“TAREA #6”
ITESO

ITESO, Universidad Jesuita de Guadalajara

MÉTODOS DE OPTIMIZACIÓN
JUAN DIEGO SÁNCHEZ TORRES

IVANNA HERRERA IBARRA 744614
ARANTZA GOMEZ HARO GAMBOA 744249
JAVIER ALEJANDRO FAJARDO LÓPEZ 740448
LUIS FERNANDO MÁRQUEZ BAÑUELOS 744489

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**Problem 1: Capital budgeting**
Reproduce the capital budget example presented at 5:20 of the video *Solver: Optimizing Financial Decisions using Analytics*. Subsequently, using only two or three lines, explain the applicability and importance of employing binary decision variables.

Las variables de decisión binarias nos permiten tomar decisiones de si y no de una manera más sencilla y directa, utilizando 1 y 0 como resultado, en este caso nosotros tenemos que el 1 significa que sí y el 0 que no. De esta forma la toma de decisiones es más rápida y eficaz, optimizando cada problema.

**Problem 2: Blackstone Mining Company**
Lee Blackstone owns the Blackstone Mining Company, which operates two coal mines in Wythe and Giles counties in South-west Virginia. Due to increased commercial and residential development in the primary areas served by these mines, Lee's forecasting team is anticipating an increase in demand for coal in the coming year. Specifically, the projections indicate a $48$-ton increase in the demand for high-grade coal, a $28$-ton increase in the demand for medium-grade coal, and a $100$-ton increase in the demand for low-grade coal. To handle this increase in demand, Lee must schedule extra shifts for workers at the mines. It costs $\$40{,}000$ per month to run an extra shift of workers at the Wythe County mine and $\$32{,}000$ per month at the Giles mine. Only one additional shift can be scheduled each month at each mine.
The amount of coal that can be produced in a month's time at each mine by a shift of workers is summarized in the following table:

| Type of Coal | Wythe Mine | Giles Mine |
|---|---|---|
| High grade | 12 tons | 4 tons |
| Medium grade | 4 tons | 4 tons |
| Low grade | 10 tons | 20 tons |

Unfortunately, the methods used to extract coal from these mines produce toxic water that enters the local groundwater aquifers. At the Wythe mine, approximately $800$ gallons of toxic water per month will be generated by running an extra shift, whereas the mine in Giles County will generate about $1{,}250$ gallons of toxic water. Although these amounts are within EPA guidelines, Lee is concerned about the environment and doesn't want to create any more pollution than is absolutely necessary. Additionally, although the company follows all OSHA safety guidelines, company records indicate that approximately $0.20$ life-threatening accidents occur per shift each month at the Wythe mine, whereas $0.45$ accidents occur per shift each month at the Giles mine. Lee knows that mining is a hazardous occupation, but she cares about the health and welfare of her workers and wants to keep the number of life-threatening accidents to a minimum.
For this problems, then, present:

1. A MOLP formulation of the problem, clearly defining the decision variables, constraints and objectives.

$x_1$: Cantidad de meses en que se activa el nuevo turno en la mina A.
$x_2$: Cantidad de meses en que se activa el nuevo turno en la mina B.

Accidentes:

$$\min\ z = 0.2x_1 + 0.45x_2$$
$$s.t.\begin{cases} 12x_1 + 4x_2 \geq 48 \\ 4x_1 + 4x_2 \geq 28 \\ 10x_1 + 20x_2 \geq 100 \\ x_1 \leq 12 \\ x_2 \leq 12 \\ x_1,\ \ x_2 \geq 0 \end{cases}$$

Agua:

$$\min\ z = 800x_1 + 1{,}250x_2$$
$$s.t.\begin{cases} 12x_1 + 4x_2 \geq 48 \\ 4x_1 + 4x_2 \geq 28 \\ 10x_1 + 20x_2 \geq 100 \\ x_1 \leq 12 \\ x_2 \leq 12 \\ x_1,\ \ x_2 \geq 0 \end{cases}$$

Costo:

$$\min\ z = 40{,}000x_1 + 32{,}000x_2$$
$$s.t.\begin{cases} 12x_1 + 4x_2 \geq 48 \\ 4x_1 + 4x_2 \geq 28 \\ 10x_1 + 20x_2 \geq 100 \\ x_1 \leq 12 \\ x_2 \leq 12 \\ x_1,\ \ x_2 \geq 0 \end{cases}$$

2. Convert each of the objectives into goals, solving the optimization problem separately for each of the objectives. Then, present the multi-objective problem as a goal programming problem.

