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Interação entre Dissipação Fracionária e Memória Não-Linear na Existência de Soluções para a Equação de Placas

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Luis Gustavo Longen

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“Mathematics is not about numbers,
equations, computations, or algorithms:
it is about understanding.”
(THURSTON, William , 2006)

RESUMO

Neste trabalho, consideramos uma equação de tipo de placas com inércia rotacional, sob os efeitos de um amortecimento fracionário e uma não-linearidade de tipo de memória. O objetico desse trabalho é encontrar o expoente crítico $\overline{p}$ que é limítrofe entre a existência e a não-existência de soluções globais para o problema dado, e entender como o amortecimento fracionário interage com a não-linearidade de memória e como essa interação pode interferir em $\overline{p}$. Com este fim, encontramos e utilizamos diversas estimativas $L^{\eta} - L^{q}$ com $1 \leq \eta \leq 2 \leq q \leq \infty$, bem como estimativas $(L^1 \cap L^p) - L^p$ para $p < 2$, um caso delicado para o qual há uma perda na taxa de decaimento. São analisados diversos cenários com base na dimensão n e nos intervalos admissíveis para θ e γ, os parâmetros que caracterizam o amortecimento fracionário e a não-linearidade de memória, respectivamente. Com esse trabalho, concluímos que, embora na maioria dos casos as taxas de decaimento obtidas sejam suficientes para alcançar o expoente crítico esperado $\overline{p}$, há uma combinação específica de intervalos envolvendo n, γ e θ para a qual a perda nas taxas de decaimento é grande o suficiente para interferir nos resultados de existência, deixando uma pequena lacuna para a qual a existência ou a não-existência de soluções globais é desconhecida.

Palavras Chave: Equação de placas. Inércia Rotacional. Dissipação fracionária. Taxas de decaimento "sharp". Não-Linearidade de tipo Memória. Estrutura de Perda de Regularidade. Dissipação efetiva. Espaço de Fourier.

ABSTRACT

In this work, we consider a plate-type equation with rotational inertia, under the effects of a fractional damping and a memory nonlinearity. The objective of this work is to find the critical exponent $\overline{p}$ that is the threshold between existence and non-existence of global solutions for the given problem, and to understand how the fractional damping interacts with the memory nonlinearity and how this interplay may interfere on $\overline{p}$. To this end, we find and employ several $L^\eta - L^q$ estimates with $1 \leq \eta \leq 2 \leq q \leq \infty$, as well as $(L^1 \cap L^p) - L^p$ estimates for $p < 2$, a delicate case for which there is a loss in the decay rate. We analyze several scenarios based on the dimension n and on the admissible ranges for θ and γ, the parameters that characterize the fractional damping and the nonlinear memory, respectively. With this work, we conclude that, though in most cases the obtained decay rates are enough to reach the expected critical exponent $\overline{p}$, there is a specific combination of ranges involving n, γ and θ for which the loss of decay is significant enough to interfere in the existence results, leaving a small gap for which existence or non-existence of global in-time solutions is uncertain.

Keywords: Plate equation. Fractional damping. Rotational Inertia. Regularity-loss structure. Sharp decay rates. Effective damping. Fourier space.

RESUMO EXPANDIDO

Introdução

Neste trabalho consideramos uma equação do tipo de placas, assim caracterizada por possuir os termos u_{tt} e $\Delta^2 u$, com inércia rotacional, descrita pelo termo $-\Delta u_{tt}$, sob os efeitos de um termo de amortecimento representado pela ação do operador laplaciano com potência fracionária $(-\Delta)^\theta u_t$ e de um termo não-linear do tipo de memória $\Gamma(1-\gamma)I^{1-\gamma}|u|^p$.

Objetivos

O objetivo do trabalho é obter taxas de decaimento do tipo $L^\eta - L^q$ com $1 \leq \eta \leq 2 \leq q \leq \infty$ para a solução e sua primeira derivada no tempo, bem como taxas de decaimento $(L^1 \cap L^p) - L^p$ para $p < 2$, para em seguida aplicá-las na equação proposta e obter resultados de existência global de soluções, a partir de dados iniciais suficientemente pequenos. Durante o procedimento, analisar-se-á de que modo os parâmetros n, θ e γ, relacionados à dimensão do espaço, à dissipação fracionária e à não-linearidade de memória, respectivamente, influenciam o expoente crítico $\overline{p}$, um valor limítrofe para a existência ou não-existência de soluções globais no tempo.

Metodologia

Através de uma revisão bibliográfica, verificamos que o expoente crítico esperado seria, *a priori*, $\overline{p}$, o maior dentre os valores p_c e γ^{-1}, sendo p_c uma generalização do notável expoente de Fujita e γ^{-1} fruto de um fenômeno decorrente da não-linearidade do tipo de memória. Contudo, a estrutura de perda de regularidade da equação dada, associada ao termo de inércia rotacional, conduziu-nos à seguinte conjectura: "Para determinadas combinações de valores de n, γ e θ, a perda nas taxas de decaimento será grande o suficiente para alterar efetivamente o valor de p para o qual se pode garantir existência global de soluções para o problema proposto."

Por outro lado, na parte dedicada a mostrar a não-existência de soluções globais no caso subcrítico, verificamos que as duas principais dificuldades diziam respeito à aplicação do método de funções teste em operadores não-locais, como é o caso do Laplaciano fracionário que aparece no termo dissipativo, e à escolha de uma função teste adequada para tratar do termo de não-linearidade de memória. Nesse sentido, baseado em trabalhos precedentes, utilizamos uma variante do método de funções teste, que se utiliza de funções com decaimento polinomial no infinito em vez de funções teste com suporte compacto, e a combinamos com o emprego de uma função teste sob medida para controlar o termo não-linear de memória.

Resultados e discussão

A equação estudada contém diversos termos cujos efeitos já foram estudados separadamente. Entre eles, a perda de efeito parabólico devida ao termo de memória, a perda de regularidade devida à inércia rotacional e a não-localidade do termo de amortecimento fracionário. Estudamos a interação desses efeitos em conjunto, coletando os resultados separadamente para dimensão $n = 1, 2, 3$ e 4, e subdividindo-os, quando necessário, com respeito à variação do parâmetro γ.

Para dimensão baixa, isto é, $n = 1, 2$, observamos que a estrutura de perda de regularidade da equação influencia a taxa de decaimento da solução em $\dot{H}^2$. Além disso, para $n = 1$, o perfil assintótico da solução em L^2 muda quando $\theta \geq 1/4$. Contudo, nenhum desses efeitos altera o expoente crítico. Para este fim, utilizamos uma imersão de Sobolev fracionária quando necessário. Outrossim, a perda de efeito parabólico não aparece em baixa dimensão, fazendo com que o expoente crítico seja p_c, para todo $\gamma \in (0, 1)$. Em particular, a própria definição de p_c torna necessário assumir que $\gamma \in \left(\frac{1}{2(1-\theta)}, 1\right)$ se $n = 1$.

Para $n = 3$, a perda de efeito parabólico aparece, fazendo com que o expoente crítico se torne γ^{-1} quando γ é próximo de zero. Para γ suficientemente distante da origem, distinguimos dois diferentes cenários: quando $p \geq 2$, utilizando a desigualdade de Gagliardo-Nirenberg, é possível obter os mesmos resultados que em dimensões menores. No caso $p < 2$, torna-se necessário aplicar estimativas $(L^1 \cap L^p) - L^p$, o que implica requerer regularidade adicional $\dot{W}^{3,p} \times \dot{W}^{2,p}$ para os dados iniciais. Utilizando-se dessa estratégia, é possível recuperar o expoente crítico esperado $\overline{p}$.

O último caso abordado, $n = 4$, traz o mesmo conjunto de efeitos que o caso $n = 3$, mas com uma principal diferença: para θ suficientemente próximo de 0 e γ suficientemente próximo de 1, verificamos de fato o que havíamos conjecturado: As taxas de decaimento produzidas pelas estimativas $(L^1 \cap L^p) - L^p$ para $p < 2$ apresentam uma perda grande o suficiente para modificar o intervalo de p para o qual se pode provar existência global de soluções.

Considerações finais

Nos casos abordados neste trabalho, mostramos ser verdadeira a conjectura que afirmava que a interação entre os parâmetros n, γ e θ pode ocasionar perdas nas taxas de decaimento grandes o suficiente para diminuir o intervalo de valores p tais que se pode provar existência de soluções globais de (p_c, ∞) para $(\tilde{p}_c, \infty)$, com $\tilde{p}_c > p_c$ no caso em questão.

Na contraparte de não-existência, a combinação do método de funções teste com decaimento polinomial e da função feita sob medida para a não-linearidade de tipo de memória se mostrou adequada, e foi possível demonstrar que não há soluções globais não-triviais no caso subcrítico, considerando-se o expoente crítico $\overline{p}$ inicialmente conjecturado.

Com esses dois resultados, percebemos uma lacuna quando $n = 4$, $\theta \in [0, \theta_0)$, $\gamma \nearrow 1$

e $p \in (p_c, \tilde{p}_c)$, em que não se foi possível concluir existência nem não-existência de soluções globais para o problema dado.

Palavras Chave: Equação de placas; Inércia Rotacional; Dissipação fracionária; Taxas de decaimento "sharp"; Não-Linearidade de tipo Memória; Estrutura de Perda de Regularidade; Dissipação efetiva; Métodos dos multiplicadores; Espaço de Fourier.

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1 INTRODUCTION

In this work, we will study the global existence of small data solutions to the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u_{tt} - \Delta u + \Delta^2 u + (-\Delta)^\theta u_t = \displaystyle\int_0^t (t-s)^{-\gamma} |u(s,\cdot)|^p \, ds \\ u(0,x) = u_0(x) \\ u_t(0,x) = u_1(x), \end{cases} \tag{1}$$

where $\theta \in \left[0, \frac{1}{2}\right)$, $\gamma \in (0,1)$, $p > 1$.

The model we are about to study can be seen as a Plate-type equation, since the solution $u(t,x)$ to the linear associated equation to (1) describes the transversal displacement of a plate under the effects of rotational inertia, characterized by the term Δu_{tt}, and fractional dissipation, characterized by $(-\Delta)^\theta u_t$.

We will look for the *critical exponent* $\overline{p} := \overline{p}(n,\gamma,\theta)$ for (1), that is, a positive value such that:

- If $p > \overline{p}$, then there exist global in-time small data solutions to (1), for a suitable choice of data and solution spaces;
- If $1 < p \leq \overline{p}$, there exist arbitrarily small initial data such that there is no global in-time solution to (1).

H. Fujita proved in 1966 (FUJITA, 1966) that the critical exponent for the classical semilinear heat equation with nonlinearity $F(u) = u^p$ is $p_F = 1 + \frac{2}{n}$. This is widely known as the *Fujita exponent*. In 2001, G. Todorova and B. Yordanov proved (TODOROVA; YORDANOV, 2001) that the *critical exponent* is still the Fujita exponent for $F(u) = |u|^p$, with the nonexistence result for the critical case $p = p_F$ being proved by Qi S. Zhang (ZHANG, Q. S., 2001).

In space dimensions $n = 1, 2$ we prove the existence of global small data solutions for $p > p_c$, where p_c is given by

$$p_c(n,\gamma,\theta) := 1 + \frac{2\,(1 + (1-\gamma)(1-\theta))}{(n - 2 + 2\gamma(1-\theta))_+}, \tag{2}$$

here we use the convention that $\frac{C}{0} = \infty$, for any finite constant C, and $p > p_c = \infty$ means that there are no global solutions for that particular case. As $\gamma \to 1$, $p_c \to 1 + 2/(n - 2\theta)$, consistently with the result obtained in (D'ABBICCO; EBERT, M. R., 2017) for evolution equations with structural damping and, previously, in (D'ABBICCO; REISSIG, 2014) for wave models.

Since the study of plate models has a special interest in space dimension $n = 2$, due to its physical background, we stress that in space dimension $n = 2$, we may write

p_c in its simpler form

$$p_c = \frac{1}{\gamma}\frac{2-\theta}{1-\theta}.$$

In particular, p_c increases from $2\gamma^{-1}$ to $3\gamma^{-1}$ as θ goes from 0 to 1/2.

Besides the assumption that the initial data of (1) are in L^1, the regularity-loss decay structure of the equation makes natural the choice of the space $H^{s_c}(\mathbb{R}^n) \times H^{s_c-1}(\mathbb{R}^n)$, for the regularity of initial data, where

$$s_c = s_c(\gamma,\theta) := 2 + 2\gamma(1-\theta). \tag{3}$$

This condition allows us to produce enough decay rate at high frequencies, to match the desired decay rate at low frequencies, for p close to the critical exponent p_c. For greater values of p, this condition can be relaxed.

We remark here that in lower dimensions, say $n = 1, 2$, we don't find great trouble to run the estimates. This is due, among others, to the fact that the influence of the nonlinear memory is not so strong for $n < 3$. For instance, if $n \geq 3$, for small values of γ, the critical exponent is expected to become γ^{-1}. This phenomenon was first investigated in 2008 by T. Cazenave, F. Dickstein and F. Weissler (CAZENAVE; DICKSTEIN; WEISSLER, 2008), who proved that the *critical exponent* for the heat equation with nonlinear memory

$$\begin{cases} v_t - \Delta v = \displaystyle\int_0^t (t-s)^{-\gamma} v(s,\cdot)^p \, \mathrm{d}s \\ v(0,x) = v_0(x) \geq 0, \end{cases} \tag{4}$$

is $\overline{p}(n,\gamma) := \max\{p_\gamma(n), \gamma^{-1}\}$, where

$$p_\gamma(n) := 1 + \frac{2(2-\gamma)}{(n-2(1-\gamma))_+}.$$

In 2014, D'Abbicco (D'ABBICCO, 2014b) proved that the same effect occurs for the damped wave equation with nonlinear memory,

$$\begin{cases} u_{tt} - \Delta u + u_t = \displaystyle\int_0^t (t-s)^{-\gamma} |u(s,\cdot)|^p \, \mathrm{d}s \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \end{cases} \tag{5}$$

and later extended this analysis to the case of a wave equation with structural damping (D'ABBICCO, 2014a).

Another interesting effect starts to appear for dimension $n \geq 3$: whereas the critical exponent given in (2) is always greater than 2, this condition is not necessarily true anymore for greater dimensions. This fact has a heavy impact and can represent a major difficulty in the task of obtaining solution results, since $L^p - L^p$ estimates are not easy to obtain for $p < 2$. In fact, we expect some loss in decay rates in this region, which can lead to a different value for $\overline{p}$ in this case.

1.1 BACKGROUND FOR PLATE MODELS

Fourth-order evolution partial differential equations arise in problems of solid mechanics as, for example, in the theory of thin plates and beams. Also, in particular formulations of problems related with the Navier-Stokes equations (see Temam (TEMAM, 1979)) appear elliptic equations of fourth-order. Models to study the vibrations of thin plates ($n = 2$) given by the full von Kármán system have been studied by several authors, in particular by Ciarlet (CIARLET, 1980), Sánchez (SÁNCHEZ, 2003), Lasiecka (LASIECKA, 1998), Lasiecka-Benabdallah (LASIECKA; BENABDALLAH, 2000), Koch-Lasiecka (KOCH; LASIECKA, 2002), Puel-Tucsnak (PUEL; TUCSNAK, 1996). Perla Menzala-Zuazua (MENZALA; ZUAZUA, 2000) considered the full von Kármán system and they proved that the Timoshenko's model

$$u_{tt} - \gamma\Delta u_{tt} + \Delta^2 u + u = 0, \quad \text{in } \mathbb{R}^2 \times (0,\infty) \tag{6}$$

may be obtained as limit of a full von Kármán system when suitable parameters tend to zero. The term $-\Delta u_{tt}$ is to absorb in the system the rotational inertia effects at the point x of the plate in a positive time t. It is well known that the plate equation (6) with $\gamma > 0$ is a hyperbolic equation with finite speed of propagation, whereas the non-rotational plate model with $\gamma = 0$ has infinite speed of propagation. The hyperbolic model with $\gamma > 0$ is more complicated to be analysed than the non-hyperbolic one. In particular, for the dissipative plate equation

$$u_{tt} - \gamma\Delta u_{tt} + (-\Delta)^{\theta} u_t + \Delta^2 u = 0,$$

with $\theta \in [0,1)$ and $\gamma > 0$, new difficulties arise, due to the property of regularity-loss decay. This fact can be observed by analysing the structure of the eigenvalues associated to the plate equation in the Fourier space (see (SUGITANI; KAWASHIMA, 2010; CHARÃO; LUZ; IKEHATA, 2013)). Due to that special structure, when we get estimates in the region of high frequencies it is necessary to impose additional regularity on the initial data to obtain the same decay estimates as in the region of low frequencies. The additional regularity necessary to obtain the result appears in the theorem in the next section. This effect does not appear if $\gamma = 0$, since the solution exponentially decays in the zone of high frequency of the Fourier space, even with fractional damping.

A more general equation to model the vibrations of a thin plate is given by

$$u_{tt} - \gamma\Delta u_{tt} + \Delta^2 u + g_0(u_t) - \operatorname{div} g_1(\nabla u_t) = 0. \tag{7}$$

Such models have been studied by several authors ((GEREDELI; LASIECKA, I., 2013; DENK; SCHNAUBELT, 2015; CHUESHOV; LASIECKA, 2008; SCHNAUBELT; VERAAR, 2010)). In (SUGITANI; KAWASHIMA, 2010), Sugitani-Kawashima obtained decay rates for a semilinear plate equation in $\mathbb{R}^n$ with $g_1 = 0$ and $g_0 = \mathrm{Id} - f$. The term u_t

represents a frictional dissipation in the plate, and the nonlinear term $f(v)$ is a smooth function of v satisfying $f(v) = O(|v|^2)$ for $v \to 0$.

Moreover, Andrade-Silva-Ma (ANDRADE, 2012) proved exponential stability for a plate equation with p-Laplacian and memory terms. To get this result they considered a structural damping of type $-\triangle u_t$. Furthermore, there are some papers in which a strong damping of type $(-\Delta)^2 u_t$ is considered in model (7), in place of the damping given in $g_0(u_t) - \operatorname{div} g_1(\nabla u_t)$ (see, e.g. (WANG, 2013; MA; YANG; ZHANG, X., 2013; XU; MA, Q., 2015) and references therein).

NOTATION

We list some notation used in this work:

- the expression $f(t) \lesssim g(t)$ denotes that there exists a constant $C > 0$, such that $f(t) \leq Cg(t)$, uniformly with respect to t;
- the expression $f(t) \approx g(t)$ denotes that $f(t) \lesssim g(t)$ and $g(t) \lesssim f(t)$, simultaneously;
- the notation $\lfloor \sigma \rfloor$ denotes the smallest integer, or floor function of σ, that is, the number $m \in \mathbb{Z}$ such that $m \leq \sigma < m + 1$;
- the notation p' denotes the conjugate exponent of $p \in [1, \infty)$, that is, the number such that $\frac{1}{p} + \frac{1}{p'} = 1$;
- the expression $\mathcal{F}f$ or $\hat{f}$ denotes the Fourier transform with respect to the x variable of f;
- the space $C_c^\infty(\Omega)$ denotes the space of *test functions,* that is, infinitely differentiable functions with compact support in Ω.
- the space $L^p(\Omega)$ represents the Lebesgue space of measurable functions f defined in an open domain $\Omega \subset \mathbb{R}^n$ with finite norm

$$\|f\|_{L^p}^p := \int_\Omega |f(x)|^p \, dx,$$

  for $1 \leq p < \infty$, quotient with functions with zero measure over Ω. For $p = \infty$, we define the space $L^\infty(\Omega)$ as the space of measurable functions f with finite norm $\|f\|_{L^\infty} := \operatorname{esssup}_{x \in \Omega} |f(x)|$;
- the space $L^p_{loc}(\Omega)$ denotes the space of locally L^p functions, that is, the space of all measurable functions $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ such that its restriction to a compact subset $K \subset \Omega$ is in $L^p(\Omega)$.

- the space $L^p(\Omega, \mu)$ represents the weighted Lebesgue space of measurable functions f such that f is in $L^p(\Omega)$ with respect to the measure μ, that is, f has finite norm
$$\left(\int_\Omega |f(x)|^p d\mu\right)^{\frac{1}{p}};$$
- the notation $\langle\xi\rangle^2$ denotes the quantity $1 + |\xi|^2$, often referred to as the *japanese norm* of ξ.
- the space $W^{s,p}(\Omega)$ represents the Sobolev space of functions f such that
$$(1 + |\xi|^s)\hat{f} \in L^p(\Omega).$$
In particular, we denote $H^s(\Omega) := W^{s,2}(\Omega)$;
- the space $\dot{W}^{s,p}(\Omega)$ represents the homogeneous Sobolev space of functions f such that $|\xi|^s\hat{f} \in L^p(\Omega)$;
- the notation $\|f\|_{\dot{H}^s}$ denotes the quantity $\left\||xi|^s\hat{f}\right\|_{L^2}$;
- the operator $(I-\Delta)^{-1}$ denotes the Bessel potential of order 2, whose action may be defined by $(I-\Delta)^{-1}f = \mathcal{F}^{-1}(\langle\xi\rangle^2\hat{f})$ for any $f \in \mathcal{S}$, and then extended by density.

2 LINEAR PROBLEM

The main argument in all of our proofs consists on a rather usual construction of solutions to nonlinear problems, based on applying Duhamel's Principle and a fixed point argument. To this end, we make use of the decay rate of solutions to the linear problem associated to (1), that is, the following linear Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u_{tt} + \Delta^2 u - \Delta u + (-\Delta)^\theta u_t = 0 \\ u(0,x) = u_0(x) \\ u_t(0,x) = u_1(x), \end{cases} \tag{8}$$

with $\theta \in \left[0, \frac{1}{2}\right)$.

Therefore, the first thing we need to do is to obtain existence results and decay estimates for the linear associated problem. Throughout the following section, we will adopt the notation $\|\cdot\|$ and $(\,\cdot\,,\cdot\,)$ to represent the norm and the inner product in L^2, respectively.

2.1 EXISTENCE OF SOLUTION TO THE LINEAR PROBLEM VIA SEMIGROUPS THEORY

Consider the following linear Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u_{tt} + \Delta^2 u - \Delta u + (-\Delta)^\theta u_t = 0, \quad t \geq 0, \ x \in \mathbb{R}^n \\ (u, u_t)(0,x) = (u_0, u_1)(x), \end{cases} \tag{9}$$

with $\theta \in [0, \frac{1}{2})$.

Formally, we make $u_t = v$ and substitute this relation in (9) to see that it is equivalent to

$$\begin{cases} u_t = v; \\ v_t - \Delta v_t + \Delta^2 u - \Delta u + (-\Delta)^\theta v = 0, \end{cases} \tag{10}$$

or yet

$$\begin{cases} u_t = v; \\ v_t = -(I-\Delta)^{-1}(\Delta^2 - \Delta + I)u + (I-\Delta)^{-1}\left(u - (-\Delta)^\theta v\right). \end{cases} \tag{11}$$

Now, this pair of equations can be seen as one vector equation: Setting $U = (u, v)$, this is the same as

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \begin{pmatrix} 0 & I \\ -A_2 & 0 \end{pmatrix} U + \begin{pmatrix} 0 \\ (I-\Delta)^{-1}(u - (-\Delta)^\theta v) \end{pmatrix}, \tag{12}$$

with

$$A_2 = (I - \Delta)^{-1}(\Delta^2 - \Delta + I). \tag{13}$$

Therefore, (9) is equivalent to the system

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} = AU + B(U); \\ U(0) = U_0, \end{cases} \tag{14}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \; U_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \; A = \begin{pmatrix} 0 & I \\ -A_2 & 0 \end{pmatrix}, \; B(U) = \begin{pmatrix} 0 \\ (I-\Delta)^{-1}(u - (-\Delta)^\theta v) \end{pmatrix}.$$

We will now define properly the operators A_2, A and $B(U)$ that appeared above and obtain some important properties about them.

2.1.1 About the Operator A_2

Let $D(A_2)$ be the subspace of $H^2(\mathbb{R}^n)$, defined as follows:

$$\begin{aligned} D(A_2) := \Big\{ u \in H^2(\mathbb{R}^n) : \exists y \in H^1(\mathbb{R}^n) \text{ such that} \\ (\Delta u, \Delta \psi) + (\nabla u, \nabla \psi) + (u, \psi) \\ = (y, \psi) + (\nabla y, \nabla \psi), \forall \psi \in H^2(\mathbb{R}^n) \Big\}. \end{aligned} \tag{15}$$

Now, define, for every $u \in D(A_2)$,

$$\begin{aligned} A_2 : D(A_2) &\to H^1(\mathbb{R}^n) \\ u &\mapsto y. \end{aligned} \tag{16}$$

Observe that, at least in a more loose sense, the equation on the definition (15) can be seen as

$$\begin{aligned} (\Delta^2 u, \psi) - (\Delta u, \psi) + (u, \psi) = (y, \psi) - (\Delta y, \psi), \quad \forall \psi \in H^2(\mathbb{R}^n), \\ \Rightarrow \Big((\Delta^2 - \Delta + I)u, \psi) \Big) = ((I - \Delta)y, \psi), \quad \forall \psi \in H^2(\mathbb{R}^n), \\ \Rightarrow A_2 u = y = (I - \Delta)^{-1}(\Delta^2 - \Delta + I)u, \quad \text{in } (H^2(\mathbb{R}^n))'. \end{aligned} \tag{17}$$

So, this definition agrees with (13). Let's prove that the operator A_2 is well-defined and later on, characterize its domain properly.

Lemma 2.1.1 *For each $u \in H^2(\mathbb{R}^n)$, there is at most one $y \in H^1(\mathbb{R}^n)$ such that*

$$(\Delta u, \Delta \psi) + (\nabla u, \nabla \psi) + (u, \psi) = (y, \psi) + (\nabla y, \nabla \psi), \forall \psi \in H^2(\mathbb{R}^n). \tag{18}$$

Proof: Assume that $y_1, y_2 \in H^1(\mathbb{R}^n)$, and that both satisfy (18). Then,

$$(y_1 - y_2, \psi) + (\nabla(y_1 - y_2), \nabla\psi) = 0, \ \ \forall\psi \in H^2(\mathbb{R}^n). \tag{19}$$

Since $C_c^\infty(\mathbb{R}^n)$ is dense on $H^2(\mathbb{R}^n)$, it holds that

$$(y_1 - y_2, \psi) + (\nabla(y_1 - y_2), \nabla\psi) = 0, \ \ \forall\psi \in C_c^\infty(\mathbb{R}^n). \tag{20}$$

Now, let $y = y_1 - y_2 \in H^1(\mathbb{R}^n)$ and consider $\{\psi_\nu\}_{\nu\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ such that $(\psi_\nu)_{\nu\in\mathbb{N}} \to y$ in $H^1(\mathbb{R}^n)$. Then,

$$\|y\|_{H^1}^2 - 2(y, \psi_\nu)_{H^1} + \|\psi_\nu\|_{H^1}^2 = \|y - \psi_\nu\|_{H^1}^2 \to 0, \quad \text{as } \nu \to \infty. \tag{21}$$

Also, because $\Big| \|\psi_\nu\|_{H^1} - \|y\|_{H^1} \Big| \leq \|\psi_\nu - y\|_{H^1}$, one has

$$\|\psi_\nu\|_{H^1} \to \|y\|_{H^1}, \quad \text{as } \nu \to \infty. \tag{22}$$

From (21) and (22),

$$\lim (y, \psi_\nu)_{H^1} = \|y\|_{H^1}^2 . \tag{23}$$

Finally, from (20) and (23),

$$0 = (y, \psi_\nu) + (\nabla y, \nabla\psi_\nu) = (y, \psi_\nu)_{H^1} \overset{\nu\to\infty}{\longrightarrow} \|y\|_{H^1}^2 ,$$

hence $\|y\|_{H^1} = 0$, that is, $y_1 = y_2$. ■

Remark 2.1.2 *From the previous lemma and the fact that $u \equiv 0 \in D(A)$ (hence it's nonempty), it follows that A_2 is well-defined.*

With the next two lemmas, we will show that $D(A_2) = H^3(\mathbb{R}^n)$.

Lemma 2.1.3 $D(A_2) \subseteq H^3(\mathbb{R}^n)$, *and there exists a constant $C > 0$ such that*

$$\|u\|_{H^3} \leq C \|A_2 u\|_{H^1} ,$$

for every $u \in D(A_2)$.

Proof: Let $u \in D(A_2)$. There exists $y \in H^1(\mathbb{R}^n)$ such that

$$(\Delta u, \Delta\psi) + (\nabla u, \nabla\psi) + (u, \psi) = (y, \psi) + (\nabla y, \nabla\psi), \tag{24}$$

for every $\psi \in H^2(\mathbb{R}^n)$. Now, define the functional $F : H^1(\mathbb{R}^n) \to \mathbb{R}$ by

$$\langle F, \psi\rangle = (y, \psi) + (\nabla y, \nabla\psi), \ \ \psi \in H^1(\mathbb{R}^n).$$

F is clearly well-defined and linear. It is also continuous:

$$\begin{aligned} |\langle F, \psi\rangle| &\leq |(y, \psi)| + |(\nabla y, \nabla\psi)| \\ &\leq \|y\| \, \|\psi\| + \|\nabla y\| \, \|\nabla\psi\| \\ &\leq 2 \|y\|_{H^1} \|\psi\|_{H^1} , \ \ \forall\, \psi \in H^1(\mathbb{R}^n), \end{aligned}$$

that is, $\|F\| \leq 2\|y\|_{H^1}$. Now that we know that F is bounded, we can rewrite the variation problem (24) as

$$(\Delta u, \Delta\psi) + (\nabla u, \nabla\psi) + (u, \psi) = \langle F, \psi\rangle, \quad \forall\psi \in H^2(\mathbb{R}^n). \tag{25}$$

Also, since $H^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, the identity

$$\Delta^2 u - \Delta u + u = F$$

holds in $\mathcal{S}'(\mathbb{R}^n)$. Applying the Fourier Transform,

$$(|\xi|^4 + |\xi|^2 + 1)\hat{u} = (1 + |\xi|^2)\hat{y}$$

$$\Rightarrow (1 + |\xi|^2)^{-\frac{1}{2}}(|\xi|^4 + |\xi|^2 + 1)\hat{u} = (1 + |\xi|^2)^{\frac{1}{2}}\hat{y}. \tag{26}$$

First we observe that

$$\begin{aligned}(1 + |\xi|^6)(1 + |\xi|^2) &= 1 + |\xi|^2 + |\xi|^6 + |\xi|^8 \\ &\leq (1 + |\xi|^2 + |\xi|^4)^2. \end{aligned} \tag{27}$$

Taking the L^2–norm in (26) and using (27), we get

$$\int_{\mathbb{R}^n}(1 + |\xi|^6)|\hat{u}|^2\,\mathrm{d}\xi \leq \int_{\mathbb{R}^n}(1 + |\xi|^2)^{-1}(1 + |\xi|^2 + |\xi|^4)^2|\hat{u}|^2\,\mathrm{d}\xi = \int_{\mathbb{R}^n}(1 + |\xi|^2)|\hat{y}|^2\,\mathrm{d}\xi. \tag{28}$$

Hence,

$$\|u\|_{H^3} \leq \|y\|_{H^1} = \|A_2 u\|_{H^1}.$$

This implies that $u \in H^3(\mathbb{R}^n)$.

∎

Lemma 2.1.4 *If $u \in H^3(\mathbb{R}^n)$, then there exists $y \in H^1(\mathbb{R}^n)$ such that*

$$-(\nabla(\Delta u), \nabla\psi) - (\Delta u, \psi) + (u, \psi) = (y, \psi) + (\nabla y, \nabla\psi), \tag{29}$$

for every $\psi \in H^2(\mathbb{R}^n)$.

Proof: Let $u \in H^3(\mathbb{R}^n)$, and define the functional $F : H^1(\mathbb{R}^n) \to \mathbb{R}$ by

$$\langle F, \psi\rangle := -(\nabla(\Delta u), \nabla\psi) - (\Delta u, \psi) + (u, \psi).$$

F is clearly well-defined, because $u \in H^3(\mathbb{R}^n)$, and it is a linear functional. Let's check its continuity:

$$\begin{aligned}|\langle F, \psi\rangle| &\leq |(\nabla(\Delta u), \nabla\psi)| + |(\Delta u, \psi)| + |(u, \psi)| \\ &\leq \|u\|_{H^3}\|\psi\|_{H_1} + \|u\|_{H^2}\|\psi\| + \|u\|\|\psi\| \\ &\leq 3\|u\|_{H^3}\|\psi\|_{H^1}, \quad \forall\psi \in H^1(\mathbb{R}^n).\end{aligned}$$

Next, define the form $a : H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \to \mathbb{R}$ as

$$a(\varphi, \psi) := (\varphi, \psi) + (\nabla\varphi, \nabla\psi).$$

The form a is well-defined, bilinear, continuous and coercive. In fact, good definition and linearity are immediate, and we check its continuity and coercivity:

- $$\begin{aligned} |a(\varphi, \psi)| &\leq |(\varphi, \psi)| + |(\nabla\varphi, \nabla\psi)| \\ &\leq \|\varphi\| \, \|\psi\| + \|\nabla\varphi\| \, \|\nabla\psi\| \\ &\leq 2 \, \|\varphi\|_{H^1} \, \|\psi\|_{H^1}, \ \ \forall \varphi, \psi \in H^1(\mathbb{R}^n); \end{aligned}$$
- $$\begin{aligned} |a(\varphi, \varphi)| &= \|\varphi\|^2 + \|\nabla\varphi\|^2 \\ &= \|\varphi\|^2_{H^1}, \ \ \forall \varphi \in H^1(\mathbb{R}^n). \end{aligned}$$

Therefore, by Lax-Milgram's Theorem, the variation problem

$$a(y, \psi) = \langle F, \psi \rangle, \quad \psi \in H^1(\mathbb{R}^n) \tag{30}$$

has a unique solution $y \in H^1(\mathbb{R}^n)$. Since (30) holds for every $\psi \in H^1(\mathbb{R}^n)$, in particular it is true for each $\psi \in H^2(\mathbb{R}^n)$, i.e., there is exactly one $y \in H^1(\mathbb{R}^n)$ such that

$$(y, \psi) + (\nabla y, \nabla\psi) = -(\nabla(\Delta u), \nabla\psi) - (\Delta u, \psi) + (u, \psi), \ \ \forall \psi \in H^2(\mathbb{R}^n),$$

or equivalently,

$$(y, \psi) + (\nabla y, \nabla\psi) = (\Delta u, \Delta\psi) - (\nabla u, \nabla\psi) + (u, \psi), \ \ \forall \psi \in H^2(\mathbb{R}^n).$$

∎

Remark 2.1.5 *Observe that the previous lemma asserts that $u \in D(A_2)$. Since $u \in H^3(\mathbb{R}^n)$ was arbitrarily given, this means that*

$$H^3(\mathbb{R}^n) \subseteq D(A_2).$$

Combining this and the result from Lemma 2.1.3, one can see that

$$D(A_2) = H^3(\mathbb{R}^n).$$

2.1.2 Returning to the Problem

Having defined properly the operator $A_2 : H^3(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$, let

$$X := H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n),$$

we'll show that the operators $A : H^3(\mathbb{R}^n) \times H^2(\mathbb{R}^n) \to X$ and $B : X \to X$ given in (14) satisfy:

- A is the infinitesimal generator of a C_0–semigroup in X;
- B is a bounded linear operator in X.

From these two statements, it follows from Semigroups Perturbation Theory that the operator $(A+B)$ generates a C_0–semigroup in X. More precisely, one can apply the following:

Theorem 2.1.6 *If A is the infinitesimal generator of a C_0–semigroup over a Banach space X and B is a linear bounded operator in X, then $A+B$ generates a infinitesimal C_0–semigroup in X.*

Now, let

$$S : [0,\infty) \to \mathcal{L}(X)$$

be the semigroup generated by $A+B$. From semigroups Theory, $U(t) := S(t)U_0$ is the strong solution of the Cauchy abstract problem (14) when $U_0 \in D(A+B) = D(A) = H^3(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$, and alternatively, it is the weak solution to (14) when $U_0 \in X$.

In other words, if $U_0 \in H^3(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$, then

$$U \in C\big([0,\infty); H^3 \times H^2\big) \cap C^1\big([0,\infty); X\big).$$

Thus, setting $U(t) = (u(t), u_t(t))$, this means that if $u_0 \in H^3(\mathbb{R}^n)$ and $u_1 \in H^2(\mathbb{R}^n)$, then the problem (9) has a unique strong solution, satisfying

$$u \in C\big([0,\infty); H^3\big) \cap C^1\big([0,\infty); H^2\big),$$
$$u_t \in C\big([0,\infty); H^2\big) \cap C^1\big([0,\infty); H^1\big),$$

that is,

$$u \in C\big([0,\infty); H^3\big) \cap C^1\big([0,\infty); H^2\big) \cap C^2\big([0,\infty); H^1\big).$$

On the other hand, if the initial data has a little bit less regularity, say $u_0 \in H^2(\mathbb{R}^n)$ and $u_1 \in H^1(\mathbb{R}^n)$, then

$$U \in C([0,\infty); X),$$

that is,

$$u \in C\big([0,\infty); H^2\big) \cap C^1\big([0,\infty); H^1\big)$$

is the unique weak solution to the linear problem (9).

So, all we have left to do to conclude this part is to prove the two following lemmas:

Lemma 2.1.7 *The operator*

$$A : H^3(\mathbb{R}^n) \times H^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$$
$$(u,v) \mapsto \begin{pmatrix} 0 & I \\ -A_2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = (v, -A_2 u)$$

is the infinitesimal generator of a C_0–semigroup in $H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$.

Proof: Let $U = (u, v) \in H^3(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$. We'll show that A satisfies the hypotheses of Lumer-Phillips' Theorem, that is, A is a densely defined, dissipative maximal operator.

The first assertion is clearly true, because $H^3(\mathbb{R}^n)$ is a dense subspace of $H^2(\mathbb{R}^n)$, and $H^2(\mathbb{R}^n)$ is a dense subspace of $H^1(\mathbb{R}^n)$. Now, let's show that A is dissipative, that is, $Re(AU, U) \leq 0$ for every $U \in D(A)$:

$$\begin{aligned}(AU, U)_{H^2 \times H^1} &= (v, u)_{H^2} + (-A_2 u, v)_{H^1} \\ &= \int_{\mathbb{R}^n} (1 + |\xi|^2 + |\xi|^4) \hat{v}\, \overline{\hat{u}}\, \mathrm{d}\xi - \int_{\mathbb{R}^n} (1 + |\xi|^2) \widehat{A_2 u}\, \overline{\hat{v}}\, \mathrm{d}\xi.\end{aligned}$$

Now, from the definition of A_2, we have $\widehat{A_2 u} = (1 + |\xi|^2)^{-1}(1 + |\xi|^2 + |\xi|^4)\hat{u}$. Hence,

$$\begin{aligned}(AU, U)_{H^2 \times H^1} &= \int_{\mathbb{R}^n} (1 + |\xi|^2 + |\xi|^4)(\hat{v}\, \overline{\hat{u}} - \hat{u}\, \overline{\hat{v}})\, \mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} (1 + |\xi|^2 + |\xi|^4) 2i\, \mathbb{I}\mathrm{m}\, (\hat{v}\, \overline{\hat{u}})\, \mathrm{d}\xi,\end{aligned}$$

which implies that $Re(AU, U)_{H^2 \times H^1} = 0$, for every $(u, v) \in D(A)$.

Now, let's prove that A is maximal, that is, $(I - A)(D(A)) = H^2 \times H^1$. Firstly, let's check that $(I - A)(D(A)) \subset H^2 \times H^1$:

Let $(f, g) \in (I - A)(D(A))$. Then, there exists $(u, v) \in D(A) = H^3 \times H^2$ such that

$$(I - A)(u, v) = (f, g).$$

Since $(u, v) \in H^3 \times H^2 \subset H^2 \times H^1$ and $A(u, v) \in H^2 \times H^1$, it follows that $(f, g) \in H^2 \times H^1$.

Now, let's prove the reverse inclusion, $H^2 \times H^1 \subset (I - A)(D(A))$.

Let $(f, g) \in H^2 \times H^1$. We need to show that there is a pair $(u, v) \in D(A) = H^3 \times H^2$ such that $(I - A)(u, v) = (f, g)$, that is, $(u - v, v + A_2 u) = (f, g)$, or yet

$$\begin{cases} u - v = f \\ v + A_2 u = g. \end{cases}$$

Finding such pair is equivalent to find $u \in H^3(\mathbb{R}^n)$ satisfying

$$u + A_2 u = f + g \tag{31}$$

and after that, define $v = u - f$. To find a solution to (31), we formally apply the operator $(I - \Delta)$ on both sides to see that we are looking for a solution to

$$(I - \Delta)u + (\Delta^2 - \Delta + I)u = (I - \Delta)(f + g).$$

Define the bilinear form

$$\begin{aligned} a : H^2(\mathbb{R}^n) \times H^2(\mathbb{R}^n) &\to \mathbb{R} \\ (\varphi, \psi) &\mapsto (\varphi, \psi)_{H^1} + (\varphi, \psi)_{H^2}. \end{aligned}$$

Then,

$$\begin{aligned}|a(\varphi,\psi)| &\leq |(\varphi,\psi)_{H^1}| + |(\varphi,\psi)_{H^2}| \\ &\leq \|\varphi\|_{H^1}\|\psi\|_{H^1} + \|\varphi\|_{H^2}\|\psi\|_{H^2} \\ &\leq 2\|\varphi\|_{H^2}\|\psi\|_{H^2},\end{aligned}$$

hence it's continuous, and

$$|a(\varphi,\varphi)| = \|\varphi\|^2_{H^1} + \|\varphi\|^2_{H^2} \geq \|\varphi\|^2_{H^2},$$

hence it's coercive.

Now, define the linear functional $F : H^2(\mathbb{R}^n) \to \mathbb{R}$ as

$$\langle F,\psi\rangle = (f+g,\psi)_{H^1}.$$

F is continuous. In fact,

$$\begin{aligned}|\langle F,\psi\rangle| &\leq \|f+g\|_{H^1}\|\psi\|_{H^1} \\ &\leq \left(\|f\|_{H^1} + \|g\|_{H^1}\right)\|\psi\|_{H^2} \\ &\leq \left(\|f\|_{H^2} + \|g\|_{H^1}\right)\|\psi\|_{H^2}, \quad \forall\psi \in H^2(\mathbb{R}^n).\end{aligned}$$

Therefore, Lax-Milgram's Theorem asserts the existence of $u \in H^2(\mathbb{R}^n)$ satisfying

$$a(u,\psi) = \langle F,\psi\rangle, \quad \forall\psi \in H^2(\mathbb{R}^n),$$

which is equivalent to

$$(u,\psi)_{H^1} + (u,\psi)_{H^2} = (f+g,\psi)_{H^1}, \quad \forall\psi \in H^2(\mathbb{R}^n)$$

$$\begin{aligned}\iff (u,\psi) + (\nabla u,\nabla\psi) + (u,\psi) + (\nabla u,\nabla\psi) + (\Delta u,\Delta\psi)& \\ = (f+g,\psi) + (\nabla(f+g),\nabla\psi),&\end{aligned} \tag{32}$$

for every $\psi \in H^2(\mathbb{R}^n)$. In particular,

$$(u,\psi) - (\nabla u,\nabla\psi) + (\Delta u,\Delta\psi) = (-u+f+g,\psi) + (\Delta(-u+f+g),\psi),$$

for every $\psi \in H^2(\mathbb{R}^n)$, with $-u+f+g \in H^1(\mathbb{R}^n)$. Therefore, $u \in D(A_2)$ and

$$A_2 u = -u + f + g.$$

As discussed before, defining $v = u - f \in H^2(\mathbb{R}^n)$, we have found a pair $(u,v) \in H^3 \times H^2$ satisfying $(I-A)(u,v) = (f,g)$. Therefore, $(f,g) \in (I-A)(D(A))$.

We have shown that A is a maximal dissipative and densely defined operator. By Lumer Phillips' Theorem, A is the infinitesimal generator of a contraction C_0–semigroup.

■

Lemma 2.1.8 *The operator*

$$\begin{aligned} B : H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) &\to H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \\ (u, v) &\mapsto \begin{pmatrix} 0 \\ (I-\Delta)^{-1}(u-(-\Delta)^{\theta}v) \end{pmatrix} \end{aligned}$$

is a linear and bounded operator.

Proof: Linearity is immediate. For the continuity,

$$\begin{aligned} \|B(U)\|^2_{H^2\times H^1} &= \left\|(I-\Delta)^{-1}(u-(-\Delta)^{\theta}v)\right\|^2_{H^1} \\ &= \int_{\mathbb{R}^n}(1+|\xi|^2)\left|\frac{\hat{u}-|\xi|^{2\theta}\hat{v}}{1+|\xi|^2}\right|^2 \mathrm{d}\xi \\ &\leq 2\int_{\mathbb{R}^n}\frac{|\hat{u}|^2}{1+|\xi|^2}\,\mathrm{d}\xi + 2\int_{\mathbb{R}^n}\frac{|\xi|^{4\theta}}{1+|\xi|^2}|\hat{v}|^2\,\mathrm{d}\xi \\ &\leq 2\int_{\mathbb{R}^n}|\hat{u}|^2\,\mathrm{d}\xi + 2\int_{\mathbb{R}^n}(1+|\xi|^2)|v|^2\,\mathrm{d}\xi \\ &\leq 2\,\|u\|^2_{H^2} + 2\,\|v\|^2_{H^1} \\ &= 2\,\|U\|^2_{H^2\times H^1}. \end{aligned}$$

To clarify, we used, on the calculations above, the fact that for $|\xi| \geq 1$ and $\theta \leq \frac{1}{2}$,

$$|\xi|^{4\theta} \leq |\xi|^2 \leq (1+2|\xi|^2+|\xi|^4) = (1+|\xi|^2)^2.$$

Therefore, $\|B(U)\|^2_{H^2\times H^1} \leq \|U\|^2_{H^2\times H^1}$, and we are done. ■

2.2 ESTIMATES AND DECAY RATES FOR THE LINEAR PROBLEM

We formally apply Fourier's Transform to the differential equation in (8) and to the initial conditions, obtaining

$$\begin{cases} \langle\xi\rangle^2\hat{u}_{tt} + |\xi|^{2\theta}\hat{u}_t + \langle\xi\rangle^2|\xi|^2\hat{u} = 0, \\ \hat{u}(0,\xi) = \hat{u}_0(\xi), \quad \hat{u}_t(0,\xi) = \hat{u}_1(\xi). \end{cases} \tag{33}$$

The characteristic equation associated to (33) is

$$\langle\xi\rangle^2\lambda^2 + |\xi|^{2\theta}\lambda + \langle\xi\rangle^2|\xi|^2 = 0. \tag{34}$$

Multiplying (34) by $\langle\xi\rangle^{-2}$,

$$\lambda^2 + \langle\xi\rangle^{-2}|\xi|^{2\theta}\lambda + |\xi|^2 = 0, \tag{35}$$

whose discriminant is given by

$$\Delta_\lambda = \langle\xi\rangle^{-4}|\xi|^{4\theta} - 4|\xi|^2.$$

We'll analyze the behavior of solutions in the low frequency (i.e., when $|\xi| \to 0$) and high frequency (when $|\xi| \to \infty$).

When $|\xi| \to 0$, we have $\langle\xi\rangle \to 1$, and so

$$\begin{cases} \Delta_\lambda > 0, & \text{if } \theta < \dfrac{1}{2} \\ \Delta_\lambda < 0, & \text{if } \theta \geq \dfrac{1}{2}. \end{cases}$$

This means that, for $|\xi| \to 0$,

i) If $\theta \in [0, \frac{1}{2})$, the eigenvalues will be real;

ii) If $\theta \in [\frac{1}{2}, 1]$, the eigenvalues will be complex.

On the other hand, if $|\xi| \to \infty$, then $\langle\xi\rangle^{-4} \approx |\xi|^{-4}$ and so

$$\Delta_\lambda > 0 \Leftrightarrow |\xi|^{-4+4\theta} - 4|\xi|^2 > 0 \Leftrightarrow -4 + 4\theta > 2 \Leftrightarrow \theta > \frac{3}{2},$$

and because of our initial assumption $\theta \in [0, 1]$, this never occurs. This means that for high frequencies, we have $\Delta_\lambda < 0$.

Observe that in the case $\theta \geq \dfrac{1}{2}$, both high and low frequencies imply that $\Delta_\lambda < 0$, that is, the solution oscillates with no decay, and hence the approach to the problem would need a completely different method. This justifies our choice of dealing only with the case $\theta \in [0, \frac{1}{2})$.

The characteristic polynomial roots are given by

$$\lambda_\pm = \frac{-|\xi|^{2\theta} \pm \sqrt{|\xi|^{4\theta} - 4\langle\xi\rangle^4|\xi|^2}}{2\langle\xi\rangle^2}. \tag{36}$$

Lemma 2.2.1 *Assume* $\theta \in [0, \frac{1}{2})$. *If* $|\xi| \to 0$*, then the eigenvalues of the characteristic equation associated to the linear problem (8) satisfy the following equivalences:*

i) $\lambda_+ \approx -|\xi|^{2(1-\theta)}$;

ii) $\lambda_- \approx -|\xi|^{2\theta}$;

iii) $\lambda_+ - \lambda_- \approx |\xi|^{2\theta}$.

Proof:

Observe that for $0 \leq |\xi| \leq 1$, we have $1 \leq \langle\xi\rangle^2 \leq 2$. Also, we notice that, because of our assumption $|\xi| \to 0$ and the fact that $\theta < \frac{1}{2}$, we can take $|\xi|$ small enough so that

$$\frac{1}{4} \leq 1 - 4\langle\xi\rangle^4|\xi|^{2-4\theta} \leq 1.$$

This implies that

$$1 \leq \frac{3}{2} \leq 1 + \sqrt{1 - 4\langle\xi\rangle^4|\xi|^{2-4\theta}} \leq 2.$$

i) We have that, multiplying and dividing by the conjugate root of the numerator,

$$\begin{aligned}\lambda_+ &= \frac{-|\xi|^{2\theta} + \sqrt{|\xi|^{4\theta} - 4\langle\xi\rangle^4|\xi|^2}}{2\langle\xi\rangle^2} \\ &= \frac{1}{2\langle\xi\rangle^2}\,\frac{|\xi|^{4\theta} - (|\xi|^{4\theta} - 4\langle\xi\rangle^4|\xi|^2)}{-|\xi|^{2\theta} - \sqrt{|\xi|^{4\theta} - 4\langle\xi\rangle^4|\xi|^2}} \\ &= -\frac{4\langle\xi\rangle^4|\xi|^2}{2\langle\xi\rangle^2|\xi|^{2\theta}(1 + \sqrt{1 - 4\langle\xi\rangle^4|\xi|^{2-4\theta}})} \\ &= -\frac{2\langle\xi\rangle^2|\xi|^2}{|\xi|^{2\theta}(1 + \sqrt{1 - 4\langle\xi\rangle^4|\xi|^{2-4\theta}})}.\end{aligned}$$

So, for $|\xi|$ small enough, we have

$$-4\frac{|\xi|^2}{|\xi|^{2\theta}} \leq \lambda_+ \leq -\frac{|\xi|^2}{|\xi|^{2\theta}},$$

that is,

$$\lambda_+ \approx -|\xi|^{2-2\theta}.$$

ii) Here, we have

$$\lambda_- = \frac{-|\xi|^{2\theta} - \sqrt{|\xi|^{4\theta} - 4\langle\xi\rangle^4|\xi|^2}}{2\langle\xi\rangle^2} = -\frac{|\xi|^{2\theta}}{2\langle\xi\rangle^2}\left(1 + \sqrt{1 - 4\langle\xi\rangle^4|\xi|^{2-4\theta}}\right).$$

Hence,

$$-|\xi|^{2\theta} \leq \lambda_- \leq -\frac{|\xi|^{2\theta}}{4},$$

that is,

$$\lambda_- \approx -|\xi|^{2\theta}.$$

iii) Directly from the definition,

$$\lambda_+ - \lambda_- = \frac{2\sqrt{|\xi|^{4\theta} - 4\langle\xi\rangle^4|\xi|^2}}{2\langle\xi\rangle^2} = |\xi|^{2\theta}\frac{\sqrt{1 - 4\langle\xi\rangle^4|\xi|^{2-4\theta}}}{\langle\xi\rangle^2}.$$

Therefore, using again the equivalences for $|\xi|$ small enough,

$$\frac{1}{4}|\xi|^{2\theta} \leq \lambda_+ - \lambda_- \leq |\xi|^{2\theta},$$

or equivalently,

$$\lambda_+ - \lambda_- \approx |\xi|^{2\theta}.$$

■

Lemma 2.2.2 *Assume* $\theta \in [0, \frac{1}{2})$. *If* $|\xi| \to \infty$, *then the complex eigenvalues of the characteristic equation (34) associated to the linear problem (8) satisfy the following equivalences:*

i) $\mathbb{R}\mathrm{e}\,\lambda_\pm \approx -|\xi|^{2(\theta-1)}$;

ii) $|\lambda_\pm| = |\xi|$;

iii) $|\lambda_+ - \lambda_-| \approx |\xi|$.

