Scalar Field Dark Matter Spectator During Inflation: The Effect of Self-interaction

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Cosmological inflation is nowadays the most accepted mechanism to explain the primordial seeds that led to the structure formation observed in the Universe. Current observations are in well agreement to initial adiabatic conditions, which imply that single-scalar-field inflation may be enough to describe the early Universe. However, there are several scenarios where the existence of more than a single field could be relevant during inflation, for instance, the situation where the so-called spectator is present. In this paper we apply the formalism of two-field inflationary models to two different spectator scenarios (jav: dos escenarios?). Firstly, to the possibility that an ultra-light scalar field dark matter coexist with the inflaton, where such ultra-light field could be auto-interacting or massive. Secondly, to the curvaton scenario (jav: ya no hay, o si?). For both cases (jav: both?) we compare the results to current cosmological data and provide constraints on the parameters describing the scalar field potential. (jav: 25 veces 'consider' se repite! – y ya cambie algunas)

PACS numbers: ????

I. INTRODUCTION

It is well accepted that the primordial seeds of the structure formation in the Universe were generated by quantum fluctuations given a scalar field (SF) during an inflationary epoch. The simplest scenario where density perturbations are carried out by a single inflaton is preferred since the initial perturbations are nearly adiabatic [1–3]. However the presence of any light field, other than the inflaton, could also fluctuate during inflation and contribute to the primordial density perturbations. A particular interest is the possibility that this extra SF can be used as the dark matter component (DM). This scenario is usually known as scalar field dark matter (SFDM) and assumes that the DM is made of bosonic excitations of an ultra-light SF. The typical mass of this field that satisfies astrophysical observations is around $m \sim 10^{-22} eV/c^2$, which might include self-interaction.

The idea of scalar fields as DM was initially introduced by [4] and since then rediscovered with different names, for instance Scalar Field Dark Matter (SFDM) [5], Fuzzy DM [6], Wave DM [7, 8], Bose-Einstein Condensate DM [9] or Ultra-light Axion DM [10, 11], amongst many others. The purpose of this SFDM is to resolve the apparent conflict, with observations, that exhibits the cold dark matter (CDM) formed of weakly-interacting massive particles (WIMPS) [12, 13]. Some of the CDM weaknesses may appear at small-scales within galaxies, e.g. cuspy halo density profiles, overproduction of satellite dwarfs within the Local Group and many others, see for example [14–18]). Since then, the SFDM model has been successfully tested by different observations; for a review on ultra-light SFDM see [19–22].

The paper is organized as follows: First, in section II we review the basics about the inflationary scenario. Then in section III we present the basic observables used to constraint our models. Once the mathematical background and the observational constrictions are given, in section IV we analyze two different models (jav: cuales?). First, we assume a massive SFDM spectator during inflation. We provide some limits for the mass of the scalar field using isocurvature constrictions and compare them with the actual constraints given by astrophysical and cosmological observations. Then we consider a self-interacting term for the SFDM. We constrain their parameters using isocurvature perturbations and compare our results with cosmological and astrophysical observations. Finally in section V our conclusions are given.

II. INFLATIONARY SCENARIO: SFDM AS AN SPECTATOR

In this section we briefly review the inflationary process for spectator-like SFs during inflation following [26] (see also (lp: Referenciar nuestro paper) for a more recent review). Throughout this paper we use natural units ($\hbar = c = 1$).

A SF ϕ_i living during the inflationary era is thought to acquire quantum fluctuations with a primordial power

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spectrum measured at Hubble exit as

$$P_{\phi_i} \simeq \left(\frac{H_*}{2\pi}\right)^2. \quad (1)$$

Here and after quantities with subscript $*$ are evaluated at the Hubble exit. Assuming that within the inflationary era there exists several scalar fields, then it is convenient to work on a rotating basis by defining the *adiabatic field* σ parallel to the trajectory of the field-space and the *entropy fields* s_i perpendicular to the same trajectory. If the background trajectory is a straight-line that evolves in the direction of the field – causing the inflationary process–, the scenario is therefore called the *inflaton scenario* (IS). In this case quantum fluctuations $\delta\sigma$ of the adiabatic field and δs_i – also called *spectator fields* –, are frozen at Hubble exit[1] and start evolving until they reenter the horizon. Considering the case with one extra SF, additional to the inflaton, then the primordial power spectrum is given by

$$P_{\sigma^*}(k) = P_s^*(k) \simeq \left(\frac{H_*}{2\pi}\right)^2. \quad (2)$$

On the other hand, the curvature and isocurvature perturbations are defined as

$$R \equiv \frac{H}{\dot{\sigma}}\delta\sigma, \quad S = \frac{H}{\dot{\sigma}}\delta s. \quad (3)$$

Then the final power spectra, evaluated at the beginning of the radiation-domination era, are thus

$$P_R = P_S \simeq P|_*, \quad (4)$$

where at linear order in slow-roll parameters

$$P|_* = \frac{1}{2\epsilon}\left(\frac{H_*}{2\pi M_{pl}}\right)^2, \quad (5)$$

with $M_{pl} = 1.221 \times 10^{19} GeV$ the Planck mass.

Gravitational waves.- Given the fact that scalar and tensor perturbations are decoupled at linear order, the amplitude of the gravitational waves spectrum and the tensor-to-scalar ratio r, in the IS, remain the same similarly to the case with no extra spectator fields during the inflationary process.

We can see then that the incorporation of these new fields during the IS will produce only isocurvature perturbations. Such perturbations can be used to constrain the free parameters of our SFDM models as we will see later.

