Unit IV - Exam B

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1.- Use the Laplace inverse to find the $y(t)$ of

$$Y(s) = \frac{2s^3 + 8s^2 + 10s + 12}{(s^2+2s+5)(s^2-1)}$$

—Laplace Inverse

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2s^3+8s^2+10s+12}{(s^2+2s+5)(s^2-1)}\right\}$$

$$= \left\{\frac{2s^3+8s^2+10s+12}{(s^2+2s+5)(s-1)(s+1)}\right\}$$

$$\boxed{s = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i}$$

—Partial fractions

$$\frac{2s^3+8s^2+10s+12}{(s^2+2s+5)(s-1)(s+1)} = \frac{As+B}{s^2+2s+5} + \frac{C}{s-1} + \frac{D}{s+1}$$

$$s_{1,2} = -1 \pm 2i$$

$$As+B = \frac{2(-1+2i)^3 + 8(-1+2i)^2 + 10(-1+2i) + 12}{(-1+2i-1)(-1+2i+1)}$$

$$As+B = \frac{22 - 4i + 24 - 32i - 10 + 20i + 12}{(\cancel{-2}+2i)(2i)} = \frac{-16i}{-4-4i} \cdot \frac{-4+4i}{-4+4i}$$

$$As+B = \frac{64 + 64i}{16+16} = \frac{64+64i}{32} = 2+2i \Rightarrow A(-1+2i)+B = 2+2i$$

$$-A + 2Ai + B = 2+2i \Rightarrow 2A = 2 \Rightarrow A = 1$$

$$-A+B = 2 \Rightarrow B = 3$$

$s = 1$

$$C = \frac{2+8+10+12}{(1+2+5)(1+1)} = \frac{32}{16} = \underline{\underline{2}}$$

$s = -1$

$$D = \frac{-2+8-10+12}{(1-2+5)(-1-1)} = \frac{8}{-8} = \underline{\underline{-1}}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s+3}{s^2+2s+5} + \frac{2}{s-1} - \frac{1}{s+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+3}{(s+1)^2+2^2}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{-1}{s+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2} + \frac{2}{(s+1)^2+2^2}\right\} + 2e^{t} - e^{-t}$$

$$= e^{-t}\cos 2t + e^{-t}\sin 2t + 2e^{t} - e^{-t}$$

$$\therefore y(t) = e^{-t}\cos 2t + e^{-t}\sin 2t + 2e^{t} - e^{-t}$$

2- Solve the ~~integral~~ given differential equation using Laplace Transform.

$y'' - 4y' = \sqrt{2}\sin(\sqrt{2}t)$ s.a $y(0)=1$ and $y'(0)=-1$

$$\mathcal{L}\{y''-4y'\} = \mathcal{L}\{\sqrt{2}\sin(\sqrt{2}t)\} =$$

$$= s^2 Y - s y(0) - y'(0) - 4[sY - y(0)] = \frac{2}{s^2+2}$$

$$Y[s^2-4s] - s + 1 + 4 = \frac{2}{s^2+2}$$

$$Y = \frac{2+(s-5)(s^2+2)}{(s^2+2)(s^2-4s)} = \frac{s^3-5s^2+2s-8}{(s^2+2)(s-4)s}$$

$$\boxed{\begin{array}{l} s_{1,2} = \pm\sqrt{2}i \\ s_3 = 4 \\ s_4 = 0 \end{array}}$$

$$= \frac{s^3-5s^2+2s-8}{(s^2+2)(s-4)s} = \frac{As+B}{s^2+2} + \frac{C}{s-4} + \frac{D}{s}$$

Making partial fractions

$s_{1,2} = \pm 2i$

$$As+B = \frac{-2\sqrt{2}i^2 - 10 + 2\sqrt{2}i - 8}{(\sqrt{2}i-4)(\sqrt{2}i)} = \frac{-2}{-2-4\sqrt{2}i} = \frac{2}{2}\left(\frac{1}{(-1-2\sqrt{2}i)}\right) \cdot \frac{-1+2\sqrt{2}i}{-1+2\sqrt{2}i}$$

$$= \frac{-1+2\sqrt{2}i}{1+8} = -\frac{1}{9} + \frac{2\sqrt{2}}{9}i = A(\sqrt{2}i)+B$$

$$A = \frac{2\sqrt{2}}{\sqrt{2}(9)} = \underline{\underline{\frac{2}{9}}} \qquad B = \underline{\underline{-\frac{1}{9}}}$$

