The Simple Assembly Line Balancing (SALB) Problem

1 The SALB problem

An assembly line is a sequence of tasks (i.e. operations) to ultimately produce a product, e.g. a car, a computer, a house, a pair of shoes, etc. Each task could have one or more precedence tasks (e.g. first a chair is assembled, then painted), and each must be assigned to one workstation.

Suppose the following example to assemble a motherboard to a computer as shown in Table 1.

Task (i)	Task name	Time in minutes (t_i)	Predecessors
1	Drill holes in the metal casing	3	–
2	Attach the motherboard	5	1
3	Mount the brackets to hold disk	2	–
4	Mount DVD and hard disk (HD)	4	2,3
5	Attach the HD controller and power supply	2	4

Table 1: An easy example to model the SALB problem.

Graphically:

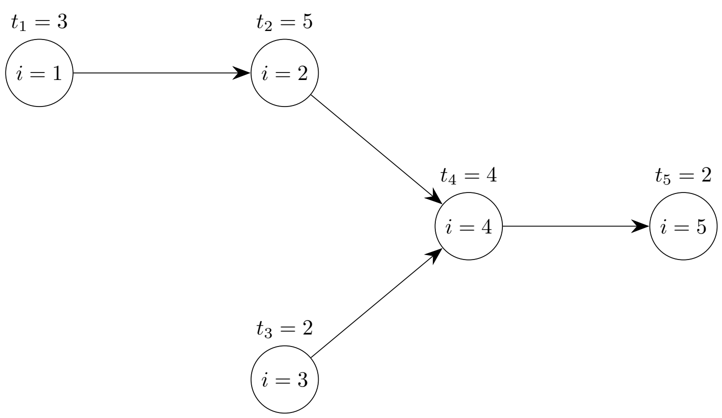

Figure 1: Graphical representation of the line balancing problem.

1.1 SALB 1: Minimise the number of workstation (s_j) given that the cycle time (C) is known.

The cycle time (C) is the maximum time a workstation must complete all the assigned tasks. In the SALB 1 problem, the C is known, and the objective is to minimise the number of workstations. The problem has N tasks ($i = 1, 2, \dots N$), M workstations ($j = 1, 2, \dots, M$), and a workstation j is represented by s_j, e.g. if the tasks $i = 2$ and $i = 4$ are assembled in workstation 3, then it is represented by $s_3 = \{i = 2, 4\}$.

Suppose that $C = 7$ minutes for the above problem; thus, a motherboard must be fully assembled every 7 minutes. Therefore, the total time of the assigned tasks to a workstation (st_j) is less than equal to 7, as shown in Eq. 1.

$$st_j = \sum_{i \in s_j} t_i \leq C \tag{1}$$

And the efficiency of the open station is computed using Eq. 2

$$E_j = \frac{\sum\limits_{i \in s_j} t_i}{C} \tag{2}$$

In the SALB 1 problem, a maximum number of stations can be open. This number is computed by Eq. 3.

$$s_{\max} = \left\lceil \frac{\sum_{i=1}^{N} t_i}{C} \right\rceil \tag{3}$$

For the problem in Fig. 1 the maximum number of stations is three, $s_{\max} = \lceil 16/7 \rceil = 3$. A possible solution to the problem is shown in Table 2. As expected every open stations must satisfy $st_j \leq 7$ (see Eq. 1). Notice that all the tasks that have predecessors are assigned to a former station; e.g. task $i = 4$ (assigned to station s_3) has two predecessors $i = 2, 3$; thus, task $i = 2$ is assigned to station s_2 and $i = 3$ is assigned to s_1. NOTE: if two tasks have a precedence relationship, the two task could be assigned to the same station, but never the predecesor task could be assigned to a later staion.

Workstation (s_j)	s_1	s_2	s_3
Tasks (i)	$i = 1, 3$	$i = 2$	$i = 4, 5$
Station time (st_j)	$t_1 + t_3 = 5 \leq 7$	$t_2 = 5 \leq 7$	$t_4 + t_5 = 6 \leq 7$
Station efficiency (E_j)	$E_1 = \frac{5}{7} = 0.71$	$E_2 = \frac{5}{7} = 0.71$	$E_3 = \frac{6}{7} = 0.86$

Table 2: A random solution to the SALB 1 problem in Fig. 1 with three stations.

The solution in Table 2 is not the only solution. Another one is shown in Table 3 with four stations. Notice that the predecessor relationships are met.

