Fachhochschule Aachen
Campus Jülich

FH AACHEN
UNIVERSITY OF APPLIED SCIENCES

Stochastic methods for uncertain performance matrices in Multi-Criteria Decision Analysis with an application to the transition process towards sustainable mobility

Bachelor's Thesis

Luis Hasenauer

Faculty of
Medizintechnik und Technomathematik
Study program Angewandte Mathematik und Informatik B.Sc.

Jülich, August 8, 2023

In many situations, decision-makers face complex challenges that require robust and effective decision-making processes. Multi Criteria Decision Analysis (MCDA) has become a valuable technique to assist decision-makers in weighing multiple competing criteria and making or analyzing decisions. MCDA methods highly rely on a performance matrix, which captures the evaluations of alternatives against various criteria. However, in many cases these methods use fixed performance matrices even though real-world decision problems often involve uncertainties in the evaluations that can significantly impact the accuracy and reliability of the performance matrix.

This bachelor thesis aims to investigate and propose solutions to address the challenges posed by uncertainties in MCDA. Here, uncertainties are modeled in a probabilistic matter and are included in the MCDA methods SAW and PROMETHEE II. Furthermore, the normal distribution and the uniform distribution are addressed specifically to model these uncertainties. On the other hand, a more general but approximate approach which uses Monte-Carlo-Simulation is developed and discussed in terms of convergence.

The presented methods are applied, discussed and visualized in various numerical experiments with different performance matrices and MCDA methods which show the need for a reliable way of uncertainty handling. Lastly, they are incorporated in a Bayesian approach on a specific example about sustainable mobility.

Contents

List of abbreviations

MCDA Multi Criteria Decision Analysis
SAW Simple Additive Weighting
PROMETHEE II Preference Ranking Organisation Method for Enrichment Evaluation II
PDF probability density function
CDF cumulative distribution function
CLT central limit theorem
MCMC Markov-Chain-Monte-Carlo

List of Figures

List of Tables

1 Introduction

1.1 Motivation

In today's quickly changing world, decision-makers face complex challenges that require robust and effective decision-making processes. Multi Criteria Decision Analysis (MCDA) [2] has become a useful technique to assist decision-makers in weighing multiple competing criteria and making educated decisions. One of the critical components in MCDA is the performance matrix, which captures the evaluations of alternatives against various criteria. However, real-world decision problems often involve uncertainties that can significantly impact the accuracy and reliability of the performance matrix.

The uncertainty in performance evaluations can arise from various sources, such as imprecise data, subjective judgments, and unpredictable environmental factors. Dealing with these uncertainties is crucial since it has a direct impact on the accuracy and reliability of the choices made based on the MCDA results. Therefore, there is a need to explore and develop methods that can effectively handle uncertain performance matrices and provide decision-makers with more reliable and actionable insights.

1.2 Objectives

This bachelor thesis aims to investigate and propose solutions to address the challenges posed by the uncertainties in MCDA. Specifically, it will focus on employing stochastic methods to account for and mitigate uncertainties in the performance matrix, thus enhancing the reliability and robustness of the decision-making process. By incorporating probabilistic approaches, the thesis seeks to enable decision-makers to analyze decisions comprehensively and handle what-if scenarios, considering the inherent uncertainties in the evaluation of alternatives.

Throughout the course of this thesis, emphasis will be placed on providing a clear and comprehensive understanding of the theoretical foundations of MCDA, expanding them using stochastic methods and analyzing their impact on decision outcomes. Furthermore, the influence of this approach will be highlighted by applying it on a real world example about sustainable mobility.

2 Multi-Criteria Decision Analysis

2.1 Introduction

MCDA encompasses methods for choosing or ranking alternatives based on multiple evaluation criteria. These methods are commonly used in various fields of application, as for example economics, social- and health science.

Let $A = \{a_1, a_2, \ldots, a_m\}$ be a set of possible alternatives, and let $\{c_1(\cdot), c_2(\cdot), \ldots, c_n(\cdot) \,|\, c : A \to \mathbb{R}\}$ be a set of evaluation criteria. Alternatives are ranked based on benefit or cost criteria. An alternative a_i is preferred over a_j, with respect to a criterion c_k, if $c_k(a_i) > c_k(a_j)$ for a benefit criterion, or $c_k(a_i) < c_k(a_j)$ for a cost criterion.

The performance matrix $\tilde{P} \in \mathbb{R}^{m \times n}$ represents the performance of alternatives with respect to the criteria:

$$\tilde{P} = \begin{pmatrix} c_1(a_1) & c_2(a_1) & \ldots & c_n(a_1) \\ c_1(a_2) & c_2(a_2) & & \\ \vdots & & \ddots & \\ c_1(a_m) & & & c_n(a_m) \end{pmatrix}.$$

The decision makers' preferences are included through a weight vector $(w_1, \ldots, w_n)^T \in S_{n-1}$, where w_i represents the relative importance of criterion c_i and

$$S_{n-1} := \left\{ w = (w_1, \ldots, w_n)^T \;\middle|\; w_i \geq 0, \sum_{i=1}^{n} w_i = 1 \right\}$$

is the standard simplex which is part of an $(n-1)$-dimensional subspace of $\mathbb{R}^n$ (see for example the standard 2-simplex in Figure 1).

Many MCDA methods compute a score φ for each alternative, allowing them to be ranked. These methods have the form $f(\tilde{P}, w) = argsort(\varphi)$, where $\varphi = Pw$ and P is a matrix obtained from $\tilde{P}$. The mapping $argsort(\cdot)$ sorts the alternatives in descending order based on their respective values in φ. With P_i^T and P_j^T being the transposed row vectors, it follows that an alternative i will be preferred over an alternative j if the following equation holds true:

$$\langle P_j^T, w \rangle \leq \langle P_i^T, w \rangle \Leftrightarrow \langle P_j^T - P_i^T, w \rangle \leq 0. \tag{2.1}$$

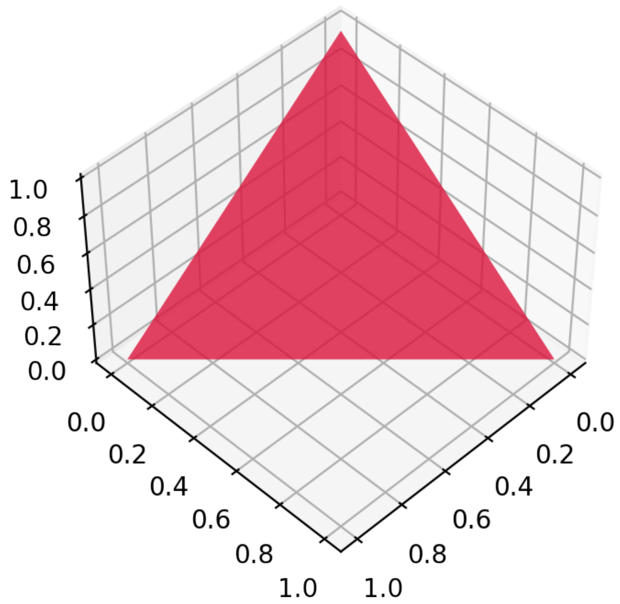

Figure 1: The standard 2-simplex [1].

2.2 Simple Additive Weighting

SAW is the oldest, most widely known and practically used method. It is also a very simple MCDA method since it does not require any other parameters than the performance matrix $\tilde{P}$ and the weight vector w [1].

To use SAW, $\tilde{P}$ needs to be transformed to a matrix P_{SAW} by normalizing it to a scale such that all criteria result in comparable units [3].

$$P_{SAW_{ij}} = \begin{cases} \dfrac{\tilde{P}_{ij}}{\max\limits_{1 \le k \le m} \tilde{P}_{kj}} & \text{if the criterion is benefit,} \\ \\ \dfrac{\min\limits_{1 \le k \le m} \tilde{P}_{kj}}{\tilde{P}_{ij}} & \text{if the criterion is cost.} \end{cases}$$

Alternatively, the criteria can also be scaled to sum up to one [4] which also leads them to be in equal units. Cost criteria can then be set to their negative values.

$$P_{SAW_{ij}} = \begin{cases} \dfrac{\tilde{P}_{ij}}{\sum_{k=1}^{m} |\tilde{P}_{kj}|} & \text{if the criterion is benefit,} \\ \\ -\dfrac{\tilde{P}_{ij}}{\sum_{k=1}^{m} |\tilde{P}_{kj}|} & \text{if the criterion is cost.} \end{cases}$$

The ranking is then created as presented in section 2.1 with $\varphi = P_{SAW}w$.

2.3 Preference Ranking Organisation Method for Enrichment Evaluation II

The PROMETHEE II is another quite popular MCDA method with more than 2300 references according to the website [5] of one of the authors [6], B. Mareschal. Unlike SAW, PROMETHEE II allows the modeler to adjust the algorithm using a few more variables based on his choice of the preference function which allows the decision maker to model the problem in a variety of ways based on their preferences.

It can be shown that PROMETHEE II is also of form $\varphi = P_{PROM2}w$ [1] with:

$$P_{PROM2_{ij}} = \frac{1}{m-1} \sum_{k=1}^{m} \left[g_j(\tilde{P}_{ij} - \tilde{P}_{kj}) - g_j(\tilde{P}_{kj} - \tilde{P}_{ij}) \right]. \tag{2.2}$$

Here, all criteria are to be maximized and if a criterion is to be minimized, its negative value is to be maximized.

The authors [6] propose a total of six different preference functions to simulate how a decision maker would choose one course of alternative over another.

Type 1, the usual criterion:

$$g(d) = \begin{cases} 0 & d \leq 0, \\ 1 & d > 0. \end{cases} \tag{2.3}$$

Type 2, the U-shape criterion:

$$g(d) = \begin{cases} 0 & d \leq q, \\ 1 & d > q. \end{cases} \tag{2.4}$$

Type 3, the V-shape criterion:

$$g(d) = \begin{cases} 0 & d < 0, \\ \dfrac{d}{p} & 0 \leq d \leq p, \\ 1 & d > p. \end{cases} \tag{2.5}$$

Type 4, the level criterion:

$$g(d) = \begin{cases} 0 & d \leq q, \\ \frac{1}{2} & q < d \leq p, \\ 1 & d > p. \end{cases} \tag{2.6}$$

Type 5, the V-shape with indifference criterion:

$$g(d) = \begin{cases} 0 & d \leq q, \\ \frac{d-q}{p-q} & q < d \leq p, \\ 1 & d > p. \end{cases}$$

Type 6, the Gaussian criterion:

$$g(d) = \begin{cases} 0 & d \leq 0, \\ 1 - e^{-\frac{d^2}{2s^2}} & d > 0. \end{cases} \tag{2.7}$$

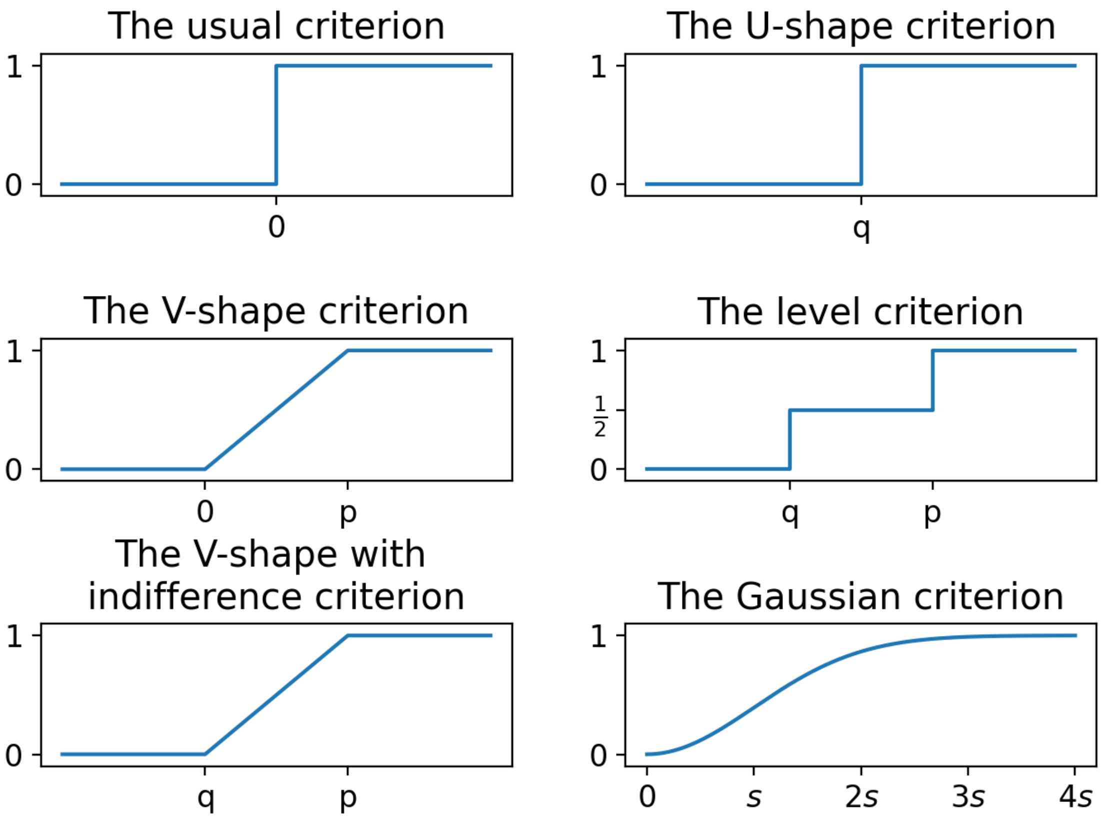

Figure 2: The preference functions.

3 Methods to face uncertainties

In the context of MCDA, the expression $\mathbb{P}(f(\tilde{P},w)=r|w)$ captures the probability of a specific ranking of alternatives $r \in A^d$ being valid, given a weight vector $w \in S_{n-1}$, when considering a possibly non-fixed matrix $\tilde{P} \in \mathbb{R}^{m\times n}$. Let the ranking be constructed in a way that all consecutive alternatives contained in it are considered superior to each other and all alternatives which are not contained in the ranking (denoted as $\tilde{r}$) are inferior to the alternative ranked last in r. Now, if the MCDA method has form $f(\tilde{P},w) = argsort(\varphi)$, like in section 2.1, equation (2.1) can be applied and we obtain:

$$\begin{aligned}\mathbb{P}(f(\tilde{P},w)=r|w) = \mathbb{P}\Big(&\langle P_{r_2}^T - P_{r_1}^T, w\rangle \leq 0,\ \ldots,\ \langle P_{r_d}^T - P_{r_{d-1}}^T, w\rangle \leq 0,\\ &\langle P_{\tilde{r}_1}^T - P_{r_d}^T, w\rangle \leq 0,\ \ldots,\ \langle P_{\tilde{r}_{n-d}}^T - P_{r_d}^T, w\rangle \leq 0|w\Big)\end{aligned}$$

Here, the terms $\langle P_{r_2}^T - P_{r_1}^T, w\rangle \leq 0,\ \ldots,\ \langle P_{r_d}^T - P_{r_{d-1}}^T, w\rangle \leq 0$ in $\mathbb{P}(f(\tilde{P},w)=r|w)$ ensure that the performance of each alternative in the ranking r is higher or equal to the performance of the alternative ranked below it. This captures the preference order specified by the ranking r for the given weights w. Additionally, the terms $\langle P_{\tilde{r}_1}^T - P_{r_d}^T, w\rangle \leq 0,\ \ldots,\ \langle P_{\tilde{r}_{n-d}}^T - P_{r_d}^T, w\rangle \leq 0$ account for the performance differences between the alternatives in $\tilde{r}$ and the alternative ranked last in r. These terms ensure that the performance of the alternatives in $\tilde{r}$ is less than or equal to the performance of the alternative ranked last in r.

