

Homework 3: Geodesics, Distance, and Metric Embedding

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Problem 1

Let $\gamma : [0, L] \rightarrow \mathcal{M}$ be an arc lenght-parametrized curve on an orientable surface $\mathcal{M} \subset \mathbb{R}^3$. Let \mathbf{v} be a tangent vector field defined along the curve, i.e., $\mathbf{v}(s) \in T_{\gamma(s)}\mathcal{M}$. If,

$$\operatorname{proj}_{T_{\gamma(s)}\mathcal{M}} \mathbf{v}'(s) = \mathbf{v}'(s) - \mathbf{n}(\gamma(s))\mathbf{n}(\gamma(s)) \cdot \mathbf{v}'(s) = 0$$

then we say that \mathbf{v} is parallel.

a) Show that the unit tangent vector field of a geodesic is parallel.

Proof - Let $\varphi : [0, 1] \rightarrow \mathcal{M}$ be a geodesic parameterized by arc length, then

$$\operatorname{proj}_{T_{\varphi(s)}\mathcal{M}} \varphi''(s) = 0$$

Its tangent vector field is $T(s) = \varphi'(s)$. For $T(s)$ to be parallel it requires to $\operatorname{proj} T'(s) = 0$, since $T'(s) = \varphi''(s)$ the proof is complete.

b) Suppose \mathbf{u} and \mathbf{v} are parallel fields along γ . Show that $\mathbf{u} \cdot \mathbf{v}$ and $\|\mathbf{u}\|_2$ are constant.

Proof - Let

$$\operatorname{proj}_{T_{\gamma(s)}\mathcal{M}} \mathbf{u}' = 0 \quad \operatorname{proj}_{T_{\gamma(s)}\mathcal{M}} \mathbf{v}' = 0$$

Let $\mathbf{u} \cdot \mathbf{v} = c$ iff $\mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}' = 0$, let's prove the later. Since $\mathbf{u} \in T_\gamma\mathcal{M}$ and $\mathbf{u}' \perp T_\gamma\mathcal{M}$ (similarly for \mathbf{v}), we have

$$\mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}' = \overrightarrow{\mathbf{u}' \cdot \mathbf{v}} + \overrightarrow{\mathbf{u} \cdot \mathbf{v}'} = 0$$

Note: Equation 1 is a first-order ODE whose solution is unique given an initial condition $\mathbf{v}(0)$. We can define the parallel transport operator P_γ by $P_\gamma \mathbf{v}(0) = \mathbf{v}(L)$, where \mathbf{v} is the unique parallel field along γ with initial condition $\mathbf{v}(0)$.

c) Use the result from b) to argue that parallel transport around a closed loop (known as holonomy) amounts to a rotation in the tangent plane.

Proof - Let a closed curve $\gamma : [0, L] \rightarrow \mathcal{M}$ such that $\gamma(0) = \gamma(L) = p$. Given a vector $\mathbf{u}_0 \in T_p \mathcal{M}$ its parallel transport is:

$$P_\gamma \mathbf{u}_0 = \mathbf{u}_L$$

And thus, both $\mathbf{u}_0, \mathbf{u}_L \in T_p \mathcal{M}$. Thus, since for any $\mathbf{v} \cdot \mathbf{u} \equiv c$, i.e., the angle is preserved, and the alternatively write the dot product as,

$$(P_\gamma \mathbf{u}) \cdot (P_\gamma \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$$

Where $P_\gamma \in SO(2)$ serves as rotation of degree two along $\mathbf{n}(\gamma(0))$.

d) Let \mathbf{v} be parallel along γ . Let $\theta(s)$ be the angle from $\gamma'(s)$ to $\mathbf{v}(s)$, measured counterclockwise about the surface normal \mathbf{n} . Show that,

$$\theta'(s) = -\kappa_g$$

where κ_g is the geodesic curvature of γ , defined by projection of the second derivative of γ into the tangent plane of the surface:

$$\underset{T_\gamma \mathcal{M}}{\text{proj}} \gamma''(s) = \kappa_g (\mathbf{n} \times \gamma'(s))$$

