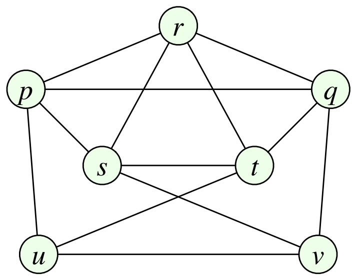
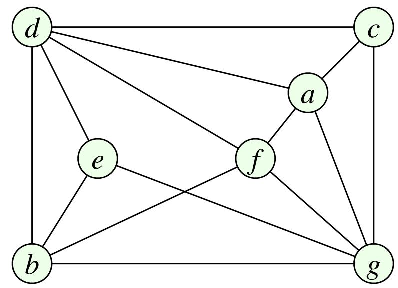
CS 111 ASSIGNMENT 5

Problem 1. Determine whether the two graphs below are planar or not. To show planarity, give a planar embedding. To show that a graph is not planar, use Kuratowski's theorem.

Graph G:



Graph H:



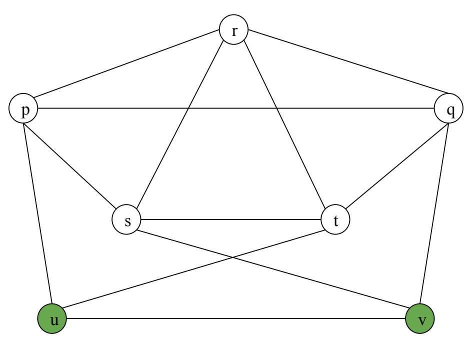
Solution 1:

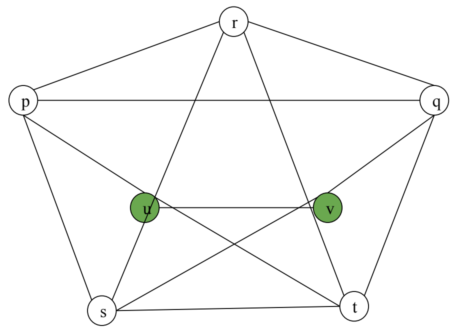
If G is a connected planar graph with $n \geq 3, m \leq 3n - 6$, we can conclude that if $(m > 3n - 6)$, the graph must be non-planar. However, if it this theorem doesn't work for said graph, we must draw a planar embedding to prove whether or not the graph is planar. Given that there are 13 edges and 7 vertices...

$$13 > 3(7) - 6$$

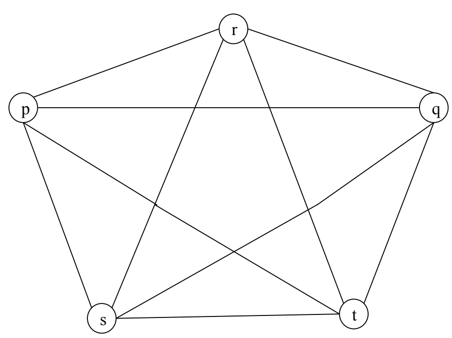
$$13 > 15$$

This is not true, so we cannot conclude that it is non-planar.

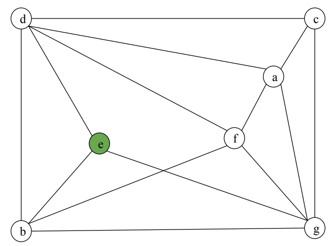


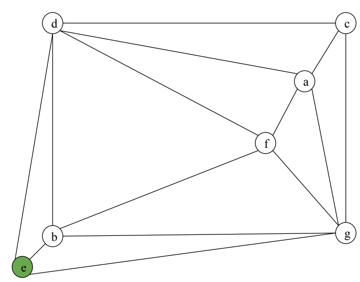


Above is an attempt at planar embedding to show that graph G is planar. However, we were able to rearrange the given diagram to form a K_5 inside. By Kuratowski's theorem, which asserts that a graph is non-planar if K_5 or $K_{3,3}$ is a sub-graph of it, we can conclude that this graph is non-planar.



Graph H: By using Euler's theorem to deduce whether or not the graph is non-planar ($m > 3n - 6$), we see that $14 > 15$, which isn't true so we cannot conclude instantly that the graph in non-planar. To advance in this problem, we must draw a planar embedding.

