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BAN402: Decision Modelling in Business

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Project 2:

Mixed Integer Linear Programming

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Contents

List of Tables

1 Part A: AirBAN

1.1 Set Covering Problem

To make sure that every country in the network of AirBAN is covered by at least one hub within the time of $\bar{t}$, we can set up a set covering problem. The model formulation for this problem would look like the following:

Sets:
$i \in I$: set of countries.

Parameters:
c_i: Cost of establishing a hub in country i.
$t_{i,j}$: Flight time from country i to country j.
$\bar{t}$: Maximum flight time in which every country has to be reached by at least one hub.
$a_{j,i} \in \{0,1\}$: 1 if country j is covered by hub i, zero otherwise.

Decision variables:
$x_i \in \{0,1\}$: 1 if a hub is established in country i, zero otherwise.

The objective function (Eq. 1) aims to minimize the costs of establishing the hubs:

$$Min\ Costs = \sum_{i \in I} c_i * x_i \tag{1}$$

This cost minimization is subject to the following constraints:
Every city has to be covered at least once:

$$\sum_{i \in I} a_{j,i} * x_i \geq 1 \quad \forall j \in J \tag{2}$$

A city is only considered as covered, if the flight time from the hub in country

i to country j is less than the defined maximum flight time. This is ensured by using the data provided for $t_{i,j}$ and $\bar{t}$ in such a way that:

$$a_{i,j} = \begin{cases} 1 & \text{if } t_{i,j} < \bar{t} \\ 0 & \text{otherwise} \end{cases} \quad \forall i, j \in I \tag{3}$$

Depending on the data for $t_{i,j}$, this model could be infeasible, for example, if there is one very remote country j which cannot be reached within the time limit $\bar{t}$ from any other country i. In such a case, $\sum_{i \in I} a_{j,i}$ would be zero for country j, which would lead to a contradiction in Eq. 2.

1.2 Facility Location Problem

When including the administrative cost in the changed situation in part A2, we have to change the model to a facility location model. This model is set up as follows:

Sets:

$i \in I$: set of countries.

Parameters:

c_i: Cost of establishing a hub in country i.

$r_{i,j}$: Administrative costs if local manager j reports to hub manager i.

m_i: Maximum capacity at hub i.

Decision variables:

$x_i \in \{0, 1\}$: 1 if a hub is established in country i, zero otherwise.

$y_{i,j} \in \{0, 1\}$: 1 if local manager j reports to hub manager i.

Our objective is still to minimize costs, but we have to include the administrative costs:

$$Min\ Costs = \sum_{i \in I} \sum_{j \in I} r_{i,j} * y_{i,j} + \sum_{i \in I} c_i * x_i \tag{4}$$

This objective function is subject to the following constraints:
The sum of local managers j who report to hub i cannot exceed the maximum capacity:

$$\sum_{j \in I} y_{i,j} \leq m_i \quad \forall i \in I \tag{5}$$

Every local manager j has to report to exactly one hub i:

$$\sum_{i \in I} y_{i,j} = 1 \quad \forall j \in I \tag{6}$$

Considering the decision variables are binary, this model is a facility location problem as a pure integer programming model. The formulation of the constraints implies that this model cannot become infeasible, because nothing is limiting the number of hubs that can be set up.

2 Part B: Developing Optimal Student Plans of Study

In this part, we work with a model set up in the paper 'Optimal Student Study Plans' (Bowman, 2021). Please note that while we adopt the names of sets, variables and parameters, for the sake of consistency, we have adapted the notation of the indices to the prevailing notation in this report. All following remarks are just additions to the model described in the Appendix of the above-mentioned paper.

2.1 Nils

Nils wants to have at most one hard course per term. This is why, we need to add an extra constraint to the model set up in the paper. This extra constraint ensures that the number of hard courses Nils takes per term cannot exceed one.

$$\sum_{i \in CHard} x_{i,t} \leq 1 \quad \forall t \in T \tag{7}$$

2.2 Lisa

Lisa can take two or more hard courses per term, as long as this situation does not occur in more than two terms over the course of her studies. To modify the model so that it is in line with that request, we introduce the auxiliary variable $z_t \in \{0, 1\}$. z_t takes value 1 if there are two or more hard courses in term t. This is ensured by the following constraint (M being a large positive parameter):

$$\sum_{i \in CHard} x_{i,t} \leq 1 + M * z_t \quad \forall t \in T \tag{8}$$

Then we can set up a second constraint restricting the sum of z_t to a maximum of two over all terms:

$$\sum_{i \in T} z_t \leq 2 \tag{9}$$

When adding these two constraint to the existing model, we can make sure that the situation of two or more hard courses per term will not occur more than twice during Lisa's studies.