3.1 Fixed performance matrix

Let $\tilde{P}$ be a fixed performance matrix and P be a matrix obtained through an arbitrary MCDA method of form $\varphi = Pw$. Furthermore, let $H_{ij} = \left\{w \mid \langle P_j^T - P_i^T, w\rangle \leq 0\right\}$ represent a half space indicating the preference of alternative i over alternative j based on their performance values. The probability of this occurring can be represented by an indicator function $\mathbb{P}(\langle P_{r_j}^T - P_{r_i}^T, w\rangle \leq 0|w) = \mathbf{1}_{H_{r_i r_j}}(w)$. Moreover, the joint probability $\mathbb{P}(f(\tilde{P},w)=r|w)$ of a weight being contained in all halfspaces is calculated using a product of indicator functions:

$$\mathbb{P}(f(\tilde{P},w)=r|w) = \prod_{i=1}^{d-1} \mathbf{1}_{H_{r_i r_{i+1}}}(w) \prod_{i=1}^{n-d} \mathbf{1}_{H_{r_d \tilde{r}_i}}(w).$$

The subsequent product of indicator functions, $\prod_{i=1}^{d-1} \mathbf{1}_{H_{r_i r_{i+1}}}(w)$, represents the condition that the weights must lie in the half spaces corresponding to the preferences between consecutive alternatives in the ranking r. Similarly, the product $\prod_{i=1}^{n-d} \mathbf{1}_{H_{r_d \tilde{r}_i}}(w)$ ensures that the weights satisfy the preference conditions between the last alternative in the ranking r and the remaining alternatives not included in r, hence $\tilde{r}$.

With $W_r := S_{n-1} \bigcap_{i=1}^{d-1} H_{r_i r_{i+1}} \bigcap_{i=1}^{n-d} H_{r_d \tilde{r}_i}$, this can be simplified to $\mathbb{P}(f(\tilde{P},w)=r|w) = \mathbf{1}_{W_r}(w)$

for $w \in S_{n-1}$ where W_r is a convex bounded polytope containing all weight vectors that result in the desired ranking [1].

3.2 Monte-Carlo-Simulation

The Monte-Carlo-Simulation provides a powerful approach to approximate $\mathbb{P}(f(\tilde{P}, w) = r|w)$ when considering a variable performance matrix and an arbitrary MCDA-method of form $\varphi = Pw$. In this context, the performance of alternatives is uncertain, and we aim to estimate the likelihood that a particular ranking holds true based on the given weights and the range of possible performance outcomes.

The Monte-Carlo approximation, denoted as $\hat{\mathbb{P}}(f(\tilde{P}, w) = r|w)_N$, allows us to calculate an estimate of the probability by generating N independent samples of the matrix P, $(P_1, \ldots, P_N)$, and evaluating the conditions that define the ranking. By aggregating the results from these samples and with W_r^i and H_{jk}^i being the respective sets from section 3.1 with $P = P_i$, we can obtain an approximation of the true probability $\mathbb{P}(f(\tilde{P}, w) = r|w)$:

$$
\begin{aligned}
\mathbb{P}(f(\tilde{P}, w) = r|w) &\overset{!}{\approx} \hat{\mathbb{P}}(f(\tilde{P}, w) = r|w)_N \\
&= \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{W_r^i}(w) \\
&= \frac{1}{N} \sum_{i=1}^{N} \left[\prod_{j=1}^{d-1} \mathbf{1}_{H^i_{r_j r_{j+1}}}(w) \prod_{j=1}^{n-d} \mathbf{1}_{H^i_{r_d \bar{r}_j}}(w) \right].
\end{aligned}
$$

To show that this Monte Carlo approximation is consistent, we need to show that it converges in expectation to the true probability $\mathbb{P}(f(\tilde{P}, w) = r|w)$ as the number of Monte Carlo samples goes to infinity. The expected value of this approximation can be computed as:

$$
\mathbb{E}\left[\frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{W_r^i}(w) \right] = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}\left[\mathbf{1}_{W_r^i}(w) \right].
$$

Since each Monte Carlo sample P_i is independently generated, the indicator function $\mathbf{1}_{W_r^i}(w)$ is an unbiased estimator of $\mathbb{P}(f(\tilde{P}, w) = r|w)$. Therefore, we have:

$$
\frac{1}{N} \sum_{i=1}^{N} \mathbb{E}\left[\mathbf{1}_{W_r^i}(w) \right] = \frac{1}{N} \sum_{i=1}^{N} \mathbb{P}(f(\tilde{P}, w) = r|w) = \frac{1}{N} \cdot N \cdot \mathbb{P}(f(\tilde{P}, w) = r|w) = \mathbb{P}(f(\tilde{P}, w) = r|w).
$$

Thus, the expected value of the Monte Carlo approximation is equal to the true probability $\mathbb{P}(f(\tilde{P}, w) = r|w)$. Since the mean almost surely converges to the expected value due to the strong law of large numbers, this demonstrates the consistency of the approximation.

Note that the consistency holds under the assumption that the Monte Carlo samples P_i are generated independently and according to the distribution that captures the uncertainty in the performance matrix $\tilde{P}$. Additionally, the convergence is in expectation, meaning that there can still be some variance or fluctuations in the estimated probabilities for a finite number of samples.

To asses convergence, one can use Chebyshev's inequality for a confidence interval [7]:

$$\mathbb{P}\left(\left|\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N-\mathbb{P}(f(\tilde{P},w)=r|w)\right|\leq\alpha\cdot\sigma\left[\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N\right]\right)\geq 1-\frac{1}{\alpha^2}$$

where $\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N$ is the Monte Carlo approximation, $\mathbb{P}(f(\tilde{P},w)=r|w)$ is the true probability, $\sigma\left[\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N\right]$ is the standard deviation of $\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N$, and α is a constant that determines the width of the confidence interval.

The maximum difference with at least probability $1-\frac{1}{\alpha^2}$ between the Monte Carlo approximation and the true probability, denoted as $\Delta\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N$, is given by:

$$\Delta\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N=\alpha\cdot\sigma\left[\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N\right]=\alpha\cdot\sigma\left[\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}_{W_r^i}(w)\right]=\alpha\frac{\sigma\left[\mathbf{1}_{W_r}(w)\right]}{\sqrt{N}}.$$

Since the standard deviation of $\mathbf{1}_{W_r}(w)$, which is Bernoulli distributed, and therefore:

$$\sigma[\mathbf{1}_{W_r}(w)]=\sqrt{\mathbb{P}(f(\tilde{P},w)=r|w)\left(1-\mathbb{P}(f(\tilde{P},w)=r|w)\right)},$$

is unknown, one can use the empirical standard deviation:

$$s_N=\sqrt{\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N\left(1-\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N\right)},$$

as an approximation. To determine the termination criterion for convergence, we can set a threshold, ϵ, and check whether the ratio of $\Delta\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N$ to $\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N$ is less than ϵ [8], i.e.:

$$\frac{\Delta\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N}{\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N}<\epsilon.$$

If this condition is met, we can consider the Monte Carlo approximation to have converged within the desired tolerance. Additionally, one should add a threshold, n_{min}, of minimum iterations since, otherwise, only observing the desired ranking in the first iterations would always result in a termination. The reason for this is that in this case:

$$\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N = \frac{1}{N}\sum_{i=1}^{N}\mathbf{1}_{W_r^i}(w) = \frac{1}{N}\sum_{i=1}^{N}1 = \frac{1}{N}N = 1.$$

Therefore, $s_N = \sqrt{\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N\left(1-\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N\right)} = 1(1-1) = 0$ and consequently:

$$\frac{\Delta\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N}{\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N} = \frac{\alpha\frac{s_N}{\sqrt{N}}}{\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N} = \frac{\alpha\frac{0}{\sqrt{N}}}{1} = 0 < \epsilon \quad \forall\epsilon > 0.$$

This approach provides a measure of confidence in the accuracy of the approximation, as the probability of the true value falling within the confidence interval increases as α becomes larger. By adjusting the values of α and ϵ, we can control the trade-off between accuracy and computational cost.

In pseudocode, an implementation could look the following:

Algorithm 1: Monte-Carlo-Simulation for MCDA

Input: w, r, α, ϵ, n_{min}

$n \leftarrow 0$;

$X \leftarrow$ `empty list`;

repeat

  $n \leftarrow n+1$;

  $\tilde{P} \leftarrow$ `sample from distribution`;

  $P \leftarrow P$ `from` $\tilde{P}$;

  $\varphi \leftarrow Pw$;

  $r_n \leftarrow$ `argsort(`φ`)`;

  if `rankingsMatch(`r_n, r`)` **then**

    `addToList(`$X, 1$`)`;

  else

    `addToList(`$X, 0$`)`;

  end

  $\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N \leftarrow$ `mean(`X`)`;

  $s_N \leftarrow$ `sqrt(`$\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N(1-\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N)$`)`;

  $\Delta\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N = \alpha\frac{s_N}{\sqrt{N}}$;

until $\frac{\Delta\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N}{\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N} < \epsilon$ *and* $n > n_{min}$;

return $\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N$

3.3 Normally distributed performance matrix

A reasonable choice to model uncertainty in the performance matrix could be a normal distribution. Therefore, let $P \in \mathbb{R}^{m \times n}$ be equal to the performance matrix, hence $P = \tilde{P}$ which is a special case of SAW, where each element P_{ij} is independently drawn from a normal distribution $\mathcal{N}(\mu_{ij}, \sigma_{ij}^2)$. Consequently, the rows of P can be regarded as multivariate normal distributions with a diagonal covariance matrix. This covariance matrix Σ_i for each row i takes the form:

$$\Sigma_i = \begin{pmatrix} \sigma_{i0}^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_{in}^2 \end{pmatrix}.$$

Additionally, the mean vector μ_i for each row i can be represented as:

$$\mu_i = (\mu_{io}, \dots, \mu_{in})^T.$$

Consequently, the transposed row vector P_i^T follows a multivariate normal distribution $\mathcal{N}_n(\mu_i, \Sigma_i)$.

An alternative i is considered superior to an alternative j if $\langle P_j^T - P_i^T, w \rangle \leq 0$. By focusing on the inner part of the dot product, we find that $P_j^T - P_i^T$ follows a multivariate normal distribution $\mathcal{N}_n(\mu_j - \mu_i, \Sigma_j + \Sigma_i)$. When considering a fixed weight vector w, it can be deduced that $\langle P_j^T - P_i^T, w \rangle$ follows a normal distribution $\mathcal{N}(w^T(\mu_j - \mu_i), w^T(\Sigma_j + \Sigma_i)w)$ [9].

Let $r \in \mathbb{R}^d$ represent the given ranking, and $\tilde{r} \in \mathbb{R}^{n-d}$ denote all alternatives not included in r. In order to satisfy all $n-1$ conditions specified by the ranking simultaneously, it is necessary to consider the joint distribution of these halfspaces. This joint distribution follows a multivariate normal distribution $\mathcal{N}_{n-1}(\hat{\mu}, \hat{\Sigma})$, where the means are given by:

$$\hat{\mu}_i = \begin{cases} w^T(\mu_{r_{i+1}} - \mu_{r_i}) & , 1 \leq i < d \\ w^T(\mu_{\tilde{r}_{i-d+1}} - \mu_{r_d}) & , d \leq i < n \end{cases}$$

The covariances, due to the independence of the entries of P, are given by:

$$\text{Cov}\left[\langle P_j^T - P_i^T, w \rangle, \langle P_l^T - P_k^T, w \rangle\right] = \begin{cases} 0 & , j \neq l \neq i \neq k \\ -\langle w^{\circ 2}, \sigma_i^2 \rangle & , j \neq l = i \neq k \\ \langle w^{\circ 2}, \sigma_i^2 \rangle & , j \neq l \neq i = k \\ \langle w^{\circ 2}, \sigma_i^2 + \sigma_j^2 \rangle & , j = l \neq i = k \end{cases} \tag{3.1}$$

with $w^{\circ 2}$ being the Hadamard power which denotes that $\tilde{w} = w^{\circ 2} \Leftrightarrow \tilde{w}_i = w_i^2 \ \forall 1 \leq i \leq n$. Furthermore, it is important to note that $\text{Cov}[X,Y] = \text{Cov}[Y,X]$ and all other cases of indices are impossible if the ranking is constructed according to the presented way.

The covariance between $\langle P_j - P_i, w\rangle$ and $\langle P_k - P_j, w\rangle$ with $i \neq j \neq k$, which corresponds to the scenario where alternative j is inferior to alternative i but superior to alternative k, can be proven as follows:

$$
\begin{aligned}
\text{Cov}\left[\langle P_j - P_i,\ w\rangle, \langle P_k - P_j, w\rangle\right] &= \text{Cov}\left[\sum_{l=1}^{n}(P_{jl} - P_{il})w_l,\ \sum_{q=1}^{n}(P_{kq} - P_{jq})w_l\right] \\
&= \sum_{l=1}^{n}\sum_{q=1}^{n} w_l w_q \text{Cov}\left[P_{jl} - P_{il},\ P_{kq} - P_{jq}\right] \\
&= \sum_{l=1}^{n}\sum_{q=1}^{n} w_l w_q \Big(\text{Cov}\left[P_{jl},\ P_{kq}\right] - \text{Cov}\left[P_{jl},\ P_{jq}\right] \\
&\qquad - \text{Cov}\left[P_{il},\ P_{kq}\right] + \text{Cov}\left[P_{il},\ P_{jq}\right]\Big)
\end{aligned}
$$

Since all entries of P are assumed to be independent, their covariances between each other will be zero. Hence, for $l \neq q$ we only obtain zero summands and consequently:

$$
\begin{aligned}
&\sum_{l=1}^{n}\sum_{q=1}^{n} w_l w_q \left(\text{Cov}\left[P_{jl},\ P_{kq}\right] - \text{Cov}\left[P_{jl},\ P_{jq}\right] - \text{Cov}\left[P_{il},\ P_{kq}\right] + \text{Cov}\left[P_{il},\ P_{jq}\right]\right) \\
&= \sum_{l=1}^{n} w_l^2 \left(\text{Cov}\left[P_{jl},\ P_{kl}\right] - \text{Cov}\left[P_{jl},\ P_{jl}\right] - \text{Cov}\left[P_{il},\ P_{kl}\right] + \text{Cov}\left[P_{il},\ P_{jl}\right]\right).
\end{aligned}
$$

Additionally, since $i \neq j \neq k$, the random variables P_{il}, P_{jl} and P_{kl} will have a covariance of zero. Therefore, the covariance of $\langle P_j - P_i,\ w\rangle$ and $\langle P_k - P_j, w\rangle$ simplifies to:

$$
\begin{aligned}
&\sum_{l=1}^{n} w_l^2 \left(\text{Cov}\left[P_{jl},\ P_{kl}\right] - \text{Cov}\left[P_{jl},\ P_{jl}\right] - \text{Cov}\left[P_{il},\ P_{kl}\right] + \text{Cov}\left[P_{il},\ P_{jl}\right]\right) \\
&= -\sum_{l=1}^{n} w_l^2 \text{Cov}\left[P_{jl},\ P_{jl}\right] \\
&= -\sum_{l=1}^{n} w_l^2 \sigma_{jl}^2 \\
&= -\langle w^{\circ 2}, \sigma_j^2\rangle.
\end{aligned}
$$

The proof for the covariance between $\langle P_j - P_i, w\rangle$ and $\langle P_k - P_i, w\rangle$ is analogous. Furthermore, when considering the covariance of a variable with itself, it simplifies to the variance $w^T(\Sigma_j + \Sigma_i)w = \langle w^{\circ 2}, \sigma_i^2 + \sigma_j^2\rangle$ since Σ_i and Σ_j are diagonal matrices.

Table 1: The covariance matrix $\hat{\Sigma}$. Here, $r_i \geq r_j$ is short for the random variable $\langle P_{r_j}^T - P_{r_i}^T, w \rangle$.