Proof: Observe that, for $|\xi| \geq 1$, it is true that

$$|\xi|^2 \leq 1 + |\xi|^2 \leq 2|\xi|^2 \Rightarrow \langle\xi\rangle^2 \approx |\xi|^2.$$

Also, for $|\xi|$ large enough, the argument of the square root that appears on (36) is negative, and so, we can explicitly write the real and imaginary parts of the eigenvalues:

$$\lambda_\pm = -\frac{|\xi|^{2\theta}}{2\langle\xi\rangle^2} \pm i\frac{\sqrt{4\langle\xi\rangle^4|\xi|^2 - |\xi|^{4\theta}}}{2\langle\xi\rangle^2}.$$

i) It follows immediately from our observation above that

$$\mathbb{R}\mathrm{e}\,\lambda_\pm \approx -\frac{1}{2}|\xi|^{2\theta-2} \Rightarrow \mathbb{R}\mathrm{e}\,\lambda_\pm \approx -|\xi|^{2(\theta-1)}.$$

ii) Also from the same observation,

$$|\lambda_\pm|^2 = \frac{|\xi|^{4\theta} + 4\langle\xi\rangle^4|\xi|^2 - |\xi|^{4\theta}}{4\langle\xi\rangle^4} = |\xi|^2$$

$$\Rightarrow |\lambda_\pm| = |\xi|.$$

iii) We have

$$\frac{|\xi|^{4\theta}}{\langle\xi\rangle^4} \leq |\xi|^{4\theta-2} \leq 1 \leq |\xi|^2 \Rightarrow -|\xi|^2 \leq -\frac{|\xi|^{4\theta}}{\langle\xi\rangle^4}.$$

With that in mind, we obtain

$$|\lambda_+ - \lambda_-|^2 = \left| i\frac{\sqrt{4\langle\xi\rangle^4|\xi|^2 - |\xi|^{4\theta}}}{\langle\xi\rangle^2} \right|^2 = \frac{4\langle\xi\rangle^4|\xi|^2 - |\xi|^{4\theta}}{\langle\xi\rangle^4} = 4|\xi|^2 - \frac{|\xi|^{4\theta}}{\langle\xi\rangle^4}$$

$$\Rightarrow 3|\xi|^2 \leq |\lambda_+ - \lambda_-|^2 \leq 4|\xi|^2$$
$$\Rightarrow |\lambda_+ - \lambda_-| \approx |\xi|.$$

■

Now, the solution of the Cauchy problem (33) is

$$\hat{u}(t,\xi) = C_1(\xi)e^{t\lambda_+} + C_2(\xi)e^{t\lambda_-}, \tag{37}$$

with $C_1(\xi), C_2(\xi)$ depending on the initial data. We have that

$$\hat{u}_0(\xi) = \hat{u}(0,\xi) = C_1(\xi) + C_2(\xi). \tag{38}$$

Differentiating (37) with respect to t,

$$\hat{u}_t(t,\xi) = \lambda_+ C_1(\xi)e^{t\lambda_+} + \lambda_- C_2(\xi)e^{t\lambda_-},$$

hence

$$\hat{u}_1(\xi) = \hat{u}_t(0,\xi) = \lambda_+ C_1(\xi) + \lambda_- C_2(\xi). \tag{39}$$

Solving the system (37)-(39), we get

$$C_1(\xi) = \frac{-\lambda_- \hat{u}_0 + \hat{u}_1}{\lambda_+ - \lambda_-},$$

$$C_2(\xi) = \frac{\lambda_+ \hat{u}_0 - \hat{u}_1}{\lambda_+ - \lambda_-}.$$

Replacing $C_1(\xi)$ and $C_2(\xi)$ on (37), we obtain

$$\hat{u}(t,\xi) = \hat{K}_0(t,\xi)\,\hat{u}_0(\xi) + \hat{K}_1(t,\xi)\,\hat{u}_1(\xi),$$

with

$$\hat{K}_0(t,\xi) = \frac{\lambda_+ e^{t\lambda_-} - \lambda_- e^{t\lambda_+}}{\lambda_+ - \lambda_-}, \quad \hat{K}_1(t,\xi) = \frac{e^{t\lambda_+} - e^{t\lambda_-}}{\lambda_+ - \lambda_-}.$$

Therefore, the solution to the linear Cauchy problem can be written as

$$u^{lin}(t,x) = K_0(t,x) * u_0(x) + K_1(t,x) * u_1(x), \tag{40}$$

with

$$K_0(t,x) = \mathcal{F}^{-1}\hat{K}_0(t,\xi), \quad K_1(t,x) = \mathcal{F}^{-1}\hat{K}_1(t,\xi). \tag{41}$$

In order to prove a theorem that gives the decay rates and regularity for the linear problem, we need to prove some estimates.

Lemma 2.2.3 *Let $\theta \in [0, \frac{1}{2})$. Then, in the Low-Frequency region, the following estimates are true, for $t > 0$:*

i) If $t|\xi|^{2\theta} \leq 1$, then $|\hat{K}_1(t,\xi)| \lesssim te^{t\lambda_-}$;

ii) If $t|\xi|^{2\theta} \geq 1$, then $|\hat{K}_1(t,\xi)| \lesssim |\xi|^{-2\theta}e^{t\lambda_+}$;

iii) $|\partial_t \hat{K}_1(t,\xi)| \lesssim e^{t\lambda_-} + |\xi|^{2(1-2\theta)} e^{t\lambda_+}$;

iv) $|\hat{K}_0(t,\xi)| \lesssim e^{t\lambda_+}$;

v) $|\partial_t \hat{K}_0(t,\xi)| \lesssim |\xi|^{2(1-\theta)} e^{t\lambda_+}$.

Proof:

i) We have that

$$\begin{aligned}|\hat{K}_1(t,\xi)| &= \left|\frac{e^{t\lambda_+} - e^{t\lambda_-}}{\lambda_+ - \lambda_-}\right| \approx |\xi|^{-2\theta}\left|e^{t\lambda_+} - e^{t\lambda_-}\right| \\ &= |\xi|^{-2\theta} e^{t\lambda_-}\left(e^{t(\lambda_+-\lambda_-)} - 1\right).\end{aligned}$$

Observe that the function $x \mapsto \dfrac{e^x - 1}{x}$ lies in $[1,2]$ for $0 \le x \le 1$.
So, our assumption $t|\xi|^{2\theta} \le 1$ implies that

$$0 \le t(\lambda_+ - \lambda_-) \le t|\xi|^{2\theta} \le 1,$$

(we remark here that the upper constant we found in the proof of Lemma 2.2.1(iii) is exactly 1). Therefore,

$$\begin{aligned}1 \le \frac{e^{t(\lambda_+-\lambda_-)} - 1}{t(\lambda_+ - \lambda_-)} \le 2 \\ \Rightarrow e^{t(\lambda_+-\lambda_-)} - 1 \approx t(\lambda_+ - \lambda_-) \approx t|\xi|^{2\theta}.\end{aligned}$$

Using this on our initial equivalence shows that

$$|\hat{K}_1(t,\xi)| \approx |\xi|^{-2\theta} e^{t\lambda_-} |\xi|^{2\theta} t = te^{t\lambda_-}.$$

ii) Assuming that $t|\xi|^{2\theta} \ge 1$, we can use the equivalence on Lemma 2.2.1(iii) to estimate

$$\begin{aligned}t(\lambda_+ - \lambda_-) \ge \frac{1}{4} t|\xi|^{2\theta} \ge \frac{1}{4} \\ \Rightarrow 1 - e^{-\frac{1}{4}} \le 1 - e^{t(\lambda_- - \lambda_+)} \le 1.\end{aligned}$$

Hence,

$$\begin{aligned}|\hat{K}_1(t,\xi)| &= \left|\frac{e^{t\lambda_+} - e^{t\lambda_-}}{\lambda_+ - \lambda_-}\right| \approx |\xi|^{-2\theta} e^{t\lambda_+}\left(1 - e^{t(\lambda_- - \lambda_+)}\right) \\ &\approx |\xi|^{-2\theta} e^{t\lambda_+}.\end{aligned}$$

iii) Differentiating the expression for $\hat{K}_1(t,\xi)$ with respect to t, we get

$$
\begin{aligned}
|\partial_t \hat{K}_1(t,\xi)| &= \left|\frac{\lambda_+ e^{t\lambda_+} - \lambda_- e^{t\lambda_-}}{\lambda_+ - \lambda_-}\right| \leq \left|\frac{\lambda_+ e^{t\lambda_+}}{\lambda_+ - \lambda_-}\right| + \left|\frac{\lambda_- e^{t\lambda_-}}{\lambda_+ - \lambda_-}\right| \\
&\lesssim \frac{|\xi|^{2(1-\theta)} e^{t\lambda_+}}{|\xi|^{2\theta}} + \frac{|\xi|^{2\theta} e^{t\lambda_-}}{|\xi|^{2\theta}} = |\xi|^{2(1-2\theta)} e^{t\lambda_+} + e^{t\lambda_-}.
\end{aligned}
$$

iv) Observe that

$$
\frac{\lambda_+}{\lambda_-} \approx \frac{|\xi|^{2(1-\theta)}}{|\xi|^{2\theta}} = |\xi|^{2(1-2\theta)} \leq 1,
$$

because $|\xi| \leq 1$ and $\theta \in [0, \frac{1}{2})$. This implies that

$$
\left|1 - \frac{\lambda_+}{\lambda_-} e^{-t(\lambda_+ - \lambda_-)}\right| \leq 1 + \left|\frac{\lambda_+}{\lambda_-} e^{-t(\lambda_+ - \lambda_-)}\right| \lesssim 1.
$$

Therefore,

$$
\begin{aligned}
|\hat{K}_0(t,\xi)| &= \left|\frac{\lambda_+ e^{t\lambda_-} - \lambda_- e^{t\lambda_+}}{\lambda_+ - \lambda_-}\right| \\
&\approx e^{t\lambda_+} |\xi|^{-2\theta} \left|\lambda_+ e^{-t(\lambda_+ - \lambda_-)} - \lambda_-\right| \\
&= e^{t\lambda_+} |\xi|^{-2\theta} |\lambda_-| \left|1 - \frac{\lambda_+}{\lambda_-} e^{-t(\lambda_+ - \lambda_-)}\right| \\
&\lesssim e^{t\lambda_+}.
\end{aligned}
$$

v) Differentiating the expression for $\hat{K}_0(t,\xi)$ with respect to t, we have

$$
\begin{aligned}
\left|\partial_t \hat{K}_0(t,\xi)\right| &= \left|\frac{\lambda_+ \lambda_- e^{t\lambda_-} - \lambda_- \lambda_+ e^{t\lambda_+}}{\lambda_+ - \lambda_-}\right| \\
&= \frac{|\lambda_+ \lambda_-|}{|\lambda_+ - \lambda_-|} e^{t\lambda_+} \left| e^{t(\lambda_- - \lambda_+)} - 1 \right| \\
&\lesssim \frac{|\xi|^{2(1-\theta)} |\xi|^{2\theta}}{|\xi|^{2\theta}} e^{t\lambda_+},
\end{aligned}
$$

since $\lambda_- - \lambda_+ < 0$ implies

$$
\begin{aligned}
& 0 \leq e^{t(\lambda_- - \lambda_+)} \leq 1 \\
& \Rightarrow |e^{t(\lambda_- - \lambda_+)} - 1| = 1 - e^{t(\lambda_- - \lambda_+)} \leq 1.
\end{aligned}
\tag{42}
$$

■

Lemma 2.2.4 *Let* $\theta \in [0, \frac{1}{2})$. *Then, in the High-Frequency region, it is true that*

$$
|\partial_t^j \hat{K}_l(t,\xi)| \lesssim |\xi|^{(j-l)} e^{t\,\mathbb{R}\mathrm{e}\,\lambda_\pm}, \quad j, l = 0, 1,
$$

for all $t > 0$.

Proof: We start by observing that, since λ_+ and λ_- are complex conjugates, we have

$$\mathbb{R}\mathrm{e}\,\lambda_+ = \mathbb{R}\mathrm{e}\,\lambda_- \quad \text{and} \quad \mathbb{I}\mathrm{m}\,\lambda_+ = -\,\mathbb{I}\mathrm{m}\,\lambda_-,$$

and hence, by Euler's Formula,

$$\begin{aligned} \left|e^{t\lambda_-} - e^{t\lambda_+}\right| &= e^{t\,\mathbb{R}\mathrm{e}\,\lambda_\pm}\left|e^{-it\,\mathbb{I}\mathrm{m}\,\lambda_+} - e^{it\,\mathbb{I}\mathrm{m}\,\lambda_+}\right| \\ &= e^{t\,\mathbb{R}\mathrm{e}\,\lambda_\pm}\left|-2i\sin\left(t\,\mathbb{I}\mathrm{m}\,\lambda_+\right)\right| \leq 2e^{t\,\mathbb{R}\mathrm{e}\,\lambda_\pm}. \end{aligned}$$

With this inequality, it follows directly from the estimates of Lemma 2.2.2 that

$$\begin{aligned} \left|\hat{K}_0(t,\xi)\right| &= \left|\frac{\lambda_+ e^{t\lambda_-} - \lambda_- e^{t\lambda_+}}{\lambda_+ - \lambda_-}\right| \\ &= \left|\frac{\lambda_+ e^{t\lambda_-} - \lambda_+ e^{t\lambda_+} + \lambda_+ e^{t\lambda_+} - \lambda_- e^{t\lambda_+}}{\lambda_+ - \lambda_-}\right| \\ &\lesssim \frac{|\lambda_+|}{|\lambda_+ - \lambda_-|}\left|e^{t\lambda_-} - e^{t\lambda_+}\right| + \left|e^{t\lambda_+}\right| \lesssim e^{t\,\mathbb{R}\mathrm{e}\,\lambda_\pm}; \\ \left|\hat{K}_1(t,\xi)\right| &= \left|\frac{e^{t\lambda_-} - e^{t\lambda_+}}{\lambda_+ - \lambda_-}\right| = \frac{1}{|\lambda_+ - \lambda_-|}\left|e^{t\lambda_-} - e^{t\lambda_+}\right| \lesssim |\xi|^{-1} e^{t\,\mathbb{R}\mathrm{e}\,\lambda_\pm}; \\ \left|\partial_t\hat{K}_0(t,\xi)\right| &= \left|\frac{\lambda_-\lambda_+ e^{t\lambda_-} - \lambda_+\lambda_- e^{t\lambda_+}}{\lambda_+ - \lambda_-}\right| = \frac{|\lambda_+\lambda_-|}{|\lambda_+ - \lambda_-|}\left|e^{t\lambda_-} - e^{t\lambda_+}\right| \lesssim |\xi|\, e^{t\,\mathbb{R}\mathrm{e}\,\lambda_\pm}; \\ \left|\partial_t\hat{K}_1(t,\xi)\right| &= \left|\frac{\lambda_- e^{t\lambda_-} - \lambda_+ e^{t\lambda_+}}{\lambda_+ - \lambda_-}\right| \\ &= \left|\frac{\lambda_- e^{t\lambda_-} - \lambda_- e^{t\lambda_+} + \lambda_- e^{t\lambda_+} - \lambda_+ e^{t\lambda_+}}{\lambda_+ - \lambda_-}\right| \\ &\lesssim \frac{|\lambda_-|}{|\lambda_+ - \lambda_-|}\left|e^{t\lambda_-} - e^{t\lambda_+}\right| + \left|e^{t\lambda_+}\right| \lesssim e^{t\,\mathbb{R}\mathrm{e}\,\lambda_\pm}. \end{aligned}$$

■

We'll use these estimates to bound L^q norms of the natural solutions K_0, K_1, as well as their spacial and time derivatives. First we'll prove an auxiliary lemma.

Lemma 2.2.5 *If $a > -n$, $b > 0$, $c > 0$, then there exists a constant $C := C(a,b,c,n)$ such that*

$$\int_{\mathbb{R}^n} |\xi|^a e^{-b|\xi|^c}\, d\xi \leq C < +\infty.$$

Proof: Integrating over shells of fixed radius,

$$\begin{aligned} \int_{\mathbb{R}^n} |\xi|^a e^{-b|\xi|^c}\, \mathrm{d}\xi &= \int_0^\infty \int_{|\xi|=\rho} \rho^a e^{-b\rho^c}\, \mathrm{d}S_\xi\, \mathrm{d}\rho \\ &= \int_0^\infty \rho^a e^{-b\rho^c}\omega_n \rho^{n-1}\, \mathrm{d}\rho \\ &= \omega_n \int_0^\infty \rho^{a+n-1} e^{-b\rho^c}\, \mathrm{d}\rho, \end{aligned}$$

where $\omega_n := \mu\{x \in \mathbb{R}^n : |x| = 1\}$ is the volume of the n–dimensional unit sphere. Now, since the function

$$f(x) = x^{a+n+1} e^{-bx^c}$$

tends to zero when $x \to +\infty$, for $b > 0, c > 0$, there exists a constant $C_1 := C_1(b, c)$ such that

$$\rho^{a+n+1} e^{-b\rho^c} < 1 \Rightarrow \rho^{a+n-1} e^{-b\rho^c} < \rho^{-2},$$

for every $\rho \geq C_1$. This implies that

$$\begin{aligned}
\int_{\mathbb{R}^n} |\xi|^a e^{-b|\xi|^c} \, d\xi &= \omega_n \left[\int_0^{C_1} \rho^{a+n-1} e^{-b\rho^c} \, d\rho + \int_{C_1}^{\infty} \rho^{a+n-1} e^{-b\rho^c} \, d\rho \right] \\
&\leq \omega_n \left[\int_0^{C_1} \rho^{a+n-1} \, d\rho + \int_{C_1}^{\infty} \rho^{-2} \, d\rho \right] \\
&= \frac{\omega_n}{a+n} \rho^{a+n} \Big|_0^{C_1} - \omega_n \lim_{\alpha \to \infty} \frac{1}{\rho} \Big|_{C_1}^{\alpha} \\
&= \frac{\omega_n C_1^{a+n}}{a+n} + \frac{\omega_n}{C_1} := C(a, b, c, n), \qquad \text{for } a + n > 0.
\end{aligned}$$

■

Remark 2.2.6 *In the following lemma, we prove several $L^\eta - L^q$ estimates for derivatives of the fundamental solutions, K_0 and K_1. In some cases, we did not choose the sharpest possible estimates, since it would lead to breaking in much more cases. Instead, we chose a more clean result which is fit for our purposes. For a precise and more general result, we address the reader to (EBERT, M.; DA LUZ; PALMA, 2020).*

2.3 LOCALIZATION OF FUNCTIONS IN THE FOURIER SPACE

Up until now, we discussed about low and high frequency regions and the need to treat them separately. In fact, the eigenvalues $\lambda_\pm$ (and consequently the fundamental solutions K_0 and K_1) behave differently depending on the size of the variable ξ. In this sense, we proved equivalences for $\lambda_\pm$ in Lemmas 2.2.1 and 2.2.2. In Lemma 2.2.1, we assumed $|\xi| < \varepsilon_0$, with $\varepsilon_0 > 0$ being the greatest value for which the eigenvalues $\lambda_\pm$ are real, that is, when the argument of the square root from the definition of $\lambda_\pm$ is non-negative. On the other hand, in Lemma 2.2.2 we assumed $|\xi| \geq 1$ to prove our estimates. One could question what happens for the intermediate frequencies, that is, when $|\xi| \in [\varepsilon_0, 1]$.

This section is dedicated to fix the localization for the low and high frequency regions as

$$\begin{cases} \text{Low-frequency: } |\xi| \leq 1, \\ \text{High-frequency: } |\xi| \geq 1, \end{cases}$$

based on the argument that, for intermediate values of $|\xi|$, one can obtain exponential decay rates for the fundamental solutions. Therefore, any result we obtain assuming $|\xi|$ very small or very large will still hold if we assume $|\xi| \leq 1$ or $|\xi| \geq 1$, respectively.

Lemma 2.3.1 *Let $\theta \in [0, \frac{1}{2})$, $\eta \in [1, 2]$, $q \in [2, +\infty]$, $k \geq 0$, $n \in \mathbb{N}$ and $j = 0, 1$. Consider $\eta' = \eta/(\eta - 1)$ and $r \in [1, \infty]$ given by*

$$\frac{1}{r} = \frac{1}{q'} - \frac{1}{\eta'} = \frac{1}{\eta} - \frac{1}{q},$$

with $q' = q/(q-1)$. If $\psi \in L^\eta(\mathbb{R}^n)$, then the following estimates for the Low-Frequency region are true:

i) $\| \, |\xi|^k \hat{K}_0(t, \cdot)\hat{\psi}\|_{L^{q'}(|\xi| \leq 1)} \lesssim (1 + t)^{-\frac{1}{2(1-\theta)}\left(\frac{n}{r}+k\right)} \|\psi\|_{L^\eta}$;

ii) $\| \, |\xi|^k \partial_t \hat{K}_0(t, \cdot)\hat{\psi}\|_{L^{q'}(|\xi| \leq 1)} \lesssim (1 + t)^{-\frac{1}{2(1-\theta)}\left(\frac{n}{r}+k\right)-1} \|\psi\|_{L^\eta}$;

iii) If $\theta \in \left(0, \frac{1}{2}\right)$, then

$$\| \, |\xi|^k \hat{K}_1(t, \cdot)\hat{\psi}\|_{L^{q'}(t|\xi|^{2\theta} \leq 1)} \lesssim (1 + t)^{1-\frac{1}{2\theta}\left(\frac{n}{r}+k\right)} \|\psi\|_{L^\eta};$$

iv) For $n\left(\frac{1}{\eta} - \frac{1}{q}\right) + k - 2\theta > 0$ and $\theta \in \left(0, \frac{1}{2}\right)$ one has

$$\| \, |\xi|^k \hat{K}_1(t, \cdot)\hat{\psi}\|_{L^{q'}(t^{-\frac{1}{2\theta}} \leq |\xi| \leq 1)} \lesssim (1 + t)^{-\frac{1}{2(1-\theta)}\left(\frac{n}{r}+k-2\theta\right)} \|\psi\|_{L^\eta}.$$

Alternatively, if $\theta = 0$,

iv') For $n\left(\frac{1}{\eta} - \frac{1}{q}\right) + k > 0$ one has

$$\| \, |\xi|^k \hat{K}_1(t, \cdot)\hat{\psi}\|_{L^{q'}(|\xi| \leq 1 \leq t)} \lesssim (1 + t)^{-\frac{1}{2}\left(\frac{n}{r}+k\right)} \|\psi\|_{L^\eta};$$

v) For $n\left(\frac{1}{\eta} - \frac{1}{q}\right) + k - 2\theta < 0$ one has

$$\| \, |\xi|^k \hat{K}_1(t, \cdot)\hat{\psi}\|_{L^{q'}(t^{-\frac{1}{2\theta}} \leq |\xi| \leq 1)} \lesssim (1 + t)^{1-\frac{1}{2\theta}\left(\frac{n}{r}+k\right)} \|\psi\|_{L^\eta};$$

vi) For $n\left(\frac{1}{\eta} - \frac{1}{q}\right) + k - 2\theta = 0$ one has

$$\| \, |\xi|^k \hat{K}_1(t, \cdot)\hat{\psi}\|_{L^{q'}(t^{-\frac{1}{2\theta}} \leq |\xi| \leq 1)} \lesssim \ln(e + t) \|\psi\|_{L^\eta}.$$

A special exception case is given when $k = 0$, $\eta = q$, namely,

vi')

$$\| \hat{K}_1(t,\cdot)\hat{\psi}\|_{L^{q'}(|\xi|\le 1\le t)} \lesssim \|\psi\|_{L^\eta};$$

vii) For $n\left(\frac{1}{\eta}-\frac{1}{q}\right)+k-2\theta \ge 0$, one has

$$\| \,|\xi|^k \partial_t \hat{K}_1(t,\cdot)\hat{\psi}\|_{L^{q'}(|\xi|\le 1)} \lesssim (1+t)^{-\frac{1}{2(1-\theta)}\left(\frac{n}{r}+k-2\theta\right)-1} \|\psi\|_{L^\eta};$$

viii) For $n\left(\frac{1}{\eta}-\frac{1}{q}\right)+k-2\theta < 0$ one has

$$\| \,|\xi|^k \partial_t \hat{K}_1(t,\cdot)\hat{\psi}\|_{L^{q'}(|\xi|\le 1)} \lesssim (1+t)^{-\frac{1}{2\theta}\left(\frac{n}{r}+k\right)} \|\psi\|_{L^\eta}.$$

Proof:

i) From Hölder's Inequality, Lemma 2.2.3(iv) and Lemma 2.2.1,

$$\begin{aligned}
\left\| |\xi|^k \hat{K}_0(t,\cdot)\hat{\psi}\right\|_{L^{q'}(|\xi|\le 1)} &\lesssim \left\| |\xi|^k \hat{K}_0(t,\cdot)\right\|_{L^r(|\xi|\le 1)} ||\hat{\psi}||_{L^{\eta'}} \\
&\lesssim \left\| |\xi|^k e^{t\lambda_+}\right\|_{L^r(|\xi|\le 1)} ||\hat{\psi}||_{L^{\eta'}} \\
&= \left(\int_{|\xi|\le 1} |\xi|^{kr} e^{tr\lambda_+}\, d\xi\right)^{\frac{1}{r}} ||\hat{\psi}||_{L^{\eta'}} \\
&\lesssim \left(\int_{|\xi|\le 1} |\xi|^{kr} e^{-Ctr|\xi|^{2(1-\theta)}}\, d\xi\right)^{\frac{1}{r}} ||\hat{\psi}||_{L^{\eta'}}.
\end{aligned}$$

First, observe that, for $|\xi| \le 1$, $\theta \in [0, \frac{1}{2})$, we have

$$\begin{aligned}
|\xi|^{2(1-\theta)} \le 1 \Leftrightarrow C|\xi|^{2(1-\theta)}r \le Cr \Leftrightarrow e^{C|\xi|^{2(1-\theta)}r} &\le e^{Cr} \\
\Leftrightarrow 1 &\le e^{Cr} e^{-C|\xi|^{2(1-\theta)}r} \\
\Leftrightarrow e^{-Ctr|\xi|^{2(1-\theta)}} &\le e^{Cr} e^{-C(1+t)r|\xi|^{2(1-\theta)}}.
\end{aligned}$$

Hence,

$$\left\| |\xi|^k \hat{K}_0(t,\cdot)\hat{\psi}\right\|_{L^{q'}(|\xi|\le 1)} \lesssim \left(\int_{|\xi|\le 1} |\xi|^{kr} e^{-C(1+t)r|\xi|^{2(1-\theta)}}\, d\xi\right)^{\frac{1}{r}} ||\hat{\psi}||_{L^{\eta'}}.$$

Now, changing variables, $\nu = (1+t)^{\frac{1}{2(1-\theta)}}\xi$, which implies $d\nu = (1+t)^{\frac{n}{2(1-\theta)}} d\xi$:

$$\begin{aligned}
&\left\| |\xi|^k \hat{K}_0(t,\cdot)\hat{\psi}\right\|_{L^{q'}(|\xi|\le 1)} \\
&\quad\lesssim \left(\int_{\mathbb{R}^n} |\nu|^{kr}(1+t)^{-\frac{kr}{2(1-\theta)}} e^{-Cr|\nu|^{2(1-\theta)}} (1+t)^{-\frac{n}{2(1-\theta)}}\, d\nu\right)^{\frac{1}{r}} ||\hat{\psi}||_{L^{\eta'}} \\
&\quad\lesssim (1+t)^{-\frac{1}{2(1-\theta)}\left(\frac{n}{r}+k\right)} \left(\int_{\mathbb{R}^n} |\nu|^{kr} e^{-Cr|\nu|^{2(1-\theta)}}\, d\nu\right)^{\frac{1}{r}} ||\hat{\psi}||_{L^{\eta'}}.
\end{aligned}$$

Therefore, using Lemma 2.2.5 and applying Hausdorff-Young, we end up with

$$\left\| |\xi|^k \hat{K}_0(t,\cdot)\hat{\psi} \right\|_{L^{q'}(|\xi|\leq 1)} \lesssim (1+t)^{-\frac{1}{2(1-\theta)}\left(\frac{n}{r}+k\right)} ||\psi||_{L^{\eta}},$$

for $kr+n>0$, which is true since we are assuming k non-negative and n, r to be positive.

ii) We will follow basically the same steps as in i), but using Lemma 2.2.3 (v);

$$\begin{aligned}
&\left\| |\xi|^k \partial_t \hat{K}_0(t,\cdot)\hat{\psi} \right\|_{L^{q'}(|\xi|\leq 1)} \\
&\qquad \lesssim \left\| |\xi|^k \partial_t \hat{K}_0(t,\cdot) \right\|_{L^{r}(|\xi|\leq 1)} ||\hat{\psi}||_{L^{\eta'}} \\
&\qquad \lesssim \left\| |\xi|^{k+2(1-\theta)} e^{t\lambda_+} \right\|_{L^{r}(|\xi|\leq 1)} ||\hat{\psi}||_{L^{\eta'}} \\
&\qquad \lesssim \left(\int_{|\xi|\leq 1} |\xi|^{kr+2r(1-\theta)} e^{-C(1+t)r|\xi|^{2(1-\theta)}} \, \mathrm{d}\xi \right)^{\frac{1}{r}} ||\hat{\psi}||_{L^{\eta'}}.
\end{aligned}$$

Making the same change of variables, $\nu = (1+t)^{\frac{1}{2(1-\theta)}}\xi$, and using Lemma 2.2.5,

$$\begin{aligned}
&\left\| |\xi|^k \partial_t \hat{K}_0(t,\cdot)\hat{\psi} \right\|_{L^{q'}(|\xi|\leq 1)} \\
&\lesssim \left(\int_{\mathbb{R}^n} |\nu|^{kr+2r(1-\theta)} (1+t)^{-\frac{kr}{2(1-\theta)}-r} e^{-Cr|\nu|^{2(1-\theta)}} (1+t)^{-\frac{n}{2(1-\theta)}} \, \mathrm{d}\nu \right)^{\frac{1}{r}} ||\hat{\psi}||_{L^{\eta'}} \\
&\lesssim (1+t)^{-\frac{1}{2(1-\theta)}\left(\frac{n}{r}+k\right)-1} \left(\int_{\mathbb{R}^n} |\nu|^{kr+2r(1-\theta)} e^{-Cr|\nu|^{2(1-\theta)}} \, \mathrm{d}\nu \right)^{\frac{1}{r}} ||\hat{\psi}||_{L^{\eta'}} \\
&\lesssim (1+t)^{-\frac{1}{2(1-\theta)}\left(\frac{n}{r}+k\right)-1} ||\psi||_{L^{\eta}}, \qquad \text{if } kr+2r(1-\theta)+n>0.
\end{aligned}$$

Again, the condition for using Lemma 2.2.5 holds, since $k, r, (1-\theta)$ are non-negative and $n \in \mathbb{N}$.

iii) Using Lemmas 2.2.3(i) and 2.2.1, and assuming $\theta > 0$,

$$\begin{aligned}
\left\| |\xi|^k \hat{K}_1(t,\cdot)\hat{\psi} \right\|_{L^{q'}(t|\xi|^{2\theta}\leq 1)} &\lesssim \left\| |\xi|^k \hat{K}_1(t,\cdot) \right\|_{L^{r}(t|\xi|^{2\theta}\leq 1)} ||\hat{\psi}||_{L^{\eta'}} \\
&\lesssim t \left\| |\xi|^k e^{t\lambda_-} \right\|_{L^{r}(t|\xi|^{2\theta}\leq 1)} ||\hat{\psi}||_{L^{\eta'}} \\
&\lesssim (1+t) \left(\int_{|\xi|\leq 1} |\xi|^{kr} e^{-C(1+t)r|\xi|^{2\theta}} \, \mathrm{d}\xi \right)^{\frac{1}{r}} ||\hat{\psi}||_{L^{\eta'}}.
\end{aligned}$$

Now, changing variables, $\nu = (1+t)^{\frac{1}{2\theta}}\xi$ and using Lemma 2.2.5,

$$\begin{aligned}
&\left\| |\xi|^k \hat{K}_1(t,\cdot)\hat{\psi} \right\|_{L^{q'}(t|\xi|^{2\theta}\le 1)} \\
&\qquad \lesssim (1+t)\left(\int_{\mathbb{R}^n} |\nu|^{kr}(1+t)^{-\frac{kr}{2\theta}} e^{-Cr|\nu|^{2\theta}}(1+t)^{-\frac{n}{2\theta}}\,\mathrm{d}\nu\right)^{\frac{1}{r}} \|\hat{\psi}\|_{L^{\eta'}} \\
&\qquad \lesssim (1+t)^{1-\frac{1}{2\theta}\left(\frac{n}{r}+k\right)}\left(\int_{\mathbb{R}^n} |\nu|^{kr} e^{-Cr|\nu|^{2\theta}}\,\mathrm{d}\nu\right)^{\frac{1}{r}} \|\hat{\psi}\|_{L^{\eta'}} \\
&\qquad \lesssim (1+t)^{1-\frac{1}{2\theta}\left(\frac{n}{r}+k\right)}\|\psi\|_{L^{\eta}}, \qquad\qquad \text{if } kr+n>0.
\end{aligned}$$

Once more, the condition $kr+n>0$ is true for our parameters k, r non-negative and $n \in \mathbb{N}$.

iv) Using Lemma 2.2.3(ii), and assuming that $\theta \in \left(0, \frac{1}{2}\right)$,

$$\begin{aligned}
&\left\| |\xi|^k \hat{K}_1(t,\cdot)\hat{\psi} \right\|_{L^{q'}\left(t^{-\frac{1}{2\theta}}\le|\xi|\le 1\right)} \\
&\qquad \lesssim \left\| |\xi|^k \hat{K}_1(t,\cdot) \right\|_{L^{r}(t^{-\frac{1}{2\theta}}\le|\xi|\le 1)} \|\hat{\psi}\|_{L^{\eta'}} \\
&\qquad \lesssim \left\| |\xi|^{k-2\theta} e^{t\lambda_+} \right\|_{L^{r}(t^{-\frac{1}{2\theta}}\le|\xi|\le 1)} \|\hat{\psi}\|_{L^{\eta'}} \\
&\qquad \lesssim \left(\int_{|\xi|\le 1} |\xi|^{(k-2\theta)r} e^{-C(1+t)r|\xi|^{2(1-\theta)}}\,\mathrm{d}\xi\right)^{\frac{1}{r}} \|\hat{\psi}\|_{L^{\eta'}}.
\end{aligned}$$

Again, making $\nu = (1+t)^{\frac{1}{2(1-\theta)}}\xi$,

$$\begin{aligned}
&\left\| |\xi|^k \hat{K}_1(t,\cdot)\hat{\psi} \right\|_{L^{q'}(t^{-\frac{1}{2\theta}}\le|\xi|\le 1)} \\
&\qquad \lesssim (1+t)^{-\frac{k-2\theta}{2(1-\theta)}}(1+t)^{-\frac{n}{2r(1-\theta)}}\left(\int_{\mathbb{R}^n} |\nu|^{(k-2\theta)r} e^{-Cr|\nu|^{2(1-\theta)}}\,\mathrm{d}\nu\right)^{\frac{1}{r}} \|\hat{\psi}\|_{L^{\eta'}} \\
&\qquad \lesssim (1+t)^{-\frac{1}{2(1-\theta)}\left(\frac{n}{r}+k-2\theta\right)}\left(\int_{\mathbb{R}^n} |\nu|^{(k-2\theta)r} e^{-Cr|\nu|^{2(1-\theta)}}\,\mathrm{d}\nu\right)^{\frac{1}{r}} \|\hat{\psi}\|_{L^{\eta'}} \\
&\qquad \lesssim (1+t)^{-\frac{1}{2(1-\theta)}\left(\frac{n}{r}+k-2\theta\right)}\|\psi\|_{L^{\eta}}, \qquad\qquad \text{if } (k-2\theta)r+n>0.
\end{aligned}$$

Observe that the condition needed for using Lemma 2.2.5 is equivalent to our assumption:

$$(k-2\theta)r+n>0 \Leftrightarrow \frac{n}{r}+k-2\theta>0 \Leftrightarrow n\left(\frac{1}{\eta}-\frac{1}{q}\right)+k-2\theta>0.$$

iv') For $\theta = 0$, observe that the condition $t^{-\frac{1}{2\theta}} \le |\xi| \le 1$, given in opposition to item iii), does not make sense. We exchange it for $|\xi| \le 1 \le t$ instead, in contrast to $(t|\xi| \le 1)$ from item iii), in the case $\theta = 0$.

The calculations are the same, though, and we find

$$\left\| |\xi|^k \hat{K}_1(t,\cdot)\hat{\psi} \right\|_{L^{q'}(|\xi|\le 1\le t)} \lesssim (1+t)^{-\frac{1}{2}\left(\frac{n}{r}+k\right)}\|\psi\|_{L^{\eta}}$$

for $kr+n>0$.

v) Here, we start the same way as in item iv):

$$\left\| |\xi|^k \hat{K}_1(t,\cdot)\hat{\psi} \right\|_{L^{q'}(t^{-\frac{1}{2\theta}} \leq |\xi| \leq 1)} \lesssim \left\| |\xi|^k \hat{K}_1(t,\cdot) \right\|_{L^r(t^{-\frac{1}{2\theta}} \leq |\xi| \leq 1)} \|\hat{\psi}\|_{L^{\eta'}}$$
$$\lesssim \left(\int_{t^{-\frac{1}{2\theta}} \leq |\xi| \leq 1} |\xi|^{(k-2\theta)r} e^{-C(1+t)r|\xi|^{2(1-\theta)}} \, \mathrm{d}\xi \right)^{\frac{1}{r}} \|\hat{\psi}\|_{L^{\eta'}}.$$

But in this case we cannot apply Lemma 2.2.5 for, as seen in the previous item, the condition $(k-2\theta)r + n > 0$ is necessary, and here we have the opposite (excluding equality, which will be dealt with in the next item).

We make the change of variables $\nu = (1+t)^{\frac{1}{2\theta}}\xi$. This choice implies that

$$|\nu|^{2\theta} = (1+t)|\xi|^{2\theta} \geq t|\xi|^{2\theta} \geq 1 \Rightarrow |\nu| \geq 1.$$

So, the monotonicity of the integral, along with the inclusion

$$\{\xi \in \mathbb{R}^n : t^{-\frac{1}{2\theta}} \leq |\xi| \leq 1\} \subset \{\nu \in \mathbb{R}^n : |\nu| \geq 1\}$$

allows us to integrate over $\{\nu \in \mathbb{R}^n : |\nu| \geq 1\}$.

Also, the change of variable imply that

$$|\xi|^{(k-2\theta)r} = (1+t)^{-\frac{(k-2\theta)r}{2\theta}} |\nu|^{(k-2\theta)r} = (1+t)^{r-\frac{kr}{2\theta}} |\nu|^{(k-2\theta)r}$$

and

$$(1+t)|\xi|^{2(1-\theta)} = (1+t)^{1-\frac{2(1-\theta)}{2\theta}} |\nu|^{2(1-\theta)} = (1+t)^{-\frac{1-2\theta}{\theta}} |\nu|^{2(1-\theta)}.$$

Hence,

$$\left\| |\xi|^k \hat{K}_1(t,\cdot)\hat{\psi} \right\|_{L^{q'}(t^{-\frac{1}{2\theta}} \leq |\xi| \leq 1)}$$
$$\lesssim (1+t)^{1-\frac{k}{2\theta}-\frac{n}{2r\theta}} \left(\int_{|\nu|\geq 1} |\nu|^{(k-2\theta)r} e^{-Cr(1+t)^{-\frac{1-2\theta}{\theta}} |\nu|^{2(1-\theta)}} \mathrm{d}\nu \right)^{\frac{1}{r}} \|\hat{\psi}\|_{L^{\eta'}}$$
$$\lesssim (1+t)^{1-\frac{1}{2\theta}\left(\frac{n}{r}+k\right)} \left(\int_{|\nu|\geq 1} |\nu|^{(k-2\theta)r} \, \mathrm{d}\nu \right)^{\frac{1}{r}} \|\hat{\psi}\|_{L^{\eta'}}.$$

And because of our assumption $n\left(\frac{1}{\eta} - \frac{1}{q}\right) + k - 2\theta < 0$, the above integral is finite:

$$\begin{aligned}
\int_{|\nu|\geq 1} |\nu|^{(k-2\theta)r} \, \mathrm{d}\nu &= \int_1^\infty \int_{|\nu|=\rho} \rho^{(k-2\theta)r} \, \mathrm{d}S_\nu \, \mathrm{d}\rho \\
&= \omega_n \int_1^\infty \rho^{(k-2\theta)r+n-1} \, \mathrm{d}\rho \\
&= \omega_n \lim_{\alpha\to+\infty} \left. \frac{\rho^{(k-2\theta)r+n}}{(k-2\theta)r+n} \right|_1^\alpha \\
&= -\frac{\omega_n}{(k-2\theta)r+n} < +\infty,
\end{aligned}$$

for $(k-2\theta)r+n<0$. Therefore, we have

$$\left\| |\xi|^k \hat{K}_1(t,\cdot)\hat{\psi} \right\|_{L^{q'}(t^{-\frac{1}{2\theta}} \leq |\xi| \leq 1)} \lesssim (1+t)^{1-\frac{1}{2\theta}(\frac{n}{r}+k)} ||\psi||_{L^{\eta}},$$

for $(k-2\theta)r+n<0$.

vi) In contrast to items iv) and v), now our assumption implies that

$$k-2\theta = -\frac{n}{r}.$$

So, starting the same way as in items iv) and v) yields

$$\begin{aligned}
&\left\| |\xi|^k \hat{K}_1(t,\cdot)\hat{\psi} \right\|_{L^{q'}(t^{-\frac{1}{2\theta}} \leq |\xi| \leq 1)} \\
&\qquad \lesssim \left\| |\xi|^k \hat{K}_1(t,\cdot) \right\|_{L^{r}(t^{-\frac{1}{2\theta}} \leq |\xi| \leq 1)} ||\hat{\psi}||_{L^{\eta'}} \\
&\qquad \lesssim \left(\int_{t^{-\frac{1}{2\theta}} \leq |\xi| \leq 1} |\xi|^{(k-2\theta)r} e^{-C(1+t)r|\xi|^{2(1-\theta)}} \, \mathrm{d}\xi \right)^{\frac{1}{r}} ||\hat{\psi}||_{L^{\eta'}} \\
&\qquad = \left(\int_{t^{-\frac{1}{2\theta}} \leq |\xi| \leq 1} |\xi|^{-n} e^{-C(1+t)r|\xi|^{2(1-\theta)}} \, \mathrm{d}\xi \right)^{\frac{1}{r}} ||\hat{\psi}||_{L^{\eta'}} \\
&\qquad \lesssim \left(\int_{t^{-\frac{1}{2\theta}} \leq |\xi| \leq 1} |\xi|^{-n} \, \mathrm{d}\xi \right)^{\frac{1}{r}} ||\hat{\psi}||_{L^{\eta'}}.
\end{aligned}$$

Again, the integral can be evaluated by integrating over shells of fixed radius:

$$\begin{aligned}
\int_{t^{-\frac{1}{2\theta}} \leq |\xi| \leq 1} |\xi|^{-n} \, \mathrm{d}\xi &= \int_{t^{-\frac{1}{2\theta}}}^{1} \int_{|\xi|=\rho} \rho^{-n} \, \mathrm{d}S_\xi \, \mathrm{d}\rho \\
&= \omega_n \int_{t^{-\frac{1}{2\theta}}}^{1} \rho^{-1} \, \mathrm{d}\rho = \omega_n \ln \rho \Big|_{t^{-\frac{1}{2\theta}}}^{1} \\
&= \omega_n \ln t^{\frac{1}{2\theta}} \\
&= \frac{\omega_n}{2\theta} \ln t, \quad \text{with } t \geq 1.
\end{aligned}$$

vi') For $\theta = 0$, $k=0$, $r=\infty$,

$$\begin{aligned}
\left\| \hat{K}_1(t,\cdot)\hat{\psi} \right\|_{L^{q'}(|\xi| \leq 1 \leq t)} &\lesssim \left\| \hat{K}_1(t,\cdot) \right\|_{L^{\infty}(|\xi| \leq 1 \leq t)} ||\hat{\psi}||_{L^{\eta'}} \\
&\lesssim \left\| e^{t\lambda_+} \right\|_{L^{\infty}(|\xi| \leq 1 \leq t)} ||\hat{\psi}||_{L^{\eta'}} \\
&\lesssim \left\| e^{-Ct|\xi|^2} \right\|_{L^{\infty}(|\xi| \leq 1 \leq t)} ||\hat{\psi}||_{L^{\eta'}} \\
&\lesssim ||\psi||_{L^{\eta}}.
\end{aligned}$$

vii) From Lemma 2.2.3(iii),

$$\begin{aligned}
&\left\||\xi|^k \partial_t \hat{K}_1(t,\cdot)\hat{\psi}\right\|_{L^{q'}(|\xi|\leq 1)} \\
&\quad\lesssim \left\||\xi|^k \partial_t \hat{K}_1(t,\cdot)\right\|_{L^r(|\xi|\leq 1)} \left\|\hat{\psi}\right\|_{L^{\eta'}} \\
&\quad\lesssim \left\||\xi|^k (e^{t\lambda_-} + |\xi|^{2(1-2\theta)} e^{t\lambda_+})\right\|_{L^r(|\xi|\leq 1)} \|\hat{\psi}\|_{L^{\eta'}} \\
&\quad\lesssim \left(\left\||\xi|^k e^{-C(1+t)|\xi|^{2\theta}}\right\|_{L^r(|\xi|\leq 1)} + \left\||\xi|^{k+2-4\theta} e^{-C(1+t)|\xi|^{2(1-\theta)}}\right\|_{L^r(|\xi|\leq 1)}\right) \|\hat{\psi}\|_{L^{\eta'}} \\
&\quad:= (N_1 + N_2)\|\hat{\psi}\|_{L^{\eta'}}.
\end{aligned}$$

Let's deal with the norms N_1 and N_2 separately.

For N_1, if $\theta \in \left(0, \frac{1}{2}\right)$, we make the change of variable $\nu = (1+t)^{\frac{1}{2\theta}}\xi$ and apply Lemma 2.2.5:

$$\begin{aligned}
N_1^r &= \left\||\xi|^k e^{-C(1+t)|\xi|^{2\theta}}\right\|^r_{L^r(|\xi|\leq 1)} \\
&= \int_{|\xi|\leq 1} |\xi|^{kr} e^{-C(1+t)r|\xi|^{2\theta}}\, \mathrm{d}\xi \\
&\lesssim (1+t)^{-\frac{kr+n}{2\theta}} \int_{\mathbb{R}^n} |\nu|^{kr} e^{-Cr|\nu|^{2\theta}}\, \mathrm{d}\nu \\
&\lesssim (1+t)^{-\frac{1}{2\theta}(n+kr)}.
\end{aligned}$$

Hence,

$$N_1 \lesssim (1+t)^{-\frac{1}{2\theta}\left(\frac{n}{r}+k\right)},$$

for $kr + n > 0$, which is true since k is nonnegative and r, n are positive.

If $\theta = 0$, we immediately obtain

$$N_1 \lesssim e^{-ct}.$$

For N_2, we make the change of variables $\nu = (1+t)^{\frac{1}{2(1-\theta)}}\xi$ and apply Lemma 2.2.5:

$$\begin{aligned}
N_2^r &= \left\||\xi|^{k+2-4\theta} e^{-C(1+t)|\xi|^{2(1-\theta)}}\right\|^r_{L^r(|\xi|\leq 1)} \\
&= \int_{|\xi|\leq 1} |\xi|^{(k+2-4\theta)r} e^{-C(1+t)r|\xi|^{2(1-\theta)}}\, \mathrm{d}\xi \\
&\lesssim (1+t)^{-\frac{(k+2-4\theta)r+n}{2(1-\theta)}} \int_{\mathbb{R}^n} |\nu|^{(k+2-4\theta)r} e^{-Cr|\nu|^{2(1-\theta)}}\, \mathrm{d}\nu \\
&\lesssim (1+t)^{-\frac{1}{2(1-\theta)}\left(n+(k+2-4\theta)r\right)}.
\end{aligned}$$

Hence,

$$N_2 \lesssim (1+t)^{-\frac{1}{2(1-\theta)}\left(\frac{n}{r}+k-2\theta\right)-1},$$

for $(k+2-4\theta)r + n > 0$, since $k \geq 0$ and r, n and $2-4\theta$ are positive.

Since we obtained two different decay rates for N_1 and N_2, we check which is the worst one.

We have the following equivalence:

$$\begin{aligned}
(1+t)^{-\frac{1}{2\theta}\left(\frac{n}{r}+k\right)} &\leq (1+t)^{-\frac{1}{2(1-\theta)}\left(\frac{n}{r}+k-2\theta\right)-1} \\
\Leftrightarrow \frac{1}{2\theta}\left(\frac{n}{r}+k\right) &\geq \frac{1}{2(1-\theta)}\left(\frac{n}{r}+k-2\theta\right)+1 \\
\Leftrightarrow 2(1-\theta)\left(\frac{n}{r}+k\right) &\geq 2\theta\left(\frac{n}{r}+k-2\theta\right)+2(1-\theta)2\theta \\
\Leftrightarrow 2\left(\frac{n}{r}+k\right)-2\theta\left(\frac{n}{r}+k\right) &\geq 2\theta\left(\frac{n}{r}+k\right)-4\theta^2+4\theta(1-\theta) \\
\Leftrightarrow 2\left(\frac{n}{r}+k\right)(1-2\theta) &\geq 4\theta(1-2\theta) \\
\left(\frac{n}{r}+k\right) &\geq 2\theta,
\end{aligned}$$

since $1-2\theta > 0$. Therefore, we get

$$\| \, |\xi|^k \partial_t \hat{K}_1(t,\cdot)\hat{\psi}\|_{L^{q'}(|\xi|\leq 1)} \lesssim (1+t)^{-\frac{1}{2(1-\theta)}\left(\frac{n}{r}+k-2\theta\right)-1}\|\psi\|_{L^\eta}.$$

viii) The argument we used on item *vii*) says that

$$\begin{aligned}
&\| \, |\xi|^k \partial_t \hat{K}_1(t,\cdot)\hat{\psi}\|_{L^{q'}(|\xi|\leq 1)} \\
&\qquad \lesssim \left((1+t)^{-\frac{1}{2\theta}\left(\frac{n}{r}+k\right)}+(1+t)^{-\frac{1}{2(1-\theta)}\left(\frac{n}{r}+k-2\theta\right)-1}\right)\|\hat{\psi}\|_{L^{\eta'}},
\end{aligned}$$

and the worst decay rate amongst those two was the latter, because of the specific assumption on that item, that $\frac{n}{r}+k-2\theta \geq 0$. Since in this item, we assume the opposite, i.e., $\frac{n}{r}+k-2\theta < 0$, the former decay rate is now the worst one, and we obtain directly

$$\| \, |\xi|^k \partial_t \hat{K}_1(t,\cdot)\hat{\psi}\|_{L^{q'}(|\xi|\leq 1)} \lesssim (1+t)^{-\frac{1}{2\theta}\left(\frac{n}{r}+k\right)}\|\psi\|_{L^\eta}.$$

■

Now we obtain the decay rates for the high-frequency region.

Lemma 2.3.2 *Let* $\theta \in [0,\frac{1}{2})$, $\eta \in [1,2]$, $q \in [2,+\infty]$, $n \in \mathbb{N}$, $k \geq 0$ *and* $j,l = 0,1$. *Consider* $\beta > 0$ *such that* $\beta < \dfrac{(1-\theta)2q}{n(q-2)_+}$, *and let* $s_l = k+j-l+\frac{1-\theta}{\beta}$. *If* $\psi \in \dot{H}^{s_l}(\mathbb{R}^n)$, *then*

$$\left\| |\xi|^k \partial_t^j \hat{K}_l(t,\cdot)\hat{\psi}\right\|_{L^{q'}(|\xi|\geq 1)} \lesssim (1+t)^{\frac{n}{2(1-\theta)}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2\beta}}\|\psi\|_{\dot{H}^{s_l}},$$

with $q' = \frac{q}{q-1}$.

Proof: From Lemma 2.2.4, we have that

$$|\partial_t^j \hat{K}_l(t,\xi)| \lesssim |\xi|^{j-l} e^{t\,\mathbb{R}\mathrm{e}\,\lambda_\pm},$$

for $j, l = 0, 1$. Also, for $\beta > 0$,

$$\begin{aligned}
e^{-Ct|\xi|^{2(\theta-1)}} &\leq e^C e^{-C(1+t)|\xi|^{2(\theta-1)}} \\
&= e^C\, C^{-\frac{1}{2\beta}} (1+t)^{-\frac{1}{2\beta}} \left(C(1+t)|\xi|^{2(\theta-1)}\right)^{\frac{1}{2\beta}} e^{-C(1+t)|\xi|^{2(\theta-1)}} |\xi|^{\frac{1-\theta}{\beta}} \\
&\lesssim (1+t)^{-\frac{1}{2\beta}} |\xi|^{\frac{1-\theta}{\beta}}, \quad \text{for } t > 0,
\end{aligned}$$

because $x \mapsto x^{\frac{1}{2\beta}} e^{-x}$ tends to zero when $x \to \infty$.

