Introdução: Complexos

Processamento Digital de Sinais

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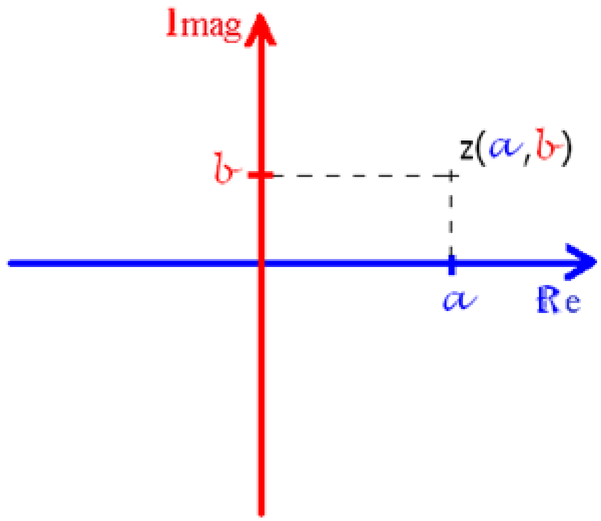
Revisão: Números complexos

- <u>Afixo do complexo</u>: $z \in \mathbb{C} : z(a,b), \quad a,b \in \mathbb{R}$
- **<u>Parte real</u>** de um complexo: $\mathrm{Re}(z) = a$
- **<u>Parte imaginária</u>** de um complexo: $\mathrm{Imag}(z) = b$
- **Unidade imaginária:** $j = \sqrt{-1}$



- **Propriedades**

$$j = \sqrt{-1}$$

$$j^2 = \left(\sqrt{-1}\right)^2 = -1$$

$$j^3 = \left(\sqrt{-1}\right)^3 = \left(\sqrt{-1}\right)^2 \sqrt{-1} = -j$$

$$j^4 = \left(\sqrt{-1}\right)^4 = \left(\sqrt{-1}\right)^2 \left(\sqrt{-1}\right)^2 = -1 \times (-1) = 1$$

$$j^5 = j^4 j = j$$

$j^p = j^r$ sendo r tal que:

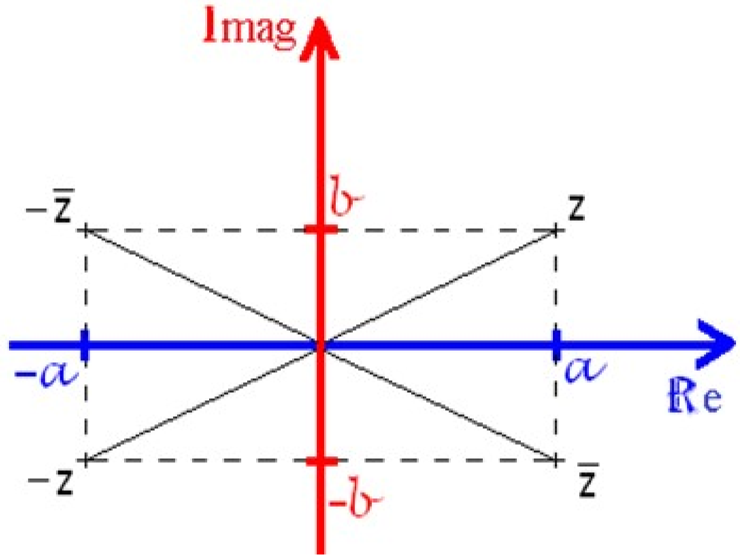
$$\begin{array}{r|l} p & 4 \\ \hline r & q \end{array}$$

pois as potências de j têm período igual a 4.

$\bar{z} = a - jb$ é o complexo conjugado de z

$-z = -a - jb$ é o complexo simétrico de z

$-\bar{z} = -a + jb$ é o simétrico do conjugado de z

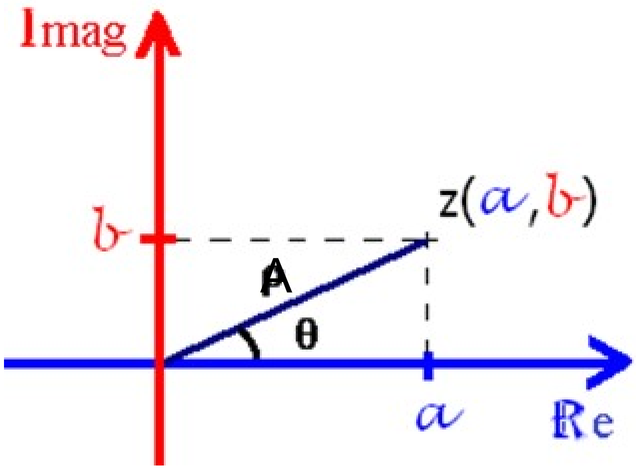


- Forma trigonométrica

$$z = A\,cis(\theta)\begin{cases} A = \sqrt{a^2 + b^2}, & \text{módulo do complexo } z \to |z| = A \\ \theta = Arg(z) = \arctan\left(\dfrac{b}{a}\right), & \text{argumento do complexo } z \end{cases}$$