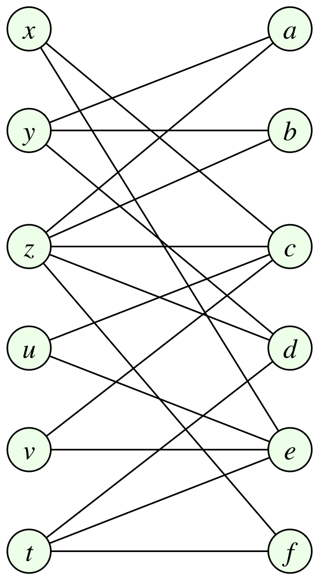




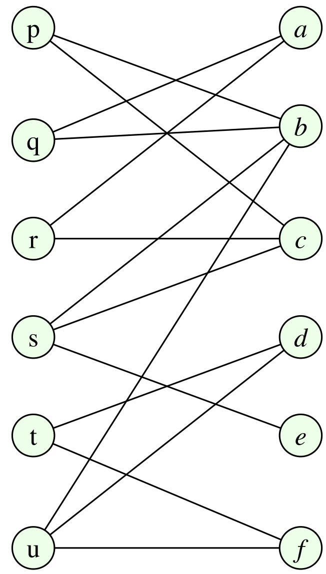
With this embedding, we can say that Graph H is planar.

Problem 2. You are given two bipartite graphs G and H below. For each graph determine whether it has a perfect matching. Justify your answer, either by listing the edges that are in the matching or using Hall's Theorem to show that the graph does not have a perfect matching.

graph G



graph H



Solution 2:

There is a perfect matching if $|L| = |R|$ and $|N(x) \geq |x||$ for each $x \subseteq L$.

Graph H's perfect matching:

$$(e, s), (d.t).(f, u), (p, b), (a, q)(c, r)$$

Graph H both has an equal cardinality for its L and R, hence meaning that it works. Going through the graph, we also cannot locate any possibilities in which there are more neighbors of vertices than the set of vertices themselves. With Hall's theorem in mind, we can conclude that there exists a perfect matching. These are matchings we found that make this graph a perfect matching. With this, we can see that Graph H has perfect matching.

Graph G's perfect matching: $|x| > |N(x)|$ means no perfect matching.

$$x = (d, b, a, f)$$

$$N(x) = (t, z, y)$$

$$|x| > |N(x)|$$

$$4 > 3$$

By Hall's theorem, graph G doesn't work; there exists no perfect matching.

Problem 3. (a) For each degree sequence below, determine whether there is a graph with 6 vertices where vertices have these degrees. If a graph exists, (i) draw it, (ii) find the chromatic number and justify. If it doesn't, justify that it doesn't exist.

Note. To give a justification for the chromatic number, you need to give a coloring and explain why it's not possible to use fewer colors.

(a1) $5, 4, 3, 2, 2, 1$.
(a2) $5, 4, 4, 3, 3, 1$.
(a3) $4, 4, 4, 3, 3, 2$.

(b) For each degree sequence below, determine whether there is a planar graph with 6 vertices where vertices have these degrees. If a graph exists, (i) draw it, (ii) find the chromatic number and justify.

(b1) $5, 5, 4, 4, 4, 2$.
(b2) $3, 3, 3, 3, 3, 3$.

Solution 3:

To find a graph, if you add all vertex degrees, it should be even. As well, the highest degree out of all nodes must be less than the total amount of nodes.

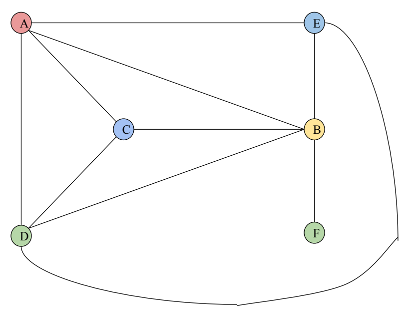
(a1).

$$5 + 4 + 2 + 2 + 3 + 1 = 17$$

This sum is not even, so therefore, a planar graph does not exist.