[1] If the trajectory is curved in the field-space, the entropy and adiabatic perturbations are correlated at Hubble exit and the perturbations continue evolving until inflation ends up [26]

III. CONSTRAINTS ON INFLATIONARY PARAMETERS

In the standard approximation the inflationary observables are given by the tensor-to-scalar ratio r, the spectral index for adiabatic perturbations n_R and the amplitude for adiabatic perturbations A_r^2. The constraints of these parameters are quoted at the pivot scale $k_0 = 0.05Mpc^{-1}$ by [1–3, 27–29](lp: Agregar referencia del nuevo Planck)

$$A_r^2(k_0) = (2.215^{+0.032}_{-0.079}) \times 10^{-9}, \quad \text{at } 68\% \text{ CL}, \quad (6a)$$

$$r_{k_0} < 0.064 \quad \text{at } 95\% \text{ CL}, \quad (6b)$$

$$n_R(k_0) = 0.968 \pm 0.006. \quad (6c)$$

Using these measurements we are able to constrain the value of the Hubble expansion rate during inflation H_* as [30, 31]

$$r = 1.6 \times 10^{-5}\left(\frac{H_{inf}}{10^{12}GeV}\right)^2. \quad (7)$$

As we have noted before, if more than one SF is present during inflation we will obtain isocurvature perturbations generated by extra scalar fields perpendicular to the trajectory on field space. Parameterizing the isocurvature power spectrum for dark matter in terms of the curvature power, we have

$$P_{DM}(k) = \frac{\beta_{iso}(k)}{1-\beta_{iso}(k)}P_R(k), \quad (8)$$

where $P_{DM} = \langle\delta\rho_{DM*}/\rho_{DM}\rangle$, $\delta\rho_{DM}$ are the isocurvature perturbations for the dark matter (DM) generated by extra scalar fields during inflation and ρ_{DM} is the initial condition of DM. The uncorrelated scale-invariant DM isocurvature is constrained by Planck data [1, 2] at pivot scale k_0 as

$$\beta_{iso}(k_0) < 0.038 \quad \text{at } 95\% \text{ CL}. \quad (9)$$

Notice that isocurvature perturbations can be used to constrain the inflationary scale, just by combining equations (7), (8) and (9).

IV. CONSTRAINING MASSIVE AND SELF-INTERACTING ULTRA-LIGHT SFDM MODELS

In this section we assume the possibility that an ultra-light SFDM candidate coexists with the inflaton during the inflationary epoch. We require that the SFDM candidate be a stable spectator field with negligible classical dynamics and energy density. Such scenario is reach by considering the inflaton scenario, i.e., within the field space, the evolution of the system is on the inflaton direction ϕ whereas the direction perpendicular to the trajectory corresponds to the SFDM ψ. Notice this requirement implies that our dark matter candidate evolves much slower than the inflaton and its density is smaller than the associated to the inflaton.

As we mentioned before it is possible to constrain the free parameters of our model when isocurvature perturbations are taken into account. For this reason in the next subsections we review the cosmological history that a massive and a self-interacting scalar field should have gone through the evolution of the Universe, to then match the present values of the field with those during the the inflationary era and therefore take into account isocurvature constrictions. We also consider different cosmological and astrophysical constraints for each SFDM model in order to compare our results.

First of all let us remember the basics about SFs. The dynamical evolution of any SF is governed by the Klein-Gordon (KG) equation

$$\Box\Psi - 2\frac{dV}{d|\Psi|^2}\Psi = 0. \tag{10}$$

When a SF is complex it is convenient to use a Madelung transformation [34]

$$\Psi = \eta \exp[i\theta], \tag{11}$$

where $\eta \equiv |\Psi|$ is the magnitude of field Ψ and θ its phase. With this new decomposition the KG equation is separated in its real and imaginary components as

$$\ddot{\eta} + 3H\dot{\eta} + 2\frac{dV}{d|\psi|^2}\eta - \omega^2\eta = 0, \tag{12a}$$

$$\dot{\omega}\eta + (2\dot{\eta} + 3H\eta)\omega = 0, \tag{12b}$$

where $\omega = \dot{\theta}$. Equation (12b) can be exactly integrated obtaining

$$a^3\eta^2\omega = Q, \tag{13}$$

where Q is a charge of the SF related to the total number of particles [37–41]. Plugging this last equation into (12a) we obtain the radial component of the scalar field follows

$$\ddot{\eta} + 3H\dot{\eta} + M^2\eta - \frac{Q^2}{\eta^3} = 0. \tag{14}$$

The term containing Q is given by the complex nature of the SF [38] and can be interpreted as a "centrifugal force" [41]; $M^2 \equiv 2(dV/d|\psi|^2)$ is seen as an effective mass term of the scalar field. Notice that if we assume Ψ is the SFDM and we take $Q^2/\eta^3 \ll 1$, by assuming the SFDM candidate fulfills the slow-roll condition during inflation, then the field η will remain frozen at value η_i until $H \sim M$. Here and for the rest of this work subindex i means values right after inflation ends. Then, when $H \sim M$ will start evolving depending on the effective mass term. In order to get the slow-roll behavior of the SFDM it is necessary that $Q \simeq 0$, as explained in (lp: Referencia paper Abril) [2]. On the other hand if Ψ is the inflaton, ϕ it is usually considered as a real field, then in this case $\theta = Q = 0$.

In this work we consider only situations where the general potential can be decomposed as $V(\phi,\psi) = V_{inf}(\phi) + V_{SFDM}(|\psi|^2)$.

