\partial r}\mathbf{e}_z + \frac{1}{r}\left(F_{rr}\mathbf{e}_r + F_{r\theta}\mathbf{e}_\theta + F_{rz}\mathbf{e}_z\right) \\
&\quad + \frac{1}{r}\frac{\partial F_{\theta r}}{\partial \theta}\mathbf{e}_r + \frac{1}{r}\frac{\partial F_{\theta\theta}}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r}\frac{\partial F_{\theta z}}{\partial \theta}\mathbf{e}_z \\
&\quad + \left(\frac{F_{\theta r}}{r}\mathbf{e}_\theta - \frac{F_{\theta\theta}}{r}\mathbf{e}_r\right) + \frac{\partial F_{zr}}{\partial z}\mathbf{e}_r + \frac{\partial F_{z\theta}}{\partial z}\mathbf{e}_\theta + \frac{\partial F_{zz}}{\partial z}\mathbf{e}_z \\
&= \mathbf{e}_r\left(\frac{\partial F_{rr}}{\partial r} + \frac{1}{r}\frac{\partial F_{\theta r}}{\partial \theta} + \frac{\partial F_{zr}}{\partial z} + \frac{F_{rr} - F_{\theta\theta}}{r}\right) \\
&\quad + \mathbf{e}_\theta\left(\frac{\partial F_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial F_{\theta\theta}}{\partial \theta} + \frac{\partial F_{z\theta}}{\partial z} + \frac{F_{\theta r}}{r} + \frac{F_{r\theta}}{r}\right) \\
&\quad + \mathbf{e}_z\left(\frac{\partial F_{rz}}{\partial r} + \frac{1}{r}\frac{\partial F_{\theta z}}{\partial \theta} + \frac{\partial F_{zz}}{\partial z} + \frac{F_{rz}}{r}\right)
\end{aligned}
$$

Part II

Stress

Chapter 9

Traction and Stress Tensor

1. On $x_1 = +a$, $\mathbf{n} = \mathbf{e}_1$

$$\begin{aligned}
\mathbf{t} &= \mathbf{e}_1 \cdot \boldsymbol{\sigma} \\
&= \sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2 + \sigma_{13}\mathbf{e}_3 \\
&= -p\left(1 - \frac{x_2^2}{a^2}\right)\mathbf{e}_1 + 2p\frac{x_2}{a}\mathbf{e}_2 + 0\mathbf{e}_3
\end{aligned}$$

On $x_1 = -a$, $\mathbf{n} = -\mathbf{e}_1$

$$\begin{aligned}
\mathbf{t} &= -\mathbf{e}_1 \cdot \boldsymbol{\sigma} \\
&= -\sigma_{11}\mathbf{e}_1 - \sigma_{12}\mathbf{e}_2 - \sigma_{13}\mathbf{e}_3 \\
&= p\left(1 - \frac{x_2^2}{a^2}\right)\mathbf{e}_1 + 2p\frac{x_2}{a}\mathbf{e}_2 + 0\mathbf{e}_3
\end{aligned}$$

Chapter 10

Principal Values of Stress

1. (a)

$$\begin{aligned} \det(\boldsymbol{\sigma} - \lambda \mathbf{I}) &= \begin{vmatrix} -(1+\lambda) & -2 & 0 \\ -2 & -\lambda & 2 \\ 0 & 2 & 1-\lambda \end{vmatrix} \\ &= \lambda^3 - 9\lambda = 0 \end{aligned}$$

Therefore the principal stresses are $\sigma_I = +3$, $\sigma_{II} = 0$ and $\sigma_{III} = -3$.

(b) To determine the principal direction corresponding to the principal stress $\sigma_I = +3$:

$$\begin{aligned} -4n_1 - 2n_2 + 0n_3 &= 0 \\ -2n_1 - 3n_2 + 2n_3 &= 0 \\ 0n_1 + 2n_2 - 2n_3 &= 0 \end{aligned}$$

The first and third equations give $n_1 = -n_2/2$ and $n_3 = n_2$. Making $\mathbf{n}$ a unit vector gives

$$\mathbf{n}_I = \mp\frac{1}{3}\mathbf{e}_1 \pm \frac{2}{3}\mathbf{e}_2 \pm \frac{2}{3}\mathbf{e}_3$$

For the principal stress $\sigma_{III} = -3$:

$$\begin{aligned} 2n_1 - 2n_2 + 0n_3 &= 0 \\ -2n_1 - 3n_2 + 2n_3 &= 0 \\ 0n_1 + 2n_2 + 4n_3 &= 0 \end{aligned}$$

The first and third equations give $n_2 = -2n_3$ and $n_1 = n_2$. Making $\mathbf{n}$ a unit vector gives

$$\mathbf{n}_{III} = \mp\frac{2}{3}\mathbf{e}_1 \mp \frac{2}{3}\mathbf{e}_2 \pm \frac{1}{3}\mathbf{e}_3$$

Choosing $\mathbf{n}_{II} = \mathbf{n}_{III} \times \mathbf{n}_I$ for a right-handed system gives

$$\mathbf{n}_{II} = -\frac{2}{3}\mathbf{e}_1 + \frac{1}{3}\mathbf{e}_2 - \frac{2}{3}\mathbf{e}_3$$

2. (a)

$$\begin{aligned}\det(\boldsymbol{\sigma} - \lambda\mathbf{I}) &= \begin{vmatrix} -\lambda & \tau\cos\theta & \tau\sin\theta \\ \tau\cos\theta & -\lambda & 0 \\ \tau\sin\theta & 0 & -\lambda \end{vmatrix} \\ &= \lambda^3 - \lambda\tau^2 = 0\end{aligned}$$

Therefore the principal stresses are $\sigma_I = +\tau$, $\sigma_{II} = 0$ and $\sigma_{III} = -\tau$.

(b) To determine the principal direction corresponding to the principal stress $\sigma_I = +\tau$:

$$\begin{aligned}-\tau n_1 + \tau\cos\theta n_2 + \tau\sin\theta n_3 &= 0 \\ \tau\cos\theta n_1 - \tau n_2 + 0n_3 &= 0 \\ \tau\sin\theta n_1 + 0n_2 - \tau n_3 &= 0\end{aligned}$$

The second and third equations give $n_2 = \cos\theta$ and $n_3 = \sin\theta$. Making $\mathbf{n}$ a unit vector gives

$$\mathbf{n}_I = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\cos\theta\mathbf{e}_2 + \frac{1}{\sqrt{2}}\sin\theta\mathbf{e}_3$$

where, for definiteness, we have taken the positive sign. For the principal stress $\sigma_{II} = 0$:

$$\begin{aligned}0n_1 + \tau\cos\theta n_2 + \tau\sin\theta n_3 &= 0 \\ \tau\cos\theta n_1 + 0n_2 + 0n_3 &= 0 \\ \tau\sin\theta n_1 + 0n_2 + 0n_3 &= 0\end{aligned}$$

The second or third equations gives $n_1 = 0$ and the first gives $n_2 = -\tan\theta n_3$. Making $\mathbf{n}$ a unit vector gives

$$\mathbf{n}_{II} = 0\mathbf{e}_1 \mp \sin\theta\mathbf{e}_2 \pm \cos\theta\mathbf{e}_3$$

Choosing $\mathbf{n}_{III} = \mathbf{n}_I \times \mathbf{n}_{II}$ for a right-handed system gives

$$\mathbf{n}_{III} = \frac{1}{\sqrt{2}}\left\{\pm\mathbf{e}_1 \mp \cos\theta\mathbf{e}_2 \mp \sin\theta\mathbf{e}_3\right\}$$

3.

$$\begin{aligned}J_2 &= \frac{1}{2}\left\{(\sigma_I')^2 + (\sigma_{II}')^2 + (\sigma_{III}')^2\right\} \\ &= \frac{1}{2}\left\{(\sigma_I - \sigma)^2 + (\sigma_{II} - \sigma)^2 + (\sigma_{III} - \sigma)^2\right\}\end{aligned}$$

where $\sigma = (\sigma_I + \sigma_{II} + \sigma_{III})/3$.

$$J_2 = \frac{1}{2}\left\{\sigma_I^2 + \sigma_{II}^2 + \sigma_{III}^2 - 3\sigma^2\right\}$$

Substituting for σ, multiplying out and collecting terms gives the result.

4. Substituting (10.11) into (10.8) gives

$$\begin{aligned}
\left(\frac{4}{3}J_2\right)^{3/2}\sin^3\alpha - J_2\sqrt{\frac{4}{3}J_2}\sin\alpha - J_3 &= 0 \\
4\sin^3\alpha - 3\sin\alpha &= \sqrt{\frac{27}{4}}\frac{J_3}{J_2^{3/2}}
\end{aligned}$$

Using the identity

$$4\sin^3\alpha - 3\sin\alpha = -\sin 3\alpha$$

gives (10.12).

5. (a)

$$\begin{aligned}
\sigma'_{II} &= \sigma_{II} - \frac{1}{3}\left\{\sigma_I + \frac{1}{2}(\sigma_I + \sigma_{III}) + \sigma_{III}\right\} \\
&= \sigma_{II} - \frac{1}{2}(\sigma_I + \sigma_{III}) = 0
\end{aligned}$$

Hence, $J_3 = 0$, $\sin\alpha = 0$ and $\alpha = 0$.

(b)

$$\begin{aligned}
\sigma'_I &= \frac{2}{3}(\sigma_I - \sigma_{III}) \\
\sigma'_{II} &= \sigma'_{III} = -\frac{1}{3}(\sigma_I - \sigma_{III})
\end{aligned}$$

Therefore

$$\begin{aligned}
J_3 &= \frac{2}{27}(\sigma_I - \sigma_{III})^3 \\
J_2 &= \frac{1}{3}(\sigma_I - \sigma_{III})^2
\end{aligned}$$

Hence $\sin 3\alpha = -1$ and $\alpha = -30\,°$.

(c)

$$\begin{aligned}
\sigma'_I &= \sigma'_{II} = \frac{1}{3}(\sigma_I - \sigma_{III}) \\
\sigma'_{III} &= -\frac{2}{3}(\sigma_I - \sigma_{III})
\end{aligned}$$

Therefore

$$\begin{aligned} J_3 &= -\frac{2}{27}\left(\sigma_I - \sigma_{III}\right)^3 \\ J_2 &= \frac{1}{3}\left(\sigma_I - \sigma_{III}\right)^2 \end{aligned}$$

Hence $\sin 3\alpha = 1$ and $\alpha = 30\,^\circ$.

6.

$$\left(\sigma_{ij} - \lambda\delta_{ij}\right) n_j = 0$$

gives the principal directions, the n_j, of the stress σ_{ij}. Substituting the deviatoric stress gives

$$\left(\sigma'_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} - \lambda\delta_{ij}\right) n_j = 0$$

Thus, the principal directions of σ'_{ij} satisfy the same equation with an altered value of λ:

$$\lambda' = \lambda + \frac{1}{3}\sigma_{kk}$$

Chapter 11

Stationary Values of Shear Traction

1. If none of the $n_i = 0$, then the terms in braces $\{\ldots\}$ must vanish. Eliminating λ leads to

$$\sigma_1^2 - 2\sigma_1 t_n = \sigma_2^2 - 2\sigma_2 t_n = \sigma_3^2 - 2\sigma_3 t_n$$

From the first equality

$$\sigma_1^2 - \sigma_2^2 = 2\left(\sigma_1 - \sigma_2\right) t_n$$

Therefore, either $\sigma_1 = \sigma_2$ or $\sigma_1 + \sigma_2 = 2t_n$. Similarly, the second and third equality leads to either $\sigma_2 = \sigma_3$ or $\sigma_2 + \sigma_3 = 2t_n$ and first and third to either $\sigma_3 = \sigma_1$ or $\sigma_1 + \sigma_3 = 2t_n$. Clearly, the first option leads immediately to $\sigma_1 = \sigma_2 = \sigma_3$. If, for example, $\sigma_1 = \sigma_2$ but

$$\sigma_2 + \sigma_3 = 2t_n$$

In this case

$$\sigma_2 + \sigma_3 = 2\left(n_1^2\sigma_1 + n_2^2\sigma_2 + n_3^2\sigma_3\right)$$

Because $\sigma_1 = \sigma_2$, this can be rewritten as

$$\begin{aligned} \frac{1}{2}\left(\sigma_1 + \sigma_3\right) &= \sigma_1\left(1 - n_3^2\right) + n_3^2\sigma_3 \\ \frac{1}{2}\left(\sigma_1 - \sigma_3\right) &= n_3^2\left(\sigma_1 - \sigma_3\right) \end{aligned}$$

Hence either $\sigma_1 = \sigma_3$ or $n_3 = \pm 1/\sqrt{2}$. In the latter case the stress state is axisymmetric ($\sigma_1 = \sigma_2$) and the maximum shear traction occurs on any plane with a normal that makes an angle of $\pi/4$ with the distinguished stress axis.

2. (a) Traction is given by

$$\begin{bmatrix} -1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -5 \\ 2 \\ 6 \end{bmatrix}$$

or

$$\mathbf{t} = \frac{1}{3} \left(-5\mathbf{e}_1 + 2\mathbf{e}_2 + 6\mathbf{e}_3 \right)$$

(b) Normal traction is

$$\begin{aligned} \mathbf{n} \cdot \mathbf{t} &= \frac{1}{3} \left(\mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3 \right) \cdot \frac{1}{3} \left(-5\mathbf{e}_1 + 2\mathbf{e}_2 + 6\mathbf{e}_3 \right) \\ &= \frac{1}{9} \left(-5 + 4 + 12 \right) = \frac{11}{9} \end{aligned}$$

Shear traction is

$$\begin{aligned} \mathbf{t}_s &= \mathbf{t} - \frac{11}{9}\mathbf{n} \\ &= -2.074\mathbf{e}_1 - 0.148\mathbf{e}_2 + 1.185\mathbf{e}_3 \end{aligned}$$

The magnitude of the shear traction is

$$\begin{aligned} t_s &= \sqrt{\mathbf{t}_s \cdot \mathbf{t}_s} \\ &= \sqrt{\mathbf{t} \cdot \mathbf{t} - t_n^2} \\ &= 2.393 \end{aligned}$$

3. (a)

$$\begin{aligned} \mathbf{t} &= \mathbf{n} \cdot \boldsymbol{\sigma} \\ &= T \left(\mathbf{n} \cdot \mathbf{u} \right) \mathbf{u} \\ &= T \cos\theta \mathbf{u} \end{aligned}$$

(b)

$$\begin{aligned} t_n &= \mathbf{n} \cdot \mathbf{t} \\ &= T \cos^2\theta \end{aligned}$$

(c)

$$\begin{aligned} t_s &= \pm\sqrt{\mathbf{t} \cdot \mathbf{t} - t_n^2} \\ &= \pm T \cos\theta \sqrt{1 - \cos^2\theta} \\ &= \pm T \cos\theta \sin\theta \end{aligned}$$

4. From Problem 10.2, the greatest and least principal stresses are τ and $-\tau$ and the corresponding principal directions are $\mathbf{n}_I$ and $\mathbf{n}_{III}$. Thus, the maximum shear stress is τ and it occurs on the plane with normal

$$\mathbf{n} = \frac{1}{\sqrt{2}}(\mathbf{n}_I + \mathbf{n}_{III}) = \pm\mathbf{e}_1$$

and

$$\mathbf{n} = \cos\theta\mathbf{e}_2 + \sin\theta\mathbf{e}_3$$

5. If the shear traction vanishes on every plane then

$$\mathbf{n}\cdot\boldsymbol{\sigma}\cdot\mathbf{s} = \mathbf{0}$$

for all unit vectors $\mathbf{n}$ and $\mathbf{s}$ such that $\mathbf{n}\cdot\mathbf{s} = \mathbf{0}$. Substituting $\boldsymbol{\sigma} = \boldsymbol{\sigma}' - p\mathbf{I}$ where $\boldsymbol{\sigma}'$ is the deviatoric stress gives

$$\mathbf{n}\cdot\boldsymbol{\sigma}'\cdot\mathbf{s} - p\mathbf{n}\cdot\mathbf{s} = 0$$

Because the second term vanishes, the first term will vanish only if the deviatoric stress $\boldsymbol{\sigma}'$ vanishes. If $\boldsymbol{\sigma} = -p\mathbf{I}$, then the first equation is satisfied automatically.

6. (a) The unit vector that makes equal angles with the principal axes is given by

$$\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$$

The traction on this plane is

$$\begin{aligned}\mathbf{t} &= \mathbf{n}\cdot\boldsymbol{\sigma}\\ &= \frac{1}{\sqrt{3}}(\sigma_1\mathbf{e}_1 + \sigma_2\mathbf{e}_2 + \sigma_3\mathbf{e}_3)\end{aligned}$$

(b) The normal component of traction is

$$\begin{aligned}t_n &= \mathbf{n}\cdot\mathbf{t}\\ &= \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)\end{aligned}$$

(c) The magnitude of the shear traction is given by

$$t_s = \sqrt{\mathbf{t}\cdot\mathbf{t} - t_n^2}$$

(d) The direction of the shear traction is

$$\begin{aligned}\mathbf{s} &= \frac{\mathbf{t} - t_n\mathbf{n}}{|\mathbf{t} - t_n\mathbf{n}|}\\ &= \frac{1}{t_s}(\mathbf{t} - t_n\mathbf{n})\\ &= \frac{1}{\sqrt{3}t_s}(\sigma_1'\mathbf{e}_1 + \sigma_2'\mathbf{e}_2 + \sigma_3'\mathbf{e}_3)\end{aligned}$$

where the σ_i' are principal values of the deviatoric stress.