	$r_1 \geq r_2$	$r_2 \geq r_3$	$\dots$	$r_{d-1} \geq r_d$	$r_d \geq r_1^C$	$r_d \geq r_2^C$	...	$r_d \geq r_{n-d}^C$
$r_1 \geq r_2$	$\langle w^{\circ 2}, \sigma_{r_1}^2 + \sigma_{r_2}^2 \rangle$	$\langle w^{\circ 2}, \sigma_{r_2}^2 \rangle$	0	0	0	0	$\dots$	0
$r_2 \geq r_3$	$\langle w^{\circ 2}, \sigma_{r_2}^2 \rangle$	$\langle w^{\circ 2}, \sigma_{r_2}^2 + \sigma_{r_3}^2 \rangle$	$\ddots$	0	0	0	$\dots$	0
$\vdots$	0	$\ddots$	$\ddots$	$\ddots$	0	0	$\dots$	0
$r_d \leq r_{d-1}$	0	0	$\ddots$	$\langle w^{\circ 2}, \sigma_{r_{d-1}}^2 + \sigma_{r_d}^2 \rangle$	$\langle w^{\circ 2}, \sigma_{r_d}^2 \rangle$	$\langle w^{\circ 2}, \sigma_{r_d}^2 \rangle$	$\dots$	$\langle w^{\circ 2}, \sigma_{r_d}^2 \rangle$
$r_d \geq r_1^C$	0	0	0	$\langle w^{\circ 2}, \sigma_{r_d}^2 \rangle$	$\langle w^{\circ 2}, \sigma_{r_d}^2 + \sigma_{r_1^C}^2 \rangle$	$-\langle w^{\circ 2}, \sigma_{r_d}^2 \rangle$	$\dots$	$-\langle w^{\circ 2}, \sigma_{r_d}^2 \rangle$
$r_d \geq r_2^C$	0	0	0	$\langle w^{\circ 2}, \sigma_{r_d}^2 \rangle$	$-\langle w^{\circ 2}, \sigma_{r_d}^2 \rangle$	$\langle w^{\circ 2}, \sigma_{r_d}^2 + \sigma_{r_2^C}^2 \rangle$	$\ddots$	$-\langle w^{\circ 2}, \sigma_{r_d}^2 \rangle$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$-\langle w^{\circ 2}, \sigma_{r_d}^2 \rangle$
$r_d \geq r_{n-d}^C$	0	0	0	$\langle w^{\circ 2}, \sigma_{r_d}^2 \rangle$	$-\langle w^{\circ 2}, \sigma_{r_d}^2 \rangle$	$-\langle w^{\circ 2}, \sigma_{r_d}^2 \rangle$	$-\langle w^{\circ 2}, \sigma_{r_d}^2 \rangle$	$\langle w^{\circ 2}, \sigma_{r_d}^2 + \sigma_{r_{n-d}^C}^2 \rangle$

Finally, it follows that $\mathbb{P}(f(\tilde{P}, w) = r | w)$ can be calculated using the multivariate normal CDF which does not have a closed form.

3.4 Uniformly distributed performance matrix

Another feasible choice to model uncertainty in the performances is the uniform distribution. Hence, let us consider a scenario where the performances are assumed to be independent and uniformly distributed. We denote this as $P_{ij} \sim \mathcal{U}(a_{ij}, b_{ij})$, where $a_{ij} < b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. The dot product of a row with a weight vector w can be expressed as the sum of independent uniform random variables: $\langle P_j^T, w \rangle = \sum_{k=1}^n P_{jk} w_k$, where $P_{jk} w_k \sim \mathcal{U}(w_k a_{jk}, w_k b_{jk})$ [10]. The condition for alternative i being superior to alternative j can also be represented as a sum of independent uniform random variables:

$$
\begin{aligned}
\langle P_j - P_i, w \rangle &\leq 0 \\
\langle P_j^T, w \rangle - \langle P_i^T, w \rangle &\leq 0 \\
\sum_{k=1}^{n} P_{jk} w_k - P_{ik} w_k &\leq 0.
\end{aligned}
$$

The individual summands will follow a uniform distribution, where $-P_{ik}w_k \sim \mathcal{U}(-w_k b_{ik}, -w_k a_{ik})$ and $P_{jk}w_k$ distributed as described previously. It is worth noting that the distribution of a sum of non-identically independent uniform random variables is known [11]. However, in this case, there is no available information regarding the joint probability between different sums which would be needed to compute the probability $\mathbb{P}(f(\tilde{P}, w) = r|w)$ for the whole ranking. Therefore, we can analyze the individual sums, but we do not have information about their joint behavior.

Since the distribution of $\langle P_j - P_i, w\rangle$ would be easier to handle if it was normally distributed, it can be tried to approximate it in this way. This approximation is made feasible by the Lyapunov central limit theorem (CLT).

Lyapunov CLT *Let $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables with finite means $\mu_k = E[X_k]$ and finite variances $\sigma_k^2 =$ Var(X_k).If for some $\delta > 0$, the following condition holds:*

$$\lim_{n\to\infty} \frac{1}{(\sigma_1^2 + \ldots + \sigma_n^2)^{\frac{2+\delta}{2}}} \sum_{k=1}^{n} \mathbb{E}\left[|X_k - \mu_k|^{2+\delta}\right] = 0,$$

then:

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{k=1}^{n} (X_k - \mu_k) \xrightarrow{d} \mathcal{N}(0, 1),$$

with $s_n^2 = \sum_{i=1}^{n} \sigma_i^2$.

We want to apply this theorem for $X_k = w_k(P_{jk} - P_{ik})$ and therefore $\mu_k = \frac{1}{2}w_k(b_{jk} + a_{jk} - b_{ik} - a_{ik})$ and $s_n^2 = \frac{1}{12}\sum_{k=1}^{n} w_k^2\left((b_{jk} - a_{jk})^2 + (b_{ik} - a_{ik})^2\right)$.

Since proving the Lyapunov condition for $\langle P_j - P_i, w\rangle$ would extend the limit of this bachelor's thesis, we investigate its asymptotic normality empirically.

To do so, 1000 samples of P_i and P_j are generated where each sample is independently drawn from a uniform distribution $P_{kl} \sim \mathcal{U}(a_{kl}, b_{kl})\ \forall 1 \leq l \leq n,\ k \in \{i, j\}$. Furthermore, the corresponding intervals are subject to the conditions $0 \leq a_{kl} < 1$ and $b_{kl} - a_{kl} \leq 1$.

Subsequently, for each sampled pair of rows P_i and P_j, the random variable $Y = \frac{1}{s_n}\sum_{k=1}^{n} w_k(P_{jk} - P_{ik}) - \mu_k$ is calculated. If the CLT would apply, this random variable Y would converge in distribution to a standard normal distribution. For the sake of investigating this, the samples of Y are tested for normality [12]. This is tested for different numbers of criteria between 10 and 100 and for two sets of weights.

The first set contains 100 weights which are uniformly distributed on the standard simplex S_{n-1}. Specifically, each weight is generated by sampling n i.i.d. samples from an exponential distribution and then scaling them by their sum. This results in the weights following a uniform distribution on S_{n-1} [13]. Since it is unlikely that a CLT holds with extreme weights on, i.e. corners of S_{n-1}, a more centered set of weights is also taken into consideration. Here, the idea is that all summands have a relatively equal influence on their weighted average or rather, there is no summand that significantly determines the result of $\langle P_j - P_i, w \rangle$ alone. Consequently, the second set contains weights which are more concentrated towards the center of S_{n-1} and are generated by sampling n i.i.d. samples from a standard uniform distribution which are then scaled by their sum.

We want to analyze the corresponding p-values of the normality test. Therefore, the average p-value of all weight vectors and the percentage of rejected normality tests using a significance level of 5% are visualized with respect to the number of criteria. Ideally, the percentage of rejected tests should approach the significance level as n approaches infinity.

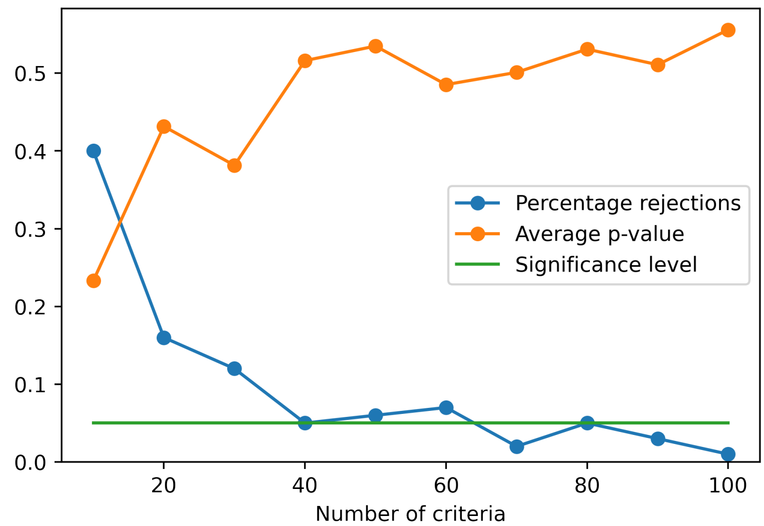

Figure 3: Results of normality test with uniformly distributed weights on S_{n-1}.

In Figure 3, it can be seen that the number of rejected null hypotheses seems to shrink as the number of criteria grows which is a good sign for assuming asymptotic normality. Furthermore the average p-value is sufficiently far away from the significance level.

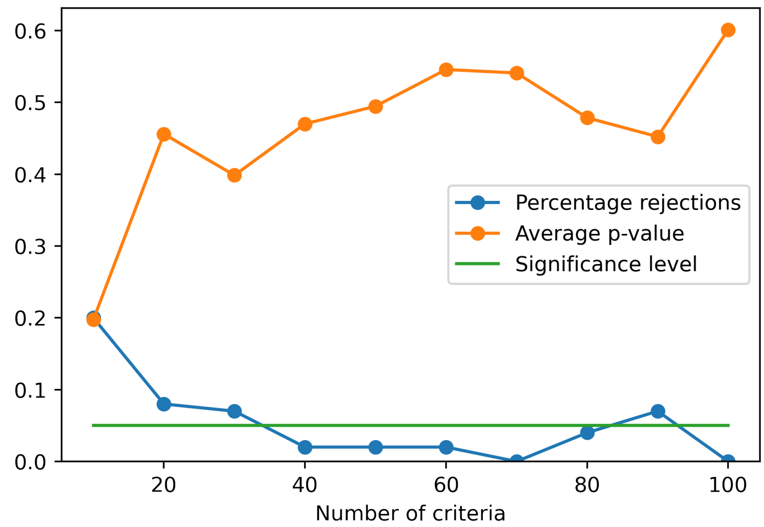

Figure 4: Results of normality test with weights closer to the center of S_{n-1}.

As expected, the results with the second sets of weights are even better than with the first set of weights. The percentage of rejected null hypotheses is really close to the significance level of 5% for almost every test (see Figure 4).

Since the CLT applies for $n \to \infty$, the test is repeated for different numbers of criteria between 100 and 1000. We observe promising results for both sets of weights (see Figures 5 & 6).

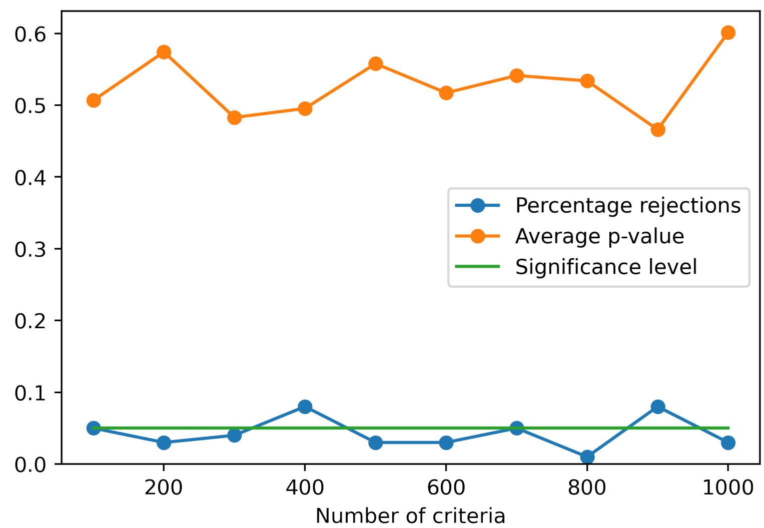

Figure 5: Results of normality test with uniformly distributed weights on S_{n-1}.

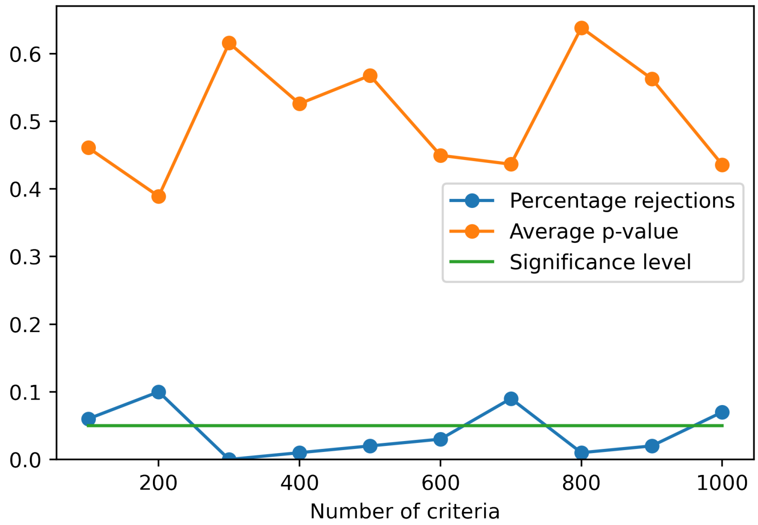

Figure 6: Results of normality test with weights closer to the center of S_{n-1}.

In summary, the empirical studies suggest that the CLT applies for the random variable $X_k = w_k(P_{jk} - P_{ik})$. Furthermore, the results were also reasonable if the number of criteria is less than 100 which is a more realistic number when applying MCDA. In our case, where $n < 100$, we take:

$$\sum_{k=1}^{n} w_k(P_{jk} - P_{ik}) \sim \mathcal{N}\left(\sum_{k=1}^{n} \mu_k,\, s_n^2\right),$$

with $\mu_k = \frac{1}{2} w_k(b_{jk} + a_{jk} - b_{ik} - a_{ik})$ and $s_n^2 = \frac{1}{12} \sum_{k=1}^{n} w_k^2 \left((b_{jk} - a_{jk})^2 + (b_{ik} - a_{ik})^2\right)$.

Lastly, it needs to be noted that this approximation may become worse if the weights move away from the center of S_{n-1} and therefore Monte-Carlo-Simulation might be a better approach in some cases if computation time is no significant factor.

3.5 Simple Additive Weighting

When using the SAW method, it is necessary to scale the criteria to comparable units. Hence, using the scaling as presented in 2.2, will result in P_{SAW} loosing its original distribution and therefore $\mathbb{P}(f(\tilde{P}, w) = r|w)$ needs to be computed by using Monte-Carlo-Simulation.

To preserve the original distribution of $\tilde{P}$ in P_{SAW}, alternatively, the scaling can be approximated by dividing each element of the performance matrix by the sum of the expected values of the corresponding column, which are assumed to be greater than zero when using SAW:

$$P_{SAW\,ij} = \frac{\tilde{P}_{ij}}{\sum_{k=1}^{m} \mathbb{E}\left[\tilde{P}_{kj}\right]}.$$

Consequently, after scaling, the sum of the expected values in each column of the scaled matrix P will be one:

$$\mathbb{E}\left[\sum_{i=1}^{m} P_{SAW\,ij}\right] = \sum_{i=1}^{m} \mathbb{E}\left[P_{SAW\,ij}\right] = \sum_{i=1}^{m} \mathbb{E}\left[\frac{\tilde{P}_{ij}}{\sum_{k=1}^{m} \mathbb{E}\left[\tilde{P}_{kj}\right]}\right] = \frac{\sum_{i=1}^{m} \mathbb{E}\left[\tilde{P}_{ij}\right]}{\sum_{k=1}^{m} \mathbb{E}\left[\tilde{P}_{kj}\right]} = 1.$$

To incorporate benefit and cost to the criteria, criteria associated to cost can be set to their negative values since this punishes high values and benefits low values which is the intended behavior.