Proof - Let $\gamma(s)$ be parameterized by arc length and $\gamma'(s), \mathbf{v}(s) \in T_{\gamma(s)} \mathcal{M}$ with $\mathbf{v}'(s) \perp T_{\gamma(s)} \mathcal{M}$, then we can represent \mathbf{v} in terms of the orthonormal basis of $T_\gamma \mathcal{M}$, $(\gamma', \mathbf{n} \times \gamma')$, with $\|\gamma'\|_2 = 1$, then

$$\mathbf{v}(s) = \alpha \cos \theta(s) \gamma'(s) + \alpha \sin \theta(s) \mathbf{n}(s) \times \gamma'(s)$$

Let's differentiate $\gamma'(s) \cdot \mathbf{v}(s) = \|\mathbf{v}\|_2 \cos \theta(s)$,

$$\gamma''(s) \cdot \mathbf{v}(s) + \gamma'(s) \cdot \mathbf{v}'(s) = -\alpha \theta'(s) \sin \theta(s)$$

Since $\mathbf{v}' \perp T_\gamma \mathcal{M}$ and $\gamma' \in T_\gamma \mathcal{M}$,

$$\begin{aligned}\gamma''(s) \cdot \mathbf{v}(s) &= -\alpha\theta' \sin \theta \\ \text{proj}_{T_\gamma \mathcal{M}} \gamma''(s) \cdot \mathbf{v}(s) &= \\ \kappa_g(\mathbf{n} \times \gamma') \cdot \mathbf{v}(s) &= \\ \kappa_g(\mathbf{n} \times \gamma') \cdot \alpha(\cos \theta \gamma' + \sin \theta \mathbf{n} \times \gamma') &= \\ \alpha \kappa_g \sin \theta &= -\alpha\theta' \sin \theta \\ \kappa_g &= -\theta'\end{aligned}$$

e) A geodesic polygon is a polygon formed from geodesic segments. Show that parallel transporting a vector around a geodesic polygon rotates it by the angle $\sum(\pi - \alpha_i)$, where α_i are the interior angles of the polygon.

Proof - Let a geodesic polygon be described by $\{\gamma_i : [0, L_i] \rightarrow \mathcal{M}\}_i^N$, such that $\gamma_i(L_i) = \gamma_{i+1}(0)$ for $i = 1, \dots, N-1$, and $\gamma_N(L_N) = \gamma_1(0)$. Thus, we can measure their exterior angles between γ_i and γ_{i+1} as the exterior angle (β_i) between their tangent vectors,

$$\cos \beta_i = \gamma'_i(L_i) \cdot \gamma'_{i+1}(0), \quad i = 1, \dots, N-1$$

since $\mathbf{v}_i = \gamma'_i$ is parallel we have

$$c_i = \mathbf{v}_i \cdot \mathbf{v}_{i+1}$$

Let transport $\mathbf{u}(0) \in T_{\gamma_1(0)} \mathcal{M}$, $P_{\gamma_1} \mathbf{u}(0) = \mathbf{u}(L_1)$, $\mathbf{u}(s) \cdot \mathbf{v}_1(s)$ is constant along γ_1 . Since $\mathbf{v}_1 \cdot \mathbf{v}_2 = \cos \beta_1$ we have $\mathbf{u}(L_1 + L_2) \cdot \mathbf{v}_2(L_2) = \cos(\beta_1 + \beta_2)$. Following this argument, after $P_\gamma = P_{\gamma_1} \circ P_{\gamma_2} \circ \dots \circ P_{\gamma_N}$ we have a total rotation of $\sum \beta_i$ of exterior angles. Corresponding to $\sum(\pi - \alpha_i)$ in interior angles.

Problem 2

Show the existence of low-distortion embeddings of arbitrary finite metric spaces.