| Solución | x1 | x2 | Accidentes | Agua | Costo |
|---|---|---|---|---|---|
| Accidentes | 10 | 0 | 2 | 8000 | 400000 |
| Agua | 4 | 3 | 2.15 | 6950 | 256000 |
| Costo | 2.5 | 4.5 | 2.525 | 7625 | 244000 |

Convertir a metas:

$$
\begin{cases}
0.2x_1 + 0.45x_2 \leq 2 \\
800x_1 + 1{,}250x_2 \leq 6{,}950 \\
40{,}000x_1 + 32{,}000x_2 \leq 244{,}000
\end{cases}
$$

$$
\begin{gathered}
12x_1 + 4x_2 \geq 48 \\
4x_1 + 4x_2 \geq 28 \\
10x_1 + 20x_2 \geq 100 \\
0 \leq x_1, x_2 \leq 12
\end{gathered}
$$

3. Regularize the goal programming problem using the procedure used in previous problems. That is, the functional is a linear combination of the slack variables. Then, state why this approach is unsuitable for finding trade-offs between objectives.
   **Hint:** Apply the *Fundamental Theorem of Linear Programming*

Regularización usando holguras:

$$\min\ z = w_1 S_1^+ + w_2 S_2^+ + w_3 S_3^+,\ \ w_i \geq 0$$

$$
s.t. \begin{cases}
0.2x_1 + 0.45x_2 - S_1^+ + S_1^- = 2 \\
800x_1 + 1{,}250x_2 - S_2^+ + S_2^- = 6{,}950 \\
40{,}000x_1 + 32{,}000x_2 - S_3^+ + S_3^- = 244{,}000 \\
12x_1 + 4x_2 \geq 48 \\
4x_1 + 4x_2 \geq 28 \\
10x_1 + 20x_2 \geq 100 \\
0 \leq x_1, x_2 \leq 12 \\
S_i^+ \geq 0,\ \ S_i^- \geq 0,\ \ i = 1, 2, 3
\end{cases}
$$

Esta forma de resolver el probelma no va a encontrar respuestas nuevas a las 3 originales, esto se debe a que se esta igualando a estas mismas respuestas pero con holgura, por lo que al minimizer el error va a buscar llegar a una de estas respuestas para que las hiolguras sean 0, y la única manera de llegar a estás respuestas es con los vertices ya encontrados, por esto esta manera de formular el problema no sirve para llegar a puntos medios pues el functional de costo sigue siendo una combinación lineal de las variables de holgura.

4. Regularize the goal programming problem using the Minimax strategy. Once the problem has been posed in the Minimax form, convert it into a linear problem using the epigraph. Note, then, that the impossibility of finding trade-offs in the strategy of the previous point is not due to the reformulation by goals but to the regularization that considers the objective function as a linear combination of the slack variables.

Regularizar:

$$s_1 = \frac{0.2x_1 + 0.45x_2 - 2}{2}$$

$$s_2 = \frac{800x_1 + 1{,}250x_2 - 6{,}950}{6{,}950}$$

$$s_3 = \frac{40{,}000x_1 + 32{,}000x_2 - 244{,}000}{244{,}000}$$

Formulación minimax:

$$\min \max \ \{s_1, \quad s_2, \quad s_3\}$$

$$s_1 = \left(\frac{0.2x_1 + 0.45x_2 - 2}{2}\right) w_1$$

$$s_2 = \left(\frac{800x_1 + 1{,}250x_2 - 6{,}950}{6{,}950}\right) w_2$$

$$s_3 = \left(\frac{40{,}000x_1 + 32{,}000x_2 - 244{,}000}{244{,}000}\right) w_3$$

$$s.t.\begin{cases} 12x_1 + 4x_2 \geq 48 \\ 4x_1 + 4x_2 \geq 28 \\ 10x_1 + 20x_2 \geq 100 \\ 0 \leq x_1, x_2 \leq 12 \end{cases}$$

Truco del epígrafe:

$$\min q$$
$$s_1 \leq q$$
$$s_2 \leq q$$
$$s_3 \leq q$$

$$s.t.\begin{cases} w_1\left(\frac{0.2x_1 + 0.45x_2 - 2}{2}\right) \leq q \\ w_2\left(\frac{800x_1 + 1{,}250x_2 - 6{,}950}{6{,}950}\right) \leq q \\ w_3\left(\frac{40{,}000x_1 + 32{,}000x_2 - 244{,}000}{244{,}000}\right) \leq q \\ 12x_1 + 4x_2 \geq 48 \\ 4x_1 + 4x_2 \geq 28 \\ 10x_1 + 20x_2 \geq 100 \\ 0 \leq x_1, x_2 \leq 12 \end{cases}$$