We first deal with the special case $q = 2$. Using the estimate above and Lemma 2.2.2,

$$\begin{aligned}
\left\| |\xi|^k \partial_t^j \hat{K}_l(t,\cdot)\hat{\psi} \right\|_{L^2(|\xi|\geq 1)} &\lesssim \left\| |\xi|^{k+j-l} e^{t\,\mathbb{R}\mathrm{e}\,\lambda_\pm} \hat{\psi} \right\|_{L^2(|\xi|\geq 1)} \\
&\lesssim \left\| |\xi|^{k+j-l} (1+t)^{-\frac{1}{2\beta}} |\xi|^{\frac{1-\theta}{\beta}} \hat{\psi} \right\|_{L^2(|\xi|\geq 1)} \\
&\lesssim (1+t)^{-\frac{1}{2\beta}} \left\| |\xi|^{k+j-l+\frac{1-\theta}{\beta}} \hat{\psi} \right\|_{L^2(|\xi|\geq 1)} \\
&\lesssim (1+t)^{-\frac{1}{2\beta}} \|\psi\|_{\dot{H}^{k+j-l+\frac{1-\theta}{\beta}}}.
\end{aligned}$$

Now, for $q > 2$, we have its conjugate $q' = \frac{q}{q-1}$ satisfying $q' \in [1, 2)$. We then estimate

$$\begin{aligned}
&\left\| |\xi|^k \partial_t^j \hat{K}_l(t,\cdot)\hat{\psi} \right\|_{L^{q'}(|\xi|\geq 1)} \\
&\lesssim \left\| |\xi|^{k+j-l} e^{t\,\mathbb{R}\mathrm{e}\,\lambda_\pm} \hat{\psi} \right\|_{L^{q'}(|\xi|\geq 1)} \\
&\lesssim \left\| |\xi|^{-\frac{1-\theta}{\beta}} e^{-C(1+t)|\xi|^{2(\theta-1)}} |\xi|^{k+j-l+\frac{1-\theta}{\beta}} \hat{\psi} \right\|_{L^{q'}(|\xi|\geq 1)} \\
&\lesssim \left\| |\xi|^{-\frac{1-\theta}{\beta}} e^{-C(1+t)|\xi|^{2(\theta-1)}} \right\|_{L^{\frac{2q'}{2-q'}}(|\xi|\geq 1)} \left\| |\xi|^{k+j-l+\frac{1-\theta}{\beta}} \hat{\psi} \right\|_{L^2(|\xi|\geq 1)} \\
&\lesssim \left(\int_{|\xi|\geq 1} |\xi|^{-\frac{1-\theta}{\beta}\frac{2q'}{2-q'}} e^{-C(1+t)\frac{2q'}{2-q'}|\xi|^{2(\theta-1)}} \, d\xi \right)^{\frac{2-q'}{2q'}} \|\psi\|_{\dot{H}^{j+k-l+\frac{1-\theta}{\beta}}}.
\end{aligned}$$

Making the change of variables $\nu = (1+t)^{\frac{1}{2(\theta-1)}} \xi$ which implies $d\nu = (1+t)^{\frac{n}{2(\theta-1)}} d\xi$, and $|\xi|^{-\frac{1-\theta}{\beta}\frac{2q'}{2-q'}} = (1+t)^{-\frac{1}{2\beta}\frac{2q'}{2-q'}} |\nu|^{-\frac{1-\theta}{\beta}\frac{2q'}{2-q'}}$, we get

$$\begin{aligned}
&\left\| |\xi|^k \partial_t^j \hat{K}_l(t,\cdot)\hat{\psi} \right\|_{L^{q'}(|\xi|\geq 1)} \\
&\lesssim (1+t)^{-\frac{n}{2(\theta-1)}\left(\frac{2-q'}{2q'}\right) - \frac{1}{2\beta}} \\
&\times \left(\int_{|\nu|\geq (1+t)^{\frac{1}{2(\theta-1)}}} |\nu|^{-\frac{1-\theta}{\beta}\frac{2q'}{2-q'}} e^{-C\frac{2q'}{2-q'}|\nu|^{2(\theta-1)}} \, d\nu \right)^{\frac{2-q'}{2q'}} \|\psi\|_{\dot{H}^{s_l}}.
\end{aligned}$$

It remains only to show that the above integral is bounded by a finite constant.

$$\begin{aligned}
\int_{|\nu|\geq(1+t)^{\frac{1}{2(\theta-1)}}} & |\nu|^{-\frac{1-\theta}{\beta}\frac{2q'}{2-q'}} e^{-C\frac{2q'}{2-q'}|\nu|^{2(\theta-1)}}\, \mathrm{d}\nu \\
&\lesssim \int_{(1+t)^{\frac{1}{2(\theta-1)}}\leq|\nu|\leq 1} |\nu|^{-\frac{1-\theta}{\beta}\frac{2q'}{2-q'}} e^{-C\frac{2q'}{2-q'}|\nu|^{2(\theta-1)}}\, \mathrm{d}\nu \\
&\quad + \int_{|\nu|\geq 1} |\nu|^{-\frac{1-\theta}{\beta}\frac{2q'}{2-q'}}\, \mathrm{d}\nu \\
&:= J_1 + J_2.
\end{aligned}$$

For the first integral J_1, we use the boundedness of $x \mapsto x^b e^{-x}$, for $b \geq 0$ and $x \geq 0$:

$$\begin{aligned}
J_1 &\lesssim \int_{|\nu|\leq 1} \left(C\frac{2q'}{(2-q')}|\nu|^{2(\theta-1)} \right)^{\frac{q'}{\beta(2-q')}} e^{-C\frac{2q'}{2-q'}|\nu|^{2(\theta-1)}}\, \mathrm{d}\nu \\
&\lesssim \int_{|\nu|\leq 1} \mathrm{d}\nu < +\infty.
\end{aligned}$$

For the second, J_2, we have

$$\begin{aligned}
J_2 &= \int_{|\nu|\geq 1} |\nu|^{-\frac{1-\theta}{\beta}\frac{2q'}{2-q'}}\, \mathrm{d}\nu = \int_1^{+\infty} \rho^{-\frac{1-\theta}{\beta}\frac{2q'}{2-q'}} \int_{|\nu|=\rho} \mathrm{d}S_\nu \mathrm{d}\rho \\
&= \omega_n \int_1^{+\infty} \rho^{-\frac{1-\theta}{\beta}\frac{2q'}{2-q'}+n-1}\, \mathrm{d}\rho < +\infty,
\end{aligned}$$

for $-\frac{1-\theta}{\beta}\frac{2q'}{2-q'} + n < 0$, that is, for $n < \frac{1-\theta}{\beta}\frac{2q'}{2-q'}$, which is equivalent to our assumption $\beta < \frac{(1-\theta)2q}{n(q-2)}$.

Therefore, we have proved the following:

$$\left\| |\xi|^k \partial_t^j \hat{K}_l(t,\cdot)\hat{\psi} \right\|_{L^{q'}(|\xi|\geq 1)} \lesssim (1+t)^{-\frac{n}{2(\theta-1)}\left(\frac{2-q'}{2q'}\right)-\frac{1}{2\beta}} \|\psi\|_{\dot{H}^{s_l}},$$

for $\beta < \frac{(1-\theta)2q}{n(q-2)}$, and with $s_l = k + j - l + \frac{1-\theta}{\beta}$.

■

Combining the estimates from the last two lemmas, that provide L^η–L^q estimates for the fundamental solutions at low-frequency and high-frequency regions, respectively, we obtain the following result:

Lemma 2.3.3 *Let* $\theta \in \left[0, \frac{1}{2}\right)$, $\eta \in [1,2]$, $q \in [2,\infty]$, $n \in \mathbb{N}$, $j, l = 0, 1$, $k \geq 0$. *Consider* $0 < \beta < \frac{(1-\theta)2q}{n(q-2)_+}$, *and let* $s_l = k + j - l + \frac{1-\theta}{\beta}$. *Then, for every* $\psi \in L^\eta(\mathbb{R}^n) \cap \dot{H}^{s_l}(\mathbb{R}^n)$,

i) $$\begin{aligned}\left\| \partial_x^k \partial_t^j K_0 * \psi \right\|_{L^q} &\lesssim (1+t)^{-\frac{1}{2(1-\theta)}\left(n\left(\frac{1}{\eta}-\frac{1}{q}\right)+k\right)-j} \|\psi\|_{L^\eta} \\ &\quad + (1+t)^{\frac{n}{2(1-\theta)}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2\beta}} \|\psi\|_{\dot{H}^{s_0}};\end{aligned}$$

ii) For $n\left(\frac{1}{\eta}-\frac{1}{q}\right)+k-2\theta>0$,

$$\left\|\partial_x^k\partial_t^j K_1*\psi\right\|_{L^q}\lesssim(1+t)^{-\frac{1}{2(1-\theta)}\left(n\left(\frac{1}{\eta}-\frac{1}{q}\right)+k-2\theta\right)-j}\|\psi\|_{L^\eta}$$
$$+(1+t)^{\frac{n}{2(1-\theta)}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2\beta}}\|\psi\|_{\dot H^{s_1}};$$

iii) For $n\left(\frac{1}{\eta}-\frac{1}{q}\right)+k-2\theta<0$,

$$\left\|\partial_x^k\partial_t^j K_1*\psi\right\|_{L^q}\lesssim(1+t)^{1-\frac{1}{2\theta}\left(n\left(\frac{1}{\eta}-\frac{1}{q}\right)+k\right)-j}\|\psi\|_{L^\eta}$$
$$+(1+t)^{\frac{n}{2(1-\theta)}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2\beta}}\|\psi\|_{\dot H^{s_1}};$$

iv) For $n\left(\frac{1}{\eta}-\frac{1}{q}\right)+k-2\theta=0$,

- $\left\|\partial_x^k K_1*\psi\right\|_{L^q}\lesssim\ln(e+t)\|\psi\|_{L^\eta}+(1+t)^{\frac{n}{2(1-\theta)}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2\beta}}\|\psi\|_{\dot H^{s_1}}$;
- $\left\|\partial_x^k\partial_t K_1*\psi\right\|_{L^q}\lesssim(1+t)^{-1}\|\psi\|_{L^\eta}+(1+t)^{\frac{n}{2(1-\theta)}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2\beta}}\|\psi\|_{\dot H^{s_1}}$.

Proof: The proof follows from Hausdorff-Young Inequality and Lemmas 2.3.1 and 2.3.2:

i) Applying Hausdorff-Young Inequality, and using Lemma 2.3.1(i),(ii) and Lemma 2.3.2,

$$\left\|\partial_x^k\partial_t^j K_0*\psi\right\|_{L^q}\lesssim\||\xi|^k\partial_t^j\hat K_0(t,\cdot)\hat\psi\|_{L^{q'}(|\xi|\le1)}+\||\xi|^k\partial_t^j\hat K_0(t,\cdot)\hat\psi\|_{L^{q'}(|\xi|\ge1)}$$
$$\lesssim(1+t)^{-\frac{1}{2(1-\theta)}\left(n\left(\frac{1}{\eta}-\frac{1}{q}\right)+k\right)-j}\|\psi\|_{L^\eta}$$
$$+(1+t)^{\frac{n}{2(1-\theta)}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2\beta}}\|\psi\|_{\dot H^{s_0}}.$$

ii) Here, we have, observing that

$$\frac{1}{2\theta}\left(\frac{n}{r}+k-2\theta\right)\ge\frac{1}{2(1-\theta)}\left(\frac{n}{r}+k-2\theta\right)\Leftrightarrow2(1-2\theta)\left(\frac{n}{r}+k-2\theta\right)\ge0,$$

so $\theta<\frac{1}{2}$ and our assumption $\frac{n}{r}+k-2\theta>0$ imply that

$$(1+t)^{-\frac{1}{2\theta}\left(\frac{n}{r}+k-2\theta\right)}\le(1+t)^{-\frac{1}{2(1-\theta)}\left(\frac{n}{r}+k-2\theta\right)},$$

and therefore item iii) of Lemma 2.3.1 gives us an even better decay rate for $j=0$. Nonetheless, we'll use the worst decay in order to summarize the results from both regions in one estimate.

Using Lemma 2.3.1(iii) (along with the above observation), Lemma 2.3.1(iv),(vii) and Lemma 2.3.2,

$$\left\|\partial_x^k\partial_t^j K_1*\psi\right\|_{L^q}\lesssim\||\xi|^k\partial_t^j\hat K_1(t,\cdot)\hat\psi\|_{L^{q'}(|\xi|\le1)}+\||\xi|^k\partial_t^j\hat K_1(t,\cdot)\hat\psi\|_{L^{q'}(|\xi|\ge1)}$$
$$\lesssim(1+t)^{-\frac{1}{2(1-\theta)}\left(n\left(\frac{1}{\eta}-\frac{1}{q}\right)+k-2\theta\right)-j}\|\psi\|_{L^\eta}$$
$$+(1+t)^{\frac{n}{2(1-\theta)}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2\beta}}\|\psi\|_{\dot H^{s_1}}.$$

iii) Again, from Hausdorff-Young Inequality, Lemma 2.3.1(iii),(v),(viii) and Lemma 2.3.2,

$$\begin{aligned}\left\|\partial_x^k\partial_t^j K_1 * \psi\right\|_{L^q} &\lesssim \left\||\xi|^k\partial_t^j\hat{K}_1(t,\cdot)\hat{\psi}\right\|_{L^{q'}(|\xi|\le 1)}+\left\||\xi|^k\partial_t^j\hat{K}_1(t,\cdot)\hat{\psi}\right\|_{L^{q'}(|\xi|\ge 1)}\\ &\lesssim (1+t)^{1-\frac{1}{2\theta}\left(n\left(\frac{1}{\eta}-\frac{1}{q}\right)+k\right)-j}\|\psi\|_{L^\eta}\\ &\quad+(1+t)^{\frac{n}{2(1-\theta)}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2\beta}}\|\psi\|_{\dot{H}^{s_1}}.\end{aligned}$$

iv) From Hausdorff-Young Inequality, Lemma 2.3.1(iii),(vi) and Lemma 2.3.2,

$$\begin{aligned}\left\|\partial_x^k K_1 * \psi\right\|_{L^q} &\lesssim \|\,|\xi|^k\hat{K}_1(t,\cdot)\hat{\psi}\|_{L^{q'}(|\xi|\le 1)}+\|\,|\xi|^k\hat{K}_1(t,\cdot)\hat{\psi}\|_{L^{q'}(|\xi|\ge 1)}\\ &\lesssim \ln(e+t)\|\psi\|_{L^\eta}+(1+t)^{\frac{n}{2(1-\theta)}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2\beta}}\|\psi\|_{\dot{H}^{s_1}}.\end{aligned}$$

For the second estimate, we use Hausdorff-Young Inequality, Lemma 2.3.1(vii) and Lemma 2.3.2 and obtain directly the result:

$$\begin{aligned}\left\|\partial_x^k\partial_t K_1 * \psi\right\|_{L^q} &\lesssim \|\,|\xi|^k\partial_t\hat{K}_1(t,\cdot)\hat{\psi}\|_{L^{q'}(|\xi|\le 1)}+\|\,|\xi|^k\partial_t\hat{K}_1(t,\cdot)\hat{\psi}\|_{L^{q'}(|\xi|\ge 1)}\\ &\lesssim (1+t)^{-1}\|\psi\|_{L^\eta}+(1+t)^{\frac{n}{2(1-\theta)}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2\beta}}\|\psi\|_{\dot{H}^{s_1}}.\end{aligned}$$

■

In particular, setting $q=2$, $\eta=1$, we can choose $\beta>0$ arbitrarily large. Then, recalling that the solution to the linear problem (8) is given by $u^{lin}=K_0*u_0+K_1*u_1$, we have the following:

Corollary 2.3.4 *Let $n\in\mathbb{N}$, $\theta\in\left[0,\frac{1}{2}\right)$, $k\ge 0$, $j=0,1$, $s\ge 1$, and assume that $u^{lin}(t,x)$ is the (global) solution to the Cauchy problem (8). Then, $u^{lin}(t,x)$ satisfies*

$$\begin{aligned}\left\|\partial_t^j u^{lin}(t,\cdot)\right\|_{\dot{H}^k} &\lesssim (1+t)^{-\frac{s-k-j}{2(1-\theta)}}\|(u_0,u_1)\|_{\dot{H}^s\times H^{s-1}}\\ &+\begin{cases}(1+t)^{-\frac{n+2k-4\theta}{4(1-\theta)}-j}\|(u_0,u_1)\|_{L^1\times L^1}, & \text{if } \frac{n}{2}+k-2\theta>0,\\ (1+t)^{1-\frac{n+2k}{4\theta}-j}\|(u_0,u_1)\|_{L^1\times L^1}, & \text{if } \frac{n}{2}+k-2\theta<0,\\ a_j(t)\|(u_0,u_1)\|_{L^1\times L^1}, & \text{if } \frac{n}{2}+k-2\theta=0,\end{cases}\end{aligned} \tag{43}$$

with

$$a_j(t)=\begin{cases}\ln(e+t), & j=0\\ (1+t)^{-1}, & j=1.\end{cases} \tag{44}$$

Now, another important set of estimates are the ones involving the operator associated with the application of Duhamel's principle to the corresponding inhomogeneous equation. These estimates are obtained taking $u_0=0$ and applying the Bessel potential

$(I-\Delta)^{-1}$ to u^{lin}. Indeed, the presence of the rotational inertia term $-\Delta u_{tt}$ and the application of Duhamel's principle, leads to gain 2 derivatives in the regularity-loss decay rate structure at high frequencies. These estimates will be a staple in the nonlinear existence argument.

We define

$$E_1(t,x) = (I-\Delta)^{-1} K_1(t,x). \tag{45}$$

Next, we prove some estimates involving convolution with E_1.

Lemma 2.3.5 *Let* $\theta \in [0, \frac{1}{2}), \eta \in [1,2], q \in [2,\infty], n \in \mathbb{N}, k \geq 0$. *Consider* $0 < \beta < \dfrac{(1-\theta)2q}{n(q-2)_+}$, *and let* $s_1 = k + j - 1 + \frac{1-\theta}{\beta}$. *Then, for every* $\psi \in L^\eta(\mathbb{R}^n) \cap \dot{H}^{s_1-2}(\mathbb{R}^n)$, *it holds that*

i) For $n\left(\frac{1}{\eta}-\frac{1}{q}\right)+k-2\theta > 0$,

$$\left\|\partial_x^k \partial_t^j E_1 * \psi\right\|_{L^q} \lesssim (1+t)^{-\frac{1}{2(1-\theta)}\left(n\left(\frac{1}{\eta}-\frac{1}{q}\right)+k-2\theta\right)-j} \|\psi\|_{L^\eta} + (1+t)^{\frac{n}{2(1-\theta)}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2\beta}} \|\psi\|_{\dot{H}^{s_1-2}};$$

ii) For $n\left(\frac{1}{\eta}-\frac{1}{q}\right)+k-2\theta < 0$,

$$\left\|\partial_x^k \partial_t^j E_1 * \psi\right\|_{L^q} \lesssim (1+t)^{1-\frac{1}{2\theta}\left(n\left(\frac{1}{\eta}-\frac{1}{q}\right)+k\right)-j} \|\psi\|_{L^\eta} + (1+t)^{\frac{n}{2(1-\theta)}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2\beta}} \|\psi\|_{\dot{H}^{s_1-2}};$$

iii) For $n\left(\frac{1}{\eta}-\frac{1}{q}\right)+k-2\theta = 0$,

- $\left\|\partial_x^k E_1 * \psi\right\|_{L^q} \lesssim \ln(e+t)\|\psi\|_{L^\eta} + (1+t)^{\frac{n}{2(1-\theta)}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2\beta}} \|\psi\|_{\dot{H}^{s_1-2}}$;
- $\left\|\partial_x^k \partial_t E_1 * \psi\right\|_{L^q} \lesssim (1+t)^{-1}\|\psi\|_{L^\eta} + (1+t)^{\frac{n}{2(1-\theta)}\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{1}{2\beta}} \|\psi\|_{\dot{H}^{s_1-2}}$.

Proof: For the low-frequency region, we observe that $\hat{E}_1 \approx \hat{K}_1$. In fact,

$$\hat{E}_1 = (1+|\xi|^2)^{-1}\hat{K}_1 \Rightarrow \frac{1}{2}\hat{K}_1 \leq \hat{E}_1 \leq \hat{K}_1.$$

This means that all the results we obtained for $\hat{K}_1$ on Lemma 2.3.1 are also true for $\hat{E}_1$. For the high-frequency, we use Lemma 2.2.4 to get

$$\begin{aligned}|\partial_t^j \hat{E}_1(t,\xi)| = |\partial_t^j (1+|\xi|^2)^{-1}\hat{K}_1(t,\xi)| &\lesssim |\xi|^{-2}|\xi|^{(j-1)} e^{t\,\mathbb{R}\mathrm{e}\,\lambda_\pm} \\ &\lesssim |\xi|^{j-3} e^{t\,\mathbb{R}\mathrm{e}\,\lambda_\pm}, \quad j=0,1.\end{aligned}$$

With this estimate, proceeding as in the proof of Lemma 2.3.2 we obtain

$$\begin{aligned}\left\||\xi|^k \partial_t^j \hat{E}_1(t,\cdot)\hat{\psi}\right\|_{L^2(|\xi|\geq 1)} &\lesssim \left\||\xi|^{k+j-3} e^{t\,\mathbb{R}\mathrm{e}\,\lambda_\pm}\hat{\psi}\right\|_{L^2(|\xi|\geq 1)} \\ &\lesssim \left\||\xi|^{k+j-3}(1+t)^{-\frac{1}{2\beta}}|\xi|^{\frac{1-\theta}{\beta}}\hat{\psi}\right\|_{L^2(|\xi|\geq 1)} \\ &\lesssim (1+t)^{-\frac{1}{2\beta}}\left\||\xi|^{k+j-3+\frac{1-\theta}{\beta}}\hat{\psi}\right\|_{L^2(|\xi|\geq 1)} \\ &\lesssim (1+t)^{-\frac{1}{2\beta}}\|\psi\|_{\dot{H}^{s_1-2}}.\end{aligned}$$

For $q > 2$, calculations are similar:

$$
\begin{aligned}
&\left\| |\xi|^k \partial_t^j \hat{E}_1(t,\cdot)\hat{\psi} \right\|_{L^{q'}(|\xi|\geq 1)} \\
&\qquad \lesssim \left\| |\xi|^{k+j-3} e^{t\,\mathbb{R}\mathrm{e}\,\lambda_\pm} \hat{\psi} \right\|_{L^{q'}(|\xi|\geq 1)} \\
&\qquad \lesssim \left\| |\xi|^{-\frac{1-\theta}{\beta}} e^{-C(1+t)|\xi|^{2(\theta-1)}} |\xi|^{k+j-3+\frac{1-\theta}{\beta}} \hat{\psi} \right\|_{L^{q'}(|\xi|\geq 1)} \\
&\qquad \lesssim \left\| |\xi|^{-\frac{1-\theta}{\beta}} e^{-C(1+t)|\xi|^{2(\theta-1)}} \right\|_{L^{\frac{2q'}{2-q'}}(|\xi|\geq 1)} \left\| |\xi|^{k+j-3+\frac{1-\theta}{\beta}} \hat{\psi} \right\|_{L^2(|\xi|\geq 1)} \\
&\qquad \lesssim \left(\int_{|\xi|\geq 1} |\xi|^{-\frac{1-\theta}{\beta}\frac{2q'}{2-q'}} e^{-C(1+t)\frac{2q'}{2-q'}|\xi|^{2(\theta-1)}} \,\mathrm{d}\xi \right)^{\frac{2-q'}{2q'}} \|\psi\|_{\dot{H}^{s_1-2}} .
\end{aligned}
$$

Making the change of variables $\nu = (1+t)^{\frac{1}{2(\theta-1)}}\xi$, we get

$$
\begin{aligned}
&\left\| |\xi|^k \partial_t^j \hat{E}_1(t,\cdot)\hat{\psi} \right\|_{L^{q'}(|\xi|\geq 1)} \\
&\qquad \lesssim (1+t)^{-\frac{n}{2(\theta-1)}\left(\frac{2-q'}{2q'}\right)-\frac{1}{2\beta}} \\
&\qquad\quad \times \left(\int_{|\nu|\geq (1+t)^{\frac{1}{2(\theta-1)}}} |\nu|^{-\frac{1-\theta}{\beta}\frac{2q'}{2-q'}} e^{-C\frac{2q'}{2-q'}|\nu|^{2(\theta-1)}} \,\mathrm{d}\nu \right)^{\frac{2-q'}{2q'}} \|\psi\|_{\dot{H}^{s_1-2}} ,
\end{aligned}
$$

and we have shown already on Lemma 2.3.2 that the integral above is finite. Therefore,

$$
\left\| |\xi|^k \partial_t^j \hat{E}_1(t,\cdot)\hat{\psi} \right\|_{L^{q'}(|\xi|\geq 1)} \lesssim (1+t)^{-\frac{n}{2(\theta-1)}\left(\frac{2-q'}{2q'}\right)-\frac{1}{2\beta}} \|\psi\|_{\dot{H}^{s_1-2}} .
$$

We conclude by combining each estimate for the low-frequency with the estimate for high-frequency, exactly in the same way that we've done on Lemma 2.3.3. ∎

Again, we shall particularize a bit to get a more clean and meaningful result for our purposes. Setting $q = 2$, $\eta = 1$, and choosing a sufficiently large $\beta > 0$, we have the following.

Corollary 2.3.6 *Let $n \in \mathbb{N}$, $\theta \in \left[0, \frac{1}{2}\right)$, $k \geq 0$, $j = 0, 1$, $s \geq 3$, and assume that $u^{lin}(t,x)$ is the (global) solution to the Cauchy problem (8) with $u_0 = 0$ and $u_1 = \varphi$. Then, we have the estimates*

$$
\begin{aligned}
\left\| \partial_t^j (I-\Delta)^{-1} u^{lin}(t,\cdot) \right\|_{\dot{H}^k} &\lesssim (1+t)^{-\frac{s-k-j}{2(1-\theta)}} \|\varphi\|_{\dot{H}^{s-3}} \\
&\quad + \begin{cases} (1+t)^{-\frac{n+2k-4\theta}{4(1-\theta)}-j} \|\varphi\|_{L^1}, & \text{if } \frac{n}{2}+k-2\theta > 0, \\ (1+t)^{1-\frac{n+2k}{4\theta}-j} \|\varphi\|_{L^1}, & \text{if } \frac{n}{2}+k-2\theta < 0, \\ a_j(t) \|\varphi\|_{L^1}, & \text{if } \frac{n}{2}+k-2\theta = 0, \end{cases}
\end{aligned}
\tag{46}
$$

with $a_j(t)$ as in (44).

2.4 SOME IMPORTANT AUXILIARY RESULTS

In our task to prove the existence of global in-time solutions for small initial data, we must estimate several norms and integrals. The following inequalities play a fundamental role in these estimates.

The first of these inequalities is Gagliardo-Nirenberg's Interpolation Inequality, with also a fractional version of it, proved in (D'ABBICCO; REISSIG, 2013).

Lemma 2.4.1 (Gagliardo-Nirenberg's Inequality) *Let $n \in \mathbb{N}$, and assume that κ is either a natural number or that it satisfies $\kappa \in (0, \frac{n}{2})$. Then, it holds that*

$$\|u\|_{L^q} \lesssim \|u\|_{L^2}^{1-\theta_\kappa(q)} \|u\|_{\dot{H}^\kappa}^{\theta_\kappa(q)}, \quad \theta_\kappa(q) = \frac{n}{\kappa}\left(\frac{1}{2} - \frac{1}{q}\right) \tag{47}$$

for any $q \geq 2$ that also satisfies

$$2 \leq q \leq \frac{2n}{n-2k},$$

in the case $\kappa \in \left(0, \frac{n}{2}\right)$.

In the special case $q = \frac{2n}{n-2\kappa}$, inequality (47) reduces to the Sobolev embedding:

$$\|u\|_{L^q} \lesssim \|u\|_{\dot{H}^\kappa}, \quad \kappa = n\left(\frac{1}{2} - \frac{1}{q}\right). \tag{48}$$

In some cases, it is also useful to use interpolation between spaces. In particular, the Bessel Potential Spaces can be interpolated as

$$\dot{H}^{s_\beta} = \left(\dot{H}^{s_0}, \dot{H}^{s_1}\right)_\beta,$$

where $s_\beta = (1-\beta)s_0 + \beta s_1$, for $\beta \in (0,1)$. In other words, if $u \in \dot{H}^{s_0} \cap \dot{H}^{s_1}$, then $u \in \dot{H}^{s_\beta}$ and it satisfies

$$\|u\|_{\dot{H}^{s_\beta}} \lesssim \|u\|_{\dot{H}^{s_0}}^{1-\beta} \|u\|_{\dot{H}^{s_1}}^{\beta}.$$

Also, in order to estimate the nonlinear part of the solution, we rely heavily on the following lemma, which we prove in Appendix A.1:

Lemma 2.4.2 *Let $\omega \in \mathbb{R}$, $\alpha > 1$, $\gamma \in (0,1)$. Then, it holds*

$$\int_0^t (1+t-\tau)^{-\omega} \int_0^\tau (\tau - s)^{-\gamma}(1+s)^{-\alpha}\, ds\, d\tau \lesssim \begin{cases} (1+t)^{-\gamma} & \omega > 1 \\ (1+t)^{-\gamma}\ln(2+t) & \omega = 1 \\ (1+t)^{1-\omega-\gamma} & \omega < 1. \end{cases}$$

3 NONLINEAR PROBLEM

Now that we have collected the necessary estimates from the linear associated problem, we return to the semilinear problem

$$\begin{cases} u_{tt} - \Delta u_{tt} + \Delta^2 u - \Delta u + (-\Delta)^{\theta} u_t = \displaystyle\int_0^t (t-s)^{-\gamma} |u(s,\cdot)|^p \, \mathrm{d}s \\ u(0,x) = u_0(x) \\ u_t(0,x) = u_1(x), \end{cases} \tag{49}$$

with $\theta \in \left[0, \frac{1}{2}\right)$, $\gamma \in (0,1)$, $p > 1$. The operator

$$F(t,u) = \int_0^t (t-s)^{-\gamma} |u(s,\cdot)|^p \, \mathrm{d}s$$

is referred to as the *nonlinear memory* of the function u. Now, for $T > 0$, we define the Banach space

$$X(T) := \begin{cases} \mathcal{C}\big([0,T], H^2\big) \cap \mathcal{C}^1\big([0,T], H^1\big), & \text{if } p \geq 2; \\ \mathcal{C}\big([0,T], H^2\big) \cap \mathcal{C}^1\big([0,T], H^1\big) \cap L^{\infty}\big([0,T], L^p\big), & \text{if } p < 2, \end{cases} \tag{50}$$

equipped with the norm

$$\|v\|_{X(T)} := \begin{cases} \sup\limits_{t\in[0,T]} \|v\|_{X_0(t)}\,, & \text{if } p \geq 2; \\ \sup\limits_{t\in[0,T]} \left(\|v\|_{X_0(t)} + h_p^*(t)^{-1} \|v(t,\cdot)\|_{L^p} \right), & \text{if } p < 2, \end{cases} \tag{51}$$

with

$$\|v\|_{X_0(t)} := \sum_{j=0}^{m} h_j(t)^{-1} \|v(t,\cdot)\|_{\dot{H}^{k_j}} + \sum_{l=0}^{1} \tilde{h}_l(t)^{-1} \|v_t(t,\cdot)\|_{\dot{H}^l}\,,$$

for some $m \geq 1$, where $0 = k_0 < \cdots < k_m = 2$. (here $\dot{H}^0 = L^2$).

Now we define the operator $G : X(T) \to X(T)$,

$$Gu(t,x) := \int_0^t \int_0^{\tau} (\tau - s)^{-\gamma} E_1(t-\tau, x) * |u(s,x)|^p \, \mathrm{d}s \, \mathrm{d}\tau, \tag{52}$$

where, we recall,

$$E_1(t,x) = (I - \Delta)^{-1} K_1(t,x).$$

Setting $v = u_t$, the semilinear problem can be rewritten as

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} = LU + H(U), \\ U(0) = U_0, \end{cases}$$

with $U = (u, v)$, $L = \begin{pmatrix} 0 & I \\ \Delta & -(I-\Delta)^{-1}(-\Delta)^{\theta} \end{pmatrix}$, $U_0 = (u_0, u_1)$ and

$$H(U) = \begin{pmatrix} 0 \\ \int_0^t (t-s)^{-\gamma}(I-\Delta)^{-1}|u(s,x)|^p \, ds \end{pmatrix}.$$

Then, by Duhamel's Principle, a function $u \in \mathcal{C}\big([0, T], H^k(\mathbb{R}^n)\big)$, where $k \geq 0$, is the global (weak) solution to (49) if, and only if, it satisfies the identity

$$u(t,\cdot) = u^{lin}(t,\cdot) + Gu(t,x), \quad \text{in} \quad H^k(\mathbb{R}^n), \tag{53}$$

where u^{lin} is the solution to the linear Cauchy problem (8).

We will be able to apply Picard's Existence Theorem and find a solution u to (53) if we prove that

i) $\left\|u^{lin}\right\|_{X(T)} \lesssim \|(u_0, u_1)\|_{\mathcal{A}}$;

ii) $\|Gu\|_{X(T)} \lesssim \|u\|^p_{X(T)}$;

iii) $\|Gu - Gv\|_{X(T)} \lesssim \|u - v\|_{X(T)} \left(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\right)$,

uniformly on $T > 0$.

3.1 CASE $n = 1$

Theorem 3.1.1 *Assume $n = 1$, $\theta \in \left[0, \frac{1}{2}\right)$, $\gamma \in \left(\frac{1}{2(1-\theta)}, 1\right)$, $p > p_c$ and $s = s_c$, with p_c as in (2) and s_c as in (3). Then, there exists $\varepsilon > 0$ such that, for initial data*

$$(u_0, u_1) \in \mathcal{A} := \left(H^s(\mathbb{R}) \cap L^1(\mathbb{R})\right) \times \left(H^{s-1}(\mathbb{R}) \cap L^1(\mathbb{R})\right), \tag{54}$$

with $\|(u_0, u_1)\|_{\mathcal{A}} \leq \varepsilon$, there exists a global solution to the problem (1), $u \in \mathcal{C}\big([0,\infty), H^2\big) \cap \mathcal{C}^1\left([0,\infty), H^1\right)$. Moreover, the following estimates hold:

$$\|u(t,\cdot)\|_{L^2} \lesssim \begin{cases} (1+t)^{-\frac{1-4\theta}{4(1-\theta)}+1-\gamma}\,\|(u_0,u_1)\|_{\mathcal{A}}, & \textit{if } \theta \in \left[0,\frac{1}{4}\right), \\ \ln(e+t)(1+t)^{1-\gamma}\,\|(u_0,u_1)\|_{\mathcal{A}}, & \textit{if } \theta = \frac{1}{4}, \\ (1+t)^{2-\frac{1}{4\theta}-\gamma}\,\|(u_0,u_1)\|_{\mathcal{A}}, & \textit{if } \theta \in \left(\frac{1}{4},\frac{1}{2}\right); \end{cases}$$
$$\|u(t,\cdot)\|_{\dot{H}^1} \lesssim (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma}\,\|(u_0,u_1)\|_{\mathcal{A}};$$
$$\|u(t,\cdot)\|_{\dot{H}^2} \lesssim (1+t)^{-\frac{1}{2(1-\theta)}+1-\gamma}\,\|(u_0,u_1)\|_{\mathcal{A}};$$
$$\|\partial_t u(t,\cdot)\|_{L^2} \lesssim \begin{cases} \ln(2+t)(1+t)^{-\gamma}\,\|(u_0,u_1)\|_{\mathcal{A}}, & \textit{if } \theta = 0, \\ (1+t)^{-\gamma}\,\|(u_0,u_1)\|_{\mathcal{A}}, & \textit{if } \theta \in \left(0,\frac{1}{4}\right), \\ \ln(2+t)(1+t)^{-\gamma}\,\|(u_0,u_1)\|_{\mathcal{A}}, & \textit{if } \theta = \frac{1}{4}, \\ (1+t)^{1-\frac{1}{4\theta}-\gamma}\,\|(u_0,u_1)\|_{\mathcal{A}}, & \textit{if } \theta \in \left(\frac{1}{4},\frac{1}{2}\right); \end{cases}$$
$$\|\partial_t u(t,\cdot)\|_{\dot{H}^1} \lesssim (1+t)^{-\frac{1}{2(1-\theta)}+1-\gamma}\,\|(u_0,u_1)\|_{\mathcal{A}}.$$

Proof: We first consider the case $\theta \in \left[0,\frac{1}{4}\right)$. We remark that, in this case, the inequality

$$\frac{n}{2}+k-2\theta > 0$$

holds for every nonnegative value of k, and therefore, when we apply Corollary 2.3.4, we will use the same kind of linear estimate for every norm.

For $T>0$, we define the Banach space

$$X(T) := \mathcal{C}\big([0,T],H^2\big)\cap\mathcal{C}^1\big([0,T],H^1\big),$$

equipped with the norm

$$\begin{aligned}\|v\|_{X(T)} := \sup_{t\in[0,T]}\Big(& h_0(t)^{-1}\|v(t,\cdot)\|_{L^2} + h_1(t)^{-1}\|v(t,\cdot)\|_{\dot{H}^1} \\ & + h_2(t)^{-1}\|v(t,\cdot)\|_{\dot{H}^2} + \tilde{h}_0(t)^{-1}\|v_t(t,\cdot)\|_{L^2} \\ & + \tilde{h}_1(t)^{-1}\|v_t(t,\cdot)\|_{\dot{H}^1}\Big),\end{aligned}$$

where

$$h_0(t) = (1+t)^{-\frac{1-4\theta}{4(1-\theta)}+1-\gamma},$$
$$h_1(t) = (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma},$$
$$\tilde{h}_0(t) = \begin{cases} (1+t)^{-\gamma} & \text{if } \theta > 0, \\ (1+t)^{-\gamma}\ln(2+t) & \text{if } \theta = 0, \end{cases}$$
$$h_2(t) = \tilde{h}_1(t) = (1+t)^{-\frac{1}{2(1-\theta)}+1-\gamma}.$$

We will now show that the solution $u^{lin}(t,x)$ to the associated linear problem (8) belongs to $X(T)$, and satisfies

$$\left\|u^{lin}\right\|_{X(T)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}}, \quad \text{uniformly with respect to } T. \tag{55}$$

To this end, we may apply Corollary 2.3.4 several times to estimate each one of the five norms that appear in the definition of $\|\cdot\|_{X(T)}$. Firstly, setting $n=1$, $j=0$, $k=0$ and $s=s_c$ in Corollary 2.3.4,

$$\begin{aligned}\left\|u^{lin}(t,\cdot)\right\|_{L^2} \lesssim (1+t)^{-\frac{1-4\theta}{4(1-\theta)}} \|(u_0,u_1)\|_{L^1\times L^1} \\ + (1+t)^{-\frac{s}{2(1-\theta)}} \|(u_0,u_1)\|_{\dot{H}^s\times \dot{H}^{s-1}}.\end{aligned} \tag{56}$$

Since $\gamma\in(0,1)$, we have $(1+t)^{-\frac{1-4\theta}{4(1-\theta)}} \le (1+t)^{-\frac{1-4\theta}{4(1-\theta)}+1-\gamma}$. Also, since $s_c>2$ and $\theta<1$, it holds that

$$\frac{s}{2(1-\theta)} \ge 1 \ge \frac{1-4\theta}{4(1-\theta)} \ge \frac{1-4\theta}{4(1-\theta)} - (1-\gamma),$$

which implies that the decay rate from the high frequency region is also faster than the decay rate we set for $h_0(t)$, that is, $(1+t)^{-\frac{s}{2(1-\theta)}} \le (1+t)^{-\frac{1-4\theta}{4(1-\theta)}+1-\gamma}$. Using this information in (56), we get

$$\left\|u^{lin}(t,\cdot)\right\|_{L^2} \lesssim (1+t)^{-\frac{1-4\theta}{4(1-\theta)}+1-\gamma} \|(u_0,u_1)\|_{\mathcal{A}}. \tag{57}$$

Now we do the same for the other norms. To estimate the norm in $\dot{H}^1$, we set $n=1$, $j=0$, $k=1$ and $s=s_c$ in Corollary 2.3.4, obtaining

$$\begin{aligned}\left\|u^{lin}(t,\cdot)\right\|_{\dot{H}^1} \lesssim (1+t)^{-\frac{3-4\theta}{4(1-\theta)}} \|(u_0,u_1)\|_{L^1\times L^1} \\ + (1+t)^{-\frac{s-1}{2(1-\theta)}} \|(u_0,u_1)\|_{\dot{H}^s\times \dot{H}^{s-1}}.\end{aligned} \tag{58}$$

Again, because $\gamma\in(0,1)$, the low-frequency part decay is obviously faster than $h_1(t)$. For the high-frequency decay rate, using that $s=s_c$ and $\frac{3-4\theta}{4(1-\theta)}<1$,

$$\frac{s-1}{2(1-\theta)} = \frac{1+2\gamma(1-\theta)}{2(1-\theta)} \ge \gamma \ge \frac{3-4\theta}{4(1-\theta)} - 1 + \gamma,$$

hence $(1+t)^{-\frac{s-1}{2(1-\theta)}} \le (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma}$, and returning to (58),

$$\left\|u^{lin}(t,\cdot)\right\|_{\dot{H}^1} \lesssim (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma} \|(u_0,u_1)\|_{\mathcal{A}}. \tag{59}$$

For the norm in $\dot{H}^2$, set $n=1$, $j=0$, $k=2$ and $s=s_c$ in Corollary 2.3.4,

$$\begin{aligned}\left\|u^{lin}(t,\cdot)\right\|_{\dot{H}^2} \lesssim (1+t)^{-\frac{5-4\theta}{4(1-\theta)}} \|(u_0,u_1)\|_{L^1\times L^1} \\ + (1+t)^{-\frac{s-2}{2(1-\theta)}} \|(u_0,u_1)\|_{\dot{H}^s\times \dot{H}^{s-1}}.\end{aligned} \tag{60}$$

We show again that the powers that appear in both low and high-frequency terms are greater (in absolute value) than the exponent in definition of $h_2(t)$. Indeed,

$$\frac{5-4\theta}{4(1-\theta)} \geq 1 \geq \frac{1}{2(1-\theta)} \geq \frac{1}{2(1-\theta)} - (1-\gamma);$$

$$\frac{s-2}{2(1-\theta)} = \gamma \geq \frac{1}{2(1-\theta)} - 1 + \gamma.$$

Using this in (60), we get

$$\left\| u^{lin}(t,\cdot) \right\|_{\dot{H}^2} \lesssim (1+t)^{-\frac{1}{2(1-\theta)}+1-\gamma} \left\| (u_0, u_1) \right\|_{\mathcal{A}}. \tag{61}$$

For the L^2–norm of the time derivative of $u^{lin}(t,x)$, we set $n = 1$, $j = 1$, $k = 0$ and $s = s_c$ in Corollary 2.3.4, which gives us

$$\begin{aligned} \left\| \partial_t u^{lin}(t,\cdot) \right\|_{L^2} &\lesssim (1+t)^{-\frac{1-4\theta}{4(1-\theta)}-1} \left\| (u_0, u_1) \right\|_{L^1 \times L^1} \\ &\quad + (1+t)^{-\frac{s-1}{2(1-\theta)}} \left\| (u_0, u_1) \right\|_{\dot{H}^s \times \dot{H}^{s-1}}. \end{aligned} \tag{62}$$

Here, both decay rates are faster than $(1+t)^{-\gamma}$, because $\gamma \in (0,1)$ and $\theta < \frac{1}{4}$:

$$\frac{1-4\theta}{4(1-\theta)} + 1 \geq 1 \geq \gamma$$

and

$$\frac{s-1}{2(1-\theta)} = \frac{1}{2(1-\theta)} + \gamma \geq \gamma.$$

Hence, (62) becomes

$$\begin{aligned} \left\| \partial_t u^{lin}(t,\cdot) \right\|_{L^2} &\lesssim (1+t)^{-\gamma} \left\| (u_0, u_1) \right\|_{\mathcal{A}} \\ &\lesssim \tilde{h}_0(t) \left\| (u_0, u_1) \right\|_{\mathcal{A}}. \end{aligned} \tag{63}$$

The last one is the $\dot{H}^1$– norm for $\partial_t u^{lin}(t,x)$. Setting $n = 1$, $j = 1$, $k = 1$ and $s = s_c$ in Corollary 2.3.4, we get

$$\begin{aligned} \left\| \partial_t u^{lin}(t,\cdot) \right\|_{\dot{H}^1} &\lesssim (1+t)^{-\frac{3-4\theta}{4(1-\theta)}-1} \left\| (u_0, u_1) \right\|_{L^1 \times L^1} \\ &\quad + (1+t)^{-\frac{s-2}{2(1-\theta)}} \left\| (u_0, u_1) \right\|_{\dot{H}^s \times \dot{H}^{s-1}}. \end{aligned} \tag{64}$$

Here, since it holds that

$$\frac{3-4\theta}{4(1-\theta)} + 1 \geq 1 \geq \frac{1}{2(1-\theta)} \geq \frac{1}{2(1-\theta)} - (1-\gamma)$$

and

$$\frac{s-2}{2(1-\theta)} = \gamma \geq \frac{1}{2(1-\theta)} - 1 + \gamma,$$

we have that, in (64),

$$\left\| \partial_t u^{lin}(t,\cdot) \right\|_{\dot{H}^1} \lesssim (1+t)^{-\frac{1}{2(1-\theta)}+1-\gamma} \left\| (u_0, u_1) \right\|_{\mathcal{A}}. \tag{65}$$

Now, we isolate $\|(u_0, u_1)\|_{\mathcal{A}}$ in estimates (57), (59), (61), (63) and (65), add them up and take the supremum over $t \in [0, T]$, obtaining

$$\left\| u^{lin} \right\|_{X(T)} \lesssim \|(u_0, u_1)\|_{\mathcal{A}}, \tag{66}$$

with a constant that does not depend on T.

Our next task is to prove that the operator G defined in (52) is well-defined, maps balls of $X(T)$ into balls of $X(T)$ for small data in $\mathcal{A}$ and it is also a contraction map in $X(T)$. In other words, we must prove the following estimates:

$$\begin{aligned} \|Gu\|_{X(T)} &\lesssim \|u\|^{p}_{X(T)}, \\ \|Gu - Gv\|_{X(T)} &\lesssim \|u - v\|_{X(T)} \left(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)} \right). \end{aligned} \tag{67}$$

To prove the first estimate in (67), we estimate $Gu(t, x)$ in each one of the five norms that compose the norm in $X(T)$.

Firstly, we estimate the norm in L^2. Setting $n = 1$, $j = 0$, $k = 0$, $\varphi = |u|^p$ and $s = 3$ in Corollary 2.3.6, we get

$$\begin{aligned} \|Gu(t, \cdot)\|_{L^2} \lesssim & \int_0^t (1 + t - \tau)^{-\frac{1-4\theta}{4(1-\theta)}} \int_0^\tau (\tau - s)^{-\gamma} \|u(s, \cdot)\|^{p}_{L^p} \, ds \, d\tau \\ & + \int_0^t (1 + t - \tau)^{-\frac{3}{2(1-\theta)}} \int_0^\tau (\tau - s)^{-\gamma} \|u(s, \cdot)\|^{p}_{L^{2p}} \, ds \, d\tau. \end{aligned} \tag{68}$$

We stress that, in contrast to the estimates from Corollary 2.3.4, the value s that appears on Corollary 2.3.6 doesn't have any relation with the regularity of the initial data (u_0, u_1).

Here, it becomes clear that we must estimate the norms $\|u(s, \cdot)\|^{p}_{L^p}$ and $\|u(s, \cdot)\|^{p}_{L^{2p}}$ that appeared in the right-hand side above. We apply Gagliardo-Nirenberg's inequality (47), with $\kappa = 1$ and $q = p$. Having in mind that $u(s, \cdot) \in X(T)$, hence its norms in L^2 and $\dot{H}^1$ are estimated by $h_0(s) \|u(s, \cdot)\|_{X(T)}$ and $h_1(s) \|u(s, \cdot)\|_{X(T)}$, respectively, we obtain

$$\begin{aligned} \|u(s, \cdot)\|^{p}_{L^p} &\lesssim \|u(s, \cdot)\|^{(1-\theta_1(p))p}_{L^2} \|u(s, \cdot)\|^{\theta_1(p)p}_{\dot{H}^1} \\ &\lesssim h_0(s)^{(1-\theta_1(p))p} h_1(s)^{\theta_1(p)p} \|u\|^{p}_{X(T)} \\ &\lesssim (1 + s)^{\left(-\frac{1-4\theta}{4(1-\theta)} + 1 - \gamma\right)(1-\theta_1(p))p + \left(-\frac{3-4\theta}{4(1-\theta)} + 1 - \gamma\right)\theta_1(p)p} \|u\|^{p}_{X(T)} \\ &:= (1 + s)^{-\alpha} \|u\|^{p}_{X(T)}. \end{aligned} \tag{69}$$

Recalling that $\theta_1(p) = \frac{1}{2} - \frac{1}{p}$, the power α in the estimate above may be rewritten as

$$\begin{aligned}\alpha &= \left(\frac{1-4\theta}{4(1-\theta)} - (1-\gamma)\right)(1-\theta_1(p))p + \left(\frac{3-4\theta}{4(1-\theta)} - (1-\gamma)\right)\theta_1(p)p \\ &= \frac{(1-4\theta)(1+\frac{p}{2}) + (3-4\theta)(\frac{p}{2}-1)}{4(1-\theta)} - (1-\gamma)p \\ &= \frac{(2-4\theta)p-2}{4(1-\theta)} - (1-\gamma)p \\ &= \frac{(1-2\theta)p-1}{2(1-\theta)} - (1-\gamma)p.\end{aligned}$$

Since our intention is to apply Lemma 2.4.2, it is important to know for which values of p the condition $\alpha > 1$ holds:

$$\begin{aligned}\alpha > 1 &\Leftrightarrow \frac{(1-2\theta)p-1}{2(1-\theta)} - (1-\gamma)p > 1 \\ &\Leftrightarrow \big(1-2\theta-2(1-\theta)(1-\gamma)\big)p > 1 + 2(1-\theta) \\ &\Leftrightarrow \big(-1+2\gamma(1-\theta)\big)p > 1 + 2(1-\theta).\end{aligned}$$

Observing that $1+2(1-\theta)$ is positive, the inequality will hold only if $-1+2\gamma(1-\theta) > 0$, that is, $\gamma > \frac{1}{2(1-\theta)}$. Assuming that $\gamma \in \left(\frac{1}{2(1-\theta)}, 1\right)$, we obtain

$$\alpha > 1 \Leftrightarrow p > \frac{3-2\theta}{-1+2\gamma(1-\theta)} = 1 + \frac{2\big(1+(1-\gamma)(1-\theta)\big)}{-1+2\gamma(1-\theta)} = p_c.$$

Doing the same for the norm in L^{2p}, that is, applying Gagliardo-Nirenberg with $\kappa = 1$ and $q = 2p$, we get

$$\begin{aligned}\|u(s,\cdot)\|_{L^{2p}}^p &\lesssim h_0(s)^{(1-\theta_1(2p))p} h_1(s)^{\theta_1(2p)p} \|u\|_{X(T)}^p \\ &:= (1+s)^{-\tilde{\alpha}} \|u\|_{X(T)}^p,\end{aligned} \tag{70}$$

with

$$\tilde{\alpha} = \alpha + \frac{1}{4(1-\theta)} > \alpha. \tag{71}$$

Notice that (71) implies that, since $\tilde{\alpha} > \alpha$, the condition $\tilde{\alpha} > 1$ is achieved automatically with the previous assumption $p > p_c$.

Using (69) and (70) in (68), we get

$$\begin{aligned}\|Gu(t,\cdot)\|_{L^2} \lesssim \Bigg[&\int_0^t (1+t-\tau)^{-\frac{1-4\theta}{4(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\alpha}\, ds\, d\tau \\ &+ \int_0^t (1+t-\tau)^{-\frac{3}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\tilde{\alpha}}\, ds\, d\tau\Bigg] \|u\|_{X(T)}^p.\end{aligned} \tag{72}$$

Applying Lemma 2.4.2 on each of the integrals above, having in mind that

$$\frac{1-4\theta}{4(1-\theta)} < 1 \quad \text{and} \quad \frac{3}{2(1-\theta)} > 1,$$

we obtain

$$\begin{aligned}\|Gu(t,\cdot)\|_{L^2} &\lesssim \left((1+t)^{-\frac{1-4\theta}{4(1-\theta)}+1-\gamma} + (1+t)^{-\gamma}\right)\|u\|^p_{X(T)} \\ &\lesssim (1+t)^{-\frac{1-4\theta}{4(1-\theta)}+1-\gamma}\|u\|^p_{X(T)}. \qquad (73)\end{aligned}$$

Similarly, we estimate the $\dot{H}^1$ and $\dot{H}^2$ norms. Setting $n=1$, $j=0$, $k=1$, $\varphi=|u|^p$ and $s=3$ in Corollary 2.3.6 yields

$$\begin{aligned}\|Gu(t,\cdot)\|_{\dot{H}^1} &\lesssim \int_0^t (1+t-\tau)^{-\frac{3-4\theta}{4(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}\|u(s,\cdot)\|^p_{L^p}\, ds\, d\tau \\ &\quad + \int_0^t (1+t-\tau)^{-\frac{2}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}\|u(s,\cdot)\|^p_{L^{2p}}\, ds\, d\tau. \qquad (74)\end{aligned}$$

Since we have already estimated the norms in L^p and L^{2p} in (69) and (70), we use these estimates and apply Lemma 2.4.2, thus obtaining

$$\begin{aligned}\|Gu(t,\cdot)\|_{\dot{H}^1} &\lesssim \left((1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma} + \ln(2+t)(1+t)^{-\gamma}\right)\|u\|^p_{X(T)} \\ &\lesssim (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma}\|u\|^p_{X(T)}, \qquad (75)\end{aligned}$$

where above we already considered the worst case scenario $\theta=0$, for which the logarithmic term appears. We observe again that on application of Lemma 2.4.2, condition $p>p_c$ implies $\alpha, \tilde{\alpha}>1$. Also, it holds that $\frac{3-4\theta}{4(1-\theta)}<1$, while $\frac{2}{2(1-\theta)}\geq 1$, so the decay rate coming from high-frequency region is fast enough so it doesn't interfere on the results.

