Parcial 2: Señales y Sistemas

Nombre: Luis Adrián Malocó Patiño.

1. Encuentre la expresión del espectro de Fourier (forma exponencial y trigonométrica) para la señal $x(t) = |6\sin(3t + \frac{\pi}{4})|^2$, con $t \in [-\pi, \pi]$. Presente las simulaciones respectivas para graficar el espectro y la reconstrucción de la señal en función del número de armónicos y el error relativo.

Solución. Forma trigonométrica.

- Graficando la señal en colab, se puede observar que la señal tiene simetría impar. Con esto se sabe que:

$$a_n = \frac{\langle x(t), \cos(n\omega_0 t)\rangle}{\|\cos(n\omega_0 t)\|_2^2} = \frac{\text{Impar} \cdot \text{Par}}{\|\cdot\|^2} = 0$$

$$b_n = \frac{\langle x(t), \text{sen}(n\omega_0 t)\rangle}{\|\text{sen}(n\omega_0 t)\|_2^2} = \frac{\text{impar} \cdot \text{impar}}{\|\cdot\|^2} = \#$$

- Para simplificar cálculos, se expresa $x(t)$ como:

$$(6\sin(3t + \tfrac{\pi}{4}))^2 = (6(\sin(3t)\cos(\tfrac{\pi}{4}) + \cos(3t)\sin(\tfrac{\pi}{4})))^2$$

$$= (6(\sin(3t) \cdot \tfrac{\sqrt{2}}{2} + \cos(3t)\tfrac{\sqrt{2}}{2}))^2$$

$$= \left(6 \cdot \frac{\sqrt{2}\sin(3t) + \sqrt{2}\cos(3t)}{2}\right)^2$$

$$= (3(\sqrt{2}\sin(3t) + \sqrt{2}\cos(3t)))^2 = (3\sqrt{2}\sin(3t) + 3\sqrt{2}\cos(3t))^2$$

$$= 18\sin^2(3t) + 36\sin(3t)\cos(3t) + 18\cos(3t)^2.$$

$$= 18 + 36\sin(3t)\cos(3t) = 18 + 18\sin(6t).$$

Entonces se tiene que:

$$x(t) = |6\sin(3t + \tfrac{\pi}{4})|^2 = 18 + 18\sin(6t)$$

$$x(t) = 18 + 18\sin(6t).$$

Ahora, se halla a_0, a_n, b_n.

a_0

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt \quad ; \quad T = t_f - t_i = \pi - (-\pi) = 2\pi$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} 18 + 18\sin(6t)\,dt = \frac{18}{2\pi}\int_{-\pi}^{\pi} 1 + \sin(6t)\,dt$$

$$a_0 = \frac{9}{\pi}\left[t\Big|_{-\pi}^{\pi} - \frac{\cos(6t)}{6}\Big|_{-\pi}^{\pi}\right] = \frac{9}{\pi}\left[2\pi - \cancel{\left[\frac{1}{6} - \frac{1}{6}\right]}^{0}\right]$$

$$a_0 = \frac{9}{\cancel{\pi}}(2\cancel{\pi}) \rightarrow a_0 = 18$$

a_n

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\,dt \quad ; \quad T = t_f - t_i = \pi - (-\pi) = 2\pi$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} (18 + 18\sin(6t))\cos(n\omega_0 t)\,dt = 0$$

b_n

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,\mathrm{sen}(n\omega_0 t)\,dt \quad ; \quad T = t_f - t_i = \pi - (-\pi) = 2\pi$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} (18 + 18\sin(6t))\,\mathrm{sen}(nt)\,dt$$

$$b_n = \frac{18}{\pi}\int_{-\pi}^{\pi} \mathrm{sen}(nt) + \mathrm{sen}(6t)\,\mathrm{sen}(nt)\,dt =$$

$$b_n = \frac{18}{\pi}\left[\frac{-\cos(nt)}{n}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos(6t - nt)}{2} - \frac{\cos(6t + nt)}{2}\,dt\right.$$

