Homework 3: Geodesics, Distance, and Metric Embedding

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Problem 1

Let $\gamma : [0, L] \to \mathcal{M}$ be an arc lenght-parametrized curve on an orientable surface $\mathcal{M} \subset \mathbb{R}^3$. Let $\mathbf{v}$ be a tangent vector field defined along the curve, i.e., $\mathbf{v}(s) \in T_{\gamma(s)}\mathcal{M}$. If,

$$\underset{T_{\gamma(s)}\mathcal{M}}{\operatorname{proj}} \mathbf{v}'(s) = \mathbf{v}'(s) - \mathbf{n}(\gamma(s))\mathbf{n}(\gamma(s)) \cdot \mathbf{v}'(s) = 0$$

then we say that $\mathbf{v}$ is parallel.

a) Show that the unit tangent vector field of a geodesic is parallel.

Proof - Let $\varphi : [0, 1] \to \mathcal{M}$ be a geodesic parameterized by arc length, then

$$\underset{T_{\varphi(s)}\mathcal{M}}{\operatorname{proj}} \varphi''(s) = 0$$

Its tangent vector field is $T(s) = \varphi'(s)$. For $T(s)$ to be parallel it requires to $\operatorname{proj} T'(s) = 0$, since $T'(s) = \varphi''(s)$ the proof is complete.

b) Suppose $\mathbf{u}$ and $\mathbf{v}$ are parallel fields along γ. Show that $\mathbf{u} \cdot \mathbf{v}$ and $\|\mathbf{u}\|_2$ are constant.

Proof - Let

$$\underset{T_{\gamma(s)}\mathcal{M}}{\operatorname{proj}} \mathbf{u}' = 0 \qquad \underset{T_{\gamma(s)}\mathcal{M}}{\operatorname{proj}} \mathbf{v}' = 0$$

Let $\mathbf{u} \cdot \mathbf{v} = c$ iff $\mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}' = 0$, let's prove the later. Since $\mathbf{u} \in T_\gamma\mathcal{M}$ and $\mathbf{u}' \perp T_\gamma\mathcal{M}$ (similarly for $\mathbf{v}$), we have

$$\mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}' = \cancelto{0}{\mathbf{u}' \cdot \mathbf{v}} + \cancelto{0}{\mathbf{u} \cdot \mathbf{v}'} = 0$$

Note: Equation 1 is a first-order ODE whose solution is unique given an initial condition $\mathbf{v}(0)$. We can define the parallel transport operator P_γ by $P_\gamma \mathbf{v}(0) = \mathbf{v}(L)$, where $\mathbf{v}$ is the unique parallel field along γ with initial condition $\mathbf{v}(0)$.

c) Use the result from *b)* to argue that parallel transport around a closed loop (known as holonomy) amounts to a rotation in the tangent plane.

Proof - Let a closed curve $\gamma : [0, L] \to \mathcal{M}$ such that $\gamma(0) = \gamma(L) = p$. Given a vector $\mathbf{u}_0 \in T_p\mathcal{M}$ its parallel transport is:

$$P_\gamma \mathbf{u}_0 = \mathbf{u}_L$$

And thus, both $\mathbf{u}_0, \mathbf{u}_L \in T_p\mathcal{M}$. Thus, since for any $\mathbf{v} \cdot \mathbf{u} \equiv c$, i.e., the angle is preserved, and the alternatively write the dot product as,

$$(P_\gamma \mathbf{u}) \cdot (P_\gamma \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$$

Where $P_\gamma \in SO(2)$ serves as rotation of degree two along $\mathbf{n}(\gamma(0))$.

d) Let $\mathbf{v}$ be parallel along γ. Let $\theta(s)$ be the angle from $\gamma'(s)$ to $\mathbf{v}(s)$, measured counterclockwise about the surface normal $\mathbf{n}$. Show that,

$$\theta'(s) = -\kappa_g$$

where κ_g is the geodesic curvature of γ, defined by projection of the second derivative of γ into the tangent plane of the surface:

$$\operatorname*{proj}_{T_\gamma\mathcal{M}} \gamma''(s) = \kappa_g(\mathbf{n} \times \gamma'(s))$$

Proof - Let $\gamma(s)$ be parameterized by arc length and $\gamma'(s), \mathbf{v}(s) \in T_{\gamma(s)}\mathcal{M}$ with $\mathbf{v}'(s) \perp T_{\gamma(s)}\mathcal{M}$, then we can represent $\mathbf{v}$ in terms of the orthonormal basis of $T_\gamma\mathcal{M}$, $(\gamma', \mathbf{n} \times \gamma')$, with $\|\gamma'\|_2 = 1$, then

$$\mathbf{v}(s) = \alpha \cos\theta(s)\gamma'(s) + \alpha \sin\theta(s)\mathbf{n}(s) \times \gamma'(s)$$