2.3 Abel

Abel prefers to have exactly two hard courses per term. To reflect these preferences in the model, we add a reward of 10 points to the objective function for every term that fulfills this condition. To do so, we introduce the two auxiliary variables $y1_t \in \{0,1\}$, $y2_t \in \{0,1\}$ and $y3_t \in \{0,1\}$to set up the following set of constraints:

$$\sum_{i \in CHard} x_{i,t} \geq 2 - M * y1_t \quad \forall t \in T \tag{10}$$

$$\sum_{i \in CHard} x_{i,t} \leq 2 + M * y3_t \quad \forall t \in T \tag{11}$$

$$y2_t + y1_t = 1 \quad \forall t \in T \tag{12}$$

$$y2_t + y3_t = 1 \quad \forall t \in T \tag{13}$$

$$\sum_{i \in CHard} x_{i,t} - 3 * y3_t \geq 0 \quad \forall t \in T \tag{14}$$

Given that M is a sufficiently large number (e.g. total number of courses), these constraints make sure, that $y2_t$ can only take value 1, if there are exactly two hard courses in a term t. Then we can add the following term to the objective function to modify it in the way requested by Abel:

$$+10 * \sum_{t \in T} y2_t \tag{15}$$

Abel now receives 10 points in his objective function for every term where he has exactly two hard courses.

3 Part C: Oil & BAN

3.1 Mixed Integer Linear Programming Model

Sets:

$j \in J$: set of crude oils.

$b \in B$: set of components.

$p \in P$: set of final products for sale.

$d \in D$: set of depots.

$k \in K$: set of markets.

$i \in I$: set of CDUs.

$bp \in BP$: set of products requiring blending.

$t \in T$: set of days.

Parameters:

$A_{j,b,i}$: refining Parameter: 1 unit of crude oil j is turned into A units of component b at CDU i.

$R_{b,bp}$: blending Parameter: 1 unit of blending product bp requires R units of component b.

E_j: fixed supply of crude oil j.

$minR$: minimum output of CDU if operational

$maxR$: maximum output of CDU if operational

$Cref_{j,i}$: cost of refining 1 unit of crude oil j at CDU i.

$Cprod_{bp}$: cost of blending 1 unit of bp.

$CMain$: cost of maintaining the CDUs.

$CTra1$: cost per unit transported from refining to blending.

$CTra2_d$: cost per unit of lowqfuel transported from refining to depot d.

$CTra3_d$: cost per unit transported from blending to depot d.

$CTra4_{d,k}$: cost per unit transported from depot d to market k.

$\delta_{p,k,t}$: demand limit for product p in market k on day t.

S_p: sales price of final product p.

$CINVJ$: inventory costs at refining department.

$CINVB$: inventory costs at blending department.

$CINVP_d$: inventory costs at depot d.

$Izero_{p,d}$: initial inventory of final product p at depot d.

$Ifinal_{p,d}$: minimum final inventory of final product p at depot d.

M: sufficiently large positive parameter.

Decision variables:

$z_{j,i,t} \geq 0$: Amount of crude oil j refined at CDU i on day t.

$u_{b,t} \geq 0$: Amount of component b obtained on day t and sent to blending.

$w_{bp,t} \geq 0$: Amount of product bp produced at the blending department on day t.

$f_{bp,d,t} \geq 0$: Amount of blended product bp sent to depot d on day t.

$fl_{d,t} \geq 0$: Amount of lowqfuel sent to depot d on day t.

$v_{p,d,k,t} \geq 0$: Amount of product p sold from depot d to market k on day t.

$IO_{j,t} \geq 0$: Inventory of oil j at refining department on day t.

$IC_{b,t} \geq 0$: Inventory of component b at blending department on day t.

$IP_{p,d,t} \geq 0$: Inventory of product p at depot d on day t.

$x_{i,t} \in \{0,1\}$: 1 if CDU i operates at day t.

$y_{i,t} \in \{0,1\}$: 1 if CDU i is maintained at day t.

Objective Function:

$$
\begin{aligned}
\text{maximize } \ profit = & \\
& \sum_{p\in P}\sum_{d\in D}\sum_{k\in K}\sum_{t\in T:12>t>0} S_p * v_{p,d,k,t} \\
& - \sum_{j\in J}\sum_{i\in I}\sum_{t\in T:t>0} z_{j,i,t} * Cref_{j,i} \\
& - \sum_{bp\in BP}\sum_{t\in T:t>0} w_{bp,t} * Cprod_{bp} \\
& - \sum_{b\in B:b\neq \text{lowqfuel}}\sum_{t\in T:t>0} u_{b,t} * CTra1 \\
& - \sum_{d\in D}\sum_{t\in T:t>0} fl_{d,t} * CTra2_d \\
& - \sum_{bp\in BP}\sum_{d\in D}\sum_{t\in T:t>0} f_{bp,d,t} * CTra3_d \\
& - \sum_{p\in P}\sum_{d\in D}\sum_{k\in K}\sum_{t\in T:t>0} v_{p,d,k,t} * CTra4_{d,k} \\
& - \sum_{i\in I}\sum_{t\in T:t>0} y_{i,t} * CMain \\
& - \sum_{j\in J}\sum_{t\in T:t>0} CINVJ * IO_{j,t} \\
& - \sum_{b\in B}\sum_{t\in T:t>0} CINVB * IC_{b,t} \\
& - \sum_{p\in P}\sum_{d\in D}\sum_{t\in T:t>0} CINVP_d * IP_{p,d,t}
\end{aligned}
\tag{16}
$$