A. Real ultra-light SFDM candidate

1. *Cosmological history*

The possibility that an ultra-light SFDM candidate could coexist with the inflaton has been recently studied in [44] considering a potential of the form

$$V(\phi, |\psi|^2) = V(\phi) + \frac{1}{2}m^2\psi^2. \tag{15}$$

and by fixing $Q = 0$ for the SFDM in Eq. (14). Such study was performed by assuming the SFDM is an axion-like particle. However their results can be easily extrapolated for mostly any massive SFDM cadidate. For this particular case we observe that $M^2 = m^2$. As mentioned above when $H \gg m$ the term with m^2 in equation (12a) can be neglected. Bering in mind the field is slowly rolling during the inflationary era, we can neglect the second derivatives in (12a), and then the field ψ remains frozen at its initial value by Hubble dragging during the early universe [42]. When the $m \sim H$ condition is approached the SFDM starts evolving and oscillates as a massive field. During its oscillation phase the dependence of ψ respect to a is $\psi \sim 1/a^{3/2}$, while its density behaves as $\rho_\psi \sim 1/a^3$ [37, 38]. So the scalar density of the SFDM can we written as

$$\rho_\psi = \begin{cases} \frac{1}{2}m^2\psi_i^2 & \text{when } H \gg m, \\ \frac{1}{2}m^2\psi_i^2\left(\frac{a_{osc}}{a}\right)^3 & \text{when } H \ll m. \end{cases} \tag{16}$$

For typical masses of a SFDM candidate, $m \sim 10^{-22}eV$, the field started oscillating during the radiation-dominated Universe. During this period the Hubble parameter evolves in terms of the scale factor as $H \propto a^{-2}$, and the KG equation (12a) can be solved exactly in terms of a. In Ref. [44] the initial conditions for the massive case were obtained (jav: they) by using the evolution of the form (16) and taking into account that the entropy of the Universe is conserved. If the total

[2] In fact the inflationary behavior is an attractor solution of the KG equation for a real field in the limit when $M^2 \ll H^2$ [35, 36]. In this limit the typical dynamics of a real SF is a stiff-like epoch, followed by an inflationary-like era.

amount of dark matter is composed of SFDM particles they obtain (jav: they?)

$$\psi_i^2 \simeq \frac{10^{34} GeV^2}{0.6} \left(\frac{g_{*osc}}{3.36}\right)^{-3/4} \left(\frac{g_{s*osc}}{3.91}\right) \left(\frac{m}{10^{-22} eV}\right)^{-1/2}. \quad (17)$$

Here g_{*osc} and g_{s*osc} are the effective degrees of freedom associated to the total particles and to the entropy of the SFDM oscillations. In particular, for the case of an ultra-light SFDM that started its oscillations during the radiation-dominated Universe $g_{osc} = 3.36$ and $g_{sosc} = 3.91$.

2. *Constrains from isocurvature perturbations*

For this case we demand the energy density contribution of the SFDM being small during inflation (DM dominates right after radiation-matter equality) and hence it is necessary that

$$\frac{m^2}{2} < \frac{V(\phi)}{\psi_i^2} \simeq \frac{H_*^2 M_p^2}{\psi_i^2}, \quad (18)$$

where during inflation our field remains frozen at value ψ_i. Notice that for an ultra-light SFDM candidate ($m \sim 10^{-22} eV$) the above expression is fulfilled by an appropriate set of initial conditions given by ψ_i. On the other hand, the isocurvature perturbations generated by a SFDM can be constrained by using Eqn. (8) or equivalently (9). The analysis was carried out, in ref [44], by noticing that the primordial spectrum can be re-expressed as $\delta\rho_\psi/\rho_\psi = 2\delta\psi/\psi_i$ ($\rho_{SFDM} \propto \psi^2$ from Eq. (16)) which implies that $P_{SFDM} = 4P_\psi/\psi_i^2$, with P_ψ given by equation (1) or (2) and ψ_i by (17), and thus compare with P_{DM} from equation (8). When such comparison is done they (jav: they?) finally obtain the result

$$\frac{m}{10^{-22}\ eV} < \left(\frac{2 \times 10^{-4}}{r}\right)^2. \quad (19)$$

Then the isocurvature constrictions allow us to constraint the mass parameter of the SFDM with the measurement of the tensor-to-scalar ratio.

The above relation for the mass parameter must be in agreement with the cosmological and astrophysical constrictions obtained by this model. We need to stress out that we cannot use all the constrictions in the literature since some of them consider different cosmological evolutions for the SFDM. For example in [37] it was studied CMB and Big Bang Nucleosynthesis (BBN) by understanding the SFDM was generated right after inflation with a stiff-like equation of state ($p \simeq \rho$). Then, this kind of restrictions are not applicable to our model.

Other constraints.- We start by considering CMB constrictions. In reference [61] the CMB was studied in the form of Planck temperature power spectrum, here they obtained $m \gtrsim 10^{-24} eV$. Considering the hydrodynamical representation of the SFDM model, Ref. [62] suggests the SFDM's quantum pressure as the origin of the offset between dark matter and ordinary matter in Abel 3827. For this purpose they required a mass $m \simeq 2 \times 10^{-24} eV$. When the model is tested with the dynamics of dwarf spheroidal galaxies (dSphs) – Fornax and Sculpture–, in reference [63] was obtained a mass constriction of $m < 0.4 \times 10^{-22} eV$ at 97.5% The constriction obtained when the survival of the cold clump in Ursa and the distribution of globular clusters in Fornax is considered requires a mass $m \sim 0.3 - 1 \times 10^{-22} eV$ [64]. Explaining the half-light mass in the ultra-faint dwarfs fits the mass term to be $m \sim 3.7 - 5.6 \times 10^{-22} eV$ [65]. The model has also been constrained by considering the process of reionization. In [66] it was obtained, using N-body simulations and demanding an ionized fraction of HI of 50% by $z = 8$, the result of $m > 2.6 \times 10^{-23} eV$. Finally, using the Lymann-α forest flux power spectrum demands that the mass parameter fulfills $m \gtrsim 20 - 30 \times 10^{-22} eV$ [67, 68].