For the norm in $\dot{H}^2$, we set $n=1$, $j=0$, $k=2$ and $s=3$ in Corollary 2.3.6, which gives us

$$\begin{aligned}\|Gu(t,\cdot)\|_{\dot{H}^2} &\lesssim \int_0^t (1+t-\tau)^{-\frac{5-4\theta}{4(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}\|u(s,\cdot)\|^p_{L^p}\, ds\, d\tau \\ &\quad + \int_0^t (1+t-\tau)^{-\frac{1}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}\|u(s,\cdot)\|^p_{L^{2p}}\, ds\, d\tau. \qquad (76)\end{aligned}$$

Using again estimates (69) and (70) for the L^p and L^{2p} norms and applying Lemma 2.4.2, we see that

$$\begin{aligned}\|Gu(t,\cdot)\|_{\dot{H}^2} &\lesssim \left((1+t)^{-\gamma} + (1+t)^{-\frac{1}{2(1-\theta)}+1-\gamma}\right)\|u\|^p_{X(T)} \\ &\lesssim (1+t)^{-\frac{1}{2(1-\theta)}+1-\gamma}\|u\|^p_{X(T)}. \qquad (77)\end{aligned}$$

Next, we estimate the two norms that concern the first order time derivative. Since $E_1(0,\cdot)\equiv 0$, we get

$$\|\partial_t(Gu)(t,\cdot)\|_{\dot{H}^k} \lesssim \int_0^t \int_0^\tau (\tau-s)^{-\gamma}\|\partial_t E_1(t-\tau,\cdot) * |u(s,\cdot)|^p\|_{\dot{H}^k}\, ds\, d\tau.$$

Hence, setting $j = 1$, $k = 0, 1$, $\varphi = |u|^p$ and $s = 3$ on Corollary 2.3.6, we get

$$\begin{aligned}\|\partial_t(Gu)(t,\cdot)\|_{\dot H^k} \lesssim &\int_0^t (1+t-\tau)^{-\frac{1+2k-4\theta}{4(1-\theta)}-1} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|_{L^p}^p \; ds\, d\tau \\ &+ \int_0^t (1+t-\tau)^{-\frac{2-k}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|_{L^{2p}} \; ds\, d\tau.\end{aligned} \tag{78}$$

We shall distinguish two cases. If $k = 0$ then, using $\theta < 1/4$ we find

$$\frac{1-4\theta}{4(1-\theta)} + 1 > 1, \qquad \frac{1}{1-\theta} \begin{cases} = 1 & \text{if } \theta = 0, \\ > 1 & \text{if } \theta > 0, \end{cases}$$

so that, applying Lemma 2.4.2 we find

$$\|\partial_t(Gu)(t,\cdot)\|_{L^2} \lesssim \tilde h_0(t) \; \|u\|_{X(T)}^p .$$

On the other hand, if $k = 1$, then, using $\theta < 1/2$ we find

$$\frac{3-4\theta}{4(1-\theta)} + 1 > 1 > \frac{1}{2(1-\theta)},$$

so that, applying Lemma 2.4.2 with

$$\omega = \min\left\{ \frac{3-4\theta}{4(1-\theta)} + 1, \; \frac{1}{2(1-\theta)} \right\} = \frac{1}{2(1-\theta)},$$

we find

$$\|\partial_t(Gu)(t,\cdot)\|_{\dot H^1} \lesssim (1+s)^{-\frac{1}{2(1-\theta)}+1-\gamma} \; \|u\|_{X(T)}^p .$$

Combining the five derived estimates, we are able to estimate

$$\|Gu\|_{X(T)} \lesssim \|u\|_{X(T)}^p . \tag{79}$$

Now it remains to show that G is a contraction in $X(T)$, that is, the second estimate in (67). We will use the Mean Value Inequality to obtain

$$||u(t,\cdot)|^p - |v(t,\cdot)|^p| \lesssim |u(t,\cdot) - v(t,\cdot)|\big(|u(t,\cdot)|^{p-1} + |v(t,\cdot)|^{p-1}\big). \tag{80}$$

Hence, by Hölder inequality and the estimate (69) for $\|\cdot\|_{L^p}$, we get

$$\begin{aligned}&\big\| |u(s,\cdot)|^p - |v(s,\cdot)|^p \big\|_{L^1} \\ &\quad\lesssim \Big\| |u(s,\cdot) - v(s,\cdot)| \big(|u(s,\cdot)|^{p-1} + |v(s,\cdot)|^{p-1}\big) \Big\|_{L^1} \\ &\quad\lesssim \|u(s,\cdot) - v(s,\cdot)\|_{L^p} \left(\|u(s,\cdot)\|_{L^p}^{p-1} + \|v(s,\cdot)\|_{L^p}^{p-1} \right) \\ &\quad\lesssim (1+s)^{-\frac{\alpha}{p}} \|u-v\|_{X(T)} (1+s)^{-\frac{(p-1)\alpha}{p}} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1} \right) \\ &\quad= (1+s)^{-\alpha} \|u-v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1} \right),\end{aligned} \tag{81}$$

and from Hölder and (70)

$$\left\| |u(s,\cdot)|^p - |v(s,\cdot)|^p \right\|_{L^2} \lesssim (1+s)^{-\tilde{\alpha}} \|u-v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1} \right). \tag{82}$$

Using these estimates, one can proceed as we did for $\|Gu(t,\cdot)\|_{X(T)}$, to get

$$\|Gu - Gv\|_{X(T)} \lesssim \|u-v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1} \right).$$

Now, for the case where $\theta \in \left[\frac{1}{4}, \frac{1}{2}\right)$, the estimate for $\|u(t,\cdot)\|_{L^2}$ changes, because condition $\frac{n}{2} + k - 2\theta > 0$ in (43) doesn't hold anymore when $k = 0$. We must then change the decay rate in the L^2–norm of both u and u_t, and also include another norm on $X(T)$ so this loss on decay won't interfere in the critical exponent p_c.

For $T > 0$, we equip $X(T)$ with the norm

$$\begin{aligned} \|v\|_{X(T)} := \sup_{t\in[0,T]} \Bigg(& h_0(t)^{-1} \|v(t,\cdot)\|_{L^2} + h_1(t)^{-1} \|v(t,\cdot)\|_{\dot{H}^{k_1}} \\ & + h_2(t)^{-1} \|v(t,\cdot)\|_{\dot{H}^1} + h_3(t)^{-1} \|v(t,\cdot)\|_{\dot{H}^2} \\ & + \tilde{h}_0(t)^{-1} \|v_t(t,\cdot)\|_{L^2} + \tilde{h}_1(t)^{-1} \|v_t(t,\cdot)\|_{\dot{H}^1} \Bigg), \end{aligned}$$

where $k_1 = \frac{1}{2} - \frac{1}{p}$, and

$$\begin{aligned} h_0(t) &= l_0(t)(1+t)^{2-\frac{1}{4\theta}-\gamma}, \\ h_1(t) &= (1+t)^{-\frac{1-\frac{1}{p}-2\theta}{2(1-\theta)}+1-\gamma}, \\ h_2(t) &= (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma}, \\ \tilde{h}_0(t) &= l_0(t)(1+t)^{-\frac{1}{4\theta}+1-\gamma}, \\ h_3(t) &= \tilde{h}_1(t) = (1+t)^{-\frac{1}{2(1-\theta)}+1-\gamma}, \end{aligned}$$

with

$$l_0(t) = \begin{cases} \ln(e+t), & \text{if } \theta = \frac{1}{4} \\ 1, & \text{otherwise.} \end{cases}$$

The linear estimates give us again (55). In fact, setting $j = 0,\ k = 0,\ k_1,\ 1, 2$ and then $j = 1,\ k = 0, 1$ with $s = s_c$ in Corollary 2.3.4 we estimate $u^{lin}(t,x)$ in the six norms that define $\|\cdot\|_{X(T)}$:

$$\begin{aligned} \left\| u^{lin}(t,\cdot) \right\|_{L^2} &\lesssim l_0(t)(1+t)^{1-\frac{1}{4\theta}} \|(u_0,u_1)\|_{L^1\times L^1} \\ &\quad + (1+t)^{-\frac{s}{2(1-\theta)}} \|(u_0,u_1)\|_{\dot{H}^s\times \dot{H}^{s-1}} \\ &\lesssim l_0(t)(1+t)^{2-\frac{1}{4\theta}-\gamma} \|(u_0,u_1)\|_{\mathcal{A}}; \end{aligned}$$

$$\begin{aligned}\left\|u^{lin}(t,\cdot)\right\|_{\dot{H}^{k_1}} &\lesssim (1+t)^{-\frac{1-\frac{1}{p}-2\theta}{2(1-\theta)}} \|(u_0,u_1)\|_{L^1\times L^1} \\ &\quad + (1+t)^{-\frac{s-\frac{1}{2}+\frac{1}{p}}{2(1-\theta)}} \|(u_0,u_1)\|_{\dot{H}^s\times\dot{H}^{s-1}} \\ &\lesssim (1+t)^{-\frac{1-\frac{1}{p}-2\theta}{2(1-\theta)}+1-\gamma} \|(u_0,u_1)\|_{\mathcal{A}};\end{aligned}$$

$$\begin{aligned}\left\|u^{lin}(t,\cdot)\right\|_{\dot{H}^{1}} &\lesssim (1+t)^{-\frac{3-4\theta}{4(1-\theta)}} \|(u_0,u_1)\|_{L^1\times L^1} \\ &\quad + (1+t)^{-\frac{s-1}{2(1-\theta)}} \|(u_0,u_1)\|_{\dot{H}^s\times\dot{H}^{s-1}} \\ &\lesssim (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma} \|(u_0,u_1)\|_{\mathcal{A}};\end{aligned}$$

$$\begin{aligned}\left\|u^{lin}(t,\cdot)\right\|_{\dot{H}^{2}} &\lesssim (1+t)^{-\frac{5-4\theta}{4(1-\theta)}} \|(u_0,u_1)\|_{L^1\times L^1} \\ &\quad + (1+t)^{-\frac{s-2}{2(1-\theta)}} \|(u_0,u_1)\|_{\dot{H}^s\times\dot{H}^{s-1}} \\ &\lesssim (1+t)^{-\frac{1}{2(1-\theta)}+1-\gamma} \|(u_0,u_1)\|_{\mathcal{A}};\end{aligned}$$

$$\begin{aligned}\left\|\partial_t u^{lin}(t,\cdot)\right\|_{L^2} &\lesssim (1+t)^{-\frac{1}{4\theta}} \|(u_0,u_1)\|_{L^1\times L^1} \\ &\quad + (1+t)^{-\frac{s-1}{2(1-\theta)}} \|(u_0,u_1)\|_{\dot{H}^s\times\dot{H}^{s-1}} \\ &\lesssim l_0(t)(1+t)^{-\frac{1}{4\theta}+1-\gamma} \|(u_0,u_1)\|_{\mathcal{A}};\end{aligned}$$

$$\begin{aligned}\left\|\partial_t u^{lin}(t,\cdot)\right\|_{\dot{H}^{1}} &\lesssim (1+t)^{-\frac{3-4\theta}{4(1-\theta)}-1} \|(u_0,u_1)\|_{L^1\times L^1} \\ &\quad + (1+t)^{-\frac{s-2}{2(1-\theta)}} \|(u_0,u_1)\|_{\dot{H}^s\times\dot{H}^{s-1}} \\ &\lesssim (1+t)^{-\frac{1}{2(1-\theta)}+1-\gamma} \|(u_0,u_1)\|_{\mathcal{A}}.\end{aligned}$$

We remark that, regarding the $\dot{H}^{k_1}$ norm, it could be the case, depending on p, that either $\frac{n}{2}+k_1-2\theta$ is positive, zero or negative, and this would change the linear estimate to be applied. However, we assert that only the first case is possible, since we are assuming $p>p_c$. In fact, observing the equivalence

$$\frac{n}{2}+k_1-2\theta\le 0 \Leftrightarrow \frac{1}{2}+\left(\frac{1}{2}-\frac{1}{p}\right)-2\theta\le 0 \Leftrightarrow p\le\frac{1}{1-2\theta},$$

and the fact that

$$p_c(1,\gamma,\theta)=1+\frac{2(1+(1-\gamma)(1-\theta))}{2\gamma(1-\theta)-1}$$

is decreasing with respect to γ (because when $\gamma\nearrow 1$, its numerator decreases and its denominator increases), we have

$$p>p_c(1,\gamma,\theta)>p_c(1,1,\theta)=1+\frac{2}{1-2\theta}>\frac{1}{1-2\theta}.$$

Therefore, our assumption $p>p_c$ implies that $\frac{1}{2}+k_1-2\theta>0$.

To prove (67), we first set $j = 0$, $k = 0, k_1, 1, 2$, $\varphi = |u|^p$ and $s = 3$ in Corollary 2.3.6, getting

$$\|Gu(t,\cdot)\|_{L^2} \lesssim \int_0^t I_0(t-\tau)(1+t-\tau)^{1-\frac{1}{4\theta}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|_{L^p}^p \ \mathrm{d}s\,\mathrm{d}\tau$$
$$+ \int_0^t (1+t-\tau)^{-\frac{3}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|_{L^{2p}}^p \ \mathrm{d}s\,\mathrm{d}\tau, \tag{83}$$

and for $k = k_1, 1, 2$,

$$\|Gu(t,\cdot)\|_{\dot{H}^k} \lesssim \int_0^t (1+t-\tau)^{-\frac{1+2k-4\theta}{4(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|_{L^p}^p \ \mathrm{d}s\,\mathrm{d}\tau$$
$$+ \int_0^t (1+t-\tau)^{-\frac{3-k}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|_{L^{2p}}^p \ \mathrm{d}s\,\mathrm{d}\tau. \tag{84}$$

To estimate the L^p norm, we use the estimate from Sobolev's Embedding. This is the reason why we included the $\dot{H}^{k_1}$–norm in $X(T)$.

$$\|u(s,\cdot)\|_{L^p}^p \lesssim \|u(s,\cdot)\|_{\dot{H}^{k_1}}^p \lesssim (1+s)^{-\alpha} \|u\|_{X(T)}^p,$$

with $\alpha := \frac{(1-2\theta)p-1}{2(1-\theta)} - (1-\gamma)p$. Then $\alpha > 1$ if, and only if, $p > p_c$.

For the L^{2p} norm, we use again Sobolev's Embedding and apply Interpolation of Bessel Potential Spaces between $\dot{H}^{k_1}$ and $\dot{H}^1$:

$$\|u(s,\cdot)\|_{L^{2p}}^p \lesssim \|u(s,\cdot)\|_{\dot{H}^{\frac{1}{2}-\frac{1}{2p}}}^p \lesssim \ \|u(s,\cdot)\|_{\dot{H}^{k_1}}^{\beta p} \|u(s,\cdot)\|_{\dot{H}^1}^{(1-\beta)p},$$

for some $\beta \in (0,1)$ that verifies $\frac{1}{2} - \frac{1}{2p} = \beta\left(\frac{1}{2} - \frac{1}{p}\right) + (1-\beta)$. Then, we obtain

$$\|u(s,\cdot)\|_{L^{2p}}^p \lesssim (1+s)^{\left(-\frac{1-\frac{1}{p}-2\theta}{2(1-\theta)}+1-\gamma\right)\beta p} (1+s)^{\left(-\frac{3-4\theta}{4(1-\theta)}+1-\gamma\right)(1-\beta)p} \|u\|_{X(T)}^p$$
$$\approx (1+s)^{-\tilde{\alpha}} \|u\|_{X(T)}^p, \tag{85}$$

with

$$\begin{aligned}
\tilde{\alpha} &:= \left(\frac{1-\frac{1}{p}-2\theta}{2(1-\theta)} - (1-\gamma)\right)\beta p + \left(\frac{3-4\theta}{4(1-\theta)} - (1-\gamma)\right)(1-\beta)p \\
&= \left(\frac{1+2\left(\frac{1}{2}-\frac{1}{p}\right)-4\theta}{4(1-\theta)} - (1-\gamma)\right)\beta p + \left(\frac{1+2-4\theta}{4(1-\theta)} - (1-\gamma)\right)(1-\beta)p \\
&= \frac{(1-4\theta)p}{4(1-\theta)} + \frac{\left(2\beta\left(\frac{1}{2}-\frac{1}{p}\right)+2(1-\beta)\right)p}{4(1-\theta)} - (1-\gamma)p \\
&= \frac{\left(1+2\left(\frac{1}{2}-\frac{1}{2p}\right)-4\theta\right)p}{4(1-\theta)} - (1-\gamma)p \\
&= \frac{(2-4\theta)p-1}{4(1-\theta)} - (1-\gamma)p \\
&= \frac{(1-2\theta)p-1}{2(1-\theta)} - (1-\gamma)p + \frac{1}{4(1-\theta)} \\
&= \alpha + \frac{1}{4(1-\theta)}.
\end{aligned} \tag{86}$$

This means again that the condition $p > p_c$ is enough so we can estimate both L^p and L^{2p} norms. Using these estimates and applying Lemma 2.4.2 in (83), observing that $I_0(t)$ is nondecreasing, we get

$$\begin{aligned}
\|Gu(t,\cdot)\|_{L^2} &\lesssim I_0(t)\bigg(\int_0^t (1+t-\tau)^{1-\frac{1}{4\theta}}\int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\alpha}\,ds\,d\tau \\
&\quad + \int_0^t (1+t-\tau)^{-\frac{3}{2(1-\theta)}}\int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\tilde{\alpha}}\,ds\,d\tau\bigg)\|u\|^p_{X(T)} \\
&\lesssim \Big(I_0(t)(1+t)^{1-\frac{1}{4\theta}+1-\gamma} + (1+t)^{-\gamma}\Big)\|u\|^p_{X(T)} \\
&\lesssim I_0(t)(1+t)^{2-\frac{1}{4\theta}-\gamma}\|u\|^p_{X(T)}, \qquad (87)
\end{aligned}$$

and applying Lemma 2.4.2 in (84) for $k = k_1, 1, 2$,

$$\begin{aligned}
\|Gu(t,\cdot)\|_{\dot{H}^{k_1}} &\lesssim \bigg(\int_0^t (1+t-\tau)^{-\frac{1+2k_1-4\theta}{4(1-\theta)}}\int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\alpha}\,ds\,d\tau \\
&\quad + \int_0^t (1+t-\tau)^{-\frac{3-k_1}{2(1-\theta)}}\int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\tilde{\alpha}}\,ds\,d\tau\bigg)\|u\|^p_{X(T)} \\
&\lesssim \bigg((1+t)^{-\frac{1-\frac{1}{p}-2\theta}{2(1-\theta)}+1-\gamma} + (1+t)^{-\gamma}\bigg)\|u\|^p_{X(T)} \\
&\lesssim (1+t)^{-\frac{1-\frac{1}{p}-2\theta}{2(1-\theta)}+1-\gamma}\|u\|^p_{X(T)}; \qquad (88)
\end{aligned}$$

$$\begin{aligned}
\|Gu(t,\cdot)\|_{\dot{H}^1} &\lesssim \bigg(\int_0^t (1+t-\tau)^{-\frac{3-4\theta}{4(1-\theta)}}\int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\alpha}\,ds\,d\tau \\
&\quad + \int_0^t (1+t-\tau)^{-\frac{2}{2(1-\theta)}}\int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\tilde{\alpha}}\,ds\,d\tau\bigg)\|u\|^p_{X(T)} \\
&\lesssim \Big((1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma} + (1+t)^{-\gamma}\Big)\|u\|^p_{X(T)} \\
&\lesssim (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma}\|u\|^p_{X(T)}; \qquad (89)
\end{aligned}$$

$$\begin{aligned}
\|Gu(t,\cdot)\|_{\dot{H}^2} &\lesssim \bigg(\int_0^t (1+t-\tau)^{-\frac{5-4\theta}{4(1-\theta)}}\int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\alpha}\,ds\,d\tau \\
&\quad + \int_0^t (1+t-\tau)^{-\frac{1}{2(1-\theta)}}\int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\tilde{\alpha}}\,ds\,d\tau\bigg)\|u\|^p_{X(T)} \\
&\lesssim \Big((1+t)^{-\gamma} + (1+t)^{-\frac{1}{2(1-\theta)}+1-\gamma}\Big)\|u\|^p_{X(T)} \\
&\lesssim (1+t)^{-\frac{1}{2(1-\theta)}+1-\gamma}\|u\|^p_{X(T)}. \qquad (90)
\end{aligned}$$

Summarizing, we get

$$\|Gu(t,\cdot)\|_{\dot{H}^{k_i}} \lesssim h_i(t)\,\|u\|^p_{X(T)}, \quad i = 0,1,2,3 \quad k_0 = 0, k_2 = 1, k_3 = 2.$$

Now we consider the two norms of $\partial_t(Gu)(t,\cdot)$. For the $\|\cdot\|_{\dot{H}^1}$ norm we may proceed as in case $\theta \in \left[0, \frac{1}{4}\right)$. On the other hand, for $\|\partial_t(Gu)(t,\cdot)\|_{L^2}$, due to $n=1$ and $\theta \in [1/4, 1/2)$, we find

$$\frac{1}{4\theta}\begin{cases} = 1 & \text{if } \theta = \frac{1}{4}, \\ < 1 & \text{if } \theta > \frac{1}{4}, \end{cases} \qquad \frac{1}{1-\theta} > 1,$$

which is the reason we included the logarithmic term $l_0(t)$ in the definition of $\tilde{h}_0(t)$. So, applying Lemma 2.4.2, we find

$$\begin{aligned} \|\partial_t(Gu)(t,\cdot)\|_{L^2} &\lesssim \left(\int_0^t (1+t-\tau)^{-\frac{1}{4\theta}} \int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\alpha}\, ds\, d\tau \right. \\ &\quad \left. + \int_0^t (1+t-\tau)^{-\frac{2}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\tilde{\alpha}}\, ds\, d\tau \right) \|u\|^p_{X(T)} \\ &\lesssim \begin{cases} \left((1+t)^{-\frac{1}{4\theta}+1-\gamma} + (1+t)^{-\gamma}\right) \|u\|^p_{X(T)}, & \theta \in \left(\frac{1}{4}, \frac{1}{2}\right), \\ (\ln(2+t)(1+t)^{-\gamma} + (1+t)^{-\gamma}) \|u\|^p_{X(T)}, & \theta = \frac{1}{4}, \end{cases} \\ &\lesssim \begin{cases} (1+t)^{1-\frac{1}{4\theta}-\gamma} \|u\|^p_{X(T)}, & \theta \in \left(\frac{1}{4}, \frac{1}{2}\right), \\ \ln(2+t)(1+t)^{-\gamma} \|u\|^p_{X(T)}, & \theta = \frac{1}{4}, \end{cases} \end{aligned} \tag{91}$$

that is,

$$\|\partial_t(Gu)(t,\cdot)\|_{L^2} \lesssim \tilde{h}_0(t)\ \|u\|^p_{X(T)}.$$

Combining the previous estimates, we get

$$\|Gu\|_{X(T)} \lesssim \|u\|^p_{X(T)}.$$

The argument that G is a contraction map in $X(T)$ is the same as in the case $\theta \in \left[0, \frac{1}{4}\right)$. Hence, we showed that, in both cases $\theta \in [0, \frac{1}{4})$ and $\theta \in [\frac{1}{4}, \frac{1}{2})$, it is true that

$$\begin{aligned} \|Gu\|_{X(T)} &\lesssim \|u\|^p_{X(T)}, \\ \|Gu - Gv\|_{X(T)} &\lesssim \|u-v\|_{X(T)} \left(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)} \right), \end{aligned}$$

and then Picard's Contraction Theorem guarantees that there exists a unique fixed point inside a sufficiently small ball of $X(T)$. In other words, there exists an $\varepsilon > 0$ such that, if $\|(u_0, u_1)\|_{\mathcal{A}} \le \varepsilon$, there exists $u \in X(T)$ such that

$$u(t,x) = u^{lin}(t,x) + Gu(t,x),$$

that is, u is the solution to (1). Moreover, since all the obtained estimates are uniform with respect to T, we can take the limit $T \to \infty$, which implies that $u(t,x)$ is a global in-time solution to (1) and satisfies all the estimates given in our theorem's statement. ∎

3.2 CASE $n = 2$

The bidimensional case $n = 2$ possess a structure that is similar to the one-dimensional case, since the decay rate coming from the low-frequency is relatively slow, and we do not have trouble to control the high-frequency decay rates so they at least match the low-frequency decay rates. In particular, the proof for $n = 2$ follows the same steps as the proof for $n = 1$ and $\theta \in \left[0, \frac{1}{4}\right)$, because in both these cases the condition

$$\frac{n}{2} + k - 2\theta > 0$$

holds for every non-negative k, leading us to use the same kind of estimates in Corollaries 2.3.4 and 2.3.6. We decided to present these results in separate theorems only to organize the work in a cleaner and easier to read structure.

Theorem 3.2.1 *Assume $n = 2$, $\theta \in \left[0, \frac{1}{2}\right)$, $\gamma \in (0, 1)$, $p > p_c$ and $s = s_c$, with p_c as in (2) and s_c as in (3). Then, there exists $\varepsilon > 0$ such that, for initial data*

$$(u_0, u_1) \in \mathcal{A} := \left(H^s(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)\right) \times \left(H^{s-1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)\right), \tag{92}$$

with $\|(u_0, u_1)\|_{\mathcal{A}} \leq \varepsilon$, there exists a global solution to the problem (1), $u \in \mathcal{C}\left([0, \infty), H^2\right) \cap \mathcal{C}^1\left([0, \infty), H^1\right)$. Moreover, the following estimates hold:

$$\begin{aligned}
\|u(t, \cdot)\|_{L^2} &\lesssim (1 + t)^{-\frac{1-2\theta}{2(1-\theta)}+1-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}, \\
\|u(t, \cdot)\|_{\dot{H}^1} &\lesssim \ln(2 + t)(1 + t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}, \\
\|u(t, \cdot)\|_{\dot{H}^2} &\lesssim (1 + t)^{-\frac{1}{2(1-\theta)}+1-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}, \\
\|\partial_t u(t, \cdot)\|_{L^2} &\lesssim \begin{cases} \ln(2 + t)(1 + t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}, & \text{if } \theta = 0, \\ (1 + t)^{-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}, & \text{if } \theta \in \left(0, \frac{1}{2}\right), \end{cases} \\
\|\partial_t u(t, \cdot)\|_{\dot{H}^1} &\lesssim (1 + t)^{-\frac{1}{2(1-\theta)}+1-\gamma} \|(u_0, u_1)\|_{\mathcal{A}}.
\end{aligned}$$

Proof: The proof for dimension $n = 2$ is similar to the first part of case $n = 1$, because the decay speed coming from the low-frequency linear estimates is still very slow. For $T > 0$. we define the Banach space

$$X(T) := \mathcal{C}\left([0, T], H^2\right) \cap \mathcal{C}^1\left([0, T], H^1\right),$$

equipped with the norm

$$\begin{aligned}
\|v\|_{X(T)} := \sup_{t\in[0,T]} \Bigg(& h_0(t)^{-1} \|v(t, \cdot)\|_{L^2} + h_1(t)^{-1} \|v(t, \cdot)\|_{\dot{H}^1} + h_2(t)^{-1} \|v(t, \cdot)\|_{\dot{H}^2} \\
& + \tilde{h}_0(t)^{-1} \|v_t(t, \cdot)\|_{L^2} + \tilde{h}_1(t)^{-1} \|v_t(t, \cdot)\|_{\dot{H}^1} \Bigg),
\end{aligned}$$

where

$$\begin{aligned}
h_0(t) &= (1+t)^{-\frac{1-2\theta}{2(1-\theta)}+1-\gamma},\\
h_1(t) &= \ln(2+t)(1+t)^{-\gamma},\\
\tilde{h}_0(t) &= \begin{cases}(1+t)^{-\gamma} & \text{if } \theta > 0,\\ (1+t)^{-\gamma}\ln(2+t) & \text{if } \theta = 0,\end{cases}\\
h_2(t) &= \tilde{h}_1(t) = (1+t)^{-\frac{1}{2(1-\theta)}+1-\gamma}.
\end{aligned}$$

To see that $u^{lin}(t,x)$ is in $X(T)$ and satisfies

$$\left\|u^{lin}\right\|_{X(T)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}},$$

we set $n=2$, $k=0,1,2$, $j=0$ and $s=s_c$ in Corollary 2.3.4, obtaining

$$\begin{aligned}
\left\|u^{lin}(t,\cdot)\right\|_{L^2} &\lesssim (1+t)^{-\frac{1-2\theta}{2(1-\theta)}}\|(u_0,u_1)\|_{L^1\times L^1} + (1+t)^{-\frac{s}{2(1-\theta)}}\|(u_0,u_1)\|_{\dot{H}^s\times\dot{H}^{s-1}}\\
&\lesssim (1+t)^{-\frac{1-2\theta}{2(1-\theta)}}\|(u_0,u_1)\|_{\mathcal{A}}; \qquad (93)
\end{aligned}$$

$$\begin{aligned}
\left\|u^{lin}(t,\cdot)\right\|_{\dot{H}^1} &\lesssim (1+t)^{-1}\|(u_0,u_1)\|_{L^1\times L^1} + (1+t)^{-\frac{s-1}{2(1-\theta)}}\|(u_0,u_1)\|_{\dot{H}^s\times\dot{H}^{s-1}}\\
&\lesssim (1+t)^{-\gamma}\|(u_0,u_1)\|_{\mathcal{A}}; \qquad (94)
\end{aligned}$$

$$\begin{aligned}
\left\|u^{lin}(t,\cdot)\right\|_{\dot{H}^2} &\lesssim (1+t)^{-\frac{3-2\theta}{2(1-\theta)}}\|(u_0,u_1)\|_{L^1\times L^1} + (1+t)^{-\frac{s-2}{2(1-\theta)}}\|(u_0,u_1)\|_{\dot{H}^s\times\dot{H}^{s-1}}\\
&\lesssim (1+t)^{-\frac{1}{2(1-\theta)}}\|(u_0,u_1)\|_{\mathcal{A}}. \qquad (95)
\end{aligned}$$

Hence, in all of the three cases above, since $\gamma\in(0,1)$, $\theta\in[0,\frac{1}{2})$ and $\ln(2+t)\geq 1$ for $t>0$, we get

$$\left\|u^{lin}(t,\cdot)\right\|_{\dot{H}^k} \lesssim h_k(t)\|(u_0,u_1)\|_{\mathcal{A}}, \quad k=0,1,2.$$

Setting $n=2, k=0,1$, $j=1$ and $s=s_c$ in Corollary 2.3.4, we get the estimates for the remaining two norms:

$$\begin{aligned}
\left\|\partial_t u^{lin}(t,\cdot)\right\|_{L^2} &\lesssim (1+t)^{-\frac{1-2\theta}{2(1-\theta)}-1}\|(u_0,u_1)\|_{L^1\times L^1} + (1+t)^{-\frac{s-1}{2(1-\theta)}}\|(u_0,u_1)\|_{\dot{H}^s\times\dot{H}^{s-1}}\\
&\lesssim (1+t)^{-\gamma}\|(u_0,u_1)\|_{\mathcal{A}}; \qquad (96)
\end{aligned}$$

$$\begin{aligned}
\left\|\partial_t u^{lin}(t,\cdot)\right\|_{\dot{H}^1} &\lesssim (1+t)^{-2}\|(u_0,u_1)\|_{L^1\times L^1} + (1+t)^{-\frac{s-2}{2(1-\theta)}}\|(u_0,u_1)\|_{\dot{H}^s\times\dot{H}^{s-1}}\\
&\lesssim (1+t)^{-\frac{1}{2(1-\theta)}+1-\gamma}\|(u_0,u_1)\|_{\mathcal{A}}. \qquad (97)
\end{aligned}$$

And again, using the fact that $\theta\in[0,\frac{1}{2})$, $\gamma\in(0,1)$, and also $ln(2+t)\geq 1$ for $t>0$, we get

$$\left\|\partial_t u^{lin}(t,\cdot)\right\|_{\dot{H}^k} \lesssim \tilde{h}_k(t)\|(u_0,u_1)\|_{\mathcal{A}}, \quad k=0,1.$$

Combining these five estimates, we are able to estimate $u^{lin}(t,x)$ in $X(T)$ as

$$\left\| u^{lin} \right\|_{X(T)} \lesssim \|(u_0, u_1)\|_{\mathcal{A}}.$$

Next, we define G in $X(T)$ as in (52) and we prove that G satisfies (67). Setting $n = 2$, $j = 0$, $k = 0,1,2$, $\varphi = |u|^p$ and $s = 3$ in Corollary 2.3.6, we have

$$\begin{aligned} \|Gu(t,\cdot)\|_{\dot{H}^k} \lesssim & \int_0^t (1+t-\tau)^{-\frac{1+k-2\theta}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|_{L^p}^p \, ds \, d\tau \\ & + \int_0^t (1+t-\tau)^{-\frac{3-k}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|_{L^{2p}}^p \, ds \, d\tau. \end{aligned} \tag{98}$$

So, again, we must estimate u in the norms L^p and L^{2p}. Applying Gagliardo-Nirenberg's inequality (47) with $n = 2$ and $\kappa = 1$, we see that

$$\begin{aligned} \|u(s,\cdot)\|_{L^p}^p &\lesssim \|u(s,\cdot)\|_{L^2}^{(1-\theta_1(p))p} \|u(s,\cdot)\|_{\dot{H}^1}^{\theta_1(p)p} \\ &\lesssim (1+s)^{\left(-\frac{1-2\theta}{2(1-\theta)}+1-\gamma\right)2} \left(\ln(2+s)(1+s)^{-\gamma}\right)^{p-2} \|u\|_{X(T)}^p \\ &\lesssim (1+s)^{-\alpha} (\ln(2+s))^{p-2} \|u\|_{X(T)}^p, \end{aligned} \tag{99}$$

where

$$\alpha = \left(\frac{1-2\theta}{2(1-\theta)} - (1-\gamma)\right) 2 + \gamma(p-2) = -\frac{1}{1-\theta} + \gamma p.$$

We remark that

$$\alpha > 1 \iff \gamma p > 1 + \frac{1}{1-\theta} \iff p > \frac{2-\theta}{\gamma(1-\theta)}.$$

So, assuming that $p > p_c$, we find that $\alpha > 1$, and choosing $\delta \in (0, \alpha - 1)$,

$$\|u(s,\cdot)\|_{L^p}^p \lesssim (1+s)^{-\alpha+\delta} \|u\|_{X(T)}^p, \tag{100}$$

with $\alpha - \delta > 1$. For the L^2 norm, applying Gagliardo-Nirenberg again, we will get an exponent $\tilde{\alpha}$ which is greater than α obtained on estimating the L^p-norm:

$$\begin{aligned} \|u(s,\cdot)\|_{L^{2p}}^p &\lesssim \|u(s,\cdot)\|_{L^2}^{(1-\theta_1(2p))p} \|u(s,\cdot)\|_{\dot{H}^1}^{\theta_1(2p)p} \\ &\lesssim (1+s)^{\left(-\frac{1-2\theta}{2(1-\theta)}+1-\gamma\right)} \left(\ln(2+s)(1+s)^{-\gamma}\right)^{p-1} \|u\|_{X(T)}^p \\ &\lesssim (1+s)^{-\tilde{\alpha}} (\ln(2+s))^{p-1} \|u\|_{X(T)}^p, \end{aligned} \tag{101}$$

where

$$\tilde{\alpha} = \frac{1-2\theta}{2(1-\theta)} - (1-\gamma) + \gamma(p-1) = -\frac{1}{2(1-\theta)} + \gamma p = \alpha + \frac{1}{2(1-\theta)}.$$

Therefore, the condition $p > p_c$ is already enough to ensure that $\tilde{\alpha} > 1$.

With these estimates, we are now ready to apply Lemma 2.4.2 in (98). We first observe that

$$\frac{1+k-2\theta}{2(1-\theta)} \begin{cases} < 1, & \text{if } k = 0 \\ = 1, & \text{if } k = 1 \end{cases}$$

and that

$$\omega = \min\left\{\frac{1+k-2\theta}{2(1-\theta)}, \frac{3-k}{2(1-\theta)}\right\} = \begin{cases} \frac{1-2\theta}{2(1-\theta)}, & k=0 \\ 1, & k=1 \\ \frac{1}{2(1-\theta)}, & k=2. \end{cases}$$

Therefore,

$$\begin{aligned} \|Gu(t,\cdot)\|_{\dot{H}^k} &\lesssim \left[\int_0^t (1+t-\tau)^{-\frac{1+k-2\theta}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\alpha+\delta}\, \mathrm{d}s\, \mathrm{d}\tau \right. \\ &\quad \left. + \int_0^t (1+t-\tau)^{-\frac{3-k}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\tilde{\alpha}+\delta}\, \mathrm{d}s\, \mathrm{d}\tau\right] \|u\|^p_{X(T)} \\ &\lesssim h_k(t)\, \|u\|^p_{X(T)}, \qquad k=0,1,2. \end{aligned} \tag{102}$$

Lastly, the two norms of $\partial_t(Gu)$. For these two norms there are no additional concerns and we get

$$\|\partial_t(Gu)(t,\cdot)\|_{L^2} \lesssim \tilde{h}_0(t)\ \|u\|^p_{X(T)};$$

$$\|\partial_t(Gu)(t,\cdot)\|_{\dot{H}^1} \lesssim (1+s)^{-\frac{1}{2(1-\theta)}+1-\gamma}\ \|u\|^p_{X(T)}.$$

Hence, just as we did in case $n=1$, we combine the five estimates for $Gu(t,x)$, obtaining

$$\|Gu\|_{X(T)} \lesssim \|u\|^p_{X(T)}.$$

Then, to prove that Gu satisfies the second estimate in (67), we argue as in (80), (81) and (82), obtaining

$$\|Gu-Gv\|_{X(T)} \lesssim \|u-v\|_{X(T)}\left(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\right).$$

Therefore, G is a contraction map in $X(T)$ and with an application of Picard's Contraction Theorem as in Theorem 3.1.1, we conclude the proof.

■

3.3 L^p ESTIMATES FOR $p<2$

Studying low-dimensional cases, we have seen that the influence of the nonlinear memory term is not so strong and its interference on the critical exponent is subtle. Also, since the obtained critical exponent is greater than the Fujita exponent

$$p_F = 1 + \frac{2}{n},$$

which equals 3 for $n=1$, and equals 2 for $n=2$, these low-dimensional cases conveniently allow us to work only in the case where $p \geq 2$. For dimension $n \geq 3$, this is no longer true, and there will be cases in which $p \in (p_c, 2)$. We will then need to obtain new estimates for L^p-norms to apply in these cases.

The approach to get these estimates consist on making use of not only the behavior of the fundamental solutions themselves, but also of their derivatives in the Fourier space.

We may divide our study in the low-frequency and high-frequency regions.

3.3.1 Low-Frequency Region

In the Low-Frequency region, we have the following estimates, proved in detail in Appendix B.1:

- $\left|\partial_\xi^\alpha \hat{K}_0(t,\xi)\right| \lesssim |\xi|^{-|\alpha|} e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}$;
- $\left|\partial_\xi^\alpha \hat{K}_1(t,\xi)\right| \lesssim |\xi|^{-2\theta-|\alpha|} e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}$;
- $\left|\partial_\xi^\alpha \hat{E}_1(t,\xi)\right| \lesssim |\xi|^{-2\theta-|\alpha|} e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}$.

We will combine these estimates for $\hat{K}_0$, $\hat{K}_1$ and $\hat{E}_1$ with a lemma to derive pointwise estimates for their inverse Fourier transforms.

Lemma 3.3.1 *Assume that $f \in C_c^\kappa(\mathbb{R}^n)$, for some $\kappa \geq 0$ integer, and that it verifies the estimates*

$$\left|\partial_\xi^\alpha f(\xi)\right| \lesssim |\xi|^{-a} \qquad \forall |\alpha| \leq \kappa,$$

for some $a < n$. Then, $g = \mathcal{F}^{-1} f$ satisfies the estimate $|g(x)| \lesssim (1+|x|)^{-\kappa}$.

Moreover, if $f \in C_c^{\kappa+1}(\mathbb{R}^n)$, and

$$\left|\partial_\xi^\alpha f(\xi)\right| \lesssim |\xi|^{-a_1} \qquad \forall |\alpha| \leq \kappa+1,$$

for some $a_1 \in [n, n+1)$, then

$$|g(x)| \lesssim \begin{cases} (1+|x|)^{-\kappa-(n-a)}, & \text{if } a > a_1 - 1, \\ (1+|x|)^{-\kappa-(n+1-a_1)}, & \text{if } a \leq a_1 - 1 \text{ and } a_1 \in (n, n+1) \\ (1+|x|)^{-\kappa-1}\log(e+|x|), & \text{if } a \leq n-1 \text{ and } a_1 = n. \end{cases}$$

Proof: Firstly, from $a < n$ and $\operatorname{supp} f$ compact, we obtain $\partial_\xi^\alpha f \in L^1$, for $|\alpha| \leq \kappa$. In fact, let M be such that $\operatorname{supp} f \subset \{|\xi| \leq M\}$. Then,

$$\int_{\mathbb{R}^n} \left|\partial_\xi^\alpha f(\xi)\right| \mathrm{d}\xi \leq C \int_{|\xi|\leq M} |\xi|^{-a}\, \mathrm{d}\xi \leq C\, \frac{\omega_n M^{n-a}}{n-a} < +\infty, \tag{103}$$

where ω_n is the volume of the n–dimensional sphere.

Now, observe that the inverse Fourier transform of any integrable function h is uniformly continuous, and $\left\|\mathcal{F}^{-1} h\right\|_{L^\infty} \lesssim \|h\|_{L^1}$. Indeed, this comes directly from

$$\left|(\mathcal{F}^{-1} h)(x)\right| = \left|(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\xi x} h(x)\, \mathrm{d}x\right| \leq (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |h(x)|\, \mathrm{d}x.$$

Therefore, since $g = \mathcal{F}^{-1} f$ and $\left(\mathcal{F}^{-1}(\partial_\xi^\kappa f)\right)(x) = (-ix)^\kappa g(x)$, it follows that

$$\begin{aligned}\left|(1+|x|)^\kappa g(x)\right| &\lesssim |g(x)| + |x|^\kappa |g(x)| = |g(x)| + |(-ix)^\kappa g(x)| \\ &=\lesssim \|f\|_{L^1} + \left\|\partial_\xi^\kappa f\right\|_{L^1} < +\infty.\end{aligned}$$

Hence,

$$|g(x)| \lesssim (1+|x|)^{-\kappa}.$$

Now, let's prove the second claim. We first observe the following identity:

$$e^{ix\xi} = -\sum_{j=1}^{n} \frac{ix_j}{|x|^2} \partial_{\xi_j} e^{ix\xi},$$

obtained by direct differentiation. Now, integrating by parts, we have

$$\begin{aligned} g(x) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} f(\xi)\, \mathrm{d}\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} -\sum_{j=1}^{n} \frac{ix_j}{|x|^2} \partial_{\xi_j} e^{ix\xi} f(\xi)\, \mathrm{d}\xi \\ &= (2\pi)^{-\frac{n}{2}} \sum_{j=1}^{n} -\frac{ix_j}{|x|^2} \int_{\mathbb{R}^n} e^{ix\xi} \partial_{\xi_j} f(\xi)\, \mathrm{d}\xi \\ &= (2\pi)^{-\frac{n}{2}} |x|^{-1} \sum_{j=1}^{n} c_j \int_{\mathbb{R}^n} e^{ix\xi} \partial_{\xi_j} f(\xi)\, \mathrm{d}\xi. \end{aligned} \tag{104}$$

We remark that no boundary terms appear, since f is compactly supported, and that $c_j = -\frac{ix_j}{|x|}$ satisfies $|c_j| \leq 1$. Integrating by parts $\kappa - 1$ more times, we obtain

$$g(x) = (2\pi)^{-n} |x|^{-\kappa} \sum_{|\beta|=\kappa} c_\beta \int_{\mathbb{R}^n} e^{ix\xi} \partial_\xi^\beta f(\xi)\, \mathrm{d}\xi, \tag{105}$$

where the constants c_β depend on each multi-index $\beta \in \mathbb{N}^n$, each one satisfying $|c_\beta| \leq 1$.

Splitting the integral in two parts and applying one more integration by parts in the latter integral, we get

$$\begin{aligned} \int_{\mathbb{R}^n} e^{ix\xi} \partial_\xi^\beta f(\xi)\, \mathrm{d}\xi = \int_{|\xi| \leq |x|^{-1}} e^{ix\xi} \partial_\xi^\beta f(\xi)\, \mathrm{d}\xi - |x|^{-1} \sum_{j=1}^{n} c_{\beta,j} \int_{|\xi|=|x|^{-1}} e^{ix\xi} \partial_\xi^\beta f(\xi)\, \mathrm{d}\xi \\ + |x|^{-1} \sum_{j=1}^{n} c_{\beta,j} \int_{|\xi| \geq |x|^{-1}} e^{ix\xi} \partial_{\xi_j} \partial_\xi^\beta f(\xi)\, \mathrm{d}\xi, \end{aligned} \tag{106}$$

where $c_{\beta,j} = c_\beta c_j$, for each $j = 1, \ldots, n$ and each $|\beta| = \kappa$.

To estimate these three integrals (in absolute value), let $\tilde{M} > |x|^{-1}$ be such that $\operatorname{supp} f \subset \left\{|\xi| \leq \tilde{M}\right\}$. Then,

$$\int_{|\xi|\leq|x|^{-1}} |\partial_\xi^\beta f(\xi)|\,\mathrm{d}\xi \lesssim \int_{|\xi|\leq|x|^{-1}} |\xi|^{-a}\,\mathrm{d}\xi \lesssim |x|^{-(n-a)}, \tag{107}$$

$$\begin{aligned} |x|^{-1}\int_{|\xi|=|x|^{-1}} |\partial_\xi^\beta f(\xi)|\,\mathrm{d}\xi &\lesssim |x|^{-1}\int_{|\xi|=|x|^{-1}} |\xi|^{-a}\,\mathrm{d}\xi \\ &\lesssim |x|^{-1}\left(|\xi|^{-a+(n-1)}\right)\Big|_{|\xi|=|x|^{-1}} = |x|^{-(n-a)}, \end{aligned} \tag{108}$$

$$\begin{aligned} |x|^{-1}\int_{|\xi|\geq|x|^{-1}} |\partial_{\xi_j}\partial_\xi^\beta f(\xi)|\,\mathrm{d}\xi &\lesssim |x|^{-1}\int_{|x|^{-1}\leq|\xi|\leq\tilde{M}} |\xi|^{-a_1}\,\mathrm{d}\xi \\ &\lesssim \begin{cases} |x|^{-(n+1-a_1)}, & \textit{if } a_1 > n, \\ |x|^{-1}\ln(\tilde{M}|x|), & \textit{if } a_1 = n. \end{cases} \end{aligned} \tag{109}$$

Using (107), (108) and (109) in (105), we get

$$|g(x)| \lesssim \begin{cases} (1+|x|)^{-\kappa-(n-a)}, & \text{if } a > a_1 - 1, \\ (1+|x|)^{-\kappa-(n+1-a_1)}, & \text{if } a \leq a_1 - 1 \text{ and } a_1 > n, \\ |x|^{-\kappa-1}\ln(e+|x|), & \text{if } a \leq n-1 \text{ and } a_1 = n. \end{cases} \tag{110}$$

∎

We are now ready to prove a set of L^p–estimates for K_0, K_1 and E_1 in the low-frequency region. In the following, we will assume $|\xi| < \varepsilon_0$, with ε_0 small enough so the estimates obtained in Appendix B.1 will hold.

Lemma 3.3.2 *Assume* $n \in \mathbb{N}$, $\theta \in [0, \frac{1}{2})$, $p > 1$, *and let the operators* $K_0(t,x), K_1(t,x), E_1(t,x)$ *be as in (41) and (45). Consider also a cut function* $\chi_0(\xi) \in C_c^\infty(\mathbb{R}^n)$ *such that* $\operatorname{supp}\chi_0 \subset B(0,\varepsilon_0)$. *Then, for any* $\delta > 0$, *the following estimates hold:*

i) $\left\|\mathcal{F}^{-1}\left(\hat{K}_0(t,\cdot)\chi_0(\cdot)\right)\right\|_{L^p} \lesssim t^{-\frac{n\left(1-\frac{1}{p}\right)}{2(1-\theta)}+\delta}$;

ii) $\left\|\mathcal{F}^{-1}\left(\hat{K}_1(t,\cdot)\chi_0(\cdot)\right)\right\|_{L^p} \lesssim t^{-\frac{n\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}+\delta}$;

iii) $\left\|\mathcal{F}^{-1}\left(\hat{E}_1(t,\cdot)\chi_0(\cdot)\right)\right\|_{L^p} \lesssim t^{-\frac{n\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}+\delta}$.

Proof: Since these estimates for $\hat{K}_0$ and $\hat{K}_1$ are similar, namely

$$\left|\partial_\xi^\alpha \hat{K}_\ell(t,\xi)\right| \lesssim |\xi|^{-2\ell\theta-|\alpha|}e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}, \qquad k = 0,1, \tag{111}$$

we can strike both cases (i) and (ii) at the same time. Also, the estimates for $\hat{K}_1$ and $\hat{E}_1$ are the same, so the result obtained for $\hat{K}_1$ will hold also for $\hat{E}_1$.

Let $\delta > 0$, and set $a = n - \delta(1-\theta) < n$. Then, for $|\xi|$ small enough, (say, $|\xi| \leq \varepsilon_0$) and for $\ell = 0, 1$,

$$\begin{aligned}
\left|\partial_\xi^\alpha \hat{K}_\ell(t,\xi)\right| &\lesssim |\xi|^{-|\alpha|-2\ell\theta} e^{-\frac{t}{2}|\xi|^{2(1-\theta)}} \lesssim |\xi|^{-(n-1)-2\ell\theta} e^{-\frac{t}{2}|\xi|^{2(1-\theta)}} \\
&\lesssim |\xi|^{-a}|\xi|^{1-2\ell\theta-\delta(1-\theta)} e^{-\frac{t}{2}|\xi|^{2(1-\theta)}} \\
&\lesssim |\xi|^{-a} \underbrace{\left(\frac{t}{2}|\xi|^{2(1-\theta)}\right)^{\frac{1-2\ell\theta}{2(1-\theta)}-\frac{\delta}{2}} e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}}_{\textit{bounded}} t^{-\frac{1-2\ell\theta}{2(1-\theta)}+\frac{\delta}{2}} \\
&\lesssim |\xi|^{-a} t^{-\frac{1-2\ell\theta}{2(1-\theta)}+\frac{\delta}{2}}, \qquad \forall |\alpha| \leq n-1.
\end{aligned} \tag{112}$$

Applying Lemma 3.3.1 with $\kappa = n-1$, $f(\xi) = t^{\frac{1-2\ell\theta}{2(1-\theta)}-\frac{\delta}{2}} \hat{K}_\ell(t,\xi)\chi_0(\xi)$,

$$\left|\mathcal{F}^{-1}\left(\hat{K}_\ell(t,\xi)\chi_0(\xi)\right)\right| \lesssim t^{-\frac{1-2\ell\theta}{2(1-\theta)}+\frac{\delta}{2}}(1+|x|)^{-(n-1)}, \qquad \ell = 0, 1. \tag{113}$$

On the other hand, for $|\alpha| \leq n-2$, we can get similar estimates. Set $a = n - \delta(1-\theta) < n$. Then, again for $|\xi| \leq \varepsilon_0$, from estimates (111), for $\ell = 0, 1$, it holds that

$$\begin{aligned}
\left|\partial_\xi^\alpha \hat{K}_\ell(t,\xi)\right| &\lesssim |\xi|^{-|\alpha|-2\ell\theta} e^{-\frac{t}{2}|\xi|^{2(1-\theta)}} \lesssim |\xi|^{-(n-2)-2\ell\theta} e^{-\frac{t}{2}|\xi|^{2(1-\theta)}} \\
&\lesssim |\xi|^{-a}|\xi|^{2-2\ell\theta-\delta(1-\theta)} e^{-\frac{t}{2}|\xi|^{2(1-\theta)}} \\
&\lesssim |\xi|^{-a} \underbrace{\left(\frac{t}{2}|\xi|^{2(1-\theta)}\right)^{\frac{2-2\ell\theta}{2(1-\theta)}-\frac{\delta}{2}} e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}}_{\textit{bounded}} t^{-\frac{2-2\ell\theta}{2(1-\theta)}+\frac{\delta}{2}} \\
&\lesssim |\xi|^{-a} t^{-\frac{2-2\ell\theta}{2(1-\theta)}+\frac{\delta}{2}}, \qquad \forall |\alpha| \leq n-2.
\end{aligned} \tag{114}$$

Therefore, another application of Lemma 3.3.1, with $\kappa = n-2$ and $f(\xi) = t^{\frac{2-2\ell\theta}{2(1-\theta)}-\frac{\delta}{2}} \hat{K}_\ell(t,\xi)\chi_0(\xi)$ yields

$$\left|\mathcal{F}^{-1}\left(\hat{K}_\ell(t,\xi)\chi_0(\xi)\right)\right| \lesssim t^{-\frac{2-2\ell\theta}{2(1-\theta)}+\frac{\delta}{2}}(1+|x|)^{-(n-2)}, \qquad \ell = 0, 1. \tag{115}$$

Now, we can interpolate (113) and (115). In fact, let $j \in (1,2)$ and take $\nu = 2-j \in (0,1)$. Then,

$$\begin{aligned}
\left|\mathcal{F}^{-1}\left(\hat{K}_\ell(t,\xi)\chi_0(\xi)\right)\right| &\lesssim \left(t^{-\frac{1-2\ell\theta}{2(1-\theta)}+\frac{\delta}{2}}(1+|x|)^{-(n-1)}\right)^\nu \left(t^{-\frac{2-2\ell\theta}{2(1-\theta)}+\frac{\delta}{2}}(1+|x|)^{-(n-2)}\right)^{1-\nu} \\
&\lesssim t^{-\frac{2-\nu-2\ell\theta}{2(1-\theta)}+\frac{\delta}{2}}(1+|x|)^{-n+2-\nu} \\
&= t^{-\frac{j-2\ell\theta}{2(1-\theta)}+\frac{\delta}{2}}(1+|x|)^{-(n-j)}, \quad \forall j \in [1,2].
\end{aligned} \tag{116}$$

With this inequality, we can estimate the desired norm:

$$\begin{aligned}
\left\|\mathcal{F}^{-1}\left(\hat{K}_\ell(t,\cdot)\chi_0(\cdot)\right)\right\|_{L^p}^p &\lesssim \int_{\mathbb{R}^n}\left|\mathcal{F}^{-1}\left(\hat{K}_\ell(t,\xi)\chi_0(\xi)\right)\right|^p \mathrm{d}x \\
&\lesssim \int_{\mathbb{R}^n} t^{\left(-\frac{j-2\ell\theta}{2(1-\theta)}+\frac{\delta}{2}\right)p}(1+|x|)^{-(n-j)p}\,\mathrm{d}x \\
&\lesssim t^{\left(-\frac{j-2\ell\theta}{2(1-\theta)}+\frac{\delta}{2}\right)p},
\end{aligned} \tag{117}$$

if, and only if, $(n-j)p > n$, or equivalently, $j < n\left(1-\frac{1}{p}\right)$.