This objective function aims at maximizing profit, i.e., revenue, consisting of the total amount of products sold times their market price, minus the sum of all costs for the observation period. These costs are made of the total refining and production costs, transport costs along the production chain, maintenance costs of the CDUs and the costs of storing excess crude oil, components or products. We exclude day 12 from the calculation of revenue, since what we would sell on day 12 would arrive to the markets on day 13, i.e., outside of our planning horizon.

Of course this profit maximization takes place within a framework of constraints reflecting the production process:

Constraints for CDU operation:

$$\sum_{t \in T : t > 6} y_{i,t} \geq 1 \quad \forall i \in I \tag{17}$$

$$y_{i,t} + x_{i,t} \leq 1 \quad \forall i \in I,\ t \in T : t > 0 \tag{18}$$

$$\sum_{i \in I} x_{i,t} \geq 1 \quad \forall t \in T : t > 0 \tag{19}$$

$$\sum_{j \in J} z_{j,i,t} \geq \mathrm{minR} \cdot x_{i,t} \quad \forall i \in I,\ t \in T : t > 0 \tag{20}$$

$$\sum_{j \in J} z_{j,i,t} \leq \mathrm{maxR} \cdot x_{i,t} \quad \forall i \in I,\ t \in T : t > 0 \tag{21}$$

$$\sum_{j \in J} z_{j,i,t} \leq M \cdot x_{i,t} \quad \forall i \in I,\ t \in T : t > 0 \tag{22}$$

The first body of constraints, equations 17 to 22, ensures that the CDUs operate within the conditions specified. Each CDU needs to undergo maintenance at least once in the last 6 days of the decision period (Eq. 17) and cannot operate while being maintained (Eq. 18). Furthermore, at least one CDU has to operate each day (Eq. 19) and refine at least the minimum (Eq. 20) and at most the maximum (Eq. 21) amount of crude oil. Eq. 22 defines that, should CDU i not operate on a given day t, it cannot refine any crude oil, with M being a sufficiently large parameter.

Constraints for Flow and Production Process:

Next, we set up a series of operational constraints to model the production process and avoid illogical flows of oil, components or products. The sum of product p sent to market k from all depots d on a given day $t-1$ cannot exceed the demand limit on day t:

$$\sum_{d \in D} v_{p,d,k,t-1} \leq \delta_{p,k,t} \quad \forall p \in P,\ k \in K,\ t \in T : t > 0 \tag{23}$$

The amount of crude oil refined at both CDUs cannot exceed the fixed supply of said oil E_j plus the inventory from the day before:

$$\sum_{i \in I} z_{j,i,t} \leq E_j + \mathrm{IO}_{j,t-1} \quad \forall j \in J,\ t \in T : t > 0 \tag{24}$$

Since it is not possible to store components at the refining department, all components (except lowqfuel) obtained must be sent to blending (arrival on day $t+1$):

$$u_{b,t} = \sum_{j \in J} \sum_{i \in I} z_{j,i,t} \cdot A_{j,b,i} \quad \forall b \in B : b \neq \text{'lowqfuel'},\ t \in T : t > 0 \tag{25}$$

Following a similar reasoning, the amount of blended products sent to depots on day t (arrival on day $t+1$) must be equal to what is produced on day t:

$$\sum_{d \in D} f_{bp,d,t} = w_{bp,t} \quad \forall bp \in BP,\ t \in T : t > 0 \tag{26}$$

Same holds for lowqfuel, but here the flow is directly from the refining department to the depots, since this product does not require blending:

$$\sum_{d \in D} fl_{d,t} = \sum_{j \in J} \sum_{i \in I} z_{j,i,t} \cdot A_{j,\text{'lowqfuel'},i} \quad \forall t \in T : t > 0 \tag{27}$$

Finally, we can only sell from the depots on day t what we have sent from the blending department on day $t-1$ plus the leftover inventory:

$$\sum_{k \in K} v_{bp,d,k,t} \leq f_{bp,d,t-1} + \mathrm{IP}_{bp,d,t-1} \quad \forall bp \in BP,\ d \in D,\ t \in T : t > 0 \tag{28}$$