Figure 1 displays the aforementioned constraints on a m *vs* r plane. Notice that we decided to plot the different constraints in two figures in order to make clearer our results. In any way both figures must be seen as if they were overlapped. Firstly, the green region (jav: cual? top, bottom?.. poner etiquetas en la figura) is fulfilled only by isocurvature observations (19), but as we can see once both figures are overlapped, these regions disappear. The dot-dashed black line corresponds to the equality values in Eq. (19). Top panel of Figure 1 displays (jav: isocurvature and ...) constrictions obtained by Lymann-α and also those allowed by isocurvature constrictions (orange region), limits from Abdel 3827 and isocurvature (dot-dashed red line), dwarf spheroidal galaxies and isocurvature (gray line), Ursa with Fornax and isocurvature (light blue line) and ultra-faint dwarfs and isocurvature (purple line). On the other hand, in the bottom panel of Figure 1, the yellow region corresponds to CMB constraints and isocurvature while the blue region is limited by CMB as well as reionization and isocurvature. We notice that isocurvature perturbations cannot constrain observations of the dynamics of dSphs galaxies given that both provide an upper limit for the mass of the SFDM. However, the detectability of gravitational waves and the different constrictions by cosmological and astrophysical observations can be used to test the massive model. For example, if we forget by the moment the dynamics of dSphs galaxies and we would like to fulfill at least observations provided by Abel 3827, we should not detect gravitational waves until $r \simeq 1.3 \times 10^{-3}$ (where the red-dashed line and the black dashed line intersect), while if we were interested in fulfilled all the other observations we should not detect gravitational waves until $r \lesssim 2.33 \times 10^{-6}$. (jav: pintar lineas)

These constraints are important given that [46] demonstrated that an ultra-light axion-like dark matter candidate must be presented during inflation. Then, if r is detected in the near future, it could represent a strong constraint for the axion-like particle model. Notice that if we relax the mechanism under this particle is created or if we add an auto-interacting component, we should expect these restrictions be less affective to the model.

We also show the actual upper limit for r in a blue dashed line. By the moment this value is not restrictive for the model since it represents an upper value for r. The only way the SFDM model can be tested with isocurvature perturbations is if r is detected.

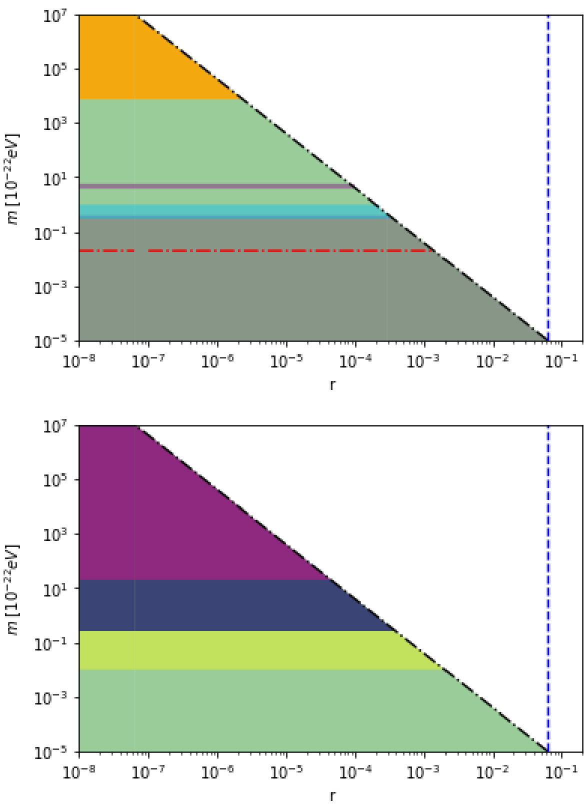

FIG. 1: Isocurvature constraints for the SFDM candidate. (jav: ya me confundio esta figura. Seria bueno poner los ejes al reves, ya que r se esta buscando y m seria un derivado. De acuerdo a esta figura, cuales son las constricciones actuales a la masa?) (jav: quitar la flecha y mejor centrar el texto, explicar regiones.)

B. Real Self-interacting SFDM Candidate

1. *Cosmological history*

(jav: ————- aqui voy ———-) In this section a self-interacting SFDM with a positive interaction is considered. This scenario is described by the general potential

$$V(\phi, |\psi|^2) = V(\phi) + \frac{1}{2}m^2\psi^2 + \frac{1}{4}\lambda\psi^4, \tag{20}$$

and fixing again $Q = 0$ for the SFDM in equation [14].

Notice that for this case $M^2 = m^2 + \lambda\psi^2$ As we have previously discussed the effective mass of the field after inflation remains constant at $M^2 = m^2 + \lambda\psi_i^2$ until $M \sim H$. Then, depending of each contribution to M^2, we can have two different dynamics.

Weakly self-interacting regime.- This limit is obtained when the constant term m^2 dominates, that is when

$$m^2 \gg \lambda\psi_i^2. \tag{21}$$

In this regime it is possible to ignore the autointeracting term in equation (14) when the oscillations of the scalar field begins. However, by ignoring this term the field behaves as a massive field and from (16) the field value always decreases. Therefore the autointeracting term never dominates and all the cosmological history remains the same as in the pure massive SFDM scenario. In fact, thanks to the decreasing behavior of this scenario we can consider that this regime is fulfilled always that $m^2 \geq \lambda|\psi_i|^2/2$ or equivalently when $\lambda \leq 2m^2/|\psi_i|^2$. If the SFDM oscillations start at the same time than the massive case (which is a good approximation since the effective mass of the SFDM is $M^2 = m^2 + \lambda\psi_i^2 \leq 2m^2$), we observe from (17) that it must be fulfilled that

$$\left(\frac{\lambda}{10^{-96}}\right) \leq 1.2\left(\frac{m}{10^{-22}eV}\right)^{5/2}. \tag{22}$$

We plot in figure 2 the weak limit obtained by our approximation. However this overestimates the maximum value of λ since the dust-like behavior is obtained when the λ term is completely negligible.