$s_3 = 4$

$$\frac{64-80+8-8}{4(16+2)} = \frac{-16}{72} = \underline{\underline{-\frac{2}{9}}} = C$$

$s = 0$

$$\frac{8}{(2)(-4)} = \frac{8}{-8} = \underline{\underline{-1}} = D$$

Substituting

$$Y = \frac{\frac{2}{9}s - \frac{1}{9}}{s^2+2} - \frac{\frac{2}{9}}{s-4} - \frac{1}{s}$$

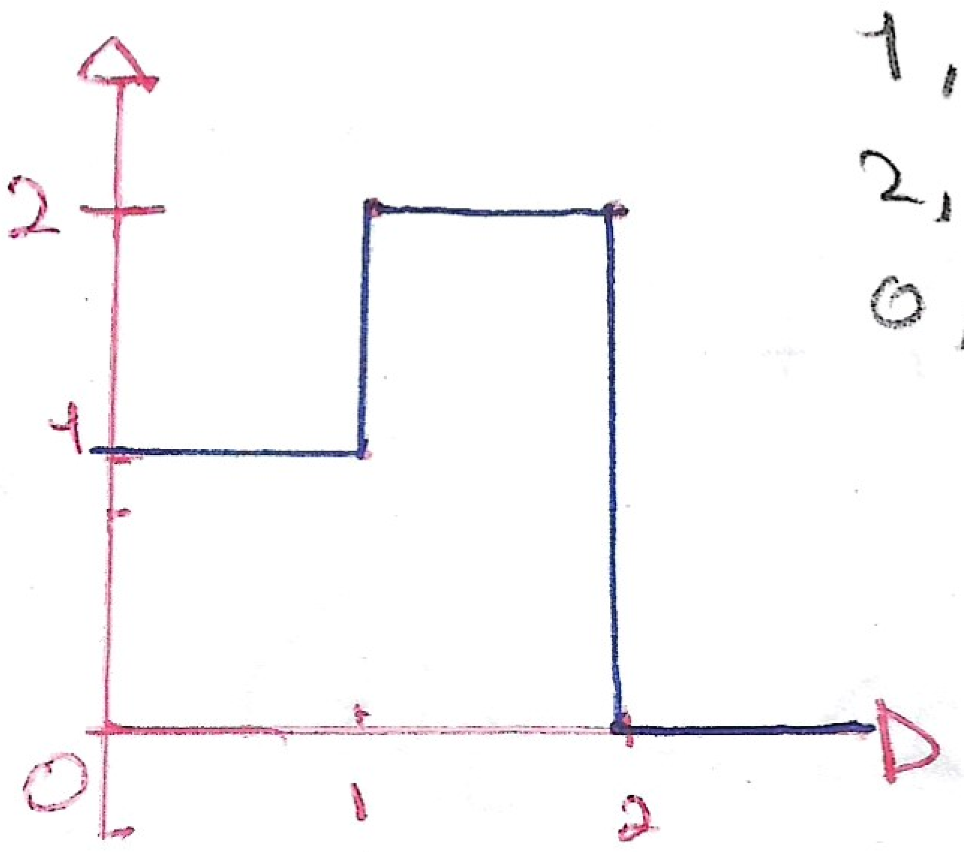
$$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{2}{9}\frac{s}{s^2+2} - \frac{1}{9(\sqrt{2})}\frac{\sqrt{2}}{s^2+2} - \frac{2}{9}\frac{1}{s-4} - \frac{1}{s}\right\}$$

$$\boxed{y(t) = \frac{2}{9}\cos\sqrt{2}t - \frac{1}{9\sqrt{2}}\sin\sqrt{2}t - \frac{2}{9}e^{4t} - 1}$$

