Unit I - Exam B

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1- Find the explicit solution of

$$5e^{3x}\frac{dy}{dx} = 6y^2 + 3y^2e^{6x}$$

$$\frac{dy}{dx} = \frac{6y^2+3y^2e^{6x}}{5e^{3x}} \Rightarrow \frac{dy}{dx} = \frac{y^2(6+3e^{6x})}{5e^{3x}}$$

$$\frac{dy}{y^2} = \left[\frac{6}{5}e^{-3x} + \frac{3}{5}e^{3x}\right]dx \qquad \begin{matrix} u=-3x \\ du=-3dx \end{matrix} \qquad \begin{matrix} w=3x \\ dw=3dx \end{matrix}$$

$$\int\frac{dy}{y^2} = \int\left[\frac{6e^{-3x}}{5} + \frac{3}{5}e^{3x}\right]dx \Rightarrow \frac{-1}{y} = \frac{-2}{5}\int e^u du + \frac{1}{5}\int e^w dw$$

$$\frac{-1}{y} = \frac{-2}{5e^{3x}} + \frac{1}{5}e^{3x} + C \Rightarrow \frac{-1}{y} = \frac{-2}{5e^{3x}} + \left(\frac{5e^{3x}}{5e^{3x}}\right)\frac{1}{5}e^{3x} + \left(\frac{5e^{3x}}{5e^{3x}}\right)C$$

$$\frac{-1}{y} = \frac{-2}{5e^{3x}} + \frac{e^{6x}}{5e^{3x}} + \frac{5e^{3x}C}{5e^{3x}} \Rightarrow \frac{-1}{y} = \frac{e^{6x}-2+e^{3x}(5C)}{5e^{3x}}$$

$5C \to C$ Constant

$$\frac{-5e^{3x}}{e^{6x}+Ce^{3x}-2} = y$$

$$\boxed{y = \frac{-5e^{3x}}{e^{6x}+Ce^{3x}-2}}$$

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I missed calculating the value of C.

2- Find the explicit solution of

$$\sin(x)\frac{dy}{dx} + \sec(x)y = \sec(x)e^{\tan(x)} \quad (1)$$

$$\frac{dy}{dx} + \underbrace{\frac{\sec(x)}{\sin(x)}}_{P(x)} y = \underbrace{\frac{\sec(x)}{\sin(x)}}_{Q(x)} e^{\tan(x)}$$

$$\int P(x)dx = \int \frac{\sec(x)}{\sin(x)}dx = \int \frac{\sec(x)}{\sin(x)}\left(\frac{\sec(x)}{\sec(x)}\right)dx = \int \frac{\sec^2(x)}{\tan(x)}dx$$

$u = \tan(x)$
$du = \sec^2(x)dx$

$$\int \frac{\sec^2(x)}{\tan(x)}dx = \int \frac{du}{u} = \ln|u| + c = \ln|\tan(x)| + c \leftarrow$$

$$w(x) = e^{\int P(x)dx} = e^{\ln|\tan(x)|} = \tan(x)$$

(1) multiplied by $w(x) \leftarrow$

$$\tan(x)\frac{dy}{dx} + \frac{\tan(x)\sec(x)}{\sin(x)} y = \frac{\tan(x)\sec(x)}{\sin(x)} e^{\tan(x)}$$

$$\tan(x)\frac{dy}{dx} + \sec^2(x)y = \sec^2(x)e^{\tan(x)}$$

$$\frac{d[\tan(x)y]}{dx} = \sec^2(x)e^{\tan(x)} \leftarrow \text{Death step}$$

$w = \tan(x)$
$dw = \sec^2(x)dx$

$$\int d[\tan(x)y] = \int \sec^2(x)e^{\tan(x)}dx$$

$$\tan(x)y = \int e^w dw \Rightarrow \tan(x)y = e^{\tan(x)} + c$$

$$y = \frac{e^{\tan(x)} + c}{\tan(x)}$$

$$\boxed{y = \frac{e^{\tan(x)} + c}{\tan(x)}}$$

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3.- Find the solution of

$$\underbrace{\left(6x^2\sqrt{y}\,e^{\sqrt{y}} + 2\cos(x)\,y^{7/2}\right)}_{M}dx + \underbrace{\left(x^3 e^{\sqrt{y}} + 6\operatorname{sen}(x)\,y^{5/2}\right)}_{N}dy = 0$$

Such as $y(0) = 100{,}000{,}000{,}000$

$$M_y = 6x^2\left[\sqrt{y}\cdot\frac{1}{2\sqrt{y}}e^{\sqrt{y}} + \frac{1}{2\sqrt{y}}e^{\sqrt{y}}\right] + 2\cdot\frac{7}{2}\cos(x)\,y^{5/2}$$

$$= \frac{3x^2 e^{\sqrt{y}}}{\sqrt{y}} + 3x^2 e^{\sqrt{y}} + 7\cos(x)\,y^{5/2}$$

$$N_x = 3x^2 e^{\sqrt{y}} + 6\cos(x)\,y^{5/2}$$

$$\frac{M_y - N_x}{N} = \frac{\frac{3x^2 e^{\sqrt{y}}}{\sqrt{y}} + \cancel{3x^2 e^{\sqrt{y}}} + 7\cos(x)y^{5/2} - \cancel{3x^2 e^{\sqrt{y}}} - 6\cos(x)y^{5/2})}{x^3 e^{\sqrt{y}} + 6\operatorname{sen}(x)\,y^{5/2}}$$

$$= \frac{\frac{3x^2 e^{\sqrt{y}}}{\sqrt{y}} + \cos(x)\,y^{5/2}}{x^3 e^{\sqrt{y}} + 6\operatorname{sen}(x)y^{5/2}}$$ ← Very hard to make division

$$\frac{N_x - M_y}{M} = \frac{3x^2 e^{\sqrt{y}} + 6\cos(x)y^{5/2} - \frac{3x^2 e^{\sqrt{y}}}{\sqrt{y}} - 3x^2 e^{\sqrt{y}} - 7\cos(x)y^{5/2}}{6x^2\sqrt{y}\,e^{\sqrt{y}} + 2\cos(x)\,y^{7/2}}$$

$$= \frac{-\cos(x)y^{5/2} - \frac{3x^2 e^{\sqrt{y}}}{\sqrt{y}}}{6x^2\sqrt{y}\,e^{\sqrt{y}} + 2\cos y^{7/2}}$$ ← Same case