- Fórmula de Euler

$$z = A\,cis\left(\theta\right) = Ae^{j\theta}$$



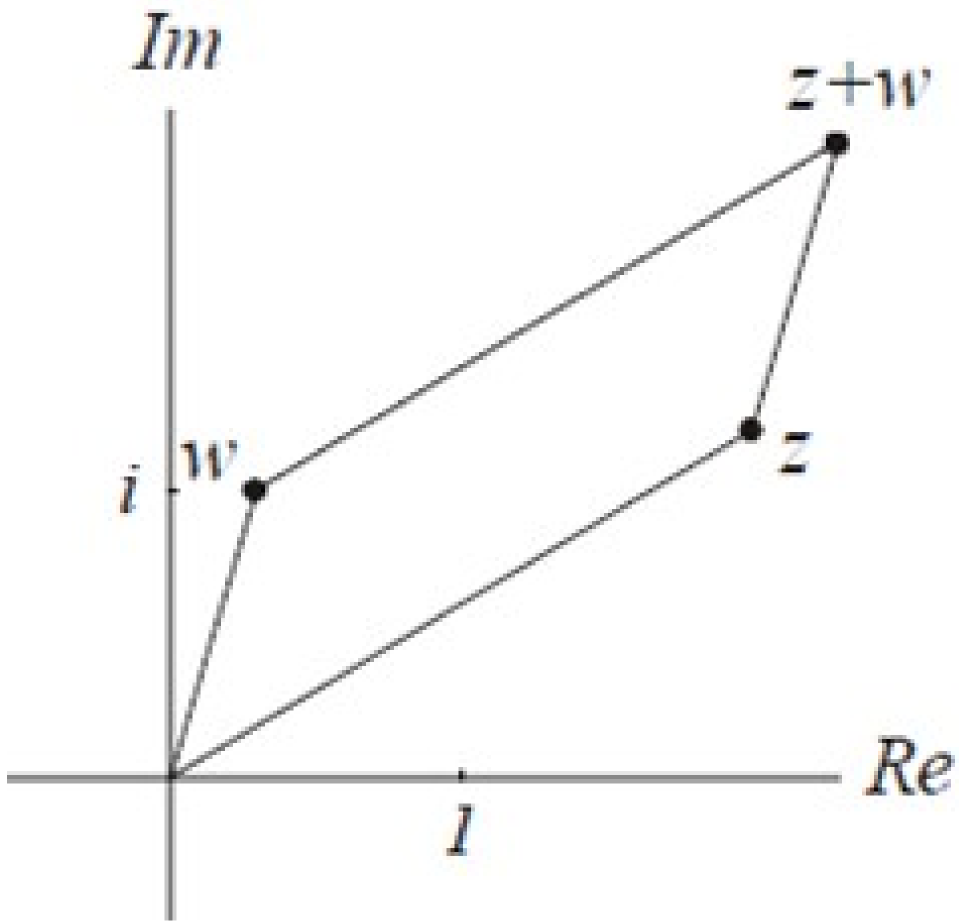
Operações com números complexos

- **Adição**

Forma mais conveniente: forma algébrica

$$\left.\begin{array}{l} z_1 = a_1 + jb_1 \\ z_2 = a_2 + jb_2 \end{array}\right\} \Rightarrow z_1 + z_2 = a_1 + a_2 + j\left(b_1 + b_2\right)$$

Exemplo



Multiplicação

Na forma algébrica:

$$\left.\begin{aligned} z_1 &= a_1 + jb_1 \\ z_2 &= a_2 + jb_2 \end{aligned}\right\} \Rightarrow z = z_1 z_2 = \underbrace{a_1 a_2 - b_1 b_2}_{\mathrm{Re}(z)} + j\left(\underbrace{a_1 b_2 + b_1 a_2}_{\mathrm{Imag}(z)}\right)$$

Na forma exponencial.

$$\left.\begin{aligned} z_1 &= A_1 e^{j\theta_1} \\ z_2 &= A_2 e^{j\theta_2} \end{aligned}\right\} \qquad z = z_1 z_2 = A_1 A_2 e^{j(\theta_1 + \theta_2)}$$

Divisão

Na forma algébrica:

$$\left.\begin{array}{l} z_1 = a_1 + jb_1 \\ z_2 = a_2 + jb_2 \end{array}\right\} \Rightarrow \frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \frac{a_1 a_2 + b_1 b_2 + j\left(b_1 a_2 - a_1 b_2\right)}{a_2^2 + b_2^2}$$

Na forma exponencial.

$$\left.\begin{array}{l} z_1 = A_1 e^{j\theta_1} \\ z_2 = A_2 e^{j\theta_2} \end{array}\right\} \rightarrow \quad z = \frac{z_1}{z_2} = \frac{A_1}{A_2} e^{j(\theta_1 - \theta_2)}$$

Potenciação

Na forma exponencial:

$$z = Ae^{j\theta} \qquad z^n = A^n e^{j(n\theta)}, \quad n \in \mathbb{Z}$$

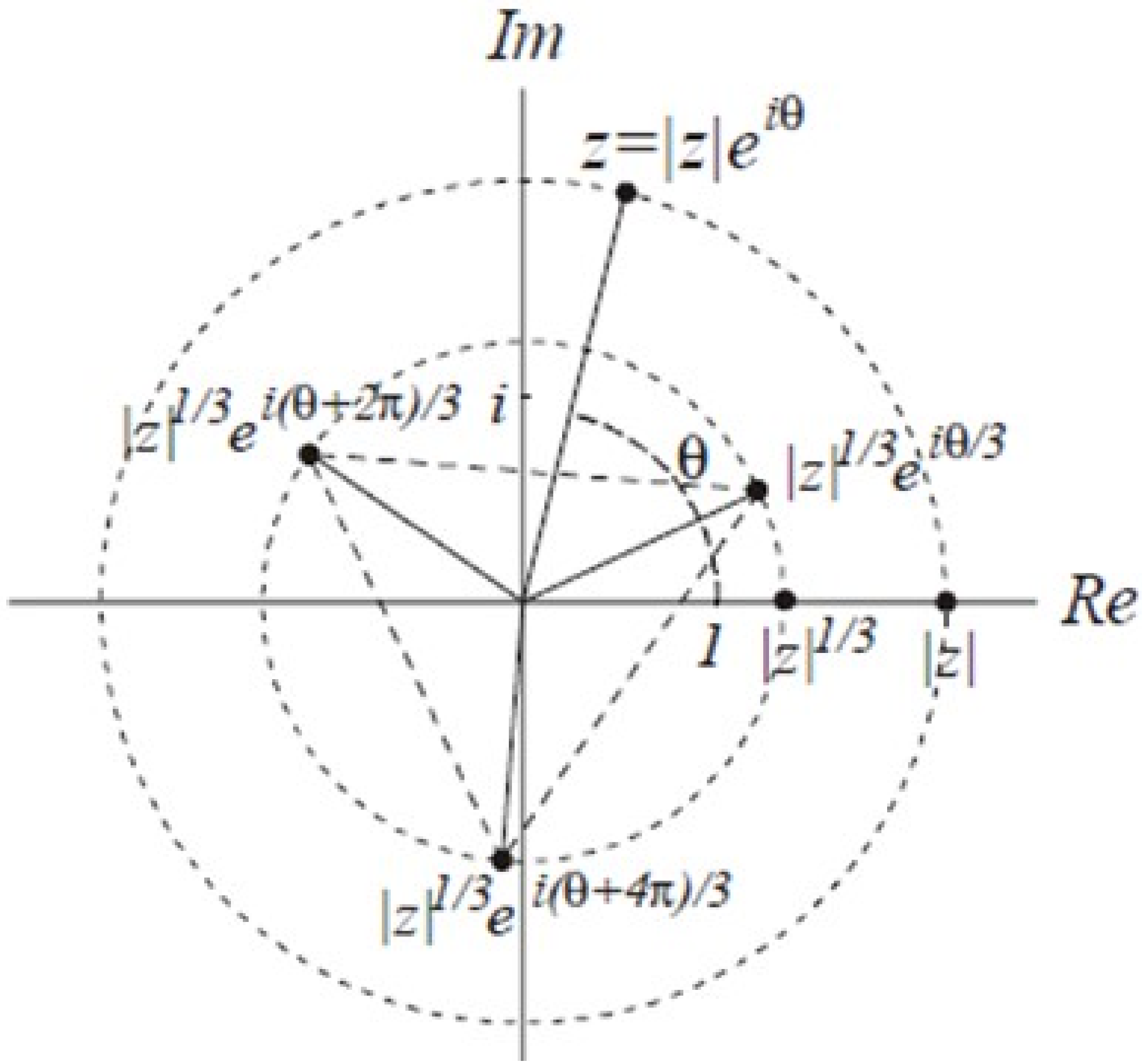
Radiciação

Na forma exponencial:

$$z = Ae^{j\theta} \qquad \sqrt[n]{z} = \sqrt[n]{A}\ e^{j\left(\frac{\theta+2k\pi}{n}\right)}$$

$$k = 0,1,\ldots,n-1$$

Exemplo



Im
Re
$z=|z|e^{i\theta}$
$|z|^{1/3}e^{i(\theta+2\pi)/3}$
$|z|^{1/3}e^{i\theta/3}$
$|z|^{1/3}e^{i(\theta+4\pi)/3}$
i
θ
1
$|z|^{1/3}$
$|z|$

- **NOTA**: De um modo geral, para efectuar operações com números complexos, deve-se escolher a forma algébrica ou a forma trigonométrica consoante a operação em causa.
- Assim são boas escolhas as seguintes:

$$a + jb \quad \text{para} \begin{cases} + \\ - \\ \times \\ \div \end{cases} \qquad\qquad Ae^{j\theta} \quad \text{para} \begin{cases} \times \\ \div \\ (\ )^n \\ \sqrt[n]{\ } \end{cases}$$

Fórmulas de Euler inversas

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Exercícios

$$\boldsymbol{x} = 4\boldsymbol{e}^{\boldsymbol{j}\frac{5\pi}{4}}$$

1. Dados os números complexos:

$$\boldsymbol{y} = \frac{3\sqrt{2}}{2} + \boldsymbol{j}\frac{3\sqrt{2}}{2}$$

a) Calcule:

$\boldsymbol{i})\ \ \boldsymbol{x}+\boldsymbol{y}\ ,\quad \boldsymbol{ii})\ \ |\boldsymbol{y}|\ ,\quad \boldsymbol{iii})\ \ \boldsymbol{arg}(\boldsymbol{y})$ ou $\angle \boldsymbol{y}$,

$\boldsymbol{iv})\ \ \boldsymbol{xy}\ ,\quad \boldsymbol{v})\ \ \dfrac{\boldsymbol{x}}{\boldsymbol{y}}\ ,\quad \boldsymbol{vi})\ \ \boldsymbol{x}^3$

b) Represente no plano complexo os afixos de ***x*** , ***y, x+y, xy e x/y.***

2. Repita o exercício anterior considerando: $\boldsymbol{x} = 2\boldsymbol{e}^{\boldsymbol{j}\frac{\pi}{3}}$

$$\boldsymbol{y} = -1 + \boldsymbol{j}\sqrt{3}$$

3. Represente graficamente, para cada inteiro $\boldsymbol{n} :\ -4 \le \boldsymbol{n} \le 1$ o módulo e o argumento (entre $-\pi$ e π) do sinal:

$$\boldsymbol{x}[\boldsymbol{n}] = (\boldsymbol{n}+1)\,\boldsymbol{e}^{\boldsymbol{j}\frac{3\pi}{4}\boldsymbol{n}}$$

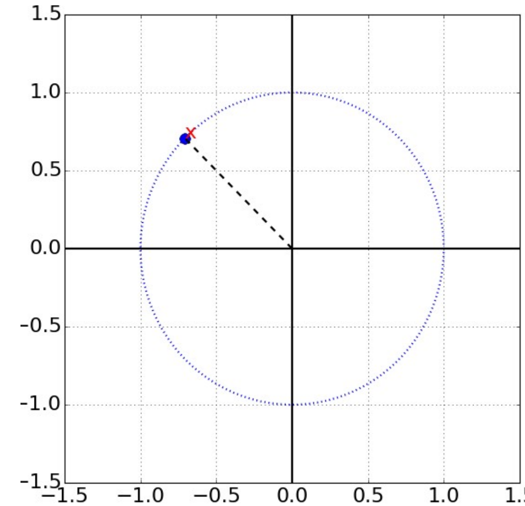
Resolução

1 – Antes de proceder às operações, escrevemos x e y ambos na forma algébrica e exponencial

$$\boldsymbol{x} = 4\boldsymbol{e}^{\boldsymbol{j}\frac{5\pi}{4}}$$

$$\boldsymbol{x} = 4\cos\left(\frac{5\pi}{4}\right) + 4\boldsymbol{j}\sin\left(\frac{5\pi}{4}\right) =$$

$$= -2.828 - \boldsymbol{j}2.828$$

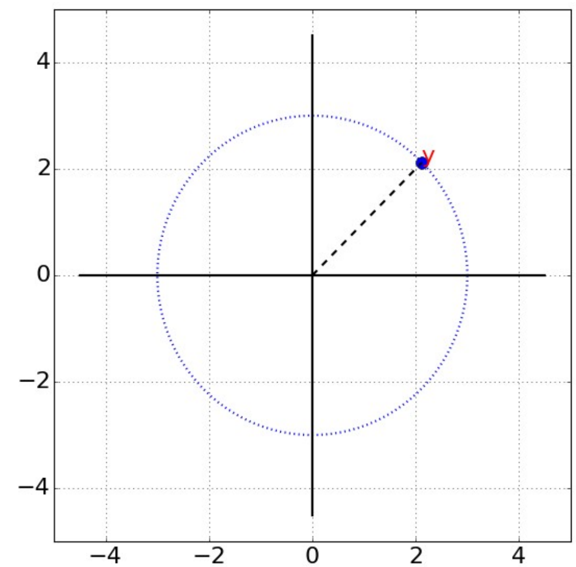


$$\boldsymbol{y} = \frac{3\sqrt{2}}{2} + \boldsymbol{j}\frac{3\sqrt{2}}{2} = 2.121 + \boldsymbol{j}2.121$$

$$|\boldsymbol{y}| = \sqrt{\left(\frac{3\sqrt{2}}{2}\right)^2 + \left(\frac{3\sqrt{2}}{2}\right)^2} = 3$$