Workstation (s_j)	s_1	s_2	s_3
Tasks (i)	$i = 1$	$i = 2, 3$	$i = 4, 5$
Station time (st_j)	$t_1 = 3 \leq 7$	$t_2 + t_3 = 7 \leq 7$	$t_4 + t_5 = 6 \leq 7$
Station efficiency (E_j)	$E_1 = \frac{3}{7} = 0.43$	$E_2 = \frac{7}{7} = 1.00$	$E_3 = \frac{6}{7} = 0.86$

Table 3: Another random solution to the SALB 1 problem in Fig. 1 with three stations.

Table 2 and 3 are solutions to the problem with three open stations. Could it be a solution with less than three stations to meet the cycle time $C = 7$ min? For this tiny problem, we realise that it is not possible.

1.2 SALB 2: Minimise the cycle time (C) given that the number of stations (s_j) is known.

In the SALB 2 problem, the objective is to minimise the cycle time C given a known number of stations s_j. The cycle time C is the maximum of the station times as shown in Eq. 4.

$$C = \max_j \{st_j\} \tag{4}$$

Suppose that two stations will be opened to solve the problem in Fig. 1, i.e. $M = 2$. A random solution is presented in Table 4 with a cycle time $C = \max\{8, 8\} = 8$ minutes. In words, with two workstations, every 8 minutes, a motherboard will be produced.

The eficiency of every station is $E_1 = 8/8 = 1.00$ y $E_2 = 8/8 = 1.00$.

Workstation (s_j)	s_1	s_2
Tasks (i)	$i = 1, 2$	$i = 3, 4, 5$
Station time (st_j)	$t_1 + t_2 = 8$	$t_3 + t_4 + t_5 = 8$

Table 4: A solution to the SALB 2 problem in Fig. 1 with two stations.

Another solution with four stations is presented in Table 5. In this case, the cycle time is $C = \max\{5, 5, 4, 2\} = 5$ minutes; thus, with four stations, every five minutes, a motherboard will be produced.

Workstation (s_j)	s_1	s_2	s_3	s_4
Tasks (i)	$i = 1, 3$	$i = 2$	$i = 4$	$i = 5$
Station time (st_j)	$t_1 + t_3 = 5$	$t_2 = 5$	$t_4 = 4$	$t_5 = 2$

Table 5: Another random solution to the SALB 2 problem in Fig. 1 with four stations.

The eficiency for workstations is $E_1 = 5/5 = 1.0$, $E_2 = 5/5 = 1.0$, $E_3 = 4/5 = 0.8$, and $E_4 = 2/5 = 0.4$.

2 Modelling the SALB problem

The mathematical model of the SALB problem has two binary decision variables.

$$x_{ij} = \begin{cases} 1, & \text{if task } i \text{ is assigned to station } j. \\ 0, & \text{otherwise} \end{cases}$$

$$y_j = \begin{cases} 1, & \text{if station } j \text{ is open.} \\ 0, & \text{otherwise} \end{cases}$$

2.1 SALB 1

In this problem, the objective is to minimise the number of station (j) but we can compute the maximum number of stations as shown in Eq. 3. Therefore $M = s_{\max}$.

$$\begin{aligned}
\text{minimize} \quad & \sum_{j=1}^{M} y_j && && \text{(5a)} \\
\text{subject to} \quad & \sum_{i=1}^{N} t_i x_{ij} \leq C y_j, && j = 1, 2, \ldots, M, && \text{(5b)} \\
& \sum_{j=1}^{M} x_{ij} = 1, && i = 1, 2, \ldots, N, && \text{(5c)} \\
& \sum_{j=1}^{M} j x_{kj} \leq \sum_{j=1}^{M} j x_{ij}, && \forall\ k \prec i, && \text{(5d)} \\
& y_{j+1} \leq y_j, && j = 1, 2, \ldots, M-1, && \text{(5e)} \\
& x_{ij} \in (0,1)\ \forall\ i, j,\ \ y_j \in (0,1)\ \forall\ j && && && \text{(5f)}
\end{aligned}$$

Eq. 5a is the objective function that minimises the number of open stations. Eq. 5b guarantees that the total time of the assigned tasks to a workstation is less than the required cycle time. Eq. 5c assures that every task is assigned to a single workstation. Eq. 5d assures that the predecessor relationships are guaranteed. Eq. 5e guarantees that only j workstations are open; i.e. if worstation 3 ($j = 3$) is not open ($y_3 = 0$); then next workstation $j = 3 + 1$ is not open ($y_{3+1} = 0$) as well.