5. Write the Minimax problem in a matrix form.

$$\min \ c^T x$$
$$s.t. \ Ax \geq b$$
$$x \geq 0$$

$$x = [x_1 \quad x_2 \quad q \quad S_1 \quad S_2 \quad S_3]^T$$
$$c^T = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]$$

$$A = \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 12 & 4 & 0 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 & 0 \\ 10 & 20 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 48 \\ 28 \\ 100 \\ -12 \\ -12 \end{bmatrix}$$

6. Solve the given problem in a Minimax form using Excel. Present at least three solutions, including their respective sensitivity analysis, and present a study. That is, which recommendation would you give to Lee Blackstone?

Solución 1:
Misma prioridad a las 3 metas.

| Tipo | x1 | x2 | Resultado | Desviación |
|---|---|---|---|---|
| Accidentes | 4.23290203 | 2.88354898 | 2.14417745 | 7.2089% |
| Agua sucia | 4.23290203 | 2.88354898 | 6990.75786 | 0.5864% |
| Costo | 4.23290203 | 2.88354898 | 261.589649 | 7.2089% |

| q |
|---|
| 0.072088725 |

Celdas de variables

| Celda | Nombre | Final Valor | Reducido Coste | Objetivo Coeficiente | Permisible Aumentar | Permisible Reducir |
|---|---|---|---|---|---|---|
| $C$4 | x1 | 4.232902033 | 0 | 0 | 0.0125 | 0.098360656 |
| $D$4 | x2 | 2.883548983 | 0 | 0 | 0.196721311 | 0.025 |
| $E$4 | q | 0.072088725 | 0 | 1 | 1E+30 | 1 |

Restricciones

| Celda | Nombre | Final Valor | Sombra Precio | Restricción Lado derecho | Permisible Aumentar | Permisible Reducir |
|---|---|---|---|---|---|---|
| $I$7 | holgura 1 total | 1.38778E-17 | 0.887245841 | 0 | 0.639344262 | 0.025819672 |
| $I$8 | holgura 2 total | 0.066224285 | 0 | 0 | 0.066224285 | 1E+30 |
| $I$9 | holgura 3 total | -1.11022E-16 | 0.112754159 | 0 | 0.025819672 | 0.194843304 |
| $I$10 | alto grado total | 62.32902033 | 0 | 48 | 14.32902033 | 1E+30 |
| $I$11 | medio grado total | 28.46580407 | 0 | 28 | 0.465804067 | 1E+30 |
| $I$12 | bajo grado total | 100 | 0.010720887 | 100 | 183.4934498 | 1.636363636 |
| $I$13 | x1 total | -4.232902033 | 0 | -12 | 7.767097967 | 1E+30 |
| $I$14 | x2 total | -2.883548983 | 0 | -12 | 9.116451017 | 1E+30 |

Solución 2:
Accidentes con peso de 10, agua sucia con peso de 5 y costo con peso de 2.

| Tipo | x1 | x2 | Resultado | Desviación |
|---|---|---|---|---|
| Accidentes | 6.02547771 | 1.98726115 | 2.09936306 | 4.9682% |
| Agua sucia | 6.02547771 | 1.98726115 | 7304.4586 | 5.1001% |
| Costo | 6.02547771 | 1.98726115 | 304.611465 | 24.8408% |

| q |
|---|
| 0.496815287 |

Celdas de variables

| Celda | Nombre | Final Valor | Reducido Coste | Objetivo Coeficiente | Permisible Aumentar | Permisible Reducir |
|---|---|---|---|---|---|---|
| $C$4 | x1 | 6.025477707 | 0 | 0 | 0.125 | 0.196721311 |
| $D$4 | x2 | 1.987261146 | 0 | 0 | 0.393442623 | 0.25 |
| $E$4 | q | 0.496815287 | 0 | 1 | 1E+30 | 1 |