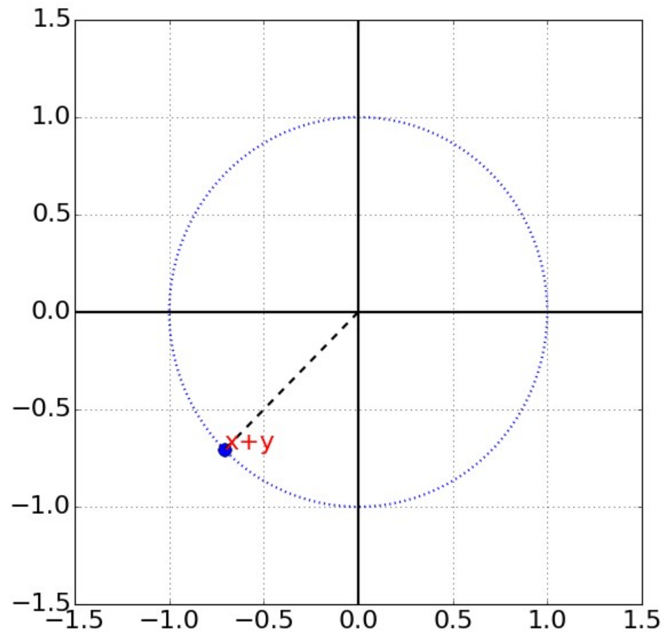
$$\angle\boldsymbol{y} = \boldsymbol{arctan}\left(\frac{\frac{3\sqrt{2}}{2}}{\frac{3\sqrt{2}}{2}}\right) = \frac{\pi}{4} + \boldsymbol{k}\pi \quad \text{e} \quad \boldsymbol{k} = 0$$

$$\boldsymbol{y} = 3\boldsymbol{e}^{\boldsymbol{j}\frac{\pi}{4}}$$



$$i)\quad x+y=-2.828-j2.828+2.121+j2.121=-0.707-j0.707=$$

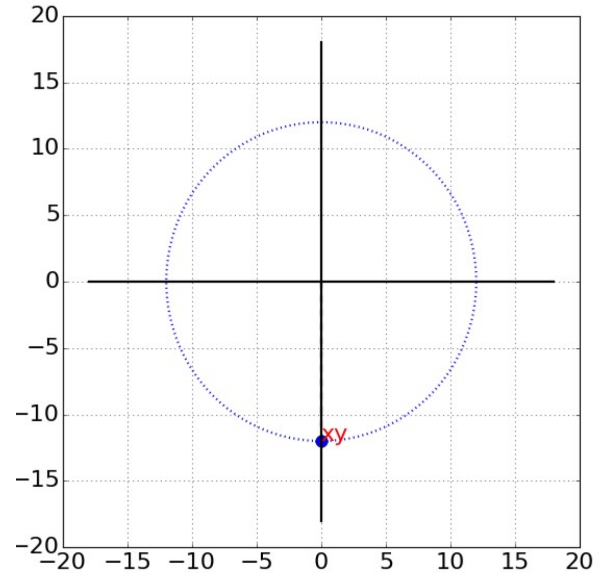
$$=1e^{-j\frac{3\pi}{4}}$$



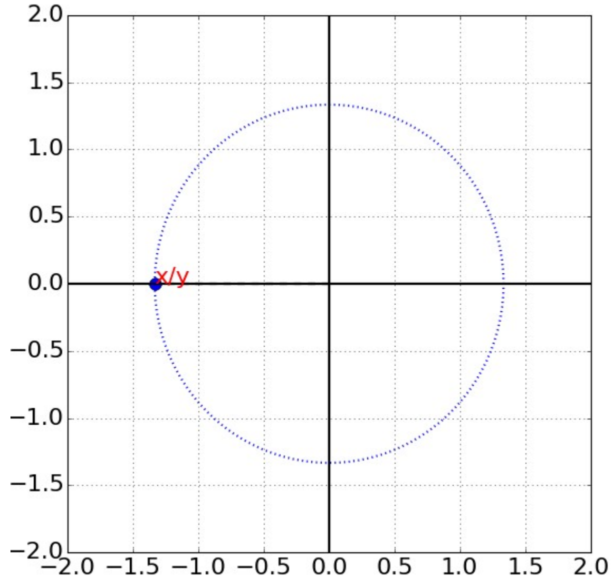
$$ii)\quad |y|=3$$

$$iii)\quad arg(y)=\frac{\pi}{4}$$

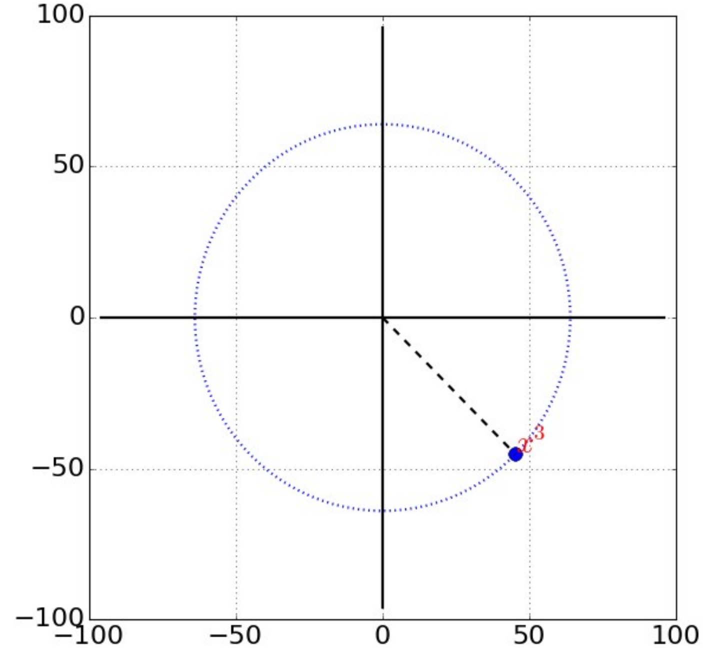
$$iv)\quad xy = 4e^{j\frac{5\pi}{4}}3e^{j\frac{\pi}{4}} = 12e^{j\frac{3\pi}{2}} = -12j$$



$$v)\quad \frac{x}{y} = \frac{4e^{j\frac{5\pi}{4}}}{3e^{j\frac{\pi}{4}}} = \frac{4}{3}e^{j\pi} = -\frac{4}{3}$$



$$vi)\quad x^3 = \left(4e^{j\frac{5\pi}{4}}\right)^3 = 64e^{j\frac{15\pi}{4}} = 64e^{-j\frac{\pi}{4}}$$