Definition Let (X, d_X) and (Y, d_Y) be a metric spaces and $f : X \rightarrow Y$ a map between them. Distortion is the smallest positive ρ such that for some c and $\forall x_1, x_2 \in X$

$$cd_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq \rho cd_X(x_1, x_2)$$

Theorem (Bourgain). For any finite metric space (X, d) of size $|X| = n$, there is an embedding $f : X \rightarrow \mathbb{R}^m$ with $m \in \Theta(\log^2 n)$ such that

$$|f_i(x) - f_i(y)| \leq d(x, y) \quad \|f(x) - f(y)\|_1 \in \Omega((\log n)d(x, y))$$

Moreover, the distortion is $O(\log n)$ for any ℓ_p metric on \mathbb{R}^m , $p \geq 1$.

a) Using Hölder's inequality, show that ℓ_1 previous results imply the conclusion for all p .

Proof - Hölder's inequality says, for any $p, q \in [1, \infty]$ with $1/p + 1/q = 1$, for any $f, g \in X$,

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Let $\mathbf{v} = f(x) - f(y) \in \mathbb{R}^m$, thanks to Hölder's inequality, we have $q = p/(p-1)$, and

$$\begin{aligned} \|\mathbf{v}\|_1 &= \|\mathbf{v}\mathbf{1}\|_1 < \|v\|_p \|\mathbf{1}\|_q \\ &= \|v\|_p \left(\sum_i^m |1|^q \right)^{\frac{1}{q}} \\ &= \|\mathbf{v}\|_p m^{\frac{1}{q}} \\ \|\mathbf{v}\|_1 &\leq m^{1-1/p} \|\mathbf{v}\|_p \end{aligned}$$

Also, in general, $\|\mathbf{x}\|_p \leq \dots \leq \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1$ for any finite \mathbf{x} . Thus, $\|\mathbf{v}\|_p \leq \|\mathbf{v}\|_1$, and we can bound $\|\mathbf{v}\|_p$ by an upper and lower limit.

$$m^{\frac{1}{p}-1} \|\mathbf{v}\|_1 \leq \|\mathbf{v}\|_p \leq \|\mathbf{v}\|_1$$

Now, using Bourgain theorem we have that there exists some $A > 0$, such that,

$$|v_i| \leq d(x, y) \quad A \log n d(x, y) \leq \|\mathbf{v}\|_1$$

Taking the first property, we can define another upperbound for $\|\mathbf{v}\|_p$ as,

$$\|\mathbf{v}\|_p = \left(\sum_i^m |v_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_i^m d^p(x, y) \right)^{\frac{1}{p}} = m^{\frac{1}{p}} d(x, y)$$

and, taking the second property, we can conclude that

$$m^{\frac{1}{p}-1} A \log n d(x, y) \leq \|\mathbf{v}\|_p \leq m^{\frac{1}{p}} d(x, y)$$

With $c = m^{\frac{1}{p}-1} A \log n$, the distortion satisfies:

$$\rho \leq \frac{m}{A \log n}$$

Since $m = \Theta(\log^2 n)$, then $\rho = O(\log n)$.

b) The map f can be built up from the distances themselves. Show that if S_i is any nonempty subset of X and

$$f_i(x) := d(x, S_i) := \min_{y \in S_i} d(x, y),$$

then $|f_i(x) - f_i(y)| \leq d(x, y)$ holds.

Proof - Let $x, y \in X$, and $S \subseteq X$ with $S \neq \emptyset$ and $z \in S$. Also let $f(x) := \min_{z \in S} d(x, z)$. Then, using the triangle inequality we have,

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \\ f(x) := \min_{z \in S} d(x, z) &\leq d(x, y) + \min_{z \in S} d(y, z) \\ f(x) &\leq d(x, y) + f(y) \\ f(x) - f(y) &\leq d(x, y) \end{aligned}$$

Similarly, we can prove $f(y) - f(x) \leq d(x, y)$, thus

$$|f(x) - f(y)| \leq d(x, y) \quad \text{for } f(s) := d(x, S)$$

c) The construction of the sets S_i and the proof of $\|f(x) - f(y)\|_1 \in \Omega(\log n d(x, y))$, rely on a probabilistic argument. The basic idea is to sample enough subsets S_i to separate any given pair of points x and y . Suppose a subset $S \subset X$ is sampled as $\mathbb{P}(x \in S) = 1/k$ independently for each $x \in X$, where $k \geq 2$. Let B be a fixed subset of X . Show that

$$\begin{aligned} |B| \geq \alpha k &\implies \mathbb{P}(B \cap S \neq \emptyset) \geq 1 - e^{-\alpha} > 0 \\ |B| \leq \alpha k &\implies \mathbb{P}(B \cap S = \emptyset) \geq 4^{-\alpha} > 0 \end{aligned}$$

