Homework 2: Surfaces and Curvature

November 11, 2025

Problem 1 - Recall the Taubin matrix for approximating mesh curvature

$$M_{\mathbf{p}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \kappa_\theta \mathbf{t}_\theta \mathbf{t}_\theta^T \; d\theta$$

a) Prove that $\mathbf{n}, \mathbf{t}_{\min}, \mathbf{t}_{\max}$ are eigenvectors of $M_{\mathbf{p}}$.

Let $\mathbf{t}_\theta = \cos\theta \; \mathbf{t}_{\min} + \sin\theta \; \mathbf{t}_{\max}$, where $\|\mathbf{t}_\theta\|_2 = 1$ and the principal directions of $T_{\mathbf{p}}(\mathcal{M})$ are $\mathbf{t}_{\min}, \mathbf{t}_{\max}$. We can prove that, with the first and second fundamental forms and the shape operators $\mathbf{I}(\mathbf{t}_1, \mathbf{t}_2), \mathbf{II}(\mathbf{t}_1, \mathbf{t}_2), S(\mathbf{t}) = -d\mathbf{n}_{\mathbf{p}}(\mathbf{t})$

$$\begin{aligned}
\mathbf{II}(\mathbf{t}_\theta, \mathbf{t}_\theta) &= \kappa_\theta \\
&= \mathbf{I}(S(\mathbf{t}_\theta), \mathbf{t}_\theta) \\
&= \mathbf{I}(\cos\theta \; S(\mathbf{t}_{\min}) + \sin\theta \; S(\mathbf{t}_{\max}), \mathbf{t}_\theta) \\
&= \cos\theta \; \mathbf{I}(S(\mathbf{t}_{\min}), \mathbf{t}_\theta) + \sin\theta \; \mathbf{I}(S(\mathbf{t}_{\max}), \mathbf{t}_\theta) \\
&= \cos\theta \; \mathbf{I}(S(\mathbf{t}_{\min}), \mathbf{t}_\theta) + \sin\theta \; \mathbf{I}(S(\mathbf{t}_{\max}), \mathbf{t}_\theta)
\end{aligned}$$

Remember that for $S : T_{\mathbf{p}}(\mathcal{M}) \to T_{\mathbf{p}}(\mathcal{M})$, $S(\mathbf{t}_{\min}) = \kappa_{\min}\mathbf{t}_{\min}$ and $S(\mathbf{t}_{\max}) = \kappa_{\max}\mathbf{t}_{\max}$. Also, $\mathbf{I}(\mathbf{t}_i, \mathbf{t}_j) = \delta_{i,j}$, thus,

$$\begin{aligned}
\mathbf{I}(S(\mathbf{t}_{\min}), \mathbf{t}_\theta) &= \mathbf{I}(\kappa_{\min}\mathbf{t}_{\min}, \cos\theta \; \mathbf{t}_{\min} + \sin\theta \; \mathbf{t}_{\max}) \\
&= \kappa_{\min} \cos\theta \; \mathbf{I}(\mathbf{t}_{\min}, \mathbf{t}_{\min}) + \sin\theta \; I(\mathbf{t}_{\min}, \mathbf{t}_{\max}) \\
&= \kappa_{\min} \cos\theta \\
\mathbf{I}(S(\mathbf{t}_{\max}), \mathbf{t}_\theta) &= \kappa_{\max} \sin\theta
\end{aligned}$$

Finally,

$$\boxed{\kappa_\theta = \kappa_{\min} \cos^2\theta + \kappa_{\max} \sin^2\theta}$$

Now, let's expand $M_{\mathbf{p}}$ as

$$
\begin{aligned}
M_{\mathbf{p}} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_\theta(\mathbf{t}_{\min}\cos\theta + \mathbf{t}_{\max}\sin\theta)(\mathbf{t}_{\min}\cos\theta + \mathbf{t}_{\max}\sin\theta)^T \; d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_\theta \cos^2\theta \; d\theta \mathbf{t}_{\min}\mathbf{t}_{\min}^T \\
&+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_\theta \cos\theta\sin\theta \; d\theta(\mathbf{t}_{\min}\mathbf{t}_{\max}^T + \mathbf{t}_{\max}\mathbf{t}_{\min}^T) \\
&+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_\theta \sin^2\theta \; d\theta \mathbf{t}_{\max}\mathbf{t}_{\max}^T \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_{\min}\cos^4\theta + \kappa_{\max}\sin^2\theta\cos^2\theta \; d\theta \mathbf{t}_{\min}\mathbf{t}_{\min}^T \\
&+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_{\min}\cos^3\theta\sin\theta + \kappa_{\max}\cos\theta\sin^3\theta \; d\theta(\mathbf{t}_{\min}\mathbf{t}_{\max}^T + \mathbf{t}_{\max}\mathbf{t}_{\min}^T) \\
&+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \kappa_{\min}\cos^2\theta\sin^2\theta + \kappa_{\max}\sin^4\theta \; d\theta \mathbf{t}_{\max}\mathbf{t}_{\max}^T \\
&= \left(\frac{3}{8}\kappa_{\min} + \frac{1}{8}\kappa_{\max}\right)\mathbf{t}_{\min}\mathbf{t}_{\min}^T \\
&+ 0(\mathbf{t}_{\min}\mathbf{t}_{\max}^T + \mathbf{t}_{\max}\mathbf{t}_{\min}^T) \\
&+ \left(\frac{1}{8}\kappa_{\min} + \frac{3}{8}\kappa_{\max}\right)\mathbf{t}_{\max}\mathbf{t}_{\max}^T \\
&= \boxed{\left(\frac{3}{8}\kappa_{\min} + \frac{1}{8}\kappa_{\max}\right)\mathbf{t}_{\min}\mathbf{t}_{\min}^T + \left(\frac{1}{8}\kappa_{\min} + \frac{3}{8}\kappa_{\max}\right)\mathbf{t}_{\max}\mathbf{t}_{\max}^T}
\end{aligned}
$$

Thus $M_{\mathbf{p}}\mathbf{t}_{\min} = \left(\frac{3}{8}\kappa_{\min} + \frac{1}{8}\kappa_{\max}\right)\mathbf{t}_{\min}$ and $M_{\mathbf{p}}\mathbf{t}_{\max} = \left(\frac{1}{8}\kappa_{\min} + \frac{3}{8}\kappa_{\max}\right)\mathbf{t}_{\max}$ by orthogonality of $\mathbf{t}_{\min}$ and $\mathbf{t}_{\max}$. Also, for $\mathbf{n}$ which is normal to any $\mathbf{t}_\theta$, we have $\mathbf{n}\cdot\mathbf{t}_{\min} = 0$ and $\mathbf{n}\cdot\mathbf{t}_{\max} = 0$, thus $M_{\mathbf{p}}\mathbf{n} = 0\mathbf{n}$.