Same logic applies to lowqfuel:

$$\sum_{k \in K} v_{\text{'lowqfuel'},d,k,t} \leq fl_{d,t-1} + \mathrm{IP}_{\text{'lowqfuel'},d,t-1} \quad \forall d \in D,\ t \in T : t > 0 \tag{29}$$

Balancing Constraints:
In addition to setting up the production process as described above, we need to define constraints for the balance of the inventory of oil, components and products as done in constraints 30 to 33. These constraints make sure that any given inventory on day t is equal to the inventory of $t-1$, plus the inflow on day t, minus the outflow on day t and are defined as described below:

$$\mathrm{IO}_{j,t} = \mathrm{IO}_{j,t-1} + E_j - \sum_{i \in I} z_{j,i,t} \quad \forall j \in J,\ t \in T : t > 0 \tag{30}$$

$$\mathrm{IC}_{b,t} = \mathrm{IC}_{b,t-1} + u_{b,t-1} - \sum_{bp \in BP} w_{bp,t} \cdot R_{b,bp} \quad \forall b \in B : b \neq \text{'lowqfuel'}, t \in T : t > 0 \tag{31}$$

$$\mathrm{IP}_{bp,d,t} = \mathrm{IP}_{bp,d,t-1} + f_{bp,d,t-1} - \sum_{k \in K} v_{bp,d,k,t} \quad \forall bp \in BP,\ d \in D,\ t \in T : t > 0 \tag{32}$$

$$\mathrm{IP}_{\text{'lowqfuel'},d,t} = \mathrm{IP}_{\text{'lowqfuel'},d,t-1} + fl_{d,t-1} - \sum_{k \in K} v_{\text{'lowqfuel'},d,k,t} \quad \forall d \in D, t \in T : t > 0 \tag{33}$$

Initial or final Flows and Inventories:
Moreover, in Equations 34 and 35, we set the minimum inventory of products on day 12 and the starting inventory on day 0.

$$\mathrm{IP}_{p,d,12} \geq \mathrm{Ifinal}_{p,d} \quad \forall d \in D,\ p \in P \tag{34}$$

$$\mathrm{IP}_{p,d,0} = \mathrm{Izero}_{p,d} \quad \forall p \in P,\ d \in D \tag{35}$$

Finally, we set all other initial flows and inventories to 0:

$$\mathrm{IC}_{b,0} = 0 \quad \forall b \in B \tag{36}$$

$$\mathrm{IO}_{j,0} = 0 \quad \forall j \in J \tag{37}$$

$$z_{j,i,0} = 0 \quad \forall j \in J,\ i \in I \tag{38}$$

$$u_{b,0} = 0 \quad \forall b \in B \tag{39}$$

$$w_{bp,0} = 0 \quad \forall bp \in BP \tag{40}$$

$$f_{bp,d,0} = 0 \quad \forall bp \in BP,\ d \in D \tag{41}$$

$$fl_{d,0} = 0 \quad \forall d \in D \tag{42}$$

$$v_{p,d,k,0} = 0 \quad \forall p \in P,\ d \in D,\ k \in K \tag{43}$$

Solving this model yields an optimal profit of \$ 3,825,571. Table 1 displays

Table 1: Operation of CDUs

Day	CDU1	CDU2
0	not operating	not operating
1	operating	operating
2	operating	operating
3	operating	operating
4	operating	operating
5	operating	operating
6	operating	operating
7	not operating (Maintenance)	operating
8	operating	operating
9	operating	operating
10	operating	operating
11	operating	not operating
12	operating	not operating (Maintenance)

the operation plan of the two CDUs at the refining department. Both CDUs are maintained once during the last six days of the decision period, CDU1 on day 7, CDU2 on day 12. While CDU1 is operating every other day within the relevant period, CDU2 is shut down on day 11 in addition to its maintenance on day 12.

The optimal production also yields some unsatisfied demand. However, this only occurs for lowqfuel. To maximize profit, we produce in a way that maximizes the amount of blended products sold, since these products yield

Table 2: Unsatisfied demand of lowqfuel

Market	1	2	3	4	5	6	7	8	9	10	11	12
K1	0	0	0	0	0	0	0	0	0	0	0	25
K2	0	0	0	0	0	0	0	0	0	0	0	0
K3	0	0	15	11.44	0	15	15	15	15	15	5	0
K4	0	0	15	15	15	15	15	15	15	5	4.84	10
K5	0	0	10	10	10	10	10	10	10	10	0	5
K6	0	0	40	40	38.23	21.93	31.07	30.07	23.87	19.58	0	0
K7	0	0	0	0	0	0	0	0	0	0	0	0
K8	0	0	15	0	0	0	0	0	0	0	0	0

higher prices. Lowqfuel can be considered more of a waste product of the refining process, the amount of which is fixed through the refining parameter $A_{j,b,i}$. We therefore accept unsatisfied demand for this product in order to be able to exploit the demand limit of the other products. Overproduction at the refining department in order to max out the demand for lowqfuel as well would not be profitable due to its low prices. The resulting amount of unsatisfied demand for lowqfuel for days 1 - 12 can be obtained from Table 2.