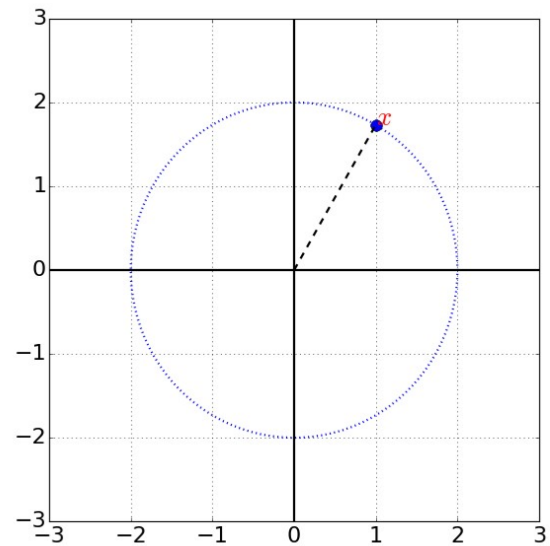
Resolução

2 – Antes de proceder às operações, escrevemos x e y ambos na forma algébrica e exponencial

$$\boldsymbol{x} = 2\boldsymbol{e}^{\boldsymbol{j}\frac{\pi}{3}}$$

$$\boldsymbol{x} = 2\cos\left(\frac{\pi}{3}\right) + 2\boldsymbol{j}\sin\left(\frac{\pi}{3}\right) =$$

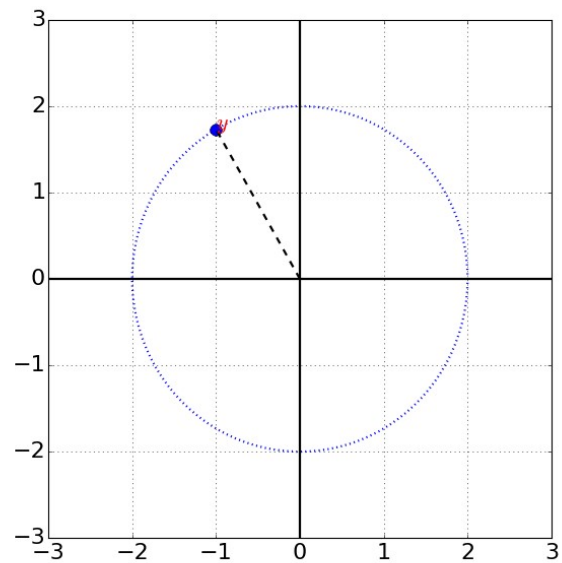
$$= 1 + \boldsymbol{j}1.732$$



$$\boldsymbol{y} = -1 + \boldsymbol{j}\sqrt{3}$$

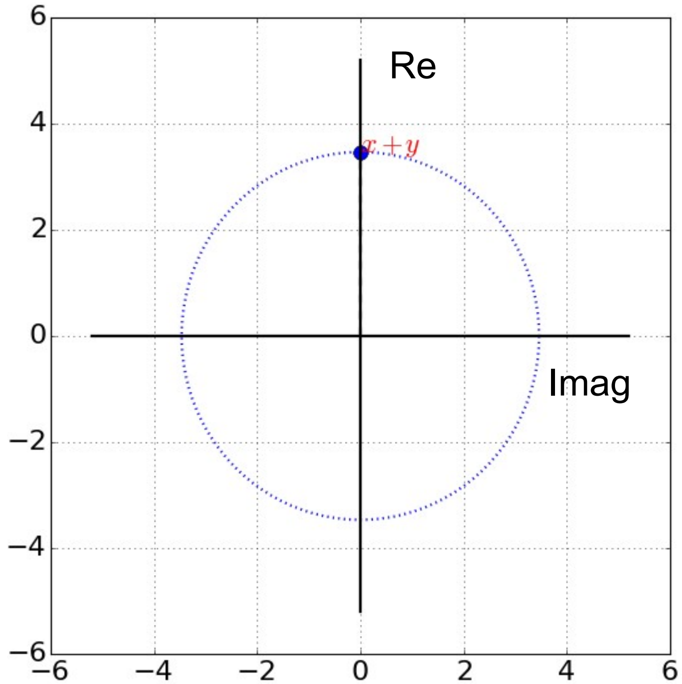
$$|\boldsymbol{y}| = \sqrt{(1)^2 + \left(\sqrt{3}\right)^2} = 2$$

$$\angle\boldsymbol{y} = \boldsymbol{arctan}\left(\frac{\sqrt{3}}{-1}\right) = \frac{2\pi}{3} + \boldsymbol{k}\pi \quad \text{e} \quad \boldsymbol{k} = 0$$



$$\boldsymbol{y} = 2\boldsymbol{e}^{\boldsymbol{j}\frac{2\pi}{3}}$$

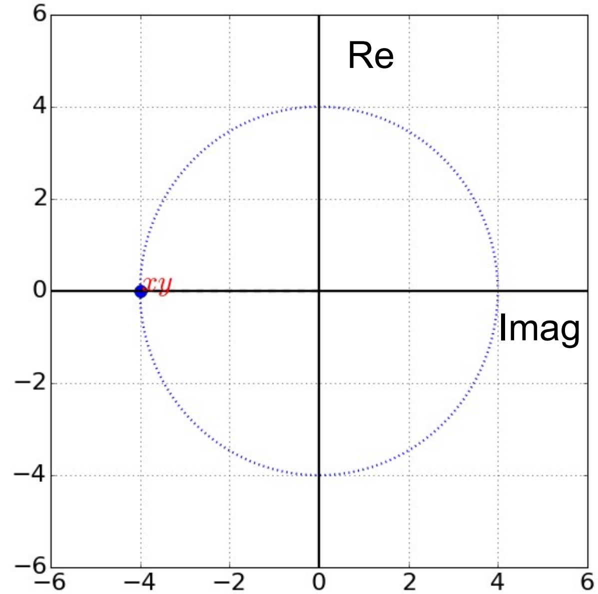
$$i)\quad x + y = 1 + j1.732 - 1 + j1.732 = j3.464 = 3.464e^{j\frac{\pi}{2}}$$