It can't be solved.

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I did not realize that I could moltiply by y/y to get a common factor and simplify the fraction

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4: $\underbrace{2x^3y}_{4} \underbrace{-x^3y}_{4}\frac{dy}{dx} = \underbrace{x^4}_{4}$ such as $y(1)=2$

It is a homogeneous D.E because all terms have power "4" combined

$\frac{1}{x^3}\left[2x^3y - x^3y\frac{dy}{dx}\right] = \left[x^4\right]\frac{1}{x^3}$

$v = \frac{y}{x}$ $\quad y = vx$

$\frac{dy}{dx} = x\frac{dv}{dx} + v$

$2y - y\frac{dy}{dx} = x$

$dy = xdv + vdx$

$2y\,dx - y\,dy = x\,dx$

$-y\,dy = x\,dx - 2y\,dx \Rightarrow -y\,dy = (x-2y)\,dx$

$-vx(xdv + vdx) = (x - 2vx)dx$

$-v\cancel{x}(xdv + vdx) = (1-2v)\cancel{x}dx$

$\Rightarrow -v(xdv + vdx) = (1-2v)dx$

$= -vxdv - v^2dx = (1-2v)dx$

$-vxdv = (v^2 - 2v + 1)dx$

$\frac{v}{v^2-2v+1}dv = -\frac{1}{x}dx \Rightarrow \int\frac{v\,dv}{(v-1)^2} = -\int\frac{dx}{x}$

$\Rightarrow \int\frac{w+1}{w^2}dw = -\int\frac{dx}{x}$

$w = v-1 \quad dw = dv$

$v = w+1$

$\Rightarrow \ln|w| - \frac{1}{w} = -\ln|x| + c$

$\Rightarrow \ln|v+1| - \frac{1}{v-1} = -\ln|x| + c$

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$\Rightarrow \ln\left|\frac{y}{x}+1\right| - \frac{1}{\frac{y}{x}-1} = -\ln|x| + c$

$x = 1 \quad y = 2$

$\ln|2+1| - \frac{1}{1} = -\ln|1| + c$

$c = -1$

$\boxed{\ln\left|\frac{y}{x}-1\right| - \frac{1}{\frac{y}{x}-1} = -\ln|x| - 1}$

$\ln\left|\frac{y}{x}-1\right| - \frac{1}{\frac{y}{x}-1} = -\ln|x| - 1 \Rightarrow \frac{x}{y-x} - \ln|y-x| = 1$ ←

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5- A tank has 50 lts of liquid mixture and initially contains 90gr of salt. Pure clean water is pumped into the tank at rate of 11 lts/min and the well mixed salt solution.

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Volume = 50 lts

Flow rate input = Flow rate output 11 lts/min

I saw 11 instead of 1, but the process I followed is the same

$Q(0) = 90gr$

$$\frac{dQ(t)}{dt} = C_iV_i - C_oV_o = \cancel{\left(0\,\frac{g}{liter}\right)\left(\frac{11\,liters}{min}\right)} - \left(\frac{Q(t)}{50\,lts}\right)\left(\frac{11\,liters}{min}\right)$$

$$\frac{dQ(t)}{dt} = -\frac{11}{50}Q(t) \Rightarrow \frac{dQ(t)}{Q(t)} = \frac{-11}{50}dt \Rightarrow \int\frac{dQ(t)}{Q(t)} = \int -\frac{11}{50}dt$$

$$\ln|Q(t)| = \frac{-11}{50}t + C \Rightarrow e^{\ln|Q(t)|} = e^{\frac{-11}{50}t + C} \Rightarrow Q(t) = Ce^{-\frac{11}{50}t}$$

$$Q(0) = C\cancel{e^{-\frac{11}{50}0}} \to 1 = 90 \Rightarrow C = 90$$

$$Q(t) = 90e^{-\frac{11}{50}t}$$

A) What is the quantity of salt after 20min?

$$Q(20) = 90e^{-\frac{11}{50}(20)} = 90e^{-\frac{22}{5}} \approx 1.104gr$$

B) How long it will take for the tank to have 1gr of salt?

$$1 = 90e^{-\frac{11}{50}t} \Rightarrow \frac{1}{90} = e^{-\frac{11}{50}t} \Rightarrow \ln|1/90| = \ln|e^{-11/50t}| \Rightarrow \ln|1/90| = -\frac{11}{50}t$$

$$t = \frac{50\ln|1/90|}{-11} \approx 20.45\,min$$

C) When the tank will be free of salt (only pure water)?

Tricky question, if we observe $\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} 90e^{-\frac{11}{50}t}$

$$= \lim_{t\to\infty}\frac{90}{e^{\frac{11}{50}t}} = \frac{90}{\infty} = 0$$

Therefore after 20.45 minutes the amount of salt will be getting closer to 0.