INSTITUTO TECNOLÓGICO DE QUERÉTARO

TECNOLÓGICO NACIONAL DE MÉXICO

ASIGNATURA: ECUACIONES DIFERENCIALES
CLAVE: ACF-0905

3ª EVALUACIÓN PARCIAL RECUPERACIÓN

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INSTRUCCIONES GENERALES: La presente evaluación consta de 6 reactivos, los cuales serán evaluados considerando el resultado y el procedimiento para llegar al mismo.

Lea cuidadosamente los reactivos antes de resolverlos.

A. Anote sus respuestas con letra clara y legible, sin borrones, enmendaduras y tachaduras, (se recomienda uso de lápiz)
B. Envíe sus respuestas en formato **PDF y en un solo archivo,** deberá subir este archivo a la plataforma de Classroom.

Utilice la definición de la transformada de Laplace $\mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st} dt$ y aplíquela en las siguientes funciones (1 punto)

1. $f(t) = t^2 e^{-2t}$
2. $f(t) = e^t cos\ t$
3. $f(t) = t\ sen\ t$

Utilice el teorema de la transformada inversa para aplicar en las siguientes funciones (1 punto)

4. $\mathcal{L}^{-1}\left[\frac{1}{(s^3+5s)}\right]$
5. $\mathcal{L}^{-1}\left[\frac{s}{(s+2)(s^2+4)}\right]$
6. $\mathcal{L}^{-1}\left[\frac{2s-4}{(s^2+s)(s^2+1)}\right]$

Resuelva las siguientes ecuaciones diferenciales aplicando derivada de la transformada de Laplace (2 puntos)

7. $x''' - x'' + x' - x = 0, \qquad x(0) = 1 \qquad x'(0) = 2 \qquad x''(0) = -1$

Resuelva la ecuación diferencial aplicando transformada de Laplace (2 puntos)

8. $t^2 x'' + 4tx' + (t^2 + 2)x = sen\ t$

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Examen de segunda oportunidad - 3 unidad

1.- $f(t) = t^2 e^{-2t}$

$$\mathcal{L}[t^2 e^{-2t}] = \int_0^\infty t^2 e^{-2t} e^{-st}\,dt$$

$$= \int_0^\infty t^2 e^{-(s+2)t}\,dt$$

$$= t^2\left(-\frac{1}{(s+2)} e^{-(s+2)t}\right)\Bigg|_0^\infty + \frac{2}{(s+2)}\int_0^\infty e^{-(s+2)t}\,t\,dt$$

$$= -\frac{t^2}{(s+2)} e^{-(s+2)t}\Bigg|_0^\infty + \frac{2}{(s+2)}\left[\frac{t e^{-(s+2)t}}{-(s+2)} + \frac{1}{(s+2)^2} e^{-(s+2)t}\right]$$

$$= -\underbrace{\frac{t^2}{s+2} e^{-(s+2)t}}_{①} - \underbrace{\frac{2t}{(s+2)^2} e^{-(s+2)t}}_{②} - \underbrace{\frac{2}{(s+2)^3} e^{-(s+2)t}}_{③}\Bigg|_0^\infty$$

Parte ①

$$= -\frac{1}{s+2}\lim_{t\to\infty} t^2 \lim_{t\to\infty} e^{-(s+2)t} + \frac{(0)}{s+2} e^0$$

usando

$$\lim_{x\to\infty} e^{kx} = 0 \quad \text{si } k<0$$

$$= -\frac{1}{s+2}\lim_{t\to\infty} t^2 (0) = 0 \quad \text{si } -(s+2)<0$$

Parte ②

$$-\frac{2}{(s+2)^2}\lim_{t\to\infty} t \lim_{t\to\infty} e^{-(s+2)t} + \frac{2(0)}{(s+2)^2} e^0 = 0$$

si $-(s+2)<0$

Parte ③

$$-\frac{2}{(s+2)^3}\lim_{t\to\infty} e^{-(s+2)t} + \frac{2}{(s+2)^3}e^{0}$$

usando

$$\lim_{t\to\infty} e^{kx} = 0, \text{ si } k<0$$

$$= -\frac{2}{(s+2)^3}(0) + \frac{2}{(s+2)^3} = \frac{2}{(s+2)^3} \quad \text{si } -(s+2)<0$$

Juntando las tres partes

$$\boxed{\mathcal{L}[t^2 e^{-2t}] = \frac{2}{(s+2)^3}}$$

2.- $f(t) = e^t \cos t$

$$\mathcal{L}[e^t\cos t] = \int_0^\infty \cos t\, e^t e^{-st}\,dt = \int_0^\infty \cos t\, e^{-(s-1)t}\,dt$$

$$= \frac{e^{-(s-1)t}}{(-(s-1))^2+1}\left(\overset{①}{-(s-1)\cos t} + \overset{②}{\operatorname{sen} t}\right)\Bigg|_0^\infty$$