$$b_n = \frac{18}{\pi}\left[\frac{-\cos(nt)}{n}\Big|_{-\pi}^{\pi} + \frac{\mathrm{sen}(t(6-n))}{2(6-n)}\Big|_{-\pi}^{\pi} - \frac{\mathrm{sen}(t(6+n))}{2(6+n)}\Big|_{-\pi}^{\pi}\right.$$

$$b_n = \frac{18}{\pi}\left[-\frac{1}{n}\cancel{(\cos(\pi n) - \cos(-\pi n))}^{0} + \frac{\mathrm{sen}(\pi(6-n)) - \mathrm{sen}(-\pi(6-n))}{2(6-n)} - \frac{\mathrm{sen}(\pi(6+n)) + \mathrm{sen}(-\pi(6+n))}{2(6+n)}\right]$$

Para $n \neq 6$, $b_n = 0$. No obstante, para $n = \pm 6$ se tiene indeterminación. por lo que se debe calcular el límite. y aproximar la indeterminación.

b6

$$b_6 = \frac{18}{\pi} \lim_{n\to 6} \frac{\operatorname{sen}(\pi(6-n)) - \operatorname{sen}(-\pi(6-n))}{2(6-n)} \to \text{Se aplica l'Hôpital.}$$

$$= \frac{18}{\pi} \lim_{n\to 6} \frac{\cos(\pi(6-n))\cdot -\pi - \cos(-\pi(6-n))\cdot \pi}{-2}$$

$$= \frac{18}{\pi} \cdot \left(\frac{-\pi-\pi}{-2}\right) = \frac{18}{\pi} \cdot \frac{-2\pi}{-2} \Rightarrow b_6 = 18$$

b-6

$$b_{-6} = \frac{18}{\pi} \lim_{n\to -6} \frac{-\operatorname{sen}(\pi(6+n)) + \operatorname{sen}(-\pi(6+n))}{2(6+n)} \to \text{Se aplica l'Hôpital.}$$

$$= \frac{18}{\pi} \lim_{n\to -6} \frac{-\pi\cos(\pi(6+n)) + \cos(-\pi(6+n))\cdot -\pi}{2}$$

$$= \frac{18}{\pi} \cdot \left(\frac{-\pi-\pi}{2}\right) = \frac{18}{\pi} \cdot \frac{-2\pi}{2} = -18$$

Ahora se halla C_n

Cn

$$C_n = \frac{a_n - j b_n}{2} \Rightarrow \frac{a_6 - j b_6}{2} = -j9 \ , \ \frac{a_{-6} - j b_{-6}}{2} = j9$$

$$C_0 = a_0 = 18$$

$$\to C_n = \begin{cases} 18 & n=0 \\ -j9 & n=6 \\ j9 & n=-6 \\ 0 & \forall n \setminus \{0,6,-6\}. \end{cases}$$

Solución: forma exponencial

$$C_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-jn w_0 t}\, dt \ ; \ T = t_f - t_i = \pi - (-\pi) = 2\pi$$

$$w_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1.$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} (18 + 18\operatorname{sen}(6t))\, e^{-jnt}\, dt = \frac{18}{2\pi}\int_{-\pi}^{\pi} e^{-jnt} + \operatorname{sen}(6t)\, e^{-jnt}\, dt$$

$$C_n = \frac{9}{\pi}\left[\frac{e^{-jnt}}{-jn}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \operatorname{sen}(6t)\, e^{-jnt}\, dt \right.$$

Scribe

- $\int_{-\pi}^{\pi} \operatorname{sen}(6t) e^{-jnt} dt$ → Aplicando integración por partes y linealidad se obtiene que:

$$\int_{-\pi}^{\pi} \operatorname{sen}(6t) e^{-jnt} dt = \frac{6e^{j\pi n}}{j^2 n^2 + 36} - \frac{6e^{-j\pi n}}{j^2 n^2 + 36}$$

$$C_n = \frac{9}{\pi}\left[-\frac{e^{-jnt}}{jn} \Big|_{-\pi}^{\pi} + \frac{6e^{j\pi n}}{j^2n^2+36} - \frac{6e^{-j\pi n}}{j^2n^2+36} \right]$$