Table 3 displays the Inventory of Crude Oils. As can be easily seen, there are 1150 barrels of Crude Oil A and 1435 barrels of Crude Oil B left on day 12. This happens because we have a fixed supply of E_j, independent of the day or the demand. For both types of crude oil, this supply adds up to a total of 1150 barrels each day, which towards the end of the planning horizon is more than we need to satisfy demand and comply with the minimum final inventory of products. That is why we have to store the excess crude oil at the refining department.

In Table 4 we display the Inventory of Components. Both for distilA and naphta2 we have an inventory of over 100 barrels at the end of planning horizon. Since we use approximately equal amounts of the two types of crude oil, we can turn to the formulation parameters $A_{j,b,i}$ and $R_{b,bp}$ in search of an explanation for this phenomenon.

Looking at the data for $A_{j,b,i}$ and $R_{b,bp}$, it is striking that we for both distilA and naphta2 we obtain more from the refining process than we need

Table 3: Inventory of Crude Oils

Days	**CrA**	**CrB**
1	263.889	0
2	0	204.494
3	0	46.519
4	0	117.089
5	0	187.658
6	32.3418	62.6582
7	465.785	79.2152
8	295	0
9	0	45
10	0	425
11	575	930
12	1150	1435

in blending. The parameter $A_{j,b,i}$ is fixed and does not leave us any decision about how much of which components we want to produce. At the same time, however, since we need the other components to be able to satisfy demand and make profit, it amounts to an overproduction of distilA and naphta2. In other words, the ratio of $A_{j,b,i}$:$R_{b,bp}$ is much higher for distilA and naphta2 than for other components. It is more profitable for us overproduce, store these components and pay the inventory costs, than to cut back our sales. This is why we see this pattern of inventory for the components.

Table 4: Inventory of Components

Days	**distilA**	**distilB**	**lowqfuel**	**naphtha1**	**naphtha2**
0	0	0	0	0	0
1	0	0	0	0	0
2	31.9537	0	0	0	0
3	130.319	0	0	0	16.8671
4	222.193	0	0	0	97.8987
5	294.334	0	0	0	104.261
6	366.475	0	0	0	110.623
7	443.63	0	0	0	65.2218
8	475.092	0	0	0	0
9	571.403	0	0	0	0
10	644.1	0	0	0	218.87
11	733.6	0	0	0	238.9
12	740.6	11.2	0	7	280.9

3.2 Introduction of Fixed Costs

To include the fixed costs for operating the CDUs as described in Part C.2 in our model, we introduce three binary auxiliary variables:

$$a1_t \in \{0, 1\},$$
$$a2_t \in \{0, 1\},$$
$$a3_t \in \{0, 1\}.$$

These variables are part of the following constraints:

$$a1_t \leq x_{CDU1,t} \quad \forall t \in T : t > 0 \tag{44}$$

$$a1_t = 1 - x_{CDU2,t} \quad \forall t \in T : t > 0 \tag{45}$$

$$a2_t \leq x_{CDU2,t} \quad \forall t \in T : t > 0 \tag{46}$$

$$a2_t = 1 - x_{CDU1,t} \quad \forall t \in T : t > 0 \tag{47}$$

$$\sum_{i \in I} x_{i,t} \leq 1 + a3_t \quad \forall t \in T : t > 0 \tag{48}$$

This body of constraints ensures that $a1_t$ only takes value 1 if CDU1 operates and CDU2 does not and $a2_t$ vice versa. $a3_t$ takes value 1 if both CDUs are operational.

Having defined these constraints, we can add the following terms to the previous objective function to guarantee that the fixed costs of CDU operation are taken into account:

$$-\sum_{t \in T : t > 0} a1_t * 60000 - \sum_{t \in T : t > 0} a2_t * 120000 - \sum_{t \in T : t > 0} a3_t * 135000 \tag{49}$$

Adding the fixed costs into our model obviously has a negative effect on the maximum profit we can achieve. The increase in costs causes the optimal profit to drop to $2,419,874.

Table 5: Operation of CDUs with fixed costs

Day	CDU1	CDU2
0	not operating	not operating
1	operating	operating
2	operating	operating
3	operating	operating
4	operating	operating
5	operating	operating
6	operating	not operating
7	not operating (Maintenance)	operating
8	operating	operating
9	operating	operating
10	operating	operating
11	operating	not operating
12	operating	not operating (Maintenance)

From table 5 we can see that the operation plan of the CDUs does not change much. That indicates that it is still more profitable to pay the $135,000 for both CDUs to operate to be able to satisfy (almost) all demand for blended products than to turn off the CDUs. Since one CDU has operate at all time, we would have to pay a minimum $60,000 at all time anyway. However, under the new cost structure, CDU2 is turned off on day 6. The maintenance still takes place at day 7 (CDU1) and day 12 (CDU2).

