Homework 3: Geodesics, Distance, and Metric Embedding

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Problem 1

Let $\gamma : [0, L] \to \mathcal{M}$ be an arc lenght-parametrized curve on an orientable surface $\mathcal{M} \subset \mathbb{R}^3$. Let $\mathbf{v}$ be a tangent vector field defined along the curve, i.e., $\mathbf{v}(s) \in T_{\gamma(s)}\mathcal{M}$. If,

$$\underset{T_{\gamma(s)}\mathcal{M}}{\text{proj}}\, \mathbf{v}'(s) = \mathbf{v}'(s) - \mathbf{n}(\gamma(s))\mathbf{n}(\gamma(s)) \cdot \mathbf{v}'(s) = 0$$

then we say that $\mathbf{v}$ is parallel.

a) Show that the unit tangent vector field of a geodesic is parallel.

Proof - Let $\varphi : [0, 1] \to \mathcal{M}$ be a geodesic parameterized by arc length, then

$$\underset{T_{\varphi(s)}\mathcal{M}}{\text{proj}}\, \varphi''(s) = 0$$

Its tangent vector field is $T(s) = \varphi'(s)$. For $T(s)$ to be parallel it requires to $\text{proj}T'(s) = 0$, since $T'(s) = \varphi''(s)$ the proof is complete.

b) Suppose $\mathbf{u}$ and $\mathbf{v}$ are parallel fields along γ. Show that $\mathbf{u} \cdot \mathbf{v}$ and $\|\mathbf{u}\|_2$ are constant.

Proof - Let

$$\underset{T_{\gamma(s)}\mathcal{M}}{\text{proj}}\, \mathbf{u}' = 0 \qquad \underset{T_{\gamma(s)}\mathcal{M}}{\text{proj}}\, \mathbf{v}' = 0$$

Let $\mathbf{u} \cdot \mathbf{v} = c$ iff $\mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}' = 0$, let's prove the later. Since $\mathbf{u} \in T_\gamma\mathcal{M}$ and $\mathbf{u}' \perp T_\gamma\mathcal{M}$ (similarly for $\mathbf{v}$), we have

$$\mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}' = \cancelto{0}{\mathbf{u}' \cdot \mathbf{v}} + \cancelto{0}{\mathbf{u} \cdot \mathbf{v}'} = 0$$

Note: Equation 1 is a first-order ODE whose solution is unique given an initial condition $\mathbf{v}(0)$. We can define the parallel transport operator P_γ by $P_\gamma \mathbf{v}(0) = \mathbf{v}(L)$, where $\mathbf{v}$ is the unique parallel field along γ with initial condition $\mathbf{v}(0)$.

c) Use the result from *b)* to argue that parallel transport around a closed loop (known as holonomy) amounts to a rotation in the tangent plane.

Proof - Let a closed curve $\gamma : [0, L] \to \mathcal{M}$ such that $\gamma(0) = \gamma(L) = p$. Given a vector $\mathbf{u}_0 \in T_p\mathcal{M}$ its parallel transport is:

$$P_\gamma \mathbf{u}_0 = \mathbf{u}_L$$

And thus, both $\mathbf{u}_0, \mathbf{u}_L \in T_p\mathcal{M}$. Thus, since $\mathbf{u}_0 \cdot \mathbf{u}_L \equiv c$ and $\|\mathbf{u}_0\|_2 = \|\mathbf{u}_1\|_2$.

d) Let $\mathbf{v}$ be parallel along γ. Let $\theta(s)$ be the angle from $\gamma'(s)$ to $\mathbf{v}(s)$, measured counterclockwise about the surface normal $\mathbf{n}$. Show that,

$$\theta'(s) = -\kappa_g$$

where κ_g is the geodesic curvature of γ, defined by projection of the second derivative of γ into the tangent plane of the surface:

$$\underset{T_\gamma\mathcal{M}}{\mathrm{proj}} \gamma''(s) = \kappa_g(\mathbf{n} \times \gamma'(s))$$

Proof - Let $\gamma(s)$ be parameterized by arc length and $\gamma'(s), \mathbf{v}(s) \in T_{\gamma(s)}\mathcal{M}$ with $\mathbf{v}'(s) \perp T_{\gamma(s)}\mathcal{M}$, then we can represent $\mathbf{v}$ in terms of the orthonormal basis of $T_\gamma\mathcal{M}$, $(\gamma', \mathbf{n} \times \gamma')$, with $\|\gamma'\|_2 = 1$, then

$$\mathbf{v}(s) = \alpha \cos\theta(s)\gamma'(s) + \alpha \sin\theta(s)\mathbf{n}(s) \times \gamma'(s)$$

Let's differentiate $\gamma'(s) \cdot \mathbf{v}(s) = \|\mathbf{v}\|_2 \cos\theta(s)$,

$$\gamma''(s) \cdot \mathbf{v}(s) + \gamma'(s) \cdot \mathbf{v}'(s) = -\alpha\theta'(s)\sin\theta(s)$$

Since $\mathbf{v}' \perp T_\gamma\mathcal{M}$ and $\gamma' \in T_\gamma\mathcal{M}$,

$$\begin{aligned}
\gamma''(s) \cdot \mathbf{v}(s) &= -\alpha\theta' \sin\theta \\
\underset{T_\gamma\mathcal{M}}{\mathrm{proj}} \gamma''(s) \cdot \mathbf{v}(s) &= \\
\kappa_g(\mathbf{n} \times \gamma') \cdot \mathbf{v}(s) &= \\
\kappa_g(\mathbf{n} \times \gamma') \cdot \alpha(\cos\theta\gamma' + \sin\theta\mathbf{n} \times \gamma') &= \\
\alpha\kappa_g \sin\theta &= -\alpha\theta' \sin\theta \\
\kappa_g &= -\theta'
\end{aligned}$$