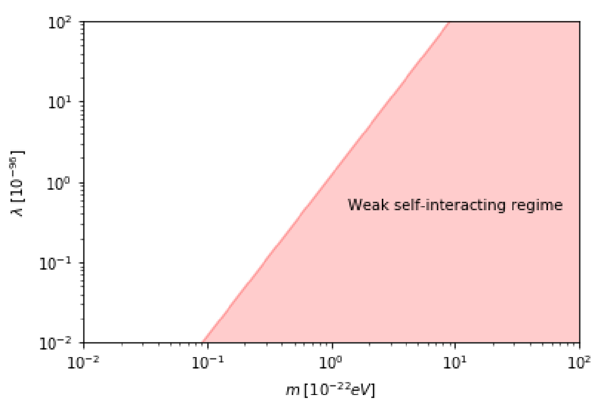

FIG. 2: Weakly self-interacting regime

Strong self-interacting regime.- This scenario is obtained when

$$m^2 \ll \lambda\psi_i^2. \tag{23}$$

Here the SFDM follows an attractor solution during the inflationary era.

Attractor behavior of the SF during inflation.- In the strong self-interacting regime the SFDM follows the attractor solution [42][3]

$$\psi_{att} = \left(2\lambda \int_{\phi}^{\phi_0} V_{,\phi}^{-1} d\phi\right)^{-1/2}, \tag{24}$$

where ϕ_0 is the value of the inflaton at the beginning of inflation. In the above expression we can identify two possible branches:

- $\psi_{att} < \sqrt{2}m/\sqrt{\lambda}$

  The SFDM follows the attractor solution until $\psi \simeq \sqrt{2}m/\sqrt{\lambda}$. Then the field reaches $\psi_i = \sqrt{2}m/\sqrt{\lambda}$ for the rest of inflation. Notice that this value corresponds to the upper limit that the weakly self-interacting regime allows. Then the field starts evolving when $H \sim M \simeq m$ behaving as a massive SF. In this way the constrictions given in the non-interacting case apply and the initial conditions are also fixed by ψ_i. Using both relations the λ value is approximated to (jav: la eqn lleva un $\sim$ en lugar de =?)(lp: Tienes razón, ya lo cambié)

$$\left(\frac{\lambda}{10^{-96}}\right) \simeq 1.2 \left(\frac{g_{*osc}}{3.36}\right)^{3/4} \left(\frac{g_{s*osc}}{3.91}\right)^{-1} \left(\frac{m}{10^{-22}eV}\right)^{5/2}. \tag{25}$$

  The mass term and the auto-interacting constant are rescale by appropriate values, that is, the mass term is measured in units of $10^{-22}eV$ while the auto-interacting constant in terms of 10^{-96}.

- $\psi_{att} > \sqrt{2}m/\sqrt{\lambda}$

  In this scenario the dynamics of the inflaton, given by (24), implies that the initial condition of the field after the inflationary period is

$$\psi_{att}^{i} = \left(2\lambda \int_{\phi_{end}}^{\phi_0} V_{,\phi}^{-1} d\phi\right)^{-1/2}, \tag{26}$$

  where ϕ_{end} is the value of the inflaton at the end of inflation. We need to stress out that this is the value of the field until its oscillation period starts (i.e. when $M \sim H$).

  In this scenario we observe that at the time the SFDM starts its oscillations its effective mass is quadratic in the field. In that regime the scalar field evolves as $\psi \sim 1/a$ and its energy density as $\rho_\psi \sim 1/a^4$, behaving as radiation. Then, when $m^2 \sim \lambda\psi_t^2$ the effective scalar field mass is now constant, obtaining the dust-like behavior already analyzed. Therefore, the history of the scalar field density is

$$\rho_\psi = \begin{cases} \frac{1}{4}\lambda^2\psi_i^4 & \text{when } H \gg \lambda\psi_i^4 \\ \frac{1}{4}\lambda\psi_i^4\left(\frac{a_{osc}}{a}\right)^4 & \text{when } H_t \leq \lambda\psi_i^4 \leq H \\ \frac{1}{2}m^2\psi_t^2\left(\frac{a_t}{a}\right)^3 & \text{when } H \leq m^2 \text{ and } \lambda\psi^2 < m^2 \end{cases} \tag{27}$$

Here sub-index t means quantities measured at transition between radiation-like to dust-like behavior of the SFDM and

$$\psi_i^2 = \left[\frac{2m^2}{\lambda}\psi_t^2\right]^{1/2} \left(\frac{a_t}{a_{osc}}\right)^2. \tag{28}$$

Notice that, for simplicity, we have taken an instantaneous transition between radiation-like to dust-like behaviors.

Since the auto-interacting KG equation cannot be solved exactly we work with approximated solutions (jav: y numericamente?)(lp: pongo que sí se podría, pero lo que me interesa es obtener la condición inicial ψ_i del campo escalar y obtenerlo como función de λ y m. Se me hace mucho más fácil utilizar estas aproximaciones semianalíticas que andar resolviendo muchas veces el campo escalar para diferentes λs y ms para luego comparar). By using a pure approximated description of the system, [38] obtained the relation (see its equation 80 and 86 and also [37])[4] (jav: si ya lo obtuvieron ellos, que obtienes nuevo?)(lp: Hasta aquí nada, sólo estoy usando su resultado para obtener la condición inicial que ando buscando. Tal vez lo nuevo es que yo ya lo estoy mentiendo considerando que el campo escalar coexistió con el inflatón, mientras que la referencia que estoy dando lo estudiaron como lo digo en el pie de página.)