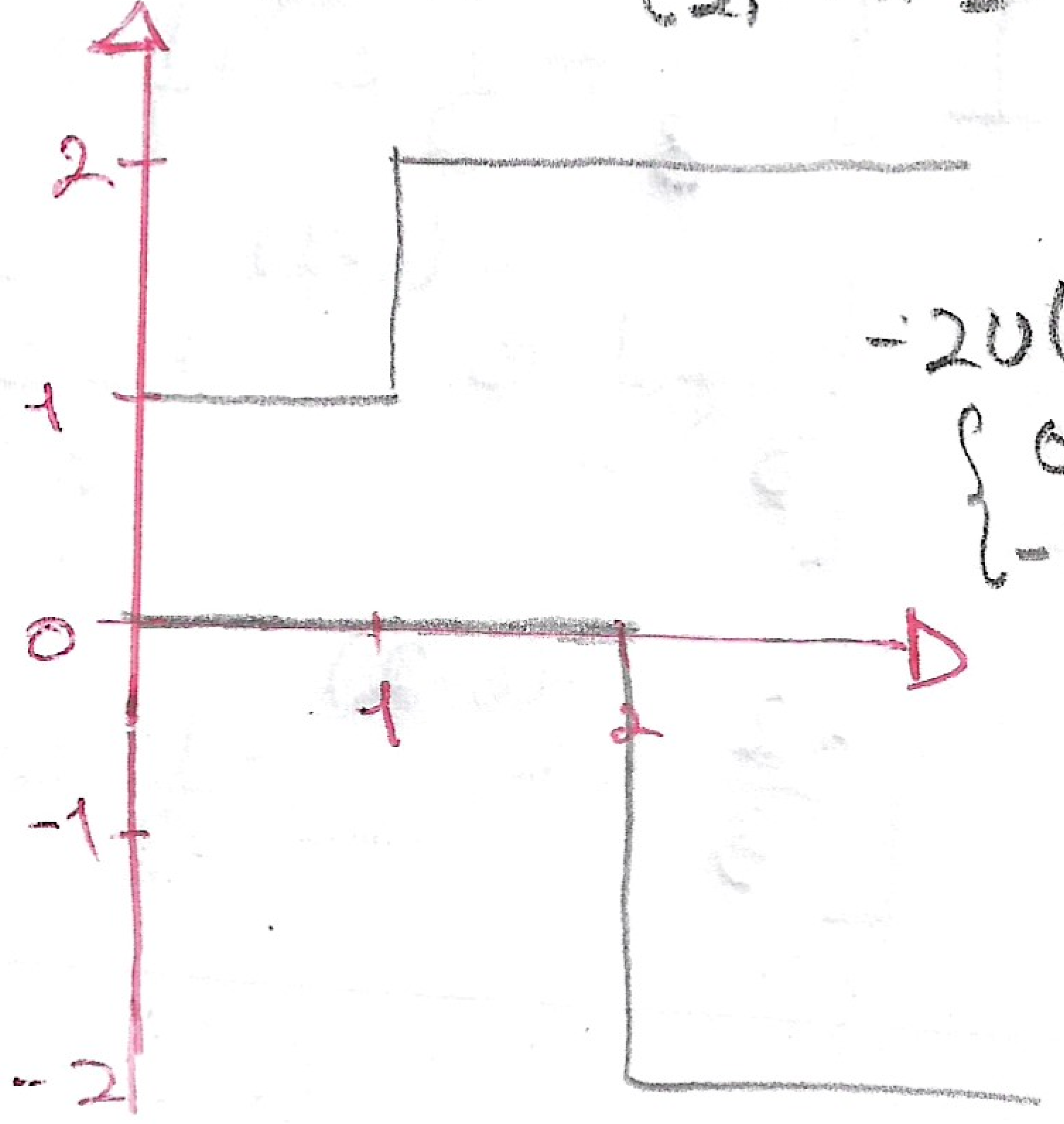
3- Solve the given diff. eq. using Laplace transform and using the step functions to define compossed functions.

$y'' + 4y' + 3y = f(t) \quad$ s.a $y'(0) = y(0) = 0$

$1 + u(t-1) = \begin{cases} 1, & 0 \le t < 1 \\ 2, & t \geq 1 \end{cases}$



$\begin{matrix} 1, & 0 \le t < 1 \\ 2, & 1 \le t < 2 \\ 0, & t \geq 2 \end{matrix}$