So, choosing $j = n\left(1-\frac{1}{p}\right) - \delta(1-\theta)$, it holds that

$$\left\|\mathcal{F}^{-1}\left(\hat{K}_\ell(t,\cdot)\chi_0(\cdot)\right)\right\|_{L^p} \lesssim t^{-\frac{n\left(1-\frac{1}{p}\right)-2\ell\theta}{2(1-\theta)}+\delta}, \qquad \ell = 0,1.$$

■

Finally, with a direct application of Young inequality, we can extract estimates for $u^{lin}(t,x)$ and convolution with E_1 in the low-frequency region.

We may write the function $u^{lin}(t,x)$ in the Fourier space as

$$\widehat{u^{lin}}(t,\xi) := \hat{K}_0(t,\xi)\hat{u}_0(\xi) + \hat{K}_1(t,\xi)\hat{u}_1(\xi). \tag{118}$$

Lemma 3.3.3 *Assume $n \in \mathbb{N}$, $\theta \in \left[0,\frac{1}{2}\right)$, $p > 1$ and $t \geq 1$. Consider $\widehat{u^{lin}}(t,\xi)$ as in* (118) *and E_1 as in* (45). *Consider also a cut function $\chi_0(\xi) \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that* $\operatorname{supp}\chi_0 \subset B(0,\varepsilon_0)$. *Then, for any $\delta > 0$, $u_0, u_1, \psi \in L^1(\mathbb{R}^n)$, the following estimates hold:*

i) $\left\|\mathcal{F}^{-1}\left(\widehat{u^{lin}}(t,\cdot)\chi_0(\cdot)\right)\right\|_{L^p} \lesssim (1+t)^{-\frac{n\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}+\delta} \|(u_0,u_1)\|_{L^1\times L^1}$;

ii) $\left\|\mathcal{F}^{-1}\left(\hat{E}_1(t,\cdot)\chi_0(\cdot)\hat{\psi}(\cdot)\right)\right\|_{L^p} \lesssim (1+t)^{-\frac{n\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}+\delta} \|\psi\|_{L^1}$.

Proof:

i) Applying Young's inequality for convolutions and Lemma 3.3.2(i)-(ii), we get

$$\begin{aligned}\left\|\mathcal{F}^{-1}\left(\widehat{u^{lin}}\chi_0\right)\right\|_{L^p} &\lesssim \left\|\mathcal{F}^{-1}\left(\hat{K}_0\chi_0\right)*u_0\right\|_{L^p} + \left\|\mathcal{F}^{-1}\left(\hat{K}_1\chi_0\right)*u_1\right\|_{L^p}\\ &\lesssim \left\|\mathcal{F}^{-1}\left(\hat{K}_0\chi_0\right)\right\|_{L^p}\|u_0\|_{L^1} + \left\|\mathcal{F}^{-1}\left(\hat{K}_1\chi_0\right)\right\|_{L^p}\|u_1\|_{L^1}\\ &\lesssim (1+t)^{-\frac{n\left(1-\frac{1}{p}\right)}{2(1-\theta)}+\delta}\|u_0\|_{L^1}\\ &\qquad + (1+t)^{-\frac{n\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}+\delta}\|u_1\|_{L^1}\\ &\lesssim (1+t)^{-\frac{n\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}+\delta}\|(u_0,u_1)\|_{L^1\times L^1}.\end{aligned} \tag{119}$$

ii) Applying Young's inequality for convolutions and Lemma 3.3.2(iii),

$$\begin{aligned}\left\|\mathcal{F}^{-1}\left(\hat{E}_1\chi_0\hat{\psi}\right)\right\|_{L^p} &= \left\|\mathcal{F}^{-1}\left(\hat{E}_1\chi_0\right)*\psi\right\|_{L^p}\\ &\lesssim \left\|\mathcal{F}^{-1}\left(\hat{E}_1\chi_0\right)\right\|_{L^p}\|\psi\|_{L^1}\\ &\lesssim (1+t)^{-\frac{n\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}+\delta}\|\psi\|_{L^1}.\end{aligned} \tag{120}$$

■

3.3.2 High-Frequency Region

For the high-frequency region, we will use again some estimates for the fundamental solutions K_0, K_1 and the operator E_1, but this time to apply a multiplier theorem for functions that vanish in a neighborhood of the origin, which is the case for the localized high-frequency functions. This multiplier theorem is given for a more general class of functions, in Hardy spaces $\mathcal{H}^p(\mathbb{R}^n)$, not to be confused with the Sobolev space $H^p(\mathbb{R}^n) = W^{p,2}(\mathbb{R}^n)$.

3.3.2.1 Multipliers on Hardy Spaces

The theory of Hardy spaces can be considered a fundamental chapter of complex function theory, with intimate connections with Fourier analysis. The definition of $\mathcal{H}^p$ can be posed for any $p \in (0,\infty)$, based on the L^p–boundedness of the Riesz transforms (see (FEFFERMAN; STEIN, 1972)). For $p \in (1,\infty)$, the Hardy space $\mathcal{H}^p(\mathbb{R}^n)$ is characterized as the space of harmonic functions $u(t,x)$ such that

$$\|u\|_{\mathcal{H}^p} := \sup_{t>0} \int_{\mathbb{R}^n} |u(t,x)|^p \,\mathrm{d}x < \infty.$$

With this definition, it is well-known (see, for instance, (STEIN; WEISS, 1971)) that $\mathcal{H}^p(\mathbb{R}^n)$ coincides with $L^p(\mathbb{R}^n)$, since $u \in \mathcal{H}^p$ if, and only if, u is the Poisson integral of an $f \in L^p$, and the Poisson integral is bounded by $\|f\|_{L^p}$ in L^p and converges to it when $t \to 0$. Therefore, the results we collect for multipliers in $\mathcal{H}^p(\mathbb{R}^n)$ will also hold for multipliers in $L^p(\mathbb{R}^n)$.

We shall consider the operator

$$f \mapsto T_m f = \mathcal{F}^{-1}\left(m(\xi)\hat{f}(\xi)\right),$$

where m is a bounded function in $\mathbb{R}^n$.

Definition 3.3.4 *A bounded function m is said to be a* Fourier multiplier *for $\mathcal{H}^p$, $p \in (0,2]$, if $T_m f \in \mathcal{H}^p$ for all $f \in \mathcal{H}^p$, and*

$$\|T_m f\|_{\mathcal{H}^p} \lesssim \|f\|_{\mathcal{H}^p}.$$

We denote $\mathcal{M}(\mathcal{H}^p)$ as the space of all the Fourier multipliers for $\mathcal{H}^p$, and define its norm $\|m\|_{\mathcal{M}(\mathcal{H}^p)}$ to be the operator norm of T_m in $\mathcal{H}^p$, that is,

$$\|m\|_{\mathcal{M}(\mathcal{H}^p)} := \|T_m f\|_{\mathcal{H}^p\to\mathcal{H}^p} = \sup \frac{\|T_m f\|_{\mathcal{H}^p}}{\|f\|_{\mathcal{H}^p}}. \tag{121}$$

Now, we can state the multiplier Theorem for Hardy spaces. For the proof, see (MIYACHI, 1980).

Theorem 3.3.5 *(Miyachi): Let* $a \geq 0$, $b \geq 0$, $0 < p_0 < 2$, $na\left(\frac{1}{p_0} - \frac{1}{2}\right) = b$, $k = \max\left\{\left[n\left(\frac{1}{p_0} - \frac{1}{2}\right)\right] + 1, \left[\frac{n}{2}\right] + 1\right\}$. *If* $m \in C^k(\mathbb{R}^n)$, $m(\xi) = 0$ *in a neighborhood of 0 and*

$$\left|\partial_\xi^\alpha m(\xi)\right| \leqslant |\xi|^{-b}\left(A|\xi|^{a-1}\right)^{|\alpha|}, |\alpha| \leqslant k$$

with $A \geqslant 1$, *then* $m \in \mathcal{M}\left(\mathcal{H}^P(\mathbb{R}^n)\right)$ *and*

$$\|m\|_{\mathcal{M}(\mathcal{H}^p)} \leqslant CA^{\left(\frac{1}{\rho} - \frac{1}{2}\right)}, \quad \text{for } p \in [p_0, 2].$$

For the high-frequency region, we have the following estimates, collected in Appendix B.2:

- $\left|\partial_\xi^\alpha\left(\hat{K_0}(t,\xi)|\xi|^{-s}\right)\right| \lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}(1+t)^{-\frac{s-b}{2(1-\theta)}}$;
- $\left|\partial_\xi^\alpha\left(\hat{K_1}(t,\xi)|\xi|^{-s}\right)\right| \lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}(1+t)^{-\frac{s+1-b}{2(1-\theta)}}$;
- $\left|\partial_\xi^\alpha\left(\hat{E_1}(t,\xi)|\xi|^{-s}\right)\right| \lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}(1+t)^{-\frac{s+3-b}{2(1-\theta)}}$.

Lemma 3.3.6 *Let* $n \in \mathbb{N}$, $\theta \in \left[0, \frac{1}{2}\right)$, α *any multi-index,* $p \in (1,2)$, *and* $t \geq 1$. *Consider also a cut function* $\chi_\infty(\xi) \in C_c^\infty(\mathbb{R}^n)$ *such that* $\operatorname{supp}\chi_\infty \subset \mathbb{R}^n \setminus B(0,1)$. *Assume that* $u_0 \in \dot{W}^{s,p}(\mathbb{R}^n)$, $u_1 \in \dot{W}^{s-1,p}(\mathbb{R}^n)$ *and* $\psi \in \dot{W}^{s-3,p}(\mathbb{R}^n)$, *for some* $s \in \mathbb{N}$ *satisfying*

$$s > n(3-2\theta)\left(\frac{1}{p} - \frac{1}{2}\right).$$

Then,

i) $\left\|\mathcal{F}^{-1}\left(\widehat{u^{lin}}(t,\cdot)\chi_\infty(\cdot)\right)\right\|_{L^p} \lesssim (1+t)^{-\frac{s}{2(1-\theta)} + \frac{n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}}\|(u_0,u_1)\|_{\dot{W}^{s,p}\times\dot{W}^{s-1,p}}$

ii) $\left\|\mathcal{F}^{-1}\left(\hat{E}_1(t,\cdot)\chi_\infty(\cdot)\hat{\psi}(\cdot)\right)\right\|_{L^p} \lesssim (1+t)^{-\frac{s}{2(1-\theta)} + \frac{n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}}\|\psi\|_{\dot{W}^{s-3,p}}$,

where

$$\|\psi\|_{\dot{W}^{r,p}} = \sum_{|\beta|=r}\left\|\partial_x^\beta\psi\right\|_{L^p}.$$

Proof: Since we are considering $t \geq 1$, we have $t^{-a} \approx (1+t)^{-a}$. for any $a \geq 0$.

i) For $\hat{K_0}(t,\xi)$, we use the estimate

$$\left|\partial_\xi^\alpha\left(\hat{K_0}(t,\xi)|\xi|^{-s}\right)\right| \lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|} t^{-\frac{s-b}{2(1-\theta)}},$$

and apply Miyachi's Theorem with $m(\xi) = t^{\frac{s-b}{2(1-\theta)}}\hat{K_0}(t,\xi)|\xi|^{-s}$,

$A = 1,\ a = 3 - 2\theta$ and $b = n(3 - 2\theta)\left(\frac{1}{p} - \frac{1}{2}\right)$. Hence, for any $\varphi \in L^p(\mathbb{R}^n)$,

$$\left\|K_0(t,\cdot) * \partial_x^{-s}\varphi\right\|_{L^p} \lesssim t^{-\frac{s}{2(1-\theta)}+\frac{n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}} \|\varphi\|_{L^p}, \tag{122}$$

for every $s > n(3-2\theta)\left(\frac{1}{p} - \frac{1}{2}\right)$. Taking $\varphi(x) = \partial_x^s u_0(x)$, we get

$$\|K_0(t,\cdot) * u_0\|_{L^p} \lesssim t^{-\frac{s}{2(1-\theta)}+\frac{n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}} \|u_0\|_{\dot{W}^{s,p}}, \tag{123}$$

for every $s > n(3-2\theta)\left(\frac{1}{p} - \frac{1}{2}\right)$.

For $\hat{K}_1(t,\xi)$, we use the estimate

$$\left|\partial_\xi^\alpha\left(\hat{K}_1(t,\xi)|\xi|^{-(s-1)}\right)\right| \lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|} t^{-\frac{s-b}{2(1-\theta)}},$$

and apply Miyachi's Theorem with $m(\xi) = t^{\frac{s-b}{2(1-\theta)}}\hat{K}_1(t,\xi)|\xi|^{-(s-1)}$,
$A = 1,\ a = 3 - 2\theta$ and $b = n(3 - 2\theta)\left(\frac{1}{p} - \frac{1}{2}\right)$, thus obtaining

$$\left\|K_1(t,\cdot) * \partial_x^{-(s-1)}\varphi\right\|_{L^p} \lesssim t^{-\frac{s}{2(1-\theta)}+\frac{n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}} \|\varphi\|_{L^p}, \tag{124}$$

for every $s > n(3-2\theta)\left(\frac{1}{p} - \frac{1}{2}\right)$. Taking $\varphi(x) = \partial_x^{s-1} u_1(x)$, we get

$$\|K_1(t,\cdot) * u_1\|_{L^p} \lesssim t^{-\frac{s}{2(1-\theta)}+\frac{n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}} \|u_1\|_{\dot{W}^{s-1,p}}, \tag{125}$$

for every $s > n(3-2\theta)\left(\frac{1}{p} - \frac{1}{2}\right)$. Recalling that

$$u^{lin}(t,x) = K_0(t,x) * u_0(x) + K_1(t,x) * u_1(x),$$

estimates (123) and (125) give us the desired result.

ii) From the estimates for $\hat{E}_1$ in the high-frequency, but plugging $s - 3$ instead of s, we have the estimate

$$\left|\partial_\xi^\alpha\left(\hat{E}_1(t,\xi)|\xi|^{-(s-3)}\right)\right| \lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|} t^{-\frac{s-b}{2(1-\theta)}},$$

for $b < s$. Applying Miyachi's Theorem with $m(\xi) = t^{\frac{s-b}{2(1-\theta)}}\hat{E}_1(t,\xi)|\xi|^{-(s-3)}$,
$A = 1,\ a = 3 - 2\theta$ and $b = n(3 - 2\theta)\left(\frac{1}{p} - \frac{1}{2}\right)$, we obtain

$$\|E_1(t,\cdot) * \psi\|_{L^p} \lesssim t^{-\frac{s}{2(1-\theta)}+\frac{n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}} \|\psi\|_{\dot{W}^{s-3,p}}, \tag{126}$$

if s satisfies $s > n(3-2\theta)\left(\frac{1}{p} - \frac{1}{2}\right)$.

■

Finally, we may present the main result of this section, as a direct consequence of the results we obtained in low-frequency and high-frequency regions, or to be more precise, in Lemmas 3.3.3 and 3.3.6:

Corollary 3.3.7 *Assume $n \in \mathbb{N}$, $\theta \in \left[0, \frac{1}{2}\right)$, $p > 1$ and $t \geq 1$. Consider $u^{lin}(t,x)$ as in* (40) *and E_1 as in* (45)*. Then, for any $\delta > 0$, $u_0 \in \dot{W}^{s,p}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, $u_1 \in \dot{W}^{s-1,p}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, $\psi \in \dot{W}^{s-3,p}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, the following estimates hold:*

i) $$\left\|u^{lin}(t,\cdot)\right\|_{L^p} \lesssim (1+t)^{-\frac{n\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}+\delta} \left\|(u_0,u_1)\right\|_{L^1\times L^1} + (1+t)^{-\frac{s}{2(1-\theta)}+\frac{n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}} \left\|(u_0,u_1)\right\|_{\dot{W}^{s,p}\times\dot{W}^{s-1,p}};$$

ii) $$\left\|E_1(t,\cdot)*\psi\right\|_{L^p} \lesssim (1+t)^{-\frac{n\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}+\delta} \left\|\psi\right\|_{L^1} + (1+t)^{-\frac{s}{2(1-\theta)}+\frac{n(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}} \left\|\psi\right\|_{\dot{W}^{s-3,p}}.$$

Proof: It suffices to write $u^{lin}(t,x)$ as

$$u^{lin}(t,x) = \mathcal{F}^{-1}\left(\widehat{u^{lin}}(t,\xi)\chi_0\right) + \mathcal{F}^{-1}\left(\widehat{u^{lin}}(t,\xi)\chi_\infty\right) \tag{127}$$

and $E_1(t,x)$ as

$$E_1(t,x) = \mathcal{F}^{-1}\left(\hat{E}_1(t,\xi)\chi_0\right) + \mathcal{F}^{-1}\left(\hat{E}_1(t,\xi)\chi_\infty\right), \tag{128}$$

and apply Lemmas 3.3.3 and 3.3.6 to bound the desired norms. ■

3.4 CASE $n = 3$

The case $n = 3$ is the first case where things start to get tricky. In fact, while in low-dimensional cases we were able to strike our problem in a single blow, for dimensions $n \geq 3$, several parameters start to show their influence, bringing some problems that we need to work around. In order to strike each of these issues in a more organized fashion, we shall break into several cases, based on the size of the memory nonlinearity parameter γ. The first region concerns small values for γ, namely $\gamma \in \left(0, \frac{1}{3}\right)$. This is a particular case of $\left(0, \frac{n-2}{n}\right)$, for which the influence of the nonlinear memory is so strong that the critical exponent switches to the value γ^{-1}, as investigated before for the heat equation with nonlinear memory in (CAZENAVE; DICKSTEIN; WEISSLER, 2008). The possible influence of the regularity-loss decay on the critical exponent is neglected by estimating the power nonlinearity at high-frequencies in $\dot{H}^1$, instead of L^2 as we did in lower dimensions.

Theorem 3.4.1 *Assume that $n = 3$, $\theta \in \left[0, \frac{1}{2}\right)$, $\gamma \in \left(0, \frac{1}{3}\right]$. Also, assume $p > \gamma^{-1}$, and $s = s_c$, with s_c as in (3). Then, there exists $\varepsilon > 0$ such that, for initial data*

$$(u_0, u_1) \in \mathcal{A} := \left(H^s(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)\right) \times \left(H^{s-1}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)\right), \tag{129}$$

there exists a global solution to the problem (1), $u \in \mathcal{C}\left([0,\infty), H^2\right) \cap \mathcal{C}^1\left([0,\infty), H^1\right)$.

Moreover, the following estimates hold:

$$\|u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma} \|(u_0,u_1)\|_{\mathcal{A}}, \tag{130}$$

$$\|u(t,\cdot)\|_{\dot{H}^2} \lesssim \begin{cases} (1+t)^{-\gamma} \|(u_0,u_1)\|_{\mathcal{A}}, & \text{if } \theta > 0, \\ (1+t)^{-\gamma} \ln(2+t) \|(u_0,u_1)\|_{\mathcal{A}}, & \text{if } \theta = 0, \end{cases} \tag{131}$$

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\gamma} \|(u_0,u_1)\|_{\mathcal{A}}, \tag{132}$$

$$\|u_t(t,\cdot)\|_{\dot{H}^1} \lesssim \begin{cases} (1+t)^{-\gamma} \|(u_0,u_1)\|_{\mathcal{A}}, & \text{if } \theta > 0, \\ (1+t)^{-\gamma} \ln(2+t) \|(u_0,u_1)\|_{\mathcal{A}}, & \text{if } \theta = 0. \end{cases} \tag{133}$$

Proof: For $T > 0$, we define the Banach space

$$X(T) := \mathcal{C}\left([0,\infty), H^2\right) \cap \mathcal{C}^1\left([0,\infty), H^1\right),$$

equipped with the norm

$$\|v\|_{X(T)} := \sup_{t\in[0,T]} \Bigg(h_0(t)^{-1} \|v(t,\cdot)\|_{L^2} + h_1(t)^{-1} \|v(t,\cdot)\|_{\dot{H}^{k_1}} + h_2(t)^{-1} \|v(t,\cdot)\|_{\dot{H}^2}$$
$$+ \tilde{h}_0(t)^{-1} \|v_t(t,\cdot)\|_{L^2} + \tilde{h}_1(t)^{-1} \|v_t(t,\cdot)\|_{\dot{H}^1} \Bigg),$$

where $k_1 = 3\left(\frac{1}{2} - \frac{1}{p}\right)$ and

$$h_0(t) = (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma},$$
$$h_1(t) = \tilde{h}_0(t) = (1+t)^{-\gamma},$$
$$h_2(t) = \tilde{h}_1(t) = \begin{cases} (1+t)^{-\gamma} & \text{if } \theta > 0, \\ (1+t)^{-\gamma} \ln(2+t) & \text{if } \theta = 0. \end{cases}$$

Setting $j = 0$, $k = 0, k_1, 2$ and $s = 2 + 2\gamma(1-\theta)$ in the linear estimates (43), we get

$$\left\|u^{lin}(t,\cdot)\right\|_{L^2} \lesssim (1+t)^{-\frac{3-4\theta}{4(1-\theta)}} \|(u_0,u_1)\|_{L^1\times L^1} + (1+t)^{-\frac{s}{2(1-\theta)}} \|(u_0,u_1)\|_{\dot{H}^s\times\dot{H}^{s-1}}$$
$$\lesssim h_0(t) \|(u_0,u_1)\|_{\mathcal{A}}; \tag{134}$$

$$\left\|u^{lin}(t,\cdot)\right\|_{\dot{H}^{k_1}} \lesssim (1+t)^{-\frac{3+2k_1-4\theta}{4(1-\theta)}} \|(u_0,u_1)\|_{L^1\times L^1} + (1+t)^{-\frac{s-k_1}{2(1-\theta)}} \|(u_0,u_1)\|_{\dot{H}^s\times\dot{H}^{s-1}}$$
$$\lesssim h_1(t) \|(u_0,u_1)\|_{\mathcal{A}}. \tag{135}$$

Here we used the fact that, since $p > \gamma^{-1} > 3$, we have $k_1 \in (1,2)$. From $k_1 > 1$, the low-frequency decay is faster than $(1+t)^{-\gamma}$:

$$\frac{3+2k_1-4\theta}{4(1-\theta)} > \frac{5-4\theta}{4(1-\theta)} > 1 > \gamma.$$

On the other hand, from $k_1 \leq 2$, the same holds for the high-frequency decay:

$$\frac{s-k_1}{2(1-\theta)} = \frac{2+2\gamma(1-\theta)-k_1}{2(1-\theta)} \geq \gamma.$$

For the $\dot{H}^2$–norm, we control the high-frequency decay by using the regularity s_c on initial data:

$$\begin{aligned} \left\| u^{lin}(t,\cdot) \right\|_{\dot{H}^2} &\lesssim (1+t)^{-\frac{7-4\theta}{4(1-\theta)}} \left\| (u_0,u_1) \right\|_{L^1\times L^1} \\ &\quad + (1+t)^{-\frac{s-2}{2(1-\theta)}} \left\| (u_0,u_1) \right\|_{\dot{H}^s\times \dot{H}^{s-1}} \\ &\lesssim h_2(t) \left\| (u_0,u_1) \right\|_{\mathcal{A}} . \end{aligned} \tag{136}$$

The norms of $u_t^{lin}(t,x)$ bring no further difficulties, since the low-frequency decay rate becomes greater than 1 and the high-frequency decay rate is not worse than the one from $\dot{H}^2$ estimate, hence also controlled by the regularity of the initial data. Setting $j=1, k=0,1$ and $s=2+2\gamma(1-\theta)$ in (43), we get

$$\left\| u_t^{lin}(t,\cdot) \right\|_{\dot{H}^k} \lesssim \tilde{h}_k(t) \left\| (u_0,u_1) \right\|_{\mathcal{A}}, \qquad k=0,1. \tag{137}$$

Therefore, $\left\| u^{lin} \right\|_{X(T)} \lesssim \left\| (u_0,u_1) \right\|_{\mathcal{A}}$. We now prove $\left\| Gu \right\|_{X(T)} \lesssim \left\| u \right\|_{X(T)}^p$. For the L^2–norm, setting $j=0$, $k=0$, $\psi = |u|^p$ and $s=3$ in Corollary 2.3.6, we get

$$\begin{aligned} \left\| Gu(t,\cdot) \right\|_{L^2} &\lesssim \int_0^t (1+t-\tau)^{-\frac{3-4\theta}{4(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \left\| u(s,\cdot) \right\|_{L^p}^p \, ds\, d\tau \\ &\quad + \int_0^t (1+t-\tau)^{-\frac{3}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \left\| u(s,\cdot) \right\|_{L^{2p}}^p \, ds\, d\tau, \end{aligned} \tag{138}$$

so we must estimate the norms $\left\| u(s,\cdot) \right\|_{L^p}^p$ and $\left\| u(s,\cdot) \right\|_{L^{2p}}^p$. For the L^p–norm, we use Sobolev's Embedding $\dot{H}^{k_1}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$,

$$\left\| u(s,\cdot) \right\|_{L^p}^p \lesssim \left\| u(s,\cdot) \right\|_{\dot{H}^{k_1}}^p \lesssim (1+s)^{-\gamma p} \left\| u \right\|_{X(T)}^p ,$$

with $\gamma p > 1$ if, and only if, $p > \gamma^{-1}$. For the L^{2p}–norm, we use the Sobolev's Embedding $\dot{H}^\kappa(\mathbb{R}^3) \hookrightarrow L^{2p}(\mathbb{R}^3)$, with $\kappa = 3\left(\frac{1}{2} - \frac{1}{2p}\right)$. Notice that $\kappa \in (k_1, 2)$, and therefore we can interpolate between $\dot{H}^{k_1}$ and $\dot{H}^2$:

$$\left\| u(s,\cdot) \right\|_{L^{2p}}^p \lesssim \left\| u(s,\cdot) \right\|_{\dot{H}^\kappa}^p \lesssim (1+s)^{-\gamma p} (\ln(2+s))^p \left\| u \right\|_{X(T)}^p ,$$

considering already the worst case scenario where $\theta = 0$, for which the logarithmic term appears. As discussed before, these logarithmic terms do not influence on the existence results, since they can be bounded by $(1+t)^{\delta}$ with $\delta > 0$ as small as we please. Returning to the L^2 estimate and applying Lemma 2.4.2,

$$\begin{aligned}\|Gu(t,\cdot)\|_{L^2} &\lesssim \left[\int_0^t (1+t-\tau)^{-\frac{3-4\theta}{4(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\gamma p}\, ds\, d\tau \right.\\ &\left. \quad + \int_0^t (1+t-\tau)^{-\frac{3}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\gamma p+\delta}\, ds\, d\tau\right] \|u\|^p_{X(T)}\\ &\lesssim (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma} \|u\|^p_{X(T)}. \end{aligned} \tag{139}$$

For the $\dot{H}^{k_1}$ and $\dot{H}^2$ norms, we may choose $s = 4$ in Corollary 2.3.6:

$$\begin{aligned}\|Gu(t,\cdot)\|_{\dot{H}^{k_1}} &\lesssim \int_0^t (1+t-\tau)^{-\frac{3+2k_1-4\theta}{4(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|^p_{L^p}\, ds\, d\tau\\ &\quad + \int_0^t (1+t-\tau)^{-\frac{4-k_1}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \||u(s,\cdot)|^p\|_{\dot{H}^1}\, ds\, d\tau. \end{aligned} \tag{140}$$

Since the L^p-norm is already estimated, we must then estimate the norm $\left\||u(s,\cdot)|^p\right\|_{\dot{H}^1}$, or equivalently, $\left\|\nabla|u(s,\cdot)|^p\right\|_{L^2}$. We first use chain rule and apply Hölder's inequality:

$$\left\|\nabla|u|^p\right\|_{L^2} \approx \left\||u|^{p-1}\nabla u\right\|_{L^2} \lesssim \|u\|^{p-1}_{L^{3(p-1)}} \|\nabla u\|_{L^6}. \tag{141}$$

The norm in $L^{3(p-1)}$ is estimated applying Sobolev's Embedding $\dot{H}^\kappa(\mathbb{R}^3) \hookrightarrow L^{3(p-1)}(\mathbb{R}^3)$, with $\kappa = 3\left(\frac{1}{2} - \frac{1}{3(p-1)}\right)$. Notice that $\kappa \in (k_1, 2)$ so we can interpolate $\dot{H}^\kappa$ between $\dot{H}^{k_1}$ and $\dot{H}^2$, obtaining

$$\|u\|^{p-1}_{L^{3(p-1)}} \lesssim \|u\|^{p-1}_{\dot{H}^\kappa} \lesssim (1+s)^{-\gamma(p-1)}\big(\ln(2+s)\big)^{p-1} \|u\|^{p-1}_{X(T)}. \tag{142}$$

On the other hand, we apply Sobolev's Embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, getting

$$\|\nabla u\|_{L^6} \lesssim \|\nabla u\|_{\dot{H}^1} \approx \|u\|_{\dot{H}^2} \lesssim (1+s)^{-\gamma} \ln(2+s) \|u\|_{X(T)}. \tag{143}$$

Using (142) and (143) in (141), we get

$$\left\|\nabla|u|^p\right\|_{L^2} \lesssim (1+s)^{-\gamma p}\big(\ln(2+s)\big)^p \|u\|^p_{X(T)}. \tag{144}$$

Returning to the $\dot{H}^{k_1}$ estimate and applying Lemma 2.4.2,

$$\begin{aligned}\|Gu(t,\cdot)\|_{\dot{H}^{k_1}} &\lesssim \left[\int_0^t (1+t-\tau)^{-\frac{3+2k_1-4\theta}{4(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\gamma p}\, ds\, d\tau \right.\\ &\left. \quad + \int_0^t (1+t-\tau)^{-\frac{4-k_1}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\gamma p+\delta}\, ds\, d\tau\right] \|u\|^p_{X(T)}\\ &\lesssim (1+t)^{-\gamma} \|u\|^p_{X(T)}. \end{aligned} \tag{145}$$

For the $\dot{H}^2$–norm, since we have already estimated both $\|u(s,\cdot)\|^p_{L^p}$ and $\left\||u(s,\cdot)|^p\right\|_{\dot{H}^1}$, we estimate directly

$$\|Gu(t,\cdot)\|_{\dot{H}^2} \lesssim \begin{cases} (1+t)^{-\gamma} \|u\|^p_{X(T)}, & \text{if } \theta > 0,\\ (1+t)^{-\gamma} \ln(2+t) \|u\|^p_{X(T)}, & \text{if } \theta = 0. \end{cases} \tag{146}$$

For the two norms of $\partial_t(Gu)$, we also choose $s = 4$ in Corollary 2.3.6, and with $\|u(s,\cdot)\|_{L^p}^p$ and $\big\||u(s,\cdot)|^p\big\|_{\dot H^1}$, we can once more apply Lemma 2.4.2 to obtain

$$\|\partial_t(Gu)(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\gamma}\,\|u\|_{X(T)}^p, \tag{147}$$

and

$$\|\partial_t(Gu)(t,\cdot)\|_{\dot H^1} \lesssim \begin{cases}(1+t)^{-\gamma}\,\|u\|_{X(T)}^p, & \text{if } \theta > 0,\\ (1+t)^{-\gamma}\ln(2+t)\,\|u\|_{X(T)}^p, & \text{if } \theta = 0.\end{cases} \tag{148}$$

Therefore, we have proved that $\|Gu\|_{X(T)} \lesssim \|u\|_{X(T)}^p$, and we conclude the proof as in cases $n = 1, 2$.

■

The second region we will explore in case $n = 3$ is for values of γ that are greater than $\frac{1}{3}$, so its influence in the critical exponent will not be so drastic. As we mentioned before, when γ tends to 1, it is possible that $p < 2$. We first show that, when this is not the case, the critical exponent remains p_c, which can be shown without great modifications on the previous proofs.

We notice the following equivalence:

$$\begin{aligned} p_c \geq 2 &\iff n + 2(1-\theta) \geq 2\,(n-2+2\gamma(1-\theta))\\ &\iff \gamma \leq \frac{6-n-2\theta}{4(1-\theta)}. \end{aligned}$$

Hence, the second result for $n = 3$ covers the case $\gamma \in \left(\frac{1}{3}, \frac{3-2\theta}{4(1-\theta)}\right]$, for which $p_c \geq 2$, and the case $\gamma \nearrow 1$, forcing the condition $p \geq 2$.

Theorem 3.4.2 *Assume $n = 3$, $\theta \in \left[0, \frac{1}{2}\right)$. Let $s = 2 + 2\gamma(1-\theta)$, and assume*

$$\begin{cases}\gamma \in \left(\frac{1}{3}, \frac{3-2\theta}{4(1-\theta)}\right] & \text{and } p > p_c, \text{ or}\\ \gamma \in \left(\frac{3-2\theta}{4(1-\theta)}, 1\right) & \text{and } p \geq 2.\end{cases}$$

Then, there exists $\varepsilon > 0$ such that, for initial data

$$(u_0, u_1) \in \mathcal{A} := \left(H^s(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)\right) \times \left(H^{s-1}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)\right),$$

there exists a global solution to the problem (1), $u \in \mathcal{C}\left([0,\infty), H^2\right) \cap \mathcal{C}^1\left([0,\infty), H^1\right)$.

Moreover, the following estimates hold:

$$\|u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma}\,\|(u_0,u_1)\|_{\mathcal{A}},$$

$$\|u(t,\cdot)\|_{\dot H^2} \lesssim \begin{cases}(1+t)^{-\gamma}\,\|(u_0,u_1)\|_{\mathcal{A}}, & \text{if } \theta > 0,\\ (1+t)^{-\gamma}\ln(2+t)\,\|(u_0,u_1)\|_{\mathcal{A}}, & \text{if } \theta = 0,\end{cases}$$

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\gamma}\,\|(u_0,u_1)\|_{\mathcal{A}},$$

$$\|u_t(t,\cdot)\|_{\dot H^1} \lesssim \begin{cases}(1+t)^{-\gamma}\,\|(u_0,u_1)\|_{\mathcal{A}}, & \text{if } \theta > 0,\\ (1+t)^{-\gamma}\ln(2+t)\,\|(u_0,u_1)\|_{\mathcal{A}}, & \text{if } \theta = 0.\end{cases}$$

Proof: For $T > 0$, we define the Banach space

$$X(T) := \mathcal{C}\big([0,\infty), H^2\big) \cap \mathcal{C}^1\big([0,\infty), H^1\big),$$

equipped with the norm

$$\|v\|_{X(T)} := \sup_{t\in[0,T]} \Bigg(h_0(t)^{-1} \|v(t,\cdot)\|_{L^2} + h_1(t)^{-1} \|v(t,\cdot)\|_{\dot{H}^{\frac{1}{2}}} + h_2(t)^{-1} \|v(t,\cdot)\|_{\dot{H}^2}$$
$$+ \tilde{h}_0(t)^{-1} \|v_t(t,\cdot)\|_{L^2} + \tilde{h}_1(t)^{-1} \|v_t(t,\cdot)\|_{\dot{H}^1} \Bigg),$$

where

$$\begin{aligned} h_0(t) &= (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma}, \\ h_1(t) &= (1+t)^{-\gamma} \ln(2+t), \\ \tilde{h}_0(t) &= (1+t)^{-\gamma}, \\ h_2(t) = \tilde{h}_1(t) &= \begin{cases} (1+t)^{-\gamma} & \text{if } \theta > 0, \\ (1+t)^{-\gamma}\ln(2+t) & \text{if } \theta = 0. \end{cases} \end{aligned}$$

First, we check $\left\|u^{lin}\right\|_{X(T)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}}$. Setting $j = 0$, $k = 0$, $\frac{1}{2}$, 2 and $s = 2+2\gamma(1-\theta)$ and after that, setting $j = 1$, $k = 0$, 1 and $s = 2+2\gamma(1-\theta)$ in Corollary 2.3.4, we estimate all five norms of Gu in $X(T)$ without any additional problem.

Now, we prove $\|Gu\|_{X(T)} \lesssim \|u\|^p_{X(T)}$. We will choose $s = 3$ in Corollary 2.3.6, for $j = 0$, $k = 0, \frac{1}{2}$, and $s = 4$ for the norms with $k = 2$, $j = 0$ and with $j = 1$, $k = 0, 1$. This will lead us again into estimating the norms $\|u(s,\cdot)\|^p_{L^p}$, $\|u(s,\cdot)\|^p_{L^{2p}}$ and $\big\||u(s,\cdot)|^p\big\|_{\dot{H}^1}$.

For the norm in L^p, we may use two different approaches, depending on p : Firstly, assume $p \in [2,3)$. Then, we apply Gagliardo-Nirenberg's inequality (47) to interpolate between L^2 and $\dot{H}^{\frac{1}{2}}$:

$$\begin{aligned} \|u(s,\cdot)\|^p_{L^p} &\lesssim \|u(s,\cdot)\|_{L^2}^{\left(1-\theta_{\frac{1}{2}}\right)p} \|u(s,\cdot)\|_{\dot{H}^{\frac{1}{2}}}^{\theta_{\frac{1}{2}}p} \\ &\lesssim (1+s)^{\left(-\frac{3-4\theta}{4(1-\theta)}+1-\gamma\right)\left(1-\theta_{\frac{1}{2}}\right)p} (1+s)^{-\gamma\theta_{\frac{1}{2}}p} (\ln(2+s))^{\theta_{\frac{1}{2}}p} \|u\|^p_{X(T)}, \end{aligned} \tag{149}$$

where $\theta_{\frac{1}{2}} = \frac{3}{\frac{1}{2}}\left(\frac{1}{2}-\frac{1}{p}\right) = 6\left(\frac{1}{2}-\frac{1}{p}\right) \in (0,1)$ due to $p < 3$. We then get

$$\|u(s,\cdot)\|^p_{L^p} \lesssim (1+s)^{-\alpha+\delta} \|u\|^p_{X(T)}, \tag{150}$$

with $\delta > 0$ sufficiently small and

$$\begin{aligned} \alpha &:= \left(\frac{3-4\theta}{4(1-\theta)} - 1 + \gamma\right)\left(1-\theta_{\frac{1}{2}}\right)p + \gamma\theta_{\frac{1}{2}}p \\ &= \frac{(3-2\theta)p-3}{2(1-\theta)} - (1-\gamma)p. \end{aligned} \tag{151}$$

We have $\alpha > 1$ if, and only if, $p > p_C$. For the case $\gamma \in \left(\frac{3-2\theta}{4(1-\theta)}, 1\right)$, this condition holds as a consequence of $p \geq 2$. In fact, from the equivalence

$$\begin{aligned} p_C \geq 2 \iff \frac{3+2(1-\theta)}{1+2\gamma(1-\theta)} \geq 2 \iff 3+2(1-\theta) \geq 2+4\gamma(1-\theta) \\ \iff 1+2(1-\theta) \geq 4\gamma(1-\theta) \iff \frac{3-2\theta}{4(1-\theta)} \geq \gamma, \end{aligned} \tag{152}$$

we have that $p \geq 2 > p_C$ will hold when $\gamma > \frac{3-2\theta}{4(1-\theta)}$.

For the case $p \geq 3$, we may use Sobolev's Embedding $\dot{H}^{\kappa}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ with $\kappa = 3\left(\frac{1}{2} - \frac{1}{p}\right)$. Notice that, for $p > 3$, this κ satisfies $\frac{1}{2} < \kappa < \frac{3}{2} < 2$, so we can interpolate between $\dot{H}^{\frac{1}{2}}$ and $\dot{H}^2$:

$$\|u(s,\cdot)\|_{L^p}^p \lesssim \|u(s,\cdot)\|_{\dot{H}^{\kappa}}^p (1+s)^{-\gamma p} (\ln(2+s))^p \|u\|_{X(T)}^p, \tag{153}$$

and since we are assuming $\gamma > \frac{1}{3}$ and $p \geq 3$, we have $\gamma p > 1$ automatically.

For the norm $\|u(s,\cdot)\|_{L^{2p}}^p$, since $2p > p$, and the map $q \mapsto 3\left(\frac{1}{2} - \frac{1}{q}\right)$ is increasing, we'll obtain a decay speed that is not worse than the one we got for the L^p–norm. Indeed, using Sobolev's Embedding $\dot{H}^{\kappa}(\mathbb{R}^3) \hookrightarrow L^{2p}(\mathbb{R}^3)$, with $\kappa = 3\left(\frac{1}{2} - \frac{1}{2p}\right)$, the condition $p \geq 2$ implies $\kappa \in \left[\frac{3}{4}, \frac{3}{2}\right) \subset (\frac{1}{2}, 2)$ and we can again interpolate between $\dot{H}^{\frac{1}{2}}$ and $\dot{H}^2$:

$$\|u(s,\cdot)\|_{L^{2p}}^p \lesssim \|u(s,\cdot)\|_{\dot{H}^{\kappa}}^p (1+s)^{-\gamma p} (\ln(2+s))^p \|u\|_{X(T)}^p. \tag{154}$$

For the norm $\left\||u(s,\cdot)|^p\right\|_{\dot{H}^1}$, we may proceed as in (141)-(143) from Theorem 3.4.1 and obtain again (144). The only slight difference is that, for the $L^{3(p-1)}$ estimate, we interpolate between $\dot{H}^{\frac{1}{2}}$ and $\dot{H}^2$, which is possible since $\kappa = 3\left(\frac{1}{2} - \frac{1}{3(p-1)}\right) \in \left[\frac{1}{2}, \frac{3}{2}\right)$ for $p \geq 2$.

We are now ready to apply Lemma 2.4.2 for all five norms inside $\|Gu\|_{X(T)}$. Naming

$$\tilde{\alpha} := \begin{cases} \frac{(3-2\theta)p-3}{2(1-\theta)} - (1-\gamma)p, & \text{if } p \in [2,3), \\ \gamma p, & \text{if } p \geq 3, \end{cases}$$

we get, for $k = 0, \frac{1}{2}$,

$$\begin{aligned} \|Gu(t,\cdot)\|_{\dot{H}^k} &\lesssim \int_0^t (1+t-\tau)^{-\frac{3+2k-4\theta}{4(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|_{L^p}^p \, ds\, d\tau \\ &\quad + \int_0^t (1+t-\tau)^{-\frac{3-k}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|_{L^{2p}}^p \, ds\, d\tau \\ &\lesssim \left[\int_0^t (1+t-\tau)^{-\frac{3+2k-4\theta}{4(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} (1+s)^{-\tilde{\alpha}+\delta} \, ds\, d\tau \right. \\ &\quad \left. + \int_0^t (1+t-\tau)^{-\frac{3-k}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} (1+s)^{-\gamma p+\delta} \, ds\, d\tau \right] \|u\|_{X(T)}^p \end{aligned} \tag{155}$$

That is,

$$\|Gu(t,\cdot)\|_{\dot H^k} \lesssim \begin{cases} (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma}\,\|u\|^p_{X(T)}, & \text{if } k=0,\\ (1+t)^{-\gamma}\ln(2+t)\,\|u\|^p_{X(T)}, & \text{if } k=\frac{1}{2}. \end{cases} \tag{156}$$

The other three norms are also now easily estimated:

$$\begin{aligned} \|Gu(t,\cdot)\|_{\dot H^2} \lesssim & \left[\int_0^t (1+t-\tau)^{-\frac{7-4\theta}{4(1-\theta)}}\int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\tilde\alpha+\delta}\,ds\,d\tau \right.\\ & \left. + \int_0^t (1+t-\tau)^{-\frac{2}{2(1-\theta)}}\int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\gamma p+\delta}\,ds\,d\tau\right]\|u\|^p_{X(T)} \\ \lesssim & \begin{cases} (1+t)^{-\gamma}\,\|u\|^p_{X(T)}, & \text{if } \theta>0,\\ (1+t)^{-\gamma}\ln(2+t)\,\|u\|^p_{X(T)}, & \text{if } \theta=0, \end{cases} \end{aligned} \tag{157}$$

and for the two norms of $\partial_t(Gu)$, we do the same and get

$$\|\partial_t(Gu)(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\gamma}, \tag{158}$$

and

$$\|\partial_t(Gu)(t,\cdot)\|_{\dot H^1} \lesssim \begin{cases} (1+t)^{-\gamma}\,\|u\|^p_{X(T)}, & \text{if } \theta>0,\\ (1+t)^{-\gamma}\ln(2+t)\,\|u\|^p_{X(T)}, & \text{if } \theta=0. \end{cases} \tag{159}$$

We conclude that $\|Gu\|_{X(T)} \lesssim \|u\|^p_{X(T)}$, and finishing the proof as in previous cases, we are done. ■

The last case for dimension $n=3$ concerns values of γ that are close to 1, and $p<2$. For this case, we make use of $(L^1\cap L^p)-L^p$ estimates that we obtained in Section 3.3. These estimates bring two additional concerns to our problem. Right away, we have the need of additional regularity $\dot W^{3,p}\times \dot W^{2,p}$ on initial data. Also, the obtained estimates have a decay speed loss when compared to the energy estimates we got for $p\geq 2$. This loss could lead to a different admissible range for p for which we can obtain global existence results. Fortunately, we are able to prove that in dimension $n=3$ this doesn't happen and we still recover the critical exponent p_c.

Theorem 3.4.3 *Assume $n=3$, $\theta\in\left[0,\frac{1}{2}\right)$. Let $\gamma\in\left(\frac{3-2\theta}{4(1-\theta)},1\right)$, $p\in(p_c,2)$ and $s=2+2\gamma(1-\theta)$. Then, there exists $\varepsilon>0$ such that, for initial data*

$$(u_0,u_1)\in\mathcal{A} := \left(H^s(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)\cap \dot W^{3,p}(\mathbb{R}^3)\right)\times\left(H^{s-1}(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)\cap \dot W^{2,p}(\mathbb{R}^3)\right),$$

there exists a global solution to the problem (1), $u\in\mathcal{C}\left([0,\infty);H^2\right)\cap\mathcal{C}^1\left([0,\infty),H^1\right)\cap L^\infty\left([0,\infty),L^p\right)$.

Moreover, the following estimates hold:

$$\|u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma} \|(u_0,u_1)\|_{\mathcal{A}},$$

$$\|u(t,\cdot)\|_{\dot{H}^2} \lesssim \begin{cases} (1+t)^{-\gamma} \|(u_0,u_1)\|_{\mathcal{A}}, & \text{if } \theta > 0,\\ (1+t)^{-\gamma} \ln(2+t) \|(u_0,u_1)\|_{\mathcal{A}}, & \text{if } \theta = 0,\end{cases}$$

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\gamma} \|(u_0,u_1)\|_{\mathcal{A}},$$

$$\|u_t(t,\cdot)\|_{\dot{H}^1} \lesssim \begin{cases} (1+t)^{-\gamma} \|(u_0,u_1)\|_{\mathcal{A}}, & \text{if } \theta > 0,\\ (1+t)^{-\gamma} \ln(2+t) \|(u_0,u_1)\|_{\mathcal{A}}, & \text{if } \theta = 0.\end{cases}$$

Proof: For $T > 0$, we define the Banach space

$$X(T) := \mathcal{C}\big([0,\infty), H^2\big) \cap \mathcal{C}^1\big([0,\infty), H^1\big) \cap L^\infty\big([0,\infty), L^p\big),$$

equipped with the norm

$$\|v\|_{X(T)} := \sup_{t\in[0,T]} \Bigg(h_0(t)^{-1} \|v(t,\cdot)\|_{L^2} + h_1(t)^{-1} \|v(t,\cdot)\|_{\dot{H}^2} + \tilde{h}_0(t)^{-1} \|v_t(t,\cdot)\|_{L^2} + \tilde{h}_1(t)^{-1} \|v_t(t,\cdot)\|_{\dot{H}^1} + h_p^*(t)^{-1} \|v(t,\cdot)\|_{L^p} \Bigg),$$

where

$$h_0(t) = (1+t)^{-\frac{3-4\theta}{4(1-\theta)}+1-\gamma},$$

$$h_1(t) = \tilde{h}_1(t) = \begin{cases} (1+t)^{-\gamma} & \text{if } \theta > 0,\\ (1+t)^{-\gamma} \ln(2+t) & \text{if } \theta = 0,\end{cases}$$

$$\tilde{h}_0(t) = (1+t)^{-\gamma},$$

$$h_p^*(t) = (1+t)^{\delta - \frac{3\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}+1-\gamma},$$

for $\delta > 0$ sufficiently small.

With exception of the L^p–norm, the estimates for $u^{lin}(t,x)$ will follow as in Theorem 3.4.2. For the L^p–norm, we may use the $u^{lin}(t,x)$ for $p < 2$, in Corollary 3.3.7:

$$\left\|u^{lin}(t,\cdot)\right\|_{L^p} \lesssim (1+t)^{\delta - \frac{3\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}} \|(u_0,u_1)\|_{L^1\times L^1} + (1+t)^{-\frac{3-3(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}} \|(u_0,u_1)\|_{\dot{W}^{3,p}\times \dot{W}^{2,p}}. \tag{160}$$

We may compare the decay rates coming from low and high-frequency regions. We notice that the condition

$$\frac{3\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)} \le \frac{3-3(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}$$

is equivalent to

$$3\left(1-\frac{1}{p}\right)-2\theta \leq 3-3(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right) \iff \frac{3}{2}-2\theta \leq 3-3(2-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)$$

$$\iff 3(2-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right) \leq \frac{3}{2}+2\theta.$$

Since we are assuming $p > p_c \geq \frac{4}{3}$ and $\theta \geq 0$, it holds that

$$3\left(\frac{1}{p}-\frac{1}{2}\right) < \frac{3}{4} \Rightarrow 3(2-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right) < \frac{3}{2} \leq \frac{3}{2}+2\theta.$$

In other words, the condition $p > p_c > \frac{4}{3}$, together with the chosen regularity $(u_0, u_1) \in \dot{W}^{3,p} \times \dot{W}^{2,p}$, implies that the slower decay rate will come from the low-frequency region, yielding

$$\left\|u^{lin}(t,\cdot)\right\|_{L^p} \lesssim (1+t)^{\delta-\frac{3\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}} \|(u_0,u_1)\|_{\mathcal{A}}, \tag{161}$$

for a sufficiently small $\delta > 0$.

Therefore, we have $\left\|u^{lin}\right\|_{X(T)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}}$. To prove $\|Gu\|_{X(T)} \lesssim \|u\|^p_{X(T)}$, we will need again estimates for the norms $\|u(s,\cdot)\|^p_{L^p}$, $\|u(s,\cdot)\|^p_{L^{2p}}$ and $\left\||u(s,\cdot)|^p\right\|_{\dot{H}^1}$.

For the L^p–norm, we may use directly the $X(T)$–norm of u :

$$\|u(s,\cdot)\|^p_{L^p} \lesssim \left(h^*_p(s)\right)^p \|u\|^p_{X(T)} = (1+s)^{-\alpha} \|u\|^p_{X(T)}, \tag{162}$$

with

$$\alpha := \frac{(3-2\theta)p-3}{2(1-\theta)} - (1-\gamma+\delta)p. \tag{163}$$

For $\delta > 0$ small enough, it holds that $\alpha > 1$ if, and only if, $p > p_c$. For the L^{2p}–norm, we apply Sobolev's Embedding $\dot{H}^\kappa(\mathbb{R}^3) \hookrightarrow L^{2p}(\mathbb{R}^3)$, with $\kappa = 3\left(\frac{1}{2}-\frac{1}{2p}\right)$. We remark that $2p > 2$, and that $\kappa \in \left(0, \frac{3}{4}, \right) \subset (0,2)$, so we can interpolate between L^2 and $\dot{H}^2$:

$$\begin{aligned}\|u(s,\cdot)\|^p_{L^{2p}} &\lesssim \|u(s,\cdot)\|^p_{\dot{H}^\kappa} \\ &\lesssim (1+s)^{\left(-\frac{3-4\theta}{4(1-\theta)}+1-\gamma\right)p} \|u\|^p_{X(T)} \\ &\lesssim (1+s)^{-\alpha} \|u\|^p_{X(T)},\end{aligned} \tag{164}$$

where in the last step we used the fact that the decay coming from L^2–norm is faster than the one coming from L^p–norm, since $p < 2$. Indeed, for $\delta > 0$ sufficiently small, we have

$$\frac{3-4\theta}{4(1-\theta)} - (1-\gamma) > -\delta + \frac{3\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)} - (1-\gamma)$$

$$\iff 3-4\theta > 6\left(1-\frac{1}{p}\right)-4\theta \iff p<2. \tag{165}$$

For the norm $\left\||u(s,\cdot)|^p\right\|_{\dot{H}^1}$, we proceed similarly to Theorem 3.4.1, using chain rule and Hölder's inequality to obtain

$$\left\|\nabla|u|^p\right\|_{L^2} \lesssim \|u\|_{L^{3(p-1)}}^{p-1}\|\nabla u\|_{L^6}.$$

The norm $\|u\|_{L^{3(p-1)}}^{p-1}$ is estimated using Sobolev's Embedding $\dot{H}^\kappa(\mathbb{R}^3) \hookrightarrow L^{3(p-1)}(\mathbb{R}^3)$, with $\kappa = 3\left(\frac{1}{2}-\frac{1}{3(p-1)}\right)$. Since $p > p_C > p_F = \frac{5}{3}$, it holds that $\kappa \in \left(0,\frac{3}{2}\right) \subset (0,2)$, and we can interpolate between L^2 and $\dot{H}^2$:

$$\begin{aligned}\|u\|_{L^{3(p-1)}}^{p-1} &\lesssim \|u\|_{\dot{H}^\kappa}^{p-1}\\ &\lesssim (1+s)^{\left(-\frac{3-4\theta}{4(1-\theta)}+1-\gamma\right)(p-1)}\|u\|_{X(T)}^{p-1}\\ &\lesssim (1+s)^{-\frac{\alpha}{p}(p-1)}\|u\|_{X(T)}^{p-1},\end{aligned} \tag{166}$$

where in the last step we used again the same reasoning as in (165) to see that the decay rate from L^2 is faster than the decay from L^p, when $p<2$.

