CS 111 ASSIGNMENT 3

Problem 1: a) Consider the following linear homogeneous recurrence relation: $R_n = 3R_{n-1} - 4R_{n-2}$. It is known that: $R_0 = 1$, $R_2 = 5$. Find R_3.

b) Determine the general solution of the recurrence equation if its characteristic equation has the following roots: 1, -3, -3, -3, 3, -5, -5.

c) Determine the general solution of the recurrence equation $A_n = 256A_{n-4}$.

d) Solve the following recurrence equation:

$$\begin{aligned} B_n &= -B_{n-1} + 5B_{n-2} - 3B_{n-3} \\ B_0 &= 0 \\ B_1 &= 6 \\ B_2 &= -4 \end{aligned}$$

Solution 1: a)

$$R_n - 3R_{n-1} + 4R_{n-2}$$

$$r^2 - 3r + 4$$

$$R_n - 3R_{n-1} + 4R_{n-2}$$

using the quadratic formula to find the roots, we find:

$$r = (\frac{3}{2} + \frac{\sqrt{7}}{2}i), (\frac{3}{2} - \frac{\sqrt{7}}{2}i)$$

$$R_n = \alpha_1(\frac{3}{2} + \frac{\sqrt{7}}{2}i)^n + \alpha_2(\frac{3}{2} - \frac{\sqrt{7}}{2}i)^n$$

$$R_o = 1$$

$$1 = \alpha_1(1) + \alpha_2(1)$$

$$5 = \alpha_1(\frac{3}{2} + \frac{\sqrt{7}}{2}i)^2 + \alpha_2(\frac{3}{2} - \frac{\sqrt{7}}{2}i)^2$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(\frac{3}{2})^2 + 2(\frac{3}{2})(\frac{\sqrt{7}}{2}i) + (\frac{\sqrt{7}}{2}i)^2$$

$$5 = \alpha_1(\frac{1}{2} + \frac{3\sqrt{7}}{2}i) + \alpha_2(\frac{1}{2} - \frac{3\sqrt{7}}{2}i)$$

$$1 = \alpha_1(1) + \alpha_2(1)$$

We now have a system of linear equations now:

$$-(\frac{1}{2} + \frac{3\sqrt{7}}{2}i) = \alpha_1(\frac{1}{2} + \frac{3\sqrt{7}}{2}i) + \alpha_2 - (\frac{1}{2} + \frac{3\sqrt{7}}{2}i)$$

$$(\frac{9}{2} - \frac{3\sqrt{7}}{2}i) = \alpha_1(0) + \alpha_2(-\frac{3\sqrt{7}}{2}i - \frac{3\sqrt{7}}{2}i)$$

$$(\frac{9}{2} - \frac{3\sqrt{7}}{2}i) = \alpha_2(-3\sqrt{7}i)$$

Now we find α_2:

$$\frac{\frac{9}{2}}{-3\sqrt{7}i} - \frac{\frac{3\sqrt{7}}{2}i}{-3\sqrt{7}i}$$

$$\frac{9}{-6\sqrt{7}i} + \frac{3\sqrt{7}i}{6\sqrt{7}i}$$

$$\alpha_2 = \frac{3}{-2\sqrt{7}i} + \frac{1}{2}$$

Now we can also solve for α_1:

$$1 = \alpha_1(1) + (\frac{1}{2} - \frac{3}{2\sqrt{7}i})$$

$$\alpha_1 = \frac{1}{2} + \frac{3}{2\sqrt{7}i}$$

Now we can finally solve for R_3

$$R_3 = \alpha_1(\frac{3}{2} + \frac{\sqrt{7}}{2}i)^3 + \alpha_2(\frac{3}{2} - \frac{\sqrt{7}}{2}i)^3$$

$$R_3 = 3$$

b)

$$(x-1)(x+3)(x+3)(x+3)(x-3)(x+5)(x+5)$$

$$(x-1)(x+3)^2(x-3)(x+5)^2$$

With this, we can write the general solution of the recurrence equation:

$$R_n = \alpha_1(1)^n + \alpha_2(-3)^n + \alpha_3 n(-3)^n + \alpha_4 n^2(-3)^n + \alpha_5(3)^n + +\alpha_6(-5)^n + \alpha_7 n(-5)^n$$

c)

$$A_n = 256A_{n-4}$$

$$A_n - 256A_{n-4} = 0$$

$$x^4 - 256$$

$$(x^2+16)(x^2-16)$$

$$(x+4)(x-4)(x^2+16)$$

With the use of the quadratic formula, we can find the roots for (x^2+16):
The roots are:

$$4, -4, 4i, -4i$$

This gives us our general solution, which is:

$$\alpha_1(4)^n + \alpha_2(-4)^n + \alpha_3(4i)^n + \alpha_4(-4i)^n$$

d)

$$B_n + B_{n-1} - 5B_{n-2} + 3B_{n-3} = 0$$

$$x^3 + x^2 - 5x + 3$$

We can use synthetic division to find the roots of this polynomial, the candidates are, 1,-1,3,-3. By doing this, we find that:

$$x = 1, (x-1)(x^2+2x-3)$$

$$, (x+3)(x-1)$$

$$x = -3, (x+3)(x^2-2x+1)$$

$$, (x-1)(x-1)$$

The roots we have found is: -3, 1, 1.

$$B_n = \alpha_1(1)^n + \alpha_2 n(1)^n + \alpha_3(-3)^n$$

$$0 = \alpha_1 + \alpha_2(0) + \alpha_3(1)$$

$$= \alpha_1 + \alpha_3$$

$$6 = \alpha_1(1) + \alpha_2(1) - \alpha_3(3)$$

$$-4 = \alpha_1(1) + \alpha_2(2) + \alpha_3(9)$$

Now we set-up our system of equations to find alpha 1, 2, 3.

$$0 = \alpha_1 + \alpha_2(0) + \alpha_3(1)$$

$$-6 = -\alpha_1(1) - \alpha_2 + \alpha_3(3)$$

$$-6 = -\alpha_2 + \alpha_3(4)$$

$$\alpha_2 = -12\alpha_3 - 10$$

lets substitute the alpha 2 we have here to solve for alpha 3.

$$-6 = -(-12\alpha_3 - 10) + 4\alpha_3$$

$$-6 = 12\alpha_3 + 4\alpha_3 + 10$$

$$-16 = 16\alpha_3$$

$$\alpha_3 = -1$$

$$0 = \alpha_1 + \alpha_3$$

$$0 = 1 - 1$$

$$\alpha_1 = 1$$

$$-6 = -1 - \alpha_2 - 3$$

$$-2 = -\alpha_2$$

$$\alpha_2 = 2$$

Now we have found alpha 1, 2, 3. so now we can plug them into our general solution:

$$B_n = (1)(1)^n + 2n(1)^n + (-1)(-3)^n$$

$$= 1 + 2n - (-3)^n$$

Problem 2: Solve the following recurrence equations:

a)

$$\begin{aligned} t_n &= 2t_{n-1} + t_{n-2} + 2^n \\ t_0 &= 0 \\ t_1 &= 4 \end{aligned}$$

b)

$$\begin{aligned} f_n &= -f_{n-1} + 5f_{n-2} - 3f_{n-3} + 2 \\ f_0 &= 0 \\ f_1 &= 3 \\ f_2 &= 2 \end{aligned}$$

Show your work (all steps: the associated homogeneous equation, the characteristic polynomial and its roots, the general solution of the homogeneous equation, computing a particular solution, the general solution of the non-homogeneous equation, using the initial conditions to compute the final solution.)

