Test for proportions

In the videos, you learnt how to perform hypothesis testing for the mean of a Gaussian population. Another very useful example is testing for a population proportion p.

An example

Imagine that you have a coin, but you don't know whether it's fair or not. The proportion you are interested in is $p = \mathbf{P}(H)$. A possible set of hypothesis for this problem is

$$H_0 : p = 0.5 \text{ vs. } H_1 : p \neq 0.5$$

Imagine you toss the coin 20 times, of which 7 turned out heads. Your random sample consists in one random variable X ="number of heads in 20 coin flips", which has a $Binomial(20, p)$ distribution. A good estimation for the proportion is the relative frequency of heads:

$$\hat{p} = \frac{X}{20}$$

Remember that under certain conditions, the Central Limit Theorem states that $\hat{p} \sim \mathrm{N}\left(p, \sqrt{\frac{p(1-p)}{20}}\right)$, or equivalently

$$Z = \frac{\frac{X}{20} - p}{\sqrt{p(1-p)}}\sqrt{20} \sim \mathrm{N}(0, 1)$$

Z **will be your test statistic.** If $H_0$is true $(p = 0.5)$, then your test statistic becomes

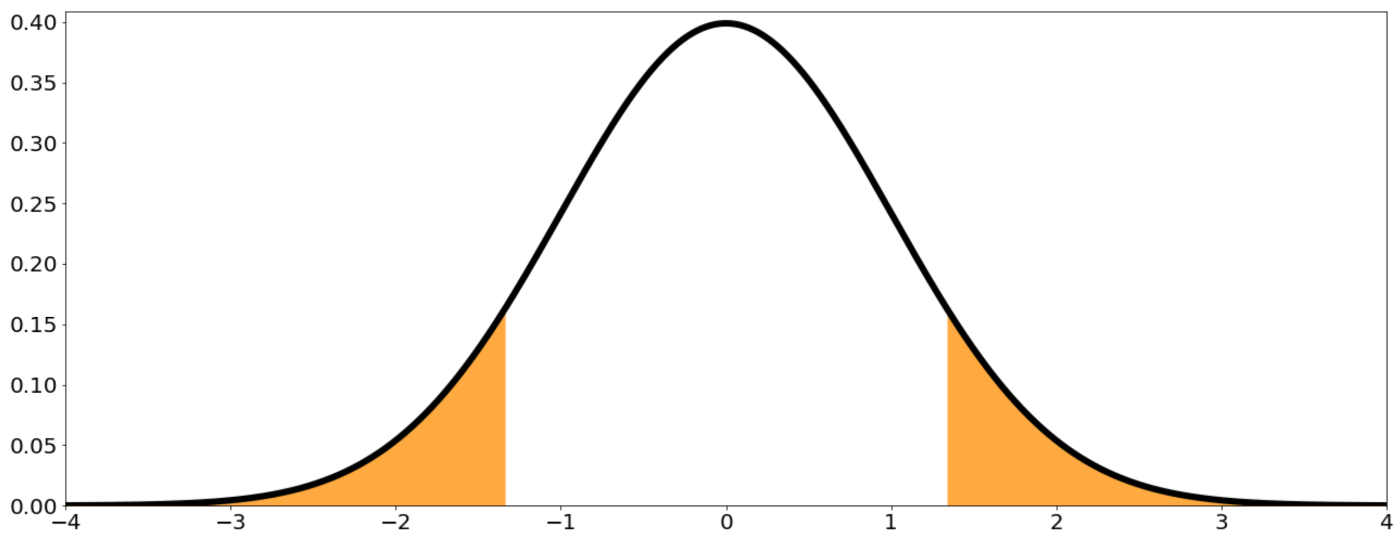
$$Z = \frac{\frac{X}{20} - 0.5}{\sqrt{0.5(1-0.5)}}\sqrt{20} = \frac{\frac{X}{20} - 0.5}{0.5}\sqrt{20} \sim \mathrm{N}(0, 1)$$

Consider a significance level $\alpha = 0.05$. Then to make a decision you need to get the *p*-value for your observed statistic. With the observed sample $x = 7$, the observed statistic is

$$z = \frac{\frac{7}{20} - 0.5}{0.5}\sqrt{20} = -1.3416$$

The *p*-value is then the probability that $Z > |z|$ or $X < -|z|$:

$$p\text{-value} = \mathbf{P}(|Z| > |z|) = \mathbf{P}(|Z| > 1.3416) = 0.1797$$



Conclusion: since the *p*-value is bigger than the significance level of 0.05, you do not have enough evidence to reject the null hypothesis that $p = 0.5$.