On the other hand, using Sobolev's Embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, we get

$$\begin{aligned}\|\nabla u\|_{L^6} &\lesssim \|\nabla u\|_{\dot{H}^1} \approx \|u\|_{\dot{H}^2} \lesssim (1+s)^{-\gamma}\ln(2+s)\|u\|_{X(T)}\\ &\lesssim (1+s)^{-\frac{\alpha}{p}}\|u\|_{X(T)}.\end{aligned} \tag{167}$$

Combining these estimates, we find

$$\left\||u(s,\cdot)|^p\right\|_{\dot{H}^1} \lesssim (1+s)^{-\alpha}\|u\|_{X(T)}^p, \tag{168}$$

with α as in (163).

Using the estimates (162), (164) and (168), we can proceed to estimate the norms of $Gu(t,x)$ in $X(T)$, exactly as in Theorem 3.4.2, with the exception of the L^p–norm. In order to estimate it, we apply the $L^p - L^p$ estimates for convolution with the operator E_1 with $p<2$, that we obtained in Section 3.3. Namely, we will apply Corollary 3.3.7, setting $n=3$, $\varphi = |u|^p$ and $s=3$, obtaining

$$\begin{aligned}\|Gu(t,\cdot)\|_{L^p} \lesssim &\int_0^t (1+t-\tau)^{\delta-\frac{3\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}}\int_0^\tau (\tau-s)^{-\gamma}\|u(s,\cdot)\|_{L^p}^p\, ds\, d\tau\\ &+\int_0^t (1+t-\tau)^{-\frac{3-3(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}}\int_0^\tau (\tau-s)^{-\gamma}\|u(s,\cdot)\|_{L^{p^2}}^p\, ds\, d\tau.\end{aligned} \tag{169}$$

We must then estimate the L^{p^2}–norm of $u(t,x)$. Noticing that $p > \frac{5}{3}$, hence $p^2 > 2$, we can apply Sobolev's Embedding $\dot{H}^\kappa(\mathbb{R}^3) \hookrightarrow L^{p^2}(\mathbb{R}^3)$, with $\kappa = 3\left(\frac{1}{2}-\frac{1}{p^2}\right)$. Also, we remark that $\kappa \in \left(\frac{21}{50}, \frac{3}{2}\right) \subset (0,2)$, due to $p > \frac{5}{3}$, and we can interpolate between L^2 and $\dot{H}^2$:

$$\begin{aligned}\|u(s,\cdot)\|^p_{L^{p2}} &\lesssim \|u(s,\cdot)\|^p_{\dot{H}^\kappa}\\ &\lesssim (1+s)^{\left(-\frac{3-4\theta}{4(1-\theta)}+1-\gamma\right)p}\|u\|^p_{X(T)}\\ &\lesssim (1+s)^{-\alpha}\|u\|^p_{X(T)}. \end{aligned} \tag{170}$$

Once more we used the argument that the decay from L^2–norm is faster than the decay speed coming from L^p–norm, according to (165).

We now return to the estimate (169) and apply Lemma 2.4.2, stressing that, due to $p > \frac{4}{3}$, the worst decay comes from the low-frequency region, and it is slower than $(1+t-\tau)^{-1}$, since

$$\frac{3\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)} < 1 \iff 3-\frac{3}{p} < 2 \iff p < 3.$$

We then obtain

$$\|Gu(t,\cdot)\|_{L^p} \lesssim (1+t)^{\delta-\frac{3\left(1-\frac{1}{p}\right)-2\theta}{2(1-\theta)}+1-\gamma}\|u\|^p_{X(T)}, \tag{171}$$

and with all the norms for Gu in $X(T)$ estimated, we finish the proof as in the previous theorems.

■

3.5 CASE $n = 4$

We have seen in the previous section that two phenomena arise for $n \geq 3$. Firstly, for small γ, the influence of the nonlinear memory changes the critical exponent to γ^{-1}. This effect appears in the same way for $n = 4$. Second, for γ close to 1, a loss in the decay speed when $p < 2$. In case $n = 3$, this loss is not great enough to actually change the critical exponent. We may see that, for $n = 4$, if the parameter γ is large enough in some sense when compared to θ, we must assume a different, greater value for $\overline{p}$ in order to prove global existence of solutions, besides the additional regularity $\dot{W}^{3,p} \times \dot{W}^{2,p}$ that was already required in the case $n = 3$.

As in case $n = 3$, we divide our results based on the size of the parameter γ. First, we show global existence for $\gamma \in \left(0, \frac{1}{2}\right]$, when $p > \gamma^{-1}$.

Theorem 3.5.1 *Assume that $n = 4$, $\theta \in \left[0, \frac{1}{2}\right)$, $\gamma \in \left(0, \frac{1}{2}\right]$. Also, assume $p > \gamma^{-1}$, and $s = s_c$, with s_c as in (3). Then, there exists $\varepsilon > 0$ such that, for initial data*

$$(u_0, u_1) \in \mathcal{A} := \left(H^s(\mathbb{R}^4) \cap L^1(\mathbb{R}^4)\right) \times \left(H^{s-1}(\mathbb{R}^4) \cap L^1(\mathbb{R}^4)\right), \tag{172}$$

there exists a global solution to the problem (1), $u \in \mathcal{C}\left([0,\infty), H^2\right) \cap \mathcal{C}^1\left([0,\infty), H^1\right)$.

Moreover, the following estimates hold:

$$\|u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\gamma} \ln(2+t)\, \|(u_0,u_1)\|_{\mathcal{A}}, \tag{173}$$

$$\|u(t,\cdot)\|_{\dot{H}^2} \lesssim \begin{cases} (1+t)^{-\gamma}\, \|(u_0,u_1)\|_{\mathcal{A}}, & \text{if } \theta > 0, \\ (1+t)^{-\gamma}\, \ln(2+t)\, \|(u_0,u_1)\|_{\mathcal{A}}, & \text{if } \theta = 0, \end{cases} \tag{174}$$

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\gamma}\, \|(u_0,u_1)\|_{\mathcal{A}}, \tag{175}$$

$$\|u_t(t,\cdot)\|_{\dot{H}^1} \lesssim \begin{cases} (1+t)^{-\gamma}\, \|(u_0,u_1)\|_{\mathcal{A}}, & \text{if } \theta > 0, \\ (1+t)^{-\gamma}\, \ln(2+t)\, \|(u_0,u_1)\|_{\mathcal{A}}, & \text{if } \theta = 0. \end{cases} \tag{176}$$

Proof: For $T > 0$, we define the Banach space

$$X(T) := \mathcal{C}\left([0,\infty), H^2\right) \cap \mathcal{C}^1\left([0,\infty), H^1\right),$$

equipped with the norm

$$\|v\|_{X(T)} := \sup_{t\in[0,T]} \Bigg(h_0(t)^{-1}\, \|v(t,\cdot)\|_{L^2} + h_1(t)^{-1}\, \|v(t,\cdot)\|_{\dot{H}^{k_1}} + h_2(t)^{-1}\, \|v(t,\cdot)\|_{\dot{H}^2} + \tilde{h}_0(t)^{-1}\, \|v_t(t,\cdot)\|_{L^2} + \tilde{h}_1(t)^{-1}\, \|v_t(t,\cdot)\|_{\dot{H}^1} \Bigg),$$

where $k_1 = 4\left(\frac{1}{2} - \frac{1}{p}\right)$ and

$$\begin{aligned} h_0(t) &= (1+t)^{-\gamma} \ln(2+t), \\ h_1(t) = \tilde{h}_0(t) &= (1+t)^{-\gamma}, \\ h_2(t) = \tilde{h}_1(t) &= \begin{cases} (1+t)^{-\gamma} & \text{if } \theta > 0, \\ (1+t)^{-\gamma} \ln(2+t) & \text{if } \theta = 0. \end{cases} \end{aligned}$$

We check that $\left\|u^{lin}\right\|_{X(T)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}}$ just as in Theorem 3.4.1. The only slight difference concerns the $\dot{H}^{k_1}$–norm. Here, since $p > \gamma^{-1} \geq 2$, we have $k_1 \in (0,2)$. But for $n = 4$ this is just what we need to control both low and high-frequency decay rates with a $(1+t)^{-\gamma}$ profile:

$$\begin{aligned} \left\|u^{lin}(t,\cdot)\right\|_{\dot{H}^{k_1}} &\lesssim (1+t)^{-1-\frac{k_1}{2(1-\theta)}}\, \|(u_0,u_1)\|_{L^1\times L^1} + (1+t)^{-\frac{s-k_1}{2(1-\theta)}}\, \|(u_0,u_1)\|_{\dot{H}^s\times\dot{H}^{s-1}} \\ &\lesssim (1+t)^{-\gamma}\, \|(u_0,u_1)\|_{\mathcal{A}}. \end{aligned}$$

To prove $\|Gu\|_{X(T)} \lesssim \|u\|^p_{X(T)}$, we also follow the same steps as in Theorem 3.4.1: To estimate the L^p–norm, we use Sobolev's Embedding $\dot{H}^{k_1}(\mathbb{R}^4) \hookrightarrow L^p(\mathbb{R}^4)$. For the L^{2p}–norm, we use Sobolev's Embedding $\dot{H}^{\kappa}(\mathbb{R}^4) \hookrightarrow L^{2p}(\mathbb{R}^4)$, with $\kappa = 4\left(\frac{1}{2} - \frac{1}{2p}\right) \in$

$(k_1, 2)$. Lastly, for the norm $\left\| |u(s,\cdot)|^p \right\|_{\dot{H}^1}$, we proceed similarly to (141)-(143), changing the dimension accordingly. From Hölder's inequality,

$$\left\| \nabla |u|^p \right\|_{L^2} \approx \left\| |u|^{p-1} \nabla u \right\|_{L^2} \lesssim \|u\|^{p-1}_{L^{4(p-1)}} \|\nabla u\|_{L^4}. \tag{177}$$

For the $L^{4(p-1)}$–norm, we apply Sobolev's Embedding $\dot{H}^\kappa(\mathbb{R}^4) \hookrightarrow L^{4(p-1)}(\mathbb{R}^4)$, with $\kappa = 4\left(\frac{1}{2} - \frac{1}{4(p-1)}\right)$. Notice that $\kappa \in (0,2)$ so we can interpolate $\dot{H}^\kappa$ between L^2 and $\dot{H}^2$, obtaining

$$\|u\|^{p-1}_{L^{4(p-1)}} \lesssim \|u\|^{p-1}_{\dot{H}^\kappa} \lesssim (1+s)^{-\gamma(p-1)} \big(\ln(2+s)\big)^{p-1} \|u\|^{p-1}_{X(T)}. \tag{178}$$

On the other hand, we apply Sobolev's Embedding $\dot{H}^1(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$, getting

$$\|\nabla u\|_{L^4} \lesssim \|\nabla u\|_{\dot{H}^1} \approx \|u\|_{\dot{H}^2} \lesssim (1+s)^{-\gamma} \ln(2+s) \|u\|_{X(T)}, \tag{179}$$

which gives us

$$\left\| |u(s,\cdot)|^p \right\|_{\dot{H}^1} \lesssim (1+s)^{-\gamma p} \big(\ln(2+s)\big)^p \|u\|^p_{X(T)}, \tag{180}$$

that is, the same estimates for $\|u\|_{L^p}$, $\|u\|_{L^{2p}}$ and $\left\| |u(s,\cdot)|^p \right\|_{\dot{H}^1}$ we got in Theorem 3.4.1. Using these estimates and applying Lemma 2.4.2, we estimate the five norms of $Gu(t,x)$, concluding that $\|Gu\|_{X(T)} \lesssim \|u\|^p_{X(T)}$. Concluding the proof as in previous cases, we are done.

■

Comparing the results for $n = 3$ and $n = 4$, the next case should cover intermediate values for γ, that is, $\gamma \in \left(\frac{n-2}{n}, \frac{6-n-2\theta}{4(1-\theta)}\right]$, for which $p_c \geq 2$. But since for $n = 4$ this never happens (because that interval is degenerate for $n > 3$) we are left with the case where γ approaches 1. Firstly, we force the condition $p \geq 2$ and apply the energy estimates, in a similar fashion to Theorem 3.4.2.

Theorem 3.5.2 *Assume $n = 4$, $\theta \in \left[0, \frac{1}{2}\right)$, $\gamma \in \left(\frac{1}{2}, 1\right)$, $p \geq 2$ and $s = 2 + 2\gamma(1-\theta)$. Then, there exists $\varepsilon > 0$ such that, for initial data*

$$(u_0, u_1) \in \mathcal{A} := \Big(H^s(\mathbb{R}^4) \cap L^1(\mathbb{R}^4)\Big) \times \Big(H^{s-1}(\mathbb{R}^4) \cap L^1(\mathbb{R}^4)\Big),$$

there exists a global solution to the problem (1), $u \in \mathcal{C}\big([0,\infty), H^2\big) \cap \mathcal{C}^1\big([0,\infty), H^1\big)$.

Moreover, the estimates (173)*,* (174)*,* (175) *and* (176) *hold.*

Proof: For $T > 0$, we define the Banach space

$$X(T) := \mathcal{C}\big([0,\infty), H^2\big) \cap \mathcal{C}^1\big([0,\infty), H^1\big),$$

equipped with the norm

$$\|v\|_{X(T)} := \sup_{t \in [0,T]} \Bigg(h_0(t)^{-1} \|v(t,\cdot)\|_{L^2} + h_1(t)^{-1} \|v(t,\cdot)\|_{\dot{H}^2} + \tilde{h}_0(t)^{-1} \|v_t(t,\cdot)\|_{L^2} + \tilde{h}_1(t)^{-1} \|v_t(t,\cdot)\|_{\dot{H}^1} \Bigg),$$

where

$$h_0(t) = (1+t)^{-\gamma}\ln(2+t),$$
$$\tilde{h}_0(t) = (1+t)^{-\gamma},$$
$$h_1(t) = \tilde{h}_1(t) = \begin{cases} (1+t)^{-\gamma} & \text{if } \theta > 0, \\ (1+t)^{-\gamma}\ln(2+t) & \text{if } \theta = 0. \end{cases}$$

Setting $j = 0, k = 0, 2$; and $j = 1, k = 0, 1$, with $s = 2 + 2\gamma(1-\theta)$, in Corollary 2.3.4, the estimate $\left\| u^{lin} \right\|_{X(T)} \lesssim \|(u_0, u_1)\|_{\mathcal{A}}$ follows.

To prove $\|Gu\|_{X(T)} \lesssim \|u\|^p_{X(T)}$, we set $j = 0, k = 0, 2$, $\varphi = |u|^p$ and $s = 4$ in Corollary 2.3.6, obtaining

$$\begin{aligned} \|Gu(t,\cdot)\|_{\dot{H}^k} \lesssim & \int_0^t (1+t-\tau)^{-1-\frac{k}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|^p_{L^p} \, ds \, d\tau \\ & + \int_0^t (1+t-\tau)^{-\frac{4-k}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \||u(s,\cdot)|^p\|_{\dot{H}^1} \, ds \, d\tau. \end{aligned} \tag{181}$$

We estimate the L^p–norm using Sobolev's Embedding $\dot{H}^\kappa(\mathbb{R}^4) \hookrightarrow L^p(\mathbb{R}^4)$, with $\kappa = 4\left(\frac{1}{2} - \frac{1}{p}\right) \in [0, 2)$, obtaining

$$\|u(s,\cdot)\|^p_{L^p} \lesssim (1+s)^{-\gamma p} (\ln(2+s))^p \|u\|^p_{X(T)}.$$

For the norm $\left\||u(s,\cdot)|^p\right\|_{\dot{H}^1}$, we proceed as in (177)-(179), obtaining again (180). Since $p \geq 2$ and $\gamma > \frac{1}{2}$, we have $\gamma p > 1$ and we can apply Lemma 2.4.2 in (181), obtaining

$$\|Gu(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\gamma} \ln(2+t) \|u\|^p_{X(T)}$$

and

$$\|Gu(t,\cdot)\|_{\dot{H}^2} \lesssim \begin{cases} (1+t)^{-\gamma} \|u\|^p_{X(T)}, & \text{if } \theta > 0 \\ (1+t)^{-\gamma} \ln(2+t) \|u\|^p_{X(T)}, & \text{if } \theta = 0. \end{cases}$$

Setting $j = 1, k = 0, 1$, $\varphi = |u|^p$ and $s = 4$ in Corollary 2.3.6, we get

$$\begin{aligned} \|\partial_t(Gu)(t,\cdot)\|_{\dot{H}^k} \lesssim & \int_0^t (1+t-\tau)^{-2-\frac{k}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|^p_{L^p} \, ds \, d\tau \\ & + \int_0^t (1+t-\tau)^{-\frac{3-k}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \||u(s,\cdot)|^p\|_{\dot{H}^1} \, ds \, d\tau \\ \lesssim & \begin{cases} (1+t)^{-\gamma} \|u\|^p_{X(T)}, & \text{if } k = 0 \text{ or } \theta > 0 \\ (1+t)^{-\gamma} \ln(2+t) \|u\|^p_{X(T)}, & \text{if } k = 1 \text{ and } \theta = 0. \end{cases} \end{aligned} \tag{182}$$

Then, with $\|Gu\|_{X(T)} \lesssim \|u\|^p_{X(T)}$, proved, we finish the proof.

■

In the case where $p < 2$, the L^p–estimates can be used to prove existence of global solutions for $p > \max\{p_c, \tilde{p}_c\}$, where

$$\tilde{p_c} = \frac{14 - 10\theta}{7 - 2\theta + 2\gamma(1-\theta)} \tag{183}$$

is obtained from the high-frequency region estimates, while p_c comes from the low-frequency region estimates. Here, we have an interplay between the parameters θ and γ. If γ is not large enough in comparison to θ, namely if

$$\gamma \leq \frac{2\theta^2 - 3\theta + 7}{8(1-\theta)^2}, \tag{184}$$

then the worst decay rate will come from the low-frequency region, and we find global solutions assuming $p > p_c$. Otherwise, we will find solutions for $p > \tilde{p_c}$.

We remark that, if $\theta \geq \theta_0$, with

$$\theta_0 := \frac{13 - \sqrt{145}}{12}, \tag{185}$$

the smallest root for $2\theta^2 - 3\theta + 7$, then the condition (184) is achieved automatically for every $\gamma \in (0,1)$.

Theorem 3.5.3 *Assume $n = 4$, and that either*

$$\begin{cases} \theta \in \big[0, \theta_0\big) \text{ and } \gamma \in \left(\frac{1}{2}, \frac{2\theta^2-3\theta+7}{8(1-\theta)^2}\right], \text{ or} \\ \theta \in [\theta_0, \frac{1}{2}) \text{ and } \gamma \in \left(\frac{1}{2}, 1\right), \end{cases}$$

with θ_0 as in (185)*. Let $p \in (p_c, 2)$ and $s = s_c$, with p_c as in* (2) *and s_c as in* (3)*. Then, there exists $\varepsilon > 0$ such that, for initial data*

$$(u_0, u_1) \in \mathcal{A} := \left(H^s(\mathbb{R}^4) \cap L^1(\mathbb{R}^4) \cap \dot{W}^{3,p}(\mathbb{R}^4)\right) \times \left(H^{s-1}(\mathbb{R}^4) \cap L^1(\mathbb{R}^4) \cap \dot{W}^{2,p}(\mathbb{R}^4)\right),$$

there exists a global solution to the problem (1), $u \in \mathcal{C}\big([0,\infty), H^2\big) \cap \mathcal{C}^1\big([0,\infty), H^1\big) \cap L^\infty\big([0,\infty), L^p\big)$.

Moreover, the estimates (173)*,* (174)*,* (175) *and* (176) *hold.*

Proof: For $T > 0$, we define the Banach space

$$X(T) := \mathcal{C}\big([0,\infty); H^2\big) \cap \mathcal{C}^1\big([0,\infty); H^1\big),$$

equipped with the norm

$$\|v\|_{X(T)} := \sup_{t\in[0,T]} \Bigg(h_0(t)^{-1} \|v(t,\cdot)\|_{L^2} + h_1(t)^{-1} \|v(t,\cdot)\|_{\dot{H}^2} + \tilde{h}_0(t)^{-1} \|v_t(t,\cdot)\|_{L^2} + \tilde{h}_1(t)^{-1} \|v_t(t,\cdot)\|_{\dot{H}^1} + h_p^*(t)^{-1} \|v(t,\cdot)\|_{L^p} \Bigg),$$

where

$$\begin{aligned} h_0(t) &= (1+t)^{-\gamma} \ln(2+t), \\ \tilde{h}_0(t) &= (1+t)^{-\gamma}, \\ h_1(t) = \tilde{h}_1(t) &= \begin{cases} (1+t)^{-\gamma} & \text{if } \theta > 0, \\ (1+t)^{-\gamma} \ln(2+t) & \text{if } \theta = 0, \end{cases} \\ h_p^*(t) &= (1+t)^{\delta - \frac{2\left(1-\frac{1}{p}\right)-\theta}{(1-\theta)} + 1 - \gamma}, \end{aligned}$$

for $\delta > 0$ sufficiently small. The norms that compose $\left\|u^{lin}\right\|_{X(T)}$ are estimated directly by setting $j = 0$, $k = 0, 2$; and $j = 1$, $k = 0, 1$, with $s = 2 + 2\gamma(1 - \theta)$ in Corollary 2.3.4, as usual. For the norm $\left\|u^{lin}(t, \cdot)\right\|_{L^p}$, we apply the L^p–estimates for $p < 2$, in Corollary 3.3.7:

$$\begin{aligned}\left\|u^{lin}(t,\cdot)\right\|_{L^p} \lesssim &(1+t)^{\delta-\frac{2\left(1-\frac{1}{p}\right)-\theta}{(1-\theta)}} \|(u_0,u_1)\|_{L^1\times L^1}\\ &+(1+t)^{-\frac{3-4(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}} \|(u_0,u_1)\|_{\dot{W}^{3,p}\times \dot{W}^{2,p}}. \end{aligned} \tag{186}$$

Comparing the low and high-frequency decay rates, we see that, for $\delta > 0$ small enough,

$$\begin{aligned} &\frac{2\left(1-\frac{1}{p}\right)-\theta}{(1-\theta)} - \delta \leq \frac{3-4(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}\\ \iff & 4\left(1-\frac{1}{p}\right)-2\theta \leq 3-4(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)\\ \iff & p \geq \frac{8(1-\theta)}{5-2\theta}. \end{aligned} \tag{187}$$

And, since we are assuming $p > p_c$, the equivalence

$$\begin{aligned} p_c \geq \frac{8(1-\theta)}{5-2\theta} \iff & 1+\frac{2\left(1+(1-\gamma)(1-\theta)\right)}{2+2\gamma(1-\theta)} \geq 1+\frac{3-6\theta}{5-2\theta}\\ \iff & \gamma \leq \frac{2\theta^2-3\theta+7}{8(1-\theta)^2} \end{aligned} \tag{188}$$

ensures us that our assumptions on p, γ and θ are just what we need to have the slowest decay rate coming from the low-frequency region, that is,

$$\left\|u^{lin}(t,\cdot)\right\|_{L^p} \lesssim (1+t)^{\delta-\frac{2\left(1-\frac{1}{p}\right)-\theta}{(1-\theta)}} \|(u_0,u_1)\|_{\mathcal{A}}. \tag{189}$$

Therefore, we have $\left\|u^{lin}\right\|_{X(T)} \lesssim \|(u_0, u_1)\|_{\mathcal{A}}$. Next, we prove that $\|Gu\|_{X(T)} \lesssim \|u\|^p_{X(T)}$.

Setting $j = 0, k = 0, 2$; and $j = 1$, $k = 0, 1$, with $\varphi = |u|^p$ and $s = 4$ in Corollary 2.3.6, we get

$$\begin{aligned}\|Gu(t,\cdot)\|_{\dot{H}^k} \lesssim &\int_0^t (1+t-\tau)^{-1-\frac{k}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|^p_{L^p}\, ds\, d\tau\\ &+\int_0^t (1+t-\tau)^{-\frac{4-k}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \||u(s,\cdot)|^p\|_{\dot{H}^1}\, ds\, d\tau, \end{aligned} \tag{190}$$

for $k = 0, 2$, and

$$\begin{aligned}\|\partial_t(Gu)(t,\cdot)\|_{\dot{H}^k} \lesssim &\int_0^t (1+t-\tau)^{-2-\frac{k}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|^p_{L^p}\, ds\, d\tau\\ &+\int_0^t (1+t-\tau)^{-\frac{3-k}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \||u(s,\cdot)|^p\|_{\dot{H}^1}\, ds\, d\tau, \end{aligned} \tag{191}$$

for $k = 0, 1$. The L^p–norm is already included in $\|\cdot\|_{X(T)}$, so we don't need to estimate it:

$$\|u(s,\cdot)\|_{L^p}^p \lesssim \left(h_p^*(s)\right)^p \|u\|_{X(T)}^p = (1+s)^{-\alpha+\delta p} \|u\|_{X(T)}^p, \tag{192}$$

with $\alpha := \frac{(2-\theta)p-2}{1-\theta} - (1-\gamma)p$. We remark that, as expected, $\alpha > 1$ if, and only if, $p > \frac{3-\theta}{1+\gamma(1-\theta)} = p_C$.

The norm $\left\||u(s,\cdot)|^p\right\|_{\dot{H}^1}$, is estimated as in (177)-(179), obtaining again (180). We must then check if $\gamma p > 1$. Indeed, since $\gamma > \frac{1}{2}$ and $p > p_C$, it holds that

$$\gamma p > \frac{\gamma(3-\theta)}{1+\gamma(1-\theta)} = 1 + \frac{2\gamma-1}{1+\gamma(1-\theta)} > 1. \tag{193}$$

Therefore, we can apply Lemma 2.4.2 and obtain the desired estimates for the first four norms that compose $\|Gu\|_{X(T)}$. The last remaining norm is $\|Gu(t,\cdot)\|_{L^p}$, for which we use the E_1–convolution estimates for $p < 2$ we obtained in Corollary 3.3.7, setting $n = 4$, $\varphi = |u|^p$ and $s = 3$:

$$\begin{aligned}\|Gu(t,\cdot)\|_{L^p} \lesssim &\int_0^t (1+t-\tau)^{\delta-\frac{2\left(1-\frac{1}{p}\right)-\theta}{(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|_{L^p}^p \, ds\, d\tau \\ &+ \int_0^t (1+t-\tau)^{-\frac{3-4(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|_{L^{p^2}}^p \, ds\, d\tau. \end{aligned}\tag{194}$$

With the L^p–norm already controlled, it remains only to estimate the L^{p^2}–norm. We observe that $p > p_C > p_F = \frac{3}{2}$, hence $p^2 > 2$, and we can apply Sobolev's Embedding $\dot{H}^\kappa(\mathbb{R}^4) \hookrightarrow L^{p^2}(\mathbb{R}^4)$, with $\kappa = 4\left(\frac{1}{2} - \frac{1}{p^2}\right) \in (0,2)$ and interpolate between L^2 and $\dot{H}^2$, finding

$$\|u(s,\cdot)\|_{L^{p^2}}^p \lesssim \|u(s,\cdot)\|_{\dot{H}^\kappa}^p \lesssim (1+s)^{-\gamma p} (\ln(2+s))^p \|u\|_{X(T)}^p.$$

As we argued before, we have $\gamma p > 1$, so we can apply Lemma 2.4.2, and from the assumption $\gamma \leq \frac{2\theta^2-3\theta+7}{8(1-\theta)^2}$, the slowest decay rate comes from the low-frequency region. It is also slower than $(1+t)^{-1}$, since

$$-\delta + \frac{2\left(1-\frac{1}{p}\right)-\theta}{1-\theta} < 1 \iff p < 2,$$

for $\delta > 0$ sufficiently small. Therefore, we find

$$\|Gu(t,\cdot)\|_{L^p} \lesssim (1+t)^{\delta-\frac{2\left(1-\frac{1}{p}\right)-\theta}{(1-\theta)}+1-\gamma} \|u\|_{X(T)}^p, \tag{195}$$

proving that $\|Gu\|_{X(T)} \lesssim \|u\|_{X(T)}^p$. The rest of the proof follows as in previous cases.

■

The last case for dimension $n = 4$ concerns small values for θ, that is $\theta \in [0, \theta_0)$, and large values for γ, namely $\gamma \in \left(\frac{2\theta^2-3\theta+7}{8(1-\theta)^2}, 1\right)$. For this range of parameters, we have $\tilde{p}_C > p_C$, so we are able to prove existence of global in-time solutions for $p > \tilde{p}_C$.

It is important to stress that in Chapter 4 we prove nonexistence of global in-time solutions only for $p < \max\{p_c, \gamma^{-1}\}$. This leaves us with a gap $p \in (p_c, \tilde{p_c})$, for which we do not know whether there are global solutions or not.

Theorem 3.5.4 *Assume* $n = 4$, $\theta \in [0, \theta_0)$, $\gamma \in \left(\frac{2\theta^2-3\theta+7}{8(1-\theta)^2}, 1\right)$, $p \in (\tilde{p}_c, 2)$ *and* $s = s_c$, *with* θ_0 *as in* (185), $\tilde{p}_c$ *as in* (183) *and* s_c *as in* (3). *Then, there exists* $\varepsilon > 0$ *such that, for initial data*

$$(u_0, u_1) \in \mathcal{A} := \left(H^s(\mathbb{R}^4) \cap L^1(\mathbb{R}^4) \cap \dot{W}^{3,p}(\mathbb{R}^4)\right) \times \left(H^{s-1}(\mathbb{R}^4) \cap L^1(\mathbb{R}^4) \cap \dot{W}^{2,p}(\mathbb{R}^4)\right),$$

there exists a global solution to the problem (1), $u \in \mathcal{C}\big([0,\infty), H^2\big) \cap \mathcal{C}^1\big([0,\infty), H^1\big) \cap L^\infty\big([0,\infty), L^p\big)$.

Moreover, the estimates (173)*,* (174)*,* (175) *and* (176) *hold.*

Proof: For $T > 0$, we define the Banach space

$$X(T) := \mathcal{C}\big([0,\infty), H^2\big) \cap \mathcal{C}^1\big([0,\infty), H^1\big),$$

equipped with the norm

$$\|v\|_{X(T)} := \sup_{t\in[0,T]} \Bigg(h_0(t)^{-1} \|v(t,\cdot)\|_{L^2} + h_1(t)^{-1} \|v(t,\cdot)\|_{\dot{H}^2} + \tilde{h}_0(t)^{-1} \|v_t(t,\cdot)\|_{L^2} + \tilde{h}_1(t)^{-1} \|v_t(t,\cdot)\|_{\dot{H}^1} + h_p^*(t)^{-1} \|v(t,\cdot)\|_{L^p} \Bigg),$$

where

$$\begin{aligned}
h_0(t) &= (1+t)^{-\gamma} \ln(2+t),\\
\tilde{h}_0(t) &= (1+t)^{-\gamma},\\
h_1(t) = \tilde{h}_1(t) &= \begin{cases} (1+t)^{-\gamma} & \text{if } \theta > 0,\\ (1+t)^{-\gamma}\ln(2+t) & \text{if } \theta = 0,\end{cases}\\
h_p^*(t) &= \begin{cases} (1+t)^{-\frac{3-4(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}+1-\gamma}, & \text{if } p < \frac{8(1-\theta)}{5-2\theta},\\ (1+t)^{\delta-\frac{2\left(1-\frac{1}{p}\right)-\theta}{(1-\theta)}+1-\gamma}, & \text{if } p \geq \frac{8(1-\theta)}{5-2\theta},\end{cases}
\end{aligned}$$

for $\delta > 0$ sufficiently small.

We observe that, since the equivalence

$$\frac{2\left(1-\frac{1}{p}\right)-\theta}{(1-\theta)} < \frac{3-4(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)} \iff p > \frac{8(1-\theta)}{5-2\theta}$$

holds, one can see the decay rate $h_p^*(t)$ as

$$h_p^*(t) = (1+t)^{-\omega+1-\gamma},$$

with $\omega = \min\left\{-\delta + \frac{2\left(1-\frac{1}{p}\right)-\theta}{(1-\theta)}, \frac{3-4(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}\right\}$, so it's slower than both the low and the high-frequency decay rates obtained in $L^p - L^p$ estimates.

As done before in Theorem 3.5.2, we estimate the four first norms that compose $\left\|u^{lin}\right\|_{X(T)}$ by setting $j = 0$, $k = 0, 2$; and $j = 1$, $k = 0, 1$, with $s = 2 + 2\gamma(1-\theta)$ in Corollary 2.3.6. As for the last norm, we use the $L^p - L^p$ estimates for u^{lin} from Corollary 3.3.7:

$$\begin{aligned}\left\|u^{lin}(t,\cdot)\right\|_{L^p} &\lesssim (1+t)^{\delta-\frac{2\left(1-\frac{1}{p}\right)-\theta}{(1-\theta)}} \|(u_0,u_1)\|_{L^1\times L^1} \\ &\quad + (1+t)^{-\frac{3-4(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}} \|(u_0,u_1)\|_{\dot W^{3,p}\times \dot W^{2,p}} \\ &\lesssim h_p^*(t)\, \|(u_0,u_1)\|_{\mathcal{A}}. \end{aligned} \tag{196}$$

Therefore, $\left\|u^{lin}\right\|_{X(T)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}}$. To prove $\|Gu\|_{X(T)} \lesssim \|u\|^p_{X(T)}$, we set $j = 0, k = 0, 2$; and $j = 1, k = 0, 1$, with $\varphi = |u|^p$ and $s = 4$ in Corollary 3.3.7, getting

$$\begin{aligned}\|Gu(t,\cdot)\|_{\dot H^k} \lesssim& \int_0^t (1+t-\tau)^{-1-\frac{k}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}\, \|u(s,\cdot)\|^p_{L^p}\; ds\, d\tau \\ &+ \int_0^t (1+t-\tau)^{-\frac{4-k}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}\, \||u(s,\cdot)|^p\|_{\dot H^1}\; ds\, d\tau \end{aligned} \tag{197}$$

and

$$\begin{aligned}\|\partial_t(Gu)(t,\cdot)\|_{\dot H^k} \lesssim& \int_0^t (1+t-\tau)^{-2-\frac{k}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}\, \|u(s,\cdot)\|^p_{L^p}\; ds\, d\tau \\ &+ \int_0^t (1+t-\tau)^{-\frac{3-k}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma}\, \||u(s,\cdot)|^p\|_{\dot H^1}\; ds\, d\tau. \end{aligned} \tag{198}$$

As for the L^p–norm, we get directly

$$\begin{aligned}\|u(s,\cdot)\|^p_{L^p} &\lesssim \left(h_p^*(s)\right)^p \|u\|^p_{X(T)} \\ &= \begin{cases} (1+s)^{-\tilde\alpha}\, \|u\|^p_{X(T)}, & \textit{if } p < \frac{8(1-\theta)}{5-2\theta} \\ (1+s)^{-\alpha+\delta p}\, \|u\|^p_{X(T)}, & \textit{if } p \geq \frac{8(1-\theta)}{5-2\theta}, \end{cases}\end{aligned} \tag{199}$$

with

$$\tilde\alpha := \left(\frac{3-4(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)} - (1-\gamma)\right) p \;\text{ and }\; \alpha := \frac{(2-\theta)p-2}{1-\theta} - (1-\gamma)p.$$

We remark that $\tilde\alpha > 1$ if, and only if,

$$\begin{aligned}\tilde\alpha > 1 \iff& 3p - 4(3-2\theta)\left(1-\frac{p}{2}\right) - 2(1-\gamma)(1-\theta)p > 2(1-\theta) \\ \iff& \big(3 + 2(3-2\theta) - 2(1-\gamma)(1-\theta)\big)p > 2(1-\theta) + 4(3-2\theta) \\ \iff& p > \frac{14-10\theta}{7-2\theta+2\gamma(1-\theta)} = \tilde p_c. \end{aligned} \tag{200}$$

On the other hand, for the case $p \geq \frac{8(1-\theta)}{5-2\theta}$, we have $\alpha > 1 \iff p > p_C$. Using the equivalence (188) and the assumption $\gamma > \frac{2\theta^2-3\theta+7}{8(1-\theta)^2}$, we have

$$p \geq \frac{8(1-\theta)}{5-2\theta} > p_C,$$

hence $\alpha > 1$.

The norm $\left\| |u(s,\cdot)|^p \right\|_{\dot{H}^1}$, is estimated as in (177)-(179), obtaining again (180). We must then check if $\gamma p > 1$. As we have seen in (193), this will hold if $p > p_C$ and $\gamma > \frac{1}{2}$. But this is true, since $\tilde{p}_C > p_C$ is equivalent to $\gamma > \frac{2\theta^2-3\theta+7}{8(1-\theta)^2}$, hence our assumption $p > \tilde{p}_C$ is stronger than what we had before.

Therefore, applying Lemma 2.4.2, we are able to estimate the first four norms inside $\|Gu\|_{X(T)}$. Lastly, for the L^p–norm,

$$\begin{aligned} \|Gu(t,\cdot)\|_{L^p} \lesssim & \int_0^t (1+t-\tau)^{\delta-\frac{2\left(1-\frac{1}{p}\right)-\theta}{(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|_{L^p}^p \, \mathrm{d}s\, \mathrm{d}\tau \\ & + \int_0^t (1+t-\tau)^{-\frac{3-4(3-2\theta)\left(\frac{1}{p}-\frac{1}{2}\right)}{2(1-\theta)}} \int_0^\tau (\tau-s)^{-\gamma} \|u(s,\cdot)\|_{L^{p^2}}^p \, \mathrm{d}s\, \mathrm{d}\tau, \end{aligned} \tag{201}$$

and using Sobolev's Embedding $\dot{H}^\kappa(\mathbb{R}^4) \hookrightarrow L^{p^2}(\mathbb{R}^4)$ with $\kappa = 4\left(\frac{1}{2}-\frac{1}{p^2}\right) \in (0,2)$, we find again

$$\|u(s,\cdot)\|_{L^{p^2}}^p \lesssim \|u(s,\cdot)\|_{\dot{H}^\kappa}^p \lesssim (1+s)^{-\gamma p} (\ln(2+s))^p \|u\|_{X(T)}^p.$$

Applying Lemma 2.4.2 and having in mind the observation from the beginning of our proof, we obtain

$$\|Gu(t,\cdot)\|_{L^p} \lesssim h_p^*(t) \|u\|_{X(T)}^p. \tag{202}$$

This concludes the argument that $\|Gu\|_{X(T)} \lesssim \|u\|_{X(T)}^p$, and we finish the proof as in all previous theorems. ∎

4 NONEXISTENCE OF GLOBAL SOLUTIONS

4.1 TEST FUNCTIONS AND WEAK SOLUTIONS

In this work, we find global in-time solutions to the problem (1) for the supercritical case, for a specific *critical exponent* candidate. The sharpness of such candidate is achieved by showing nonexistence of global solutions in the subcritical case.

This nonexistence counterpart is usually derived by using a classic test function method. However, since the Laplace operator has a nonlocal behavior, such method is not applicable (at least directly), since it relies on the compactness of the test functions' support under the action of these operators.

Having this in mind, we use a modified test function to obtain our results. To deal with the nonlocality of the fractional Laplace operators, we will replace the usual compactly supported test functions by some suitable test functions with polynomial decay. To this end, we introduce a class of functions and a definition of a weak solution to problem (1) which is adequate to our purposes.

Definition 4.1.1 *Let $\theta \in (0,1)$ be a fixed number, and fix $q = n + 2\theta$. We define the space $\mathcal{C}_q^\infty(\mathbb{R}^n)$ as the subspace of infinitely differentiable functions φ such that $\langle x\rangle^q\varphi$ is bounded, and for any $\sigma > 0$ with σ integer or $\sigma - \lfloor\sigma\rfloor \in [\theta, 1)$, the function $\langle x\rangle^q(-\Delta)^\sigma\varphi$ is also bounded.*

The choice of this specific space $C_q^\infty(\mathbb{R}^n)$, with $q = n + 2\theta$ is justified by the following lemma and its corollary. With these, we may show that a function in $C_q^\infty(\mathbb{R}^n)$ remains in the same space after the action of the Fractional Laplace operator $(-\Delta)^\theta$. First, we recall the definition of the fractional Laplace operator and an alternative definition for non-integer powers, that is equivalent with the usual definition when the domain is the whole space $\mathbb{R}^n$.

Definition 4.1.2 *For any $\sigma > 0$, we may define the fractional Laplace operator $(-\Delta)^\sigma : H^{2\sigma} \to L^2$ as*

$$(-\Delta)^\sigma f = \mathcal{F}^{-1}\left(|\xi|^{2\sigma}\hat{f}\right). \tag{203}$$

If $\sigma \in (0,1)$, then the operator $(-\Delta)^\sigma$ admits an integral representation. The identity

$$(-\Delta)^\sigma f(x) = -C_\sigma \int_{\mathbb{R}^n} \frac{f(x+y) + f(x-y) - 2f(x)}{|y|^{n+2\sigma}}\, dy \tag{204}$$

holds for any $f \in \mathcal{S}$. Also, the constant C_σ is given by

$$C_\sigma = \frac{1}{2}\left(\int_{\mathbb{R}^n} \frac{1-\cos(y_1)}{|y|^{n+2\sigma}}\, dy\right)^{-1} > 0.$$

Remark 4.1.3 *The identity given in Definition 4.1.2 is shown in (NEZZA; PALATUCCI; VALDINOCI, 2012).*

The fractional Laplace operator can be conveniently extended to more general spaces. In particular, it may be extended by duality to the tempered distribution space $\mathcal{S}'$. Using the above definitions, one can prove that the function $\langle x\rangle^{-q}$ remains in $C_q^\infty(\mathbb{R}^n)$ after the action of the fractional Laplacian, as done in (D'ABBICCO; FUJIWARA, 2021). We provide a proof for the following Lemma in Appendix C.

Lemma 4.1.4 *Assume $f \in C^2$ bounded, with bounded derivatives. If there exists a constant C_0 such that the estimates*

$$|f(y)| \leq C_0|f(x)|, \quad \sup_{|\alpha|=2} |\partial^\alpha f(y)| \leq C_0 \sup_{|\alpha|=2} |\partial^\alpha f(x)|$$

hold when $|x| \leq |y|$, then for $|x| > 1$, the following pointwise estimate holds:

$$\begin{aligned} |(-\Delta)^\sigma f(x)| \leq C|x|^{-n-2\sigma} \int_{|y|<3|x|} |f(y)|dy + C|f(x)||x|^{-2\sigma} \\ + C|x|^{2-2\sigma} \sum_{|\alpha|=2} \frac{|\alpha|}{\alpha!}\left|\partial^\alpha f\left(\tfrac{x}{2}\right)\right|, \end{aligned} \tag{205}$$

for any $\sigma \in (0,1)$.

Proof: See Appendix C. ∎

As a consequence of Lemma 4.1.4, we derive the following corollary, which bounds the action of the fractional Laplace operator on $\langle x\rangle^{-q}$ pointwisely. For the ease of reading, we prove it also in Appendix C

Corollary 4.1.5 *Let $f(x) = \langle x\rangle^{-q}$, for $q > n$, and let $\sigma > 0$. We set $s = \sigma - [\sigma]$. Then,*

$$\left|(-\Delta)^\sigma f(x)\right| \leq C\langle x\rangle^{-q_\sigma}, \qquad \forall\, x \in \mathbb{R}^n,$$

where $q_\sigma = q + 2\sigma$ if σ is an integer, or $q = n + 2s$ otherwise.

Proof: See Appendix C. ∎

Remark 4.1.6 *We observe that, choosing $q = n + 2\theta$, with $\theta \in [0,1)$ Corollary 4.1.5 says that $f(x) = \langle x\rangle^{-q} \in C_q^\infty(\mathbb{R}^n)$, and that $(-\Delta)^\sigma\langle x\rangle^{-q} \in C_q^\infty(\mathbb{R}^n)$ for every $\sigma > 0$.*

Having defined the appropriate function space to our problem, we may introduce a definition of a weak solution to problem (1), locally and globally with respect to the time variable.

Definition 4.1.7 *Fix $q = n+2\theta$, and fix $T \in [0,\infty)$. We say that $u \in L^p_{loc}\big([0,T), L^p\left(\mathbb{R}^n, \langle x\rangle^q\, dx\right)\big)$ is a weak solution to (1) if for any function $\psi \in \mathcal{C}_c^2([0,T])$ satisfying $\psi(0) = 1$, $\psi(T) =$*

$\psi_t(T) = 0$, *and for any* $\varphi \in \mathcal{C}_q^\infty(\mathbb{R}^n)$, *it holds*

$$\int_0^T \psi(t) \int_{\mathbb{R}^n} F(t,u)\varphi(x)\, dx\, dt \tag{206}$$

$$\begin{aligned}
&= \int_0^T \psi(t) \int_{\mathbb{R}^n} u(t,x)(-\Delta\varphi)(x)\, dx\, dt \\
&+ \int_0^T \psi(t) \int_{\mathbb{R}^n} u(t,x)(\Delta^2\varphi)(x)\, dx\, dt \\
&- \int_0^T \psi_t(t) \int_{\mathbb{R}^n} u(t,x)((-\Delta)^\theta\varphi)(x)\, dx\, dt \\
&+ \int_0^T \psi_{tt}(t) \int_{\mathbb{R}^n} u(t,x)\varphi(x)\, dx\, dt \\
&+ \int_0^T \psi_{tt}(t) \int_{\mathbb{R}^n} u(t,x)(-\Delta\varphi)(x)\, dx\, dt \\
&- \int_{\mathbb{R}^n} u_0(x)((-\Delta)^\theta\varphi)(x)\, dx \\
&- \int_{\mathbb{R}^n} u_1(x)\varphi(x)\, dx - \int_{\mathbb{R}^n} u_1(x)(-\Delta\varphi)(x)\, dx \\
&+ \psi_t(0)\left(\int_{\mathbb{R}^n} u_0(x)\varphi(x)\, dx + \int_{\mathbb{R}^n} u_0(x)(-\Delta\varphi)(x)\, dx\right).
\end{aligned} \tag{207}$$

We say that the weak solution is locally-in-time defined if $T < \infty$ *and is globally-in-time defined if* $T = \infty$.

Equivalently, a function $u \in L^p_{loc}\big([0,T), L^p\left(\mathbb{R}^n, \langle x\rangle^q\, dx\right)\big)$ *is a global in-time solution if, and only if,* $u\big|_{[0,T)\times\mathbb{R}^n}$ *is a local in-time weak solution, for any* $T > 0$.

Remark 4.1.8 *We remark that it holds*

$$L^p_{loc}\big([0,T), L^p\left(\mathbb{R}^n, \langle x\rangle^q\, dx\right)\big) \subset L^p_{loc}\big([0,T), L^p\left(\mathbb{R}^n\right)\big),$$

so the above defined weak solutions space is properly contained in a more conventional solution space.

We may show with ease that classical solutions to problem (1) are also weak solutions.

Proposition 4.1.9 *Assume that* $u_0, u_1 \in \mathcal{S}(\mathbb{R}^n)$. *Also, assume that* $u \in \mathcal{C}^2\left([0,T), \mathcal{S}\right)$ *is a "classical" solution to problem (1). Then, u is also a weak solution to (1), according to Definition 4.1.7.*

Proof: Multiplying the equation in (1) by $\psi(t)\varphi(x)$ and integrating in $[0, T] \times \mathbb{R}^n$, we get

$$\begin{aligned}\int_0^T \psi(t) \int_{\mathbb{R}^n} F(t,u)\varphi(x)\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \psi(t) \int_{\mathbb{R}^n} (-\Delta u)(t,x)\varphi(x)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \psi(t) \int_{\mathbb{R}^n} (\Delta^2 u)(t,x)\varphi(x)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \psi(t) \int_{\mathbb{R}^n} ((-\Delta)^\theta u_t)(t,x)\varphi(x)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \psi(t) \int_{\mathbb{R}^n} u_{tt}(t,x)\varphi(x)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \psi(t) \int_{\mathbb{R}^n} (-\Delta u_{tt})(t,x)\varphi(x)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{208}$$

Integrating by parts in space, due to $u, u_t, u_{tt} \in \mathcal{S}$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\int_0^T \psi(t) \int_{\mathbb{R}^n} F(t,u)\varphi(x)\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \psi(t) \int_{\mathbb{R}^n} u(t,x)(-\Delta\varphi)(x)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \psi(t) \int_{\mathbb{R}^n} u(t,x)(\Delta^2\varphi)(x)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \psi(t) \int_{\mathbb{R}^n} u_t(t,x)((-\Delta)^\theta\varphi)(x)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \psi(t) \int_{\mathbb{R}^n} u_{tt}(t,x)\varphi(x)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \psi(t) \int_{\mathbb{R}^n} u_{tt}(t,x)(-\Delta\varphi)(x)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{209}$$

Now, integrating by parts in time as many times as needed to get rid of all time derivatives of u, recalling that $\psi(0) = 1$, $\psi(T) = \psi_t(T) = 0$, we obtain

$$\begin{aligned}&\int_0^T \psi(t) \int_{\mathbb{R}^n} F(t,u)\varphi(x)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \psi(t) \int_{\mathbb{R}^n} u(t,x)(-\Delta\varphi)(x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \psi(t) \int_{\mathbb{R}^n} u(t,x)(\Delta^2\varphi)(x)\,\mathrm{d}x\,\mathrm{d}t \\ &- \int_{\mathbb{R}^n} u(0,x)((-\Delta)^\theta\varphi)(x)\,\mathrm{d}x - \int_0^T \psi_t(t) \int_{\mathbb{R}^n} u(t,x)((-\Delta)^\theta\varphi)(x)\,\mathrm{d}x\,\mathrm{d}t \\ &- \int_{\mathbb{R}^n} u_t(0,x)\varphi(x)\,\mathrm{d}x + \int_{\mathbb{R}^n} \psi_t(0)u(0,x)\varphi(x)\,\mathrm{d}x \\ &+ \int_0^T \psi_{tt}(t) \int_{\mathbb{R}^n} u(t,x)\varphi(x)\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}^n} u_t(0,x)(-\Delta\varphi)(x)\,\mathrm{d}x \\ &+ \int_{\mathbb{R}^n} \psi_t(0)u(0,x)(-\Delta\varphi)(x)\,\mathrm{d}x \\ &+ \int_0^T \psi_{tt}(t) \int_{\mathbb{R}^n} u(t,x)(-\Delta\varphi)(x)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{210}$$

Lastly, applying the boundary conditions $u(0,x) = u_0(x)$ and $u_t(0,x) = u_1(x)$ we arrive exactly at (206), thus concluding the proof.

■

The idea of using polynomially decaying test functions instead of the usual compactly supported test functions is quite recent, from D'Abbicco and Fujiwara in 2021 (D'ABBICCO; FUJIWARA, 2021). In the problem addressed in (D'ABBICCO; FUJIWARA, 2021), the nonlinearity is a p–power time-derivative of u, that is, $|\partial_t^\ell u|^p$, with ℓ positive integer. Since in our case the nonlinearity is a memory term, we must change the argument a little, specifically on the time-related part. To this end, we introduce the fractional and differential operators as follows, as well as an adequate time-dependent function. For the fractional and differential operators, we may follow the definition given in (SAMKO; KILBAS; MARICHEV, 1993). We refer the reader to (SAMKO; KILBAS; MARICHEV, 1993) for more properties related to these operators.

Definition 4.1.10 *Let $\alpha \in (0,1)$ and fix $T > 0$. We define, for any function $f \in L^1(0,T)$,*

$$J^{\alpha}_{0|t} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{-(1-\alpha)} f(s)\, \mathrm{d}s, \tag{211}$$

$$J^{\alpha}_{t|T} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{-(1-\alpha)} f(s)\, \mathrm{d}s, \tag{212}$$

called the left-sided and right-sided Riemann-Liouville fractional integrals of the order α.

Definition 4.1.11 *Let $\alpha \in (0,1)$ and fix $T > 0$. We define, for any function $f \in AC([0,T])$, the space of all absolutely continuous functions on $[0,T]$,*

$$D^{\alpha}_{0|t} f(t) := \partial_t J^{1-\alpha}_{0|t} f(t), \tag{213}$$

$$D^{\alpha}_{t|T} f(t) := -\partial_t J^{1-\alpha}_{t|T} f(t), \tag{214}$$

which are called the left-sided and right-sided Riemann-Liouville fractional derivatives.

The following theorem asserts that the Riemann-Liouville Integral and Derivative are, in some sense, inverse of each other. For its proof and a more detailed explanation, the reader is addressed to (SAMKO; KILBAS; MARICHEV, 1993).

Theorem 4.1.12 ((SAMKO; KILBAS; MARICHEV, 1993), p.44) *Let $\alpha \in (0,1)$. Then the equality*

$$D^{\alpha}_{0|t} J^{\alpha}_{0|t} f(t) = f(t) \tag{215}$$

is valid for any $f \in L^1([0,T])$., Also, the equality

$$J^{\alpha}_{0|t} D^{\alpha}_{0|t} f(t) = f(t) \tag{216}$$

is satisfied for $f(t) \in J^{\alpha}_{0|t}(L^1([0,T]))$.

With this essential property, we can also prove a generalization of the integration by parts formula.

