CS111 Spring'24 ASSIGNMENT 1

Names: Luis Barrios, Andy Payan

Problem 1: Give an asymptotic estimate for the number $D(n)$ of "D"s printed by Algorithm PRINT_DS below. Your solution *must* consist of the following steps:

(a) First express $D(n)$ using the summation notation $\sum$.

(b) Next, give a closed-form expression[1] for $D(n)$.

(c) Finally, give the asymptotic value of $D(n)$ using the Θ-notation.

Show your work and include justification for each step.

Algorithm PRINT_DS (n : **integer**)
  for $i \leftarrow 3$ **to** $4n$ **do**
    for $j \leftarrow 1$ **to** $(i+2)^2$ **do** print("D")
  for $i \leftarrow 1$ **to** $n+1$ **do**
    for $j \leftarrow 3$ **to** $2i$ **do** print("D")

Note: If you need any summation formulas for this problem, you are allowed to look them up. You do not need to prove them, you can just state in the assignment when you use them.

Solution 1:

1. $\sum_{i=3}^{4n}(i+2)^2 + \sum_{i=1}^{n+1}(2i-2)$ (Two summations derived from each nested loop respectively)

2. $\sum_{i=3}^{4n}(i^2+4i+4) + \sum_{i=1}^{n+1}(2i-2)$ (Assessing the squared summation first)
$= \sum_{i=3}^{4n} i^2 + \sum_{i=3}^{4n} 4i + \sum_{i=3}^{4n} 4 + \sum_{i=1}^{n+1}(2i-2)$ (Summation properties to expand)
$= \sum_{i=3}^{4n} i^2 + 4\sum_{i=3}^{4n} i + \sum_{i=3}^{4n} 4 + \sum_{i=1}^{n+1}(2i-2)$
$= \sum_{i=3}^{4n} i^2 + 4\sum_{i=3}^{4n} i + \sum_{i=3}^{4n} 4 + \sum_{i=1}^{n+1}(2i-2)$
$= (\sum_{i=1}^{4n} i^2 - \sum_{i=1}^{2} i^2) + 4(\sum_{i=1}^{4n} i - \sum_{i=1}^{2} i) + (\sum_{i=1}^{4n} 4 - \sum_{i=1}^{2} 4) + \sum_{i=1}^{n+1}(2i-2)$
(Summation property to convert i=3 to i=1)
$= (\sum_{i=1}^{4n} i^2 - 5) + 4(\sum_{i=1}^{4n} i - 3) + (\sum_{i=1}^{4n} 4 - 8) + \sum_{i=1}^{n+1}(2i-2)$
(Summation rules to obtain integers from summations)
$= (\frac{4n(4n+1)(8n+1)}{6} - 5) + 4(\frac{4n(4n+1)}{2} - 3) + (16n-8) + \sum_{i=1}^{n+1}(2i-2)$
(Summation rules to find the summation equivalents)
$= (\frac{2n(4n+1)(8n+1)}{3} - 5) + 4(2n(4n+1) - 3) + (16n-8) + \sum_{i=1}^{n+1}(2i-2)$
$= (\frac{2n(4n+1)(8n+1)}{3} - 5) + 4(8n^2+2n-3) + (16n-8) + \sum_{i=1}^{n+1}(2i-2)$
$= (\frac{2n(4n+1)(8n+1)}{3} - 5) + (32n^2+8n-12) + (16n-8) + \sum_{i=1}^{n+1}(2i-2)$
$= (\frac{(8n^2+2n)(8n+1)}{3} - 5) + (32n^2+8n-12) + (16n-8) + \sum_{i=1}^{n+1}(2i-2)$
$= (\frac{64n^3+8n^2+16n^2+2n}{3} - 5) + (32n^2+8n-12) + (16n-8) + \sum_{i=1}^{n+1}(2i-2)$
$= (\frac{64n^3+8n^2+16n^2+2n}{3} - 5) + (32n^2+8n-12) + (16n-8) + \sum_{i=1}^{n+1}(2i-2)$
$= (\frac{64n^3+24n^2+2n}{3} - 5) + (32n^2+8n-12) + (16n-8) + \sum_{i=1}^{n+1}(2i-2)$
$= (\frac{64}{3}n^3 + 8n^2 + \frac{2}{3}n - 5) + (32n^2+8n-12) + (16n-8) + \sum_{i=1}^{n+1}(2i-2)$
$= (\frac{64}{3}n^2 + 40n^2 + \frac{74}{3}n - 25) + \sum_{i=1}^{n+1}(2i-2)$ (Combining like terms)
$= (\frac{64}{3}n^2 + 40n^2 + \frac{74}{3}n - 25) + (\sum_{i=1}^{n+1} 2i - \sum_{i=1}^{n+1} 2)$ (Assessing second summation)
$= (\frac{64}{3}n^2 + 40n^2 + \frac{74}{3}n - 25) + (2\sum_{i=1}^{n+1} i - \sum_{i=1}^{n+1} 2)$ (Summation rules to find summation equivalents)
$= (\frac{64}{3}n^2 + 40n^2 + \frac{74}{3}n - 25) + (2(\frac{(n+1)(n+2)}{2}) - (2n+2))$