4 Part D: Earthquake Relief in Momiss

4.1 Minimizing Maximum Travel Time

Sets and Indexes:

$i/j \in C$: set of cities.

$i/j \in S$: set of cities where operation centers can be installed.

$k \in K$: set of trucks.

Parameters:

${T_{i,j}}^{DIST}$: travel time from city i to city j.

T_i^{LOAD}: loading time at city i, $i \in S$.

Decision Variables:

$x_{i,k} \in \{0,1\}, i \in S$: 1 if truck k is placed in city i, 0 otherwise.

$y_{k,i,j} \in \{0,1\},\ i,j \in C$: 1 if truck k drives from city i to city j, 0 otherwise.

$u_{i,k} \in \{0,1\}$: 1 if truck k visits city i, 0 otherwise.

t_k: time truck k needs for the whole route.

The total travel time for each truck is given by time from one city to the other plus loading time at the operation center:

$$t_k = \sum_{i \in C} \sum_{j \in C: j \neq i} y_{k,i,j} * {T_{i,j}}^{DIST} + \sum_{i \in S} x_{i,k} * T_i^{LOAD} \quad \forall k \in K \tag{50}$$

This variable is necessary, since we are not minimizing the total travel time by all trucks combined, but instead the maximum travel of one truck. To find the optimal routes for this objective we formulate the following objective function:

$$min \max_{k \in K} t_k \tag{51}$$

The routes of the trucks are subject to the following constraints. There must be only one incoming trip to each city i of any truck k from any other city j and one outgoing trip of any truck k from city i to any city j:

$$\sum_{k\in K}\sum_{j\in C:j\neq i} y_{k,j,i} = 1 \quad \forall i \in C \tag{52}$$

$$\sum_{k\in K}\sum_{j\in C:j\neq i} y_{k,i,j} = 1 \quad \forall i \in C \tag{53}$$

Every truck must visit exactly four cities, including the operation center it starts from:

$$\sum_{i\in C}\sum_{j\in C:j\neq i} y_{k,i,j} = 4 \quad \forall k \in K \tag{54}$$

Every truck must have exactly one operation center on its route, which is the starting point of every route:

$$\sum_{i\in S} x_{i,k} = 1 \quad \forall k \in K \tag{55}$$

If truck k is placed in operation center i, this truck must make a trip from this operation center to any other city j (Eq. 56), and this truck must make a trip from any city j to this operation center i (Eq. 57). This constraint is a greater or equal constraint, since truck k can still visit another city in S, even if it is not chose as the operation center on his route (only four of seven possible cities are chosen as operation center):

$$\sum_{j\in C:j\neq i} y_{k,i,j} \geq x_{i,k} \quad \forall k \in K, i \in S \tag{56}$$

$$\sum_{j\in C:j\neq i} y_{k,j,i} \geq x_{i,k} \quad \forall k \in K, i \in S \tag{57}$$

In reality, the trucks would obviously start from their operation center to be loaded. For the modelling however, we refrain from setting the order in which the trucks visit the cities in this task, since it is not important for the objective value. Once an optimal route is determined, it will be the same length regardless of where of the four cities the truck starts from. We just assume that the trucks start from the chosen operation centers. However, we must make sure that the route of the truck is a sequence of connected trips. In other words, if truck k visits city i, there must be a trip from this city (eq. 58) and to this city (eq. 59) for truck k:

$$\sum_{j \in C: j \neq i} y_{k,i,j} = u_{i,k} \quad \forall k \in K, i \in C \tag{58}$$

$$\sum_{j \in C: j \neq i} y_{k,j,i} = u_{i,k} \quad \forall k \in K, i \in C \tag{59}$$

Lastly, we must eliminate subtours within each route. Since we are already setting the visits equal to the trips and making sure that each city is driven to and from only once, there is only one possibility of subtour to eliminate. So far, the trucks could drive back and forth between two cities in two separate tours. To eliminate this, we introduce the follwoing constraint:

$$y_{k,i,j} + y_{k,j,i} <= 1 \quad \forall k \in K, i \in C, j \in C : j \neq i \tag{60}$$

Solving the model in AMPL, the shortest maximum travel time by any truck is 50.88 hours for truck 4, including the loading time at the operation center. The other trucks drive for 47.54, 36.22 and 47.4 hours respectively. The optimal routes are displayed in figure 1. The cities chosen as operation centers are colored in yellow: S4, C1, S5, and N1. The longest trip starts at operation center N1, drives to cities N2, N3, N4, and returns to N1. Please note that even though we explicitly name the trucks in our model from 1 to 4, it is not said that it is always truck 1 which is travelling the longest time. The set of trucks in our model denotes the fact that we want to have four independent subtours. Mathematically, however, the solver does not care

