# Homework 3: Geodesics, Distance, and Metric Embedding

Luis David Fuentes Juvera

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## Problem 1

Let $\gamma : [0, L] \to \mathcal{M}$ be an arc lenght-parametrized curve on an orientable surface $\mathcal{M} \subset \mathbb{R}^3$. Let $\mathbf{v}$ be a tangent vector field defined along the curve, i.e., $\mathbf{v}(s) \in T_{\gamma(s)}\mathcal{M}$. If,

$$\underset{T_{\gamma(s)}\mathcal{M}}{\text{proj}}\, \mathbf{v}'(s) = \mathbf{v}(s) - \mathbf{n}(\gamma(s))\mathbf{n}(\gamma(s)) \cdot \mathbf{v}'(s) = 0$$

then we say that $\mathbf{v}$ is parallel.

*a)* Show that the unit tangent vector field of a geodesic is parallel.

**Proof** - Let $\varphi : [0, 1] \to \mathcal{M}$ be a geodesic parameterized by arc length, then

$$\underset{T_{\varphi(s)}\mathcal{M}}{\text{proj}}\, \varphi''(s) = 0$$

Its tangent vector field is $T(s) = \varphi'(s)$. For $T(s)$ to be parallel it requires to $\text{proj}T'(s) = 0$, since $T'(s) = \varphi''(s)$ the proof is complete.

*b)* Suppose $\mathbf{u}$ and $\mathbf{v}$ are parallel fields along $\gamma$. Show that $\mathbf{u} \cdot \mathbf{v}$ and $\|\mathbf{u}\|_2$ are constant.