(lp: Aquí empieza lo que usé de Abril y Chavanis)

$$\left(\frac{a_t}{a_{osc}}\right)^2 = \frac{3}{7^{1/3} f^2\left(\frac{a_s}{r_S}\right)}, \tag{29a}$$

where

$$f(\sigma) = \frac{1}{s^{1/3}(1+4s)^{1/6}}, \tag{29b}$$

[3] The study done in reference [42] was for a curvaton-like scalar field in a chaotic-like inflationary scenario. However their results can be used as well in this context were the attractor behavior can be easily obtained for whichever inflationary potetial.

[4] The reference [38] obtained this relation by considering a Universe with only a SFDM content. However the result that we are using here can be used in a Universe with several types of matter contents as they explain in their work.

with

$$s = \frac{4\sigma - 1 + \sqrt{(4\sigma-1)^2 + 12\sigma}}{6}. \tag{29c}$$

Additionally $r_S = 2mG/c^2$ and $a_s = \hbar^2\lambda/4\pi m$. Then it follows that $a_s/r_S = \lambda M_p^2/m^2$. Rearranging the expression in a more convenient way we have

$$\sigma \simeq 5.93 \times 10^2 \left(\frac{m}{10^{-22}eV}\right)^{-2} \left(\frac{\lambda}{10^{-96}}\right). \tag{30}$$

(lp: Aquí termina)

Notice that when $a_t/a_{osc} \simeq 1$ i.e. $3/(7^{1/3}f^2(\sigma)) \sim 1$, there is no radiation-like epoch. This scenario should match with the non-interacting scenario that we present previously. Inserting equation (29a) into (28) yields to

$$\psi_i^2 = \frac{3}{7^{1/3}f^2(\sigma)} \left[\frac{2m^2}{\lambda}\psi_t^2\right]^{1/2}. \tag{31}$$

The relation (31) matches the field at ψ_t with the value it has right after inflation ends. Then if we obtain the value of ψ_t by comparing with quantities at present, with the above expresion we can also obtain the value of ψ_i. On the other hand, notice that at a_t the scalar field behaves as dust with an effective mass $M^2 = m^2 + \lambda\psi_t^2$. This implies that dust-like oscillations of the SF began a little before than in the non-interacting case. If we allow m to be ultra-light ($m \sim 10^{22}eV$) and using the fact that m^2 is about the same order that $\lambda\psi_t^2$ (jav: de donde se ve esto?)(lp: Anteriormente definí ψ_t como el valor del campo donde se cumple que $m^2 \sim \lambda\psi_t^2$) we get that such oscillations start during the same epoch than in the non-interacting case. In fact because the decreasing behavior of the SF at that period (jav: which period)(lp: Cuando comienzas las dust-like oscillations) ($\psi \sim 1/a^{3/2}$) the auto-interacting term contribution quickly vanishes and then the dynamics of the field is described only by the mass term m. Thus, once the dust-like behavior starts, the dynamics is described similarly to the non-interacting case, in such case the condition (17) is fulfilled by the SF as well, but interchanging subindex i with t[5]. (jav: creo que las ideas estan un poco mezcladas, por que al final siempre terminas con la contribucion $m^2\phi^2$)(lp: Siempre debo terminar con esa contribución porque al final necesito que el campo escalar se comporte como polvo. Entonces lo único que estoy haciendo aquí es obtener cual es la contribución de la época de radiación. Digo que durante el período de polvo se cumple lo que ya habíamos obtenido para un campo masivo y eso lo uso junto con la ecuación 27 y 28a para obtener la contribución que da la época de radiación y con ello la condición inicial durante inflación)

[5] In fact this is a lower limit for the strong auto-interacting case.

2. *Constraining isocurvature perturbations for a self-interacting SFDM model*

As we have shown in the last section we have 2 different scenarios for this model: a weak self-interacting and a strong self-interacting. In the weak limit our SFDM behaves effectively as a massive field without auto-interaction, and in such case the constrictions for the massive field applies to this scenario as well. On the other hand when the auto-interacting term is big enough, the SFDM will have a new period with a behavior similar to a radiation-like fluid. In this way the constrictions we obtained before will not apply to this model anymore (jav: por que?) (jav: osea que el weak interacting, no aporta nada?)(lp: El weak interacting es básicamente de nuevo el campo masivo, mientras que en el otro ya tenemos un período en el que el SFDM se comporta diferente).

In the strong self-interacting regime, during the inflationary era, the SFDM follows the attractor solution (24). The value that the homogeneous field acquired after inflation depends on the condition $\psi_{att} \lessgtr \sqrt{2}m/\sqrt{\lambda} \equiv \psi_t$. For $\psi_{att} < \psi_t$ the field follows the attractor solution until $\psi \simeq \psi_t$. Then the SFDM is frozen at that value and starts oscillating as a massive field when $m \sim H$. We can constraint this scenario by noticing that it is the same than the massive case but with the initial condition $\psi_i = \psi_t$. Matching Eq. (17) with ψ_t and considering the constriction (19) we obtain

$$\left(\frac{\lambda}{10^{-96}}\right) \leq 1.2 \left(\frac{2\times10^{-4}}{r}\right)^5. \tag{32}$$

In figure 3 we have plotted the above condition that is valid in the strong (jav: weak? sure? al inicio del parrafo dice strong)(lp: Me equivoqué, ya lo corregí) self-interacting regime when $\psi_{att} < \psi_t$. The pink region corresponds with the region allowed by isocurvature perturbations in this limit. As we can observe the self-interacting term for this model can be constrained in a simmilar way than the mass parameter in the just massive case. This scenario must fulfilled the relation (19) as well since its cosmological evolution after inflation is only like a massive SFDM.