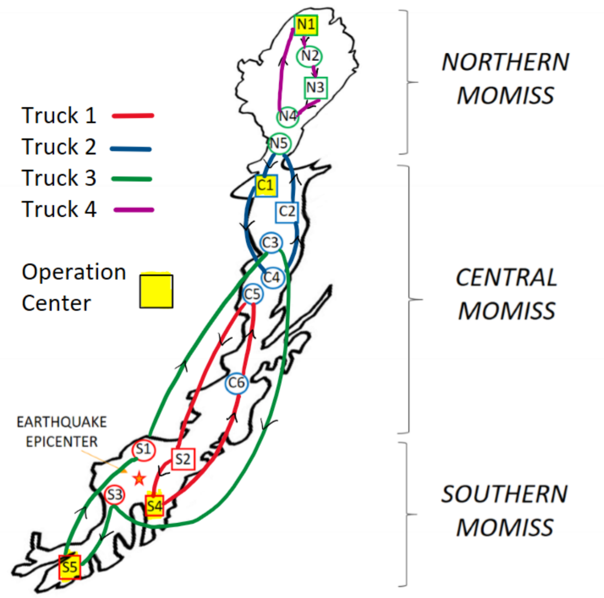

Figure 1: Optimal routes minimizing maximum travel time

about the other tours, once it has found the optimal longest tour. In figure one can also visually see that the optimal solution remains the same when changing the operation center for truck 2 from C1 to C2. When running the code in AMPL, one will could encounter different routes. The longest route and the objective value, however, will remain the same.

4.2 Emergency Score Is All That Counts

In task D.2, we now need to take into account in which order the cities are visited, because the place in which a city is visited determines the emergency score, which we want to maximize. We therefore introduce a new set N, which denotes the place in which a city is visited. Variables x, y, u from the previous section are now indexed to N additionally. Variable t is dropped for this task.

New Sets:
$n \in N : \{1, 2, 3, 4\}$: set of "places" in which a city is visited, representing the order of visitation.

New Parameters:
$E_{i,n}$: emergency score when city i is visted in nth *place*.

Note that we do not need the parameters from task D.1 for this model. The objective function thus changes to the following, maximizing the overall emergency score:

$$max \sum_{k \in K} \sum_{i \in C} \sum_{n \in N} u_{i,k,n} * E_{i,n} \tag{61}$$

We need to modify all of the constraints to incorporate the new set N. Adjusting incoming and outgoing trips for the new set N:

$$\sum_{k \in K} \sum_{j \in C: j \neq i} \sum_{n \in N} y_{k,j,i,n} = 1 \quad \forall i \in C \tag{62}$$

$$\sum_{k \in K} \sum_{j \in C: j \neq i} \sum_{n \in N} y_{k,i,j,n} = 1 \quad \forall i \in C \tag{63}$$

Adjusting the number of trips for each truck for the new set N:

$$\sum_{i \in C} \sum_{j \in C: j \neq i} \sum_{n \in N} y_{k,i,j,n} = 4 \quad \forall k \in K \tag{64}$$

Adjusting the constraint of exactly one operation center per route:

$$\sum_{i \in S} \sum_{n \in N: n=1} x_{i,k,n} = 1 \quad \forall k \in K \tag{65}$$

Instead of saying "if truck k is placed in operation center i, there must be a link from this operation center and to this operation center for truck k" (see eq. 56 and 57), we now explicitly set the starting point ($n = 1$, eq. 66) and the last trip ($n = 4$, eq. 67) for every truck to the city where the operation center is placed.

$$u_{i,k,1} = x_{i,k,1} \quad \forall k \in K, i \in S \tag{66}$$

$$\sum_{j \in C : j \neq i} y_{k,j,i,4} = x_{i,k,1} \quad \forall k \in K, i \in S \tag{67}$$

Adjusting the sequence constraints to the new set N:

$$\sum_{j \in C : j \neq i} y_{k,i,j,n} = u_{i,k,n} \quad \forall k \in K, i \in C, n \in N \tag{68}$$

$$\sum_{j \in C : j \neq i} y_{k,j,i,n} = u_{i,k,n-1} \quad \forall k \in K, i \in C, n \in N : n > 1 \tag{69}$$

Adjusting the subtour elimination constraint for the new set N:

$$y_{k,i,j,n} + y_{k,j,i,n+1} <= 1 \quad \forall k \in K, i \in C, j \in C, n \in N : n < 4, j \neq i \tag{70}$$

Solving the model in AMPL, we obtain an optimal emergency score of 1460. This score is composed of scores of 420, 250, 370, and 420 for trucks 1, 2, 3, and 4 respectively. Cities S2, S4, C1, and C2 are selected as operation centers. Figure 2 visualizes the routes for this solution. As we can see, the routes would not really make sense in a real world example, since this model does not take into account the distance or travel time at all. We will tackle this issue in the next task.