Restricciones

| Celda | Nombre | Final Valor | Sombra Precio | Restricción Lado derecho | Permisible Aumentar | Permisible Reducir |
|---|---|---|---|---|---|---|
| $I$7 | holgura 1 total | -1.11022E-16 | 0.611464968 | 0 | 1.278688525 | 0.651639344 |
| $I$8 | holgura 2 total | 0.2418091 | 0 | 0 | 0.2418091 | 1E+30 |
| $I$9 | holgura 3 total | -1.11022E-15 | 0.388535032 | 0 | 0.651639344 | 0.310065221 |
| $I$10 | alto grado total | 80.25477707 | 0 | 48 | 32.25477707 | 1E+30 |
| $I$11 | medio grado total | 32.05095541 | 0 | 28 | 4.050955414 | 1E+30 |
| $I$12 | bajo grado total | 100 | 0.07388535 | 100 | 19.02439024 | 4.953271028 |
| $I$13 | x1 total | -6.025477707 | 0 | -12 | 5.974522293 | 1E+30 |
| $I$14 | x2 total | -1.987261146 | 0 | -12 | 10.01273885 | 1E+30 |

Solución 3:
Accidentes con peso de 2, agua sucia con peso de 4 y costo con peso de 8.

| Tipo | x1 | x2 | Resultado | Desviación |
|---|---|---|---|---|
| Accidentes | 3.5248 | 3.4752 | 2.2688 | 13.4400% |
| Agua sucia | 3.5248 | 3.4752 | 7163.84 | 3.0768% |
| Costo | 3.5248 | 3.4752 | 252.1984 | 3.3600% |

| q |
|---|
| 0.2688 |

Celdas de variables

| Celda | Nombre | Final Valor | Reducido Coste | Objetivo Coeficiente | Permisible Aumentar | Permisible Reducir |
|---|---|---|---|---|---|---|
| $C$4 | x1 | 3.5248 | 0 | 0 | 0.25 | 0.262295082 |
| $D$4 | x2 | 3.4752 | 0 | 0 | 0.262295082 | 0.25 |
| $E$4 | q | 0.2688 | 0 | 1 | 1E+30 | 1 |

Restricciones

| Celda | Nombre | Final Valor | Sombra Precio | Restricción Lado derecho | Permisible Aumentar | Permisible Reducir |
|---|---|---|---|---|---|---|
| $I$7 | holgura 1 total | -5.55112E-17 | 0.512 | 0 | 0.243442623 | 0.14321267 |
| $I$8 | holgura 2 total | 0.145726619 | 0 | 0 | 0.145726619 | 1E+30 |
| $I$9 | holgura 3 total | -1.11022E-16 | 0.488 | 0 | 0.525 | 0.243442623 |
| $I$10 | alto grado total | 56.1984 | 0 | 48 | 8.1984 | 1E+30 |
| $I$11 | medio grado total | 28 | 0.1856 | 28 | 2.082236842 | 0.599697123 |
| $I$12 | bajo grado total | 104.752 | 0 | 100 | 4.752 | 1E+30 |
| $I$13 | x1 total | -3.5248 | 0 | -12 | 8.4752 | 1E+30 |
| $I$14 | x2 total | -3.4752 | 0 | -12 | 8.5248 | 1E+30 |

La solución que recomendamos es la primera, pues tiene el valor de q más chico y se le da la misma importancia a todas las metas por lo que distribuye el error de major manera. Esta solución es muy Buena pues el nievel de contaminación se mantiene casi al mínimo y tanto los accidents como el costo solo están 7% por encima del mínimo, siendo así una solución muy útil para la empresa. Además los precios sombre de sus restricciones son muy bajos por lo que se pueden mover un poco las variables sin afectar tanto el funcional de costo manteniendo las metas muy cerca del mínimo.

**Problem 3: Burnit Goal Programming**

The Leon Burnit Advertising Agency is trying to determine a TV advertising schedule for Priceler Auto Company. Priceler has three goals:

1. Goal 1: Its ads should be seen by at least 40 million high-income men (HIM).
2. Goal 2: Its ads should be seen by at least 60 million low-income people (LIP).
3. Goal 3: Its ads should be seen by at least 35 million high-income women (HIW).

Leon Burnit can purchase two types of ads: those shown during football games and those shown during soap operas. At most, $600,000 can be spent on ads. The advertising costs and potential audiences of a one-minute ad of each type are shown in the following table:

| | Millions of Viewers |
|---|---|

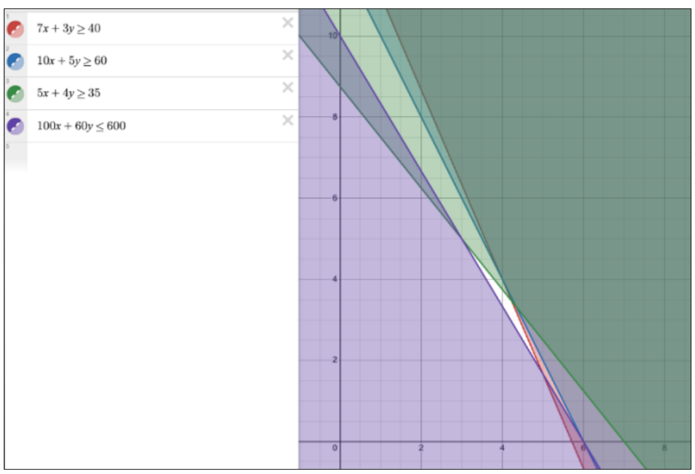
| Ad | HIP | LIP | HIW | Cost ($) |
|---|---|---|---|---|
| Football | 7 | 10 | 5 | 100,000 |
| Soap opera | 3 | 5 | 4 | 60,000 |

*Cost and Number of Viewers of Ads for Priceler*

Leon Burnit must determine how many football ads and soap opera ads to purchase for Priceler.
Let $x_1$ the number of minutes of ads shown during football games and $x_2$ number of minutes of ads shown during soap operas.

Hence, we would meet Priceler's goals:
$7x_1 + 3x_2 \geq 40$ (HIM constraint)
$10x_1 + 5x_2 \geq 60$ (LIP constraint)
$5x_1 + 4x_2 \geq 35$ (HIW constraint)
$100x_1 + 60x_2 \leq 600$ (Budget constraint)
$x_1, x_2 \geq 0$

Draw the feasible set and note that no point that satisfies the budget constraint meets all three of Priceler's goals. Thus, the previous goal set has no feasible solution.




Since it is impossible to meet all of Priceler's goals, Burnit might ask Priceler to identify, for each goal, a cost (per-unit short of meeting each goal) that is incurred for failing to meet the goal. Suppose Priceler determines that

- Each million exposures by which Priceler falls short of the HIM goal costs Priceler a $\$200{,}000$ penalty because of lost sales.
- Each million exposures by which Priceler falls short of the LIP goal costs Priceler a $\$100{,}000$ penalty because of lost sales.
- Each million exposures by which Priceler falls short of the HIW goal costs Priceler a $\$50{,}000$ penalty because of lost sales.

As a first approach, we can rewrite the first three goals as

$$7x_1 + 3x_2 + S_1^- - S_1^+ = 40 \text{ (HIM constraint)}$$
$$10x_1 + 5x_2 + S_2^- - S_2^+ = 60 \text{ (LIP constraint)}$$
$$5x_1 + 4x_2 + S_3^- - S_3^+ = 35 \text{ (HIW constraint)}$$

where the slack variable $S_i^+$ is amount by which we numerically exceed the $i$-th goal and $S_i^-$ is amount by which we are numerically under the $i$-th goal.

Finally, the following regularized problem represents how Burnit can minimize the penalty from Priceler's lost sales by solving an LP:

$$\min\ z = 200S_1^- + 100S_2^- + 50S_3^-$$

$$s.t.\begin{cases} 7x_1 + 3x_2 + S_1^- - S_1^+ = 40\ (HIM\ constraint) \\ 10x_1 + 5x_2 + S_2^- - S_2^+ = 60\ (LIP\ constraint) \\ 5x_1 + 4x_2 + S_3^- - S_3^+ = 35\ (HIW\ constraint) \\ 100x_1 + 60x_2 \le 600\ (Budget\ constraint) \end{cases}$$

All variables nonnegative

Use the previous information to propose an alternative regularization using the Minimax approach. Solve it in Excel, including the sensitivity analysis. Then, recommend what to do in this case to Burnit.

Do not forget to write the Minimax problem in a matrix form before solving it in Excel.

$$\min \max\ \{200S_1^-,\ \ 100S_2^-,\ \ 50S_3^-\}$$

$$\min q$$
$$200S_1^- \le q$$
$$100S_2^- \le q$$
$$50S_3^- \le q$$

$$s.t.\begin{cases} 7x_1 + 3x_2 = 40 + S_1^+ - S_1^- \\ 10x_1 + 5x_2 = 60 + S_2^+ - S_2^- \\ 5x_1 + 4x_2 = 35 + S_3^+ - S_3^- \\ 100x_1 + 60x_2 \le 600 \\ x_1,\ \ x_2 \ge 0 \\ S_i^+,\ \ S_i^- \ge 0 \end{cases}$$

$$\min\ c^T x$$

$$s.t.\begin{cases} Ax = b \\ Dx \ge E \\ x \ge 0 \end{cases}$$

$$x = [q \quad x_1 \quad x_2 \quad S_1^+ \quad S_1^- \quad S_2^+ \quad S_2^- \quad S_3^+ \quad S_3^-]^T$$

$$c^T = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$A = \begin{bmatrix} 0 & 7 & 3 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 10 & 5 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 5 & 4 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 40 \\ 60 \\ 35 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & -200 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -100 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -50 \\ 0 & -100 & -60 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -600 \end{bmatrix}$$