The SALB 1 mathematical model for the problem in Fig. 1 is shown below. The maximum number of stations is $s_{\max} = \lceil 16/7 \rceil = 3$; and suppose $C = 7$ minutes; thus,

$$
\begin{aligned}
&\min \quad y_1 + y_2 + y_3 \\
\text{s.t.} \quad & \\
&3x_{11} + 5x_{21} + 2x_{31} + 4x_{41} + 2x_{51} \leq 7y_1 \\
&3x_{12} + 5x_{22} + 2x_{32} + 4x_{42} + 2x_{52} \leq 7y_2 \\
&3x_{13} + 5x_{23} + 2x_{33} + 4x_{43} + 2x_{53} \leq 7y_3 \\
&x_{11} + x_{12} + x_{13} = 1 \\
&x_{21} + x_{22} + x_{23} = 1 \\
&x_{31} + x_{32} + x_{33} = 1 \\
&x_{41} + x_{42} + x_{43} = 1 \\
&x_{51} + x_{52} + x_{53} = 1 \\
&x_{11} + 2x_{12} + 3x_{13} \leq x_{21} + 2x_{22} + 3x_{2,3} \\
&x_{21} + 2x_{22} + 3x_{23} \leq x_{41} + 2x_{42} + 3x_{43} \\
&x_{31} + 2x_{32} + 3x_{33} \leq x_{41} + 2x_{42} + 3x_{43} \\
&x_{41} + 2x_{42} + 3x_{43} \leq x_{51} + 2x_{52} + 3x_{53} \\
&y_2 \leq y_1 \\
&y_3 \leq y_2 \\
&x_{ij} \in (0,1) \ i = 1, \ldots, 5; \ j = 1,2,3; \ y_j \in (0,1) \ j = 1,2,3
\end{aligned}
$$

2.2 SALB 2

In this problem, the objective is to minimise the cycle time C given a known number of stations; i.e., we will open the M number of stations.

$$
\begin{aligned}
&\text{minimize} \quad C && && \text{(6a)} \\
&\text{subject to} \quad \sum_{i=1}^{N} t_i x_{ij} \leq C, && j = 1,2,\ldots,M, && \text{(6b)} \\
&\sum_{j=1}^{M} x_{ij} = 1, && i = 1,2,\ldots,N, && \text{(6c)} \\
&\sum_{j=1}^{M} j x_{kj} \leq \sum_{j=1}^{M} j x_{ij}, && \forall\ k \prec i, && \text{(6d)} \\
&x_{ij} \in (0,1)\ \forall\ i,j, \quad C \geq 0 && && \text{(6e)}
\end{aligned}
$$

Eq. 6a minimises the cycle time. Eq. 6b, Eq. 6c, and Eq. 6d have the same meaning as Eq. 5b, Eq. 5c, and Eq. 5d, respectivelly.

Suppose that the number of workstations to be open is 4; thus, $j = 1, 2, 3, 4$.

$$\min \quad C$$

s.t.

$$
\begin{aligned}
&3x_{11} + 5x_{21} + 2x_{31} + 4x_{41} + 2x_{51} \leq C \\
&3x_{12} + 5x_{22} + 2x_{32} + 4x_{42} + 2x_{52} \leq C \\
&3x_{13} + 5x_{23} + 2x_{33} + 4x_{43} + 2x_{53} \leq C \\
&3x_{14} + 5x_{24} + 2x_{34} + 4x_{44} + 2x_{54} \leq C \\
&x_{11} + x_{12} + x_{13} + x_{14} = 1 \\
&x_{21} + x_{22} + x_{23} + x_{24} = 1 \\
&x_{31} + x_{32} + x_{33} + x_{34} = 1 \\
&x_{41} + x_{42} + x_{43} + x_{44} = 1 \\
&x_{51} + x_{52} + x_{53} + x_{54} = 1 \\
&x_{11} + 2x_{12} + 3x_{13} \leq x_{21} + 2x_{22} + 3x_{2,3} \\
&x_{21} + 2x_{22} + 3x_{23} \leq x_{41} + 2x_{42} + 3x_{43} \\
&x_{31} + 2x_{32} + 3x_{33} \leq x_{41} + 2x_{42} + 3x_{43} \\
&x_{41} + 2x_{42} + 3x_{43} \leq x_{51} + 2x_{52} + 3x_{53} \\
&x_{ij} \in (0,1) \ i = 1, \ldots, 5; \ j = 1, 2, 3, 4; \ C \geq 0
\end{aligned}
$$