(a2).

$$5 + 3 + 3 + 3 + 3 + 1 = 20$$



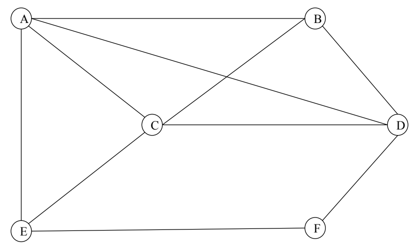
This sum is even and $5 < 6$, so a graph exists.
To find a max bound for a graph with these qualities, we know by theorem that this graph can be colored with at most 4 colors since it is planar. So, for here we need AT MOST 4 colors.
To find a minimum bound, we observe that our vertices can create a K_4 sub-graph. As this is the highest K graph we can create, we know we need AT LEAST 4 colors. Hence, the chromatic number for said graph is 4.

Conclusion: We know by the 4-color-theorem and by an existing K_4 sub-graph that the chromatic number is 4.

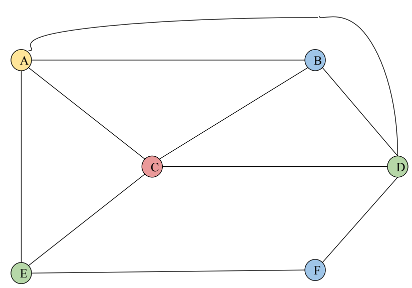
(a3).

$$4 + 4 + 4 + 3 + 3 + 2 = 20, 4 < 6$$

With this, we have sufficient proof that we can create a graph:



We can always assert that the max degree is $d+1$, so $4+1=5$, hence creating a max bound. However, as this is a planar graph, we know at most we simply need 4 colors.
Min Bound: We can form a K_4 graph through the vertices (a, c, b, d), which indicates that there the minimum amount of colors (chromatic color) is 4. The diagram is shown below.



(b1):

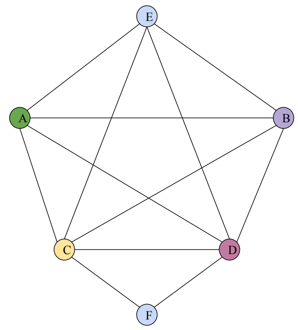
$$5+5+4+4+2+4=24; 5<6$$

Since 24 is even and d is less than the total amount of nodes, the graph works.

$$m > 3n-6$$

$$n=6, m=12 => 12 > 18-6 => 12 > 12$$

Since 12 isn't greater than 12, this doesn't work, we can't conclude it's non-planar, we try planar embedding:



The graph contains K_5, which is not planar and requires 5 colors to be graphed. 5 vertices and 10 edges:

$$10 > 3 * 5 - 6 = 9$$

$$10 > 9$$

With this, there exists another proof that the graph is non-planar.
Since K_5 is formed, we need a minimum of 5 colors to color the graph. The max degree is $d + 1 = 5 + 1 = 6$, being our max bound.
As shown by the minimum bound, the chromatic color is 5. We can disregard the max.

(b2):

$$3 < 6$$

$$3 + 3 + 3 + 3 + 3 + 3 = 18$$

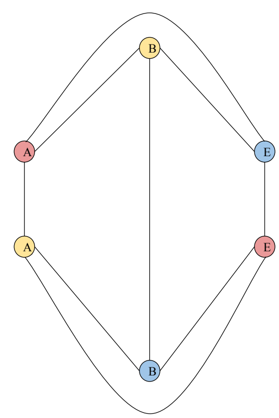
A graph can be formed as both conditions are fulfilled. Highest degree is $d + 1 = 4$.

$$m > 3n - 6$$

$$18/2 = 9$$

$$9 > 3(6) - 6$$

$9 > 12$ doesn't work, so we look for a planar embedding to exist and meet the requirements:
This has the $k_{3,3}$, which is non-planar. However, we are searching for a planar embedding in this case. So, despite knowing that $k_{3,3}$ is bipartite and is in these vertices, we disregard this and only look for the coloring for planar embeddings of these vertices. Below is a variation of how these vertices and edges can form a planar graph.



With this planar embedding, we see that there exists a K_3 graph, which would be the minimum bound for the coloring. As that is the minimum bound, it will also be the chromatic color.

Academic integrity declaration. This homework was a collaboration between Andy Payan and Luis Barrios. The only resources we used for this assignment were simply the lecture/discussion slides and office hours with Biqian.

Submission. To submit the homework, you need to upload the pdf file to Gradescope. If you submit with a partner, you need to put two names on the assignment and submit it as a group assignment. Remember that only LaTeX papers are accepted.