Parte ①

$$= \frac{-(s-1)}{(s-1)^2+1}\lim_{t\to\infty} e^{-(s-1)t}\cos t + \frac{s-1}{(s-1)^2+1}e^{0}\cos(0)$$

Analizando el limite

$$-1 \le \cos t \le 1$$

$$-e^{-(s-1)} \le e^{-(s-1)}\cos t \le e^{-(s-1)}$$

$$-\lim_{t\to\infty} e^{-(s-1)} \le \lim_{t\to\infty} e^{-(s-1)}\cos t \le \lim_{t\to\infty} e^{-(s-1)}$$

$$0 \le \lim_{t\to\infty} e^{-(s-1)}\cos t \le 0$$

Entonces

$$\lim_{t \to \infty} e^{-(s-1)} \cos t = 0 \quad , \text{si} \quad -(s-1) < 0$$

Entonce

$$\frac{-(s-1)}{(s-1)^2+1} \lim_{t \to \infty} e^{-(s-1)} \cos t + \frac{s-1}{(s-1)^2+1} = 0 + \frac{s-1}{(s-1)^2+1}$$

Parte 2

$$\frac{1}{(s+1)^2+1} \lim_{t \to \infty} e^{-(s-1)t} \operatorname{sen} t - \frac{1}{(s-1)^2+1} e^{0} \operatorname{sen}(0)$$

$$-\lim_{t \to \infty} e^{-(s-1)t} \leq \lim_{t \to \infty} e^{-(s-1)t} \operatorname{sen} t \leq \lim_{t \to \infty} e^{-(s-1)t}$$

$$\therefore \lim_{t \to \infty} e^{-(s-1)t} \operatorname{sen} t = 0 \quad , \text{si} \quad -(s-1) < 0$$

$$= \frac{1}{(s+1)^2+1} (0) - 0 = 0$$

Juntando resultados

$$\boxed{\mathcal{L}[e^{t} \cos t] = \frac{s-1}{(s-1)^2+1}}$$

3: $f(t) = t\,sen\,t$

$$\int_0^\infty t\,sen\,t\,e^{-st}\,dt =$$

$$= t\left[\frac{e^{-st}}{s^2+1}(-s\,sen\,t - \cos t)\right]\Bigg|_0^\infty - \frac{1}{s^2+1}\int_0^\infty e^{-st}(-s\,sen\,t - \cos t)\,dt$$

① ②

Parte ②

$$-\frac{1}{s^2+1}\int_0^\infty e^{-st}(-s\,sen\,t - \cos t)\,dt$$

$$= -\frac{1}{(s^2+1)}\left[-s\int_0^\infty e^{-st}\,sen\,t - \int_0^\infty e^{-st}\cos t\,dt\right]$$

$$= -\frac{1}{s^2+1}\left[\frac{-s}{s^2+1}\left[e^{-st}(-s\,sen\,t - \cos t)\right] - \frac{e^{-st}}{s^2+1}(-s\cos t + sen\,t)\right]\Bigg|_0^\infty$$

$$= -\frac{1}{s^2+1}\left[\frac{e^{-st}}{s^2+1}\left[s^2\,sen\,t + s\cos t + s\cos t - sen\,t\right]\right]\Bigg|_0^\infty$$

$$= -\frac{1}{s^2+1}\left[\frac{s^2}{s^2+1}\lim_{t\to\infty} e^{-st}\,sen\,t + \frac{2s}{s^2+1}\lim_{t\to\infty} e^{-st}\cos t + \frac{2s}{s^2+1} - \frac{1}{s^2+1}\lim_{t\to\infty} e^{-st}\,sen\right]$$

$$= -\frac{1}{s^2+1}\left[\frac{s^2}{s^2+1}(0) + \frac{2s}{s^2+1}(0) - \frac{2s}{s^2+1} - \frac{1}{s^2+1}(0)\right] \text{ si } -s<0$$

$$= -\frac{1}{s^2+1}\left(-\frac{2s}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