7. The normal traction is

$$\begin{aligned} t_n &= \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} \\ &= \sigma_1 n_1^2 + \frac{1}{2}(\sigma_1 + \sigma_3) n_2^2 + \sigma_3 n_3^2 \end{aligned}$$

Using $n_2^2 = 1 - \left(n_1^2 + n_3^2\right)$ and setting equal to $\frac{1}{2}(\sigma_1 + \sigma_3)$ gives

$$\left(n_1^2 - n_3^2\right)(\sigma_1 - \sigma_3) = 0$$

Because $\sigma_2 > 0$, $\sigma_1 \neq \sigma_3$, and therefore $n_1^2 = n_3^2$ and

$$n_2^2 + 2n_1^2 = 1$$

Calculating

$$\begin{aligned} \mathbf{t} \cdot \mathbf{t} &= \sigma_1^2 n_1^2 + \left[\frac{1}{2}(\sigma_1 + \sigma_3)\right]^2 n_2^2 + \sigma_3^2 n_3^2 \\ &= \left(\sigma_1^2 + \sigma_3^2\right) n_1^2 + \frac{1}{4}(\sigma_1 + \sigma_3)^2 \left(1 - 2n_1^2\right) \\ &= \frac{1}{2}(\sigma_1 - \sigma_3) n_1^2 + \frac{1}{4}(\sigma_1 + \sigma_3)^2 \end{aligned}$$

Computing the magnitude of the shear traction

$$\begin{aligned} t_s &= \sqrt{\mathbf{t} \cdot \mathbf{t} - t_n^2} \\ &= \frac{1}{\sqrt{2}} n_1 (\sigma_1 - \sigma_3) \end{aligned}$$

Setting equal to $(\sigma_1 - \sigma_3)/4$ gives

$$n_1 = n_3 = \frac{1}{2\sqrt{2}}$$

and

$$n_2 = \sqrt{1 - 2n_1^2} = \frac{\sqrt{3}}{2}$$

Therefore the normal to the plane is

$$\mathbf{n} = \frac{1}{2\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_3) + \frac{\sqrt{3}}{2}\mathbf{e}_2$$

Chapter 12

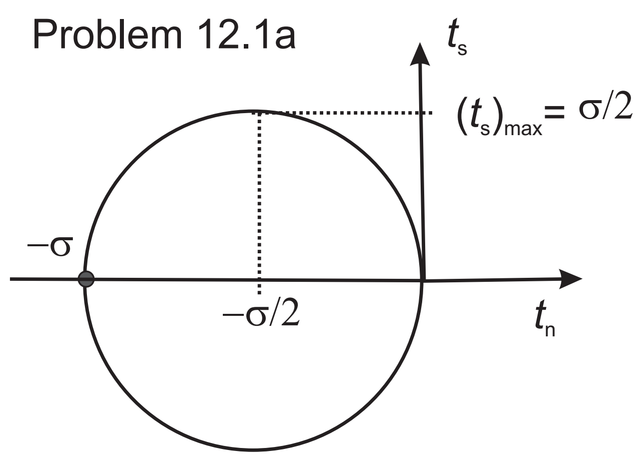
Mohr's Circle

1. (a) Maximum shear stress is

$$\frac{1}{2}\left(0-(-\sigma)\right)=\frac{\sigma}{2}$$

and the normal stress on the plane of maximum shear is

$$\frac{1}{2}\left(0+(-\sigma)\right)=-\frac{\sigma}{2}$$

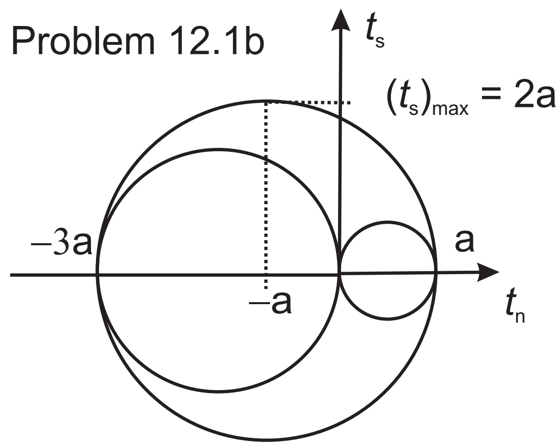


(b) Maximum shear stress is

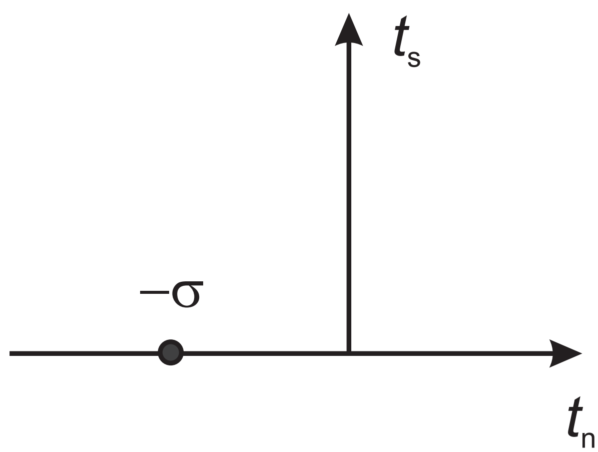
$$\frac{1}{2}\left(a-(-3a)\right)=2a$$

and the normal stress on the plane of maximum shear is

$$\frac{1}{2}\left(a+(-3a)\right)=-a$$



(c) Maximum shear stress is 0 and the normal stress on the plane of maximum shear is $-\sigma$.

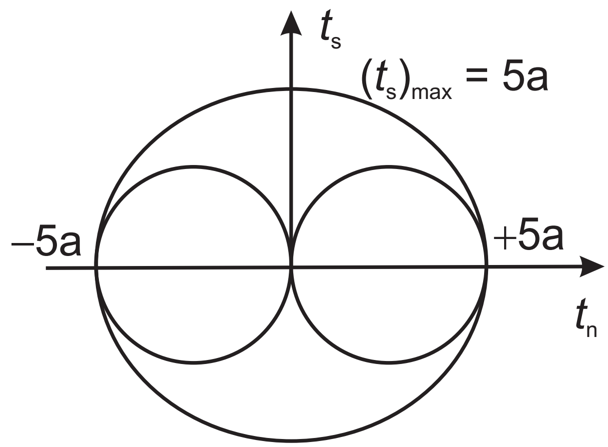


(d) Principal stresses are $-5a$, 0, $+5a$. Therefore, maximum shear stress is

$$\frac{1}{2}\left(5a - (-5a)\right) = 5a$$

and the normal stress on the plane of maximum shear is

$$\frac{1}{2}\left(5a + (-5a)\right) = 0$$

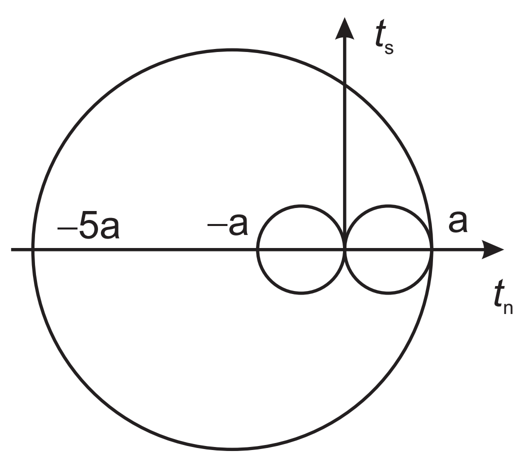


(e) Maximum shear stress is

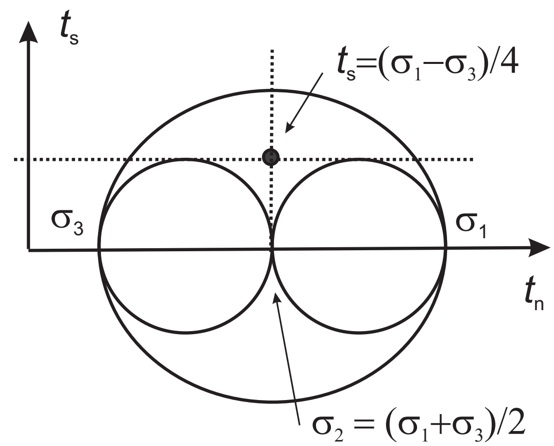
$$\frac{1}{2}\left(a-(-5a)\right)=3a$$

and the normal stress on the plane of maximum shear is

$$\frac{1}{2}\left(a+(-5a)\right)=-2a$$



2. See graphical construction in the Figure.



3. Failure occurs when the Mohr's circle first becomes tangent to the failure condition

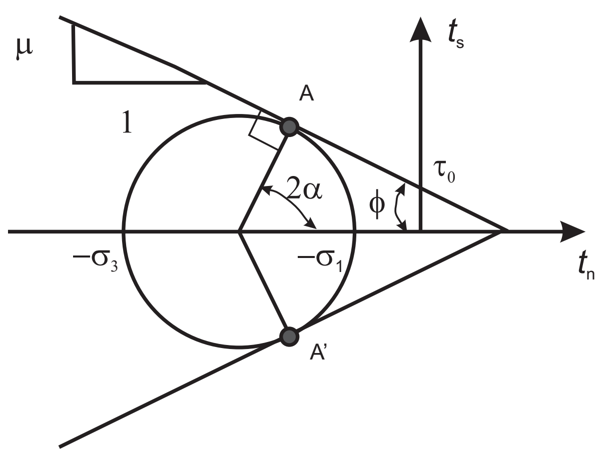
$$|t_s| = \tau_0 - \mu t_n$$

Because $2\alpha = (\pi/2) - \phi$, the magnitude of the shear traction at points A and A' is

$$\begin{aligned} |t_s| &= \frac{1}{2}(\sigma_3 - \sigma_1)\sin\left[\pm\left(\frac{\pi}{2} - \phi\right)\right] \\ &= \frac{1}{2}(\sigma_3 - \sigma_1)\cos\phi \end{aligned}$$

and the normal traction is

$$\begin{aligned} |t_s| &= \frac{1}{2}(\sigma_3 + \sigma_1) + \frac{1}{2}(\sigma_3 - \sigma_1)\cos\left[\pm\left(\frac{\pi}{2} - \phi\right)\right] \\ &= \frac{1}{2}(\sigma_3 + \sigma_1) + \frac{1}{2}(\sigma_3 - \sigma_1)\sin\phi \end{aligned}$$



Part III

Motion and Deformation

Chapter 13

Current and Reference Configuration

1. (a) Motion: $x_1 = X_1 + \gamma(t)X_2$, $x_2 = X_2$, $x_3 = X_3$.
   (b) Lagrangian description of velocity
   $$V_1(\mathbf{X}, t) = \dot{\gamma}(t)X_2,\ V_2(\mathbf{X}, t) = V_3(\mathbf{X}, t) = 0$$
   Eulerian description of velocity
   $$v_1(\mathbf{x}, t) = \dot{\gamma}(t)x_2,\ v_2(\mathbf{x}, t) = v_3(\mathbf{x}, t) = 0$$
   (c) $X_1 = x_1 - \gamma(t)x_2$, $X_2 = x_2$, $X_3 = x_3$
2. (a)
   $$\begin{aligned} a_i &= \frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} \\ &= x_i \frac{(1-k)}{(1+kt)^2} \end{aligned}$$
   (b)
   $$\begin{aligned} \frac{d\theta}{dt} &= \frac{\partial \theta}{\partial t} + v_k \frac{\partial \theta}{\partial x_k} \\ &= kx_2 + v_1 + v_2 kt \\ &= kx_2 + \frac{x_1 + x_2 kt}{1 + kt} \end{aligned}$$
3. (a) Lagrangian:
   $$V_i = X_i k$$

To get Eulerian velocity, invert the motion:

$$X_i = \frac{x_i}{(1+kt)}$$

Substituting into the Lagrangian velocity gives:

$$v_i = \frac{kx_i}{(1+kt)}$$

Note also that differentiating the inverted motion gives

$$\frac{d}{dt}X_i = \frac{\dot{x}_i}{(1+kt)} - \frac{kx_i}{(1+kt)^2}$$

The left hand side is zero because position is fixed in the reference configuration and $\dot{x}_i$ is the velocity. Hence solving for $\dot{x}_i$ gives the Eulerian description of the velocity; i.e.

$$v_i = \dot{x}_i = \frac{kx_i}{(1+kt)}$$

(b) $A_i = 0$

(c) Substitution: $a_i = 0$. Calculation from material derivative:

$$\begin{aligned} a_i &= \frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} \\ &= \frac{-k^2 x_i}{(1+kt)^2} + \frac{kx_k}{(1+kt)} \frac{k\delta_{ik}}{(1+kt)} \\ &= 0 \end{aligned}$$

4. (a) Lagrangian description of velocity

$$V_1 = X_2 e^t,\ V_2 = -X_1 e^{-t},\ V_3 = 0$$

Lagrangian description of acceleration:

$$A_1 = X_2 e^t,\ A_2 = X_1 e^{-t},\ V_3 = 0$$

(b) First invert the motion to get

$$\begin{aligned} X_1 &= \frac{x_1 - x_2\left(e^t - 1\right)}{e^t + e^{-t} - 1} \\ X_2 &= \frac{x_2 + x_1\left(1 - e^t\right)}{e^t + e^{-t} - 1} \end{aligned}$$

Then substitute into (a) to get

$$\begin{aligned} v_1 &= \frac{x_2 e^t + x_1\left(e^t - 1\right)}{e^t + e^{-t} - 1} \\ v_2 &= \frac{-x_1 e^{-t} + x_2\left(1 - e^{-t}\right)}{e^t + e^{-t} - 1} \end{aligned}$$

(c) Substitute the inverted motion into the answer from (a) to get

$$\begin{aligned} a_1 &= \frac{x_2 e^t + x_1 \left(e^t - 1\right)}{e^t + e^{-t} - 1} \\ a_2 &= \frac{x_1 e^{-t} - x_2 \left(1 - e^{-t}\right)}{e^t + e^{-t} - 1} \end{aligned}$$

Use the material derivative

$$a_i = \frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k}$$

to get the same result (after much algebra).

Chapter 14

Rate of Deformation

1. Principal values λ are given by

$$\begin{aligned}
\det \begin{bmatrix} -\lambda & W_{12} & W_{13} \\ -W_{12} & -\lambda & W_{23} \\ -W_{13} & -W_{23} & -\lambda \end{bmatrix} &= 0 \\
-\lambda \left\{ \lambda^2 + \left(W_{12}^2 + W_{13}^2 + W_{23}^2 \right) \right\} &= 0
\end{aligned}$$

Thus, $\lambda = 0$ is one principal value and because $(\ldots) > 0$, the remaining two are purely imaginary.

2. The components of the axial vector are given by

$$\begin{aligned}
w_m &= -\frac{1}{2}\epsilon_{ilm} W_{il} \\
&= -\frac{1}{2}\epsilon_{ilm} \left(b_i a_l - a_i b_l \right) \\
&= -\frac{1}{2} \left(\mathbf{b} \times \mathbf{a} \right)_m + \frac{1}{2} \left(\mathbf{a} \times \mathbf{b} \right)_m
\end{aligned}$$

Therefore

$$\mathbf{w} = \mathbf{a} \times \mathbf{b}$$

3. (a) The axial vector of the anti-symmetric tensor $\mathbf{W}$ satisfies

$$\mathbf{W} \cdot \mathbf{u} = \mathbf{w} \times \mathbf{u}$$

for all $\mathbf{u}$. Setting $\mathbf{u} = \mathbf{p}$ gives $\mathbf{w} \times \mathbf{p} = \mathbf{0}$. Because the cross product of $\mathbf{w}$ and $\mathbf{p}$ vanishes, they must be parallel and $\mathbf{w} = w\mathbf{p}$.

(b) From (a)

$$\mathbf{W} \cdot \mathbf{u} = w\mathbf{p} \times \mathbf{u}$$

Using $\mathbf{p} = \mathbf{q} \times \mathbf{r}$ gives

$$\begin{aligned}
\mathbf{W} \cdot \mathbf{u} &= w \left(\mathbf{q} \times \mathbf{r} \right) \times \mathbf{u} \\
&= -w\mathbf{u} \times \left(\mathbf{q} \times \mathbf{r} \right)
\end{aligned}$$

From exercise 4.5.a

$$\begin{aligned}\mathbf{W}\cdot\mathbf{u} &= -\mathbf{u}\cdot w\,(\mathbf{rq}-\mathbf{qr})\\ &= \mathbf{u}\cdot w\,(\mathbf{qr}-\mathbf{rq})\end{aligned}$$

Writing the left side as $\mathbf{u}\cdot\mathbf{W}^T$ and noting that the relation applies for all $\mathbf{u}$ gives

$$\mathbf{W}^T = w\,(\mathbf{qr}-\mathbf{rq})$$

or

$$\mathbf{W} = w\,(\mathbf{rq}-\mathbf{qr})$$

Alternatively, using the result of (a) and Exercise 4.9 gives

$$W_{ij} = \epsilon_{ijk} w p_k$$

Substituting $\mathbf{p} = \mathbf{q}\times\mathbf{r}$ in index form

$$p_k = \epsilon_{klm} q_l r_m$$

gives

$$W_{ij} = \epsilon_{kji}\epsilon_{klm} w q_l r_m$$

Using the $\epsilon-\delta$ identity (4.13) gives

$$W_{ij} = w\,(r_i q_j - q_i r_j)$$

Converting to dyad form gives the result.

