Problem 4

The study of the butterfly effect. The Lorenz system is a system of ordinary differential equations first studied by Edward Lorenz. It is notable for having chaotic solutions for certain parameter values and initial conditions. In popular media the 'butterfly effect' stems from the real-world implications of the Lorenz system, i.e. that in any physical system, in the absence of perfect knowledge of the initial conditions (even the minuscule disturbance of the air due to a butterfly flapping its wings), our ability to predict its future course will always fail. The Lorenz system is given by a 3-dimensional nonlinear ODE:

$$\frac{dx}{dt} = \sigma(y - x) \tag{1}$$

$$\frac{dy}{dt} = x(\rho - z) - y \tag{2}$$

$$\frac{dz}{dt} = xy - \beta z \tag{3}$$

where σ, ρ, β are modeling parameters. You can now use the mathematical method you learnt in this class to quantitatively study the bufferfly effect of the Lorenz system. Please read Section 5.9 and develop a numerical method to solve the Lorenz system numerically. With this numerical solver, you should finish the following tasks:

a) We first set $\sigma = 5, \beta = 3, \rho = 0.5$. Please choose an arbitrary initial condition (x(0),y(0),z(0)) and solve the Lorenz system numerically. With the same parameter setting and a slightly perturbed initial condition $(x(0) + \delta_1, y(0) + \delta_2, z(0) + \delta_3)$, where $\delta_1, \delta_2, \delta_3$ are constants with small absolute value such as 0.00001, -0.00003, 0.0002, solve the Lorenz system numerically and compare the perturbed solution with the non-perturbed solution. Please use plots to show whether the system exhibits the butterfly effect. i.e. The perturbed solution deviates from the non-perturbed solution dramatically as time $t \to +\infty$.

b) Repeat what you have done for a new parameter setting $\sigma = 1, \beta = 2.5, \rho = 10$. Can you observe the butterfly effect?

c) Repeat what you have done for a new parameter setting $\sigma = 10, \beta = 8/3, \rho = 28$. Can you observe the butterfly effect?

d) Plot your numerical solution $[x(t), y(t), z(t)]$ in 3D with the parameter setting $\sigma = 10, \beta = 8/3, \rho = 28$. What does the trajectory look like?

Solution

In this assignment we study the behavior of the solutions to a system of non-linear differential equations. In doing so we get insight into deterministic chaos; defined by Edward Lorenz as "When the present determines the future, but the approximate present does not approximately determine the future." Specifically, we study the solutions to the Lorenz system. Which as implied, was first studied by Edward Lorenz.

In order to approximate a solution to the Lorenz system we used the four-step Adams-Bashforth explicit method. To generate the first w_1, w_2, and w_3 values we use the classic Runge–Kutta method, or more formally known as Runge–Kutta's 4th order method. Furthermore, we initially used Euler's method but when compared to the four-step Adams-Bashforth Explicit method it was obvious the latter scheme was much more powerful.

a) Using parameters $\sigma = 5, \beta = 3, \rho = 0.5$, bounds interval $[0, 3]$, number of sub-intervals = 1000, and initial conditions $[x(0) = 1, y(0) = 100, z(0) = 1]$ we graph the solution in 3D and obtain the plot on the left of Figure 1. Furthermore, when we perturb the initial conditions by a factor of 1×10^{-5}, resulting in the initial conditions being $[1 + (1 * 10^{-}5), 100 + (-2 * 10^{-}5), 1 + (3 * 10^{-}5)]$, we can observe that there is no chaotic behavior for the solution to the Lorenz's system by plotting both solutions on the same graph (i.e graphs overlap). In fact, if we perturb the initial conditions from $[1, 100, 1]$ to $[0, 96, 0]$ there is very little change in the solutions to the Lorenz system, as in the right plot of Figure 1.

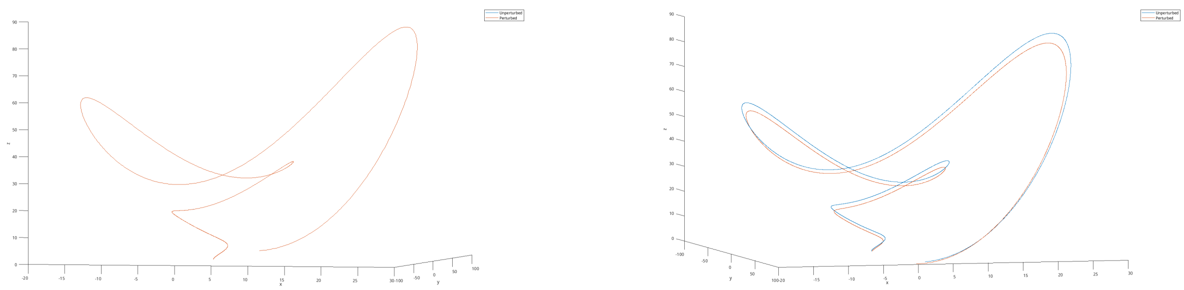

Figure 1: Lorenz systems with $\sigma = 5, \beta = 3, \rho = 0.5$.
Unperturbed system in blue; perturbed system in red.
Left: Interval $[0, 3], \delta = 10^{-5}[1, 1, 1]$.
Right: Interval $[0, 3], \delta = [-1, -4, -1]$.