Proposition 4.1.13 (Integration by Parts Formula) *Assume that* $\alpha \in (0,1)$, $\varphi \in L^p([0,T])$, $\psi \in L^q([0,T])$, *and* $\frac{1}{p} + \frac{1}{q} \le 1 + \alpha$. *Then, the formula for fractional integration by parts,*

$$\int_0^T \varphi(t)(J^\alpha_{0|t}\psi)(t)\,dt = \int_0^T \psi(t)(J^\alpha_{t|T}\varphi)(t)\,dt \tag{217}$$

is valid.

Proof: The proof is based in interchanging the order of integration, by making use of Fubini's Theorem. To justify its application, we argue as follows. First, from the Hardy-Littlewood Theorem with limiting exponent (see (SAMKO; KILBAS; MARICHEV, 1993),Th.3.5), the fractional integration operator $J^\alpha_{0|t}$ is bounded from $L^p([0,T])$ into $L^q([0,T])$, with $q = \frac{p}{1-\alpha p}$. Therefore, one can apply Hölder inequality in (217) and see that both integrals are absolutely convergent, and thus we can use Fubini's Theorem. We have then

$$\begin{aligned}
\int_0^T \varphi(t)(J^\alpha_{0|t}\psi)(t)\,\mathrm{d}t &= \int_0^T \varphi(t)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\psi(s)\mathrm{d}s\,\mathrm{d}t \\
&= \frac{1}{\Gamma(\alpha)}\int_0^T\int_0^T \varphi(t)\chi_{[0,t]}(s)\,(t-s)^{\alpha-1}\psi(s)\,\mathrm{d}s\mathrm{d}t \\
&= \frac{1}{\Gamma(\alpha)}\int_0^T \psi(s)\int_0^T \varphi(t)\chi_{[s,T]}(t)\,(t-s)^{\alpha-1}\,\mathrm{d}t\mathrm{d}s \\
&= \frac{1}{\Gamma(\alpha)}\int_0^T \psi(s)\int_s^T \varphi(t)(t-s)^{\alpha-1}\,\mathrm{d}t\mathrm{d}s \\
&= \int_0^T \psi(s)(J^\alpha_{t|T}\varphi)(s)\,\mathrm{d}s,
\end{aligned}$$

where $\chi_A(x)$ represents the indicator function of $x \in A$, that is, $\chi_A(x) = 1$, if $x \in A$ and $\chi_A(x) = 0$, if $x \notin A$. ■

As a consequence, we can prove also a version of the integration by parts formula with Riemann-Liouville Integrals in the argument of integration over $[0,T]$.

Proposition 4.1.14 *Assume that* $\alpha \in (0,1)$, $f \in J^\alpha_{0|t}(L^p([0,T]))$, $g \in J^\alpha_{t|T}(L^q([0,T]))$ *with* $\frac{1}{p} + \frac{1}{q} \le 1 + \alpha$. *Then, the formula*

$$\int_0^T (D^\alpha_{0|t}f)(t)g(t)\,dt = \int_0^T f(t)(D^\alpha_{t|T}g)(t)\,dt \tag{218}$$

holds.

Proof: Let $\varphi(t) = D^{\alpha}_{0|t}f(t) \in L^p([0,T])$ and $\psi(t) = D^{\alpha}_{t|T}g(t) \in L^q([0,T])$. Then, using (216) and integration by parts, we get

$$\begin{aligned}
\int_0^T (D^{\alpha}_{0|t}f)(t)g(t)\,\mathrm{d}t &= \int_0^T \varphi(t)g(t)\,\mathrm{d}t \\
&= \int_0^T \varphi(t)\left(J^{\alpha}_{t|T}D^{\alpha}_{t|T}g\right)(t)\,\mathrm{d}t \\
&= \int_0^T \varphi(t)(J^{\alpha}_{t|T}\psi)(t)\,\mathrm{d}t \\
&= \int_0^T (J^{\alpha}_{0|t}\varphi)(t)\psi(t)\,\mathrm{d}t \\
&= \int_0^T \left(J^{\alpha}_{0|t}D^{\alpha}_{0|t}f\right)(t)\psi(t)\,\mathrm{d}t \\
&= \int_0^T f(t)(D^{\alpha}_{t|T}g(t))\,\mathrm{d}t.
\end{aligned}$$

■

Definition 4.1.15 *We define the auxiliary function, for a fixed $T > 0$:*

$$\omega_T(t) = \begin{cases} 1 - \frac{t}{T}, & t \in [0,T] \\ 0, & t > T. \end{cases} \tag{219}$$

Remark 4.1.16 *We remark that* $\operatorname{supp}\omega_T = [0,T]$ *and that* $(\omega_T(t))^{\beta} \in \mathcal{C}^k_c([0,\infty))$, *for any* $\beta > k$. *It also has the following interaction with the fractional derivative, which is of our interest:*

Lemma 4.1.17 *For any* $\alpha \in (0,1)$, *it follows that*

$$D^{\alpha}_{t|T}\omega_T(t)^{\beta} = C_{\alpha,\beta}T^{-\alpha}\omega_T(t)^{\beta-\alpha}, \tag{220}$$

for every $\beta > \alpha$, *where*

$$C_{\alpha,\beta} = \frac{\Gamma(\beta+1)}{(\beta+2-\alpha)\Gamma(\beta-\alpha)}.$$

Proof: We have initially that, making the change of variables $\tau = \frac{s-t}{T-t}$,

$$\begin{aligned}
\Gamma(1-\alpha)J^{1-\alpha}_{t|T}\omega_T(t)^{\beta} &= \int_t^T (s-t)^{-\alpha}\left(1-\tfrac{s}{T}\right)^{\beta}\,\mathrm{d}s \\
&= \int_0^1 \tau^{-\alpha}(T-t)^{-\alpha}T^{-\beta}(T-t)^{\beta}(1-\tau)^{\beta}(T-t)\,\mathrm{d}\tau \\
&= (T-t)^{-\alpha+\beta+1}T^{-\beta}\int_0^1 \tau^{-\alpha}(1-\tau)^{\beta}\,\mathrm{d}\tau \\
&= \omega_T(t)^{-\alpha+\beta+1}T^{1-\alpha}\,\frac{\Gamma(1-\alpha)\Gamma(1+\beta)}{\Gamma(\beta+2-\alpha)}.
\end{aligned}$$

Hence,

$$J^{1-\alpha}_{t|T}\omega_T(t)^\beta = \frac{\Gamma(1+\beta)}{\Gamma(\beta+2-\alpha)}\, T^{1-\alpha}\;\omega_T(t)^{-\alpha+\beta+1}. \tag{221}$$

We now differentiate with respect to t, obtaining

$$\partial_t\omega_T(t)^{-\alpha+\beta+1} = (-\alpha+\beta+1)\left(1-\tfrac{t}{T}\right)^{-\alpha+\beta}(-T^{-1}). \tag{222}$$

Using (222) in (221), we get

$$\begin{aligned} D^{\alpha}_{t|T}\omega_T(t)^\beta &= \frac{\Gamma(1+\beta)}{\Gamma(\beta+2-\alpha)}\, T^{1-\alpha}(-\alpha+\beta+1)\left(1-\tfrac{t}{T}\right)^{-\alpha+\beta} T^{-1} \\ &= \frac{\Gamma(\beta+1)}{(\beta+2-\alpha)\Gamma(\beta-\alpha)}\; T^{-\alpha}\,\omega_T(t)^{\beta-\alpha}. \end{aligned}$$

■

We conclude this section remarking that the function $D^{\alpha}_{t|T}\omega_T(t)^\beta$ satisfies all the requirements to be chosen as a time-dependent test function according to Definition 4.1.7, for an appropriately chosen β. Indeed, supp $D^{\alpha}_{t|T}\omega_T(t)^\beta = [0,T]$, and it is in $\mathcal{C}^2$, for any β such that $\beta-\alpha > 2$, or equivalently, $\beta > 2+\alpha$. Also, from the last lemma, we have $D^{\alpha}_{t|T}\omega_T(0)^\beta = 0$, and the same is true for its derivative, as we'll see later in (228).

4.2 NONEXISTENCE RESULTS

Since the Fractional Laplacian is a nonlocal operator, it is not easy, in general, to prove the optimality of $\overline{p}$ by applying the test function method. This section is dedicated to state the nonexistence theorem for the subcritical case.

Theorem 4.2.1 *Let p_c be as in Definition* (2)*, define*

$$\overline{p} = \begin{cases} \gamma^{-1}, & \text{, if } \gamma \in \left(0, \frac{n-2}{n}\right] \\ p_c, & \text{if } \gamma \in \left(\frac{n-2}{n}, 1\right) \end{cases}$$

and fix $q = n + 2\theta$. Assume that $u_0, u_1 \in L^1(\mathbb{R}^n, \langle x\rangle^q dx)$. Moreover, assume the sign condition

$$\int_{\mathbb{R}^n} u_1\,dx > 0.$$

If there exists a global in-time (nontrivial) weak solution

$$u \in L^p\left([0,\infty), L^p(\mathbb{R}^n, \langle x\rangle^{-q}dx)\right)$$

to problem (1), then $p \geq \overline{p}$.

Proof: We assume $p < \overline{p}$, and that u is a nontrivial global in-time weak solution to (1). We put $\alpha = 1 - \gamma$, fix $\beta > (\alpha + 2)p'$ and fix suitable test functions depending on a parameter $R \gg 1$ as follows.

Consider the function ω_T as in Definition 4.1.15, and let $\varphi \in \mathcal{C}^\infty$ be defined as $\varphi(x) := \langle x\rangle^{-q}$.

For any $R \gg 1$, and for a fixed $T > 0$, we define

$$\varphi_R(x) = \varphi(R^{-1}x), \quad \psi_T(t) = D^{\alpha}_{t|T}\left(\omega_T(t)^{\beta}\right).$$

First of all, we observe that, for any $\sigma \geq 0$,

$$\left((-\Delta)^{\sigma}\varphi_R\right)(x) = R^{-2\sigma}(-\Delta)^{\sigma}\varphi(x).$$

Since u is a global weak solution to (1), it satisfies, for any $T > 0$,

$$\begin{aligned}
&\int_0^T \psi_T(t)\int_{\mathbb{R}^n} F(t,u)\varphi_R(x)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= \int_0^T \psi_T(t)\int_{\mathbb{R}^n} u(t,x)(-\Delta\varphi_R)(x)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+ \int_0^T \psi_T(t)\int_{\mathbb{R}^n} u(t,x)(\Delta^2\varphi_R)(x)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad- \int_0^T \partial_t\psi_T(t)\int_{\mathbb{R}^n} u(t,x)((-\Delta)^{\theta}\varphi_R)(x)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+ \int_0^T \partial_{tt}\psi_T(t)\int_{\mathbb{R}^n} u(t,x)\varphi_R(x)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+ \int_0^T \partial_{tt}\psi_T(t)\int_{\mathbb{R}^n} u(t,x)(-\Delta\varphi_R)(x)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad- \int_{\mathbb{R}^n} u_0(x)(-\Delta)^{\theta}\varphi_R(x)\,\mathrm{d}x\\
&\quad- \int_{\mathbb{R}^n} u_1(x)\varphi_R(x)\,\mathrm{d}x - \int_{\mathbb{R}^n} u_1(x)(-\Delta\varphi_R)(x)\,\mathrm{d}x.
\end{aligned} \tag{223}$$

Firstly, we obtain the following identity: Since $u_0, u_1 \in L^1(\langle x\rangle^q dx)$ and φ_R, $(-\Delta)^{\theta}\varphi_R, \Delta\varphi_R \in L^\infty(\langle x\rangle^{-q}dx)$, Lebesgue's Dominant Convergence Theorem ensures that

$$\begin{aligned}
&\lim_{R\to\infty}\left(\int_{\mathbb{R}^n} u_0((-\Delta)^{\theta}\varphi_R)\,\mathrm{d}x + \int_{\mathbb{R}^n} u_1\varphi_R\,\mathrm{d}x + \int_{\mathbb{R}^n} u_1(-\Delta\varphi_R)\,\mathrm{d}x\right)\\
&\quad= \int_{\mathbb{R}^n} u_0 \lim_{R\to\infty}((-\Delta)^{\theta}\varphi_R)\,\mathrm{d}x\\
&\qquad+ \int_{\mathbb{R}^n} u_1 \lim_{R\to\infty}\varphi_R\,\mathrm{d}x + \int_{\mathbb{R}^n} u_1 \lim_{R\to\infty}(-\Delta\varphi_R)\,\mathrm{d}x\\
&\quad= \int_{\mathbb{R}^n} u_0 \lim_{R\to\infty} R^{-2\theta}(-\Delta)^{\theta}\varphi\,\mathrm{d}x + \int_{\mathbb{R}^n} u_1 \lim_{R\to\infty}\varphi_R\,\mathrm{d}x\\
&\qquad+ \int_{\mathbb{R}^n} u_1 \lim_{R\to\infty} R^{-2}(-\Delta\varphi)\,\mathrm{d}x\\
&\quad= \int_{\mathbb{R}^n} u_1\,\mathrm{d}x.
\end{aligned} \tag{224}$$

Due to the sign assumption from hypothesis, the latter is positive. Hence, for R large enough,

$$\begin{aligned}\int_0^T \psi_T(t)\int_{\mathbb{R}^n} F(t,u)\varphi_R(x)\,\mathrm{d}x\,\mathrm{d}t < &\int_0^T \psi_T(t)\int_{\mathbb{R}^n} u(t,x)(-\Delta\varphi_R)(x)\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^T \psi_T(t)\int_{\mathbb{R}^n} u(t,x)(\Delta^2\varphi_R)(x)\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^T \partial_t\psi_T(t)\int_{\mathbb{R}^n} u(t,x)((-\Delta)^\theta\varphi_R)(x)\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^T \partial_{tt}\psi_T(t)\int_{\mathbb{R}^n} u(t,x)\varphi_R(x)\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^T \partial_{tt}\psi_T(t)\int_{\mathbb{R}^n} u(t,x)(-\Delta\varphi_R)(x)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}\tag{225}$$

Now, from the properties from Propositions 4.1.12 and 4.1.14, since $F(t,u) = \Gamma(\alpha)J^{\alpha}_{0|t}|u|^p$, the left-hand side can be seen as

$$\begin{aligned}\int_0^T \psi_T(t)\int_{\mathbb{R}^n} F(t,u)\varphi_R(x)\,\mathrm{d}x\,\mathrm{d}t &= \Gamma(\alpha)\int_0^T\int_{\mathbb{R}^n}\omega_T(t)^\beta|u(t,x)|^p\varphi_R(x)\,\mathrm{d}x\,\mathrm{d}t\\ &:= \Gamma(\alpha)I(u).\end{aligned}$$

On the other hand, we estimate the five terms in the right-hand side. To do so, we integrate in $[0,T]$ in time and in B_R in space, and after that we control the speeds at which we let T and R go to infinity.

We recall here the following estimates:

$$\psi_T(t) \le CT^{-\alpha}\omega_T(t)^{\beta-\alpha}, \qquad |((-\Delta)^\sigma\varphi_R)(x)| \le CR^{-2\sigma}|\varphi_R(x)|.$$

Applying Young's inequality,

$$\begin{aligned}&\left|\int_0^T \psi_T(t)\int_{B_R} u(t,x)(-\Delta\varphi_R)(x)\,\mathrm{d}x\,\mathrm{d}t\right|\\ &\qquad\le \varepsilon\int_0^T\int_{B_R}\omega_T(t)^\beta|u|^p\varphi_R\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\quad + C_\varepsilon\int_0^T\int_{B_R}\psi_T^{p'}\omega_T^{-\frac{\beta p'}{p}}|(-\Delta\varphi_R)(x)|^{p'}|\varphi_R|^{-\frac{p'}{p}}\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\le \varepsilon I(u) + C_\varepsilon\int_0^T\int_{B_R} CT^{-\alpha p'}\omega_T^{\beta-\alpha p'}R^{-2p'}|\varphi_R|\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Since $\omega_T(t) \le 1$, $|\varphi_R(x)| \le 1$ and $\beta - \alpha p' > 0$, we get

$$\begin{aligned}&\left|\int_0^T \psi_T(t)\int_{B_R} u(t,x)(-\Delta\varphi_R)(x)\,\mathrm{d}x\,\mathrm{d}t\right|\\ &\qquad\le \varepsilon I(u) + C_\varepsilon T^{-\alpha p'}R^{-2p'}\int_0^T\int_{B_R}\mathrm{d}x\,\mathrm{d}t\\ &\qquad\le \varepsilon I(u) + C_\varepsilon T^{1-\alpha p'}R^{n-2p'}.\end{aligned}\tag{226}$$

For the second integral in the right-hand side of (225), we proceed similarly, obtaining

$$\begin{aligned}\left|\int_0^T \psi_T(t)\int_{\mathbb{R}^n} u(t,x)(\Delta^2\varphi_R)(x)\,\mathrm{d}x\,\mathrm{d}t\right|&\\ \le \varepsilon\int_0^T\int_{B_R}\omega_T(t)^\beta|u|^p\varphi_R\,\mathrm{d}x\,\mathrm{d}t&\\ + C_\varepsilon\int_0^T\int_{B_R}\psi_T^{p'}\omega_T^{-\frac{\beta p'}{p}}|(\Delta^2\varphi_R)(x)|^{p'}|\varphi_R|^{-\frac{p'}{p}}\,\mathrm{d}x\,\mathrm{d}t&\\ \le \varepsilon I(u) + C_\varepsilon\int_0^T\int_{B_R} CT^{-\alpha p'}\omega_T^{\beta-\alpha p'}R^{-4p'}|\varphi_R|\,\mathrm{d}x\,\mathrm{d}t.&\end{aligned} \tag{227}$$

Now, the third integral has a time derivative of $\psi_T(t)$. Differentiating (220) with respect to t, we get

$$\begin{aligned}\partial_t\psi_T(t) &= C_{\alpha,\beta}T^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left(1-\tfrac{t}{T}\right)^{\beta-\alpha}\\ &= -C_{\alpha,\beta}T^{-\alpha-1}\omega_T(t)^{\beta-\alpha-1}.\end{aligned} \tag{228}$$

With this identity, we estimate the third integral in the RHS of (225):

$$\begin{aligned}\left|\int_0^T \partial_t\psi_T\int_{\mathbb{R}^n} u((-\Delta)^\theta\varphi_R)\,\mathrm{d}x\,\mathrm{d}t\right|&\\ \le \varepsilon\int_0^T\int_{B_R}\omega_T(t)^\beta|u|^p\varphi_R\,\mathrm{d}x\,\mathrm{d}t&\\ + C_\varepsilon\int_0^T\int_{B_R}\partial_t\psi_T^{p'}\omega_T^{-\frac{\beta p'}{p}}|((-\Delta)^\theta\varphi_R)(x)|^{p'}|\varphi_R|^{-\frac{p'}{p}}\,\mathrm{d}x\,\mathrm{d}t&\\ \le \varepsilon I(u) + C_\varepsilon\int_0^T\int_{B_R} CT^{-(\alpha+1)p'}\omega_T^{\beta-(\alpha+1)p'}R^{-2\theta p'}|\varphi_R|\,\mathrm{d}x\,\mathrm{d}t&\\ \le \varepsilon I(u) + C_\varepsilon T^{1-(\alpha+1)p'}R^{n-2\theta p'},&\end{aligned} \tag{229}$$

since $\beta-(\alpha+1)p' > 0$. For the last two integrals, we differentiate (220) twice, obtaining

$$\partial_{tt}\psi_T(t) = C_{\alpha,\beta}T^{-\alpha-2}\omega_T(t)^{\beta-\alpha-2},$$

and thus we obtain

$$
\begin{aligned}
&\left|\int_0^T \partial_{tt}\psi_T \int_{\mathbb{R}^n} u\varphi_R \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad\leq \varepsilon \int_0^T \int_{B_R} \omega_T(t)^\beta |u|^p \varphi_R \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad+ C_\varepsilon \int_0^T \int_{B_R} \partial_{tt}\psi_T^{p'} \omega_T^{-\frac{\beta p'}{p}} |\varphi_R|^{p'} |\varphi_R|^{-\frac{p'}{p}} \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\leq \varepsilon I(u) + C_\varepsilon \int_0^T \int_{B_R} CT^{-(\alpha+2)p'} \omega_T^{\beta-(\alpha+2)p'} |\varphi_R| \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\leq \varepsilon I(u) + C_\varepsilon T^{1-(\alpha+2)p'} R^n,
\end{aligned} \tag{230}
$$

and lastly, using that $\beta - (\alpha+2)p' > 0$,

$$
\begin{aligned}
&\left|\int_0^T \partial_{tt}\psi_T \int_{\mathbb{R}^n} u(-\Delta\varphi_R) \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad\leq \varepsilon \int_0^T \int_{B_R} \omega_T(t)^\beta |u|^p \varphi_R \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad+ C_\varepsilon \int_0^T \int_{B_R} \partial_{tt}\psi_T^{p'} \omega_T^{-\frac{\beta p'}{p}} |\varphi_R|^{p'} |(-\Delta)\varphi_R|^{-\frac{p'}{p}} \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\leq \varepsilon I(u) + C_\varepsilon \int_0^T \int_{B_R} CT^{-(\alpha+2)p'} \omega_T^{\beta-(\alpha+2)p'} R^{-2p'} |\varphi_R| \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\leq \varepsilon I(u) + C_\varepsilon T^{1-(\alpha+2)p'} R^{n-2p'}.
\end{aligned} \tag{231}
$$

Now, using (226), (227), (229),(230) and (231) in (225), we find

$$
\begin{aligned}
&(\Gamma(\alpha) - 5\varepsilon) \int_0^T \int_{B_R} \omega_T(t)^\beta |u(t,x)|^p \varphi_R(x) \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad < C_\varepsilon \Big(T^{1-\alpha p'} R^{n-2p'} + T^{1-\alpha p'} R^{n-4p'} \\
&\qquad\qquad + T^{1-(\alpha+1)p'} R^{n-2\theta p'} + T^{1-(\alpha+2)p'} R^n + T^{1-(\alpha+2)p'} R^{n-2p'} \Big).
\end{aligned} \tag{232}
$$

Choosing $\varepsilon > 0$ small enough we get, for R, T large enough,

$$
\begin{aligned}
&\int_0^T \int_{B_R} \omega_T(t)^\beta |u(t,x)|^p \varphi_R(x) \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad < C_{\alpha,\varepsilon} \Big(T^{1-\alpha p'} R^{n-2p'} + T^{1-\alpha p'} R^{n-4p'} \\
&\qquad\qquad + T^{1-(\alpha+1)p'} R^{n-2\theta p'} + T^{1-(\alpha+2)p'} R^n + T^{1-(\alpha+2)p'} R^{n-2p'} \Big).
\end{aligned} \tag{233}
$$

Now, first consider the case where $\gamma > \frac{n-2}{n}$. We stress that this is always true for $n = 1, 2$. For $\eta \geq 0$, we set $T = R^\eta$. Then, we have

$$
\begin{aligned}
&\int_0^{R^\eta} \int_{B_R} \omega_T(t)^\beta |u(t,x)|^p \varphi_R(x) \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad < C_{\alpha,\varepsilon} \Big(R^{-(\alpha\eta+2)p'+n+\eta} + R^{-(\alpha\eta+4)p' n+\eta} \\
&\qquad\qquad + R^{-((\alpha+1)\eta+2\theta)p'+n+\eta} + R^{-(\alpha+2)\eta p'+n+\eta} \\
&\qquad\qquad + R^{-((\alpha+2)\eta+2)p'+n+\eta} \Big).
\end{aligned} \tag{234}
$$

Now, we define, for $\eta \geq 0$,

$$\begin{aligned} g(\eta) &= \min\left\{\alpha\eta + 2, \alpha\eta + 4, (\alpha+1)\eta + 2\theta, (\alpha+2)\eta, (\alpha+2)\eta + 2\right\} \\ &= \min\left\{\alpha\eta + 2, (\alpha+1)\eta + 2\theta, (\alpha+2)\eta\right\} \\ &= \begin{cases} (3-\gamma)\eta, & \text{if } \eta \in [0, 2\theta] \\ (2-\gamma)\eta + 2\theta, & \text{if } \eta \in (2\theta, 2(1-\theta)], \\ (1-\gamma)\eta + 2, & \text{if } \eta \in (2(1-\theta), \infty), \end{cases} \end{aligned} \tag{235}$$

where in the last step we used that $\alpha = 1 - \gamma$. This function $g(\eta)$ gives the fastest possible decay for which we can control all five terms in (234). Below, we see a graphical representation of it.

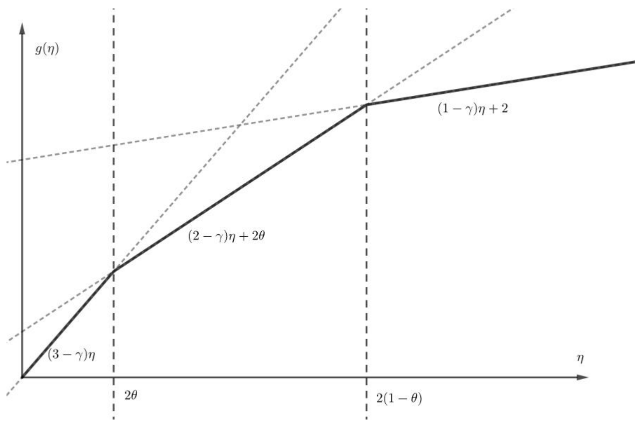

Figure 1 – Function $g(\eta)$

Therefore, we have that

$$\int_0^{R^\eta} \int_{B_R} \omega_T(t)^\beta |u(t,x)|^p \varphi_R(x)\,\mathrm{d}x\,\mathrm{d}t < 5C_{\alpha,\varepsilon} R^{-g(\eta)p' + n + \eta}. \tag{236}$$

To conclude the argument, it is sufficient that the exponent above is negative. This will happen if

$$g(\eta)p' - (n+\eta) > 0, \tag{237}$$

or equivalently,

$$p' > \frac{n+\eta}{g(\eta)} \iff p < \frac{n+\eta}{n+\eta-g(\eta)} = 1 + \frac{g(\eta)}{n+\eta-g(\eta)} := h(\eta). \tag{238}$$

In particular, for $\eta = 2(1-\theta)$, we have

$$\begin{aligned} h(2(1-\theta)) &= 1 + \frac{2(1-\gamma)(1-\theta) + 2}{n + 2(1-\theta) - 2(1-\gamma)(1-\theta) - 2} \\ &= 1 + \frac{2 + 2(1-\gamma)(1-\theta)}{n - 2 + 2\gamma(1-\theta)} = p_c. \end{aligned} \tag{239}$$

Moreover, we can see that p_c is the greatest value for which we can prove nonexistence in this case. In other words, $h(\eta)$ reaches its maximum in $\gamma = 2(1-\theta)$, assuming that $\gamma > \frac{n-2}{n}$.

Indeed, since

$$h'(\eta) = \frac{-g(\eta) + (n+\eta)g'(\eta)}{(n+\eta-g(\eta))^2},$$

its sign is determined by the sign of its numerator $-g(\eta) + (n+\eta)g'(\eta)$, which is given by

$$\operatorname{sgn}(-g(\eta) + (n+\eta)g'(\eta)) = \begin{cases} n(3-\gamma) > 0, & \text{if } \eta \in (0, 2\theta) \\ n(2-\gamma) - 2\theta > 0 & \text{if } \eta \in [2\theta, 2(1-\theta)) \\ -2 + (1-\gamma)n, & \text{if } \eta \in (2(1-\theta), \infty), \end{cases}$$

and in the last case, we have

$$-2 + (1-\gamma)n < 0 \iff \gamma > \frac{n-2}{n}.$$

Since $h(\eta)$ is continuous by parts, it has a local maximum point at $\eta = 2(1-\theta)$ when $\gamma > \frac{n-2}{n}$.

Returning to (236), if $p < p_c$, then applying Beppo-Levi's Monotone Convergence Theorem, since $\omega_T(t), \varphi_R(x) \nearrow 1$ when $T, R \nearrow \infty$, we obtain

$$\int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^p \, \mathrm{d}x \, \mathrm{d}t \leq 0 \Rightarrow u \equiv 0, \tag{240}$$

a contradiction.

Now, assume that $\gamma \leq \frac{n-2}{n}$. In this case, the auxiliary function $h(\eta)$ is always non-decreasing. Since

$$\lim_{\eta \to \infty} h(\eta) = \gamma^{-1},$$

it is expected that γ^{-1} will replace p_c in this case. Below, we show both cases for $h(\eta)$; the first shows the case where p_c is the maximum of $h(\eta)$, that is, when $\gamma > \frac{n-2}{n}$; the second illustrates the case where $h(\eta)$ does not achieve its maximum, but monotonically tends to its supremum γ^{-1}.

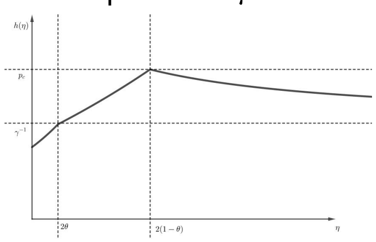

Figure 2 – $h(\eta)$, $\gamma > \frac{n-2}{n}$

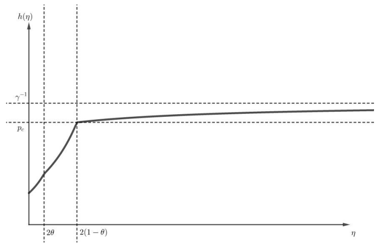

Figure 3 – $h(\eta)$, $\gamma \leq \frac{n-2}{n}$

We set $R = \ln T$ in (233), thus obtaining

$$\int_0^T \int_{B_R} \omega_T(t)^\beta |u(t,x)|^p \varphi_R(x)\,\mathrm{d}x\,\mathrm{d}t$$
$$< C_{\alpha,\varepsilon} T^{1-\alpha p'} \Big((\ln T)^{n-2p'} + (\ln T)^{n-4p'} + T^{-p'} (\ln T)^{n-2\theta p'} + T^{-2p'} (\ln T)^n + T^{-2p'} (\ln T)^{n-2p'} \Big). \tag{241}$$

For T large enough, the logarithmic terms are bounded by $C_\delta T^\delta$, for any $\delta > 0$. Hence,

$$\int_0^T \int_{B_R} \omega_T(t)^\beta |u(t,x)|^p \varphi_R(x)\,\mathrm{d}x\,\mathrm{d}t < C_{\alpha,\varepsilon,\delta} T^{1-\alpha p'+\delta}. \tag{242}$$

If $p < \frac{1}{\gamma}$, then $\alpha p' > 1$. Then, choosing $\delta \in (0, \alpha p' - 1)$, we obtain again (240), which contradicts the assumption that u is nontrivial.

■

5 FINAL REMARKS

In this thesis, we proposed and analyzed a class of evolution PDEs that possess several terms that are already difficult to deal with separately, and these difficulties add up and even overlap in some cases. The presence of the rotational inertia term, $-\Delta u_{tt}$, makes it so one does not have exponential decay rates in the high-frequency region, which makes us having to worry about these high-frequency L^p estimates for the fundamental solutions u_0, u_1 and for the convolution with the operator $(I-\Delta)^{-1}K_1$. The fractional dissipation term, $(-\Delta)^{\theta}u_t$, interferes in the eigenvalues's profile, which changes the whole set of estimates we obtained through all of our work. It is important to remark that the case where $\theta \in \left(\frac{1}{2}, 1\right)$, often called the *non-effective damping* case, leads to a very different approach. Lastly, the memory nonlinearity has its own particular influence in the problem, leading to a completely different critical exponent in some cases.

In order to summarize the results and strategies employed in the several cases considered, we provide a table that displays the different ranges for the parameters θ, γ and p, as well as the correspondent spaces required for the initial data u_0 and u_1. Here, we recall

$$p_c = p_c(n,\gamma,\theta) := 1 + \frac{2\,(1+(1-\gamma)(1-\theta))}{n-2+2\gamma(1-\theta)}, \qquad s_c = s_c(\gamma,\theta) := 2 + 2\gamma(1-\theta)$$

$$\theta_0 := \frac{13-\sqrt{145}}{12} \qquad \gamma_0 = \gamma_0(\theta) := \frac{2\theta^2-3\theta+7}{8(1-\theta)^2}$$

$$\tilde{p}_c := \frac{14-10\theta}{7+2\gamma(1-\theta)-2\theta}.$$

n	θ	γ	p	u_0	u_1
1	$\left[0, \frac{1}{2}\right)$	$\left(\frac{1}{2(1-\theta)}, 1\right)$	(p_c, ∞)	$H^{s_c} \cap L^1$	$H^{s_c-1} \cap L^1$
2	$\left[0, \frac{1}{2}\right)$	$(0, 1)$	(p_c, ∞)	$H^{s_c} \cap L^1$	$H^{s_c-1} \cap L^1$
3	$\left[0, \frac{1}{2}\right)$	$\left(0, \frac{1}{3}\right]$	$\left(\frac{1}{\gamma}, \infty\right)$	$H^{s_c} \cap L^1$	$H^{s_c-1} \cap L^1$
3	$\left[0, \frac{1}{2}\right)$	$\left(\frac{1}{3}, \frac{3-2\theta}{4(1-\theta)}\right]$	(p_c, ∞)	$H^{s_c} \cap L^1$	$H^{s_c-1} \cap L^1$
3	$\left[0, \frac{1}{2}\right)$	$\left(\frac{3-2\theta}{4(1-\theta)}, 1\right)$	$(p_c, 2)$	$H^{s_c} \cap L^1 \cap \dot{W}^{3,p}$	$H^{s_c-1} \cap L^1 \cap \dot{W}^{2,p}$
3	$\left[0, \frac{1}{2}\right)$	$\left(\frac{3-2\theta}{4(1-\theta)}, 1\right)$	$[2, \infty)$	$H^{s_c} \cap L^1$	$H^{s_c-1} \cap L^1$
4	$\left[0, \frac{1}{2}\right)$	$\left(0, \frac{1}{2}\right]$	$\left(\frac{1}{\gamma}, \infty\right)$	$H^{s_c} \cap L^1$	$H^{s_c-1} \cap L^1$
4	$\left[0, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 1\right)$	$[2, \infty)$	$H^{s_c} \cap L^1$	$H^{s_c-1} \cap L^1$
4	$[0, \theta_0)$	$\left(\frac{1}{2}, \gamma_0\right]$	$(p_c, 2)$	$H^{s_c} \cap L^1 \cap \dot{W}^{3,p}$	$H^{s_c-1} \cap L^1 \cap \dot{W}^{2,p}$
4	$[\theta_0, \frac{1}{2})$	$\left(\frac{1}{2}, 1\right)$	$(p_c, 2)$	$H^{s_c} \cap L^1 \cap \dot{W}^{3,p}$	$H^{s_c-1} \cap L^1 \cap \dot{W}^{2,p}$
4	$[0, \theta_0)$	$(\gamma_0, 1)$	$(\tilde{p}_c, 2]$	$H^{s_c} \cap L^1 \cap \dot{W}^{3,p}$	$H^{s_c-1} \cap L^1 \cap \dot{W}^{2,p}$

Table 1 – Obtained Results

The strategies we used to estimate norms in each theorem are briefly described below:

- In Theorems 3.1.1 and 3.2.1, we used Gagliardo-Nirenberg, with $\kappa = 1$; For the special case $n = 1, \theta \geq \frac{1}{4}$, we used Sobolev Embedding into L^p instead; We also applied Sobolev Embedding into $L^{q(p-1)}$ and L^r with $q \to \infty$ and $r \to 2$, in order to improve the decay rate;
- In Theorems 3.4.1 and 3.5.1, we used Sobolev embedding into L^p, as well as Sobolev embeddings into $L^{n(p-1)}$ and $L^{2n/(n-2)}$;
- In Theorem 3.4.2, we used fractional Gagliardo-Nirenberg with $\kappa = \frac{1}{2}$ to reach the critical exponent, and Sobolev embedding into L^p for larger values of p. Sobolev embeddings into $L^{3(p-1)}$ and L^6 were also necessary (as in Theorem 3.4.1);
- In Theorem 3.4.3, we asked for regularity in L^p for the initial data, and applied Sobolev embedding into L^{2p} and L^{p^2};
- In Theorem 3.5.2, we used Sobolev embedding into L^p and Sobolev embeddings into $L^{4(p-1)}$ and L^4 (by the same reason as in Theorems 3.4.1 and 3.4.2);

- In Theorems 3.5.3 and 3.5.4, we again required additional L^p regularity for initial data, and applied Sobolev embedding into L^{p^2} and into $L^{4(p-1)}$ and L^4.

There are several possible generalizations and extensions to this problem that can be explored in the future. For instance, changing the rotational inertia term into $(-\Delta)^\sigma u_{tt}$, with $\sigma \in (0,1)$, would yield similar results without great modifications in the approach. Adding and changing terms concerning Laplace operators $(-\Delta)^\delta u$ can also be done, since the estimates depend only on the smallest of such δ for the low-frequency region and on the greatest δ for the high-frequency region.

One can also investigate cases of higher dimensions and obtain directly some partial results with the same techniques, although some new problems may arise; for examples, the fact that for $n > 4$, the critical exponent is no longer greater than $\sqrt{2}$, hence the argument used in the last two theorems can not be replicated. Also, the nonlinearity can be changed to $F(t, u_t)$ instead of $F(t, u)$. This modification changes the profile of the solution, leading to different estimates and results.

Lastly, one can study generalizations in the order of time-dependent derivatives of u, leading to a problem like

$$\partial_t^m u + \sum_{j=0}^{m-1} \Lambda_j u + (-\Delta)^\theta u_t = Fu,$$

where m is a non-negative integer, $\Lambda_j = a_j(-\Delta)^{\sigma_j}$, $a_j \in \mathbb{R}$, $\sigma_j \geq 0$, and Fu can represent any of $\Gamma(1-\gamma)\ J^{1-\gamma}(|u|^p)$ or $\Gamma(1-\gamma)\ J^{1-\gamma}(|u_t|^p)$, with J being the Riemann-Liouville fractional integral of order $1-\gamma$ as defined in the text. This generalization includes many classical PDEs, like the heat, wave and plate equations, as well as models for many problems in solid mechanics and for the vibrations of thin plates as the full von Karmán system and the Timoshenko's model.

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APPENDIX A – A LEMMA FOR BOUNDING INTEGRALS

A.1 AN AUXILIARY LEMMA

Lemma A.1.1 *Let* $\omega \in \mathbb{R},\ \alpha > 1,\ \gamma \in (0,1)$. *Then, it holds*

$$\int_0^t (1+t-\tau)^{-\omega} \int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\alpha}\, ds\, d\tau \lesssim \begin{cases} (1+t)^{-\gamma}, & \omega > 1 \\ (1+t)^{-\gamma}\log(2+t), & \omega = 1 \\ (1+t)^{1-\omega-\gamma}, & \omega < 1. \end{cases}$$

Proof:

We first deal with the inner integral, and show that

$$\int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\alpha}\, \mathrm{d}s \leq C_0(1+\tau)^{-\gamma},$$

for some positive constant C_0.

- if $\tau \leq 2$, then

$$\int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\alpha}\, \mathrm{d}s \leq \int_0^\tau (\tau-s)^{-\gamma}\, \mathrm{d}s = \frac{\tau^{1-\gamma}}{1-\gamma} \leq \frac{2^{1-\gamma}}{1-\gamma}. \tag{243}$$

  From here, one can easily estimate the above constant by the desired continuous function $(1+\tau)^{-\gamma}$ over a compact set $[0,2]$:

$$\tau \leq 2 \Rightarrow 3^{-\gamma} \leq (1+\tau)^{-\gamma} \Rightarrow 1 \leq 3^{\gamma}(1+\tau)^{-\gamma},$$

$$\overset{(243)}{\Rightarrow} \int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\alpha}\, \mathrm{d}s \leq \frac{2^{1-\gamma}}{1-\gamma}3^{\gamma}\ (1+\tau)^{-\gamma}, \quad \text{if } \tau \leq 2.$$

- if $\tau \geq 2$,

$$\int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-\alpha}\, \mathrm{d}s = \int_0^{\frac{\tau}{2}} + \int_{\frac{\tau}{2}}^{\tau}.$$

  For the first integral, we have

$$\begin{aligned} \int_0^{\frac{\tau}{2}} (\tau-s)^{-\gamma}(1+s)^{-\alpha}\, \mathrm{d}s &\leq \left(\frac{\tau}{2}\right)^{-\gamma} \int_0^{\frac{\tau}{2}} (1+s)^{-\alpha}\, \mathrm{d}s \\ &\leq \left(\frac{\tau}{2}\right)^{-\gamma} \int_0^{\infty} (1+s)^{-\alpha}\, \mathrm{d}s \\ &= \left(\frac{\tau}{2}\right)^{-\gamma} \frac{1}{\alpha-1}, \end{aligned} \tag{244}$$

here we used the fact that $\tau - s \geq \frac{\tau}{2}$, and $\gamma > 0$. The partial result comes from straightforward calculations:

$$\tau \geq 2 \Rightarrow 3\,\frac{\tau}{2} = \tau + \frac{\tau}{2} \geq 1+\tau \Rightarrow \left(\frac{\tau}{2}\right)^{-\gamma} \leq 3^{\gamma}(1+\tau)^{-\gamma},$$

$$\overset{(244)}{\Rightarrow} \int_0^{\frac{\tau}{2}} (\tau - s)^{-\gamma}(1+s)^{-\alpha}\, \mathrm{d}s \le \frac{3^\gamma}{\alpha - 1}\ (1+\tau)^{-\gamma}, \quad \text{if } \tau \ge 2.$$

For the second integral,

$$\begin{aligned}
\int_{\frac{\tau}{2}}^{\tau} (\tau - s)^{-\gamma}(1+s)^{-\alpha}\, \mathrm{d}s &\le \left(1 + \frac{\tau}{2}\right)^{-\alpha} \int_{\frac{\tau}{2}}^{\tau} (\tau - s)^{-\gamma}\, \mathrm{d}s \\
&= \left(1 + \frac{\tau}{2}\right)^{-\alpha} \frac{1}{1-\gamma} \left(\frac{\tau}{2}\right)^{1-\gamma} \\
&\le \frac{1}{1-\gamma} \left(\frac{\tau}{2}\right)^{1-\gamma-\alpha}. \qquad (245)
\end{aligned}$$

Once more, a little bit of computation gives us the partial result:

$$\tau \ge 2 \Rightarrow \left(\frac{\tau}{2}\right)^{1-\alpha} \le 1^{1-\alpha} = 1,$$

$$\overset{(245)}{\Rightarrow} \int_{\frac{\tau}{2}}^{\tau} (\tau - s)^{-\gamma}(1+s)^{-\alpha}\, \mathrm{d}s \le \frac{1}{1-\gamma}\ \left(\frac{\tau}{2}\right)^{-\gamma} \le \frac{3^\gamma}{1-\gamma}\ (1+\tau)^{-\gamma}, \quad \text{if } \tau \ge 2.$$

Choosing $C_0 = \max\left\{ \frac{2^{1-\gamma}3^\gamma}{1-\gamma}, \frac{3^\gamma}{\alpha-1} + \frac{3^\gamma}{1-\gamma} \right\}$, we obtain

$$\int_0^t (1+t-\tau)^{-\omega} \int_0^\tau (\tau - s)^{-\gamma}(1+s)^{-\alpha}\, \mathrm{d}s\, \mathrm{d}\tau \le C_0 \int_0^t (1+t-\tau)^{-\omega}(1+\tau)^{-\gamma}\, \mathrm{d}\tau.$$

We now proceed on a similar way as above to bound the integral

$$\int_0^t (1+t-\tau)^{-\omega}(1+\tau)^{-\gamma}\, \mathrm{d}\tau.$$

The case $\omega < 0$ is immediate, because the function $s \mapsto (1+s)^{-\omega}$ is increasing, hence

$$\begin{aligned}
\int_0^t (1+t-\tau)^{-\omega} &\int_0^\tau (\tau - s)^{-\gamma}(1+s)^{-\alpha}\, \mathrm{d}s\, \mathrm{d}\tau \\
&\le C_0 (1+t)^{-\omega} \int_0^t (1+\tau)^{-\gamma}\, \mathrm{d}\tau \\
&= \frac{C_0}{1-\gamma} (1+t)^{-\omega} (1+\tau)^{1-\gamma}\Big|_0^t \\
&\le \frac{C_0}{1-\gamma} (1+t)^{1-\omega-\gamma}. \qquad (246)
\end{aligned}$$

Now, assume $\gamma \ge 0$. If $t \le 2$, then

$$\int_0^t (1+t-\tau)^{-\omega}(1+\tau)^{-\gamma}\, \mathrm{d}\tau \le \int_0^t\ \mathrm{d}\tau = t \le 2,$$

and now, we must only bound this constant 2 by the functions we desire in our result, which is only a matter of finding the right constants:

- $t \leq 2 \Rightarrow 2(1+t)^{\omega} \leq 2 \cdot 3^{\omega} \Rightarrow 2 \leq \left(2 \cdot 3^{\omega}\right)(1+t)^{-\omega}$;
- $t \leq 2 \Rightarrow \dfrac{2 \cdot 3^{\omega}}{\log 2} \geq \dfrac{2(1+t)^{\omega}}{\log(2+t)} \Rightarrow 2 \leq \left(\dfrac{2 \cdot 3^{\omega}}{\log 2}\right) \log(2+t)(1+t)^{-\omega}$;
- $0 \leq t \leq 2 \Rightarrow 1 \leq (1+t) \leq 3 \Rightarrow$

$$\begin{cases} 1 \leq (1+t)^{1-\omega-\gamma} \Rightarrow 2 \leq 2(1+t)^{1-\omega-\gamma}, & \text{if } 1-\omega-\gamma \geq 0 \\ 3^{1-\omega-\gamma} \leq (1+t)^{1-\omega-\gamma} \Rightarrow 2 \leq \left(2 \cdot 3^{-1+\omega+\gamma}\right)(1+t)^{1-\omega-\gamma}, & \text{if } 1-\omega-\gamma < 0. \end{cases}$$

Hence, for $t \leq 2$, we have

$$\int_0^t (1+t-\tau)^{-\omega}(1+\tau)^{-\gamma}\, d\tau \leq \begin{cases} C_1\,(1+t)^{-\omega}, & \omega > 1 \\ C_1 \log(2+t)\,(1+t)^{-\omega}, & \omega = 1 \\ C_1\,(1+t)^{1-\omega-\gamma}, & \omega < 1. \end{cases} \tag{247}$$

If $t \geq 2$, we break the integration interval in two halves again,

$$\int_0^t (1+t-\tau)^{-\omega}(1+\tau)^{-\gamma}\, d\tau = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^{t}.$$

For the first integral,

$$\begin{aligned} \int_0^{\frac{t}{2}} (1+t-\tau)^{-\omega}(1+\tau)^{-\gamma}\, d\tau &\leq \left(\frac{t}{2}\right)^{-\omega} \int_0^{\frac{t}{2}} (1+\tau)^{-\gamma}\, d\tau \\ &\leq \frac{1}{1-\gamma}\left(\frac{t}{2}\right)^{-\omega}\left(1+\frac{t}{2}\right)^{1-\gamma}. \end{aligned} \tag{248}$$

Since the functions $\left(\frac{t}{2}\right)$, $\left(1+\frac{t}{2}\right)$ and $(1+t)$ are equivalent for large values of t, the partial result follows:

$$\begin{aligned} &\int_0^{\frac{t}{2}} (1+t-\tau)^{-\omega}(1+\tau)^{-\gamma}\, d\tau \\ &\qquad \leq C_2(1+t)^{1-\omega-\gamma} \quad \leq \begin{cases} C_2\,(1+t)^{-\gamma}, & \omega > 1 \\ C_2 \log(2+t)\,(1+t)^{-\gamma}, & \omega = 1 \\ C_2\,(1+t)^{1-\omega-\gamma}, & \omega < 1, \end{cases} \end{aligned} \tag{249}$$

fot $t \geq 2$. For the second integral,

$$
\begin{aligned}
&\int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\omega} (1 + \tau)^{-\gamma} \, d\tau \\
&\qquad \leq \left(1 + \frac{t}{2}\right)^{-\gamma} \int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\omega} \, d\tau \\
&\qquad = \left(1 + \frac{t}{2}\right)^{-\gamma} \int_{0}^{\frac{t}{2}} (1 + s)^{-\omega} \, ds \\
&\qquad \leq \begin{cases} \left(1 + \frac{t}{2}\right)^{-\gamma} \int_{0}^{\infty} (1 + s)^{-\omega} \, ds = \frac{1}{\omega - 1} \left(1 + \frac{t}{2}\right)^{-\gamma}, & \omega > 1 \\ \left(1 + \frac{t}{2}\right)^{-\gamma} \log(1 + \frac{t}{2}), & \omega = 1 \\ \frac{1}{1 - \omega} \left(1 + \frac{t}{2}\right)^{1 - \omega - \gamma}, & \omega < 1. \end{cases}
\end{aligned} \tag{250}
$$

Again, the equivalence of the functions $\left(\frac{t}{2}\right)$, $\left(1 + \frac{t}{2}\right)$ and $(1 + t)$, for large t gives us

$$
\int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\omega} (1 + \tau)^{-\gamma} \, d\tau \leq \begin{cases} C_3 \frac{1}{\omega - 1} (1 + t)^{-\gamma}, & \omega > 1 \\ C_3 \log(2 + t) \, (1 + t)^{-\gamma}, & \omega = 1 \\ C_3 \frac{1}{1 - \omega} (1 + t)^{1 - \omega - \gamma}, & \omega < 1, \end{cases} \tag{251}
$$

for $t \geq 2$.

Finally, combining (247) with the sum of (249) and (251), we are done.

■

APPENDIX B – ESTIMATES FOR FUNDAMENTAL SOLUTIONS AND THEIR DERIVATIVES IN FOURIER SPACE

We recall from vector calculus that, for any $\alpha = (\alpha_1, ...\alpha_\nu)$ multi-index,

$$\left|\partial_\xi^\alpha |\xi|^k\right| \lesssim |\xi|^{k-|\alpha|}. \tag{252}$$

We also recall the multivariate version of the Leibniz's rule,

$$\partial_\xi^\alpha(fg) = \sum_{\beta\leq\alpha} C_\beta \left(\partial_\xi^\beta f\right)\left(\partial_\xi^{\alpha-\beta} g\right), \tag{253}$$

where $\beta \leq \alpha$ is in the sense of multi-indexes, that is, $\beta_i \leq \alpha_i$ for all i. Another useful identity is Faà di Bruno's formula for higher order derivatives: Let $y = g(\xi)$, $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$. Then,

$$\partial_\xi^\alpha f(y) := \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}\xi_1 ... \partial^{\alpha_n}\xi_n} f(y) = \sum_{\pi\in\Pi} f^{(|\pi|)}(y) \cdot \prod_{B\in\pi} \frac{\partial^{|B|}}{\prod_{j\in B}\partial\xi_j} y, \tag{254}$$

where π runs through the set Π of all the partitions of the set $\{1, ..., |\alpha|\}$ and "$B \in \pi$" means that B runs through the list of all blocks of the partition π. Observe that $|\alpha| = \alpha_1 + ... + \alpha_n$, while $|\pi|$ is the number of blocks in the partition π and $|B|$ is the size of the block B.

Applying Faà di Bruno with $y(\xi) = |\xi|^2$ and $f(y) = (1 + y)^{\frac{k}{2}}$, we get

$$\left|\partial_\xi^\alpha \langle\xi\rangle^k\right| \lesssim \begin{cases} |\xi|^{k-|\alpha|}, & |\xi| \geq 1, \\ |\xi|^{-|\alpha|}, & |\xi| \leq 1. \end{cases} \tag{255}$$

Lastly, we'll need Young's Inequality for products: If $a, b \geq 0$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{256}$$

Our objective is to collect estimates for the derivatives of the fundamental solutions K_0, K_1 defined in (41) and the operator E_1 defined in (45), which are given in terms of $|\xi|$ and $\langle\xi\rangle$. The different behavior of $\langle\xi\rangle$ in low and high-frequency regions makes it necessary for us to look for estimates separately in these two regions.