$$C_n = \frac{9}{\pi}\left[-\frac{e^{-j\pi n}}{jn} + \frac{e^{j\pi n}}{jn} + \frac{6e^{j\pi n}}{j^2n^2+36} - \frac{6e^{-j\pi n}}{j^2n^2+36} \right]$$

$$C_n = \frac{9}{\pi}\left[\frac{1}{jn}\left[-e^{-j\pi n} + e^{j\pi n}\right] + \frac{6}{j^2n^2+36}\left[e^{j\pi n} - e^{-j\pi n}\right] \right]$$

$$\rightarrow e^{j\pi n} - e^{-j\pi n} = \cos(\pi n) + j\operatorname{sen}(\pi n) - \cos(\pi n) + j\operatorname{sen}(\pi n)$$

$$e^{j\pi n} - e^{-j\pi n} = j2\operatorname{sen}(\pi n)$$

$$C_n = \frac{9}{\pi}\left[\frac{1}{jn} \cdot j2\operatorname{sen}(\pi n) - \frac{6}{n^2-36} \cdot j2\operatorname{sen}(\pi n) \right]$$

$$C_n = \frac{9}{\pi}\left[\frac{2\operatorname{sen}(\pi n)}{n} - \frac{j12\operatorname{sen}(\pi n)}{n^2-36} \right]$$

Para $n \neq 0, n \neq \pm 6$, $n = 0$. No obstante para $n=0, n=\pm 6$ se tiene indeterminación, por lo que se debe calcular el límite y aproximar la indeterminación.

C_0

$$C_0 = \frac{9}{\pi} \lim_{n\to 0} \frac{2\operatorname{sen}(\pi n)}{n} = \frac{9}{\pi} \lim_{n\to 0} 2\pi\cos(\pi n) = \frac{9}{\pi} \cdot 2\pi = 18$$

C_6

$$C_6 = -\frac{9}{\pi} \lim_{n\to 6} \frac{j12\operatorname{sen}(\pi n)}{n^2-36} = -\frac{9}{\pi} \lim_{n\to 6} \frac{j12\pi\cos(\pi n)}{2n} = -\frac{9}{\pi} \cdot \frac{j6\pi}{6} = -j9$$

C_{-6}

$$C_{-6} = -\frac{9}{\pi} \lim_{n\to -6} \frac{j12\operatorname{sen}(\pi n)}{n^2-36} = -\frac{9}{\pi} \lim_{n\to -6} \frac{j12\pi\cos(\pi n)}{2n} = -\frac{9}{\pi} \cdot \frac{j6\pi}{-6} = j9$$

$$C_n = \begin{cases} 18 & n=0 \\ -j9 & n=6 \\ j9 & n=-6 \\ 0 & \forall n \setminus \{0,6,-6\} \end{cases}$$

- El error relativo se calcula según:

$$E_r[\%] = \left[1 - \frac{1}{P_x}\sum_{n=-N}^{N}|C_n|^2\right]\cdot 100\%$$

→ $P_x = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}|x(t)|^2\,dt$ y $T = t_f - t_i = \pi - (-\pi) = 2\pi$

$$P_x = \frac{1}{2\pi}\int_{-\pi}^{\pi}(18+18\,sen(6t))^2\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}18^2 + 2\cdot 18^2\,sen(6t) + 18^2\,sen^2(6t)\,dt$$

$$P_x = \frac{18^2}{2\pi}\int_{-\pi}^{\pi}1 + 2\,sen(6t) + sen^2(6t)\,dt$$

$$P_x = \frac{324}{2\pi}\int_{-\pi}^{\pi}1 + 2\,sen(6t) + \frac{1}{2} - \frac{\cos(12t)}{2}\,dt$$

$$P_x = \frac{162}{\pi}\left[\,t\,\Big|_{-\pi}^{\pi} - \frac{2\cos(6t)}{6}\Big|_{-\pi}^{\pi} + \frac{t}{2}\Big|_{-\pi}^{\pi} - \frac{sen(12t)}{24}\Big|_{-\pi}^{\pi}\right]$$

$$P_x = \frac{162}{\pi}\left[\pi + \pi - \cancel{\frac{\cos(6\pi)}{3}} + \cancel{\frac{\cos(-6\pi)}{3}} + \frac{\pi}{2} + \frac{\pi}{2} - \cancel{\frac{sen(12\pi)}{24}}^{0} + \cancel{\frac{sen(-12\pi)}{24}}^{0}\right]$$

$$P_x = \frac{162}{\pi}[2\pi + \pi] \rightarrow P_x = \frac{162}{\cancel{\pi}}[3\cancel{\pi}] \rightarrow P_x = 486$$