Furthermore, if we assume that the elements of $\tilde{P}$ are normally distributed, which was discussed in section 3.3, we can express the distribution of $P_{SAW\,ij}$ as follows:

$$P_{SAW\,ij} \sim \mathcal{N}\left(\frac{\mu_{ij}}{\sum_{k=1}^{m} \mu_{kj}}, \frac{\sigma_{ij}^2}{(\sum_{k=1}^{m} \mu_{kj})^2}\right).$$

Here, μ_{ij} represents the mean and σ_{ij}^2 represents the variance associated with $\tilde{P}_{ij}$. Similarly, if we assume a uniform distribution, as in section 3.4, the distribution of $P_{SAW\,ij}$ can be expressed as:

$$P_{SAW\,ij} \sim \mathcal{U}\left(\frac{a_{ij}}{\sum_{k=1}^{m} \mu_{kj}}, \frac{b_{ij}}{\sum_{k=1}^{m} \mu_{kj}}\right). \tag{3.2}$$

In this case, a_{ij} and b_{ij} represent the lower and upper bounds of the uniform distribution, respectively. If one would try to approximate this by a normal distribution under the assumption that a CLT holds, the tests in section 3.4 used the conditions $\frac{a_{ij}}{\sum_{k=1}^{m} \mu_{kj}} < 1$ and $\frac{b_{ij}}{\sum_{k=1}^{m} \mu_{kj}} - \frac{a_{ij}}{\sum_{k=1}^{m} \mu_{kj}} \leq 1$. For the first condition, it can be obtained that:

$$\frac{a_{ij}}{\sum_{k=1}^{m} \mu_{kj}} < \frac{a_{ij}}{\mu_{ij}} = \frac{a_{ij}}{\frac{a_{ij}+b_{ij}}{2}} < \frac{a_{ij}}{a_{ij}} = 1.$$

Furthermore, the second condition results in:

$$\begin{aligned}
& \frac{b_{ij}}{\sum_{k=1}^{m} \mu_{kj}} - \frac{a_{ij}}{\sum_{k=1}^{m} \mu_{kj}} \leq 1 \\
\Leftrightarrow \quad & \frac{b_{ij}}{\sum_{k=1}^{m} \mu_{kj}} \leq 1 + \frac{a_{ij}}{\sum_{k=1}^{m} \mu_{kj}}
\end{aligned}$$

$$\Leftrightarrow \qquad b_{ij} \leq \sum_{k=1}^{m} \mu_{kj} + a_{ij}.$$

Therefore, the upper bound b_{ij} of each entry $\tilde{P}_{ij}$ needs to be smaller than the sum of means of criterion j and its lower bound a_{ij}. This condition will most likely be fulfilled if the number of alternatives is reasonably large.

These considerations highlight the flexibility of the SAW method in handling uncertain performance matrices, allowing for different scaling approaches and distribution assumptions.

3.6 Preference Ranking Organisation Method for Enrichment Evaluation II

While utilizing PROMETHEE II, the entries of P_{PROM2}, as in (2.2), are defined by:

$$P_{PROM2_{ij}} = \frac{1}{m-1} \sum_{k=1}^{m} \Big[g_j(\tilde{P}_{ij} - \tilde{P}_{kj}) - g_j(\tilde{P}_{kj} - \tilde{P}_{ij}) \Big].$$

Inserting this equation into the probability of an alternative i being preferred over an alternative j results in:

$$\begin{aligned}
&\mathbb{P}\Big(\langle P^T_{PROM2_j} - P^T_{PROM2_i}, w \rangle \leq 0 \Big| w \Big) \\
&= \mathbb{P}\left(\sum_{l=1}^{n} w_l (P_{PROM2_{jl}} - P_{PROM2_{il}}) \leq 0 \Big| w \right) \\
&= \mathbb{P}\left(\frac{1}{m-1} \sum_{l=1}^{n} w_l \sum_{k=1}^{m} \Big[g_l(\tilde{P}_{jl} - \tilde{P}_{kl}) - g_l(\tilde{P}_{kl} - \tilde{P}_{jl}) - g_l(\tilde{P}_{il} - \tilde{P}_{kl}) + g_l(\tilde{P}_{kl} - \tilde{P}_{il}) \Big] \leq 0 \Big| w \right) \\
&= \mathbb{P}\left(\sum_{l=1}^{n} w_l \sum_{k=1}^{m} \Big[g_l(\tilde{P}_{jl} - \tilde{P}_{kl}) - g_l(\tilde{P}_{kl} - \tilde{P}_{jl}) - g_l(\tilde{P}_{il} - \tilde{P}_{kl}) + g_l(\tilde{P}_{kl} - \tilde{P}_{il}) \Big] \leq 0 \Big| w \right).
\end{aligned}$$

To investigate this probability, let us begin with the pairwise differences $\tilde{P}_{ij} - \tilde{P}_{kl}$. Assuming the performances are normally distributed with $\tilde{P}_{ij} \sim \mathcal{N}(\mu_{ij}, \sigma^2_{ij})$ and $\tilde{P}_{kl} \sim \mathcal{N}(\mu_{kl}, \sigma^2_{kl})$, it follows that $\tilde{P}_{ij} - \tilde{P}_{kl} \sim \mathcal{N}(\mu_{ij} - \mu_{kl},\ \sigma^2_{ij} + \sigma^2_{kl})$.

On the other hand, if the performances are uniformly distributed with $\tilde{P}_{ij} \sim \mathcal{U}(a_{ij}, b_{ij})$ and $\tilde{P}_{kl} \sim \mathcal{U}(a_{kl}, b_{kl})$, further investigations need to be done. The difference $\tilde{P}_{ij} - \tilde{P}_{kl}$ can be written as $\tilde{P}_{ij} + (-\tilde{P}_{kl})$ where $-\tilde{P}_{kl} \sim \mathcal{U}(-b_{kl}, -a_{kl})$. Therefore, their distribution can be found by calculating their convolution [14, Theorem 5.5.1].

To calculate the convolution of two uniform random variables, let $X \sim \mathcal{U}(a,b)$, $Y \sim \mathcal{U}(c,d)$

and $Z = X + Y$. The convolution of their respective densities is then defined as:

$$\begin{aligned}
f_Z(z) &= (f_X * f_Y)(z) \\
&= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx \\
&= \int_{-\infty}^{\infty} \frac{\mathbf{1}_{(a,b)}(x)}{b-a} \cdot \frac{\mathbf{1}_{(c,d)}(z-x)}{d-c} dx \\
&= \int_{a}^{b} \frac{1}{b-a} \cdot \frac{\mathbf{1}_{(c,d)}(z-x)}{d-c} dx \\
&= \int_{\max(a,z-d)}^{\min(b,z-c)} \frac{1}{b-a} \cdot \frac{1}{d-c} dx \\
&= \frac{\min(b,z-c)}{(b-a)(d-c)} - \frac{\max(a,z-d)}{(b-a)(d-c)} \\
&= \frac{\min(b,z-c) - \max(a,z-d)}{(b-a)(d-c)}.
\end{aligned}$$

To eliminate the $\min$ and $\max$ from this PDF, let us investigate $\min(b, z-c)$ first. We observe that:

$$\min(b, z-c) = \begin{cases} z-c & a+c \leq z \leq b+c \\ b & b+c \leq z \leq b+d \end{cases}.$$

Additionally, for $\max(a, z-d)$ it can be seen that:

$$\max(a, z-d) = \begin{cases} a & a+c \leq z \leq a+d \\ z-d & a+d \leq z \leq b+d \end{cases}.$$

To resolve $\min(b,z-c) - \max(a,z-d)$, these intervals need to be intersected. For the sake of this, let us assume that $a+d \leq b+c$, which can always be achieved by swapping the respective values of X and Y. Consequently, $f_Z(z)$ can be written as:

$$f_Z(z) = \begin{cases} 0 & , z < a+c \\ \dfrac{z-c-a}{(b-a)(d-c)} & , a+c \leq z \leq a+d \\ \dfrac{1}{b-a} & , a+d \leq z \leq b+c \\ \dfrac{-z+b+d}{(b-a)(d-c)} & , b+c \leq z \leq b+d \\ 0 & , b+d < z \end{cases}.$$

Hence, by integration, the CDF is defined as:

$$F_Z(z) = \begin{cases} 0 & ,z < a+c \\ \dfrac{0.5z^2 - (a+c)z + 0.5(a+c)^2}{(b-a)(d-c)} & ,a+c \leq z \leq a+d \\ \dfrac{0.5(a+d)^2 - (a+c)(a+d) + 0.5(a+c)^2}{(b-a)(d-c)} + \dfrac{z-(a+d)}{b-a} & ,a+d \leq z \leq b+c \\ \left[\dfrac{0.5(a+d)^2 - (a+c)(a+d) + 0.5(a+c)^2}{(b-a)(d-c)} + \dfrac{(b+c)-(a+d)}{b-a} \right. & \\ \left. + \dfrac{-0.5z^2 + (b+d)z + 0.5(b+c)^2 - (b+d)(b+c)}{(b-a)(d-c)}\right] & ,b+c \leq z \leq b+d \\ 1 & ,b+d < z \end{cases},$$

which can be simplified to:

$$F_Z(z) = \begin{cases} 0 & ,z < a+c \\ \dfrac{0.5z^2 - (a+c)z + 0.5(a+c)^2}{(b-a)(d-c)} & ,a+c \leq z \leq a+d \\ \dfrac{0.5(d-c)}{b-a} + \dfrac{z-(a+d)}{b-a} & ,a+d \leq z \leq b+c \\ \dfrac{-a+b+0.5(c-d)}{b-a} + \dfrac{-0.5\left(z^2-(b+c)^2\right) + (b+d)(z-(b+d))}{(b-a)(d-c)} & ,b+c \leq z \leq b+d \\ 1 & ,b+d < z \end{cases}.$$

Therefore, the CDF of $\tilde{P}_{ij} - \tilde{P}_{kl}$ will be a piecewise linear or quadratic function on the interval $(a_{ij} - b_{kl},\ b_{ij} - a_{kl})$.

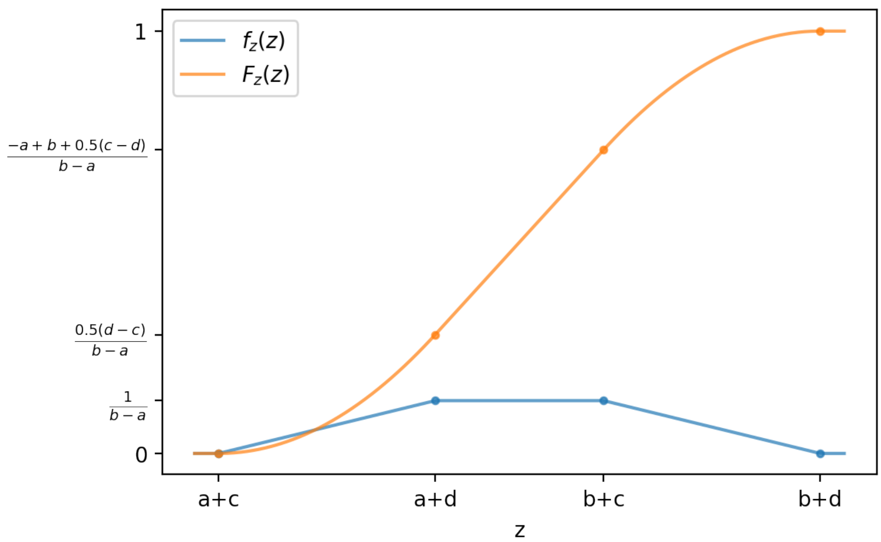

Figure 7: PDF and CDF of the sum of two uniformly distributed random variables.

To make further statements about $\mathbb{P}\left(\langle P^T_{PROM2_j} - P^T_{PROM2_i}, w\rangle \leq 0 \middle| w\right)$, the preference functions needs to be addressed.

Lemma: The probability of an alternative i being preferred over an alternative j in PROMETHEE II, denoted as $\mathbb{P}\left(\langle P^T_{PROM2_j} - P^T_{PROM2_i}, w\rangle \leq 0 \middle| w\right)$, is piecewise constant when the preference functions have a discrete codomain.

Proof: To investigate this probability, let us first reconsider the pairwise differences $\tilde{P}_{ij} - \tilde{P}_{kl}$. As shown before, if the performances are normally distributed, $\tilde{P}_{ij} \sim \mathcal{N}(\mu_{ij}, \sigma^2_{ij})$, and $\tilde{P}_{kl} \sim \mathcal{N}(\mu_{kl}, \sigma^2_{kl})$, then $\tilde{P}_{ij} - \tilde{P}_{kl} \sim \mathcal{N}(\mu_{ij} - \mu_{kl},\ \sigma^2_{ij} + \sigma^2_{kl})$. On the other hand, if the performances are uniformly distributed, $\tilde{P}_{ij} \sim \mathcal{U}(a_{ij}, b_{ij})$ and $\tilde{P}_{kl} \sim \mathcal{U}(a_{kl}, b_{kl})$, then the distribution of $\tilde{P}_{ij} - \tilde{P}_{kl}$ can be obtained by calculating their convolution, as previously shown.

However, since we are considering preference functions with a discrete codomain, the probability of $g(\tilde{P}_{ij} - \tilde{P}_{kl})$ taking a particular value, e.g. (0, 0.5 or 1), will be constant, independent of the specific distribution of $\tilde{P}_{ij} - \tilde{P}_{kl}$ and determined by the corresponding CDF.

If preference function one (2.3) is used, we observe that:

$$\mathbb{P}\left(g\left(\tilde{P}_{ik} - \tilde{P}_{jk}\right) = 0\right) = \begin{cases} 1 & , i = j \\ \mathbb{P}\left(\tilde{P}_{ik} - \tilde{P}_{jk} \leq 0\right) & , i \neq j \end{cases},$$

and $\mathbb{P}\left(g\left(\tilde{P}_{ik} - \tilde{P}_{jk}\right) = 1\right) = 1 - \mathbb{P}\left(g\left(\tilde{P}_{ik} - \tilde{P}_{jk}\right) = 0\right)$.

For preference function two (2.4), it can be seen that:

$$\mathbb{P}\left(g\left(\tilde{P}_{ik} - \tilde{P}_{jk}\right) = 0\right) = \begin{cases} 1 & , i = j \text{ and } q \leq 0 \\ 0 & , i = j \text{ and } q > 0 \\ \mathbb{P}\left(\tilde{P}_{ik} - \tilde{P}_{jk} \leq q\right) & , i \neq j \end{cases},$$

and also $\mathbb{P}\left(g\left(\tilde{P}_{ik} - \tilde{P}_{jk}\right) = 1\right) = 1 - \mathbb{P}\left(g\left(\tilde{P}_{ik} - \tilde{P}_{jk}\right) = 0\right)$.

Finally, using the fourth preference function (2.6), which is the last function with a discrete codomain, the probability can be calculated as:

$$\mathbb{P}\left(g\left(\tilde{P}_{ik} - \tilde{P}_{jk}\right) = 0\right) = \begin{cases} 1 & , i = j \text{ and } q \leq 0 \\ 0 & , i = j \text{ and } q > 0 \\ \mathbb{P}\left(\tilde{P}_{ik} - \tilde{P}_{jk} \leq q\right) & , i \neq j \end{cases}.$$

Furthermore, the probability of observing $\frac{1}{2}$ is given by:

$$\mathbb{P}\left(g\left(\tilde{P}_{ik}-\tilde{P}_{jk}\right)=\frac{1}{2}\right)=\begin{cases}1 & ,i=j \text{ and } q<0\leq p\\ 0 & ,i=j \text{ and } (q\geq 0 \text{ or } p<0)\\ \mathbb{P}\left(\tilde{P}_{ik}-\tilde{P}_{jk}\leq p\right)-\mathbb{P}\left(\tilde{P}_{ik}-\tilde{P}_{jk}<q\right) & ,i\neq j\end{cases},$$

and, lastly, the probability of obtaining 1 will be:

$$\mathbb{P}\left(g\left(\tilde{P}_{ik}-\tilde{P}_{jk}\right)=1\right)=\begin{cases}1 & ,i=j \text{ and } p<0\\ 0 & ,i=j \text{ and } p\geq 0\\ \mathbb{P}\left(\tilde{P}_{ik}-\tilde{P}_{jk}<q\right) & ,i\neq j\end{cases}.$$

Therefore, under the assumption that the preference functions have a discrete codomain, one could calculate the probabilities of all combinations of rows P_i and P_j which result in $\langle P^T_{PROM2_j}-P^T_{PROM2_i},w\rangle\leq 0$ and join their probability to obtain $\mathbb{P}(\langle P^T_{PROM2_j}-P^T_{PROM2_i},w\rangle\leq 0\big|w)$. It follows that only the number of possible combinations for P_i and P_j is dependent on w but not the associated probabilities, hence $\mathbb{P}(\langle P^T_{PROM2_j}-P^T_{PROM2_i},w\rangle\leq 0\big|w)$ will be piecewise constant. The specific probabilities associated with the pairwise differences, P_j-P_i, do not affect the piecewise constant property of the overall probability. □

Corollary: If PROMETHEE II is used and all preference functions have a discrete codomain, $\mathbb{P}(f(\tilde{P},w)=r|w)$ will be piecewise constant.