4. (a) From example 13.1, the Eulerian description of the velocity is

$$\mathbf{v}(\mathbf{x},t) = \dot{\mathbf{Q}}(t)\cdot\mathbf{Q}^{-1}\cdot\{\mathbf{x}-\mathbf{c}(t)\}+\dot{\mathbf{c}}(t)$$

Therefore the velocity gradient is $\mathbf{L} = \dot{\mathbf{Q}}(t)\cdot\mathbf{Q}^{-1}$. The rate of deformation is

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L}+\mathbf{L}^T\right) = \frac{1}{2}\left\{\dot{\mathbf{Q}}(t)\cdot\mathbf{Q}^{-1}+\left(\dot{\mathbf{Q}}(t)\cdot\mathbf{Q}^{-1}\right)^T\right\}$$

and the spin tensor is

$$\mathbf{W} = \frac{1}{2}\left(\mathbf{L}-\mathbf{L}^T\right) = \frac{1}{2}\left\{\dot{\mathbf{Q}}(t)\cdot\mathbf{Q}^{-1}-\left(\dot{\mathbf{Q}}(t)\cdot\mathbf{Q}^{-1}\right)^T\right\}$$

(b) If $\mathbf{D} = 0$, then the distance between any two points is constant

$$|\mathbf{x}_p-\mathbf{x}_q| = |\mathbf{X}_p-\mathbf{X}_q|$$

Substituting the motion yields $\mathbf{Q}^T\cdot\mathbf{Q} = \mathbf{I}$ and therefore $\mathbf{Q}^{-1} = \mathbf{Q}^T$. Alternatively, $\mathbf{D} = 0$ gives

$$\dot{\mathbf{Q}}\cdot\mathbf{Q}^{-1}+\left(\dot{\mathbf{Q}}\cdot\mathbf{Q}^{-1}\right)^T = 0$$

Multiplying from the left by $\mathbf{Q}^T$ and from the right by $\mathbf{Q}$ gives

$$\begin{aligned}\mathbf{Q}^T \cdot \dot{\mathbf{Q}} + \mathbf{Q}^T \cdot \left(\mathbf{Q}^{T^{-1}} \cdot \dot{\mathbf{Q}}^T\right) \cdot \mathbf{Q} &= 0 \\ \frac{d}{dt}\left\{\mathbf{Q}^T \cdot \mathbf{Q}\right\} &= 0\end{aligned}$$

Hence $\mathbf{Q}^T \cdot \mathbf{Q}$ is a constant. Setting the constant equal to $\mathbf{I}$ again gives $\mathbf{Q}^{-1} = \mathbf{Q}^T$. If $\mathbf{D} = 0$

$$\dot{\mathbf{Q}} \cdot \mathbf{Q}^{-1} = -\left(\dot{\mathbf{Q}} \cdot \mathbf{Q}^{-1}\right)^T$$

and $\mathbf{W} = \dot{\mathbf{Q}} \cdot \mathbf{Q}^{-1}$ is antisymmetric.

(c) The velocity can be written

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{W} \cdot \{\mathbf{x} - \mathbf{c}(t)\} + \dot{\mathbf{c}}(t)$$

From example 13.1, the acceleration is

$$\mathbf{a}(\mathbf{x}, t) = \ddot{\mathbf{Q}} \cdot \mathbf{Q}^{-1} \cdot \{\mathbf{x} - \mathbf{c}(t)\} + \ddot{\mathbf{c}}(t)$$

Writing

$$\dot{\mathbf{Q}} = \mathbf{W} \cdot \mathbf{Q}$$

and differentiating gives

$$\begin{aligned}\ddot{\mathbf{Q}} &= \dot{\mathbf{W}} \cdot \mathbf{Q} + \mathbf{W} \cdot \dot{\mathbf{Q}} \\ &= \dot{\mathbf{W}} \cdot \mathbf{Q} + \mathbf{W} \cdot \mathbf{W} \cdot \mathbf{Q}\end{aligned}$$

Substituting into the expression for the acceleration gives the result.

(d) Follows from definition of the axial vector of an anti-symmetric tensor.

5. Position in a rigid body is given by

$$\mathbf{x}(t) = \mathbf{a}_p X_p + \mathbf{c}(t)$$

where the X_p are fixed because the body is rigid. Solve for the X_p by forming the scalar product with $\mathbf{a}_q$

$$\begin{aligned}\mathbf{a}_q \cdot \mathbf{x}(t) &= \mathbf{a}_q \cdot \mathbf{a}_p X_p + \mathbf{a}_q \cdot \mathbf{c}(t) \\ &= \delta_{qp} X_p + \mathbf{a}_q \cdot \mathbf{c}(t) \\ &= X_q + \mathbf{a}_q \cdot \mathbf{c}(t)\end{aligned}$$

or

$$X_q = \mathbf{a}_q \cdot \{\mathbf{x}(t) - \mathbf{c}(t)\}$$

The Lagrangian velocity is given by

$$\mathbf{v}(\mathbf{X}, t) = \dot{\mathbf{a}}_p X_p + \dot{\mathbf{c}}(t)$$

To obtain the Eulerian description, substitute for the X_p to get

$$\mathbf{v}(\mathbf{X}, t) = \dot{\mathbf{a}}_p \mathbf{a}_p \cdot \{\mathbf{x}(t) - \mathbf{c}(t)\} + \dot{\mathbf{c}}(t)$$

Hence, the velocity gradient is $\mathbf{L} = \dot{\mathbf{a}}_p \mathbf{a}_p$ and the anti-symmetric part is

$$W = \frac{1}{2} \{\dot{\mathbf{a}}_p \mathbf{a}_p - \mathbf{a}_p \dot{\mathbf{a}}_p\}$$

The expression for the axial vector follows from exercise 14.2.

Chapter 15

Geometric Measures of Deformation

1.

$$
\begin{aligned}
\mathbf{C}^T &= \left(\mathbf{F}^T \cdot \mathbf{F}\right)^T \\
&= \mathbf{F}^T \cdot \mathbf{F}^{T^T} \\
&= \mathbf{F}^T \cdot \mathbf{F} = \mathbf{C}
\end{aligned}
$$

2.

$$
\begin{aligned}
\det \mathbf{F} &= \det (\mathbf{R} \cdot \mathbf{U}) \\
&= \det \mathbf{R} \det \mathbf{U} \\
&= \det \mathbf{U}
\end{aligned}
$$

where the last line follows because $\mathbf{R}$ is an orthogonal tensor and, hence, has a determinant equal to one, as long as the rotation is from right-handed to right-handed.

3. (a)

$$
\begin{aligned}
F_{11} &= \frac{\partial x_1}{\partial X_1} = \alpha \cos \theta \\
F_{12} &= \frac{\partial x_1}{\partial X_2} = \beta \sin \theta \\
F_{21} &= \frac{\partial x_2}{\partial X_1} = -\alpha \sin \theta \\
F_{22} &= \frac{\partial x_2}{\partial X_2} = \beta \cos \theta \\
F_{13} &= F_{31} = F_{23} = F_{32} = 0 \\
F_{33} &= 1
\end{aligned}
$$

(b) $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$ gives in matrix notation

$$\begin{aligned}
[C] &= [F]^T [F] \\
&= \begin{bmatrix} \alpha\cos\theta & -\alpha\sin\theta & 0 \\ \beta\sin\theta & \beta\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha\cos\theta & \beta\sin\theta & 0 \\ -\alpha\sin\theta & \beta\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

or, in dyadic notation,

$$\mathbf{C} = \alpha^2 \mathbf{e}_1\mathbf{e}_1 + \beta^2 \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3$$

(c)

$$\begin{aligned}
\mathbf{U} &= \sqrt{\mathbf{C}} \\
&= \alpha\mathbf{e}_1\mathbf{e}_1 + \beta\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3
\end{aligned}$$

(d)

$$\mathbf{R} = \mathbf{F} \cdot \mathbf{U}^{-1}$$

or, in matrix notation,

$$\begin{aligned}
[R] &= [F]\,[U]^{-1} \\
&= \begin{bmatrix} \alpha\cos\theta & \beta\sin\theta & 0 \\ -\alpha\sin\theta & \beta\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha^{-1} & 0 & 0 \\ 0 & \beta^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

or

$$\mathbf{R} = \cos\theta\,(\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2) + \sin\theta\,(\mathbf{e}_2\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_2) + \mathbf{e}_3\mathbf{e}_3$$

(e)

$$\begin{aligned}
\mathbf{n}_I &= \mathbf{R} \cdot \mathbf{e}_1 \\
&= \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2 \\
\mathbf{n}_{II} &= \mathbf{R} \cdot \mathbf{e}_2 \\
&= -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2 \\
\mathbf{n}_{III} &= \mathbf{e}_3
\end{aligned}$$

4.

$$
\begin{aligned}
\nabla_{\mathbf{X}}\mathbf{u} &= \left(\mathbf{e}_i \frac{\partial}{\partial X_i}\right)(u_j \mathbf{e}_j) \\
&= \frac{\partial u_j}{\partial X_i}\mathbf{e}_i\mathbf{e}_j \\
&= \frac{\partial u_j}{\partial x_k}\frac{\partial x_k}{\partial X_i}\mathbf{e}_i\mathbf{e}_j \\
&= \left(\frac{\partial x_k}{\partial X_i}\mathbf{e}_i\mathbf{e}_k\right)\cdot\left(\frac{\partial u_j}{\partial x_l}\mathbf{e}_l\mathbf{e}_j\right) \\
&= \mathbf{F}^T \cdot \nabla_{\mathbf{x}}\mathbf{u}
\end{aligned}
$$

5. Forming the scalar product of each side of Nanson's formula with itself gives

$$
\begin{aligned}
(da)^2 &= J^2 (dA)^2 \left(\mathbf{N}\cdot\mathbf{F}^{-1}\right)\cdot\left(\mathbf{N}\cdot\mathbf{F}^{-1}\right) \\
&= J^2 (dA)^2\, \mathbf{N}\cdot\left(\mathbf{F}^{-1}\cdot\mathbf{F}^{-1^T}\right)\cdot\mathbf{N} \\
&= J^2 (dA)^2 \left(\mathbf{N}\cdot\mathbf{C}^{-1}\cdot\mathbf{N}\right)
\end{aligned}
$$

or

$$
\frac{da}{dA} = J\sqrt{\mathbf{N}\cdot\mathbf{C}^{-1}\cdot\mathbf{N}}
$$

6.

$$
\begin{aligned}
dS_A dS_B \cos\Theta &= d\mathbf{X}_A \cdot d\mathbf{X}_B \\
&= \left(\mathbf{F}^{-1}\cdot d\mathbf{x}_A\right)\cdot\left(\mathbf{F}^{-1}\cdot d\mathbf{x}_B\right) \\
&= d\mathbf{x}_A\cdot\left(\mathbf{F}^{-1^T}\cdot\mathbf{F}^{-1}\right)\cdot d\mathbf{x}_B \\
&= ds_A ds_B \mathbf{n}_A\cdot\mathbf{B}^{-1}\cdot\mathbf{n}_B
\end{aligned}
$$

Rearranging gives

$$
\begin{aligned}
\cos\Theta &= \frac{\mathbf{n}_A\cdot\mathbf{B}^{-1}\cdot\mathbf{n}_B}{(dS_A/ds_A)\,(dS_B/ds_B)} \\
&= \lambda_A^{-1}\lambda_B^{-1}\mathbf{n}_A\cdot\mathbf{B}^{-1}\cdot\mathbf{n}_B
\end{aligned}
$$

7. Multiplying both sides of Nanson's formula from the right by $\mathbf{F}$ gives

$$
J\mathbf{N}dA = \mathbf{n}\cdot\mathbf{F}da
$$

Forming the scalar product of each side with itself gives

$$
\begin{aligned}
J^2 dA^2 &= (\mathbf{n}\cdot\mathbf{F})\cdot(\mathbf{n}\cdot\mathbf{F})\,da^2 \\
&= \mathbf{n}\cdot\left(\mathbf{F}\cdot\mathbf{F}^T\right)\cdot\mathbf{n}da^2
\end{aligned}
$$

Rearranging gives

$$\left(\frac{dA}{da}\right)^2 = J^{-2}\mathbf{n}\cdot\mathbf{B}\cdot\mathbf{n}$$

8. See exercise 6.10.

9.

$$\begin{aligned}
d\mathbf{x} = \mathbf{n}ds &= \mathbf{F}\cdot d\mathbf{X} \\
&= \mathbf{F}\cdot\frac{d\mathbf{X}}{dS}dS \\
\mathbf{n}\frac{ds}{dS} &= \mathbf{F}\cdot\mathbf{N} \\
\Lambda\mathbf{n} &= \mathbf{F}\cdot\mathbf{N}
\end{aligned}$$

10.

$$\mathbf{F} = \gamma\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3$$

$$\mathbf{C} = \mathbf{e}_1\mathbf{e}_1 + \gamma\left(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1\right) + \left(1+\gamma^2\right)\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3$$

11. Unit normals in the directions of the diagonals are given by

$$\mathbf{N}_+ = \frac{1}{\sqrt{2}}\left(\mathbf{e}_1 + \mathbf{e}_2\right)$$

and

$$\mathbf{N}_- = \frac{1}{\sqrt{2}}\left(-\mathbf{e}_1 + \mathbf{e}_2\right)$$

The stretches are

$$\begin{aligned}
\Lambda_+ &= \sqrt{\mathbf{N}_+\cdot\mathbf{C}\cdot\mathbf{N}_+} \\
&= \sqrt{1+\gamma+\frac{1}{2}\gamma^2}
\end{aligned}$$

and

$$\begin{aligned}
\Lambda_- &= \sqrt{\mathbf{N}_-\cdot\mathbf{C}\cdot\mathbf{N}_-} \\
&= \sqrt{1-\gamma+\frac{1}{2}\gamma^2}
\end{aligned}$$

12. (a)

$$[F] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$
\begin{aligned}
[C] &= [F]^T [F] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\det(C - cI) = (1 - c)\{(1 - c)(2 - c) - 1\} = 0
$$

with solutions

$$
\begin{aligned}
c_I &= \frac{1}{2}\left(3 + \sqrt{5}\right) = 2.618 \\
c_{II} &= 1 \\
c_{III} &= \frac{1}{2}\left(3 - \sqrt{5}\right) = 0.382
\end{aligned}
$$

(b)

$$
\begin{aligned}
\Lambda_I &= \sqrt{c_I} = 1.618 \\
\Lambda_{II} &= \sqrt{c_{II}} = 1 \\
\Lambda_{III} &= \sqrt{c_{III}} = 0.618
\end{aligned}
$$

(c) For the principal direction $\mathbf{N}^I$ corresponding to c_I

$$
\begin{aligned}
(1 - c_I)(N_I)_1 + (N_I)_2 &= 0 \\
(N_I)_1 + (2 - c_I)(N_I)_2 &= 0 \\
(1 - c_I)(N_I)_3 &= 0
\end{aligned}
$$

The third equation gives $(N_I)_3 = 0$. From the first or second equation

$$
\tan\theta = \frac{(N_I)_2}{(N_I)_1} = 1.618
$$

or $\theta = 58.3\,°$.

13. (a)

$$
[F] = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{aligned}
[C] &= [F]^T [F] = \begin{bmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1 + \gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$\det(C - cI) = (1-c)\left\{(1-c)\left(1-c+\gamma^2\right) - \gamma^2\right\} = 0$$

with solutions $c_{III} = 1$ and

$$c_{\pm} = 1 + \frac{1}{2}\gamma^2 \pm \gamma\sqrt{1 + (\gamma/2)^2}$$

Note that since $\det(C) = 1$, $c_+ c_- = 1$. The principal stretches are

$$\begin{aligned} \Lambda_I &= \sqrt{c_+} = \alpha \\ \Lambda_{II} &= \sqrt{c_-} = \alpha^{-1} \\ \Lambda_{III} &= 1 \end{aligned}$$

(b) The components of the principal direction corresponding to c_+ satisfy

$$\begin{bmatrix} 1-c_+ & \gamma & 0 \\ \gamma & 1+\gamma^2-c_+ & 0 \\ 0 & 0 & 1-c_+ \end{bmatrix} \begin{bmatrix} (N_I)_1 \\ (N_I)_2 \\ (N_I)_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{aligned} (1-c_+)(N_I)_1 + \gamma (N_I)_2 &= 0 \\ \gamma (N_I)_1 + \left(1+\gamma^2-c_+\right)(N_I)_2 &= 0 \\ (1-c_+)(N_I)_3 &= 0 \end{aligned}$$

From the third equation $N_3 = 0$. From the first

$$(N_I)_2 = -\frac{1-c_+}{\gamma}(N_I)_1 = \alpha (N_I)_1$$

Therefore the principal direction can be written as

$$\mathbf{N}_I = \cos\Theta\,\mathbf{e}_1 + \sin\Theta\,\mathbf{e}_2$$

where $\tan\Theta = \alpha$. Because $c_{III} = 1$ the corresponding principal direction is given by $\mathbf{N}_{III} = \mathbf{e}_3$. Then $\mathbf{N}_{II} = \mathbf{N}_{III} \times \mathbf{N}_I$:

$$\mathbf{N}_{II} = -\sin\Theta\,\mathbf{e}_1 + \cos\Theta\,\mathbf{e}_2$$

Therefore $\mathbf{N}_K = \mathbf{R}^L \cdot \mathbf{e}_k$ (Note that L is not an index here and K and k denote corresponding Roman and Arabic numbers) implies

$$\mathbf{R}^L = \cos\Theta\,(\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2) + \sin\Theta\,(\mathbf{e}_2\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_2)$$

(c) The deformation tensor is given in principal axis form by

$$\mathbf{U} = \alpha\mathbf{N}_I\mathbf{N}_I + \alpha^{-1}\mathbf{N}_{II}\mathbf{N}_{II} + \mathbf{e}_3\mathbf{e}_3$$