Let's differentiate $\gamma'(s) \cdot \mathbf{v}(s) = \|\mathbf{v}\|_2 \cos\theta(s)$,

$$\gamma''(s) \cdot \mathbf{v}(s) + \gamma'(s) \cdot \mathbf{v}'(s) = -\alpha\theta'(s)\sin\theta(s)$$

Since $\mathbf{v}' \perp T_\gamma\mathcal{M}$ and $\gamma' \in T_\gamma\mathcal{M}$,

$$
\begin{aligned}
\gamma''(s)\cdot\mathbf{v}(s) &= -\alpha\theta'\sin\theta \\
\underset{T_\gamma\mathcal{M}}{\mathrm{proj}}\gamma''(s)\cdot\mathbf{v}(s) &= \\
\kappa_g(\mathbf{n}\times\gamma')\cdot\mathbf{v}(s) &= \\
\kappa_g(\mathbf{n}\times\gamma')\cdot\alpha(\cos\theta\gamma' + \sin\theta\mathbf{n}\times\gamma') &= \\
\alpha\kappa_g\sin\theta &= -\alpha\theta'\sin\theta \\
\kappa_g &= -\theta'
\end{aligned}
$$

e) A geodesic polygon is a polygon formed from geodesic segments. Show that parallel transporting a vector around a geodesic polygon rotates it by the angle $\sum(\pi - \alpha_i)$, where α_i are the interior angles of the polygon.

Proof - Let a geodesic polygon be described by $\{\gamma_i : [0, L_i] \to \mathcal{M}\}_i^N$, such that $\gamma_i(L_i) = \gamma_{i+1}(0)$ for $i = 1, \dots, N-1$, and $\gamma_N(L_N) = \gamma_1(0)$. Thus, we can measure their exterior angles between γ_i and γ_{i+1} as the exterior angle (β_i) between their tangent vectors,

$$
\cos\beta_i = \gamma_i'(L_i)\cdot\gamma_{i+1}'(0), \qquad i = 1, \dots, N-1
$$

since $\mathbf{v}_i = \gamma_i'$ is parallel we have

$$
c_i = \mathbf{v}_i\cdot\mathbf{v}_{i+1}
$$

Let transport $\mathbf{u}(0) \in T_{\gamma_1(0)}\mathcal{M}$, $P_{\gamma_1}\mathbf{u}(0) = \mathbf{u}(L_1)$, $\mathbf{u}(s)\cdot\mathbf{v}_1(s)$ is constant along γ_1. Since $\mathbf{v}_1\cdot\mathbf{v}_2 = \cos\beta_1$ we have $\mathbf{u}(L_1+L_2)\cdot\mathbf{v}_2(L_2) = \cos(\beta_1+\beta_2)$. Following this argument, after $P_\gamma = P_{\gamma_1}\circ P_{\gamma_2}\circ\cdots\circ P_{\gamma_N}$ we have a total rotation of $\sum\beta_i$ of exterior angles. Corresponding to $\sum(\pi - \alpha_i)$ in interior angles.

Problem 2

Show the existence of low-distortion embeddings of arbitrary finite metric spaces.

Definition Let (X, d_X) and (Y, dY) be a metric spaces and $f : X \to Y$ a map between them. Distortion is the smallest positive ρ such that for some c and $\forall x_1, x_2 \in X$

$$
cd_X(x_1, x_2) \le d_Y(f(x_1), f(x_2)) \le \rho cd_X(x_1, x_2)
$$

Theorem (Bourgain). For any finite metric space (X, d) of size $|X| = n$, there is an embedding $f : X \to \mathbb{R}^m$ with $m \in \Theta(\log^2 n)$ such that

$$|f_i(x) - f_i(y)| \leq d(x, y) \qquad \|f(x) - f(y)\|_1 \in \Omega((\log n)d(x, y))$$

Moreover, the distortion is $O(\log n)$ for any ℓ_p metric on $\mathbb{R}^m$, $p \geq 1$.

a) Using Hölder's inequality, show that ℓ_1 previous results imply the conslusion for all p.

Proof - Hölder's inequality says, for any $p, q \in [1, \infty]$ with $1/p + 1/q = 1$, for any $f, g \in X$,

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Let $\mathbf{v} = f(x) - f(y) \in \mathbb{R}^m$, thanks to Hölder's inequality, we have $q = p/(p-1)$, and

$$\begin{aligned}
\|\mathbf{v}\|_1 = \|\mathbf{v}\mathbf{1}\|_1 &< \|v\|_p \|\mathbf{1}\|_q \\
&= \|v\|_p \left(\sum_i^m |1|^q \right)^{\frac{1}{q}} \\
&= \|\mathbf{v}\|_p m^{\frac{1}{q}} \\
\|\mathbf{v}\|_1 &\leq m^{1-1/p} \ \|\mathbf{v}\|_p
\end{aligned}$$