Solution 2: a)

$$2t_{n-1} + t_{n-2} + 2^n, t_0 = 0, t_1 = 4$$

We can use the quadratic formula to find the roots of this polynomial:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{2 \pm \sqrt{4 - 4(1)(-1)}}{2(1)}$$

$$Roots = (1 + \sqrt{2}), (1 - \sqrt{2})$$

With this, our general solution is:

$$t'n = \alpha_1(1 + \sqrt{2})^n + \alpha_2(1 - \sqrt{2})^n$$

Now we find alpha one and two:

$$t_0 = \alpha_1(1 + \sqrt{2})^0 + \alpha_2(1 - \sqrt{2})^0$$

$$t_0 = \alpha_1 + \alpha_2 = 0$$

$$t_1 = \alpha_1(1 + \sqrt{2})^1 + \alpha_2(1 - \sqrt{2})^1 = 4$$

$$t''n = \beta \cdot 2^n$$

$$\beta \cdot 2^n = \beta \cdot 2 \cdot 2^{n-1} + \beta \cdot 2^{n-2} + 2^n$$

$$\beta \cdot 2^2 = \beta \cdot 2 \cdot 2^1 + \beta \cdot 2^0 + 2^2$$

$$4\beta = 4\beta + \beta + 4$$

$$-\beta = 4$$

$$\beta = -4$$

Now lets find alpha 1 and 2.

$$0 = \alpha_1 + \alpha_2 - 4$$

$$4 = \alpha_1(1 + \sqrt{2}) + \alpha_2(1 - \sqrt{2}) - 4$$

$$\alpha_1 = -\alpha_2 + 4$$
$$4 = (-\alpha_2 + 4)(1+\sqrt{2}) + \alpha_2(1-\sqrt{2}) - 4$$
$$4 = -\alpha_2 - \alpha_2\sqrt{2} + 4 + 4\sqrt{2} + \alpha_2 - \alpha_2\sqrt{2} - 4$$
$$4 = -2\sqrt{2}\alpha_2 + 4\sqrt{2}$$
$$\frac{4-4\sqrt{2}}{-2\sqrt{2}} = \alpha_2$$
$$\alpha_2 = 2 - \sqrt{2}$$

$$\alpha_1 = -(2-\sqrt{2}) + 4$$
$$\alpha_1 = \sqrt{2} - 2 + 4$$
$$\alpha_1 = 2 + \sqrt{2}$$

With this, our particular solution becomes:

$$t'n = (2+\sqrt{2})(1+\sqrt{2})^n + (2-\sqrt{2})(1-\sqrt{2})^n - 4$$

b)

$$-f_{n-1} + 5f_{n-2} - 3f_{n-3} + 2$$
$$f_0 = 0, f_1 = 3, f_2 = 2$$
$$f_n + f_{n-1} - 5f_{n-2} + 3f_{n-3} = 0$$
$$x^3 + x^2 - 5x + 3$$

The candidates for the roots of this polynomial are, ± 3 and ± 1. We use the multiples of 3 while having it divided by the coefficient of the highest degree term, which is one from x^3. With using these, we can use the synthetic division method. With these we find:

$$(x-1)(x^2+2x-3)$$

$$(x+3)(x-1)$$
$$x = -3, 1, 1$$

With this, our general solution takes the form:

$$f_n = \alpha_1(-3)^n + \alpha_2(1)^n + \alpha_3 n(1)^n$$

Now lets find Beta:

$$f''n = b \cdot n^2$$
$$b \cdot n^2 = -b(n-1)^2 + 5b(n-2)^2 - 3b(n-3)^2 + 2$$
$$(n-1)(n-1) = n^2 - 2n + 1$$
$$(n-2)(n-2) = n^2 - 4n + 4$$
$$(n-3)(n-3) = n^2 - 6n + 9$$
$$bn^2 = -bn^2 + 2bn - b + 5bn^2 - 20bn + 20b - 3bn^2 + 18bn - 27b + 2$$
$$bn^2 = bn^2 - 0bn - 8b + 2$$

$$0 = -8b + 2$$
$$8b = 2$$
$$b = 1/4$$
$$f"n = \frac{1}{4}n^2$$

$$f_n = \alpha_1(-3)^n + \alpha_2(1)^n + \alpha_3 n(1)^n + \frac{1}{4}n^2$$
$$0 = \alpha_1 + \alpha_2$$
$$3 = -3\alpha_1 + \alpha_2 + \alpha_3 + \frac{1}{4}$$
$$2 = 9\alpha_1 + \alpha_2 + 2\alpha_3 + 1$$