Proof of corollary: Instead of considering all combinations of two rows, the previous idea can be extended to all combinations of the whole matrix since $\mathbb{P}(f(\tilde{P},w)=r|w)$ just implies $\langle P^T_{PROM2_j}-P^T_{PROM2_i},w\rangle\leq 0$ for $n-1$ pairs of rows simultaneously. □

Concluding, since the number of combinations rises rapidly, as the number of alternatives and criteria increases, this approach might not be feasible for large performance matrices and Monte-Carlo-Simulation could have a better trade-off between accuracy and computational time. Additionally, this approach is not applicable for preference functions with a continuous codomain and therefore not suitable for a wide variety of possible preference functions.

4 Numerical experiments

4.1 Convergence investigations

For the sake of investigating the convergence speed of the Monte-Carlo-Simulation, let $P = \tilde{P}$ with mean being:

$$\mathbb{E}[P] = \begin{pmatrix} 1.0 & 0.3 \\ 0.8 & 0.8 \\ 0.7 & 0.9 \end{pmatrix}.$$

Furthermore, let the standard deviation be ten percent of the mean, hence $\sigma[P] = 0.1 \cdot \mathbb{E}[P]$. Additionally, let the entries of P be normally distributed with $P_{ij} \sim \mathcal{N}(\mathbb{E}\left[P_{ij}\right], \sigma^2\left[P_{ij}\right])$.

When using $w = (0.3,\ 0.7)^T$ as a weight vector and setting the second alternative superior to the other ones, hence $r = (a_2)$, we observe the following by comparing the real probability $\mathbb{P}(f(\tilde{P}, w) = r|w) \approx 0.329$ with 50 runs of Monte-Carlo-Simulation and its 88.89% confidence interval:

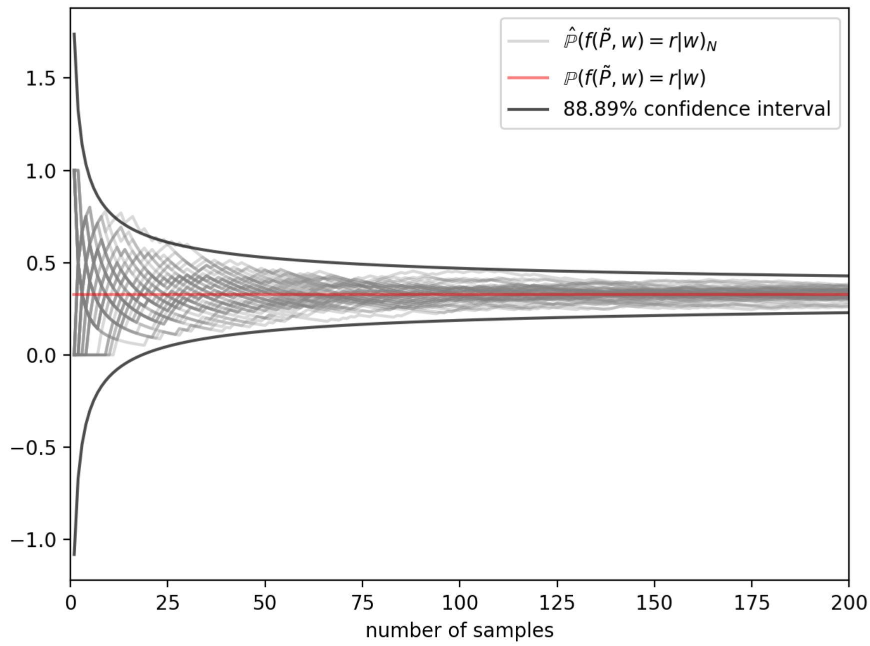

Figure 8: Comparison of 50 Monte-Carlo simulations and the real probability, example 1.

On the other hand, using $w = (0.8,\ 0.2)^T$ as a weight vector and using $r = (a_3)$ as ranking results in $\mathbb{P}(f(\tilde{P}, w) = r|w) \approx 0.056$ and:

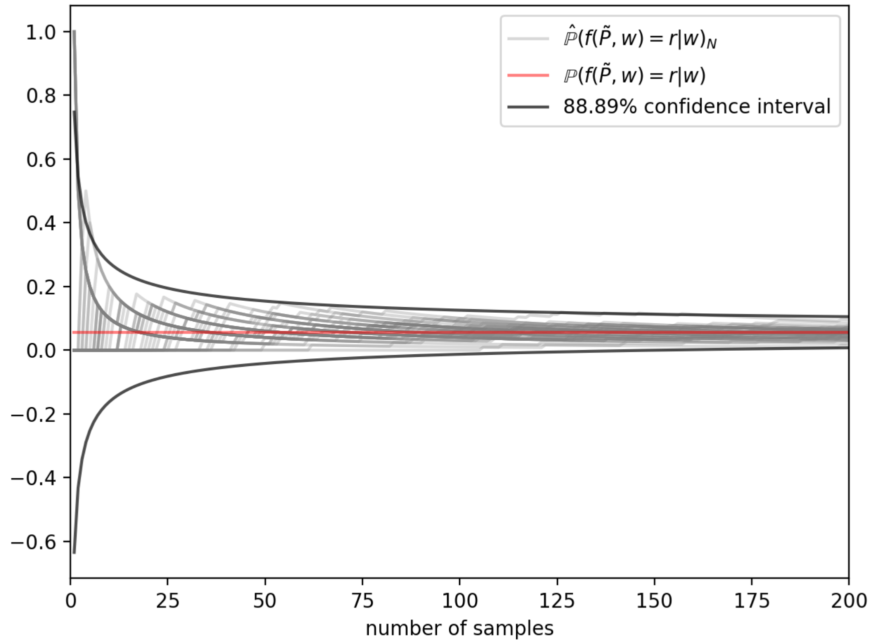

Figure 9: Comparison of 50 Monte-Carlo simulations and the real probability, example 2.

It can be seen that the second confidence interval is smaller than the first one and therefore, for a given number of samples, its Monte-Carlo estimate is expected to be more accurate in terms of the absolute value. This can be explained by the fact that, for a given N and α:

$$\begin{aligned}\Delta\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N &= \alpha\frac{\sigma\left[\mathbf{1}_{W_r}(w)\right]}{\sqrt{N}}\\ &= \alpha\frac{\sqrt{\mathbb{P}(f(\tilde{P},w)=r|w)(1-\mathbb{P}(f(\tilde{P},w)=r|w))}}{\sqrt{N}}\\ &\propto \sqrt{\mathbb{P}(f(\tilde{P},w)=r|w)(1-\mathbb{P}(f(\tilde{P},w)=r|w))}.\end{aligned}$$

After investigating this equation with respect to the probability $\mathbb{P}(f(\tilde{P},w)=r|w)$ (see Figure 10), it can be seen that this standard deviation maximises at a probability of $\mathbb{P}(f(\tilde{P},w)=r|w)=0.5$ with a value of 0.5 and constantly shrinks to zero towards $\mathbb{P}(f(\tilde{P},w)=r|w)=0$ and $\mathbb{P}(f(\tilde{P},w)=r|w)=1$. Due to this, the confidence interval in Figure 9, where the real probability is rather small, is less wide than the one in Figure 8, since there the real probability is close to 0.5.

Additionally, it can be seen that the width of the confidence interval asymptotically converges to zero as N gets bigger. The reason for this is that for a fixed probability $\mathbb{P}(f(\tilde{P},w)=r|w)$ and a fixed α:

$$\Delta\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N = \alpha\frac{\sigma\left[\mathbf{1}_{W_r}(w)\right]}{\sqrt{N}} = \alpha\frac{\sqrt{\mathbb{P}(f(\tilde{P},w)=r|w)(1-\mathbb{P}(f(\tilde{P},w)=r|w))}}{\sqrt{N}} \propto \frac{1}{\sqrt{N}}.$$

If follows that, in order to reduce the width of the confidence interval by half, the number of samples needs to be quadrupled.

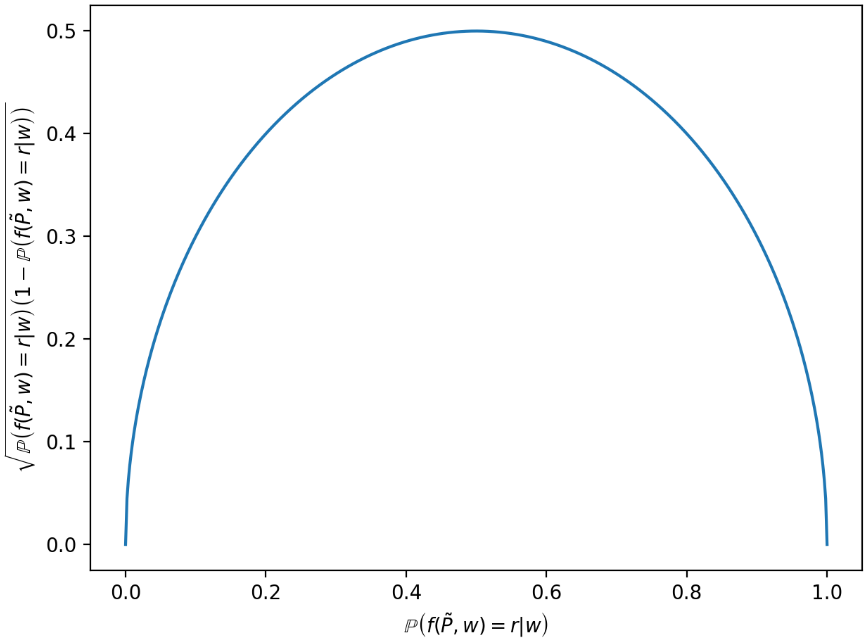

Figure 10: Standard deviation of Monte-Carlo estimates.

Lastly, it can be observed that the confidence interval given by Chebyshev's inequality exceeds the interval [0,1] for a low number of samples which does not make sense when approximating a probability. Therefore it may be feasible to adjust this intervals lower bound to:

$$\max\left(0,\ \mathbb{P}(f(\tilde{P},w)=r|w)-\alpha\cdot\sigma\left[\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N\right]\right),$$

and its upper bound to:

$$\min\left(1,\ \mathbb{P}(f(\tilde{P},w)=r|w)+\alpha\cdot\sigma\left[\hat{\mathbb{P}}(f(\tilde{P},w)=r|w)_N\right]\right).$$

4.2 Comparison of fixed and variable performance matrices

To compare the likelihood function $\mathbb{P}(f(\tilde{P},w)=r|w)$ using a fixed and a variable performance matrix, let $\tilde{P}_{var}$ be normally distributed with:

$$\mathbb{E}\left[\tilde{P}_{var}\right]=\begin{pmatrix}1.0 & 0.3\\ 0.8 & 0.8\\ 0.7 & 0.9\end{pmatrix} \tag{4.1}$$

and $\sigma\left[\tilde{P}_{var}\right] = 0.05 \cdot \mathbb{E}\left[\tilde{P}_{var}\right]$. Furthermore, let both criteria be associated to benefit and the second alternative superior to the first and third, hence $r = (a_2)$. The comparison will be done for SAW and PROMETHEE II separately.

4.2.1 SAW

When using SAW and $\tilde{P}_{fix} = \mathbb{E}\left[\tilde{P}_{var}\right]$, we obtain a matrix P_{SAW} with:

$$P_{SAW} = \begin{pmatrix} 1.0 & \frac{1}{3} \\ 0.8 & \frac{8}{9} \\ 0.7 & 1 \end{pmatrix}.$$

The likelihood for the variable performance matrix was obtained using Monte-Carlo-Simulation with parameters $\alpha = 3$ and $\epsilon = 0.05$. Therefore, the error is less than 5% with at least a probability of 88.89%.

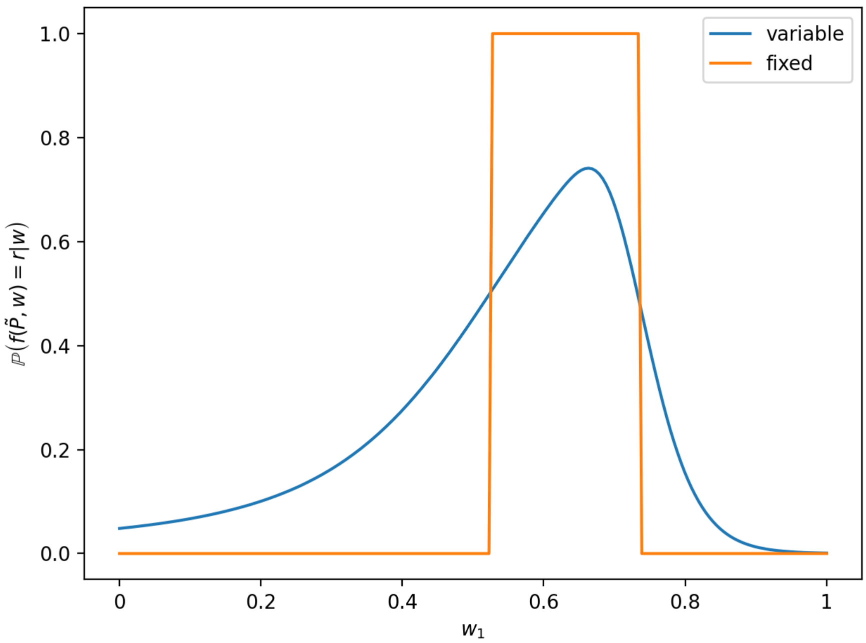

Figure 11: Comparison of a normal and a fixed performance matrix using SAW.

The sampling took about 225 seconds in python on a notebook with an Intel Core i7 processor and 16GB of RAM. The value of w_2 can be obtained by $w_2 = 1 - w_1$. It can be seen that the probability of observing the associated ranking changes quite a lot after only adding an uncertainty with a standard deviation of 5%.

4.2.2 PROMETHEE II

When using PROMETHEE II on the same performance matrix as before (4.1) and preference function type six (2.7) for both criteria, with $s = \frac{1}{10} \max_{1 \le i \le m} \left[\tilde{P}_{fix_{ij}}\right]$, we obtain the following preference function for criterion j:

$$g_j(x) = 1 - \exp\left(-50\left(\frac{x}{\max_{1 \le i \le m}\left[\tilde{P}_{fix_{ij}}\right]}\right)^2\right).$$

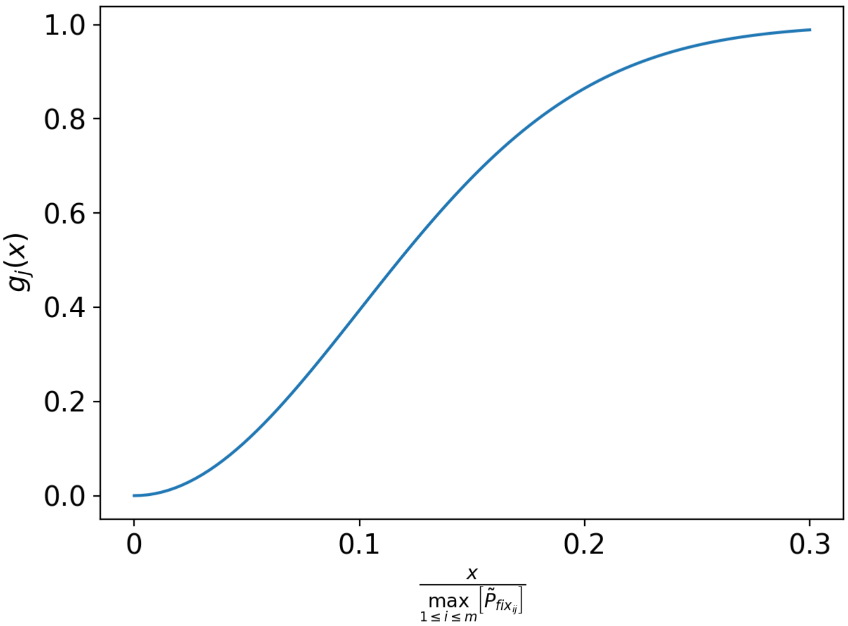

Figure 12: The used preference function for criterion j.