→ Así:

$$E_r[\%] = \left[1 - \frac{|C_{-6}|^2 + |C_0|^2 + |C_6|^2}{P_x}\right]\cdot 100\%$$

$$E_r[\%] = \left[1 - \frac{9^2 + 18^2 + 9^2}{486}\right]\cdot 100\%$$

$$E_r[\%] = [1 - 1]\cdot 100\%$$

$$E_r[\%] = 0 \rightarrow E_r = 0\%$$

2. Sea la señal portadora $c(t) = A_c\cos(2\pi F_c t)$, con $A_c, F_c \in \mathbb{R}$, y la señal de mensaje $m(t) \in \mathbb{R}$. Encuentre el espectro en frecuencia de la señal modulada en amplitud (AM), $y(t) = (1 + m(t)/A_c)\cdot c(t)$. Luego, descargue desde Youtube, 5 segundos de su canción favorita (capturando del segundo 20 al 25). Presente una simulación de modulación por amplitud AM (tomando como mensaje el fragmento de la canción escogida y con un índice de modulación de 1). Gráfique las señales en tiempo y frecuencia (magnitud) de la señal mensaje, portadora y modulada. Reproduzca los fragmentos de audio del mensaje, portadora y señal modulada.

Solución.

- La transformada de la señal modulada se puede encontrar como:

$$Y(\omega) = \mathcal{F}\{y(t)\} = \mathcal{F}\left\{1 + \frac{m(t)}{A_c}\right. \quad = \mathcal{F}\{c(t)\} + \frac{1}{A_c}\mathcal{F}\{m(t)\cdot c(t)\}$$

- Utilizando tablas de Fourier.

$$C(\omega) = \mathcal{F}\{c(t)\} = \mathcal{F}\{A_c\cos(2\pi F_c t)\} = A_c\mathcal{F}\left\{\frac{e^{2\pi F_c t} + e^{-2\pi F_c t}}{2}\right\}$$

como: $\mathcal{F}\{e^{\pm j\omega_0 t}\} = 2\pi\delta(\omega \mp \omega_0)$

$$\rightarrow C(\omega) = \frac{A_c 2\pi}{2}\left(\delta(\omega - 2\pi F_c) + \delta(\omega + 2\pi F_c)\right)$$

$$C(\omega) = A_c\pi\left(\delta(\omega - 2\pi F_c) + \delta(\omega + 2\pi F_c)\right)$$

- De forma similar.

$$\frac{1}{A_c}\mathcal{F}\{m(t)c(t)\} = \frac{1}{A_c}\mathcal{F}\{m(t)A_c\cos(2\pi F_c t)\} = \mathcal{F}\{m(t)\cos(2\pi F_c t)\}$$

$$= \mathcal{F}\left\{\frac{m(t)e^{2\pi F_c t} + m(t)e^{-2\pi f_c t}}{2}\right\}$$

como:
$$\mathcal{F}\{x(t)e^{\pm j\omega_0 t}\} = X(\omega \mp \omega_0)$$

$$\rightarrow \frac{1}{A_c}\mathcal{F}\{m(t)c(t)\} = \frac{1}{2}\left(M(\omega - 2\pi F_c) + M(\omega + 2\pi F_c)\right)$$

$$Y(\omega) = A_c\pi\left(\delta(\omega - 2\pi F_c) + \delta(\omega + 2\pi F_c)\right) + \frac{1}{2}\left(M(\omega - 2\pi F_c) + M(\omega + 2\pi F_c)\right)$$