Note: As we mentioned in task D.1, there are multiple possible solutions with the same optimal objective value. Depending on how and in which order the constraints are formulated, and how the model as a whole is defined, a solver might come to the same solution with different routes.

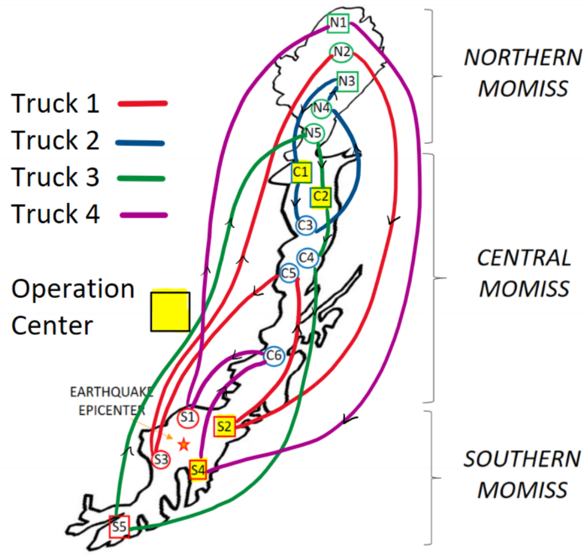

Figure 2: Optimal routes maximizing emergency score

4.3 Tradeoff: Travel Time And Emergency Score

The optimal emergency score of our solution in the first task is 1160 (350 for truck 1, 300 for truck 2, 410 for truck 3 and 0 for truck 4). This score being much lower than the optimal objective value for task D.2 shows that with the optimal routes it will not be possible to reach the highest emergency score. On the other hand, maximizing only for the emergency score produces unnecessarily long routes with trucks driving from far south to far north. In this last task, we will incorporate both objectives in our model, to see what the longest travel times for different emergency scores will be.
For this model, we start with the model from task D.2 and conduct the following changes.

New Sets and Indexes:
$\alpha \in I : \{1.00, 0.95, 0.90, 0.85, 0.80, 0.75\}$: values of alpha for which the model is solved.

New Parameters:
$T_{i,j}^{DIST}$: travel time from city i to city j (same as in D.1).
T_i^{LOAD}: loading time at city i, $i \in S$ (same as in D.1)
$EScoreMin$: minimum emergency score to be reached (alpha*1460).

New variables:
t_k: total time needed by truck k (same as in D.1).

The objective function is the same as in task D.1, minimizing the maximum travel time for any truck. We add eq. 50 from task D.1 to the model in D.2. Adjusting it for the set N, we get eq. 71:

$$t_k = \sum_{i \in C} \sum_{j \in C: j \neq i} \sum_{n \in N} y_{k,i,j,n} * T_{i,j}^{DIST} + \sum_{i \in S} \sum_{n \in N} x_{i,k,n} * T_i^{LOAD} \quad \forall k \in K \tag{71}$$

Furthermore, we have to make sure that the routes are determined to meet the minimum required emergency score:

$$\sum_{k \in K} \sum_{i \in C} \sum_{n \in N} u_{i,k,n} \cdot E_{i,n} \geq EScoreMin \tag{72}$$

All other constraints remain the same as formulated in task D.2.

In the .run-file in AMPL, we loop over set I, calculate the parameter $EScoreMin$ for every $\alpha \in I$, and solve the model for each of the iterations. The results we obtain are summarized in table 6.

As the heading for this section states, the data clearly shows the trade-off between minimized travel time and overall emergency score. To reach the maximum possible emergency score, we must accept an increase in the minimized longest travel time of almost 48%. However, this increase is exponentially reduced for lower values of alpha. The optimal trade-off between

Table 6: Emergency score and longest travel time per alpha

Alpha	Overall Emergency Score	Longest Time (hours)
100%	1,460	75.18
95%	1,390	61.18
90%	1,340	55.14
85%	1,300	53.04
80%	1,190	50.88
75%	1,190	50.88

the two objectives is probably at an alpha-value between 90% and 85%. At an alpha-value of 90% we still achieve a relatively high emergency score while accepting an increase in longest travel time of only around 5%. Our advice to the government of Momiss deciding on the best distribution of relief supplies would thus be to choose the alpha-value around 90%. This way we can ensure that supplies reach many of the most impacted cities first, while reaching every city in a reasonable amount of time.

References

Bowman, R. A. (2021). Developing optimal student plans of study. *INFORMS Journal on Applied Analytics*, *51*(6), 409–421.