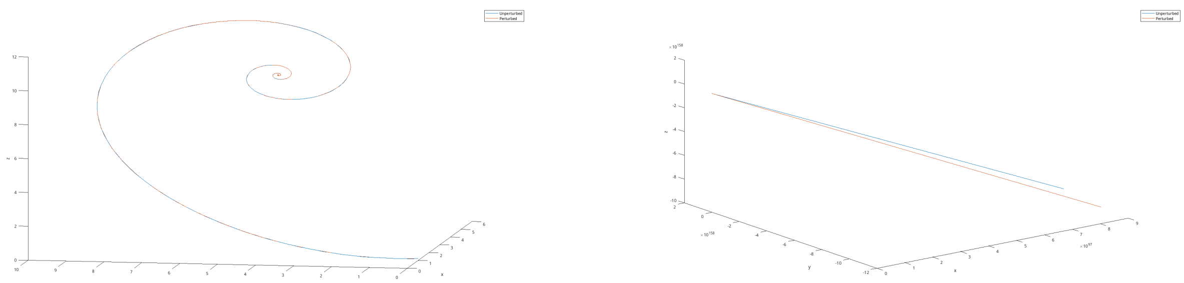

Figure 2: Lorenz systems with $\sigma = 1, \beta = 2.5, \rho = 10$.
Unperturbed system in blue; perturbed system in red.
Left: Interval $[0, 100], \delta = 0.001[1, 1, 1]$.
Right: Interval $[0, 10000], \delta = 0.01[1, 1, 1]$.

b) As seen on the left of Figure 2, the butterfly effect is not exhibited by the behavior of the Lorenz system with the parameters $\sigma = 1, \beta = 2.5, \rho = 10$. In this case, the initial condition was $[1, 0, 0.1]$, and the system was perturbed by a positive 0.001 on each axis. Extending the interval from 100 to 10,000 and increasing the perturbation from 0.001 to 0.01, we obtain the plot on the right in Figure 2. We do see some deviation between solutions, but they are insignificant and with a high enough perturbation in the initial condition that they need not be attributed to the butterfly effect and the system is not chaotic.

c) Now when using parameters $\sigma = 10, \beta = 8/3, \rho = 28$, bounds interval $[0, 37]$, number of sub-intervals = 10000, and initial conditions $[x(0) = 1, y(0) = 100, z(0) = 1]$ we graph the solution in 3D and obtain the plot int Figure 3. Furthermore, when we perturb the initial conditions by a factor of 1×10^{-5}, resulting in the initial conditions being $[1+(1*10^-5), 100+(-2*10^-5), 1+(3*10^-5)]$, we can observe chaotic behavior. Although the chosen initial conditions don't make for the best Lorenz attractor plot I decided to use these initial conditions, and perturbed initial conditions, because it obviously displays how the perturbed solution deviates from the non-perturbed solution as $t \to \infty$.

d) When graphing solutions to the Lorenz system with parameters $\sigma = 10, \beta = 8/3, \rho = 28$, we get an interesting figure which is known as the Lorenz attractor. The graph resembles a "butterfly", and has a non integer dimension.

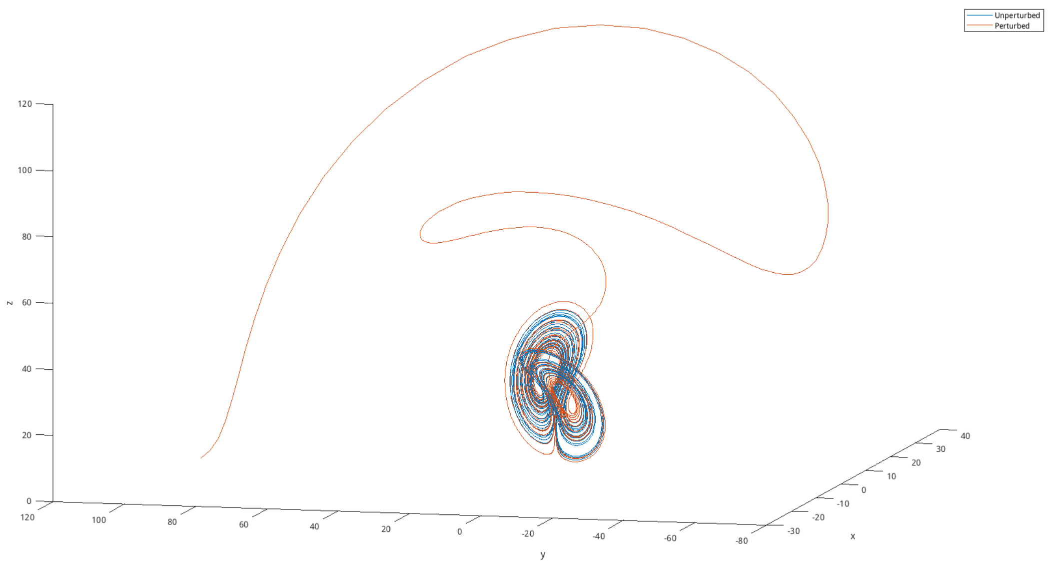

Figure 3: Lorenz systems with $\sigma = 10, \beta = 8/3, \rho = 28$.
Unperturbed system in blue, perturbed system in red.
Interval $[0, 37], \delta = 10^{-5}[1, -2, 3]$.

Contributors:

- Luis Diego Andrade
- Elio Bahena
- Charles Parkinson
- Christopher Tapia
- Joshua Vuong