Additionally, in this scenario, the inflationary potential fulfills the condition

$$\left(\int_\phi^{\phi_0} V_{,\phi}^{-1} d\phi\right)^{-1/2} < 2m. \tag{33}$$

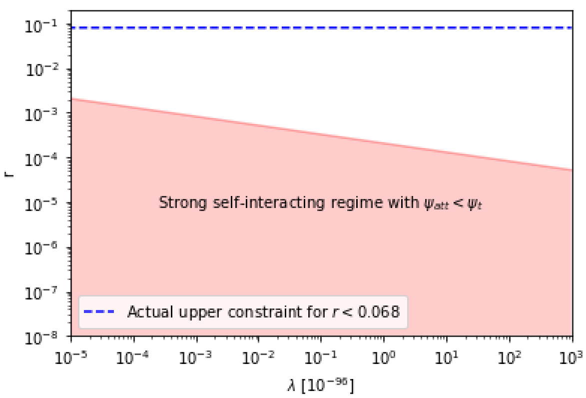

FIG. 3: Isocurvature constraints for the weakly self-interacting term. (jav: weak?)

We can see that it is very difficult to obtain this relation for an ultra-light SFDM candidate. For example, if we consider a chaotic-like inflationary potential, $V(\phi) = \frac{1}{2}M_{inf}^2\phi^2$, the above conditions implies that

$$\left(\log\frac{\phi_0}{\phi}\right)^{-1/2} < 2\frac{m}{M_{inf}}. \tag{34}$$

However, for this potential the mass M_{inf} of the inflaton that best matches the observations[6] is of order $M_{inf} \sim 10^{12}GeV$ [48]. If now we assume an ultra-light SFDM candidate with a mass $m \sim 10^{-22}eV$, the above conditions implies that the logarithmic part of the expression should be lower that $\sim 10^{-43}$. The inflationary behavior for a chaotic-like inflaton ends when $\phi_{end} \simeq 2M_{pl}$ [42, 48]. Moreover as it is explained in [42], the initial condition of the inflaton cannot be arbitrarily large since the stochastic behavior is significant for $\dot{\phi}H^{-1} < H/2\pi$. If the Universe starts when the inflaton escapes from this behavior we have that its initial condition should be

$$\phi_0 \sim 10^5 \times M_{pl}\left(\frac{10^{13}GeV}{M_{inf}}\right)^{1/2}. \tag{35}$$

where we can easily see that the condition given in (34) cannot be fulfilled (jav: y si no se satisface, que pasa?)(lp: No podría estar el campo escalar auto-interactuante en el régimen débil con el inflatón. Esto no representa un gran problema ya que entonces significaría que el campo escalar debería estar en el régimen fuerte). This should implies then that if we consider that an ultra-light self-interacting SFDM candidate coexist with the inflaton, it should be in the strong regime since their conditions are easier to be fulfilled.

[6] This chaotic-like inflationary potential is ruled-out now for observations, however we use it as an example in order to obtain general constraints for our models.

When $\psi_{att} > \psi_t$ we have that the field follows the attractor solution during all the period of inflation. In this way the initial condition for the SFDM is given by (26)

$$\psi_{att}^i = \left(2\lambda\int_{\phi_{end}}^{\phi_0} V_{,\phi}^{-1}d\phi\right)^{-1/2}. \tag{36}$$

Then the SFDM remains frozen at value ψ_{att}^i until $M \sim H$ and starts oscillating with a quartic potential. In this scenario the SFDM density behaves as $\rho_{SFDM} \propto \psi^4$ and in such case we can write $\delta\rho_\psi/\rho_\psi = 4\delta\psi/\psi_i$. In this way the primordial isocurvature perturbations for a strong self-interacting SFDM is given by

$$P_{SFDM}(k) = \left(\frac{2H_*}{\pi\psi_i}\right)^2. \tag{37}$$

In the last section we showed the relation of the initial condition with the value of the field today. Using eqs. (31) and (17), with $g_{*osc} = 3.36$ and $g_{s*osc} = 3.91$ and considering appropriate units we obtain

$$r < \frac{1.172\times10^{-4}}{7^{1/3}f^2(\sigma)}\left[\frac{2\left(\frac{m}{10^{-22}eV}\right)^{3/2}}{\left(\frac{\lambda}{10^{-96}}\right)}\right]^{1/2}. \tag{38}$$

Similar than in the massive case the above relation must be compared with other constrictions that the strong self-interacting SFDM scenario has. However in the literature we reviewed we couldn't find so much constrictions for the self-interacting regime. Then, let us review what we found.

Other constraints.-Testing the model at galactic levels by assuming that its ground state corresponds with the minimum galaxy halo (Fornax) [38] it is possible to constraint the ratio λ/m^2. If we use observations coming from the Bullet Cluster [49] it is possible to set each of the free parameters of the model (jav: no entendi)(lp: Quiero decir que con FORNAX se obtiene el radio entre λ y m y con bullet cluster se fijan los valores que m y λ deben tene). In [38] it was obtained that for a strong self-interacting SFDM model we have

$$m \simeq 1.10\times10^{-3}eV, \qquad \lambda \simeq 2.46\times10^{-17}, \tag{39}$$

This result corresponds to the upper bound of the mass of the SFDM while the non-interacting case should corresponds with the lower bound. Therefore the mass of the SFDM should be in the range $2.92\times10^{-22} \lesssim m \lesssim 1.10\times10^{-3}eV$ while the self-interacting term in the range $0 \lesssim \lambda \lesssim 1.69\times10^{-17}$. On the other hand when the model is tested at cosmological levels using the CMB and the abundances of light elements produced by the BBN it is obtained the values [37, 38]

$$m \simeq 3\times10^{-21}eV, \qquad \lambda \simeq 1.69\times10^{-87}. \tag{40}$$

Notice that these expressions are in agreement with the ones given by galactic constrictions.