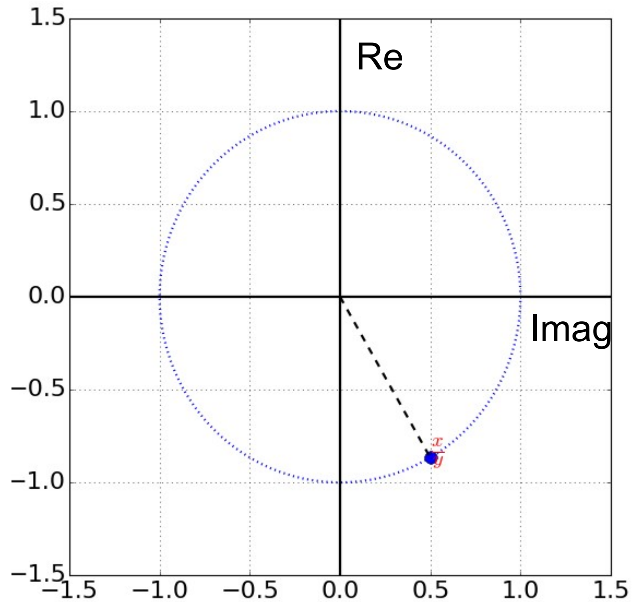
$$ii)\quad |y| = 2$$

$$iii)\quad arg(y) = \frac{2\pi}{3}$$

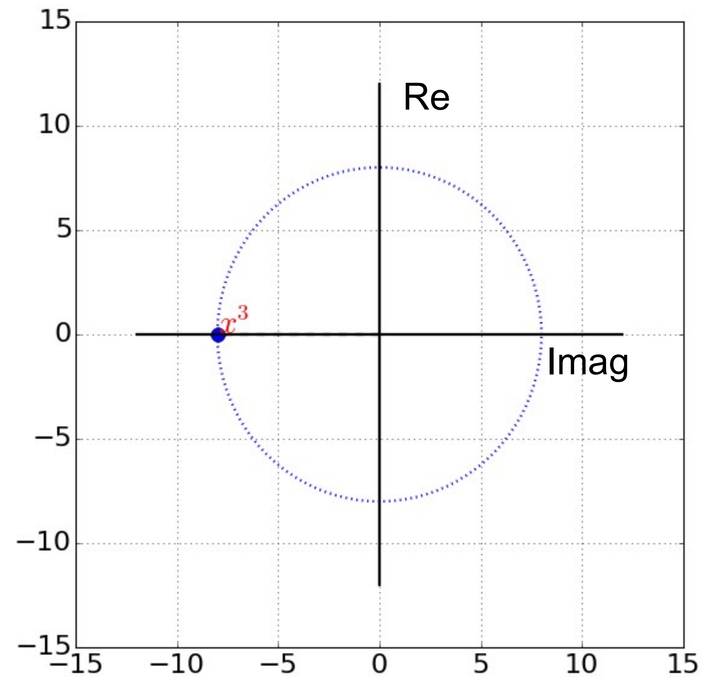
$$iv)\quad xy = 2e^{j\frac{\pi}{3}}2e^{j\frac{2\pi}{3}} = 4e^{j\pi} = -4$$



$$v)\quad \frac{x}{y} = \frac{2e^{j\frac{\pi}{3}}}{2e^{j\frac{2\pi}{3}}} = 1e^{-j\frac{\pi}{3}}$$



$$vi)\quad x^3 = \left(2e^{j\frac{\pi}{3}}\right)^3 = 8e^{j\pi} = -8$$



3 -

$$x[-4] = (-4+1)e^{j\frac{3\pi}{4}(-4)} = -3e^{-j3\pi} = -3e^{-j\pi} = 3$$

$$x[-3] = (-3+1)e^{j\frac{3\pi}{4}(-3)} = -2e^{-j\frac{9\pi}{4}} = 2e^{j\pi}e^{-j\frac{9\pi}{4}} =$$

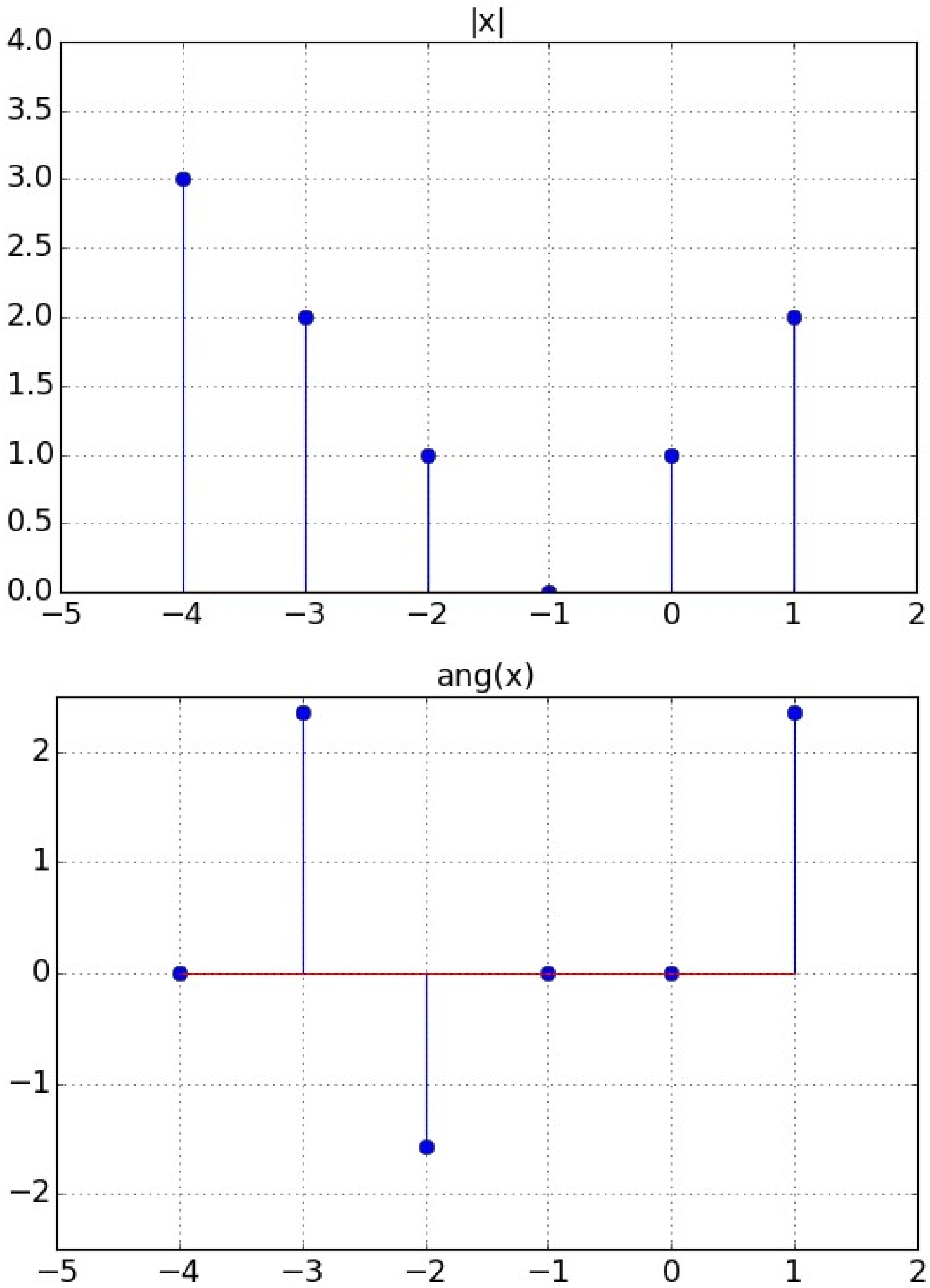
$$= 2e^{-j\frac{5\pi}{4}} = 2e^{j\left(-\frac{5\pi}{4}+2\pi\right)} = 2e^{j\frac{3\pi}{4}}$$