[1] A closed-form expression is a formula that can be evaluated in some fixed number of arithmetic operations, independent of n. For example, $3n^5 + n - 1$ and $n2^n + 5n^2$ are closed-form expressions, but $\sum_{i=1}^{n} i^2$ is not, as it involves $n-1$ additions.

$= (\frac{64}{3}n^2 + 40n^2 + \frac{74}{3}n - 25) + (n^2 + 3n + 2 - 2n + 2)$
$= (\frac{64}{3}n^2 + 40n^2 + \frac{74}{3}n - 25) + (n^2 + 2)$

3. $(\frac{64}{3}n^3 + 41n^2 + \frac{77}{3}n - 25) = O(n^3)$

We can conclude that, just by the function being a polynomial, $\Theta(n^3)$ is the answer, as the function is our best estimate. However, if we want to prove it with the concept of Big O and Big Ω, we can simply show that $\frac{64}{3}n^3 + 41n^2 + \frac{77}{3}n - 25 \geq \frac{64}{3}n^3$. for n $\geq$ 1, we observe that c = $\frac{64}{3}$ and that the resultant it $\Omega(n^3)$. To prove Big O, we simply can show the inequality $\frac{64}{3}n^3 + 41n^2 + \frac{77}{3}n - 25 \leq \frac{64}{3}n^3 + 41n^3 + \frac{77}{3}n^3 - 25n^3$. when n $\geq$ 0. This can also be shown by $41n^2 + \frac{77}{3}n - 25 \leq 41n^2 + \frac{77}{3}n + 25n \leq 41n^3$. c is 41, n $\geq$ 2. Since we have O(n^3) and $\Omega(n^3)$, we can conclude this is $\Theta(n^3)$.

Problem 2: (a) Use properties of quadratic functions to prove that $5x^2 \geq (x+1)^2$ for all real $x \geq 1$.

(b) Use mathematical induction and the inequality from part (a) to prove that $3 \cdot 5^n \geq 4^{n+1} + n^2$ for all integers $n \geq 2$.

(c) Let $g(n) = 4^{n+1} + n^2$ and $h(n) = 5^n$. Using the inequality from part (b), prove that $g(n) = O(h(n))$. You need to give a rigorous proof derived directly from the definition of the O-notation, without using any theorems from class. (First, give a complete statement of the definition. Next, show how $g(n) = O(h(n))$ follows from this definition.)

Solution 2:

(a) $5x^2 \geq (x+1)^2$ when x $\geq$ 1
$5x^2 \geq (x^2 + 2x + 1)$
$4x^2 - 2x - 1 \geq 0$
Using the quadratic formula (A = 4, B = -2, C = -1)...
$\frac{-(-2) \pm \sqrt{(-2)^2 - 4(4)(-1)}}{2(4)}$
$\frac{2 \pm \sqrt{20}}{8}$
$\frac{2}{8} \pm 2\frac{\sqrt{5}}{8}$
$\frac{1}{4} \pm \frac{\sqrt{5}}{4}$

These are the roots. We can also simply test that, by plugging in 1 into $4x^2 - 2x - 1$, we can see that we obtain $1 \geq 0$. Therefore, this must be true for all $x \geq 1$. As we know, this is a parabola. Since we solved for the roots, as long as x isn't between $(\frac{1}{4} - \frac{\sqrt{5}}{4}, \frac{1}{4} + \frac{\sqrt{5}}{4})$ the value will be positive. Showing that x $\geq$ 1 by simply plugging in 1 further proves that the number will reach infinity as x increases.