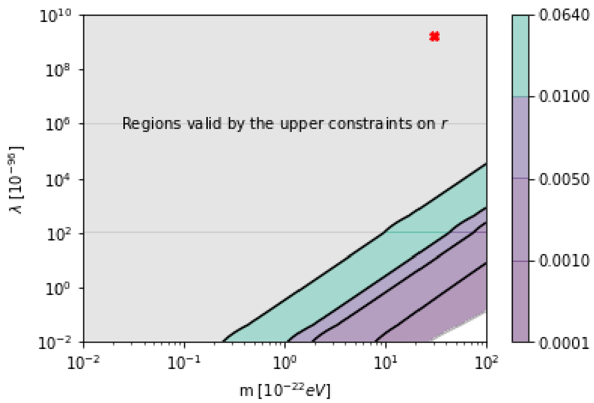

FIG. 4: Isocurvature constraints for the strong auto-interacting scenario. (jav: explicar bien)

Considering (40) as the most acceptable values for the mass of the SFDM and its auto-interacting term we can easily see from Figure 2 that we should be in the strong regime. Then in this scenario the constrictions given in Eq. (38) should apply.

In figure 4 we have plotted contour levels of the right side of the relation (38). The grey region corresponds with values higher than 0.064 which is the actual upper constraint on tensor-to-scalar ratio. That means that in that region we are certain that (38) is fulfilled and by the momment this region is completely allowed by observations. The white region corresponds with the weak limit. We observe that the parameters obtained by BBN and CMB are easily accepted by the model (red cross) which implies that the isocurvature perturbations are small enough that can avoid actual upper constrictions. In this way isocurvature perturbations does not represent any problem for the strong self-interacting model that matches with cosmological observations.

Remark: This scenario is of special interest given that the attractor solution justify the initial conditions for the SFDM model and because it is natural to avoid isocurvature perturbations when the auto-interacting term of the SF is big enough.

Similarly to the above description we can compute general constraints for the inflationary potential that should generate inflation on these kind of scenarios. First we have that

$$\left(\int_{\phi}^{\phi_0} V_{,\phi}^{-1} d\phi\right)^{-1/2} > 2m, \tag{41}$$

which is very easy to fulfill as we saw in the chaotic-like example. Using isocurvature constriction we also have

$$r < \frac{0.6\times 10^{40}}{\left(\frac{\lambda}{10^{-96}}\right)}\left(\frac{1}{\int_{\phi_{end}}^{\phi_0} V_{,\phi}^{-1} d\phi}\right), \tag{42}$$

that can also be satisfied as far as the auto-interacting term and the integral are small enough; for example in the chaotic-like scenario by using (34) and (35) and taking $M_{inf}\simeq 10^{-6}M_{pl}$, we can obtain the constriction

$$\left(\frac{\lambda}{10^{-96}}\right) < \frac{0.3288\times 10^{84}}{r}. \tag{43}$$

that is easily satisfied for whichever value of λ of our interest.

If now we compare (26) and (31) we have

$$\left(\int_{\phi_{end}}^{\phi_0} V_{,\phi}^{-1} d\phi\right)^{-1} = \frac{6}{7^{1/3} f^2(\sigma)}\left(2m^2\lambda|\psi_t|^2\right)^{1/2}. \tag{44}$$

This relation is interpreted as follows: consider that the auto-interacting SFDM candidate coexists with the inflaton, and suppose there are several measurements constraining the mass parameter m as well as the auto-interacting parameter λ, therefore such constraints are translated into restrictions to the inflationary potential.

It is also necessary to be careful that the SFDM does not come to dominate the inflationary period (jav: por que?)(lp: Si domina tendríamos que considerar inflación de 2 campos. Además tendríamos que justificar como es que le hizo el campo escalar para luego volverse subdominante y volver a dominar en igualdad materia-radiación). This is guarantee by demanding that

$$\lambda < \frac{H_*^2 M_p^2}{|\psi_i|^4}, \tag{45}$$

or in terms of (36)

$$\lambda > \left(4H_*^2 M_{pl}^2\left(\int_{\phi_{end}}^{\phi_0} V_{,\phi}^{-1} d\phi\right)^2\right)^{-1}. \tag{46}$$

Considering the chaotic-like example and using $H_* = 10^{14}GeV$, we obain the constriction

$$\frac{\lambda}{10^{-96}} > 5.777\times 10^{77}. \tag{47}$$

Notice that the above expresion fulfills astrophysical constrictions but it does not fulfills CMB and BBN. This is not a problem for auto-interacting SFDM models since Eq. (47) is obtained for a chaotic-like inflationary potential and, as we already know, that inflationary model is ruled-out by the actual constraints given by Planck (lp: Referencia Planck 2018).

C. Complex SFDM generalization

As we have seen at the begining of this section, when we consider a complex scalar field its dynamics is modified only by the centrifugal term (see eq. (14)). However as it was mentioned in [38] such term does not affect the dynamics of the field at cosmological levels, obtaining then that a complex scalar field and a real scalar field have the same cosmological history in the Universe. In this way if we take that our complex SFDM fulfilled slow-roll conditions during inflation then its constrictions for isocurvature perturbations must be the same than in the real field analogue.

V. CONCLUSIONS

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