With this we can solve for alpha 1, 2, and 3.

$$\alpha_1 = -\alpha_2$$
$$-3(-\alpha_2) + \alpha_2 + \alpha_3 + \frac{1}{4} = 3$$
$$9(-\alpha_2) + \alpha_2 + 2\alpha_3 + 1 = 2$$
$$4\alpha_2 + \alpha_3 + \frac{1}{4} = 3$$
$$-8\alpha_2 + 2\alpha_3 + 1 = 2$$

$$8\alpha_2 + 2\alpha_3 + \frac{2}{4} = 6$$
$$-8\alpha_2 + 2\alpha_3 + 1 = 2$$

$$4\alpha_3 + \frac{6}{4} = 8$$
$$4\alpha 3 = \frac{26}{4}$$
$$\alpha_3 = \frac{13}{8}$$

$$4\alpha_2 + \frac{13}{8} + \frac{1}{4} = 3$$
$$4\alpha_2 = \frac{9}{8}$$
$$\alpha_2 = \frac{9}{32}$$

$$\alpha_1 = -\alpha_2$$
$$\alpha_1 = -\frac{9}{32}$$

With this our final particular solution is:

$$f_n = -\frac{9}{32}(-3)^n + \frac{9}{32}(1)^n + \frac{13}{8}n(1)^n + \frac{1}{4}n^2$$

Problem 3: We want to tile an $n \times 1$ strip with 1×1 tiles that are green (G), blue (B), and red (R), 2×1 purple (P) and 2×1 orange (O) tiles. Blue cannot be next to purple and orange tiles, and there should be no two purple or orange or no three blue tiles in a row (for ex., BBPOPORGP and RRBOBP is allowed, but BBBOGR, PBPGP, OOBBPR and OOPBPRO are not). Give a formula for the number of such tilings. Your solution must include a recurrence equation (with initial conditions!), and a full justification. You do not need to solve it.

Solution 3:

From the given constraints of number 3, such that blue cannot be next to purple or orange, and there should be no two purple/orange in a row, as well as 3 blue tiles in the row. By breaking down these constraints to find the possible combinations, we use a tree to find a general solution:

$$T_{n-1} + T_{n-1} + T_{n-2} + T_{n-2} + T_{n-3} + T_{n-3} + T_{n-3} + T_{n-3} + T_{n-3} + T_{n-3}$$

We can further simplify:

$$2T_{n-1} + 2T_{n-2} + 6T_{n-3}$$

$$S_0 = 1$$

We concluded so, as there is only one case in which 0 tiles exist.

$$S_1 = 3;$$

$S_1 = 3$, as there are only 3 1x1 tiles to choose for a 1x1 tile case.

$$S_2 = 11$$

$S_2 = 11$, as there are 2 2x1 tiles, specifically P and O, along with 9 1x1 tiles. We know this because we can combine these elements 9 different ways, as well as have the two 2x1 tiles, it brings our condition of S_2 to 11.'

$$S_3 = 26 + 8 = 34$$

$S_3 = 34$, as there are 3^3 combinations, which makes 27, but since we cannot have BBB, this makes the combinations 26. We can also include the combinations from P and O that do not violate the constraints given, which is a total of 8. These 8 include PG, OG, PR, OR, RP, RO, GP, and GO. This means our $S_3 = 34$.

Academic integrity declaration. This assignment was a collaboration between **Luis Barrios** and **Andy Payan**. The only resources we had used throughout this assignment include TA office hours (with Biqian) and reviewing slides and notes given from lecture and discussion. As well, we watched videos to review on how to solve linear equations.