(b) Using mathematical induction and the inequality from part (a) to prove $3 \cdot 5^n \geq 4^{n+1} + n^2$ for all integers $n \geq 2$, we start with proving the base case. Let's use $n = 2$.

$$3(5^2) \geq 4^3 + 2^2 = 75 \geq 68$$

We next need to include our base case. Our inductive hypothesis will be: n=k,

$$3 \cdot 5^{k+1} \geq 4^{k+2} + (k+1)^2$$

The inequality holds for k = 4, so now we move onto the induction step. where k = k+1.

$$3 \cdot 5^{k+1} \geq 4^{k+2} + (k+1)^2$$

$$3 \cdot 5 \cdot 5^k \geq 4^{k+2} + (k+1)^2$$

As...

$$3 \cdot 5^k = 4^{k+1} + k^2$$

We can continue with our induction by showing

$$5 \cdot 3 \cdot 5^k \geq 5 \cdot (4^{k+1} + k^2)$$

$$5 \cdot 3 \cdot 5^k \geq 5 \cdot 4^{k+1} + 5 \cdot k^2$$

We can continue by the fact that the inequality...

$$5 \cdot 4^{k+1} + 5 \cdot k^2 \geq 4 \cdot 4^{k+1} + 4 \cdot k^2$$

$$5 \cdot 3 \cdot 5^k \geq 4 \cdot 4^{k+1} + 4 \cdot k^2$$

$$5 \cdot 3 \cdot 5^k \geq 4 \cdot (4^{k+1} + k^2)$$

$$5 \cdot (3 \cdot 5^k) \geq 4 \cdot (3 \cdot 5^k)$$

From the inequality proven true from part A, we saw that $5x^2 \geq (x+1)^2$ for all $x \geq 1$, so using this fact, we can say the same thing for $5n^2 \geq (n+1)^2$ for all $n \geq 2$. We can clearly see that 5 is greater than 4, considering they both share other exact terms. Therefore, by induction, we can surely say that $3 \cdot 5^n \geq 4^{n+1} + n^2$ for all integers $n \geq 2$.

(c) The definition of O notation is a way to describe a function's growth rate. To be more specific, the O notation describes the growth rate of the upper bound of a function.

If we say the functions g(n) and h(n) to be real numbers, then lets say g(n) = O(h(n)) if and only if there are positive constants c and n0 such that $|g(n)| \leq c|h(n)|$ for all n $\geq$ n0.

$$g(n) = 4^{n+1} + n^2$$

$$h(n) = 5^n$$

$$3 \cdot 5^n \geq 4^{n+1} + n^2, \forall n \geq 2$$

$$c \cdot h(n) \geq g(n), \forall n \geq n0$$

$$c = 3, n0 = 2$$

Since the constants are positive, $g(n) \leq c \cdot h(n)$ where $g(n) = 4^{n+1} + n^2$ and $h(n) = 5^n$ for all $n \geq 2$. So, g(n) = O(h(n)). $4^{n+1} + n^2 = O(5^n)$.

Problem 3: Give asymptotic estimates, using the Θ-notation, for the following functions:

(a) $7n^4 + 5n^3 + 2n + 3$
(b) $n^2 \log^7 n + n^{2.5} \log^4 n + 5n^2 \log_4 n$
(c) $5n^3 \log^5 n + 4n^3\sqrt{n} + 3n^3 + 2n^2\sqrt{n}$
(d) $2n^{4.5} + n^3 \log^2 n + n \cdot (3.5)^n$
(e) $n^5 + n^3 \cdot 5^n + n^2 \cdot 7^n$

Justify your answer, using asymptotic relations between the basic reference functions: n^b, $\log n$, and c^n.