Substituting for $\mathbf{N}_I$ and $\mathbf{N}_{II}$ from (b) and multiplying out the dyads gives

$$\begin{aligned}\mathbf{U} &= \alpha\left\{\cos^2\Theta\mathbf{e}_1\mathbf{e}_1+\cos\Theta\sin\Theta\left(\mathbf{e}_2\mathbf{e}_1+\mathbf{e}_1\mathbf{e}_2\right)+\sin^2\Theta\mathbf{e}_2\mathbf{e}_2\right\}\\ &\quad+\alpha^{-1}\left\{\sin^2\Theta\mathbf{e}_1\mathbf{e}_1-\cos\Theta\sin\Theta\left(\mathbf{e}_2\mathbf{e}_1+\mathbf{e}_1\mathbf{e}_2\right)+\cos^2\Theta\mathbf{e}_2\mathbf{e}_2\right\}\\ &\quad+\mathbf{e}_3\mathbf{e}_3\end{aligned}$$

Collecting the coefficients of the dyads gives

$$\begin{aligned}U_{11} &= \alpha\cos^2(\Theta)+\alpha^{-1}\sin^2(\Theta)\\ U_{12} &= U_{21}=\left(\alpha-\alpha^{-1}\right)\cos(\Theta)\sin(\Theta)\\ U_{22} &= \alpha^{-1}\cos^2(\Theta)+\alpha\sin^2(\Theta)=\\ U_{33} &= 1\end{aligned}$$

and after some algebra

$$\begin{aligned}U_{11} &= \frac{2\alpha}{\alpha^2+1}\\ U_{12} &= U_{21}=\frac{\alpha^2-1}{\alpha^2+1}\\ U_{22} &= \frac{\alpha^3+\alpha^{-1}}{\alpha^2+1}\\ U_{33} &= 1\end{aligned}$$

14. First find the principal directions in the current state. From Exercise 15.9,

$$\Lambda_K\mathbf{n}_K=\mathbf{F}\cdot\mathbf{N}_K,\ \text{(no sum on } K\text{)}$$

Therefore

$$\Lambda_I\mathbf{n}_I=\mathbf{F}\cdot\mathbf{N}_I$$

giving

$$\mathbf{n}_I=\frac{1}{\alpha\sqrt{1+\alpha^2}}\left\{(1+\gamma\alpha)\,\mathbf{e}_1+\alpha\mathbf{e}_2\right\}$$

Using $1+\alpha\gamma=\alpha^2$ gives

$$\mathbf{n}_I=\frac{1}{\sqrt{1+\alpha^2}}\left\{\alpha\mathbf{e}_1+\mathbf{e}_2\right\}$$

Similarly

$$\mathbf{n}_{II}=\frac{1}{\sqrt{1+\alpha^2}}\left\{-\mathbf{e}_1+\alpha\mathbf{e}_2\right\}$$

Noting that $\mathbf{R}^E=\mathbf{n}_K\mathbf{e}_K$ or comparing with the result of Exercise 6.14 give

$$\mathbf{R}^E=\left(\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2\right)\cos\theta+\left(\mathbf{e}_2\mathbf{e}_1-\mathbf{e}_1\mathbf{e}_2\right)\sin\theta+\mathbf{e}_3\mathbf{e}_3$$

where

$$\tan\theta = \alpha^{-1}$$

(see Figure below). Recall from Exercise 15.13 that

$$\tan\Theta = \alpha$$

and, hence, $\theta = \pi/2 - \Theta$. Because $\mathbf{R}$ rotates the $\mathbf{N}_K$ into the $\mathbf{n}_K$,

$$\mathbf{R} = \mathbf{n}_K\mathbf{N}_K$$

Substituting for the $\mathbf{n}_K$ and $\mathbf{N}_K$ gives

$$\mathbf{R} = \frac{2\alpha}{1+\alpha^2}\left(\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2\right) - \frac{\alpha^2-1}{1+\alpha^2}\left(\mathbf{e}_2\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_2\right) + \mathbf{e}_3\mathbf{e}_3$$

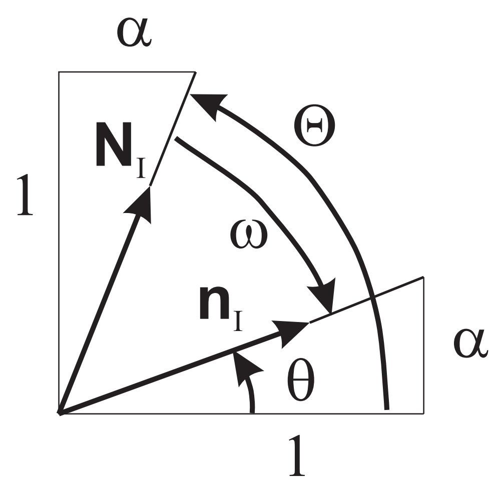
(Alternatively, $\mathbf{R}$ can be found from $\mathbf{F}\cdot\mathbf{U}^{-1}$.) Identifying

$$\begin{aligned}\cos\omega &= \frac{2\alpha}{1+\alpha^2}\\ \sin\omega &= \frac{\alpha^2-1}{1+\alpha^2}\end{aligned}$$

gives $\tan\omega = \gamma/2$. Note the minus sign preceding $\sin\omega$ in $\mathbf{R}$ indicates that ω is positive for a clockwise rotation about the $\mathbf{e}_3$. Using the trigonometric identity

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

and the expressions $\tan\Theta = \alpha$ and $\tan\theta = \alpha^{-1}$ gives $\omega = \Theta - \theta$ (See figure below).



The figure shows the relation between the rotations. The principal axis in the reference configuration $\mathbf{N}_I$ is rotated an angle ω clockwise about the $\mathbf{e}_3$ axis into the principal axis in the current configuration $\mathbf{n}_I$.

Chapter 16

Strain Tensors

1. Writing position in the reference configuration as $X_m = x_m - u_m$ gives

$$F_{ki}^{-1} = \frac{\partial X_k}{\partial x_i} = \delta_{ki} - \frac{\partial u_k}{\partial x_i}$$

Substituting into (16.11)

$$\begin{aligned} e_{ij}^A &= \frac{1}{2}\left\{\delta_{ij} - \left(\delta_{ki} - \frac{\partial u_k}{\partial x_i}\right)\left(\delta_{kj} - \frac{\partial u_k}{\partial x_j}\right)\right\} \\ &= \frac{1}{2}\left\{\delta_{ij} - \delta_{ij} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right\} \\ &= \frac{1}{2}\left\{\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right\} \end{aligned}$$

2. From (16.15), $\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1}$. From $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$

$$\dot{\mathbf{F}} = \dot{\mathbf{R}} \cdot \mathbf{U} + \mathbf{R} \cdot \dot{\mathbf{U}}$$

and

$$\begin{aligned} \mathbf{F}^{-1} = (\mathbf{R} \cdot \mathbf{U})^{-1} &= \mathbf{U}^{-1} \cdot \mathbf{R}^{-1} \\ &= \mathbf{U}^{-1} \cdot \mathbf{R}^{T} \end{aligned}$$

Substituting

$$\begin{aligned} \mathbf{L} &= \dot{\mathbf{R}} \cdot \mathbf{U} \cdot \mathbf{U}^{-\mathbf{1}} \cdot \mathbf{R}^{\mathbf{T}} + \mathbf{R} \cdot \dot{\mathbf{U}} \cdot \mathbf{U}^{-1} \cdot \mathbf{R}^{T} \\ &= \dot{\mathbf{R}} \cdot \mathbf{R}^{\mathbf{T}} + \mathbf{R} \cdot \dot{\mathbf{U}} \cdot \mathbf{U}^{-1} \cdot \mathbf{R}^{T} \end{aligned}$$

3. (a)

$$\begin{aligned}
\mathbf{E}^{(-2)} &= \frac{1}{2}\sum_K \left(1-\Lambda_K^{-2}\right)\mathbf{N}_K\mathbf{N}_K \\
&= \frac{1}{2}\left\{\sum_K \mathbf{N}_K\mathbf{N}_K - \sum_K \frac{1}{\Lambda_K}\mathbf{N}_K\mathbf{N}_K \cdot \sum_L \frac{1}{\Lambda_L}\mathbf{N}_L\mathbf{N}_L\right\} \\
&= \frac{1}{2}\left(\mathbf{I}-\mathbf{U}^{-2}\right)
\end{aligned}$$

$$\mathbf{C}^{-1} = \left(\mathbf{U}^T\cdot\mathbf{U}\right)^{-1} = \mathbf{U}^{-1}\cdot\mathbf{U}^{T^{-1}} = \mathbf{U}^{-2}$$

(b)

$$\begin{aligned}
F_{mn}^{-1} &= \frac{\partial X_m}{\partial x_n} = \frac{\partial}{\partial x_n}\left\{x_m - u_m\right\} \\
&= \delta_{mn} - \frac{\partial u_m}{\partial x_n}
\end{aligned}$$

$$\mathbf{C}^{-1} = \left(\mathbf{F}^T\cdot\mathbf{F}\right)^{-1} = \mathbf{F}^{-1}\cdot\mathbf{F}^{T^{-1}}$$

$$\begin{aligned}
C_{kl}^{-1} &= F_{km}^{-1}F_{ml}^{T^{-1}} = F_{km}^{-1}F_{lm}^{-1} \\
&= \left(\delta_{km} - \frac{\partial u_k}{\partial x_m}\right)\left(\delta_{lm} - \frac{\partial u_l}{\partial x_m}\right) \\
&= \delta_{kl} - \frac{\partial u_k}{\partial x_l} - \frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_m}\frac{\partial u_l}{\partial x_m}
\end{aligned}$$

$$E_{ij}^{(-2)} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_m}\frac{\partial u_j}{\partial x_m}\right)$$

(c)

$$\begin{aligned}
\mathbf{e}^A &= \frac{1}{2}\left(\mathbf{I} - \mathbf{F}^{-1^T}\cdot\mathbf{F}^{-1}\right) \\
&= \frac{1}{2}\left\{\mathbf{I} - (\mathbf{R}\cdot\mathbf{U})^{-1^T}\cdot(\mathbf{R}\cdot\mathbf{U})^{-1}\right\} \\
&= \frac{1}{2}\left\{\mathbf{I} - \left(\mathbf{U}^{-1}\cdot\mathbf{R}^T\right)^T\cdot\left(\mathbf{U}^{-1}\cdot\mathbf{R}^T\right)\right\} \\
&= \frac{1}{2}\left\{\mathbf{I} - \mathbf{R}\cdot\mathbf{U}^{-1^T}\cdot\mathbf{U}^{-1}\cdot\mathbf{R}^T\right\} \\
&= \mathbf{R}\cdot\left\{\frac{1}{2}\left(\mathbf{I} - \mathbf{U}^{-1^T}\cdot\mathbf{U}^{-1}\right)\right\}\cdot\mathbf{R}^T \\
&= \mathbf{R}\cdot\mathbf{E}^{(-2)}\cdot\mathbf{R}^T
\end{aligned}$$

4. (a)

$$\begin{aligned}
\mathbf{e}^{(-2)} &= \frac{1}{2}\sum_K \left(\lambda_K^{-2}-1\right)\mathbf{n}_K\mathbf{n}_K \\
&= \frac{1}{2}\left\{\sum_K \frac{1}{\lambda_K}\mathbf{n}_K\mathbf{n}_K \cdot \sum_L \frac{1}{\lambda_L}\mathbf{n}_L\mathbf{n}_L - \sum_K \mathbf{n}_K\mathbf{n}_K\right\} \\
&= \frac{1}{2}\left(\mathbf{V}^2-\mathbf{I}\right) \\
&= \frac{1}{2}\left(\mathbf{F}\cdot\mathbf{F}^T-\mathbf{I}\right) \\
&= \frac{1}{2}\left(\mathbf{B}-\mathbf{I}\right)
\end{aligned}$$

(b)

$$\begin{aligned}
e_{ij}^{(-2)} &= \frac{1}{2}\left(F_{ik}F_{jk}-\delta_{ij}\right) \\
&= \frac{1}{2}\left\{\left(\delta_{ik}+\frac{\partial u_i}{\partial X_k}\right)\left(\delta_{jk}+\frac{\partial u_j}{\partial X_k}\right)-\delta_{ij}\right\} \\
&= \frac{1}{2}\left\{\frac{\partial u_i}{\partial X_j}+\frac{\partial u_j}{\partial X_i}+\frac{\partial u_i}{\partial X_k}\frac{\partial u_j}{\partial X_k}\right\}
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{e}^{(-2)} &= \frac{1}{2}\left(\mathbf{F}\cdot\mathbf{F}^T-\mathbf{I}\right) \\
&= \frac{1}{2}\left\{(\mathbf{R}\cdot\mathbf{U})\cdot(\mathbf{R}\cdot\mathbf{U})^T-\mathbf{I}\right\} \\
&= \frac{1}{2}\left\{(\mathbf{R}\cdot\mathbf{U})\cdot\left(\mathbf{U}^T\cdot\mathbf{R}^T\right)-\mathbf{I}\right\} \\
&= \mathbf{R}\cdot\left\{\frac{1}{2}\left(\mathbf{U}^2-\mathbf{I}\right)\right\}\cdot\mathbf{R}^T \\
&= \mathbf{R}\cdot\mathbf{E}^G\cdot\mathbf{R}^T
\end{aligned}$$

5.

$$\ln\Lambda = \Lambda - 1 - \frac{1}{2}(\Lambda-1)^2 + \frac{1}{3}(\Lambda-1)^3 + \dots$$

Therefore

$$\begin{aligned}
\mathbf{E}^{(0)} &= \ln\mathbf{U} = \sum_K \ln(\Lambda_K)\mathbf{N}_K\mathbf{N}_K \\
&= \sum_K (\Lambda_K-1)\,\mathbf{N}_K\mathbf{N}_K - \sum_K \frac{1}{2}(\Lambda_K-1)^2\,\mathbf{N}_K\mathbf{N}_K + \dots \\
&= (\mathbf{U}-\mathbf{I}) - \frac{1}{2}(\mathbf{U}-\mathbf{I})^2 + \frac{1}{3}(\mathbf{U}-\mathbf{I})^3 + \dots
\end{aligned}$$

6. Principal values of the Green-Lagrange strain are

$$\begin{aligned} E_K^G &= \mathbf{N}_K \cdot \mathbf{E}^G \cdot \mathbf{N}_K \\ &= \mathbf{N}_K \cdot \left\{ \frac{1}{2} (\mathbf{C} - \mathbf{I}) \right\} \cdot \mathbf{N}_K \\ &= \frac{1}{2} (\mathbf{N}_K \cdot \mathbf{C} \cdot \mathbf{N}_K - \mathbf{1}) \\ &= \frac{1}{2} \left(\Lambda_K^2 - \mathbf{1} \right) \end{aligned}$$

Solving for Λ_K^2 gives

$$\Lambda_K^2 = 1 + 2E_K^G$$

Prinipal values of the Almansi strain are

$$\begin{aligned} e_K^A &= \mathbf{n}_K \cdot \mathbf{e}^A \cdot \mathbf{n}_K \\ &= \mathbf{n}_K \cdot \left\{ \frac{1}{2} \left(\mathbf{I} - \mathbf{B}^{-1} \right) \right\} \cdot \mathbf{n}_K \\ &= \frac{1}{2} \left\{ 1 - \mathbf{n}_K \cdot \mathbf{B}^{-1} \cdot \mathbf{n}_K \right\} \\ &= \frac{1}{2} \left\{ 1 - \frac{1}{\Lambda_K^2} \right\} \end{aligned}$$

Solving for Λ_K^2 gives

$$\Lambda_K^2 = \frac{1}{1 - 2e_K^A}$$

Equating the two expressions for Λ_K^2 gives the result.