$$x[-2] = (-2+1)e^{j\frac{3\pi}{4}(-2)} = -e^{-j\frac{3\pi}{2}} = e^{-j\frac{3\pi}{2}}e^{j\pi} = e^{-j\frac{\pi}{2}}$$

$$x[-1] = (-1+1)e^{-j\frac{3\pi}{4}} = 0$$

$$x[0] = (0+1)e^{j\frac{3\pi}{4}0} = 1$$

$$x[1] = (1+1)e^{j\frac{3\pi}{4}1} = 2e^{j\frac{3\pi}{4}}$$


|x|
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
−5
−4
−3
−2
−1
0
1
2
ang(x)
2
1
0
−1
−2
−5
−4
−3
−2
−1
0
1
2

Exercícios

4. Converta para a forma polar (fórmula de Euler):

$$\boldsymbol{a})\ \ \boldsymbol{z} = 10\boldsymbol{j}\ \ ,\ \ \boldsymbol{b})\ \ \boldsymbol{z} = -10\ \ ,\ \ \boldsymbol{c})\ \ \boldsymbol{z} = (-10, -10)\ \ ,$$

$$\boldsymbol{d})\ \ \boldsymbol{z} = -2 + 2\boldsymbol{j}\ \ ,\ \ \boldsymbol{e})\ \ \boldsymbol{z} = -3 + \sqrt{3}\boldsymbol{j}\ \ ,\ \ \boldsymbol{f})\ \ \boldsymbol{z} = 20$$

5. Converta para a forma algébrica e determine as partes real e imaginária de cada um dos seguintes complexos:

$$\boldsymbol{a})\ \ \boldsymbol{z} = 3\sqrt{2}\boldsymbol{e}^{-\boldsymbol{j}\frac{3\pi}{4}} \qquad \boldsymbol{c})\ \ \boldsymbol{z} = 4\angle\left(\frac{\pi}{3}\right)$$ (módulo 4 e argumento $\frac{\pi}{3}$)

$$\boldsymbol{b})\ \ \boldsymbol{z} = 5\boldsymbol{e}^{\boldsymbol{j}\frac{\pi}{2}} \qquad \boldsymbol{d})\ \ \boldsymbol{z} = 5\angle(-61\pi)$$

4

a) $z = 10j = 0 + 10j = 10e^{j\pi/2}$

b) $z = -10 = 10e^{j\pi}$

c) $z = -10 - 10j \longrightarrow |z| = \sqrt{(-10)^2 + (-10)^2} = \sqrt{200} = 10\sqrt{2} = 14.14$

$= 10\sqrt{2}\, e^{-j\frac{3\pi}{4}}$ $\quad \measuredangle z = \operatorname{arctg}\left(\underset{\substack{\uparrow \\ 3^{\circ}Q}}{\frac{-10}{-10}}\right) = \pi/4 - \pi = -\frac{3\pi}{4}$

d) $z = -2 + 2j \longrightarrow |z| = \sqrt{(-2)^2 + 2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$= 2\sqrt{2}\, e^{j\frac{3\pi}{4}}$ $\quad \measuredangle z = \operatorname{arctg}\left(\underset{\substack{\uparrow \\ 2^{\circ}Q}}{\frac{2}{-2}}\right) = \frac{3\pi}{4}$

e) $z = -3 + \sqrt{3}j \longrightarrow |z| = \sqrt{(-3)^2 + (\sqrt{3})^2} = \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}$

$= 2\sqrt{3}\, e^{j\frac{5\pi}{6}}$ $\quad \measuredangle z = \operatorname{arctg}\left(\frac{\sqrt{3}}{-3}\right) = \operatorname{arctg}\left(\frac{1}{-\sqrt{3}}\right) = \frac{5\pi}{6}$

f) $z = 20 = 20e^{j0}$

5

a) $z = 3\sqrt{2}\,e^{-j\frac{3\pi}{4}} = 3\sqrt{2}\cos\left(\frac{3\pi}{4}\right) - 3\sqrt{2}\,j\sin\left(\frac{3\pi}{4}\right) = -3-3j$

b) $z = 5e^{j\pi/2} = 5\cos\pi/2 + 5j\sin\pi/2 = 5j$

c) $z = 4e^{j\pi/3} = 4\cos\pi/3 + 4j\sin\pi/3 = 2+j2\sqrt{3} = 2+j\,3.464$

d) $z = 5e^{-j61\pi} = 5e^{-j60\pi}\cdot e^{-j\pi} = -5$

6

a) $\bar{z}_1 = -2-2j$

b) $j z_2 = 3j\,e^{-j\frac{3\pi}{4}} = 3\,e^{j\pi/2}\,e^{-j\frac{3\pi}{4}} = 3e^{-j\pi/4} = \frac{3\sqrt{2}}{2} - j\frac{3\sqrt{2}}{2}$