Parte ① $t\left[\frac{e^{-st}}{s^2+1}(-s\,\text{sen}t-\cos t)\right]\Big|_0^{\infty}$

$$=-\frac{s}{s^2+1}\lim_{t\to\infty} t \lim_{t\to\infty} \text{sen}t\, e^{-st}-\frac{1}{s^2+1}\lim_{t\to\infty} t \lim_{t\to\infty} e^{-st}\cos t+(0)[\ ]$$

$$=-\frac{s}{s^2+1}\lim_{t\to\infty} t\ (0)-\frac{1}{s^2+1}\lim_{t\to\infty} t\ (0) \qquad \text{si } -s<0$$

$$=0$$

Juntando todo

$$\mathcal{L}[t\,\text{sen}t]=0+\frac{2s}{(s^2+1)^2}$$

$$\mathcal{L}[t\,\text{sen}t]=\frac{2s}{(s^2+1)^2}$$

4.- $\mathcal{L}^{-1}\left[\dfrac{1}{(s^3+5s)}\right]$

$$\frac{1}{(s^3+5s)} = \frac{1}{s(s^2+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+5}$$

$$1 = As^2 + 5A + Bs^2 + Cs$$

Igualando termino a termino

$A + B = 0 \rightarrow B = -A$

$C = 0$

$5A = 1 \rightarrow A = \frac{1}{5} \therefore B = -\frac{1}{5}$

$$\mathcal{L}^{-1}\left[\frac{1}{s^3+5s}\right] = \frac{1}{5}\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{1}{5}\mathcal{L}^{-1}\left[\frac{s}{s^2+5}\right]$$

usando

$\mathcal{L}^{-1}\left[\frac{a}{s}\right] = a \qquad \mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

$$\boxed{\mathcal{L}^{-1}\left[\frac{1}{s^3+5s}\right] = \frac{1}{5} - \frac{1}{5}\cos\sqrt{5}\,t}$$

5.- $\mathcal{L}^{-1}\left[\dfrac{s}{(s+2)(s^2+4)}\right]$

$$\frac{s}{(s+2)(s^2+4)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+4}$$

$s = A(s^2+4) + (Bs+C)(s+2)$

$s = A2^2 + 4A + Bs^2 + Cs + 2Bs + 2C$

$-2 = A(8) + (-2B+C)(-2+2)$

$-2 = 8A$

$A = -\frac{2}{8} = -\frac{1}{4}$

$s = 0$

$$0 = 4A + C(0+2)$$

$$0 = 4\left(-\frac{1}{4}\right) + 2C$$

$$0 = -1 + 2C$$

$$2C = 1 \quad \therefore \quad C = \frac{1}{2}$$

$s = 1$

$$1 = A(1+4) + (B+C)(1+2)$$

$$1 = 5A + 3B + 3C$$

$$1 = -\frac{5}{4} + 3B + \frac{3}{2}$$

$$-\frac{1}{2} + \frac{5}{4} = 3B$$

$$\frac{3}{4} = 3B \quad \therefore \quad B = \frac{1}{4}$$

$$\mathcal{L}^{-1}\left[\frac{s}{(s+2)(s^2+4)}\right] = -\frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{4}\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] + \frac{1}{2}\mathcal{L}\left[\frac{1}{s^2+}\right.$$

usando

$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at} \qquad \mathcal{L}^{-1}\frac{s}{s^2+a^2} = \cos at \qquad \mathcal{L}\frac{a}{s^2+a^2} = \operatorname{sen} at$$

$$\mathcal{L}^{-1}\left[\frac{s}{(s+2)(s^2+4)}\right] = -\frac{1}{4}e^{-2t} + \frac{1}{4}\cos 2t + \frac{1}{4}\operatorname{sen} 2t$$

6.- $\mathcal{L}^{-1}\left[\dfrac{2s-4}{(s^2+9)(s^2+1)}\right]$

$$\frac{2s-4}{(s^2+s)(s^2+1)} = \frac{2s-4}{s(s+1)(s^2+1)}$$

$$\frac{2s-4}{s(s+1)(s^2+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

$$2s-4 = (As+A)(s^2+1) + Bs(s^2+1) + (Cs+D)(s+1)s$$
$$2s-4 = As^3 + As + As^2 + A + Bs^3 + Bs + Cs^3 + Cs^2 + Ds^2 + Ds$$
$$2s-4 = (A+B+C)s^3 + (A+C+D)s^2 + (A+B+D)s + A$$

Igualando terminos

$A+B+C=0$

$A+C+D=0$

$A+B+D=2$

$A=-4$

sustituyendo A

$-(B+C=4)$ $\qquad -B-C=-4$

① $C+D=4$ $\Rightarrow$ $\qquad C+D=4$

② $B+D=6$ $\qquad \cancel{B}+D=6$

$2D=6$

$\leftarrow$ $D=3$

sustituyendo en ① y ②

$C+3=4 \quad \therefore \quad C=1$

$B+3=6 \quad \therefore \quad B=3$

$$\mathcal{L}^{-1}\left[\frac{2s-4}{(s^2+s)(s^2+1)}\right] = -4\,\mathcal{L}^{-1}\left[\frac{1}{s}\right] + 3\,\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\frac{s}{s^2+1} + 3\,\mathcal{L}^{-1}\frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left[\frac{2s-4}{(s^2+s)(s^2+1)}\right] = \boxed{-4 + 3e^{-t} + \cos t + 3\,\mathrm{sen}\,t}$$