7. Differentiating

$$\mathbf{R} \cdot \mathbf{R}^T = \mathbf{I}$$

gives

$$\begin{aligned} \dot{\mathbf{R}} \cdot \mathbf{R}^T + \mathbf{R} \cdot \dot{\mathbf{R}}^T &= 0 \\ \dot{\mathbf{R}} \cdot \mathbf{R}^T &= -\mathbf{R} \cdot \dot{\mathbf{R}}^T \\ &= -\left(\dot{\mathbf{R}} \cdot \mathbf{R}^T \right) \end{aligned}$$

8.

$$\mathbf{E}^{(\ln)} = \sum_K \ln(\Lambda_K) \mathbf{N}_K \mathbf{N}_K$$

If the principal axes of $\mathbf{U}$ (i.e., the $\mathbf{N}_K$) do not rotate

$$\dot{\mathbf{E}}^{(\ln)} = \sum_K \frac{\dot{\Lambda}_K}{\Lambda_K} \mathbf{N}_K \mathbf{N}_K$$

and the $\mathbf{N}_K$ can be replaced by $\mathbf{n}_K$. But

$$\begin{aligned}
\frac{\dot{\Lambda}_K}{\Lambda_K} &= \frac{1}{ds/dS}\frac{d}{dt}\frac{ds}{dS} \\
&= \frac{1}{ds}\frac{d}{dt}ds
\end{aligned}$$

Therefore $\dot{\Lambda}_K/\Lambda_K$ are the principal values of $\mathbf{D}$ and

$$\dot{\mathbf{E}}^{(\ln)} = \mathbf{D}$$

9.

$$\dot{J} = \epsilon_{ijk}\dot{F}_{i1}F_{j2}F_{k3} + \epsilon_{ijk}F_{i1}\dot{F}_{j2}F_{k3} + \epsilon_{ijk}F_{i1}F_{j2}\dot{F}_{k3}$$

Use

$$\dot{\mathbf{F}} = \mathbf{L}\cdot\mathbf{F}$$

and rearrange the indices to get

$$\dot{J} = \dot{F}_{p1}F_{q2}F_{r3}h_{pqr}$$

where

$$h_{pqr} = \epsilon_{iqr}L_{ir} + \epsilon_{pjr}L_{jq} + \epsilon_{pqk}L_{kr}$$

10.

$$\begin{aligned}
\frac{d}{dt}(\mathbf{n}da) &= \frac{d}{dt}\left(J\mathbf{N}dA\cdot\mathbf{F}^{-1}\right) \\
&= \mathbf{N}dA\left(\dot{J}\cdot\mathbf{F}^{-1} + J\cdot\dot{\mathbf{F}}^{-1}\right)
\end{aligned}$$

Using $\dot{J} = J\mathrm{tr}L$, from the preceding problem and $\dot{\mathbf{F}}^{-1} = -\mathbf{F}^{-1}\cdot\mathbf{L}$ give

$$\begin{aligned}
\frac{d}{dt}(\mathbf{n}da) &= \left(J\mathbf{N}dA\cdot\mathbf{F}^{-1}\right)\mathrm{tr}L - \left(J\mathbf{N}dA\cdot\mathbf{F}^{-1}\right)\cdot\mathbf{L} \\
&= \mathbf{n}da\,\mathrm{tr}L - \mathbf{n}da\cdot\mathbf{L}
\end{aligned}$$

Forming the scalar product with $\mathbf{n}$ and noting that $\mathbf{n}\cdot\mathbf{n} = 1$ gives

$$\mathbf{n}\cdot\frac{d}{dt}(\mathbf{n}da) = da\,(\mathrm{tr}L - \mathbf{n}\cdot\mathbf{L}\cdot\mathbf{n})$$

The left side gives

$$\frac{d}{dt}(\mathbf{n}da) = \dot{\mathbf{n}}da + \mathbf{n}\frac{d}{dt}(da)$$

Forming the scalar product with $\mathbf{n}$ and noting that $\mathbf{n}\cdot\mathbf{n} = 1$ and $\dot{\mathbf{n}}\cdot\mathbf{n} = \mathbf{0}$ gives

$$\frac{d}{dt}(da) = da\,(\mathrm{tr}L - \mathbf{n}\cdot\mathbf{L}\cdot\mathbf{n})$$

$$\begin{aligned} d\mathbf{x} &= \boldsymbol{F} \cdot d\mathbf{X} \\ \frac{ds}{dS}\frac{d\mathbf{x}}{ds} &= \boldsymbol{F} \cdot \frac{d\mathbf{X}}{dS} \end{aligned}$$

but

$$\begin{aligned} \frac{ds}{dS} &= \Lambda \\ \boldsymbol{n} &= \frac{d\mathbf{x}}{ds} \\ \boldsymbol{N} &= \frac{d\mathbf{X}}{dS} \end{aligned}$$

yielding the desired result.

11. (a) Differentiate $\Lambda\mathbf{n} = \mathbf{F} \cdot \mathbf{N}$ (Exercise 15.9) to get

$$\dot{\Lambda}\boldsymbol{n} + \Lambda\dot{\boldsymbol{n}} = \dot{\boldsymbol{F}} \cdot \boldsymbol{N} \tag{16.11.1}$$

Multiply both sides of $\Lambda\mathbf{n} = \mathbf{F} \cdot \mathbf{N}$ by $\boldsymbol{F}^{-1}$ to get

$$\boldsymbol{N} = \Lambda\boldsymbol{F}^{-1} \cdot \boldsymbol{n} \tag{16.11.2}$$

and substitute into the precding equation:

$$\dot{\Lambda}\boldsymbol{n} + \Lambda\dot{\boldsymbol{n}} = \Lambda\boldsymbol{L} \cdot \boldsymbol{n}$$

where $\boldsymbol{L} = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}$. Dot both sides with $\boldsymbol{n}$

$$\dot{\Lambda}\boldsymbol{n} \cdot \boldsymbol{n} + \Lambda\boldsymbol{n} \cdot \dot{\boldsymbol{n}} = \Lambda\boldsymbol{n} \cdot \boldsymbol{L} \cdot \boldsymbol{n}$$

But since $\boldsymbol{n}$ is a unit vector

$$\boldsymbol{n} \cdot \boldsymbol{n} = 1 \tag{16.11.4}$$

and.

$$\boldsymbol{n} \cdot \dot{\boldsymbol{n}} = 0 \tag{16.11.5}$$

Since $\boldsymbol{n} \cdot \boldsymbol{W} \cdot \boldsymbol{n} = 0$ (because $\boldsymbol{W} = -\boldsymbol{W}^T$). $\boldsymbol{n} \cdot \boldsymbol{L} \cdot \boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{D} \cdot \boldsymbol{n}$ Converting to index notation and dividing through by Λ gives the result.

(b) Differentiate $d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}$ to get

$$d\mathbf{v} = \dot{\boldsymbol{F}} \cdot d\mathbf{X} = \boldsymbol{L} \cdot d\mathbf{x}$$

where the second equality follows from the definition of $\boldsymbol{L}$. Differentiate again to get

$$d\mathbf{a} = \ddot{\boldsymbol{F}} \cdot d\mathbf{X} = \dot{\boldsymbol{L}} \cdot d\mathbf{x} + \boldsymbol{L} \cdot d\mathbf{v} \tag{16.11.5}$$

where $\mathbf{a}$ is the acceleration. Using $d\mathbf{X} = \boldsymbol{F}^{-1} \cdot d\mathbf{x}$ in the first equality of (16.11.5) yields

$$d\mathbf{a} = \ddot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} \cdot d\mathbf{x} = \boldsymbol{Q} \cdot d\mathbf{x} \tag{16.11.6}$$

and gives

$$\boldsymbol{Q} = \ddot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} \tag{16.11.7}$$

Using $d\mathbf{v} = \boldsymbol{L} \cdot d\mathbf{x}$ in the second equality of (16.11.5) yields an expression for $\boldsymbol{Q}$ in terms of $\boldsymbol{L}$

$$\boldsymbol{Q} = \dot{\boldsymbol{L}} + \boldsymbol{L} \cdot \boldsymbol{L}$$

Differentiating (16.11.1) gives

$$\ddot{\Lambda}\boldsymbol{n} + 2\dot{\Lambda}\dot{\boldsymbol{n}} + \Lambda\ddot{\boldsymbol{n}} = \ddot{\boldsymbol{F}} \cdot (\Lambda \boldsymbol{F}^{-1} \cdot \boldsymbol{n}) \tag{16.11.8}$$

where (16.11.2) has been used for $\boldsymbol{N}$. Dotting both sides of (16.11.8) with $\boldsymbol{n}$ and using (16.11.7), (16.11.5) and (16.11.4) yield

$$\ddot{\Lambda} + \Lambda\ddot{\boldsymbol{n}} \cdot \boldsymbol{n} = \Lambda\boldsymbol{n} \cdot \boldsymbol{Q} \cdot \boldsymbol{n} \tag{16.11.9}$$

Differentiating (16.11.4) yields

$$\ddot{\boldsymbol{n}} \cdot \boldsymbol{n} + \dot{\boldsymbol{n}} \cdot \dot{\boldsymbol{n}} = 0$$

Using $\ddot{\boldsymbol{n}} \cdot \boldsymbol{n} = -\dot{\boldsymbol{n}} \cdot \dot{\boldsymbol{n}}$ in (16.11.9), dividing through by Λ and converting to index notation gives the result.

Another approach is to begin with

$$2ds\frac{d}{dt}(ds) = d\mathbf{v} \cdot d\mathbf{x} + d\mathbf{x} \cdot d\mathbf{v}$$

Differentiating again gives

$$2\left(\frac{d}{dt}(ds)\right)^2 + 2ds\frac{d^2}{dt^2}(ds) = d\mathbf{a} \cdot d\mathbf{x} + d\mathbf{x} \cdot d\mathbf{a} + 2d\mathbf{v} \cdot d\mathbf{v}$$

Using the last equality in (16.11.6), noting that $\boldsymbol{Q}$ is symmetric and dividing through by $2(ds)^2$ gives

$$\frac{1}{ds}\frac{d^2}{dt^2}(ds) = \boldsymbol{n} \cdot \boldsymbol{Q} \cdot \boldsymbol{n} + \left(\frac{1}{ds}\right)^2 d\mathbf{v} \cdot d\mathbf{v} - \left(\frac{1}{ds}\frac{d}{dt}(ds)\right)^2 \tag{16.11.10}$$

Multiplying the top and bottom of the left side by dS gives $\ddot{\Lambda}/\Lambda$ and the last term on the right is $(\mathbf{n} \cdot \mathbf{D} \cdot \mathbf{n})^2$. Differentiate $d\mathbf{x} = \boldsymbol{n}ds$ to get

$$d\mathbf{v} = \dot{\boldsymbol{n}}ds + \boldsymbol{n}\frac{d}{dt}(ds)$$

Dividing through by ds and dotting the result with itself give

$$\frac{d\mathbf{v}}{ds} \cdot \frac{d\mathbf{v}}{ds} = \dot{\boldsymbol{n}} \cdot \dot{\boldsymbol{n}} + \left(\frac{1}{ds}\frac{d}{dt}(ds)\right)^2$$

Substituting back into (16.11.10) gives the desired result.

Chapter 17

Linearized Displacement Gradients

1. Using $\sin\gamma \approx \gamma$, (17.8) and (17.11) in (17.13) gives

$$\gamma \approx \frac{\mathbf{N}_A \cdot \mathbf{N}_B + 2\mathbf{N}_A \cdot \boldsymbol{\varepsilon} \cdot \mathbf{N}_B}{(1 + \mathbf{N}_A \cdot \boldsymbol{\varepsilon} \cdot \mathbf{N}_A)\,(1 + \mathbf{N}_B \cdot \boldsymbol{\varepsilon} \cdot \mathbf{N}_B)}$$

Because $\mathbf{N}_A$ and $\mathbf{N}_B$ are orthogonal $\mathbf{N}_A \cdot \mathbf{N}_B = 0$. To first order in $\boldsymbol{\varepsilon}$

$$\frac{1}{(1 + \mathbf{N}_A \cdot \boldsymbol{\varepsilon} \cdot \mathbf{N}_A)} \approx 1 - \mathbf{N}_A \cdot \boldsymbol{\varepsilon} \cdot \mathbf{N}_A + \ldots$$

and similarly for the other term in the denominator. Retaining only terms that are linear in $\boldsymbol{\varepsilon}$ gives (17.14).

2. Because

$$\mathbf{U} = \sum_K \Lambda_K \mathbf{N}_K \mathbf{N}_K$$

the inverse of $\mathbf{U}$ is given by

$$\mathbf{U}^{-1} = \sum_K \frac{1}{\Lambda_K} \mathbf{N}_K \mathbf{N}_K$$

Because

$$\mathbf{C} \approx \mathbf{I} + 2\boldsymbol{\varepsilon}$$

$\mathbf{C}$, $\mathbf{U}$ and $\boldsymbol{\varepsilon}$ have the same principal directions. Because $\Lambda = 1 + \mathbf{N} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{N}$, the principal stretch ratios are

$$\Lambda_K = 1 + \varepsilon_K$$

where the ε_K are the principal values of $\boldsymbol{\varepsilon}$. Therefore

$$\frac{1}{\Lambda_K} = \frac{1}{1 + \varepsilon_K} \approx 1 - \varepsilon_K$$

Substituting back into the expression for $\mathbf{U}^{-1}$ gives

$$\begin{aligned}\mathbf{U}^{-1} &= \sum_K \mathbf{N}_K\mathbf{N}_K - \sum_K \varepsilon_K \mathbf{N}_K\mathbf{N}_K \\ &= \mathbf{I} - \boldsymbol{\varepsilon}\end{aligned}$$

3. Because $F_{ij} = \delta_{ij} + \partial u_i/\partial x_j$

$$F_{ij}^{-1} \approx \delta_{ij} - \frac{\partial u_i}{\partial x_j}$$

Also, $J \approx 1 + \mathrm{tr}\boldsymbol{\varepsilon}$. Substituting into Nanson's formula gives

$$\begin{aligned}n_i da &\approx (1 + \mathrm{tr}\boldsymbol{\varepsilon}) \left\{ N_k \left(\delta_{ki} - \frac{\partial u_k}{\partial x_i} \right) \right\} dA \\ &\approx (1 + \mathrm{tr}\boldsymbol{\varepsilon})\, N_i dA - N_k \frac{\partial u_k}{\partial x_i} dA \\ &\approx (1 + \mathrm{tr}\boldsymbol{\varepsilon})\, N_i dA - N_k \left(\varepsilon_{ki} + \Omega_{ki}\right) dA\end{aligned}$$

4. The infinitesimal rotation vector

$$\begin{aligned}w_m &= \frac{1}{2}\epsilon_{ijm}\Omega_{ij} \\ &= \frac{1}{2}\epsilon_{ijm}\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\right] \\ &= \frac{1}{4}\left(\epsilon_{ijm}\frac{\partial u_i}{\partial x_j} + \epsilon_{jim}\frac{\partial u_i}{\partial x_j}\right) \\ &= \frac{1}{2}\epsilon_{ijm}\frac{\partial u_i}{\partial x_j}\end{aligned}$$

Taking the divergence

$$\frac{\partial w}{\partial x_m} = \frac{1}{2}\epsilon_{ijm}\frac{\partial^2 u_i}{\partial x_j \partial x_m}$$

Because ϵ_{ijm} is skew symmetric and $\partial^2 u_i/\partial x_j \partial x_m$ is symmetric with respect interchange of j and m, the product vanishes.

5. Setting the curl of the integrand in (17.32) equal to zero gives

$$\begin{aligned}\epsilon_{mni}\partial_n \left\{\varepsilon_{ij} + \epsilon_{jik} w_k\right\} &= 0 \\ \epsilon_{mni}\varepsilon_{ij,n} + \epsilon_{mni}\epsilon_{jik} w_k &= 0 \\ \epsilon_{mni}\varepsilon_{ij,n} + \epsilon_{imn}\epsilon_{ikj} w_{k,n} &= 0 \\ \epsilon_{mni}\varepsilon_{ij,n} + \left\{\delta_{mk}\delta_{nj} - \delta_{mj}\delta_{nk}\right\} &= 0 \\ \epsilon_{mni}\varepsilon_{ij,n} + w_{m,j} - \delta_{mj} w_{k,k} &= 0\end{aligned}$$

Relabelling indices gives (17.33)

Part IV

Balance of Mass, Momentum, and Energy

Chapter 18

Transformation of Integrals

1.

$$\begin{aligned}
\int\int_A \frac{\partial u_x}{\partial x} dxdy &= \int_e^d dy \int_{x_1(y)}^{x_2(y)} \frac{\partial u_x}{\partial x} dx \\
&= \int_e^d \{u_x[x_2(y), y] - u_x[x_1(y), y]\} \, dy \\
&= \int_e^d u_x[x_2(y), y] dy + \int_d^e u_x[x_1(y), y] dy \\
&= \int_C u_x dy
\end{aligned}$$

From Figure 18.2

$$dy = \frac{dy}{ds} ds = \cos\alpha ds$$

2. (a)

$$\begin{aligned}
\int_S \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{v} dS &= \int_S n_i \sigma_{ij} v_j dS \\
&= \int_V \frac{\partial}{\partial x_i} (\sigma_{ij} v_j) \, dV \\
&= \int_V \left\{ \frac{\partial \sigma_{ij}}{\partial x_i} v_j + \sigma_{ij} \frac{\partial v_j}{\partial x_i} \right\} dV
\end{aligned}$$

If σ_{ij} is stress and v_j is velocity then the second term can be replaced by $\sigma_{ij} D_{ji}$, because $\sigma_{ij} = \sigma_{ji}$ and D_{ji} is the symmetric part of $\partial v_j / \partial x_i$.

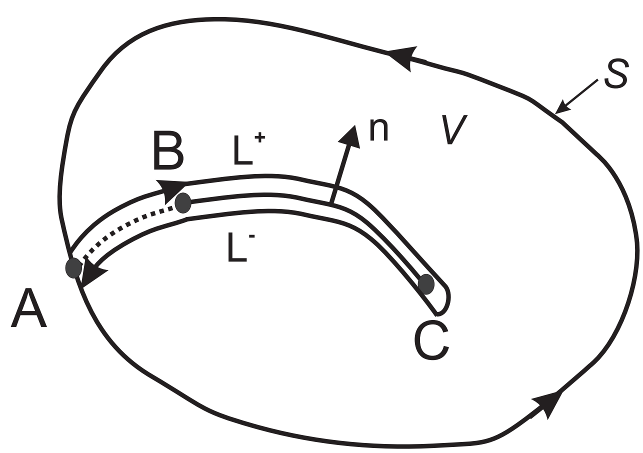
(b)

$$\begin{aligned}
\int_S \epsilon_{rms} x_m \sigma_{js} n_j dS &= \int_V \frac{\partial}{\partial x_j} \left(\epsilon_{rms} x_m \sigma_{js} \right) dV \\
&= \epsilon_{rms} \int_V \left\{ \frac{\partial x_m}{\partial x_j} \sigma_{js} + x_m \frac{\partial \sigma_{js}}{\partial x_j} \right\} dV \\
&= \int_V \left\{ \epsilon_{rms} \delta_{mj} \sigma_{js} + \epsilon_{rms} x_m \frac{\partial \sigma_{js}}{\partial x_j} \right\} dV \\
&= \int_V \left\{ \epsilon_{rjs} \sigma_{js} + \epsilon_{rms} x_m \frac{\partial \sigma_{js}}{\partial x_j} \right\} dV
\end{aligned}$$

If $\sigma_{js} = \sigma_{sj}$, then the first term vanishes because $\epsilon_{rjs} = -\epsilon_{rsj}$.