Solución:

| Tipo | x1 | x2 | Resultado | Desviación |
|---|---|---|---|---|
| HIM | 5 | 1.66666667 | 40 | 0 |
| LIP | 5 | 1.66666667 | 58.3333333 | -1.6666667 |
| HIW | 5 | 1.66666667 | 31.6666667 | -3.3333333 |

| q |
|---|
| 166.666667 |

Celdas de variables

| Celda | Nombre | Final Valor | Reducido Coste | Objetivo Coeficiente | Permisible Aumentar | Permisible Reducir |
|---|---|---|---|---|---|---|
| $D$14 | x1 | 5 | 0 | 0 | 1E+30 | 1E+30 |
| $E$14 | x2 | 1.666666667 | 0 | 0 | 1E+30 | 1E+30 |

Restricciones

| Celda | Nombre | Final Valor | Sombra Precio | Restricción Lado derecho | Permisible Aumentar | Permisible Reducir |
|---|---|---|---|---|---|---|
| $L$4 | HIM total | 40 | 0 | 40 | 7.58758E-15 | 16.25 |
| $L$5 | LIP total | 60 | 0 | 60 | 9.05625E-15 | 1E+30 |
| $L$6 | HIW total | 35 | 0 | 35 | 6.32298E-15 | 3.095238095 |
| $L$7 | holgura 1 total | 0 | 0 | 0 | 0 | 1E+30 |
| $L$8 | holgura 2 total | 0 | 0 | 0 | 0 | 1E+30 |
| $L$9 | holgura 3 total | 0 | 0 | 0 | 0 | 1E+30 |
| $L$10 | Presupuesto total | -600 | 0 | -600 | 5.8366E-14 | 1E+30 |

Utilizando el método de minimax obtenemos que lo mejor opción es dedicarle 5 minutos de publicidad durante el football y 1.66 minutos durante la opera, de esta forma llegamos a toda la audiencia de HIM, para LIP llegamos a 1.66 millones de personas menos y para HIW llegamos a 3.33 millones de personas menos, lo cual no es un error muy grave pues HIW es donde menos cuesta no llegar a las personas.

**Problem 4: Bond allocation**

A bond portfolio manager has $\$100{,}000$ to allocate to two different bonds: a corporate bond and a government bond.

These bonds have the following yield, risk level, and maturity:

| Bond | Yield | Risk level | Maturity |
|---|---|---|---|
| Corporate | 4% | 2 | 3 years |

| | | | |
|---|---|---|---|
| Government | 3% | 1 | 4 years |

In the first scenario, the portfolio manager would like to allocate funds so that the average risk level of the portfolio is at most $1.5$ years and the average maturity is at most $3.6$ years. Any amount not invested in the bonds will be kept in a cash account that is assumed to generate no interest and does not contribute to the average risk level or maturity. In other words, assume cash has zero yield, risk level, and maturity.
Assume the portfolio can only include long positions. In this case, how should the manager allocate funds to the two bonds to maximize yield?
In a second scenario, assume that yield enhancement and risk reduction are two fundamental objectives of fund allocation. In this case, a MOLP formulation will be set up, and this problem will be solved using the Minimax strategy.
Additionally, in both cases, assume that all money must be invested for ease of calculations.

Formulación MOLP:

$x_1$: Cantidad de dinero para bonos corporativos.
$x_2$: Cantidad de dinero para bonos de gobierno.

Riesgo:

$$\min\ z = 2x_1 + x_2$$
$$s.t.\begin{cases} x_1 + x_2 = 100 \\ x_1,\ x_2 \geq 0 \end{cases}$$

Yield:

$$\max\ z = 4x_1 + 3x_2$$
$$s.t.\begin{cases} x_1 + x_2 = 100 \\ x_1,\ x_2 \geq 0 \end{cases}$$

Maduración:

$$\min\ z = 3x_1 + 4x_2$$
$$s.t.\begin{cases} x_1 + x_2 = 100 \\ x_1,\ x_2 \geq 0 \end{cases}$$

| | Riesgo | Yield | Maturity |
|---|---|---|---|
| (0,0) | 0 | 0 | 0 |
| (0,100) | 100 | 300 | 400 |
| (100,0) | 200 | 400 | 300 |