$-2u(t-2)$

$\begin{cases} 0, & 0 \le t < 2 \\ -2, & t \geq 2 \end{cases}$

$f(t) = 1 + u(t-1) - 2u(t-2)$

$$\mathcal{L}\{y'' + 4y' + 3y\} = \mathcal{L}\{1 + u(t-1) - 2u(t-2)\}$$

$$s^2 Y - s\cancel{y(0)}^0 - \cancel{y'(0)}^0 + 4[sY - \cancel{y(0)}^0] + 3Y = \frac{1}{s} + \frac{e^{-s}}{s} - \frac{2e^{-2s}}{s}$$

$$Y[s^2 + 4s + 3] = \frac{1}{s} + \frac{e^{-s}}{s} - \frac{2e^{-2s}}{s}$$

$$Y = \frac{1}{s(s^2+4s+3)} + \frac{e^{-s}}{s(s^2+4s+3)} - \frac{2e^{-2s}}{s(s^2+4s+3)}$$

Making partial fractions

$$Y = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3} + \left[\frac{D}{s} + \frac{E}{s+1} + \frac{F}{s+3}\right]e^{-s} - 2\left[\frac{G}{s} + \frac{H}{s+1} + \frac{I}{s+3}\right]e^{-2s}$$

$s_1 = 0 \qquad A = \frac{1}{(0+1)(0+3)} = \frac{1}{3} \qquad D = \frac{1}{3} \qquad G = -\frac{2}{3} = -\frac{2}{3}$

$s_2 = -1 \qquad E = -\frac{1}{2} \qquad H = -2 \cdot -\frac{1}{2} = 1$

$s_3 = -3 \qquad B = \frac{1}{-(-1+3)} = -\frac{1}{2} \qquad F = \frac{1}{6} \qquad I = -\frac{2}{6} = -\frac{1}{3}$

$C = \frac{1}{-3(-3+1)} = \frac{1}{6}$

Substituting

$$Y = \frac{\frac{1}{3}}{s} - \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{6}}{s+3} + \left[\frac{\frac{1}{3}}{s} - \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{6}}{s+3}\right]e^{-s} + \left[\frac{-\frac{2}{3}}{s} + \frac{1}{s+1} - \frac{\frac{1}{3}}{s+3}\right]e^{-2s}$$

$$\mathcal{L}^{-1}\{Y\} = \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t} +$$

$$\left[\frac{1}{3} - \frac{1}{2}e^{-(t-1)} + \frac{1}{6}e^{-3(t-1)}\right]u(t-1) +$$

$$\left[\frac{-2}{3} + e^{-(t-2)} - \frac{1}{3}e^{-3(t-2)}\right]u(t-2) = y(t)$$

4- Solve the integro-differential equation using Laplace Transform

$$\frac{df(t)}{dt} = t + \int_0^t \cos(\tau) f(t-\tau)\,d\tau$$

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \mathcal{L}\left\{t + \int_0^t \cos(\tau) f(t-\tau)\,d\tau\right\}$$

$$\Delta F(\Delta) - f(0) = \frac{1}{\Delta^2} + \frac{\Delta}{\Delta^2+1} F(\Delta)$$

$$\Delta F(\Delta) - 4 = \frac{1}{\Delta^2} + \frac{\Delta}{\Delta^2+1} F(\Delta) \Rightarrow \left(\Delta - \frac{\Delta}{\Delta^2+1}\right) F(\Delta) = \frac{1}{\Delta^2} + 4$$