Rounded to two decimal places, this results in a matrix P_{PROM2} with:

$$P_{PROM2} = \begin{pmatrix} 0.93 & -1 \\ -0.24 & 0.27 \\ -0.69 & 0.73 \end{pmatrix}.$$

Here, the sampling took 270 seconds. Therefore, the sampling took longer than using SAW which is due to the fact that the computation of P_{PROM2} is more computationally intensive than P_{SAW}.

The result can be seen in Figure 13 and we observe that the likelihood changes drastically with a small uncertainty in the performance matrix. This highlights the necessity of a stochastic approach to MCDA.

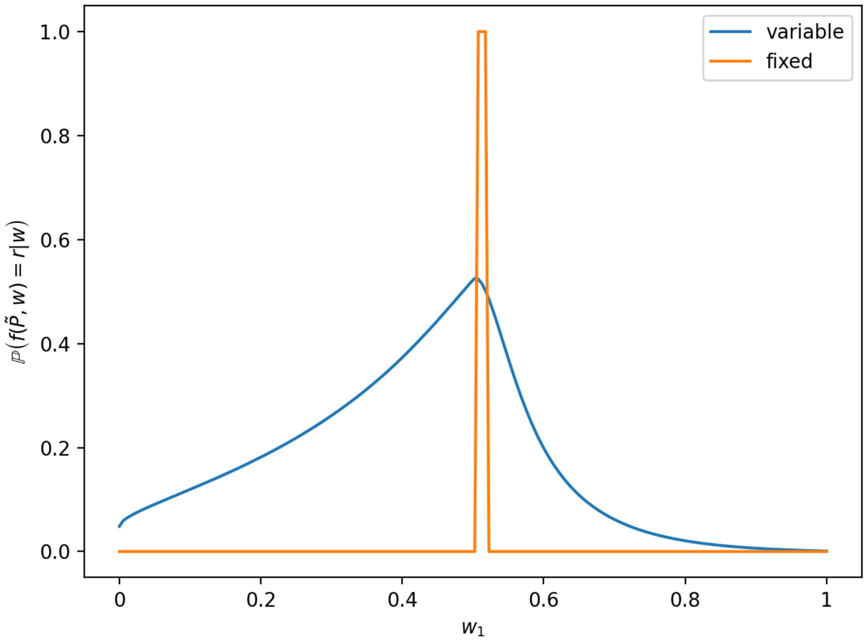

Figure 13: Comparison of a normal and a fixed performance matrix using PROMETHEE II.

4.3 Scaling methods with SAW

To compare the different scaling approaches for SAW, let $\tilde{P}$ be normally distributed with expected value:

$$\mathbb{E}\left[\tilde{P}\right] = \begin{pmatrix} 100 & 3 \\ 150 & 1.75 \\ 125 & 2 \\ 200 & 1 \end{pmatrix},$$

and standard deviation $\sigma\left[\tilde{P}\right] = 0.05 \cdot \mathbb{E}\left[\tilde{P}\right]$. In this case, $\tilde{P}$ consists of four alternatives and two criteria which are expressed in different units.

When associating both criteria to benefit and computing the likelihood for all possible rankings containing only one alternative (see Figure 14), we observe that scaling the entries of $\tilde{P}$ by their expected values results in mostly similar results as scaling them by their sum using Monte-Carlo-Simulation. On the other hand, the scaling by minimum and maximum shows some differences at sections where the probability is not nearly constant.

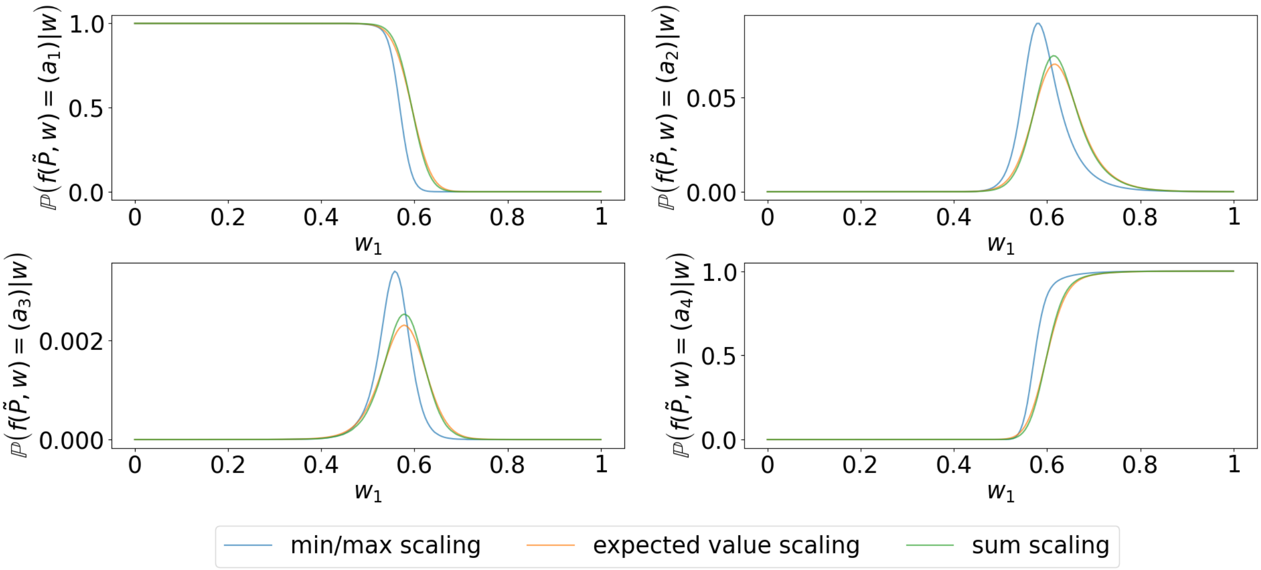

Figure 14: Comparison of scaling methods for SAW using benefit criteria.

Moreover, if both criteria were associated to cost, the difference between the scaling by expected value and the scaling by sum can be neglected, but the scaling by minimum and maximum results in major differences in the likelihood function.

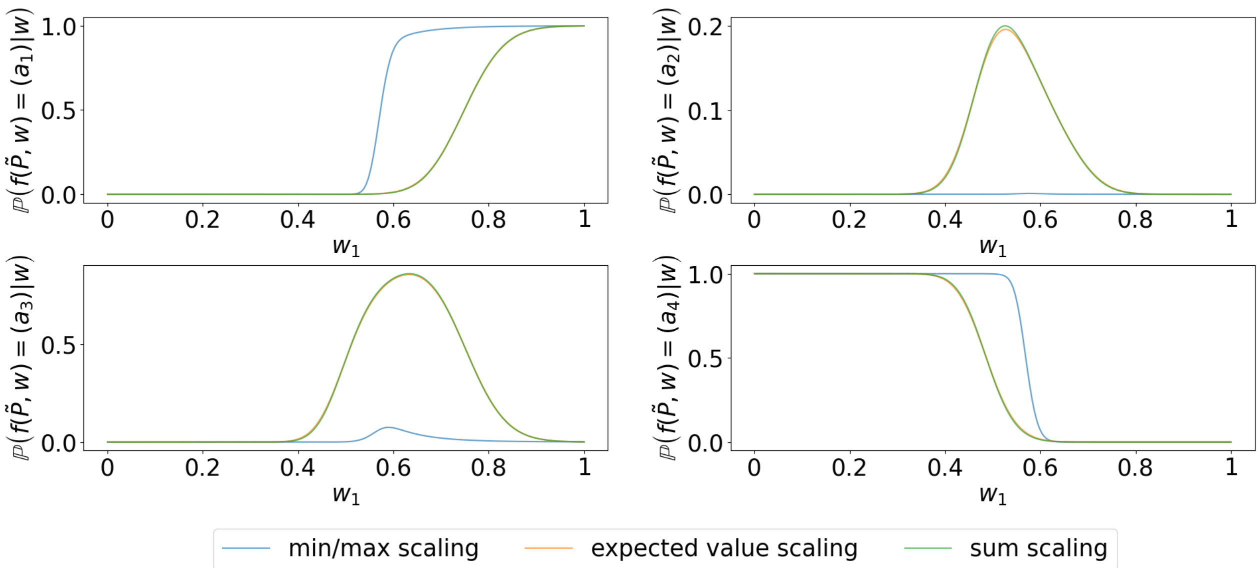

Figure 15: Comparison of scaling methods for SAW using cost criteria.

4.4 Analyzing the structure of PROMETHEE likelihoods

In section 3.6 it was shown that only using discrete preference functions will result in a piecewise constant likelihood function. To illustrate this, let $\tilde{P}$ be uniformly distributed with:

$$\mathbb{E}\left[\tilde{P}\right] = \begin{pmatrix} 100 & 3 & 4000 \\ 150 & 1.75 & 3000 \\ 125 & 2 & 4000 \\ 200 & 1 & 3300 \end{pmatrix},$$

and $\tilde{P}_{ij} \sim \mathcal{U}(0.95 \cdot \mathbb{E}\left[\tilde{P}_{ij}\right], 1.05 \cdot \mathbb{E}\left[\tilde{P}_{ij}\right])$. Additionally, let the preference function have a hard threshold at 10% of the criteria mean.

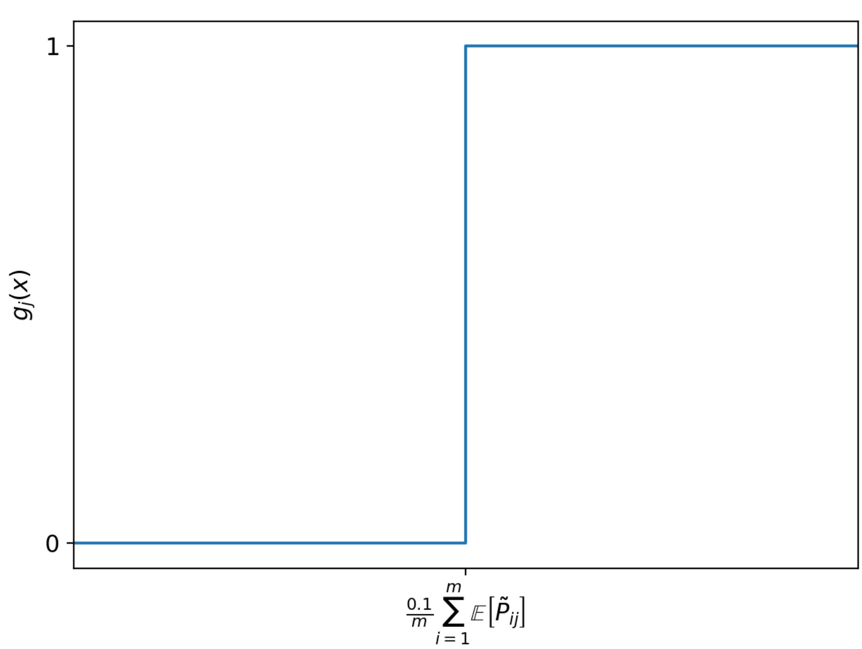

Figure 16: The hard threshold preference function for criterion j.

We want to investigate $\mathbb{P}(f(\tilde{P}, w) = (a_4, a_2)|w)$, hence the fourth alternative should be preferred over the second alternative which should be superior to all other alternatives.

As proven, the resulting likelihood function is piecewise constant (see Figure 17). It can be seen that the probability decreases gradually when moving away from $(1, 0, 0)$.

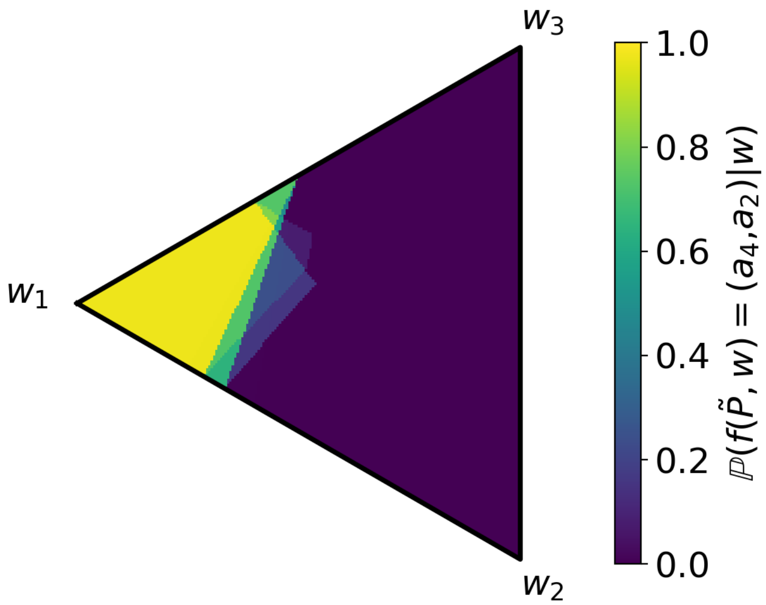

Figure 17: The piecewise constant likelihood function.

To put that in contrast, let us compare this to the the likelihood with the same distributions but the preference function from Figure 12 used for all criteria.

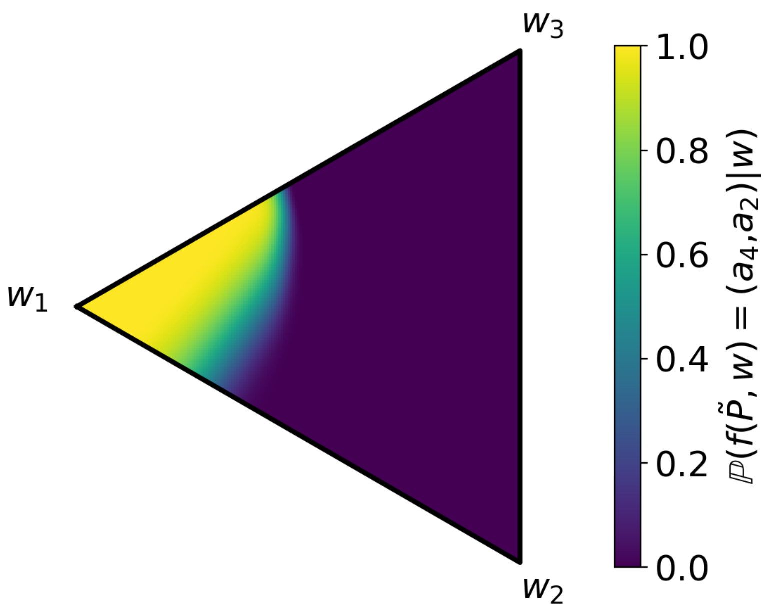

Figure 18: The likelihood function with gaussian criterion.

Now, the likelihood function is no longer piecewise constant since the preference functions do not have discrete codomains.

4.5 Approximating a uniform distribution with SAW

In section 3.4 it was suggested that a normal distribution might be a good alternative to approximate $\mathbb{P}(\langle P_j^T - P_i^T, w\rangle \leq 0|w)$ if $\tilde{P}$ is uniformly distributed. To asses the quality of this approximation, a probability–probability plot can be used.

For the sake of this experiment, let us consider the distribution of $\langle {P_{SAW}}_2^T - {P_{SAW}}_1^T, w\rangle$. Furthermore, let $\tilde{P}_{ij} \sim \mathcal{U}(a_{ij}, b_{ij})$ with:

$$A = \begin{pmatrix} 1 & 2 & 4 & 8 & 16 & 32 & 64 & 128 & 256 & 512 \\ 0.9 & 2.2 & 4.5 & 7 & 15 & 35 & 70 & 120 & 220 & 500 \end{pmatrix},$$

and $B = 1.1 \cdot A$. Additionally, when using (3.2), we obtain:

$$P_{SAW\,ij} \sim \mathcal{U}\left(\frac{a_{ij}}{\sum_{k=1}^{10} \frac{a_{kj}+b_{kj}}{2}}, \frac{b_{ij}}{\sum_{k=1}^{10} \frac{a_{kj}+b_{kj}}{2}}\right). \tag{4.2}$$

Let these new upper and lower bounds be $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$, respectively. When approximating $\langle {P_{SAW}}_2^T - {P_{SAW}}_1^T, w\rangle$ with a normal distribution, this results in:

$$\langle {P_{SAW}}_2^T - {P_{SAW}}_1^T, w\rangle \sim \mathcal{N}\left(\sum_{k=1}^{10} \mu_k,\ s_{10}^2\right),$$

with $\mu_k = \sum_{k=1}^{10} \frac{1}{2} w_k(\tilde{b}_{2k} + \tilde{a}_{2k} - \tilde{b}_{1k} - \tilde{a}_{1k})$ and $s_{10}^2 = \frac{1}{12}\sum_{k=1}^{10} w_k^2\left((\tilde{b}_{2k} - \tilde{a}_{2k})^2 + (\tilde{b}_{1k} - \tilde{a}_{1k})^2\right)$.