Metas:

$$\frac{2x_1 + x_2}{100} \leq 1$$
$$\frac{4x_1 + 3x_2}{100} \geq 4$$

$$\frac{3x_1 + 4x_2}{100} \leq 3$$

$$S_1 = \frac{2x_1 + x_2 - 100}{100}$$
$$S_2 = \frac{-4x_1 - 3x_2 + 400}{400}$$
$$s_3 = \frac{3x_1 + 4x_2 - 300}{300}$$

Minimax:
$\min q$
$w_1 S_1 \leq q$
$w_2 S_2 \leq q$
$w_3 S_3 \leq q$
$x_1 + x_2 = 100$
$x_1,\ x_2 \geq 0$

Minimax:
$\min q$
$w_1 S_1 \leq q \qquad q - w_1 S_1 \geq 0$
$w_2 S_2 \leq q \qquad q - w_2 S_2 \geq 0$
$w_3 S_3 \leq q \qquad q - w_3 S_3 \geq 0$
$x_1 + x_2 \leq 100 \qquad -x_1 - x_2 \geq -100$
$-x_1 - x_2 \leq -100 \qquad x_1 + x_2 \geq 100$
$x_1,\ x_2 \geq 0$

Comment making a comparison between both scenarios, considering future investment recommendations.

Escenario 1 (Maximizar yield):

| x1 | x2 |
|---|---|
| 100 | 0 |

| | Riesgo | Yield | Maturity |
|---|---|---|---|
| (100,0) | 200 | 400 | 300 |

Escenario 2 (Maximizar yield minimizando el riesgo):

| Tipo | x1 | x2 | Resultado | Desviación |
|---|---|---|---|---|
| riesgo | 25 | 75 | 125 | 25.0000% |
| yield | 25 | 75 | 325 | 18.7500% |
| maduración | 25 | 75 | 375 | 25.0000% |

| q |
|---|
| 0.25 |

En el escanario uno si bien maximizamos el yield de la inversión, también estamos corriendo el mayor riesgo posible. Para la segunda opción nuestro yield es 18.75% menor al maximo pero el riesgo que se corre es solo 25% mayor al mínimo posible.

Con esto podemos recomendar el escenario 2 como una mejor opción de inversión, pues a pesar de ganar menos el riesgo que se corre es bajo sin sacrificar tanto la ganancia. Usar el método de multiobjetivo es bueno para futuras inversiones pues a largo plazo es importate no correr riesgos altos con el dinero invertido.

**Problem 5: Pond Island Bank investments**

A trust officer at Pond Island Bank needs to determine what percentage of the bank's investible funds to place in each of the following investments.

| Investment | Yield | Maturity | Risk |
|---|---|---|---|
| A | 11% | 8 | 5 |
| B | 8% | 1 | 2 |
| C | 8.5% | 7 | 1 |
| D | 10% | 6 | 5 |
| E | 9% | 2 | 3 |

The column labeled Yield represents each investment's annual yield. The Maturity column indicates the year's funds must be placed in each investment. The column labeled Risk indicates an independent financial analyst's assessment of each investment's risk. In general, the trust officer wants to maximize the weighted average yield on the funds in these investments while minimizing the weighted average maturity and risk.

1. Formulate a MOLP model for this problem and implement the model in an Excel spreadsheet. Write the problem in a matrix form before solving it in Excel

Formulación MOLP:

$x_1$: Cantidad de dinero invertido en el fondo A.
$x_2$: Cantidad de dinero invertido en el fondo B.
$x_3$: Cantidad de dinero invertido en el fondo C.
$x_4$: Cantidad de dinero invertido en el fondo D.
$x_5$: Cantidad de dinero invertido en el fondo E.

Riesgo:

$$\min\ z = 5x_1 + 2x_2 + x_3 + 5x_4 + 3x_5$$

$$s.t. \begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 1 \\ x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0 \end{cases}$$

Yield:

$$\max\ z = 11x_1 + 8x_2 + 8.5x_3 + 10x_4 + 9x_5$$

$$s.t.\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 1 \\ \quad x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0 \end{cases}$$