3. Connect L to the boundary by a surface AB as shown in the Figure. Then apply the divergence theorem to the volume enclosed by the outer surface S and a contour that wraps around ABC as shown. Because the traction $\mathbf{n} \cdot \boldsymbol{\sigma}$ continuous on AB, the contribution from this portion of the surface cancels because the segment AB is traversed in opposite directions. Because $\mathbf{n} \cdot \boldsymbol{\sigma}$ is discontinuous on L (BC) there are different contributions from the + and − sides. The result is

$$\int_V \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} dV = \int_S \mathbf{n} \cdot \boldsymbol{\sigma} dS + \int_B^C \left\{ (\mathbf{n} \cdot \boldsymbol{\sigma})^+ - (\mathbf{n} \cdot \boldsymbol{\sigma})^- \right\} dS$$



Chapter 19

Conservation of Mass

1.

$$\begin{aligned} I &= \frac{d}{dt}\int_V \rho \mathcal{A} J dV \\ &= \int_V \left\{ \frac{d\rho}{dt}\mathcal{A}J + \rho\frac{d\mathcal{A}}{dt}J + \rho\mathcal{A}\frac{dJ}{dt} \right\} dV \end{aligned}$$

Use

$$\frac{dJ}{dt} = J\,\mathrm{tr}\mathbf{L} = J\boldsymbol{\nabla}\cdot\mathbf{v}$$

and rearrange to get

$$I = \int_V \left\{ \left[\frac{d\rho}{dt} + \rho\boldsymbol{\nabla}\cdot\mathbf{v} \right] \mathcal{A} + \rho\frac{d\mathcal{A}}{dt} \right\} JdV$$

The term $[\ldots]$ is zero by mass conservation leaving

$$I = \int_V \rho\frac{d\mathcal{A}}{dt}JdV$$

2.

$$\frac{d}{dt}\left(\frac{4}{3}\pi R^3\right) = 4\pi R^2\dot{R}$$

Chapter 20

Conservation of Momentum

1.

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} &= \mathbf{e}_k \frac{\partial}{\partial x_k} \cdot \{T(\mathbf{x}) n_i n_j\} \\
&= \mathbf{e}_k \frac{\partial T(\mathbf{x})}{\partial x_k} \cdot \mathbf{nn} \\
&= \boldsymbol{\nabla} T(\mathbf{x}) \cdot \mathbf{nn}
\end{aligned}$$

Because equilibrium (in the absence of body forces) requires that this vanish and the unit vector $\mathbf{n}$ is not zero,

$$\boldsymbol{\nabla} T(\mathbf{x}) \cdot \mathbf{n} = 0$$

Hence, the gradient of $T(\mathbf{x})$ is orthogonal to $\mathbf{n}$ since their scalar product vanishes.

2.

$$\begin{aligned}
\frac{\partial}{\partial x_1}\left[-p\frac{\left(x_1^2 - x_2^2\right)}{a^2}\right] + \frac{\partial}{\partial x_2}\left[2p\frac{x_1 x_2}{a^2}\right] &= \\
-2p\frac{x_1}{a^2} + 2p\frac{x_1}{a^2} &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial x_2}\left[p\frac{\left(x_1^2 - x_2^2\right)}{a^2}\right] + \frac{\partial}{\partial x_1}\left[2p\frac{x_1 x_2}{a^2}\right] &= \\
-2p\frac{x_2}{a^2} + 2p\frac{x_2}{a^2} &= 0
\end{aligned}$$

Therefore, equilibrium is satisfied.

3.

$$
\begin{aligned}
\frac{\partial}{\partial t}(\rho v_j) + \frac{\partial}{\partial x_i}(\rho v_i v_j) &= \\
\rho \frac{\partial v_j}{\partial t} + v_j \frac{\partial \rho}{\partial t} + v_j \frac{\partial}{\partial x_i}(\rho v_i) + (\rho v_i)\frac{\partial v_j}{\partial x_i} &= \\
v_j \left\{ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) \right\} + \rho \left\{ \frac{\partial v_j}{\partial t} + v_i \frac{\partial v_j}{\partial x_i} \right\} &= \rho \frac{d v_j}{dt}
\end{aligned}
$$

where the first term vanishes because of mass conservation. The balance of linear momentum for a control volume fixed in space expresses that the work of the surface tractions on the boundary and the body forces in the volume

$$\int_a \mathbf{t} da + \int_v \rho \mathbf{b} dv$$

plus the flux of momentum through the boundary into the volume

$$-\int_a \mathbf{n} \cdot \mathbf{v} \, (\rho \mathbf{v}) \, da$$

(where the minus sign appears because the normal is *outward*) equals the rate of change of momentum in the volume

$$\frac{\partial}{\partial t} \int_v \rho \mathbf{v} dv$$

The time derivative can be moved inside the integral because the volume is instantaneously fixed in space. Writing the traction as $\mathbf{t} = \mathbf{n} \cdot \boldsymbol{\sigma}$, using the divergence theorem on this term and the term for the momentum flux and using the result from the first part of the problem gives the equation of motion.

Chapter 21

Conservation of Energy

1.
$$
\begin{aligned}
\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}\cdot\mathbf{v}) &= \mathbf{e}_k\frac{\partial}{\partial x_k}\cdot[(\sigma_{ij}\mathbf{e}_i\mathbf{e}_j)\cdot v_l\mathbf{e}_l] \\
&= \delta_{ki}\left\{\frac{\partial\sigma_{ij}}{\partial x_k}v_l+\sigma_{ij}\frac{\partial v_l}{\partial x_k}\right\}\delta_{jl} \\
&= \frac{\partial\sigma_{kl}}{\partial x_k}v_l+\sigma_{ij}v_{j,l} \\
&= \left\{\mathbf{e}_k\frac{\partial}{\partial x_k}\cdot(\sigma_{ij}\mathbf{e}_i\mathbf{e}_j)\right\}\cdot v_l\mathbf{e}_l+(\sigma_{ij}\mathbf{e}_i\mathbf{e}_j)\cdot\cdot v_{l,k}\mathbf{e}_l\mathbf{e}_k \\
&= (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\cdot\mathbf{v}+\boldsymbol{\sigma}\cdot\cdot(\boldsymbol{\nabla}\mathbf{v})^T
\end{aligned}
$$

Because $\boldsymbol{\sigma}=\boldsymbol{\sigma}^T$, $\boldsymbol{\nabla}\mathbf{v}$ can be replaced by its symmetric part $\mathbf{L}$.

2. Substitute $\boldsymbol{\sigma}=-p\mathbf{I}$ into (21.6) to get
$$\rho\frac{du}{dt}=-p\,\mathrm{tr}\mathbf{L}-\boldsymbol{\nabla}\cdot\mathbf{q}+\rho r$$
But
$$\mathrm{tr}\mathbf{L}=\boldsymbol{\nabla}\cdot\mathbf{v}=-\frac{1}{\rho}\frac{d\rho}{dt}$$
where the last equality follows from mass conservation (19.10). Substituting gives the result.

3. If the material is rigid, $\boldsymbol{\sigma}\cdot\cdot\mathbf{L}=0$ and
$$\frac{du}{dt}=\frac{\partial u}{\partial t}$$
in (21.6). If the energy is only a function of the temperature
$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial t}=c\frac{\partial\theta}{\partial t}$$
Substituting this and Fourier's law and rearranging gives the result.

4. Energy conservation is given by (21.1). The power input express in terms of the reference configuration is

$$P = \int_A \mathbf{t}^0 \cdot \mathbf{v} dA + \int_V \rho_0 \mathbf{b}_0 \cdot \mathbf{v} dV$$

The first term can be rewritten as

$$\int_A \mathbf{t}^0 \cdot \mathbf{v} dA = \int_A \mathbf{N} \cdot \mathbf{T}^0 \cdot \mathbf{v} dA = \int_V \boldsymbol{\nabla}_{\mathbf{X}} \cdot \left(\mathbf{T}^0 \cdot \mathbf{v}\right) dV$$

Working out the integrand yields

$$\boldsymbol{\nabla}_{\mathbf{X}} \cdot \left(\mathbf{T}^0 \cdot \mathbf{v}\right) = \left(\boldsymbol{\nabla}_{\mathbf{X}} \cdot \mathbf{T}^0\right) \cdot \mathbf{v} + \mathbf{T}^0 \cdot \cdot \dot{\mathbf{F}}$$

Thus the power input can be rewritten as

$$P = \int_V \left\{\boldsymbol{\nabla}_{\mathbf{X}} \cdot \mathbf{T}^0 + \rho_0 \mathbf{b}_0\right\} \cdot \mathbf{v} dV + \int_V \mathbf{T}^0 \cdot \cdot \dot{\mathbf{F}} dV$$

The equation of motion (20.13) can be used in the first term to give

$$P = \int_V \rho_0 \frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{v} dV + \int_V \mathbf{T}^0 \cdot \cdot \dot{\mathbf{F}} dV$$

and then the first term can be rearranged as follows:

$$P = \int_V \frac{\partial}{\partial t} \left(\frac{1}{2} \rho_0 \mathbf{v} \cdot \mathbf{v}\right) dV + \int_V \mathbf{T}^0 \cdot \cdot \dot{\mathbf{F}} dV$$

The rate of heat input is

$$\dot{Q} = -\int_A \mathbf{N} \cdot \mathbf{Q} dA + \int_V \rho_0 R dV$$

where the first term is the flux of heat through the surface area (and the minus sign occurs because the normal N points out of the body) and the second term is the heat generated within the volume of the body. Using the divergence theorem on the first term gives

$$\dot{Q} = -\int_V \boldsymbol{\nabla}_{\mathbf{X}} \cdot \mathbf{Q} dV + \int_V \rho_0 R dV$$

The rate of change of the total energy is

$$\dot{E}_{Total} = \frac{\partial}{\partial t} \int_V \rho_0 u dV + \frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \rho_0 \mathbf{v} \cdot \mathbf{v}\right) dV$$

where the first term is the rate of change of the internal energy and the second is the rate of change of the kinetic energy. Because the volume is fixed in the reference state the time derivatives can be taken inside the integrals. Substituting into (21.1), cancelling the common term and rearranging gives

$$\int_V \left\{ \rho_0 \frac{\partial u}{\partial t} - \mathbf{T}^0 \cdot\cdot \dot{\mathbf{F}} + \boldsymbol{\nabla}_{\mathbf{X}} \cdot \mathbf{Q} - \rho_0 R \right\} dV$$

Because this must apply for any volume, the integrand must vanish which gives (21.7). Because the heat flux must be the same whether referred to the current configuration or the reference configuration

$$\mathbf{N} \cdot \mathbf{Q} dA = \mathbf{n} \cdot \mathbf{q} da$$

and using Nanson's formula (15.12) gives the relation between $\mathbf{Q}$ and $\mathbf{q}$.

5.

$$\begin{aligned}
\mathbf{E}^{(-2)} &= \frac{1}{2}\left(\mathbf{I} - \mathbf{U}^{-2}\right) \\
&= \frac{1}{2}\left(\mathbf{I} - \mathbf{C}^{-1}\right) \\
&= \frac{1}{2}\left(\mathbf{I} - \mathbf{F}^{-1} \cdot \mathbf{F}^{-1T}\right)
\end{aligned}$$

Therefore

$$\dot{\mathbf{E}}^{(-2)} = -\frac{1}{2}\left(\dot{\mathbf{F}}^{-1} \cdot \mathbf{F}^{-1T} + \mathbf{F}^{-1} \cdot \dot{\mathbf{F}}^{-1T}\right)$$

To compute $\dot{\mathbf{F}}^{-1}$ differentiate $\mathbf{F} \cdot \mathbf{F}^{-1} = \mathbf{I}$ to get

$$\dot{\mathbf{F}} \cdot \mathbf{F}^{-1} + \mathbf{F} \cdot \dot{\mathbf{F}}^{-1} = 0$$

Rearranging gives

$$\begin{aligned}
\dot{\mathbf{F}}^{-1} &= -\mathbf{F}^{-1} \cdot \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} \\
&= -\mathbf{F}^{-1} \cdot \mathbf{L}
\end{aligned}$$

and the transpose is

$$\dot{\mathbf{F}}^{-1T} = -\mathbf{L}^T \cdot \mathbf{F}^{-1T}$$

Substituting into

$$\begin{aligned}
\dot{\mathbf{E}}^{(-2)} &= \frac{1}{2}\left\{\mathbf{F}^{-1} \cdot \mathbf{L} \cdot \mathbf{F}^{-1T} + \mathbf{F}^{-1} \cdot \mathbf{L}^T \cdot \mathbf{F}^{-1T}\right\} \\
&= \mathbf{F}^{-1} \cdot \frac{1}{2}\left\{\mathbf{L} + \mathbf{L}^T\right\} \cdot \mathbf{F}^{-1T} \\
&= \mathbf{F}^{-1} \cdot \mathbf{D} \cdot \mathbf{F}^{-1T}
\end{aligned}$$

or

$$\mathbf{D} = \mathbf{F} \cdot \dot{\mathbf{E}}^{(-2)} \cdot \mathbf{F}^T$$

Substituting into the stress working equality

$$\begin{aligned}
\mathbf{S}^{(-2)} \cdot \cdot \dot{\mathbf{E}}^{(-2)} &= J\boldsymbol{\sigma} \cdot \cdot \mathbf{D} \\
&= J\boldsymbol{\sigma} \cdot \cdot \left(\mathbf{F} \cdot \dot{\mathbf{E}}^{(-2)} \cdot \mathbf{F}^T \right) \\
&= \left(\mathbf{F}^T \cdot J\boldsymbol{\sigma} \cdot \mathbf{F} \right) \cdot \cdot \dot{\mathbf{E}}^{(-2)}
\end{aligned}$$

Hence

$$\mathbf{S}^{(-2)} = \mathbf{F}^T \cdot J\boldsymbol{\sigma} \cdot \mathbf{F}$$

6.

$$\begin{aligned}
J\boldsymbol{\sigma} \cdot \cdot \mathbf{D} &= \mathbf{S}^{(1)} \cdot \cdot \dot{\mathbf{E}}^{(1)} \\
&= \mathbf{S}^{(1)} \cdot \cdot \dot{\mathbf{U}}
\end{aligned}$$

$$\begin{aligned}
\mathbf{L} &= \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} \\
&= \left(\dot{\mathbf{R}} \cdot \mathbf{U} + \mathbf{R} \cdot \dot{\mathbf{U}} \right) \cdot (\mathbf{R} \cdot \mathbf{U})^{-1} \\
&= \dot{\mathbf{R}} \cdot \mathbf{R}^T + \mathbf{R} \cdot \dot{\mathbf{U}} \cdot \mathbf{U}^{-1} \cdot \mathbf{R}^T
\end{aligned}$$

Taking the transpose

$$\mathbf{L}^T = \mathbf{R} \cdot \dot{\mathbf{R}}^T + \mathbf{R} \cdot \mathbf{U}^{-1T} \cdot \dot{\mathbf{U}}^T \cdot \mathbf{R}^T$$

Noting that $\mathbf{R} \cdot \dot{\mathbf{R}}^T$ is anti-symmetric and forming

$$\mathbf{D} = \mathbf{R} \cdot \left[\frac{1}{2} \left(\dot{\mathbf{U}} \cdot \mathbf{U}^{-1} + \mathbf{U}^{-1T} \cdot \dot{\mathbf{U}}^T \right) \right] \cdot \mathbf{R}^T$$

Substituting in the work-rate equality

$$\begin{aligned}
\mathbf{S}^{(1)} \cdot \cdot \dot{\mathbf{U}} &= J\boldsymbol{\sigma} \cdot \cdot \mathbf{D} \\
&= J\boldsymbol{\sigma} \cdot \cdot \left\{ \mathbf{R} \cdot \left[\frac{1}{2} \left(\dot{\mathbf{U}} \cdot \mathbf{U}^{-1} + \mathbf{U}^{-1T} \cdot \dot{\mathbf{U}}^T \right) \right] \cdot \mathbf{R}^T \right\} \\
&= \frac{1}{2} \left\{ \mathbf{U}^{-1} \cdot \left(\mathbf{R}^T \cdot J\boldsymbol{\sigma} \cdot \mathbf{R} \right) + \left(\mathbf{R}^T \cdot J\boldsymbol{\sigma} \cdot \mathbf{R} \right) \cdot \mathbf{U}^{-1T} \right\} \cdot \cdot \dot{\mathbf{U}}
\end{aligned}$$

Therefore

$$\mathbf{S}^{(1)} = \frac{1}{2} \left\{ \mathbf{U}^{-1} \cdot \left(\mathbf{R}^T \cdot J\boldsymbol{\sigma} \cdot \mathbf{R} \right) + \left(\mathbf{R}^T \cdot J\boldsymbol{\sigma} \cdot \mathbf{R} \right) \cdot \mathbf{U}^{-1T} \right\}$$

Part V

Ideal Constitutive Relations

Chapter 22

Fluids

1. For $\mathbf{v} = v(x_2)\mathbf{e}_1$, the flow is isochoric, $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ and $\mathbf{v} \cdot \boldsymbol{\nabla}\mathbf{v} = 0$. For steady flow $\partial \mathbf{v}/\partial t = 0$ and for no body force $\rho \mathbf{b} = 0$. The remaining terms in equation (23.12) are
$$\mu \frac{\partial^2 v}{\partial x_2^2} + \frac{dp}{dx_1} = 0$$
where $dp/dx_1 = \text{constant} < 0$ for flow in the positive x_1 direction. Integrating yields
$$v(x_2) = \frac{1}{2\mu}\frac{dp}{dx_1}x_2^2 + Ax_2 + B$$
where A and B are constants. Because the velocity of the fluid must equal the velocity of the boundary at $x_2 = \pm h$,
$$v(\pm h) = 0$$
Solving for the constants yields $A = 0$ and
$$B = -\frac{h^2}{2\mu}\frac{dp}{dx_1}$$
Substituting back into the velocity yields
$$v(x_2) = -\frac{1}{2\mu}\frac{dp}{dx_1}\left(h^2 - x_2^2\right)$$
The only non-zero component of shear stress is
$$\tau = \mu \frac{dv}{dx_2} = \frac{dp}{dx_1}x_2$$