$$7.-\ x''' - x'' + x' - x = 0; \quad x(0) = 1 \quad x'(0) = 2 \quad x''(0) = -1$$

$$\mathcal{L}[x'''] = s^3 \mathcal{L}[x] - x_0 s^2 - x'_0 s - x''_0 = s^3 L - s^2 - 2s + 1$$

$$\mathcal{L}[x''] = s^2 \mathcal{L}[x] - x_0 s - x_0' = s^2 L - s - 2$$

$$\mathcal{L}[x'] = s\mathcal{L}[x] - x_0 = sL - 1$$

sustituyendo

$$s^3 L - s^2 - 2s + \cancel{1} + s^2 L + s + 2 + sL - \cancel{1} - L = 0$$

$$s^3 L - s^2 L + sL - L = s^2 + 2s + s + 2$$

$$L(s^3 - s^2 + s + 1) = s^2 + s - 2$$

$$L = \frac{s^2 + s - 2}{s^3 - s^2 + s + 1} = \frac{(s+2)(s-1)}{s^2(s-1) + (s-1)}$$

$$= \frac{(s+2)\cancel{(s-1)}}{(s^2+1)\cancel{(s-1)}}$$

$$L = \frac{s+2}{s^2+1}$$

$$x = \mathcal{L}^{-1} \frac{s}{s^2+1} + 2\mathcal{L}^{-1} \frac{1}{s^2+1}$$

$$x = \cos t + 2 \operatorname{sen} t$$

usando

$$\mathcal{L}^{-1} \frac{s}{s^2 + a^2} = \cos at \qquad \mathcal{L}^{-1} \frac{a}{s^2 + a^2} = \operatorname{sen} at$$

8.- $t^2x'' + 4tx' + (t^2+2)x = \text{sent}$

$$\mathcal{L}[t^2x''] = \frac{d^2}{ds^2}\mathcal{L}[x''] = \frac{d^2}{ds^2}(s^2L - sx_0 - x'_0)$$

$$= \frac{d}{ds}(2sL + s^2L' - x_0 - 0)$$

$$\mathcal{L}[t^2x''] = 2L + 2sL' + 2sL' + s^2L'' = 2L + 4sL' + s^2L''$$

$$\mathcal{L}[tx'] = -\frac{d}{ds}\mathcal{L}[x'] = -\frac{d}{ds}(sL - x_0)$$

$$= -L - sL'$$

$\mathcal{L}[x] = L$

$$\mathcal{L}[t^2x] = \frac{d^2}{ds^2}\mathcal{L}[x] = L''$$

$$\mathcal{L}[\text{sent}] = \frac{1}{s^2+1} \qquad \mathcal{L}[\text{senat}] = \frac{a}{s^2+a^2}$$

sustituyendo

$$\cancel{2L} + \cancel{4sL'} + s^2L'' - \cancel{4L} - \cancel{4sL'} + L'' + \cancel{2L} = \frac{1}{s^2+1}$$

$$s^2L'' + L'' = \frac{1}{s^2+1}$$

$$L''(s^2+1) = \frac{1}{(s^2+1)}$$

$$L'' = \frac{1}{(s^2+1)^2} \qquad L'' = \mathcal{L}t^2x$$

$$\boxed{\mathcal{L}[t^2x] = \frac{1}{(s^2+1)^2}}$$

$$t^2x = \mathcal{L}^{-1}\frac{1}{(s^2+1)^2}$$

usando convolución

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)(s^2+1)}\right] = \mathcal{L}^{-1}\frac{1}{(s^2+1)} * \mathcal{L}^{-1}\frac{1}{s^2+1}$$

$$= \operatorname{sen} t * \operatorname{sen} t$$

$$= \int_0^t \operatorname{sen} x \,(\operatorname{sen}(t-x)\, dx$$

usando identidades

$$= \frac{1}{2}\int_0^t [\cos(x-t+x) - \cos(x+t-x)]\, dx$$

$$= \frac{1}{2}\left[\int_0^t \cos(2x-t)\, dx - \int_0^t \cos t\, dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\operatorname{sen}(2x-t)\Big|_0^t - \cos t\; x\Big|_0^t\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\operatorname{sen} t - \frac{1}{2}\operatorname{sen}(-t) - t\cos t\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\operatorname{sen} t + \frac{1}{2}\operatorname{sen} t - t\cos t\right]$$

$$= \frac{1}{2}\operatorname{sen} t - \frac{1}{2}t\cos t$$

$$\therefore \quad t^2 x = \frac{1}{2}\operatorname{sen} t - \frac{1}{2}t\cos t$$

$$x = \frac{1}{2}t^{-2}\operatorname{sen} t - \frac{1}{2t}\cos t$$