Now, to investigate the quality of this approximation, 1000 samples from (4.2) are taken and $\langle {P_{SAW}}_2^T - {P_{SAW}}_1^T, w\rangle$ is calculated for each sample. Then, the empirical CDF of the standardized samples $\frac{1}{s_{10}}\left(\langle {P_{SAW}}_2^T - {P_{SAW}}_1^T, w\rangle - \sum_{k=1}^{10} \mu_k\right)$ is compared to the theoretical CDF of the standard normal distribution.

This is done twice with two different weight vectors with the first weight vector being $w = \frac{1}{55} \cdot (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)^T$ (see Figure 19) and the second weight vector containing the same values but in reversed order (see Figure 20).

The x-axis represents the theoretical CDF while the y-axis represents the empirical CDF. If the distributions were the same, the plot would be a straight line from (0,0) to (1,1) which is colored in red. The green marker highlights the point where $\langle {P_{SAW}}_2^T - {P_{SAW}}_1^T, w\rangle \leq 0$.

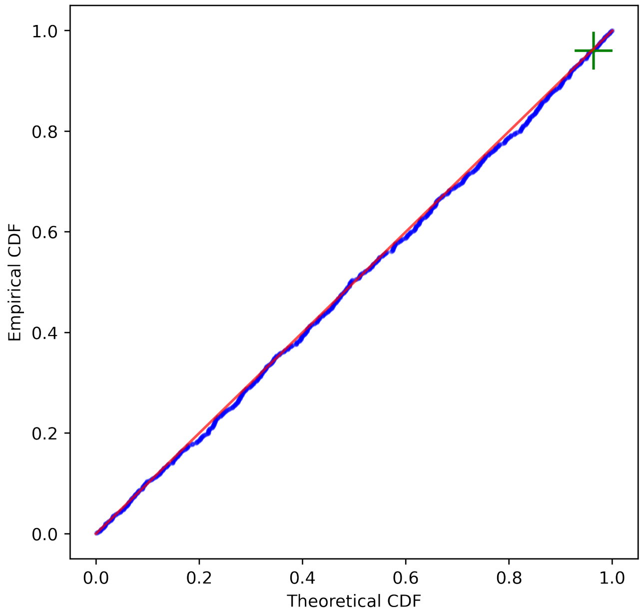

Figure 19: The probability–probability plot for the first weight vector.

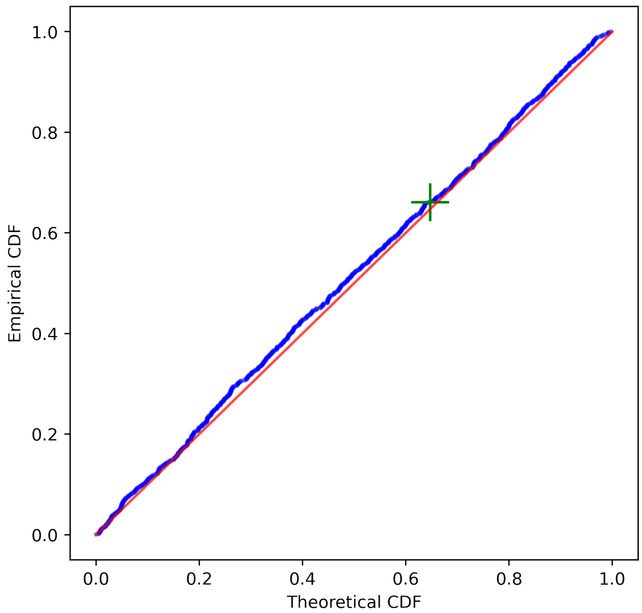

Figure 20: The probability–probability plot for the second weight vector.

The results show that the respective CDFs only differ slightly which leads to the fact that approximating the distribution of $\langle P_{SAW}{}_2^T - P_{SAW}{}_1^T, w\rangle$ by a normal distribution may be a good idea in this particular case. However, this might still not be the case for different performance matrices or weight vectors.

5 Application

When applying the presented methods to an example about the transition process towards sustainable mobility, Bayesian statistics can be used to gain insight on which factors lead to customers buying specific types of electrical cars on the German market. This approach is based on Bayes' theorem which states that:

$$p(w|f(\tilde{P},w)=r)=\frac{p(w)\cdot p(f(\tilde{P},w)=r|w)}{p(f(\tilde{P},w)=r)}.$$

Here, $p(w)$ is called the prior distribution which contains previous knowledge about the weight vector. Furthermore, the likelihood $p(f(\tilde{P},w)=r|w)$ is the conditional probability of observing a specific ranking of alternatives to a given weight vector (see chapter 3) expressed as a function in w. Lastly, $p(f(\tilde{P},w)=r)$ is the unknown but constant model evidence and $p(w|f(\tilde{P},w)=r)$ is called the posterior distribution which is a PDF representing the probability of the weight vector conditioned on the given ranking, hence the updated prior beliefs.

Table 2: Performance matrix $\tilde{P}$ with corresponding criteria from [15].

	Price-performance ratio	Price	Comfort	Family friendliness	Environmental friendliness	Resale value	Image/prestige	Safety	Other
Mini cars	3.5	3.9	3.4	1.5	3.6	2.9	2.0	3.8	5.0
Small cars	4.2	3.6	3.0	2.7	3.4	3.7	2.0	4.0	5.0
Medium cars	4.1	3.1	3.9	3.2	3.1	3.7	3.0	4.1	5.0
Large cars	3.8	2.2	4.3	4.2	2.8	3.8	3.0	4.6	5.0
Executive cars	3.1	1.0	4.0	3.3	2.2	3.7	4.0	4.5	5.0
Luxury cars	3.2	1.0	4.7	3.6	1.1	4.1	5.0	3.9	5.0
SUVs	4.0	3.1	3.5	3.4	2.1	4.2	4.5	4.0	5.0
Off-road vehicles	3.9	2.5	3.8	3.6	1.7	3.8	4.5	4.3	5.0
Mini vans	2.9	1.8	3.2	4.7	1.6	3.1	2.0	4.7	5.0

In the following, the posterior was calculated twice. Firstly, it was calculated with a fixed performance matrix [1], and secondly, using a variable performance matrix. This was done to highlight the importance of considering uncertainties in the performances. For the fixed case, the matrix from Table 2 was used, and for the variable case, the same matrix was used as the mean, and the entries of $\tilde{P}$, except for "Other," were independently normally distributed with a standard deviation of 5% of their mean. The last criterion, "Other," was fixed for both cases to prevent a possible bias.

Furthermore, PROMETHEE II was used as MCDA method, and a V-shape preference function (2.5), with $p = \max_{1\leq i\leq m} \mathbb{E}[\tilde{P}_{ij}] - \min_{1\leq i\leq m} \mathbb{E}[\tilde{P}_{ij}]$, was used for every criterion j to compare it to previous calculations [1]. Additionally, all criteria were associated with benefit since $\tilde{P}$ is on a scale such that higher values corresponded to a better performance. For the ranking, the following order was chosen:

1. Small cars,
2. Medium cars,
3. Mini cars,
4. Large cars.

Table 3: Discrete choice experiment data for the criteria weights.

criterion	$\tilde{w}_1$	$\tilde{w}_2$	...
Price-performance ratio	10.04547	8.66926	...
Price	29.49723	6.51079	...
Comfort	13.63339	10.34713	...
Family friendliness	9.99244	10.47944	...
Environmental friendliness	4.72200	5.87986	...
Resale value	1.41605	2.96719	...
Image/prestige	3.53576	5.52158	...
Safety	1.10667	7.30710	...
Drive technology	12.23415	18.51638	...
Range	2.35857	8.43138	...
Charging/refuel duration PHEV	3.43427	1.97538	...
Charging duration BEV	5.55045	8.80491	...
Fixed operating costs	2.47352	4.58960	...

To formalize the prior distribution, a data-based prior was used. The data contains 420 data points and was obtained by a discrete choice experiment which means that the participants' preferences were elicited without directly asking them to state their preferred options. Before setting the prior distribution, the data was normalized to the standard simplex. Then, the prior was chosen to be a multivariate normal PDF with its mean and variance as the corresponding sample statistics. Additionally, all mismatching criteria between Tables 2 and 3 were summed up to ”Other”.

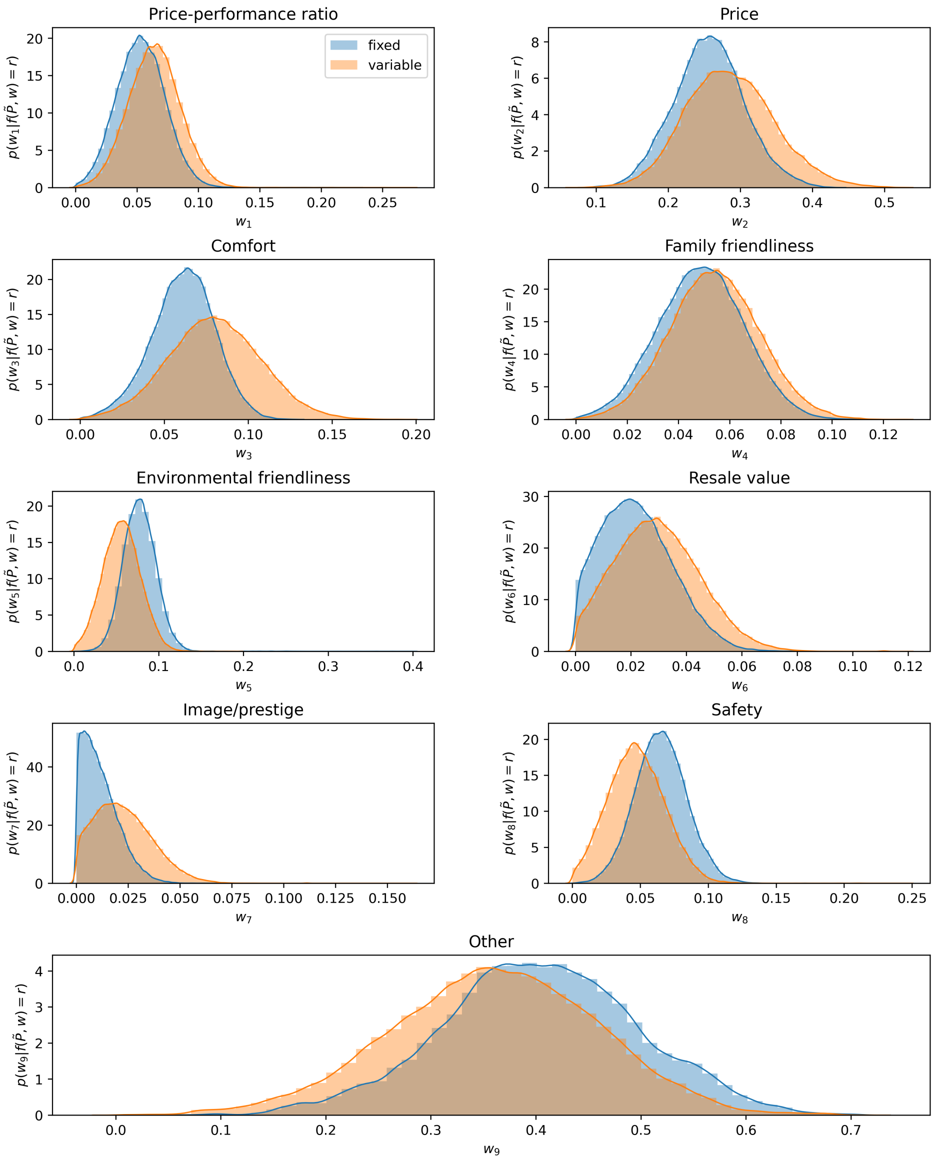

Figure 21: Estimated marginal posterior densities with likelihood functions to fixed and variable performance matrices using a V-shape preference function.

The marginal posteriors were not calculated directly but instead obtained using kernel density estimation with Markov-Chain-Monte-Carlo (MCMC) samples. Here, the samples were generated by hopsy, a Python wrapper for the sampling toolbox HOPS [16] which has adapted MCMC algorithms to convex, bounded polytopes. Furthermore, the samples were generated using four Markov chains with 100,000 samples each and the $\hat{R}$ statistic [17] was used to assure convergence.

The results in Table 4 show that the price tends to be the most important known factor for people who prefer cars the same way as conditioned in the ranking since it has the highest expected value besides ”Other”. Sadly, the weight for ”Other” tends to be the biggest in general which still leaves some uncertainty about the factors leading to the specified decision.

Table 4: Expected values of posteriors using Monte-carlo Integration (rounded) and using a V-shape preference function.

	Price-performance ratio	Price	Comfort	Family friendliness	Environmental friendliness	Resale value	Image/ prestige	Safety	Other
Fixed	0.053	0.257	0.061	0.049	0.078	0.022	0.012	0.065	0.403
Variable	0.064	0.286	0.08	0.054	0.056	0.03	0.023	0.047	0.36

Table 5: Variances of posteriors using Monte-carlo Integration (rounded, $\sigma^2 \cdot 10^2$) and using a V-shape preference function.

	Price-performance ratio	Price	Comfort	Family friendliness	Environmental friendliness	Resale value	Image/ prestige	Safety	Other
Fixed	0.038	0.24	0.035	0.028	0.039	0.016	0.008	0.036	0.892
Variable	0.044	0.375	0.075	0.03	0.047	0.023	0.019	0.042	0.978

When focusing on the means and variances, it can be seen that the variances are higher when using a variable performance matrix which makes sense because more uncertainty was added. It is also interesting to notice that all expected values rose, except for ”Other”, ”Environmental friendliness” and ”Safety”.

When focusing on the correlations between criteria (see Figure 22), it can be investigated that all criteria are negatively correlated to ”Other”. Specifically ”Price” and ”Comfort” show a strong negative correlation to ”Other” which means that increasing the weight for these criteria will most likely result in a reduction for the weight of ”Other” when achieving the specified ranking. Furthermore, there seems to be a quite strong positive correlation between ”Comfort” and ”Price”. Additionally, the strong correlation between ”Safety” and ”Price” shrinks a lot when adding uncertainty to the performances. Lastly, even though the variances rose with a variable performance matrix, most correlations seem to have shrunk.