2. For velocity only in the θ direction and only a function of radial distance $\mathbf{v} = v(r)\mathbf{e}_\theta$. From problem 8.13, the only nonzero component of $\mathbf{D}$ is
$$D_{r\theta} = \frac{1}{2}r\frac{d}{dr}\left(\frac{v}{r}\right)$$

and the shear stress is

$$\sigma_{r\theta} = 2\mu D_{r\theta} = \mu r \frac{d}{dr}\left(\frac{v}{r}\right)$$

From the answer to problem 8.14, the only non-trivial equilibrium equation is

$$\frac{d}{dr}\sigma_{r\theta} + \frac{2}{r}\sigma_{r\theta} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\sigma_{r\theta}\right) = 0$$

Substituting for $\sigma_{r\theta}$ gives

$$\frac{d}{dr}\left\{r^3\frac{d}{dr}\left(\frac{v}{r}\right)\right\} = 0$$

Integrating yields

$$v(r) = \frac{A}{r} + Br$$

Because the fluid velocity must equal the velocity of the boundary

$$v(r = a) = \Omega a$$

and

$$v(r = b) = 0$$

Solving for the constants yields

$$B = -\Omega\frac{a^2}{b^2 - a^2}$$

and

$$A = \Omega\frac{a^2 b^2}{b^2 - a^2}$$

The shear stress at $r = a$

$$\sigma_{r\theta}(r = a) = -2\Omega\frac{b^2}{b^2 - a^2}$$

where this is the negative of the force per unit area on the cylinder. The torque exerted on the cylinder is

$$\begin{aligned} T &= -2\pi a^2 \Delta z \sigma_{r\theta}(r = a) \\ &= 4\pi\Delta z\Omega\frac{a^2 b^2}{b^2 - a^2} \end{aligned}$$

where Δz is the distance out of plane (along the axis of the cylinder).Alternate method: the only non-zero term in equation (23.12) is

$$\mu\nabla^2\mathbf{v} = 0$$

From the answer to problem 8.10

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

Applying to the form of the velocity field gives

$$v''(r) + \frac{1}{r}v'(r) - \frac{1}{r^2}v(r) = 0$$

where $v'(r) = dv/dr$. Looking for a solution of the form r^n gives $n = \pm 1$ and a velocity field of the same form as before.

3. The solution is again given by (23.15). The boundary conditions are now

$$v(y \to \infty, t) = V$$

and

$$v(y \to 0, t) = 0$$

The second corresponds to $\xi \to 0$ and gives $B = 0$. The first corresponds to $\xi \to \infty$ and gives

$$A = \frac{2}{\sqrt{\pi}}$$

Hence, the velocity is

$$v(y,t) = V\,\mathrm{erf}\left(\frac{y}{\sqrt{4\eta t}}\right)$$

At a distance of 1 mm the velocity 1 s is 14.5%, 52% and 6.6% of V for air, water and SAE 30 oil.

4. Look for a solution of the form

$$v(y,t) = V_0 \exp(ky)\exp(i\omega t)$$

and take the real part so that $v(0,t)$ satisfies

$$v(0,t) = V_0\cos(\omega t)$$

Substituting into equation (23.13) and cancelling common terms gives

$$k^2 = i\omega/\eta$$

or

$$k = \sqrt{i}\sqrt{\omega/\eta}$$

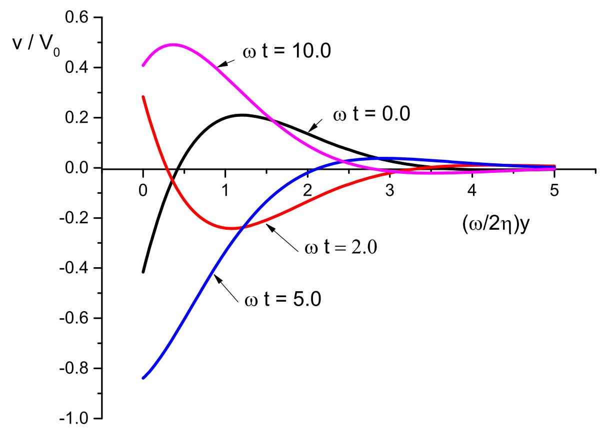
where

$$\sqrt{i} = \exp(i\pi/4)\,,\ \exp\left(\frac{i}{2}\left(\frac{\pi}{2} + 2\pi\right)\right)$$

Take the second solution so that $v(y,t) \to 0$ as $y \to \infty$.Therefore,

$$v(y,t) = V_0 \exp\left(-\sqrt{\omega/2\eta}\right) \mathcal{R}\left[\exp i\left(\omega t - \sqrt{\omega/2\eta y}\right)\right]$$

The solution is plotted in the Figure below.



Velocity $v(y,t)/V_0$ vs. nondimensional distance $\omega y/2\eta$ for four nondimensional times $\omega t = 0.0, 2.0, 5.0, 10.0$.

Chapter 23

Elasticity

1. If the strain energy is regarded as a function of $\mathbf{F}$

$$\frac{\partial W}{\partial F_{mn}} = S_{ij}\frac{\partial E_{ij}}{\partial F_{mn}}$$

where we have used (23.8). The components of the Green-Lagrange strain are given by (16.4)

$$E_{ij} = \frac{1}{2}\left\{F_{ki}F_{kj} - \delta_{ij}\right\}$$

Calculating the derivative

$$\begin{aligned}
\frac{\partial E_{ij}}{\partial F_{mn}} &= \frac{1}{2}\left\{\frac{\partial F_{ki}}{\partial F_{mn}}F_{kj} + F_{ki}\frac{\partial F_{kj}}{\partial F_{mn}}\right\} \\
&= \frac{1}{2}\left\{\delta_{km}\delta_{in}F_{kj} + F_{ki}\delta_{km}\delta_{jn}\right\} \\
&= \frac{1}{2}\left\{\delta_{in}F_{mj} + F_{mi}\delta_{jn}\right\}
\end{aligned}$$

Substituting into the first equation above gives

$$\begin{aligned}
\frac{\partial W}{\partial F_{mn}} &= \frac{1}{2}S_{ij}\left\{\delta_{in}F_{mj} + F_{mi}\delta_{jn}\right\} \\
&= \frac{1}{2}\left\{S_{nj}F_{mj} + S_{in}F_{mi}\right\} \\
&= S_{nj}F^T_{jm} = T^0_{nm}
\end{aligned}$$

where the last line follows because $S_{ij} = S_{ji}$.

2. (a) Regarding $\mathbf{C}$ as a function of $\mathbf{E}$ gives

$$S_{ij} = \frac{\partial W}{\partial E_{ij}} = \frac{\partial W}{\partial C_{kl}}\frac{\partial C_{kl}}{\partial E_{ij}}$$

Because $C_{kl} = \delta_{kl} + 2E_{kl}$ the second term is

$$\frac{\partial C_{kl}}{\partial E_{ij}} = 2\delta_{ik}\delta_{jl}$$

This gives

$$S_{ij} = 2\frac{\partial W}{\partial C_{ij}}$$

Because the material is isotropic, W is a function only of the invariants of $\mathbf{C}$.

$$\begin{aligned} S_{ij} &= 2\left\{\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial C_{ij}} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial C_{ij}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial C_{ij}}\right\} \\ &= 2\left\{W_1\frac{\partial I_1}{\partial C_{ij}} + W_2\frac{\partial I_2}{\partial C_{ij}} + W_3\frac{\partial I_3}{\partial C_{ij}}\right\} \end{aligned}$$

The forms of the first two invariants are given by (7.9) and (7.10). For $I_3 = \det(\mathbf{C})$ the form is given by problem 7.3. The derivatives of the invariants can be computed as follows. For I_1

$$\frac{\partial I_1}{\partial C_{ij}} = \frac{\partial C_{kk}}{\partial C_{ij}} = \delta_{ki}\delta_{kj} = \delta_{ij}$$

For I_2

$$\begin{aligned} \frac{\partial I_2}{\partial C_{ij}} &= \frac{\partial}{\partial C_{ij}}\left\{\frac{1}{2}\left[C_{lk}C_{kl} - C_{kk}C_{ll}\right]\right\} \\ &= \frac{1}{2}\left\{\delta_{li}\delta_{jk}C_{kl} + C_{lk}\delta_{ki}\delta_{lj} - 2C_{kk}\delta_{li}\delta_{lj}\right\} \\ &= \frac{1}{2}\left\{C_{ji} + C_{ij} - 2I_1\delta_{ij}\right\} = C_{ij} - I_1\delta_{ij} \end{aligned}$$

For I_3

$$\begin{aligned} \frac{\partial I_3}{\partial C_{ij}} &= \frac{\partial}{\partial C_{ij}}\left\{\frac{1}{3}C_{kl}C_{lm}C_{mk} - I_1 I_2 - \frac{1}{3}I_1^3\right\} \\ &= \frac{1}{3}\left(\delta_{ki}\delta_{lj}C_{lm}C_{mk} + C_{kl}\delta_{li}\delta_{mj}C_{mk} + C_{kl}C_{lm}\delta_{mi}\delta_{kj}\right) \\ &\quad -\frac{\partial I_1}{\partial C_{ij}}I_2 - I_1\frac{\partial I_2}{\partial C_{ij}} - I_1^2\frac{\partial I_1}{\partial C_{ij}} \\ &= \frac{1}{3}\left(C_{jm}C_{mi} + C_{ki}C_{jk} + C_{jl}C_{li}\right) \\ &\quad -\delta_{ij}I_2 - I_1\left(C_{ij} - I_1\delta_{ij}\right) - I_1^2\delta_{ij} \\ &= C_{ik}C_{kj} - \delta_{ij}I_2 - I_1 C_{ij} \end{aligned}$$

where the last line uses the symmetry of $\mathbf{C}$. Substituting into $S_{ij} = 2\partial W/\partial C_{ij}$ gives

$$\begin{aligned} S_{ij} &= 2\left\{W_1\delta_{ij} + W_2\left(C_{ij} - I_1\delta_{ij}\right) + W_3\left(C_{ik}C_{kj} - \delta_{ij}I_2 - I_1 C_{ij}\right)\right\} \\ &= 2\left\{\delta_{ij}\left[W_1 - I_1W_2 - I_2W_3\right] + C_{ij}\left[W_2 - I_1W_3\right] + W_3C_{ik}C_{kj}\right\} \end{aligned}$$

or

$$\mathbf{S} = 2\left\{\mathbf{I}\left[W_1 - I_1 W_2 - I_2 W_3\right] + \mathbf{C}\left[W_2 - I_1 W_3\right] + W_3 \mathbf{C}\cdot\mathbf{C}\right\}$$

(b) The relation between the Cauchy stress $\boldsymbol{\sigma}$ and the second Piola-Kirchhoff stress is given by (21.12) Substituting the result of (a) gives

$$\begin{aligned}\boldsymbol{\sigma} &= \frac{2}{J}\{\mathbf{F}\cdot\mathbf{I}\cdot\mathbf{F}^T\left[W_1 - I_1 W_2 - I_2 W_3\right] + \mathbf{F}\cdot\mathbf{C}\cdot\mathbf{F}^T\left[W_2 - I_1 W_3\right] \\ &\quad \ldots + W_3 \mathbf{F}\cdot\mathbf{C}\cdot\mathbf{C}\cdot\mathbf{F}^T\}\end{aligned}$$

Each of the terms can be expressed in terms of $\mathbf{B}$ as follows

$$\begin{aligned}\mathbf{F}\cdot\mathbf{I}\cdot\mathbf{F}^T &= \mathbf{F}\cdot\mathbf{F}^T = \mathbf{B} \\ \mathbf{F}\cdot\mathbf{C}\cdot\mathbf{F}^T &= \mathbf{F}\cdot\left(\mathbf{F}^T\cdot\mathbf{F}\right)\cdot\mathbf{F}^T = \left(\mathbf{F}\cdot\mathbf{F}^T\right)\cdot\left(\mathbf{F}\cdot\mathbf{F}^T\right) = \mathbf{B}\cdot\mathbf{B} \\ \mathbf{F}\cdot\mathbf{C}\cdot\mathbf{C}\cdot\mathbf{F}^T &= \left(\mathbf{F}\cdot\mathbf{F}^T\right)\cdot\left(\mathbf{F}\cdot\mathbf{F}^T\right)\cdot\left(\mathbf{F}\cdot\mathbf{F}^T\right) = \mathbf{B}\cdot\mathbf{B}\cdot\mathbf{B}\end{aligned}$$

But because $\mathbf{B}$ and $\mathbf{C}$ have the same principal values, they have the same invariants. Therefore

$$\mathbf{B}^3 = \mathbf{B}\cdot\mathbf{B}\cdot\mathbf{B} = I_1\mathbf{B}\cdot\mathbf{B}{+}I_2\mathbf{B} + I_3 I$$

Also

$$J^2 = \det(\mathbf{F}\cdot\mathbf{F}^T) = \det(\mathbf{F}\cdot\mathbf{F}^T) = \det\left(\mathbf{B}\right)$$

Substituting into the first equation of (b) gives the result.

(c) From problem 7.4.a

$$\mathbf{B}\cdot\mathbf{B} = I_1\mathbf{B}{+}I_2\mathbf{I}{+}I_3\mathbf{B}^{-1}$$

Substituting this into the result of (b) gives the result.

3. With $\boldsymbol{\tau} = J\boldsymbol{\sigma}$, (21.12) gives

$$\boldsymbol{\tau} = \mathbf{F}\cdot\mathbf{S}\cdot\mathbf{F}^T$$

or, in index notation

$$\tau_{ij} = F_{ik} S_{kl} F_{jl}$$

To linearize write the stresses as $\tau_{ij} = \bar{\sigma}_{ij} + \delta\tau_{ij}$ and $S_{ij} = \bar{\sigma}_{ij} + \delta S_{ij}$ where $\bar{\sigma}_{ij}$ is the Cauchy stress in the reference state. The components of the deformation gradient are

$$F_{ij} = \delta_{ij} + u_{i,j}$$

where $u_{i,j} = \partial u_i/\partial X_j$. Substituting and retaining only terms of first order gives

$$\begin{aligned}\bar{\sigma}_{ij} + \delta\tau_{ij} &= \left(\delta_{ij} + u_{i,k}\right)\left(\bar{\sigma}_{kl} + \delta S_{kl}\right)\left(\delta_{jl} + u_{j,l}\right) \\ &= \bar{\sigma}_{ij} + S_{ij} + u_{i,k}\,\bar{\sigma}_{kj} + \bar{\sigma}_{il} u_{j,l} \\ &= \bar{\sigma}_{ij} + C_{ijkl}\varepsilon_{kl} + u_{i,k}\,\bar{\sigma}_{kj} + \bar{\sigma}_{il} u_{j,l}\end{aligned}$$

Cancelling $\bar{\sigma}_{ij}$ yields

$$\delta\tau_{ij} = C_{ijkl}\varepsilon_{kl} + u_{i,k}\,\bar{\sigma}_{kj} + \bar{\sigma}_{il}u_{j,l}$$

The result is symmetric with respect to interchange of i and j. Substitute

$$u_{k,l} = \varepsilon_{kl} + \Omega_{kl}$$

and rearrange to give

$$\delta\tau_{ij} - \Omega_{il}\bar{\sigma}_{lj} - \Omega_{jl}\bar{\sigma}_{il} = \left\{C_{ijkl} + \delta_{ki}\bar{\sigma}_{lj} + \delta_{kj}\bar{\sigma}_{il}\right\}\varepsilon_{kl}$$

Therefore

$$\delta\tau^{*}_{ij} = C^{\tau}_{ijkl}\varepsilon_{kl}$$

where

$$C^{\tau}_{ijkl} = C_{ijkl} + \delta_{ki}\bar{\sigma}_{lj} + \delta_{kj}\bar{\sigma}_{il}$$

and

$$\delta\tau^{*}_{ij} = \delta\tau_{ij} - \Omega_{il}\bar{\sigma}_{lj} - \Omega_{jl}\bar{\sigma}_{li}$$

is the Jaumann increment of the Kirchhoff stress (Prager, Introduction to Mechanics of Continua, Dover, p. 155). This is the rate as computed by an observer rotating with the material. Because $\varepsilon_{kl} = \varepsilon_{lk}$, C^{τ}_{ijkl} can be written symmetrically on k and l:

$$C^{\tau}_{ijkl} = C_{ijkl} + \frac{1}{2}\left\{\delta_{ki}\bar{\sigma}_{lj} + \delta_{li}\bar{\sigma}_{kj} + \delta_{kj}\bar{\sigma}_{il} + \delta_{lj}\bar{\sigma}_{ik}\right\}$$

The result is also symmetric on interchange of ij and kl. To linearize

$$\tau_{ij} = J\sigma_{ij}$$

note from Section 17.1.3 that

$$J = \det(\mathbf{F}) \approx 1 + u_{k,k}$$

Substituting and writing the stresses as the sum of the Cauchy stress in the reference state $\bar{\sigma}_{ij}$and an increment gives

$$\begin{aligned}\delta\tau_{ij} + \bar{\sigma}_{ij} &= \left(\delta\sigma_{ij} + \bar{\sigma}_{ij}\right)\left(1 + u_{k,k}\right)\\ &= \bar{\sigma}_{ij} + \delta\sigma_{ij} + \bar{\sigma}_{ij}u_{k,k}\end{aligned}$$

Cancelling the reference stress on both sides gives

$$\begin{aligned}\delta\tau_{ij} &= C^{\tau}_{ijkl}\varepsilon_{kl}\\ &= \left(C^{\sigma}_{ijkl} + \bar{\sigma}_{ij}\delta_{kl}\right)\varepsilon_{kl}\end{aligned}$$

or

$$C^{\sigma}_{ijkl} = C^{\tau}_{ijkl} - \bar{\sigma}_{ij}\delta_{kl}$$

Note that $C^{\sigma}_{ijkl} \neq C^{\sigma}_{klij}$.