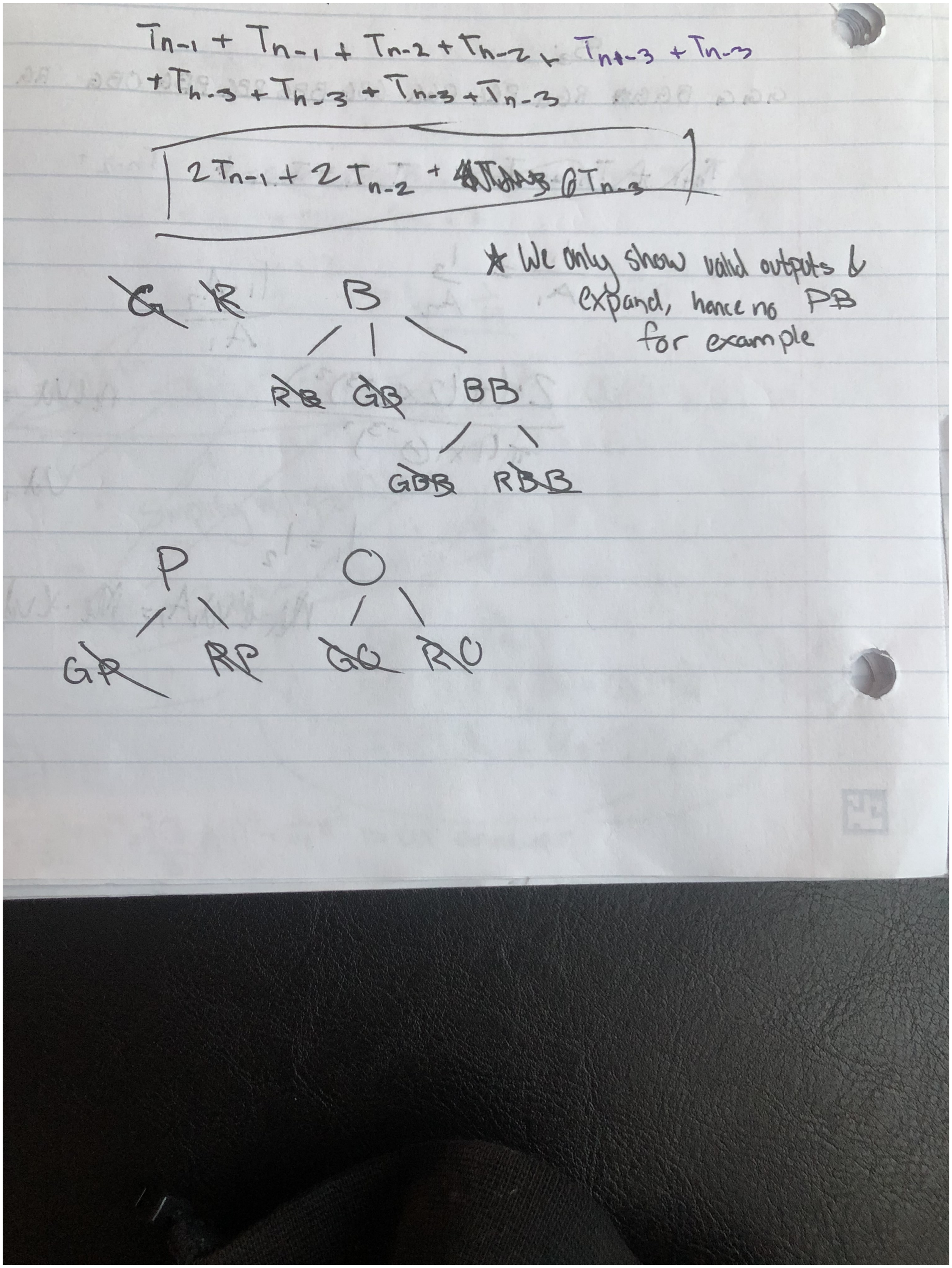

Figure 1: Trees for number 3