Maduración:

$$\min\ z = 8x_1 + x_2 + 7x_3 + 6x_4 + 2x_5$$

$$s.t.\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 1 \\ \quad x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0 \end{cases}$$

| Riesgo | | | | | | |
|---|---|---|---|---|---|---|
| x1 | x2 | x3 | x4 | x5 | Total | Restricción |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

| x1 | x2 | x3 | x4 | x5 | Min |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 | 1 |

| Yield | | | | | | |
|---|---|---|---|---|---|---|
| x1 | x2 | x3 | x4 | x5 | Total | Restricción |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

| x1 | x2 | x3 | x4 | x5 | Max |
|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 11 |

| Maturity | | | | | | |
|---|---|---|---|---|---|---|
| x1 | x2 | x3 | x4 | x5 | Total | Restricción |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

| x1 | x2 | x3 | x4 | x5 | Min |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 1 |

Metas:

$$5x_1 + 2x_2 + x_3 + 5x_4 + 3x_5 \leq 1$$

$$11x_1 + 8x_2 + 8.5x_3 + 10x_4 + 9x_5 \geq 11$$

$$8x_1 + x_2 + 7x_3 + 6x_4 + 2x_5 \leq 1$$

$$S_1 = \frac{5x_1 + 2x_2 + x_3 + 5x_4 + 3x_5 - 1}{1}$$

$$S_2 = \frac{11 - 11x_1 + 8x_2 + 8.5x_3 + 10x_4 + 9x_5}{11}$$

$$S_3 = \frac{8x_1 + x_2 + 7x_3 + 6x_4 + 2x_5 - 1}{1}$$

Minimax:

$$\min q$$

$$w_1 S_1 \leq q$$

$$w_2 S_2 \leq q$$

$$w_3 S_3 \leq q$$

$$x_1 + x_2 + x_3 + x_4 + x_5 = 1$$

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0$$

Minimax:

$\min q$

$w_1 S_1 \leq q \qquad q - w_1 S_1 \geq 0$

$w_2 S_2 \leq q \qquad q - w_2 S_2 \geq 0$

$w_3 S_3 \leq q \qquad q - w_3 S_3 \geq 0$

$x_1 + x_2 + x_3 + x_4 + x_5 \leq 1$

$x_1 + x_2 + x_3 + x_4 + x_5 \geq 1$

$x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0$

Forma matricial:

$\min\ c^T x$

$s.t.\ \ Ax \geq b$

$x \geq 0$

$x = [q \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad S_1 \quad S_2 \quad S_3]^T$

$c^T = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -w_1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -w_2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -w_3 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

2. Determine the best possible value for each objective in the problem.

| Tipo | x1 | x2 | x3 | x4 | x5 | Resultado | Desviación |
|---|---|---|---|---|---|---|---|
| riesgo | 0 | 0.8571429 | 0.1428571 | 0 | 0 | 1.8571429 | 85.71% |
| yield | 0 | 0.8571429 | 0.1428571 | 0 | 0 | 8.0714286 | 26.62% |
| maduración | 0 | 0.8571429 | 0.1428571 | 0 | 0 | 1.8571429 | 85.71% |

| q |
|---|
| 1.0392157 |

Como podemos ver, al darle el mismo peso a cada una de las metas, tenemos que tanto para los fondos A, D y E no se debe invertir nada, mientras que en el fondo B, se debe invertir el 85.71% del total mientras que en el C el 14.28% restante. En este caso, corremos con un 85% más de riesgo, tenemos un 26% de yield menos de lo que se esperaba y una maduración 85% mayor de lo que queríamos.

3. Suppose management considers minimizing the average maturity twice as important as minimizing average risk and maximizing average yield twice as important as minimizing average maturity. What solution does this suggest?

| Tipo | x1 | x2 | x3 | x4 | x5 | Resultado | Desviación |
|---|---|---|---|---|---|---|---|
| riesgo | 0 | 0.8235294 | 0.0686275 | 0 | 0.1078431 | 2.0392157 | 103.92% |
| yield | 0 | 0.8235294 | 0.0686275 | 0 | 0.1078431 | 8.1421569 | 25.98% |
| maduración | 0 | 0.8235294 | 0.0686275 | 0 | 0.1078431 | 1.5196078 | 51.96% |

| q |
|---|
| 1.0392157 |

Si hacemos los cambios para que lo que más importe sea el Yield, y que la madurez sea el doble de importante que el riesgo, obtenemos que al fondo A y D no se debe meter nada, mientras que, al B, el 82.35% del total, al C el 6.86% y al

E el 10.78%. De esta forma tenemos que el riesgo es un 103% mayor al que en un inicio buscábamos, la maduración es 51% más alta y el yield es 25.98% más bajo de lo que se buscaba.

En este caso dependiendo de las necesidades de la empresa es el escenario que se debería tomar, to,ando en cuenta las prioridades que las empresas les pongan a cada meta y la forma en como quieren se cumplan cada una de ellas.