Also, in general, $\|\mathbf{x}\|_p \leq \cdots \leq \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1$ for any finite $\mathbf{x}$. Thus, $\|\mathbf{v}\|_p \leq \|\mathbf{v}\|_1$, and we can bound $\|\mathbf{v}\|_p$ by an upper and lower limit.

$$m^{\frac{1}{p}-1} \|\mathbf{v}\|_1 \leq \|\mathbf{v}\|_p \leq \|\mathbf{v}\|_1$$

Now, using Bourgain theorem we have that there exists some $A > 0$, such that,

$$|v_i| \leq d(x, y) \qquad A \log n \ d(x, y) \leq \|\mathbf{v}\|_1$$

Taking the first property, we can define another upperbound for $\|\mathbf{v}\|_p$ as,

$$\|\mathbf{v}\|_p = \left(\sum_i^m |v_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_i^m d^p(x, y) \right)^{\frac{1}{p}} = m^{\frac{1}{p}} d(x, y)$$

and, taking the second property, we can conclude that

$$m^{\frac{1}{p}-1} A \log n \ d(x, y) \leq \|\mathbf{v}\|_p \leq m^{\frac{1}{p}} d(x, y)$$

With $c = m^{\frac{1}{p}-1} A \log n$, the distortion satisfies:

$$\rho \leq \frac{m}{A \log n}$$

Since $m = \Theta(\log^2 n)$, then $\rho = O(\log n)$.

b) The map f can be built up from the distances themselves. Show that if S_i is any nonempty subset of X and

$$f_i(x) := d(x, S_i) := \min_{y \in S_i} d(x, y),$$

then $|f_i(x) - f_i(y)| \leq d(x, y)$ holds.

Proof - Let $x, y \in X$, and $S \subseteq X$ with $S \neq \emptyset$ and $z \in S$. Also let $f(x) := \min_{z \in S} d(x, z)$. Then, using the triangle inequality we have,

$$\begin{aligned} d(x,z) &\leq d(x,y) + d(y,z) \\ f(x) := \min_{z \in S} d(x,z) &\leq d(x,y) + \min_{z \in S} d(y,z) \\ f(x) &\leq d(x,y) + f(y) \\ f(x) - f(y) &\leq d(x,y) \end{aligned}$$

Similarly, we can prove $f(y) - f(x) \leq d(x, y)$, thus

$$|f(x) - f(y)| \leq d(x,y) \qquad \text{for } f(s) := d(x, S)$$

c) The construction of the sets S_i and the proof of $\|f(x) - f(y)\|_1 \in \Omega(\log n \, d(x, y))$, rely on a probabilistic argument. The basic idea is to sample enough subsets S_i to separate any given pair of points x and y. Suppose a subset $S \subset X$ is sampled as $\mathbb{P}(x \in S) = 1/k$ independently for each $x \in X$, where $k \geq 2$. Let B be a fixed subset of X. Show that

$$\begin{aligned} |B| \geq \alpha k &\Longrightarrow \mathbb{P}(B \cap S \neq \emptyset) \geq 1 - e^{-\alpha} > 0 \\ |B| \leq \alpha k &\Longrightarrow \mathbb{P}(B \cap S = \emptyset) \geq 4^{-\alpha} > 0 \end{aligned}$$

Proof - Each element in S is included independently, with probability $1/k$ the probability that $x \in S \wedge x \notin B$ is

$$\mathbb{P}(B \cap S = \emptyset) = \left(1 - \frac{1}{k}\right)^{|B|}$$

whereas the probability that $x \in S \wedge x \in B$

$$\mathbb{P}(B \cap S \neq \emptyset) = 1 - \left(1 - \frac{1}{k}\right)^{|B|}$$

First, we can assume that if $|B| \geq \alpha k$,

$$\left(1 - \frac{1}{k}\right)^{|B|} \leq \left(1 - \frac{1}{k}\right)^{\alpha k} \leq e^{-\alpha}$$

using the e^{α} definition. Thus,

$$\mathbb{P}(B \cap S \neq \emptyset) = 1 - \left(1 - \frac{1}{k}\right)^{|B|} \geq 1 - e^{-\alpha} > 0$$

Second, if $|B| \leq \alpha k$ then

$$\mathbb{P}(B \cap S = \emptyset) = \left(1 - \frac{1}{k}\right)^{|B|} \geq \left(1 - \frac{1}{k}\right)^{\alpha k}$$

For $k \geq 2$,

$$\left(1 - \frac{1}{k}\right)^{k} \geq \left(1 - \frac{1}{2}\right)^{2} = \frac{1}{4}$$

and finally,

$$\mathbb{P}(B \cap S = \emptyset) \geq 4^{-\alpha}$$