Solution 3:

a) $7n^4 + 5n^3 + 2n + 3$
O-notation(upper bound)

$$7n^4 + 5n^3 + 2n + 3 \leq 7n^4 + 5n^4 + 2n^4 + 3n^4$$

$$5n^3 + 2n + 3 \leq 5n^3 + 2n + 3n \leq 5n^4 = O(n^4)$$

$$O(n^4), C = 5; n \geq 2$$

The Ω-notation(lower bound)

$$7n^4 + 5n^3 + 2n + 3 \geq 7n^4$$

$$f(n) = \Omega(n^4)$$

$$\Omega(n^4), C = 7; n \geq 0$$

The asymptotic estimate is $\Theta(\mathrm{n}^4)$, since $\mathrm{f(n)=O(n^4)}$ and $\mathrm{f(n)=\Omega(n^4)}$.

b) $n^2\log^7 n + n^{2.5}\log^4 n + 5n^2\log_4 n$

$$n^2log^7n + n^{2.5}log^4n + 5n^2log_4n$$

$$n^2(log^7n + \sqrt{n}log^4n + 5n^2log_4n)$$

$$f(n) \geq n^{2.5}log^4n$$

$$f(n) = \Omega(n^{2.5}log^4n)$$

$$c = 1, n \geq 1$$

$$n^2log^7n = n^2(log^3n)(log^4n)$$

By theorem (b), we can convert the $log^3(n)$ to $\mathrm{O}(n^{1/2})$

$$O(n^2)O(n^{\frac{1}{2}})O(log^4n)$$

$$f(n) = O(n^{2.5}log^4n)$$

$$c = 1; n \geq 1$$

The asymptotic estimate is $\Theta(n^{2.5}log^4n)$, since $\mathrm{f(n)=O}(n^{2.5}log^4n)$ and $\mathrm{f(n)=\Omega}(n^{2.5}log^4n)$.

c) $5n^3\log^5 n + 4n^3\sqrt{n} + 3n^3 + 2n^2\sqrt{n}$

$$5n^3\log^5 n + 4n^{3.5} + 3n^3 + 2n^{2.5}$$

$$f(n) \geq 4n^{3.5}$$

$$f(n) = \Omega(n^{3.5})$$

$$c = 4; n \geq 1$$

$$n^{2.5} \leq (n^{3.5})$$

Already inherently true, so...

$$f(n) = O(n^{3.5})$$

$$n \geq 1; c = 1$$

The asymptotic estimate is $\Theta(n^{3.5})$, since $\mathrm{f(n)=O}(n^{3.5})$ and $\mathrm{f(n)=\Omega}(n^{3.5})$.

d) $2n^{4.5} + n^3 \log^2 n + n \cdot (3.5)^n$

$$n^{4.5} + n^3 \log^2 n + n \cdot (3.5)^n$$
$$f(n) \geq n(3.5^n)$$
$$f(n) \geq \Omega(n(3.5^n))$$
$$c = 1; n \geq 1$$
$$f(n) = O(n(3.5^n))$$

By theorem (c), we can convert $n^{3.5}$ to $O(3.5^n)$

$$= n \cdot n^{3.5} -> n^{3.5} -> O(3.5^n)$$
$$n^3 log^2 n$$

By theorem (b), we can convert the log into a term of our convenience.

$$n^3 -> O(3.5^n) log^2 n$$
$$log^2 n = O(n)$$
$$O(n) \cdot O(3.5^n) = O(n(3.5^n))$$

The asymptotic estimate is $\Theta(n(3.5^n))$, since f(n)=O$(n(3.5^n))$ and f(n)=$\Omega(n(3.5^n))$.

e) $n^5 + n^3 \cdot 5^n + n^2 \cdot 7^n$

$$f(n) \geq (n^2 \cdot 7^n)$$
$$= \Omega(n^2 \cdot 7^n)$$
$$c = 1, n \geq 0$$
$$f(n) = O(n^2 \cdot 7^n)$$

By theorem (c), we can turn n^3 into $O(7^n)$

$$n^5 = n^2 \cdot n^3 = n^2 O(7^n)$$
$$O(n^2) \cdot O(7^n) = O(n^2 \cdot 7^n)$$

The asymptotic estimate is $\Theta(n^2 \cdot 7^n)$, since f(n)=O$(n^2 \cdot 7^n)$ and f(n)=$\Omega(n^2 \cdot 7^n)$.

Academic integrity declaration. This assignment was a collaboration between Luis Barrios and Andy Payan. The only resources we had used throughout this assignment include TA office hours (with Biqian, Ezekiel, and Alice), searching up summation rules for reference, and reviewing slides and notes given from lecture and discussion.

Submission. To submit the homework, you need to upload the pdf file to Gradescope. If you submit with a partner, you need to put two names on the assignment and submit it as a group assignment.

Reminders. Remember that only LaTeX papers are accepted.