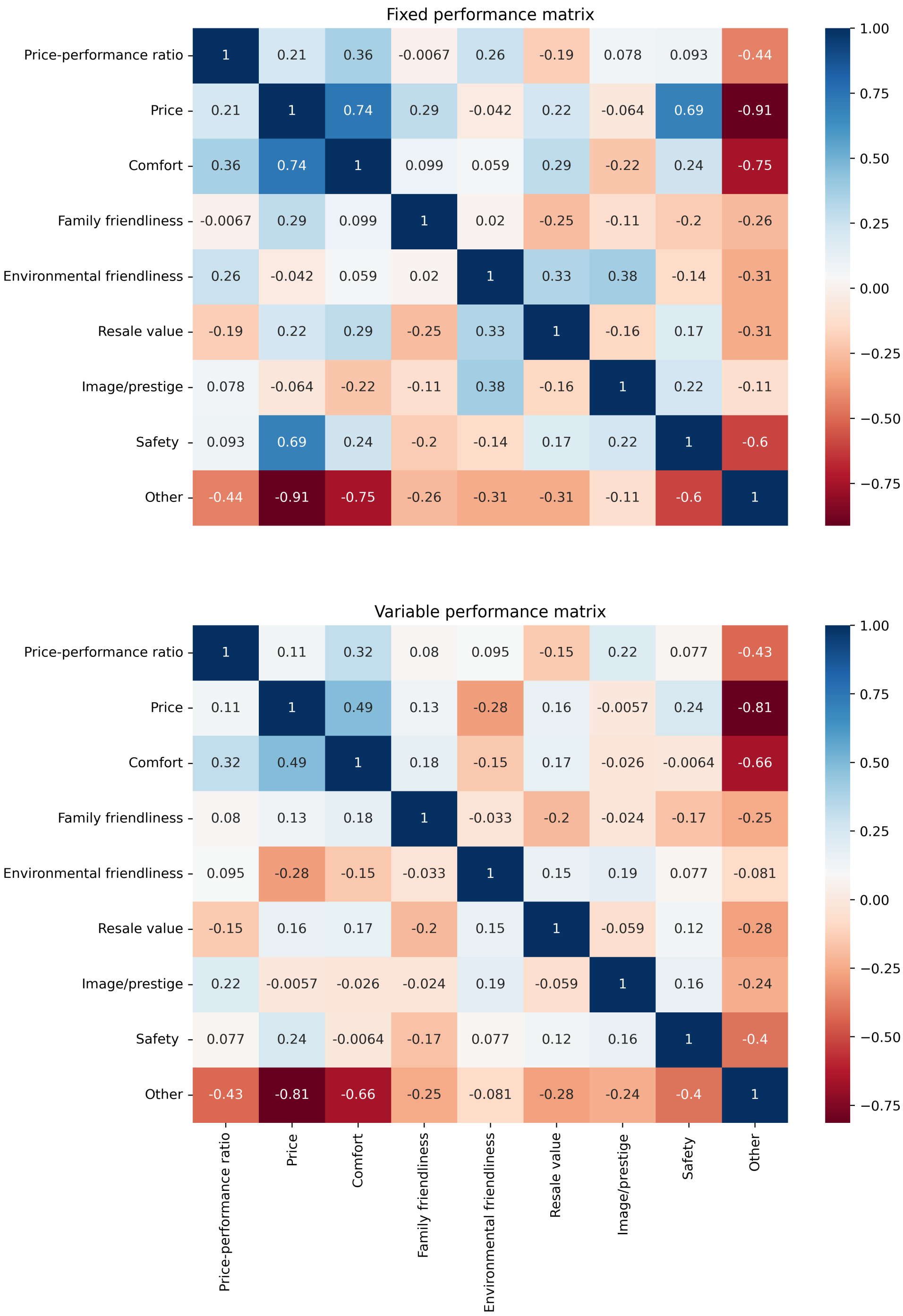

Figure 22: The correlations of the posterior samples using a V-shape preference function.

Since the V-shape preference function was not used in any of the previous chapters, the calculations were repeated using a Gaussian preference function with $s = \frac{1}{10} \max_{1 \leq i \leq m} \left[\tilde{P}_{f i x_{ij}} \right]$ (see Figure 12) to preserve consistency throughout the use of preference functions. The resulting marginal densities (see Figure 23) show similar results even though the difference between the posteriors for "Other" and "Price" are more significant than in the previous example while the difference for "Resale value" became less.

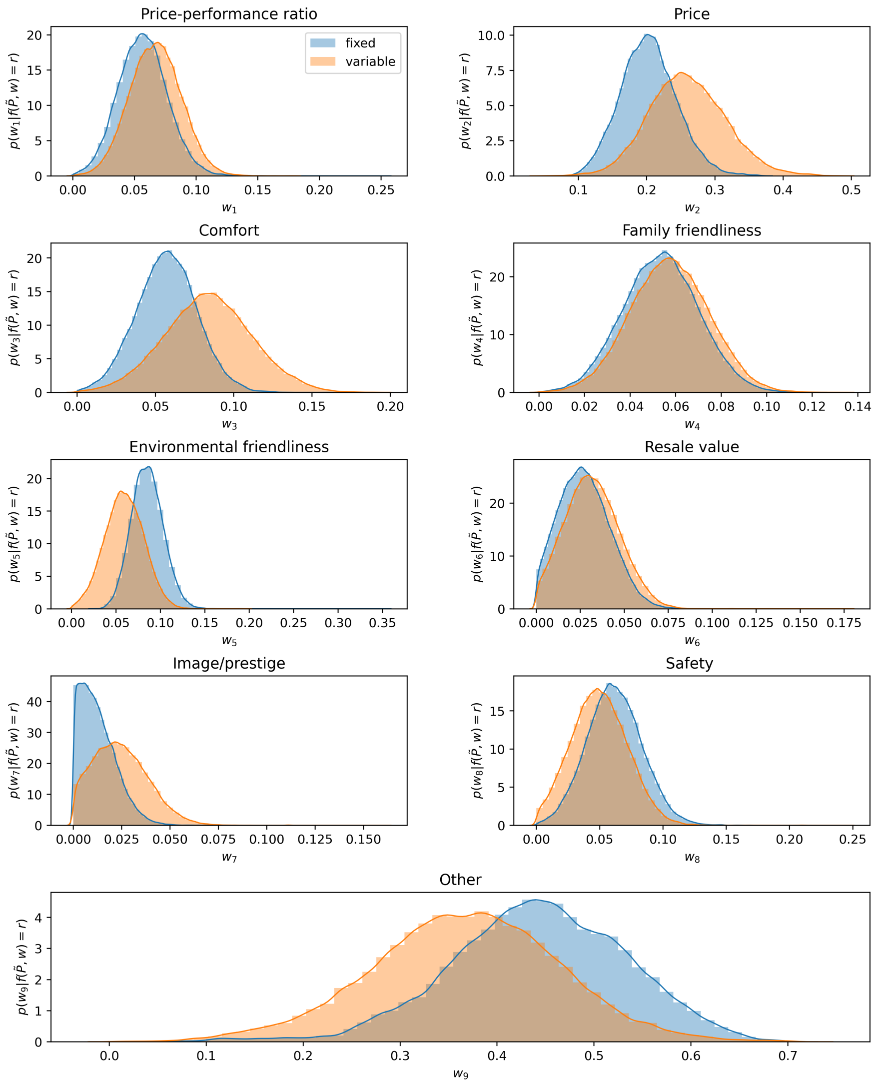

Figure 23: Estimated marginal posterior densities with likelihood functions to fixed and variable performance matrices using a Gaussian preference function.

Table 6: Expected values of posteriors using Monte-carlo Integration (rounded) and using a Gaussian preference function.

	Price-performance ratio	Price	Comfort	Family friendliness	Environmental friendliness	Resale value	Image/ prestige	Safety	Other
Fixed	0.057	0.203	0.057	0.054	0.085	0.028	0.013	0.062	0.442
Variable	0.067	0.26	0.084	0.057	0.059	0.032	0.024	0.05	0.367

Table 7: Variances of posteriors using Monte-carlo Integration (rounded, $\sigma^2 \cdot 10^2$) and using a Gaussian preference function.

	Price-performance ratio	Price	Comfort	Family friendliness	Environmental friendliness	Resale value	Image/ prestige	Safety	Other
Fixed	0.04	0.169	0.037	0.028	0.034	0.02	0.009	0.047	0.845
Variable	0.043	0.313	0.077	0.029	0.047	0.023	0.02	0.048	0.953

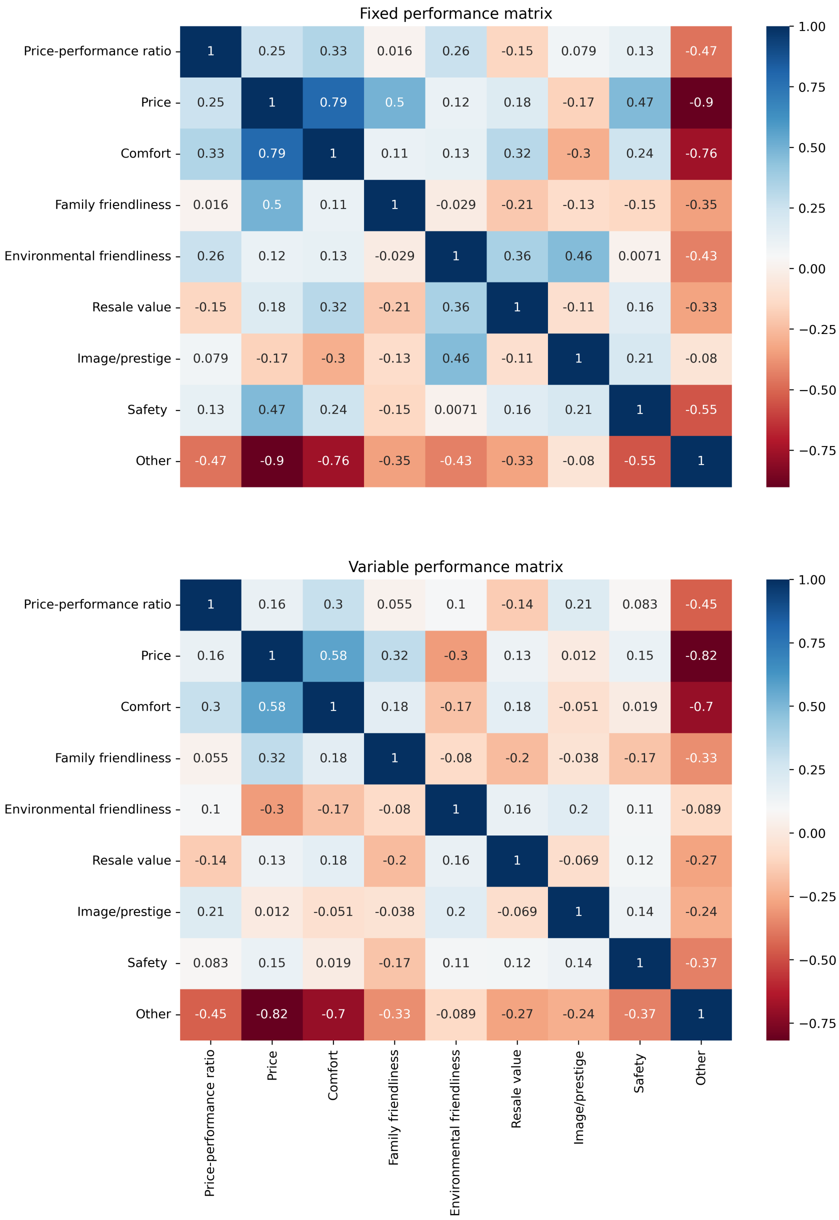

Figure 24: The correlations of the posterior samples using a Gaussian preference function.

The expected values (see Table 6) show a similar behaviour to the previous example and the variances (see Table 7) also seem to rise when adding uncertainty to the performance matrix. Additionally, the correlations (see Figure 24) seem to mostly shrink.

In summary, adding uncertainties to a Bayesian approach on analyzing criteria weights for electrical cars on the German car market shows a strong influence on the posterior distribution. Therefore, the necessity of addressing uncertainties for this example can not be neglected.

6 Conclusion

In conclusion, this bachelor's thesis has dealt with the area of MCDA and its application in the face of ever-evolving challenges in the modern world. As decision-makers deal with complex problems, the need for robust and effective decision-making processes has become more important than ever before. The investigation has centered on a critical component of MCDA, the performance matrix, which serves as the foundation for evaluating alternatives against a range of criteria.

To address these challenges, this thesis sought to explore and develop methods that can effectively handle uncertain performance matrices, empowering decision-makers with more reliable and actionable insights. The primary focus has been on using stochastic methods, which embrace the probabilistic nature of uncertainties, to enhance the reliability and robustness of the decision-making process. By adopting such approaches, decision-makers can now analyze decisions comprehensively and explore what-if scenarios while considering the inherent uncertainties in the evaluation of alternatives.

The theoretical foundations of MCDA have been thoroughly examined and extended through the incorporation of stochastic methods. Beyond theory, the practical implications of this research have been demonstrated through a real-world example concerning sustainable mobility. The application of stochastic methods in this context showcased their ability to provide actionable insights for transitioning towards sustainable and environmentally conscious mobility solutions.

Nevertheless, it needs to be acknowledged that this research still has some limitations. The fact that the discussed methods are only applicable under the condition of independent criteria creates opportunities for additional research and advancement. Possible ways to build up on this research could be adding correlations between different criteria or even connections between different stakeholders since decisions are rarely made under strict independence.

In summary, this bachelor's thesis points to the importance of acknowledging and addressing uncertainties in decision-making processes. By integrating stochastic methods into MCDA, the groundwork for more informed and reliable decisions has been laid, empowering decision-makers to handle the complexities of evaluating certain criteria.

References

[1] HASENAUER, Luis: Bayesian statistics in the context of Multi Criteria Decision Analysis. (2023)

[2] GRECO, Salvatore ; FIGUEIRA, Jose ; EHRGOTT, Matthias: *Multiple criteria decision analysis*. Bd. 37. Springer, 2016

[3] PODVEZKO, Valentinas: The comparative analysis of MCDA methods SAW and COPRAS. In: *Engineering Economics* (2011)

[4] *Kapitel* Multi-criteria Approaches to Ancillary Effects: The Example of E-Mobility. In: VÖGELE, Stefan ; BALL, Christopher ; KUCKSHINRICHS, Wilhelm: *Ancillary Benefits of Climate Policy: New Theoretical Developments and Empirical Findings”.* Cham : Springer International Publishing, 2020. – ISBN 978–3–030–30978–7, p. 157–178

[5] MARESCHAL, B.: *The PROMETHEE Bibliographical Database*. `http://en.promethee-gaia.net/bibliographical-database.html`, 2020

[6] BRANS, J.P. ; VINCKE, Ph. ; MARESCHAL, B.: How to select and how to rank projects: The PROMETHEE method. In: *European Journal of Operational Research* 24 (1985)

[7] BALLIO, Francesco ; GUADAGNINI, Alberto: Convergence assessment of numerical Monte Carlo simulations in groundwater hydrology. In: *Water Resources Research* 40 (2004), No. 4. `http://dx.doi.org/https://doi.org/10.1029/2003WR002876`. – DOI https://doi.org/10.1029/2003WR002876

[8] WIEN, TU: *Konvergenzverhalten, Abschätzung des Simulationsfehlers*. `https://www.iue.tuwien.ac.at/phd/harlander/node31.html`, 2023

[9] SIEGRIST, Kyle: *The Multivariate Normal Distribution*. `https://stats.libretexts.org/Bookshelves/Probability_Theory/Probability_Mathematical_Statistics_and_Stochastic_Processes_(Siegrist)/05%3A_Special_Distributions/5.07%3A_The_Multivariate_Normal_Distribution`, 2022

[10] ROSS, Kevin: *Transformations of random variables*. `https://bookdown.org/kevin_davisross/probsim-book/transformations-of-random-variables.html`, 2022

[11] BRADLEY, David M. ; GUPTA, Ramesh C.: On the distribution of the sum of n non-identically distributed uniform random variables. In: *Ann Inst Stat Math* 54 (2002), No. 3, p. 689–700

[12] SCIPY: *Normaltest*. https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.normaltest.html, 2005

[13] VENKATASUBRAMANIAN, Suresh: *Sampling from the simplex*. http://blog.geomblog.org/2005/10/sampling-from-simplex.html, 2005

[14] TSUN, Alex: *Multiple Random Variables*. https://courses.cs.washington.edu/courses/cse312/20su/files/student_drive/5.5.pdf, 2023

[15] VÖGELE, Stefan ; BROSKA, Lisa H. ; OTTE, Sebstian ; RÜBBELKE, Dirk: Why the trend towards gas-guzzlers? a closer look at the complex effects of social norms on german car buyers. In: *Energy Research & Social Science* (2021)

[16] JADEBECK, Johann F. ; THEORELL, Axel ; LEWEKE, Samuel ; NÖH, Katharina: HOPS: high-performance library for (non-)uniform sampling of convex-constrained models. In: *Bioinformatics* 37 (2020), 10, No. 12. https://doi.org/10.1093/bioinformatics/btaa872

[17] GELMAN, Andrew ; RUBIN, Donald B.: Inference from iterative simulation using multiple sequences. In: *Statistical Science* 7 (1992)