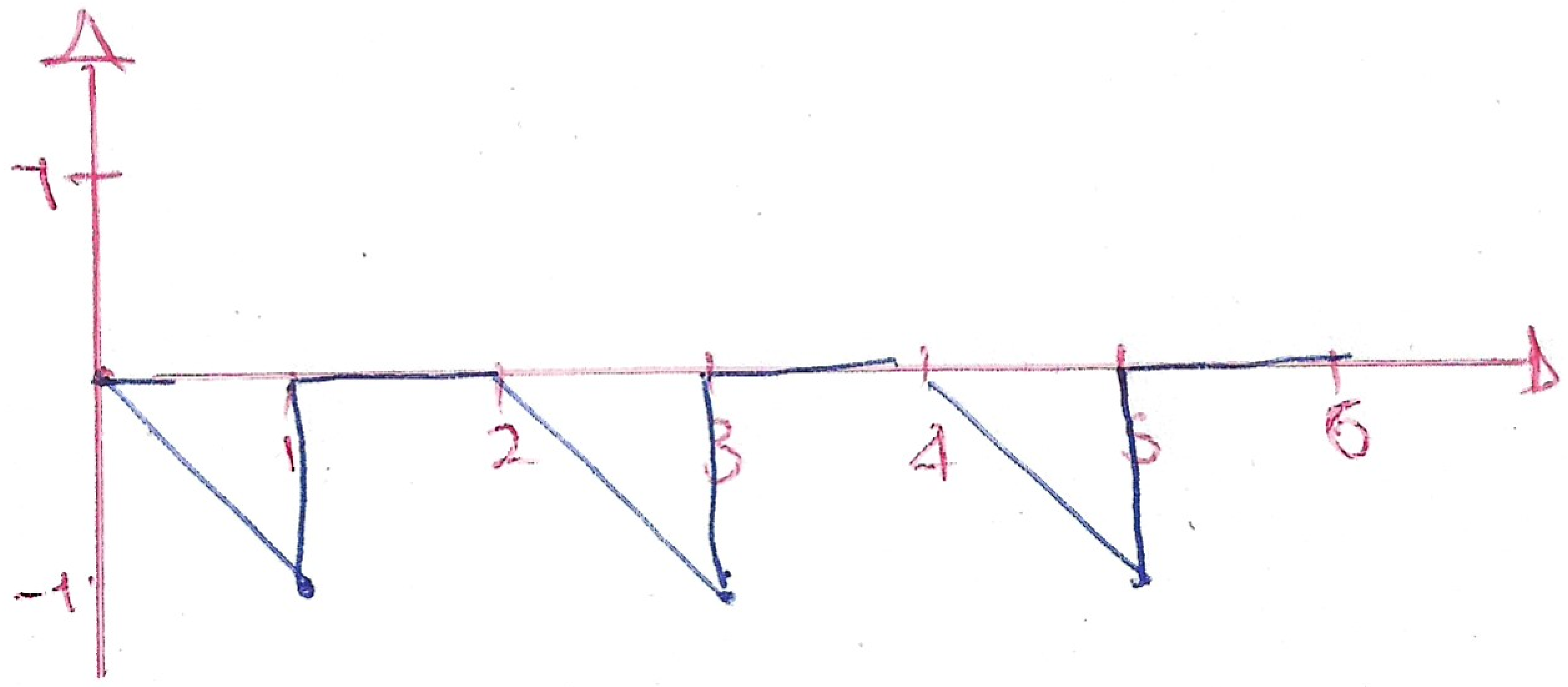
$$\left(\frac{\Delta^3 + \Delta - \Delta}{\Delta^2+1}\right) F(\Delta) = \frac{1+4\Delta^2}{\Delta^2} \Rightarrow F(\Delta) = \frac{(1+4\Delta^2)(\Delta^2+1)}{\Delta^5}$$

$$F(\Delta) = \frac{\Delta^2 + 1 + 4\Delta^4 + 4\Delta^2}{\Delta^5} = \frac{4\Delta^4 + 5\Delta^2 + 1}{\Delta^5}$$

$$\mathcal{L}^{-1}\{F(\Delta)\} = \mathcal{L}\left\{\frac{4}{\Delta} + \frac{5}{\Delta^3} + \frac{1}{\Delta^5}\right\}$$

$$y(t) = 4 + \frac{5t^2}{2} + \frac{t^4}{24}$$

5- Find the Laplace transform of the periodic function



$T = 2 \qquad F_T(t) = \begin{cases} -t, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}$

$$\mathcal{L}\{F_T(t)\} = \frac{\int_0^T e^{-st} f_T(t)\,dt}{1 - e^{-2s}} = \frac{1}{1-e^{-2s}}\left[\int_0^1 -t e^{-st}\,dt + \cancel{\int_1^2 0\, e^{-st}\,dt}\right]$$

$u = -t \qquad v = \frac{-1}{s} e^{-st}$

$du = -dt \qquad dv = e^{-st}\,dt$

$$\left[\frac{1}{s} t e^{-st} - \int \frac{-1}{s} e^{-st}(-dt)\right]_0^1 = \left[\frac{1}{s} t e^{-st} + \frac{1}{s^2} e^{-st}\right]_0^1$$

$$= \left[\frac{(st+1)e^{-st}}{s^2}\right]_0^1 = \frac{(s+1)e^{-st}}{s^2} - \frac{1}{s^2}$$

$$\mathcal{L}\{F_T(t)\} = \frac{(s+1)e^{-st} - 1}{s^2(1-e^{-2s})}$$