B.1 ESTIMATES FOR THE LOW-FREQUENCY REGION

Lemma B.1.1 *For any multi-index α, $|\xi|$ small enough and $A, B \in \mathbb{R}$, $\theta \in \left[0, \frac{1}{2}\right)$*

i) $\left|\partial_\xi^\alpha\left(|\xi|^A\langle\xi\rangle^B\right)\right| \lesssim |\xi|^{A-|\alpha|}$;

ii) $\left|\partial_\xi^\alpha\left(4|\xi|^2 - |\xi|^{4\theta}\langle\xi\rangle^{-4}\right)^{\frac{1}{2}}\right| \lesssim |\xi|^{2\theta-|\alpha|}$;

Proof:

i) From Leibniz's rule (253), with $f(\xi) = |\xi|^A$ and $g(\xi) = \langle\xi\rangle^B$, and from (252) and (255), it follows that

$$\begin{aligned}\left|\partial_\xi^\alpha\left(|\xi|^A\langle\xi\rangle^B\right)\right| &\leq \sum_{\beta\leq\alpha} C_\beta\left|\partial_\xi^\beta|\xi|^A\right|\left|\partial_\xi^{\alpha-\beta}\langle\xi\rangle^B\right| \\ &\lesssim \sum_{\beta\leq\alpha}|\xi|^{A-|\beta|}|\xi|^{-|\alpha-\beta|} \\ &\lesssim |\xi|^{A-|\alpha|}. \end{aligned} \tag{257}$$

ii) Observe that, for $f(y) = y^{\frac{1}{2}}$,

$$f^{(|\pi|)}(y) = \left((-1)^{|\pi|}\frac{\prod_{k=0}^{|\pi|-1}2k-1}{2^{|\pi|}}\right)y^{-\frac{2|\pi|-1}{2}} := C_\pi y^{-\frac{2|\pi|-1}{2}}$$

Also, observe that, taking ξ small enough so that $4|\xi|^{2-4\theta} < \frac{1}{8}$, we have

$$|\xi|^{4\theta}\langle\xi\rangle^{-4} - 4|\xi|^2 = |\xi|^{4\theta}\left(\langle\xi\rangle^{-4} - 4|\xi|^{2-4\theta}\right) \geq \frac{1}{8}|\xi|^{4\theta},$$

therefore, for any $k \geq 0$,

$$\left(|\xi|^{4\theta}\langle\xi\rangle^{-4} - 4|\xi|^2\right)^{-\frac{k}{2}} \lesssim |\xi|^{-2k\theta}$$

With these estimates in mind, we'll apply (254) with $y(\xi) = 4|\xi|^2 - |\xi|^{4\theta}\langle\xi\rangle^{-4}$ and $f(y) = y^{\frac{1}{2}}$:

$$\begin{aligned}\left|\partial_\xi^\alpha f(y)\right| &= \left|\sum_{\pi\in\Pi}C_\pi\left(4|\xi|^2 - |\xi|^{4\theta}\langle\xi\rangle^{-4}\right)^{-\frac{2|\pi|-1}{2}}\prod_{B\in\pi}\frac{\partial^{|B|}}{\prod_{j\in B}\partial x_j}\left(4|\xi|^2 - |\xi|^{4\theta}\langle\xi\rangle^{-4}\right)\right| \\ &\lesssim \sum_{\pi\in\Pi}|\xi|^{-4\theta|\pi|+2\theta}\prod_{B\in\pi}\left(|\xi|^{4\theta-|B|} + |\xi|^{2-|B|}\right) \\ &\lesssim \sum_{\pi\in\Pi}|\xi|^{-4\theta|\pi|+2\theta}\prod_{B\in\pi}|\xi|^{4\theta-|B|} \\ &\lesssim \sum_{\pi\in\Pi}|\xi|^{-4\theta|\pi|+2\theta}|\xi|^{4\theta|\pi|-|\alpha|} \\ &\lesssim |\xi|^{2\theta-|\alpha|}. \end{aligned} \tag{258}$$

Notice that, for each fixed $\pi \in \Pi$, there are $|\pi|$ terms inside the product, and that the sum of the $|B|$'s as B runs through π is equal to $|\alpha|$, by the definition of a partition. ■

We recall the definition of the eigenvalues for problem (1) in the Fourier space,

$$\lambda_\pm = -\frac{1}{2}|\xi|^{2\theta}\langle\xi\rangle^{-2} \pm \frac{1}{2}i\left(4|\xi|^2 - |\xi|^{4\theta}\langle\xi\rangle^{-4}\right)^{\frac{1}{2}}. \tag{259}$$

Therefore, using the previous lemma, we can bound the derivatives of $\lambda_\pm$.

Lemma B.1.2 *Let $\lambda_\pm$ as in (259), α any multi-index and $\xi \in \mathbb{R}^n$ with $|\xi| \leq \varepsilon_0$. Then,*

$$\left|\partial_\xi^\alpha \lambda_\pm\right| \lesssim |\xi|^{2\theta-|\alpha|}.$$

Proof: Directly from the estimates of the previous lemma, it follows that

$$\begin{aligned}\left|\partial_\xi^\alpha \lambda_\pm\right| &\lesssim \left|\partial_\xi^\alpha \left(|\xi|^{2\theta}\langle\xi\rangle^{-2}\right)\right| + \left|\partial_\xi^\alpha \left(4|\xi|^2 - |\xi|^{4\theta}\langle\xi\rangle^{-4}\right)^{\frac{1}{2}}\right| \\ &\lesssim |\xi|^{2\theta-|\alpha|}. \end{aligned} \tag{260}$$

■

Lemma B.1.3 *Let $\lambda_\pm$ as in (259), α any multi-index and $\xi \in \mathbb{R}^n$ with $|\xi| \leq \varepsilon_0$. Then,*

i) $\left|\partial_\xi^\alpha (\lambda_+ - \lambda_-)\right| \lesssim |\xi|^{2\theta-|\alpha|}$.

ii) $\left|\partial_\xi^\alpha (\lambda_+ - \lambda_-)\right|^{-1} \lesssim |\xi|^{-2\theta-|\alpha|}$.

Proof:

i) The result follows immediately from Lemma B.1.1 and the fact that

$$(\lambda_+ - \lambda_-) = i\left(4|\xi|^2 - |\xi|^{4\theta}\langle\xi\rangle^{-4}\right)^{\frac{1}{2}}.$$

ii) We'll apply (254) with $f(y) = y^{-1}$ and $y(\xi) = \lambda_+ - \lambda_-$.

Here, we have

$$f^{(|\pi|)}(y) = \left((-1)^{|\pi|}|\pi|!\right)\ y^{-|\pi|-1} = C_\pi y^{-|\pi|-1},$$

so we get, using item i),

$$\begin{aligned}\left|\partial_\xi^\alpha (\lambda_+ - \lambda_-)^{-1}\right| &= \left|\sum_{\pi\in\Pi} C_\pi (\lambda_+ - \lambda_-)^{-|\pi|-1} \prod_{B\in\pi} \frac{\partial^{|B|}}{\prod_{j\in B}\partial\xi_j}(\lambda_+ - \lambda_-)\right| \\ &\lesssim \sum_{\pi\in\Pi} |\lambda_+ - \lambda_-|^{-|\pi|-1} \prod_{B\in\pi} |\xi|^{2\theta-|B|} \\ &\lesssim \sum_{\pi\in\Pi} |\xi|^{-2\theta|\pi|-2\theta}|\xi|^{|2\theta\pi|-|\alpha|} \\ &\lesssim |\xi|^{-2\theta-|\alpha|}. \end{aligned} \tag{261}$$

■

Now, we proceed similarly to estimate the exponential terms that appear in the definition of $\hat{E}_1$.

Lemma B.1.4 *Let $\lambda_\pm$ as in (259), α any multi-index and $\xi \in \mathbb{R}^n$ with $|\xi| \leq \varepsilon_0$. Then,*

$$\left|\partial_\xi^\alpha e^{t\lambda_\pm}\right| \lesssim |\xi|^{-|\alpha|} e^{-\frac{t}{2}\lambda_\pm}$$

Proof: We will work separately with λ_+ and λ_-.

Applying (254), with $y(\xi) = t\lambda_+$ and $f(y) = e^y$:

$$\begin{aligned}
\left|\partial_\xi^\alpha e^{t\lambda_+}\right| &\lesssim \sum_{\pi\in\Pi} C_\pi e^{t\lambda_+} \prod_{B\in\pi} t|\xi|^{2(1-\theta)-|B|} \\
&\lesssim \sum_{\pi\in\Pi} e^{t\lambda_+} t^{|\pi|}|\xi|^{2(1-\theta)|\pi|-|\alpha|} \\
&\lesssim e^{t\lambda_+}|\xi|^{-|\alpha|} \sum_{\pi\in\Pi} t^{|\pi|}|\xi|^{2(1-\theta)|\pi|} \\
&\lesssim \begin{cases} |\xi|^{-|\alpha|}e^{t\lambda_+}, & t|\xi|^{2(1-\theta)} \leq 1 \\ |\xi|^{(1-2\theta)|\alpha|}t^{|\alpha|}e^{t\lambda_+}, & t|\xi|^{2(1-\theta)} \geq 1. \end{cases}
\end{aligned} \tag{262}$$

Since it holds that

$$t^{|\alpha|}e^{t\lambda_+}|\xi|^{(1-2\theta)|\alpha|} \lesssim \left(\frac{t}{2}|\xi|^{2-2\theta}\right)^{|\alpha|} e^{-\frac{t}{2}|\xi|^{2(1-\theta)}} e^{\frac{t}{2}\lambda_+}|\xi|^{-|\alpha|} \lesssim |\xi|^{-|\alpha|}e^{\frac{t}{2}\lambda_+},$$

we have

$$\left|\partial_\xi^\alpha e^{t\lambda_+}\right| \lesssim |\xi|^{-|\alpha|}e^{\frac{t}{2}\lambda_+}.$$

For λ_-, calculations are similar. We apply (254) with $y(\xi) = t\lambda_-$ and $f(y) = e^y$, resulting in

$$\left|\partial_\xi^\alpha e^{t\lambda_-}\right| \lesssim \begin{cases} |\xi|^{-|\alpha|}e^{t\lambda_-}, & t|\xi|^{2\theta} \leq 1 \\ |\xi|^{(2\theta-1)|\alpha|}t^{|\alpha|}e^{t\lambda_-}, & t|\xi|^{2\theta} \geq 1. \end{cases} \tag{263}$$

And since $t^{|\alpha|}e^{t\lambda_-}|\xi|^{(2\theta-1)|\alpha|} \lesssim \left(\frac{t}{2}|\xi|^{2\theta}\right)^{|\alpha|} e^{-\frac{t}{2}|\xi|^{2\theta}}|\xi|^{-|\alpha|}e^{\frac{t}{2}\lambda_+} \lesssim |\xi|^{-|\alpha|}e^{\frac{t}{2}\lambda_+}$, we get also

$$\left|\partial_\xi^\alpha e^{t\lambda_-}\right| \lesssim |\xi|^{-|\alpha|}e^{\frac{t}{2}\lambda_-}.$$

■

Summarizing, we have until now proved the following estimates in the low-frequency region:

- $\left|\partial_\xi^\alpha\lambda_\pm\right| \lesssim |\xi|^{2\theta-|\alpha|}$;
- $\left|\partial_\xi^\alpha e^{t\lambda_\pm}\right| \lesssim |\xi|^{-|\alpha|}e^{-\frac{t}{2}\lambda_\pm}$;
- $\left|\partial_\xi^\alpha(\lambda_+ - \lambda_-)\right|^{-1} | \lesssim |\xi|^{-2\theta-|\alpha|}$.

Our goal is to use these estimates to bound the derivatives of $\hat{K}_0$, $\hat{K}_1$ and $\hat{E}_1$:

Lemma B.1.5 *Let $\lambda_\pm$ as in (259), α any multi-index and $\xi \in \mathbb{R}^n$ with $|\xi| \leq \varepsilon_0$. Then,*

i) $\left|\partial_\xi^\alpha\hat{K}_0(t,\xi)\right| \lesssim |\xi|^{-|\alpha|}e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}$;

ii) $\left|\partial_\xi^\alpha \hat{K}_1(t,\xi)\right| \lesssim |\xi|^{-2\theta-|\alpha|} e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}$;

iii) $\left|\partial_\xi^\alpha \hat{E}_1(t,\xi)\right| \lesssim |\xi|^{-2\theta-|\alpha|} e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}$.

Proof: Using (253) and the estimates above, we have:

i)

$$\begin{aligned}
\left|\partial_\xi^\alpha \hat{K}_0(t,\xi)\right| &\lesssim \left|\partial_\xi^\alpha \lambda_+ e^{t\lambda_-}(\lambda_+-\lambda_-)^{-1}\right| \\
&\quad + \left|\partial_\xi^\alpha \lambda_- e^{t\lambda_+}(\lambda_+-\lambda_-)^{-1}\right| \\
&\lesssim \sum_{\beta\le\alpha} \left|\partial_\xi^\beta \lambda_+ e^{t\lambda_-}\right| \left|\partial_\xi^{\alpha-\beta}(\lambda_+-\lambda_-)^{-1}\right| \\
&\quad + \sum_{\beta\le\alpha} \left|\partial_\xi^\beta \lambda_- e^{t\lambda_+}\right| \left|\partial_\xi^{\alpha-\beta}(\lambda_+-\lambda_-)^{-1}\right| \\
&\lesssim \sum_{\beta\le\alpha}\sum_{\gamma\le\beta} |\xi|^{2\theta-|\gamma|}|\xi|^{-|\beta|+|\gamma|}|\xi|^{-2\theta-|\alpha|+|\beta|}\left(e^{\frac{t}{2}\lambda_-}+e^{\frac{t}{2}\lambda_+}\right) \\
&\lesssim |\xi|^{-|\alpha|}\left(e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}+e^{-\frac{t}{2}|\xi|^{2\theta}}\right) \\
&\lesssim |\xi|^{-|\alpha|}e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}.
\end{aligned} \tag{264}$$

ii)

$$\begin{aligned}
\left|\partial_\xi^\alpha \hat{K}_1(t,\xi)\right| &\lesssim \left|\partial_\xi^\alpha e^{t\lambda_-}(\lambda_+-\lambda_-)^{-1}\right| + \left|\partial_\xi^\alpha e^{t\lambda_+}(\lambda_+-\lambda_-)^{-1}\right| \\
&\lesssim \sum_{\beta\le\alpha} \left|\partial_\xi^\beta e^{t\lambda_-}\right| \left|\partial_\xi^{\alpha-\beta}(\lambda_+-\lambda_-)^{-1}\right| \\
&\quad + \sum_{\beta\le\alpha} \left|\partial_\xi^\beta e^{t\lambda_+}\right| \left|\partial_\xi^{\alpha-\beta}(\lambda_+-\lambda_-)^{-1}\right| \\
&\lesssim |\xi|^{-2\theta-|\alpha|}\left(e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}+e^{-\frac{t}{2}|\xi|^{2\theta}}\right) \\
&\lesssim |\xi|^{-2\theta-|\alpha|}e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}.
\end{aligned} \tag{265}$$

iii) Using item ii), we have

$$\begin{aligned}
\left|\partial_\xi^\alpha \hat{E}_1(t,\xi)\right| &= \left|\partial_\xi^\alpha \langle\xi\rangle^{-2}\hat{K}_1(t,\xi)\right| \\
&\lesssim \sum_{\beta\le\alpha} \left|\partial_\xi^\beta \langle\xi\rangle^{-2}\right|\left|\partial_\xi^{\alpha-\beta}\hat{K}_1(t,\xi)\right| \\
&\lesssim |\xi|^{-|\beta|}|\xi|^{-2\theta-|\beta|+|\alpha|}e^{-\frac{t}{2}|\xi|^{2(1-\theta)}} \\
&\lesssim |\xi|^{-2\theta-|\alpha|}e^{-\frac{t}{2}|\xi|^{2(1-\theta)}}.
\end{aligned} \tag{266}$$

■

B.2 ESTIMATES FOR THE HIGH-FREQUENCY REGION

Lemma B.2.1 *For any multi-index* α, $|\xi| \geq 1$ *and* $A, B \in \mathbb{R}$, $\theta \in \left[0, \frac{1}{2}\right)$

i) $\left|\partial_\xi^\alpha \left(|\xi|^A \langle\xi\rangle^B\right)\right| \lesssim |\xi|^{A+B-|\alpha|}$;

ii) $\left|\partial_\xi^\alpha \left(4|\xi|^2 - |\xi|^{4\theta}\langle\xi\rangle^{-4}\right)^{\frac{1}{2}}\right| \lesssim |\xi|^{1-|\alpha|}$;

Proof:

i) From Leibniz's rule (253), with $f(\xi) = |\xi|^A$ and $g(\xi) = \langle\xi\rangle^B$, and from (252) and (255), it follows that

$$\begin{aligned}\left|\partial_\xi^\alpha \left(|\xi|^A \langle\xi\rangle^B\right)\right| &\leq \sum_{\beta\leq\alpha} C_\beta \left|\partial_\xi^\beta |\xi|^A\right| \left|\partial_\xi^{\alpha-\beta}\langle\xi\rangle^B\right| \\ &\lesssim \sum_{\beta\leq\alpha} |\xi|^{A-|\beta|}|\xi|^{B-|\alpha-\beta|} \\ &\lesssim |\xi|^{A+B-|\alpha|}. \end{aligned} \tag{267}$$

ii) We'll apply (254) with $y(\xi) = 4|\xi|^2 - |\xi|^{4\theta}\langle\xi\rangle^{-4}$ and $f(y) = y^{\frac{1}{2}}$. Observe that

$$f^{(|\pi|)}(y) = \left((-1)^{|\pi|}\frac{\prod_{k=0}^{|\pi|-1} 2k-1}{2^{|\pi|}}\right) y^{-\frac{2|\pi|-1}{2}} := C_\pi y^{-\frac{2|\pi|-1}{2}},$$

therefore

$$\left|\partial_\xi^\alpha f(y)\right| = \left|\sum_{\pi\in\Pi} C_\pi \left(4|\xi|^2 - |\xi|^{4\theta}\langle\xi\rangle^{-4}\right)^{-\frac{2|\pi|-1}{2}} \prod_{B\in\pi} \frac{\partial^{|B|}}{\prod_{j\in B}\partial x_j}\left(4|\xi|^2 - |\xi|^{4\theta}\langle\xi\rangle^{-4}\right)\right|. \tag{268}$$

Now, since $|\xi| \geq 1$ and $\theta \in \left[0, \frac{1}{2}\right)$,

$$\left(1 - \frac{1}{4}|\xi|^{4\theta-2}\langle\xi\rangle^{-4}\right) \geq \frac{3}{4} \Rightarrow 4|\xi|^2 - |\xi|^{4\theta}\langle\xi\rangle^{-4}$$
$$= 4|\xi|^2\left(1 - \frac{1}{4}|\xi|^{4\theta-2}\langle\xi\rangle^{-4}\right) \geq 3|\xi|^2,$$

which means that, for any nonnegative k,

$$\left(4|\xi|^2 - |\xi|^{4\theta}\langle\xi\rangle^{-4}\right)^{-\frac{k}{2}} \lesssim |\xi|^{-k}.$$

Also, using item *i*), one can estimate

$$\begin{aligned}\left|\partial_\xi^\alpha \left((4|\xi|^2 - |\xi|^{4\theta}\langle\xi\rangle^{-4}\right)\right| &\lesssim \left|\partial_\xi^\alpha |\xi|^2\right| + \left|\partial_\xi^\alpha |\xi|^{4\theta}\langle\xi\rangle^{-4}\right| \\ &\lesssim |\xi|^{2-|\alpha|} + |\xi|^{4\theta-4-|\alpha|} \\ &\lesssim |\xi|^{2-|\alpha|}. \end{aligned} \tag{269}$$

Using these two on (268), one gets

$$\left|\partial_\xi^\alpha f(y)\right| \lesssim \sum_{\pi\in\Pi} |\xi|^{-2|\pi|+1} \prod_{B\in\pi} |\xi|^{2-|B|}. \tag{270}$$

Notice that, for each fixed $\pi \in \Pi$, there are $|\pi|$ terms inside the product, and that the sum of the $|B|$'s as B runs through π is equal to $|\alpha|$, by the definition of a partition. Thus,

$$\begin{aligned}\left|\partial_\xi^\alpha \left((4|\xi|^2 - |\xi|^{4\theta}\langle\xi\rangle^{-4}\right)\right| &\lesssim \sum_{\pi\in\Pi} |\xi|^{-2|\pi|+1}|\xi|^{2|\pi|-|\alpha|} \\ &\lesssim |\xi|^{1-|\alpha|}.\end{aligned} \tag{271}$$

■

Using the previous lemma, we can bound the derivatives of $\lambda_\pm$.

Lemma B.2.2 *Let* $\lambda_\pm$ *as in (259),* α *any multi-index,* $\theta \in \left[0, \frac{1}{2}\right)$ *and* $\xi \in \mathbb{R}^n$ *with* $|\xi| \geq 1$. *Then,*

$$\left|\partial_\xi^\alpha \lambda_\pm\right| \lesssim |\xi|^{1-|\alpha|}.$$

Proof: Directly from the estimates of the previous lemma, it follows that

$$\begin{aligned}\left|\partial_\xi^\alpha \lambda_\pm\right| &\lesssim \left|\partial_\xi^\alpha \left(|\xi|^{2\theta}\langle\xi\rangle^{-2}\right)\right| + \left|\partial_\xi^\alpha \left(4|\xi|^2 - |\xi|^{4\theta}\langle\xi\rangle^{-4}\right)^{\frac{1}{2}}\right| \\ &\lesssim |\xi|^{2\theta-2-|\alpha|} + |\xi|^{1-|\alpha|} \\ &\lesssim |\xi|^{1-|\alpha|}.\end{aligned} \tag{272}$$

■

Lemma B.2.3 *Let* $\lambda_\pm$ *as in (259),* α *any multi-index,* $\theta \in \left[0, \frac{1}{2}\right)$ *and* $\xi \in \mathbb{R}^n$ *with* $|\xi| \geq 1$. *Then,*

i) $\left|\partial_\xi^\alpha (\lambda_+ - \lambda_-)\right| \lesssim |\xi|^{1-|\alpha|}$.

ii) $\left|\partial_\xi^\alpha (\lambda_+ - \lambda_-)^{-1}\right| \lesssim |\xi|^{-1-|\alpha|}$.

Proof:

i) Since

$$(\lambda_+ - \lambda_-) = i\left(4|\xi|^2 - |\xi|^{4\theta}\langle\xi\rangle^{-4}\right)^{\frac{1}{2}},$$

the result follows immediately from Lemma B.2.1.

ii) We'll apply (254) with $f(y) = y^{-1}$ and $y(\xi) = \lambda_+ - \lambda_-$.

Here, we have

$$f^{(|\pi|)}(y) = \left((-1)^{|\pi|}|\pi|!\right)\ y^{-|\pi|-1} = C_\pi y^{-|\pi|-1},$$

so we get, using item i),

$$\begin{aligned}\left|\partial_\xi^\alpha (\lambda_+ - \lambda_-)^{-1}\right| &= \left|\sum_{\pi\in\Pi} C_\pi (\lambda_+ - \lambda_-)^{-|\pi|-1} \prod_{B\in\pi} \frac{\partial^{|B|}}{\prod_{j\in B}\partial\xi_j}(\lambda_+ - \lambda_-)\right| \\ &\lesssim \left|\sum_{\pi\in\Pi} (\lambda_+ - \lambda_-)^{-|\pi|-1} \prod_{B\in\pi} |\xi|^{1-|B|}\right| \\ &\lesssim \sum_{\pi\in\Pi} |\lambda_+ - \lambda_-|^{-|\pi|-1} \prod_{B\in\pi} |\xi|^{1-|B|} \\ &\lesssim \sum_{\pi\in\Pi} |\xi|^{-|\pi|-1}|\xi|^{|\pi|-|\alpha|} \\ &\lesssim |\xi|^{-1-|\alpha|}. \end{aligned} \tag{273}$$

■

Now, we proceed similarly to estimate the exponential terms that appear in the definition of $\hat{E}_1$.

Lemma B.2.4 *Let $\lambda_\pm$ as in (259), α any multi-index, $\theta \in \left[0, \frac{1}{2}\right)$ and $\xi \in \mathbb{R}^n$ with $|\xi| \geq 1$. Then,*

i) $\left|\partial_\xi^\alpha e^{t\lambda_\pm}\right| \lesssim \left(t^{|\alpha|} + |\xi|^{-|\alpha|}\right) e^{-t|\xi|^{2(\theta-1)}}$

ii) $\left|\partial_\xi^\alpha \left(e^{t\lambda_+} - e^{t\lambda_-}\right)\right| \lesssim \left(t^{|\alpha|} + |\xi|^{-|\alpha|}\right) e^{-t|\xi|^{2(\theta-1)}}$

Proof:

i) Applying again (254), with $f(y) = e^{ty}$ and $y(\xi) = \lambda_\pm$,

$$\begin{aligned}\left|\partial_\xi^\alpha e^{t\lambda_\pm}\right| &= \left|\sum_{\pi\in\Pi} t^{|\pi|} e^{t\lambda_\pm} \prod_{B\in\pi} \frac{\partial^{|B|}}{\prod_{j\in B}\partial\xi_j}\lambda_\pm\right| \\ &\lesssim \sum_{\pi\in\Pi} t^{|\pi|} e^{t\lambda_\pm} \prod_{B\in\pi} |\xi|^{1-|B|} \\ &= \sum_{\pi\in\Pi} t^{|\pi|} e^{t\lambda_\pm} |\xi|^{|\pi|-|\alpha|}. \end{aligned} \tag{274}$$

Now, from (256) with $a = t^{|\pi|}$, $b = |\xi|^{|\pi|-|\alpha|}$, $p = \frac{|\alpha|}{|\pi|}$ and $q = \frac{|\alpha|}{|\alpha|-|\pi|}$, it holds that

$$t^{|\pi|}|\xi|^{|\pi|-|\alpha|} \leq \frac{|\pi|}{|\alpha|} t^{|\alpha|} + \frac{|\alpha| - |\pi|}{|\alpha|}|\xi|^{-|\alpha|},$$

for every $|\pi| < |\alpha|$. (for $|\pi| = |\alpha|$, the inequality above still holds trivially, $t^{|\alpha|} \leq t^{|\alpha|}$.) Thus, one gets

$$\begin{aligned}\left|\partial_\xi^\alpha e^{t\lambda_\pm}\right| &\lesssim \left(t^{|\alpha|} + |\xi|^{-|\alpha|}\right) e^{t\lambda_\pm} \\ &\lesssim \left(t^{|\alpha|} + |\xi|^{-|\alpha|}\right) e^{-t|\xi|^{2(\theta-1)}}. \end{aligned} \tag{275}$$

ii) Directly from item i),

$$\left|\partial_\xi^\alpha\left(e^{t\lambda_+}-e^{t\lambda_-}\right)\right|\le\left|\partial_\xi^\alpha e^{t\lambda_+}\right|+\left|\partial_\xi^\alpha e^{t\lambda_-}\right|\lesssim\left(t^{|\alpha|}+|\xi|^{-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}}.$$

■

Lemma B.2.5 *Let $\lambda_\pm$ as in (259), α any multi-index, $\theta\in\left[0,\frac{1}{2}\right)$ and $\xi\in\mathbb{R}^n$ with $|\xi|\ge 1$. Also, let $\hat{K}_0,\hat{K}_1,\hat{E}_1$ be as in (41) and (45). Then,*

i) $\left|\partial_\xi^\alpha\hat{K}_0(t,\xi)\right|\lesssim\left(t^{|\alpha|}+|\xi|^{-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}}$;

ii) $\left|\partial_\xi^\alpha\hat{K}_1(t,\xi)\right|\lesssim|\xi|^{-1}\left(t^{|\alpha|}+|\xi|^{-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}}$;

iii) $\left|\partial_\xi^\alpha\hat{E}_1(t,\xi)\right|\lesssim|\xi|^{-3}\left(t^{|\alpha|}+|\xi|^{-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}}$.

Proof:

i) Applying (253) and using Lemmas B.2.3 and B.2.4,

$$\begin{aligned}
|\partial_\xi^\alpha\hat{K}_0(t,\xi)| &= \left|\partial_\xi^\alpha\left(\lambda_+e^{t\lambda_-}\right)(\lambda_+-\lambda_-)^{-1}-\partial_\xi^\alpha\left(\lambda_-e^{t\lambda_+}\right)(\lambda_+-\lambda_-)^{-1}\right|\\
&\lesssim\sum_{\beta\le\alpha}\left|\partial_\xi^\beta\left(\lambda_+e^{t\lambda_-}\right)\right|\left|\partial_\xi^{\alpha-\beta}(\lambda_+-\lambda_-)^{-1}\right|\\
&\qquad+\sum_{\beta\le\alpha}\left|\partial_\xi^\beta\left(\lambda_-e^{t\lambda_+}\right)\right|\left|\partial_\xi^{\alpha-\beta}(\lambda_+-\lambda_-)^{-1}\right|\\
&\lesssim\sum_{\gamma\le\beta}\sum_{\beta\le\alpha}|\partial_\xi^\gamma\lambda_+|\left|\partial_\xi^{\beta-\gamma}e^{t\lambda_-}\right|\left|\partial_\xi^{\alpha-\beta}(\lambda_+-\lambda_-)^{-1}\right|\\
&\qquad+\sum_{\gamma\le\beta}\sum_{\beta\le\alpha}|\partial_\xi^\gamma\lambda_-|\left|\partial_\xi^{\beta-\gamma}e^{t\lambda_+}\right|\left|\partial_\xi^{\alpha-\beta}(\lambda_+-\lambda_-)^{-1}\right|\\
&\lesssim\sum_{\gamma\le\beta}\sum_{\beta\le\alpha}|\xi|^{1-|\gamma|}\left(t^{|\beta|-|\gamma|}+|\xi|^{-|\beta|+|\gamma|}\right)e^{-t|\xi|^{2(\theta-1)}}|\xi|^{-1-|\alpha|+|\beta|}\\
&\lesssim\sum_{\gamma\le\beta}\sum_{\beta\le\alpha}\left(|\xi|^{-|\alpha|+|\beta|-|\gamma|}t^{|\beta|-|\gamma|}+|\xi|^{-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}}\\
&\lesssim\left(t^{|\alpha|}+|\xi|^{-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}}.
\end{aligned}\tag{276}$$

ii) Applying (253) and using Lemmas B.2.3 and B.2.4,

$$\begin{aligned}
|\partial_\xi^\alpha\hat{K}_1(t,\xi)| &= \left|\partial_\xi^\alpha\left(e^{t\lambda_+}-e^{t\lambda_-}\right)(\lambda_+-\lambda_-)^{-1}\right|\\
&\lesssim\sum_{\beta\le\alpha}\left|\partial_\xi^\beta\left(e^{t\lambda_+}-e^{t\lambda_-}\right)\right|\left|\partial_\xi^{\alpha-\beta}(\lambda_+-\lambda_-)^{-1}\right|\\
&\lesssim\sum_{\beta\le\alpha}\left(t^{|\beta|}+|\xi|^{-|\beta|}\right)e^{-t|\xi|^{2(\theta-1)}}|\xi|^{-1-|\alpha-\beta|}\\
&\lesssim\sum_{\beta\le\alpha}\left(t^{|\beta|}|\xi|^{-1-|\alpha|+|\beta|}+|\xi|^{-1-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}}\\
&\lesssim\left(t^{|\alpha|}|\xi|^{-1}+|\xi|^{-1-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}}\\
&=|\xi|^{-1}\left(t^{|\alpha|}+|\xi|^{-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}}
\end{aligned}\tag{277}$$

iii) Another application of (253), making use of Lemma B.2.1 and item i), yields

$$
\begin{aligned}
|\partial_\xi^\alpha \hat{E}_1(t,\xi)| &= \left|\partial_\xi^\alpha \langle\xi\rangle^{-2}\hat{K}_1(t,\xi)\right| \\
&\leq \sum_{\beta\leq\alpha} \left|\partial_\xi^\beta \langle\xi\rangle^{-2}\right| \left|\partial_\xi^{\alpha-\beta} K_1(\hat{t},\xi)\right| \\
&\lesssim \sum_{\beta\leq\alpha} |\xi|^{-2-|\beta|}|\xi|^{-1}\left(t^{|\alpha-\beta|}+|\xi|^{-|\alpha-\beta|}\right)e^{-t|\xi|^{2(\theta-1)}} \\
&= \sum_{\beta\leq\alpha} |\xi|^{-3-|\beta|}\left(t^{|\alpha|-|\beta|}+|\xi|^{-|\alpha|+|\beta|}\right)e^{-t|\xi|^{2(\theta-1)}} \\
&= \sum_{\beta\leq\alpha} \left(t^{|\alpha|-|\beta|}|\xi|^{-3-|\beta|}+|\xi|^{-1-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}} \\
&\lesssim \left(t^{|\alpha|}|\xi|^{-3}+|\xi|^{-3-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}} \\
&= |\xi|^{-3}\left(t^{|\alpha|}+|\xi|^{-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}}
\end{aligned} \tag{278}
$$

iv) Now, using (253) and item ii),

$$
\begin{aligned}
|\partial_\xi^\alpha \hat{E}_1(t,\xi)|\xi|^{-s}| &\leq \sum_{\beta\leq\alpha} \left|\partial_\xi^\beta \hat{E}_1(t,\xi)\right| \left|\partial_\xi^{\alpha-\beta}|\xi|^{-s}\right| \\
&\lesssim \sum_{\beta\leq\alpha} |\xi|^{-3}\left(t^{|\beta|}+|\xi|^{-|\beta|}\right)e^{-t|\xi|^{2(\theta-1)}}|\xi|^{-s-|\alpha-\beta|} \\
&= \sum_{\beta\leq\alpha} |\xi|^{-s-3}\left(t^{|\beta|}|\xi|^{-|\alpha|+|\beta|}+|\xi|^{-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}} \\
&\lesssim |\xi|^{-s-3}\left(t^{|\alpha|}+|\xi|^{-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}}
\end{aligned} \tag{279}
$$

■

Lemma B.2.6 *Let $\lambda_\pm$ as in (259), α any multi-index, $\theta \in \left[0,\frac{1}{2}\right)$, $\xi \in \mathbb{R}^n$ with $|\xi| \geq 1$ and $s \in \mathbb{R}$. Also, let $\hat{K_0}, \hat{K_1}, \hat{E_1}$ be as in (41) and (45). Then,*

i) $\left|\partial_\xi^\alpha \hat{K_0}(t,\xi)|\xi|^{-s}\right| \lesssim |\xi|^{-s}\left(t^{|\alpha|}+|\xi|^{-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}}$;

ii) $\left|\partial_\xi^\alpha \hat{K_1}(t,\xi)|\xi|^{-s}\right| \lesssim |\xi|^{-s-1}\left(t^{|\alpha|}+|\xi|^{-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}}$;

iii) $\left|\partial_\xi^\alpha \hat{E_1}(t,\xi)|\xi|^{-s}\right| \lesssim |\xi|^{-s-3}\left(t^{|\alpha|}+|\xi|^{-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}}$.

Proof: The previous Lemma gave us three similar estimates for $\hat{K_0}$, $\hat{K_1}$ and $\hat{E_1}$, in the form

$$
\left|\partial_\xi^\alpha \hat{f}(\xi)\right| \lesssim |\xi|^{-k}\left(t^{|\alpha|}+|\xi|^{-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}}, \tag{280}
$$

with $k = 0, 1, 3$, for $\hat{K_0}, \hat{K_1}, \hat{E_1}$, respectively. We'll strike all cases at once with a generic a.

Assume that (280) holds for some $\hat{f}(\xi)$. Using again (253) and item ii),

$$
\begin{aligned}
|\partial_\xi^\alpha \hat{f}(\xi)|\xi|^{-s}| &\leq \sum_{\beta\leq\alpha} \left|\partial_\xi^\beta \hat{f}(\xi)\right| \left|\partial_\xi^{\alpha-\beta}|\xi|^{-s}\right| \\
&\lesssim \sum_{\beta\leq\alpha} |\xi|^{-k}\left(t^{|\beta|}+|\xi|^{-|\beta|}\right)e^{-t|\xi|^{2(\theta-1)}}|\xi|^{-s-|\alpha-\beta|} \\
&= \sum_{\beta\leq\alpha} |\xi|^{-s-k}\left(t^{|\beta|}|\xi|^{-|\alpha|+|\beta|}+|\xi|^{-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}} \\
&\lesssim |\xi|^{-s-k}\left(t^{|\alpha|}+|\xi|^{-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}}. \qquad (281)
\end{aligned}
$$

■

Lemma B.2.7 *Let $k \geq 0$, α any multi-index, $\theta \in \left[0, \frac{1}{2}\right)$, $\xi \in \mathbb{R}^n$ with $|\xi| \geq 1$. Assume that*

$$\left|\partial_\xi^\alpha \hat{f}(\xi)\right| \lesssim |\xi|^{-s-k}\left(t^{|\alpha|}+|\xi|^{-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}},$$

for every $t \geq 1$. Then, for every $b < s+k$, the following estimate holds

$$\left|\partial_\xi^\alpha \hat{f}(\xi)\right| \lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}(1+t)^{-\frac{s+k-b}{2(1-\theta)}}.$$

Proof: We have, using the fact that $x \mapsto (2x+1)^m e^{-x}$ and $x \mapsto x^m e^{-x}$ are bounded for $m > 0$,

$$
\begin{aligned}
\left|\partial_\xi^\alpha \hat{f}(\xi)\right| &\lesssim |\xi|^{-s-k}\left(t^{|\alpha|}+|\xi|^{-|\alpha|}\right)e^{-t|\xi|^{2(\theta-1)}} \\
&\lesssim |\xi|^{-s-k}\left(t+|\xi|^{-1}\right)^{|\alpha|}e^{-t|\xi|^{2(\theta-1)}} \\
&\lesssim |\xi|^{-s-k}e^{-\frac{t}{2}|\xi|^{2(\theta-1)}}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}\left(t|\xi|^{2(\theta-1)}+|\xi|^{2\theta-3}\right)^{|\alpha|}e^{-\frac{t}{2}|\xi|^{2(\theta-1)}} \\
&\lesssim |\xi|^{-s-k}e^{-\frac{t}{2}|\xi|^{2(\theta-1)}}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}\underbrace{\left(t|\xi|^{2(\theta-1)}+1\right)^{|\alpha|}e^{-\frac{t}{2}|\xi|^{2(\theta-1)}}}_{\textit{bounded}} \\
&\lesssim |\xi|^{-s-k}e^{-\frac{t}{2}|\xi|^{2(\theta-1)}}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}. \qquad (282)
\end{aligned}
$$

Now, let $b < s+k$. Then,

$$
\begin{aligned}
\left|\partial_\xi^\alpha \hat{f}(\xi)\right| &\lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}\left[|\xi|^{-s-k+b}e^{-\frac{t}{2}|\xi|^{2(\theta-1)}}\right] \\
&\lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}\left[\left(\frac{t}{2}|\xi|^{2(\theta-1)}\right)^{\frac{-s-k+b}{2(\theta-1)}}e^{-\frac{t}{2}|\xi|^{2(\theta-1)}}\right]t^{-\frac{-s-k+b}{2(\theta-1)}} \\
&\lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}t^{-\frac{s+k-b}{2(1-\theta)}}, \qquad (283)
\end{aligned}
$$

provided that the exponent of the term inside the square brackets is positive, that is,

$$\frac{-s-k+b}{2(\theta-1)} > 0 \iff s+k-b > 0 \iff b < s+k.$$

■

Finally, a direct application of the previous two lemmas provides us the following estimates:

Corollary B.2.8 *Let $\lambda_\pm$ as in (259), α any multi-index, $\theta \in \left[0, \frac{1}{2}\right)$, $\xi \in \mathbb{R}^n$ with $|\xi| \geq 1$ and $s \in \mathbb{R}$. Also, let $\hat{K_0}, \hat{K_1}, \hat{E_1}$ be as in (41) and (45). Then, for every $t \geq 1$,*

i) $\left|\partial_\xi^\alpha \hat{K_0}(t,\xi)|\xi|^{-s}\right| \lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}(1+t)^{-\frac{s-b}{2(1-\theta)}}$;

ii) $\left|\partial_\xi^\alpha \hat{K_1}(t,\xi)|\xi|^{-s}\right| \lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}(1+t)^{-\frac{s+1-b}{2(1-\theta)}}$;

iii) $\left|\partial_\xi^\alpha \hat{E_1}(t,\xi)|\xi|^{-s}\right| \lesssim |\xi|^{-b}\left(|\xi|^{2(1-\theta)}\right)^{|\alpha|}(1+t)^{-\frac{s+3-b}{2(1-\theta)}}$.

APPENDIX C – FRACTIONAL LAPLACIAN AND ITS ACTION ON THE TEST FUNCTION

C.1 POINTWISE CONTROL OF A TEST FUNCTION UNDER THE ACTION OF THE FRACTIONAL LAPLACE OPERATOR

The following Lemma and its Corollary ensure that the test function $\phi = \langle x\rangle^{-q}$ and the space C_q^∞ defined in the text are suitable for our purposes in the task of proving the non-existence results.

Lemma C.1.1 *Assume $f \in C^2$ bounded, with bounded derivatives. If there exists a constant C_0 such that the estimates*

$$|f(y)| \leq C_0|f(x)|, \quad \sup_{|\alpha|=2} |\partial^\alpha f(y)| \leq C_0 \sup_{|\alpha|=2} |\partial^\alpha f(x)|$$

hold when $|x| \leq |y|$, then for $|x| > 1$, the following pointwise estimate holds:

$$\begin{aligned} |(\Delta)^\sigma f(x)| \leq C|x|^{-n-2\sigma} \int_{|y|<3|x|} |f(y)|dy + C|f(x)||x|^{-2\sigma} \\ + C|x|^{2-2\sigma} \sum_{|\alpha|=2} \frac{|\alpha|}{\alpha!} \left|\partial^\alpha f\left(\tfrac{x}{2}\right)\right|, \end{aligned} \tag{284}$$

for any $\sigma \in (0,1)$.

Proof: Using Definition 4.1.2 with $\sigma \in (0.1)$, we get

$$(-\Delta)^\sigma f(x) = -C_\sigma \int_{\mathbb{R}^n} \frac{f(x+y) - 2f(x) + f(x-y)}{|y|^{n+2\sigma}}\, \mathrm{d}y.$$

From Taylor's theorem, we have

$$f(x+y) - f(x) = \nabla f(x) \cdot y + \sum_{|\alpha|=2} \frac{|\alpha|}{\alpha!} y^\alpha \int_0^1 (1-\theta)\partial^\alpha f(x+\theta y)\, \mathrm{d}\theta,$$

and

$$f(x-y) - f(x) = -\nabla f(x) \cdot y + \sum_{|\alpha|=2} \frac{|\alpha|}{\alpha!} (-y)^\alpha \int_0^1 (1-\theta)\partial^\alpha f(x-\theta y)\, \mathrm{d}\theta.$$

Therefore, by the symmetry, $(-\Delta)^\sigma$ is given by

$$\begin{aligned} (-\Delta)^\sigma f(x) = 2C_\sigma \int_{|y|>r} \frac{f(x) - f(x+y)}{|y|^{n+2\sigma}}\, \mathrm{d}y \\ - 2C_\sigma \sum_{|\alpha|=2} \frac{|\alpha|}{\alpha!} \frac{y^\alpha}{y^{n+2\sigma}} \int_0^1 (1-\theta)\partial^\alpha f(x+2\theta y)\, \mathrm{d}\theta\, \mathrm{d}y, \end{aligned} \tag{285}$$

for any $r > 0$. We set $r = \frac{|x|}{2}$. We now estimate the terms of the right-hand side of (285). For the first term, we break the integral in two and estimate both separately:

$$\int_{|y|>\frac{|x|}{2}} \frac{f(x)}{|y|^{n+2\sigma}}\, \mathrm{d}y = f(x) \int_{|y|>\frac{|x|}{2}} |y|^{-n-2\sigma}\, \mathrm{d}y.$$

Evaluating the latter integral over shells of fixed radii, we obtain

$$
\begin{aligned}
\int_{|y|>\frac{|x|}{2}} |y|^{-n-2\sigma}\,\mathrm{d}y &= \int_{\frac{|x|}{2}}^{\infty}\int_{|y|=\rho} \rho^{-n-2\sigma}\,\mathrm{d}S_y \mathrm{d}\rho \\
&= \int_{\frac{|x|}{2}}^{\infty} \rho^{-n-2\sigma}\mu(B_n)\rho^{n-1}\,\mathrm{d}\rho \\
&= \mu(B_n)\int_{\frac{|x|}{2}}^{\infty} \rho^{-2\sigma-1}\,\mathrm{d}\rho \\
&= \frac{\mu(B_n)}{2\sigma 2^{2\sigma}}|x|^{-2\sigma},
\end{aligned}
$$

where $\mu(B_n)$ denotes the volume of the unit ball in $\mathbb{R}^n$. Therefore, we get

$$
\int_{|y|>\frac{|x|}{2}} \frac{f(x)}{|y|^{n+2\sigma}}\,\mathrm{d}y = C_1 f(x)|x|^{-2\sigma}, \tag{286}
$$

with $C_1 = \frac{\mu(B_n)}{2\sigma 2^{2\sigma}}$. On the other hand, we have the estimate

$$
\begin{aligned}
\int_{|y|>\frac{|x|}{2}} \frac{|f(x+y)|}{|y|^{n+2\sigma}}\,\mathrm{d}y &\le \int_{\frac{|x|}{2}<|y|<2|x|} \frac{|f(x+y)|}{|y|^{n+2\sigma}}\,\mathrm{d}y + \int_{|y|>2|x|} \frac{|f(x+y)|}{|y|^{n+2\sigma}}\,\mathrm{d}y \\
&:= I_1 + I_2.
\end{aligned} \tag{287}
$$

For I_1, we change variables, $y \leftrightarrow x+y$. We remark that $|y| < 2|x|$ implies $|x+y| \le |x|+|y| < 3|x|$, and that $\frac{|x|}{2} < |y|$ implies $|y|^{-n-2\sigma} < 2^{-n-2\sigma}|x|^{-n-2\sigma}$. Therefore,

$$
I_1 \le C|x|^{-n-2\sigma}\int_{|y|<3|x|} |f(y)|\,\mathrm{d}y. \tag{288}
$$

For I_2, we change variables again, $y \leftrightarrow x+y$. But here, $|y| > 2|x|$ implies $|x+y| > |y|-|x| > |x|$, and we can use the estimate from the assumption over f to obtain

$$
I_2 \le C_0|f(x)|\int_{|y|>|x|} |y|^{-n-2\sigma}\,\mathrm{d}y \le CC_0|f(x)||x|^{-2\sigma}. \tag{289}
$$

Using (288) and (289) in (287),

$$
\begin{aligned}
\int_{|y|>\frac{|x|}{2}} \frac{|f(x+y)|}{|y|^{n+2\sigma}}\,\mathrm{d}y \le C|x|^{-n-2\sigma}&\int_{|y|<3|x|} |f(y)|\,\mathrm{d}y \\
&+ CC_0|f(x)||x|^{-2\sigma}.
\end{aligned} \tag{290}
$$

Now, we estimate the second term of (285). Since for any $\theta \in (0,1)$ and $|y| < \frac{|x|}{2}$ it holds that $|x+\theta y| \ge |x| - \theta|y| > \frac{|x|}{2}$, we can apply the second assumption over f to obtain the estimate

$$
\left|\int_0^1 (1-\theta)\partial^\alpha f(x+\theta y)\,\mathrm{d}\theta\right| \le C_0\left|\partial^\alpha f\left(\tfrac{x}{2}\right)\right|\int_0^1 1-\theta\,\mathrm{d}\theta = \frac{C_0}{2}\left|\partial^\alpha f\left(\tfrac{x}{2}\right)\right|,
$$

hence

$$\left| \sum_{|\alpha|=2} \frac{|\alpha|}{\alpha!} \int_{|y|<\frac{|x|}{2}} \frac{y^\alpha}{|y|^{n+2\sigma}} \int_0^1 (1-\theta)\partial^\alpha f(x+\theta y)\, \mathrm{d}\theta \mathrm{d}y \right| \\ \leq C \sum_{|\alpha|=2} \frac{|\alpha|}{\alpha!} \left| \partial^\alpha f\left(\tfrac{x}{2}\right) \right| \int_{|y|<\frac{|x|}{2}} |y|^{-n-2\sigma+2}\, \mathrm{d}y,$$

And evaluating the latter integral over shells again, we obtain

$$\begin{aligned} \int_{|y|<\frac{|x|}{2}} |y|^{-n-2\sigma+2}\, \mathrm{d}y &= \int_0^{\frac{|x|}{2}} \int_{|y|=\rho} \rho^{-n-2\sigma+2} \mathrm{d}S_y \mathrm{d}\rho \\ &= \int_0^{\frac{|x|}{2}} \rho^{-n-2\sigma+2} \mu(B_n) \rho^{n-1}\, \mathrm{d}\rho \\ &= \frac{2^{2\sigma-2}}{2-2\sigma} \mu(B_n) |x|^{2-2\sigma}, \end{aligned}$$

which gives us

$$\left| \sum_{|\alpha|=2} \frac{|\alpha|}{\alpha!} \int_{|y|<\frac{|x|}{2}} \frac{y^\alpha}{|y|^{n+2\sigma}} \int_0^1 (1-\theta)\partial^\alpha f(x+\theta y)\, \mathrm{d}\theta \mathrm{d}y \right| \\ \leq C_2 |x|^{2-2\sigma} \sum_{|\alpha|=2} \frac{|\alpha|}{\alpha!} \left| \partial^\alpha f\left(\tfrac{x}{2}\right) \right|, \tag{291}$$

with $C_2 = \frac{2^{2\sigma-2}}{2-2\sigma} \mu(B_n)$.

Finally, using (286), (290) and (291) in (285), we get the desired result, that is

$$\begin{aligned} \left|(-\Delta)^\sigma f(x)\right| &\leq 2C_\sigma C_1 |f(x)|\, |x|^{-2\sigma} + 2C_\sigma C C_0 |f(x)|\, |x|^{-2\sigma} \\ &+ 2C_\sigma C C_0 \int_{|y|>3|x|} |f(y)| \mathrm{d}y \;\; |x|^{-n-2\sigma} \\ &+ 2C_\sigma C_2 \sum_{|\alpha|=2} \frac{|\alpha|}{\alpha!} \left| \partial^\alpha f\left(\tfrac{x}{2}\right) \right| |x|^{2-2\sigma} \\ &:= \tilde{C} |f(x)|\, |x|^{-2\sigma} + \tilde{C} \int_{|y|>3|x|} |f(y)| \mathrm{d}y \;\; |x|^{-n-2\sigma} \\ &+ \tilde{C} \sum_{|\alpha|=2} \frac{|\alpha|}{\alpha!} \left| \partial^\alpha f\left(\tfrac{x}{2}\right) \right| |x|^{2-2\sigma}, \end{aligned}$$

where $\tilde{C} = \max\left\{2C_\sigma C_1 + 2C_\sigma C C_0, 2C_\sigma C C_0, 2C_\sigma C_2\right\}$. ■

Corollary C.1.2 *Let $f(x) = \langle x \rangle^{-q}$, for $q > n$, and let $\sigma > 0$. We set $s = \sigma - [\sigma]$. Then,*

$$\left|(-\Delta)^\sigma f(x)\right| \leq C \langle x \rangle^{-q_\sigma}, \qquad \forall\, x \in \mathbb{R}^n,$$

where $q_\sigma = q + 2\sigma$ if σ is an integer, or $q = n + 2s$ otherwise.

Proof: Initially, we have the identity

$$
\begin{aligned}
-\Delta f(x) &= q\sum_{j=1}^{n} \partial_{x_j}\left(x_j\langle x\rangle^{-q-2}\right)\\
&= q\sum_{j=1}^{n}\left(\langle x\rangle^{-q-2} - (q+2)x_j^2\langle x\rangle^{-q-4}\right)\\
&= qn\langle x\rangle^{-q-2} - q(q+2)|x|^2\langle x\rangle^{-q-4}\\
&= qn\langle x\rangle^{-q-2} - q(q+2)\langle x\rangle^{-q-2} + q(q+w)\langle x\rangle^{-q-4}\\
&= q(q+2-n)\langle x\rangle^{-q-2} + q(q+2)\langle x\rangle^{-q-4}\\
&:= c_0\langle x\rangle^{-q-2} + c_1\langle x\rangle^{-q-2-2},
\end{aligned}
$$

with c_0, c_1 depending only on n, σ, q. After $[\sigma]$ iterations of this identity, we get

$$
(-\Delta)^{[\sigma]} f(x) = \sum_{k=0}^{[\sigma]} c_k\langle x\rangle^{-q-2[\sigma]-2k}, \tag{292}
$$

for some $c_k = c_k(n, \sigma, q) \in \mathbb{R}$.

Now, if we assume that σ is an integer, then $[\sigma] = \sigma$ and $s = 0$. Therefore, (292) implies

$$
\left|(-\Delta)^{\sigma} f(x)\right| \leq C\langle x\rangle^{-q-2\sigma},
$$

which is what we wanted to prove in this case. Next, assume σ non-integer, that is, $s \in (0,1)$. We have

$$
\begin{aligned}
\left|(-\Delta)^{\sigma} f(x)\right| &= \left|(-\Delta)^{s}\left((-\Delta)^{[\sigma]} f(x)\right)\right|\\
&= \left|(-\Delta)^{s}\sum_{k=0}^{[\sigma]} c_k\langle x\rangle^{-q-2[\sigma]-2k}\right|\\
&\leq \sum_{k=0}^{[\sigma]} |c_k|\left|(-\Delta)^{s}\langle x\rangle^{-q-2[\sigma]-2k}\right|.
\end{aligned} \tag{293}
$$

Then, applying Lemma 4.1.4 for each $g(x) = \langle x\rangle^{-q-2[\sigma]-2k}$ with $k = 0, ...[\sigma]$ (plugging s in the place of σ), we obtain the estimates

$$
\begin{aligned}
\left|(-\Delta)^{s}\langle x\rangle^{-q-2[\sigma]-2k}\right| \leq{}& C|x|^{-n-2s}\int_{|y|<3|x|}|g(y)|\mathrm{d}y + C|g(y)||x|^{-2s}\\
&+ C|x|^{-2s+2}\sum_{|\alpha|=2}\left|\partial^{\alpha} g\left(\tfrac{x}{2}\right)\right|,
\end{aligned}
$$

for every $|x| > 1$. Now, since g and $\partial^{\alpha} g$ are bounded, we must show only that the integral on the right-hand side is bounded. But this is a consequence of $\langle x\rangle^{-a} \in L^1(\mathbb{R}^n)$

for $a > n$. Indeed,

$$\begin{aligned}\int_{|y|<3|x|} |g(y)| \,\mathrm{d}y &\leq \int_{\mathbb{R}^n} |g(y)| \,\mathrm{d}y \\ &\leq \int_{|y|<1} |g(y)| \,\mathrm{d}y + \int_{|y|>1} |g(y)| \,\mathrm{d}y \\ &\leq \int_{|y|<1} \mathrm{d}y + \int_{|y|>1} |y|^{-a} \mathrm{d}y < +\infty, \text{ for } a > n.\end{aligned}$$

Since for $|x| > 1$ we have $|x| \sim \langle x \rangle$, we get the estimates

$$\left| (-\Delta)^s \langle x \rangle^{-q-2[\sigma]-2k} \right| \leq C \langle x \rangle^{-n-2s}, \tag{294}$$

for $k = 0, 1, ...[\sigma]$. Therefore, using (294) in (293), we obtain

$$\left| (-\Delta)^\sigma f(x) \right| \leq C \langle x \rangle^{-n-2s}, \tag{295}$$

which concludes our proof. ■