Proof - Each element in S is included independently, with probability $1/k$ the probability that $x \in S \wedge x \notin B$ is

$$\mathbb{P}(B \cap S = \emptyset) = \left(1 - \frac{1}{k}\right)^{|B|}$$

whereas the probability that $x \in S \wedge x \in B$

$$\mathbb{P}(B \cap S \neq \emptyset) = 1 - \left(1 - \frac{1}{k}\right)^{|B|}$$

First, we can assume that if $|B| \geq \alpha k$,

$$\left(1 - \frac{1}{k}\right)^{|B|} \leq \left(1 - \frac{1}{k}\right)^{\alpha k} \leq e^{-\alpha}$$

using the e^α definition. Thus,

$$\boxed{\mathbb{P}(B \cap S \neq \emptyset) = 1 - \left(1 - \frac{1}{k}\right)^{|B|} \geq 1 - e^{-\alpha} > 0}$$

Second, if $|B| \leq \alpha k$ then

$$\mathbb{P}(B \cap S = \emptyset) = \left(1 - \frac{1}{k}\right)^{|B|} \geq \left(1 - \frac{1}{k}\right)^{\alpha k}$$

For $k \geq 2$,

$$\left(1 - \frac{1}{k}\right)^k \geq \left(1 - \frac{1}{2}\right)^2 = \frac{1}{4}$$

and finally,

$$\boxed{\mathbb{P}(B \cap S = \emptyset) \geq 4^{-\alpha}}$$

d) For each $i \in \{1, \dots, \log_2 n\}$, sample a subset S_i as follows: $\mathbb{P}(x \in S_i) = 2^{-i}$ independently for each $x \in X$. Defining $f_i(x) = d(x, S_i)$ as above, show that for a fixed pair $x, y \in X$,

$$\mathbb{E} \left[\sum_i |f_i(x) - f_i(y)| \right] \in \Omega(d(x, y)),$$

where the expectation is taken over the sets S_i .

Proof - Let $x, y \in X$ be fixed, thus $d(x, y)$ is fixed. X can be covered by dyadic balls $B_x^j = \{z \in X, d(x, z) < 2^j d(x, y)\}$, which double their radius. Being $n = |X|$, we can cover X with at most a ball with $j = \log_2 n$. We can also define disjoint annular sets $A_x^j = B_x^j \setminus B_x^{j-1}$ with $A_x^0 = B_x^0$. Thus,

$$A_0 := \{z \in X, d(x, z) < d(x, y)\}$$

and,

$$A_j := \{z \in X, \quad 2^{j-1}d(x, y) \leq d(x, z) < 2^j d(x, y)\}$$

Recall that $\forall z, w \in X, |f_i(z) - f_i(w)| \leq d(z, w)$, with $f_i(z) := d(z, S_i)$, with S_i being a random sampling of X with independent probability 2^{-i} . Let's fix j and look at the contribution to $|f_i(x) - f_i(y)|$ arising from points $z \in A_x^j$.

- If $S_i \cap A_x^j \neq \emptyset$, then $f_i(x)$ is at most $2^j d(x, y)$. Also, if S_i contains no element at a distance closer to 2^{j-1} from y , then

$$|f_i(x) - f_i(y)| < 2^{j-1}d(x, y)$$

because $f_i(x) \leq 2^r d(x, y)$ and $f_i(y) \geq 2^{j-1} d(x, y)$, with symmetry in x, y .

Let the size of B_x^j and B_y^j be the probabilities that S_i hits one of x, y but not the other. From previous problem, having $k = 2^i$,

- If $|B_x^j| \geq \alpha \cdot 2^i$, and $|B_y^j| \leq \alpha \cdot 2^i$,

$$\mathbb{P}(S_i \cap B_x^j \neq \emptyset \wedge S_i \cap B_y^j = \emptyset) \geq (1 - e^{-\alpha}) \cdot 4^{-\alpha}$$