4. The Kirchhoff stress $\boldsymbol{\tau}$ is related to the Cauchy stress $\boldsymbol{\sigma}$ by

$$\boldsymbol{\tau} = J\boldsymbol{\sigma}$$

Writing $\boldsymbol{\tau} = \bar{\boldsymbol{\sigma}} + \delta\boldsymbol{\tau}$ and $\boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}} + \delta\boldsymbol{\sigma}$ gives

$$\bar{\boldsymbol{\sigma}} + \delta\boldsymbol{\tau} = J\left(\bar{\boldsymbol{\sigma}} + \delta\boldsymbol{\sigma}\right)$$

where $\bar{\boldsymbol{\sigma}}$ is the Cauchy stress in the reference state. $J = \det(\mathbf{F})$ can be written as

$$J = \det\left(\delta_{ij} + \frac{\partial u_i}{\partial X_j}\right)$$

where u_i is the component of displacement from the reference state. Linearizing for small displacement gradients (Sec. 17.1.3), $|\partial u_i/\partial X_j| \ll 1$, gives

$$J \simeq 1 + \mathrm{tr}(\boldsymbol{\varepsilon})$$

Substituting and cancelling $\bar{\boldsymbol{\sigma}}$ on both sides gives

$$\delta\boldsymbol{\tau} = \delta\boldsymbol{\sigma} + \bar{\boldsymbol{\sigma}}\,\mathrm{tr}\,\boldsymbol{\varepsilon}$$

For $\mathrm{tr}\,\boldsymbol{\varepsilon}$ sufficiently small so that the product $\bar{\boldsymbol{\sigma}}\,\mathrm{tr}\,\boldsymbol{\varepsilon}$ is small compared to the stress increment, the Cauchy stress and Kirchhoff stress are approximately equal. Alternatively, multiplying and summing (23.20) with the infinitesimal strain ε_{ij} gives

$$C^{\sigma}_{ijkl}\varepsilon_{kl} = C^{\tau}_{ijkl}\varepsilon_{kl} - \bar{\sigma}_{ij}\varepsilon_{kk}$$

If the incremental volume strain (multiplied by the stress in the reference state) is sufficiently small, the incremental moduli for the Cauchy and Kirchhoff stress are the same.

5. The thermal conductivity tensor can be written as

$$K_{ij} = \frac{\partial q_i}{\partial T_{,j}}$$

where $T_{,j} = \partial T/\partial x_j$.

(a) If the heat flux can be written

$$q_i = \frac{\partial G}{\partial T_{,j}}$$

then

$$K_{ij} = \frac{\partial^2 G}{\partial T_{,j}\,\partial T_{,i}} = \frac{\partial^2 G}{\partial T_{,i}\,\partial T_{,j}} = K_{ji}$$

because the derivatives can be taken in either order.

(b) If the $x_1 x_e$ plane is a plane of symmetry, then

$$K'_{ij} = A_{ki} A_{lj} K_{kl} = K_{ij}$$

for a reversal of the x_2 axis corresponding to $A_{11} = A_{33} = -A_{22} = 1$. Therefore

$$K_{12} = A_{i1} A_{j2} K_{ij} = A_{11} A_{22} K_{12} = -K_{12}$$

and, hence, $K_{12} = 0$ and, similarly, $K_{23} = 0$.

(c) If the $x_1 x_2$ plane is a plane of symmetry, the same argument as in (b) with $A_{11} = A_{22} = -A_{33} = 1$ gives $K_{13} = 0$.

(d) If the material is isotropic, then $\mathbf{K}$ must be an isotropic tensor, $\mathbf{K} = k\mathbf{I}$ or $K_{ij} = k\delta_{ij}$.

6. Consider a rotation through an angle θ around the x_3 axis. The components of a tensor in the primed (rotated) system are given in terms of the components in the unprimed system by the answer to Problem 6.15b:

$$\begin{aligned}
F'_{11} &= F_{11} c^2 + 2cs F_{12} + F_{22} s^2 \\
F'_{12} &= (F_{22} - F_{11})\, cs + F_{12}\left(c^2 - s^2\right) \\
F'_{13} &= F_{13} c + F_{23} s \\
F'_{22} &= F_{11} s^2 - 2cs F_{12} + F_{22} c^2 \\
F'_{23} &= -F_{13} s + F_{23} c \\
F'_{33} &= F_{33}
\end{aligned}$$

where F_{ij} is symmetric and $c = \cos\theta$ and $s = \sin\theta$. First show that $C_{44} = C_{55}$. From above

$$\begin{aligned}
\sigma'_{13} &= c\sigma_{13} + s\sigma_{23} \\
&= 2cC_{55}\varepsilon_{13} + 2sC_{44}\varepsilon_{23}
\end{aligned}$$

Also

$$\begin{aligned}
\sigma'_{13} &= 2C'_{55}\varepsilon'_{13} \\
&= 2C'_{55}\left\{c\varepsilon_{13} + s\varepsilon_{23}\right\}
\end{aligned}$$

Equating the coefficient of ε_{23} in the two expressions for σ'_{13} gives $C_{44} = C'_{55}$ and equating the coefficients of ε_{13} gives $C'_{55} = C_{55}$. Therefore $C_{44} = C_{55}$. Now consider

$$\sigma'_{33} = \sigma_{33}$$

with

$$\sigma_{33} = C_{13}\varepsilon_{11} + C_{23}\varepsilon_{22} + C_{33}\varepsilon_{33}$$

and

$$\begin{aligned}
\sigma'_{33} &= C'_{13}\varepsilon'_{11} + C'_{23}\varepsilon'_{22} + C_{33}\varepsilon'_{33} \\
&= C'_{13}\left\{\varepsilon_{11}c^2 + 2cs\varepsilon_{12} + \varepsilon_{22}s^2\right\} \\
&\quad + C'_{23}\left\{\varepsilon_{11}s^2 - 2cs\varepsilon_{12} + \varepsilon_{22}c^2\right\} \\
&\quad + C'_{33}\varepsilon_{33}
\end{aligned}$$

Equating the coefficients of ε_{12} gives $C'_{13} = C'_{23}$. Equating the coefficients of ε_{11} and ε_{22} gives

$$c^2C'_{13} + s^2C'_{23} = C_{13}$$

and

$$s^2C'_{13} + c^2C'_{23} = C_{23}$$

respectively. With $C'_{13} = C'_{23}$ we conclude that $C'_{13} = C_{13} = C_{23}$. Now examine

$$\sigma'_{12} = (\sigma_{22} - \sigma_{11})\,cs + \sigma_{12}\left(c^2 - s^2\right)$$

The left side can be written as

$$\begin{aligned}
\sigma'_{12} &= 2C'_{66}\varepsilon'_{12} \\
&= 2C'_{66}\left\{(\varepsilon_{22} - \varepsilon_{11})\,cs + \varepsilon_{12}\left(c^2 - s^2\right)\right\}
\end{aligned}$$

and the right side as

$$\begin{aligned}
(\sigma_{22} - \sigma_{11})\,cs + \sigma_{12}\left(c^2 - s^2\right) &= cs\left\{(C_{12} - C_{11})\,\varepsilon_{11} + (C_{22}{-}C_{12})\,\varepsilon_{22}\right\} \\
&\quad + 2C_{66}\left(c^2 - s^2\right)\varepsilon_{12}
\end{aligned}$$

Again equating coefficients of the strain components gives

$$\begin{aligned}
C_{11} &= C_{22} \\
C'_{66} &= C_{66} = \frac{1}{2}\,(C_{11} - C_{12})
\end{aligned}$$

The results are independent of the particular value of θ and hence apply for any rotation about the x_3 axis.

7. Consider a rotation of the axis through an angle θ about the x_3. Then

$$\sigma'_{22} = s^2\sigma_{11} + c^2\sigma_{22} - 2cs\sigma_{12}$$

where $c = \cos\theta$ and $s = \sin\theta$. But

$$\begin{aligned}
\sigma_{11} &= C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{12}\varepsilon_{33} \\
\sigma_{22} &= C_{12}\varepsilon_{11} + C_{11}\varepsilon_{22} + C_{12}\varepsilon_{33} \\
\sigma_{12} &= 2C_{44}\varepsilon_{12}
\end{aligned}$$

and

$$\begin{aligned}
\sigma'_{22} &= C_{12}\varepsilon'_{11} + C_{11}\varepsilon'_{22} + C\varepsilon'_{33} \\
&= C_{12}\left(c^2\varepsilon_{11} + 2cs\varepsilon_{12} + s^2\varepsilon_{22}\right) \\
&\quad + C_{11}\left(s^2\varepsilon_{11} - 2cs\varepsilon_{12} + c^2\varepsilon_{22}\right) \\
&\quad + C_{12}\varepsilon_{33}
\end{aligned}$$

Substituting and equating the coefficients of ε_{12} give

$$C_{44} = \frac{1}{2}\left(C_{11} - C_{12}\right)$$

8. From (23.32) with only $\sigma_{11} \neq 0$:

$$2\mu\varepsilon_{11} = \frac{2\left(\lambda + \mu\right)}{3\lambda + 2\mu}\sigma_{11}$$

Therefore

$$\sigma_{11} = \mu\frac{(3\lambda + 2\mu)}{(\lambda + \mu)}\varepsilon_{11} = E\varepsilon_{11}$$

gives (23.33). Equation (23.32) with only $\sigma_{11} \neq 0$ also gives

$$2\mu\varepsilon_{22} = -\frac{\lambda}{(3\lambda + 2\mu)}\sigma_{11}$$

Substituting for σ_{11} from above and solving for ε_{22} gives

$$\varepsilon_{22} = -\frac{\lambda}{2\left(\lambda + \mu\right)}\varepsilon_{11}$$

Comparing with the equation preceding (23.34) yields (23.34).

9. Equation (23.32) for only $\sigma_{12} \neq 0$ gives $2\mu\varepsilon_{12} = \sigma_{12}$ The equation following (23.34) gives

$$\varepsilon_{12} = \frac{(1 + \nu)}{E}\sigma_{12}$$

Equating the expressions for ε_{12} gives (23.35). Eliminating E from (23.35) and (23.33) gives (23.36).

10. From (23.31) with only $\varepsilon_{11} \neq 0$

$$\begin{aligned}
\sigma_{11} &= (\lambda + 2\mu)\,\varepsilon_{11} \\
\sigma_{22} &= \lambda\varepsilon_{11}
\end{aligned}$$

Therefore $M = (\lambda + 2\mu)$ and

$$\frac{\sigma_{22}}{\sigma_{11}} = \frac{\lambda}{\lambda + 2\mu}$$

Using (23.36) gives

$$M = 2\mu \frac{1-\nu}{1-2\nu}$$

and

$$\frac{\sigma_{22}}{\sigma_{11}} = \frac{\nu}{1-2\nu}$$

11. (a) From the equation following (23.34)

$$\varepsilon_{33} = 0 = \frac{(1+\nu)}{E}\sigma_{33} - \frac{\nu}{E}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)$$

which gives

$$\sigma_{33} = \nu\left(\sigma_{11} + \sigma_{22}\right)$$

(b) Substituting the result from (a) back into the equation following (23.34) gives

$$\begin{aligned} \varepsilon_{\alpha\beta} &= \frac{(1+\nu)}{E}\sigma_{\alpha\beta} - \frac{\nu}{E}\delta_{\alpha\beta}\left(\sigma_{11} + \sigma_{22} + \nu\left(\sigma_{11} + \sigma_{22}\right)\right) \\ &= \frac{(1+\nu)}{E}\left\{\sigma_{\alpha\beta} - \delta_{\alpha\beta}\nu\left(\sigma_{11} + \sigma_{22}\right)\right\} \end{aligned}$$

12. From (23.26) $W = (1/2)\varepsilon_{ij}C_{ijkl}\varepsilon_{kl} = (1/2)\sigma_{ij}\varepsilon_{ij}$. Substituting (23.31) gives

$$\begin{aligned} W &= \frac{1}{2}\left(\lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}\right)\varepsilon_{ij} \\ &= \frac{1}{2}\left\{\lambda\left(\varepsilon_{kk}\right)^2 + 2\mu\varepsilon_{ij}\varepsilon_{ij}\right\} \end{aligned}$$

Writing $\varepsilon_{ij} = \varepsilon'_{ij} + (1/3)\delta_{ij}\varepsilon_{kk}$, where ε'_{ij} is the deviatoric strain, and carrying out the product gives (23.39).

13. From equation preceding (23.32) $K = \lambda + (2/3)$. Using (23.36) gives

$$K = 2\mu\frac{(1+\nu)}{3(1-2\nu)}$$

Then $K > 0$ and $\mu > 0$ give (23.40). Using $\nu > -1$ and $\mu > 0$ in (23.35) gives $E > 0$.

14. Setting $P_a = m/a^3$ and taking the stated limits gives $B = 0$ and $A = m/2$. Therefore $u = u_r = m/2r^2$, $\sigma_{rr} = -m/r^3$, $u_i = mX_i/2r^3$ and

$$\sigma_{ij} = \frac{m}{2r^3}\left(\delta_{ij} - \frac{3X_iX_j}{r^2}\right)$$

15. (a) The initial and final volumes are $V_{initial} = (4/3)\,\pi a^3$ and $V_{final} = (4/3)\,\pi\,(a+\varepsilon_0)^3$. For small (infinitesimal) strain

$$\begin{aligned}\varepsilon_{kk} &= \frac{V_{final} - V_{initial}}{V_{initial}} \\ &= (1+\varepsilon_0)^3 - 1 \\ &\approx 3\varepsilon_0\end{aligned}$$

where other terms are at least as small as $(\varepsilon_0)^2$. Therefore ε_0 is equal to one-third of the volume strain.

(b) From (23.3.2), the radial displacement for the general spherically symmetric solution has the form

$$u_r = u(r) = \frac{A}{r^2} + Br$$

where A and B are constants. If the elastic constants are included in calculating the form of the stresses

$$\sigma_{ij} = (3\lambda + 2\mu)\,\delta_{ij} B + 2\mu \frac{A}{r^2}\left(\delta_{ij} - \frac{3X_iX_j}{r^2}\right)$$

and

$$\sigma_{rr} = (3\lambda + 2\mu)\,B - 4\mu\frac{A}{r^2}$$

To calculate the pressure needed to restore the spherical region to its original size, note that $A = 0$, $u(r = a) = -\varepsilon_0 a$ and

$$\sigma_{rr}(r = a) = -p_0$$

These conditions give $B = -\varepsilon_0$ and $p_0 = (3\lambda + 2\mu)\,\varepsilon_0$. To calculate the stress and displacement due to removing the force layer, note that $B = 0$ for $r > a$ and $A = 0$ for $r < a$. Therefore, for $r > a$

$$\begin{aligned}u^+(r) &= \frac{A}{r^2} \\ \sigma^+_{rr}(r) &= -4\mu\frac{A}{r^3}\end{aligned}$$

and for $r < aB$

$$\begin{aligned}u^-(r) &= Br \\ \sigma^-_{rr}(r) &= B\,(3\lambda + 2\mu)\end{aligned}$$

The constants A and B are determined by the conditions of continuity of displacements

$$u^+(a) = u^-(a)$$

and that the jump in σ_{rr} equal minus the pressure due to the force layer

$$\sigma_{rr}^{+}(a) + p_0 = \sigma_{rr}^{-}(a)$$

Solving for A and B gives

$$\begin{aligned} B &= \varepsilon_0 \frac{\lambda + 2\mu/3}{\lambda + 2\mu} \\ A &= Ba^3 \end{aligned}$$

Since $B = u^{-}(a)/a$, the is the actual strain of the inclusion is

$$\varepsilon = u^{-}(a)/a = B = \varepsilon_0 \frac{\lambda + 2\mu/3}{\lambda + 2\mu}$$

The pressure in the inclusion is the pressure p_a due to restoring the size of the inclusion due minus the pressure due to removing the force layer $\sigma_{rr}^{-}(r)$:

$$p_0 - \sigma_{rr}^{-}(r) = -\sigma_{rr}^{+}(a) = 4\mu\varepsilon$$

16. The form of the solutions for $u(r)$ and $\sigma_{rr}(r)$ are the same as in the preceding problem. But now $u_{-}(r)$ and $u^{+}(r)$ are the total displacements from the unstressed state. Hence they satisfy the following condition at $r = a$:

$$u^{+}(a) = u^{-}(a) + \varepsilon_0 a$$

The tractions are now continuous at $r = a$:

$$\sigma_{rr}^{+}(a) = \sigma_{rr}^{-}(a)$$

Using these conditions to solve for A and B gives

$$\begin{aligned} A &= \varepsilon_0 a^3 \frac{\lambda + 2\mu/3}{\lambda + 2\mu} \\ B &= -\frac{4\mu}{3\,(\lambda + 2\mu)} \end{aligned}$$

Recognizing that the net strain of the inclusion is $u^{+}(a)/a$ and the pressure is $-\sigma_{rr}^{-}(a)$ gives the same results as the preceding question.