General case:

p is the population proportion of individuals in a particular category (i.e. probability of the coin landing heads)

p_0 is the population proportion under the null hypothesis (i.e. $p_0 = 0.5$)

x is the observed number of individuals in the sample from the specified category (i.e. number of heads)

n is the sample size (i.e. number of coin toss)

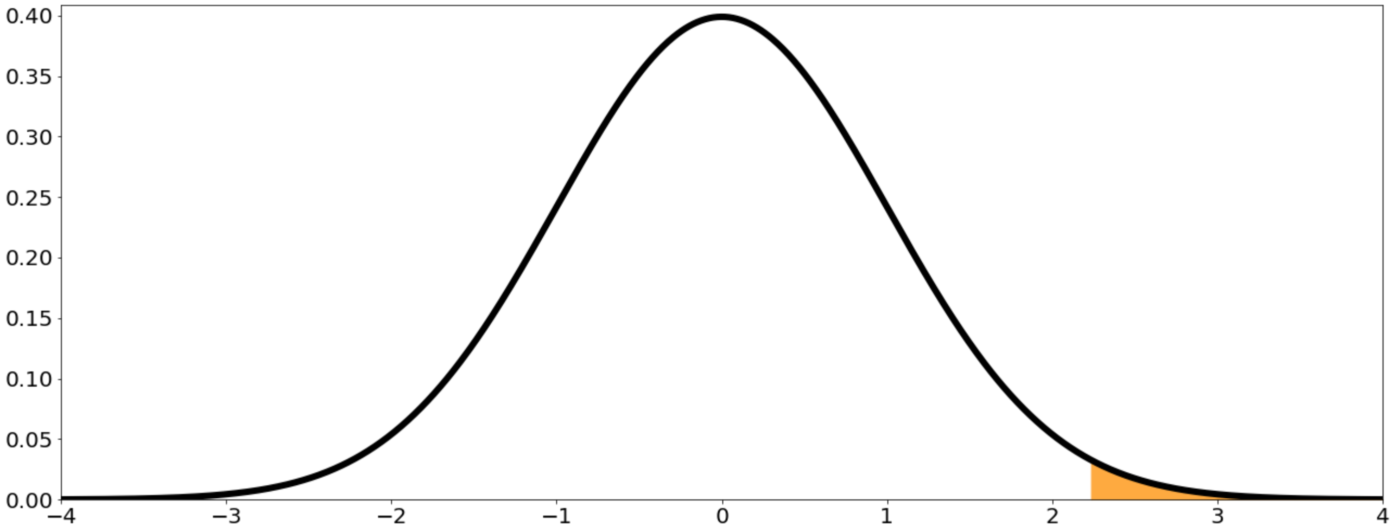
$\hat{p} = \frac{x}{n}$ is the sample proportion for the observed sample x.

Then, $Z = \frac{\frac{X}{n} - p_0}{\sqrt{p_0(1-p_0)}}\sqrt{n} \sim \mathrm{N}(0, 1)$ is the test statistic for comparing proportions, and $z = \frac{\frac{x}{n} - p_0}{\sqrt{p_0(1-p_0)}}\sqrt{n}$ is the observed statistic.

Depending on the type of hypothesis, you have different expressions for the *p*-value:

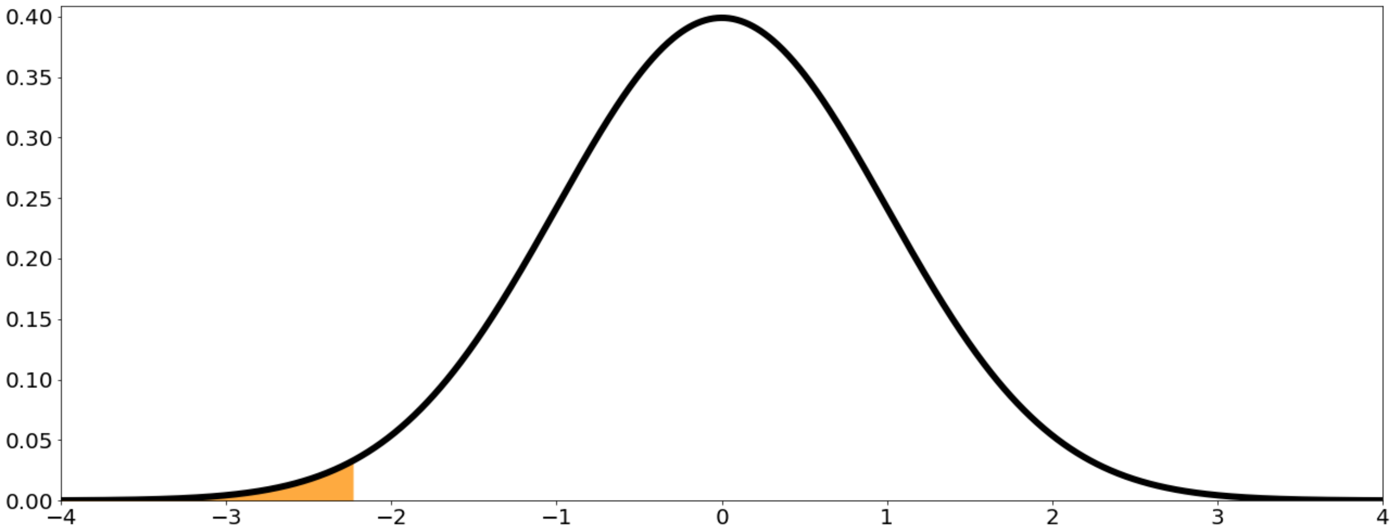
Right-tailed test: $H_0 : p = p_0$ vs. $H_1 : p > p_0$:

$$p\text{-value} = \mathbf{P}(Z > z)$$



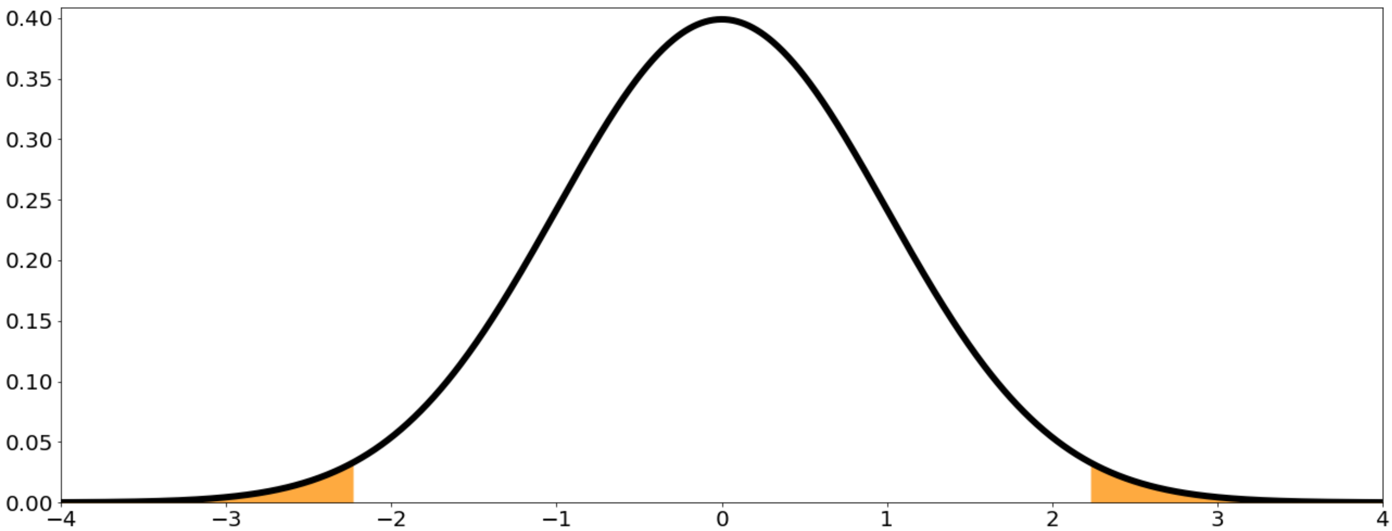
Left-tailed test: $H_0 : p = p_0$ vs. $H_1 : p < p_0$

$$p\text{-value} = \mathbf{P}(Z < z)$$



Two-tailed test: $H_0 : p = p_0$ vs. $H_1 : p \neq p_0$

$$p\text{-value} = \mathbf{P}(|Z| > |z|)$$



For this results to be valid, the following conditions need to be satisfied:

The population size needs to be at least 20 times bigger than the sample size. This is necessary to ensure that all samples are independent. This condition is not needed in situation like the coin toss, where independence is

inherent to the experiment.
The individuals in the population can be divided into two categories: whether they belong to the specified category or they don't
The values $np_0 > 10$ and $n(1-p_0) > 10$. This condition needs to be verified so that the Gaussian approximation holds when the assumption that H_0 is true.