c) $\frac{z_2}{z_1} = \frac{3e^{-j\frac{3\pi}{4}}}{2\sqrt{2}\,e^{j\frac{3\pi}{4}}} = \frac{3}{2\sqrt{2}}e^{-j\frac{3\pi}{2}} = \frac{3}{2\sqrt{2}}e^{j\pi/2} = j\,1.061$

d) $z_2^2 = 3^2 e^{-j2\times\frac{3\pi}{4}} = 9e^{-j\frac{3\pi}{2}} = 9e^{j\pi/2} = 9j$

e) $1/z_1 = \frac{1}{2\sqrt{2}\,e^{j\frac{3\pi}{4}}} = \frac{1}{2\sqrt{2}}e^{-j\frac{3\pi}{4}} = \frac{1}{2\cancel{\sqrt{2}}}\frac{\cancel{\sqrt{2}}}{2} - \frac{1}{2\cancel{\sqrt{2}}}\frac{\cancel{\sqrt{2}}}{2}j = \frac{1}{4} - \frac{1}{4}j$

Exercícios

6. Considere os seguintes complexos:

$$z_1 = -2 + 2j \quad \text{e} \quad z_2 = 3e^{-j\frac{3\pi}{4}}$$

Determine:

a) z_1^*	d) z_2^2	g) $z_1 + z_2^*$
b) jz_2	e) $z_1^{-1} = 1/z_1$	h) $\lvert z_2 \rvert^2 = z_2 z_2^*$
c) z_2/z_1	f) $z_1 z_2$	i) $z_2 + z_2^*$

7. Simplifique, apresentando o resultado na forma polar de Euler:

a) $z_a = e^{-j\frac{5\pi}{3}} + 2e^{j\frac{5\pi}{6}}$ b) $z_a = \sqrt{2}e^{j\frac{\pi}{4}} + \sqrt{2}e^{-j\frac{\pi}{4}} - 1$

6.

a) $\bar{z}_1 = -2-2j$

b) $j z_2 = 3j\, e^{-j\frac{3\pi}{4}} = 3\, e^{j\pi/2}\, e^{-j\frac{3\pi}{4}} = 3e^{-j\pi/4} = \frac{3\sqrt{2}}{2} - j\frac{3\sqrt{2}}{2}$

c) $\frac{z_2}{z_1} = \frac{3e^{-j\frac{3\pi}{4}}}{2\sqrt{2}\, e^{j\frac{3\pi}{4}}} = \frac{3}{2\sqrt{2}} e^{-j\frac{3\pi}{2}} = \frac{3}{2\sqrt{2}} e^{j\pi/2} = j\, 1.061$

d) $z_2^2 = 3^2 e^{-j2\times\frac{3\pi}{4}} = 9e^{-j\frac{3\pi}{2}} = 9e^{j\pi/2} = 9j$

e) $1/z_1 = \frac{1}{2\sqrt{2}\, e^{j\frac{3\pi}{4}}} = \frac{1}{2\sqrt{2}} e^{-j\frac{3\pi}{4}} = \frac{1}{2\cancel{\sqrt{2}}}\frac{\cancel{\sqrt{2}}}{2} - \frac{1}{2\cancel{\sqrt{2}}}\frac{\cancel{\sqrt{2}}}{2} j = \frac{1}{4} - \frac{1}{4}j$

f) $z_1 z_2 = 2\sqrt{2}\, \cancel{e^{j\frac{3\pi}{4}}} \times 3\, \cancel{e^{-j\frac{3\pi}{4}}} = 6\sqrt{2} = 8.485$

g) $z_1 + \bar{z}_2 = -2+2j - 2.121 + 2.121j = -4.121 + 4.121j$

h) $|z_2|^2 = z_2 \bar{z}_2 = 3\cancel{e^{-j\frac{3\pi}{4}}} \times 3\cancel{e^{j\frac{3\pi}{4}}} = 9$

i) $z_2 + \bar{z}_2 = -3\frac{\sqrt{2}}{2} - \cancel{3\frac{\sqrt{2}}{2}j} - 3\frac{\sqrt{2}}{2} + \cancel{3\frac{\sqrt{2}}{2}j} = -3\sqrt{2} = -4.243$

7

a) $z_a = e^{-j\frac{5\pi}{3}} + 2e^{j\frac{5\pi}{6}} = \cos\frac{5\pi}{3} - j\sin\frac{5\pi}{3} + 2\cos\frac{5\pi}{6} + j2\sin\frac{5\pi}{6} =$

$= \frac{1}{2} + j\frac{\sqrt{3}}{2} - \cancel{2}\frac{\sqrt{3}}{\cancel{2}} + j\cancel{2}\frac{1}{\cancel{2}} = -1.232 + 1.866j$

$|z_a| = \sqrt{(-1.232)^2 + (1.866)^2} = 2.236$

$\sphericalangle z_a = \arctan\frac{1.866}{-1,232} = 0.686\pi$

$2^o q$

$z_a = 2.236\, e^{j0.686\pi} = 2.236\, e^{j2.155}$

b) $Z_a = \sqrt{2}\,e^{j\pi/4} + \sqrt{2}\,e^{-j\pi/4} - 1 = \sqrt{2}\cos\pi/4 + \cancel{\sqrt{2}j\sin\pi/4} + \sqrt{2}\cos\pi/4 - \cancel{j\sqrt{2}\sin\pi/4}$

$- 1 = \sqrt{2} \times \frac{\sqrt{2}}{2} + \sqrt{2}\frac{\sqrt{2}}{2} - 1 = 1 = 1\,e^{j0}$

Exercícios

8. Considere o seguinte excerto de um script em Python onde se apresentam formas de representar e operar números complexos:

```
# -*- coding: utf-8 -*-
"""
Created on Sun Feb 28 17:10:04 2016

@author: Isabel
"""

import numpy as np
import matplotlib.pyplot as plt

z1=np.complex(2,3)
z2=np.sqrt(2)*np.exp((1j)*(np.pi/3))
z3=2-5j
z4=np.conjugate(z3)
print z1 , z2 , z2+z3 , z4
```

Escreva scripts em python para efectuar os cálculos indicados nos problemas anteriores.

Confirme os resultados obtidos analiticamente.