Algorithms III

Dynamic Programming

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What is Dynamic Programming?

Definition

- DP can be defined as a tool that uses previously calculated values to obtain a new value.
- In other words, it uses what is already know in time t to answer a question in time $t + 1$.

Characteristics

Most of the DP problems have these two following properties:

1. A recursive function. A way to express a new value using previously obtained values.
2. Memory usage. Is common the use of arrays and multidimensional arrays to store the information needed to compute a new value.

DP Examples

Factorial

The factorial of number of a number n is defined as:

$$n! = 1 \times 2 \times \cdots \times n - 1 \times n \tag{1}$$
$$= (n-1)! \times n \tag{2}$$

If we write this as a programmatic function it would look like this:

$$f(n) = f(n-1) \cdot n$$

where $f(n)$ is a function that returns the factorial of n.

Factorial Implementation

```
int f(unsigned int n) {
  if (n == 0) {
    return 1;
  }

  return n * f(n - 1);
}
```

Fibonacci Sequence

The First two elements of the Fibonacci sequence are $1, 1$, after that the next elements are calculated by the sum of their two previous elements. So the first 10 numbers of the Fibonacci sequence are:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55$$

Fibonacci Sequence as Function

If we see the Fibonacci sequence as a function, it would look like this:

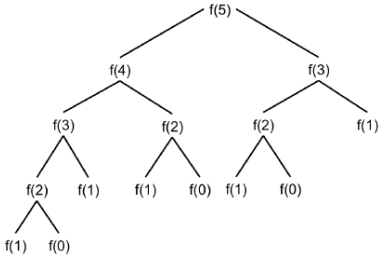
$$fibo(n) = \begin{cases} 1 & n \in \{1, 2\}, \\ f(n-1) + f(n-2) & n > 2 \end{cases}$$

Fibonacci Implementation

```
int fibo(int n) {
  if (n == 0 || n == 1) {
    return 1;
  }

  return fibo(n - 1) + fibo(n - 2);
}
```

Fibonacci - Is recursion the right approach?



Fancy Thing from *Source et al.*

- Some
- Interesting
- Points

A better approach

Use DP. Store into memory all Fibonacci values computed, that way

```
fibo[n] = fibo[n-1] + fibo[n-2]
```

where ***fibo*** is a vector or array.

Exercises

Domino Tiles

There are two sizes of domino pieces, 2×1, and 2×2, in how many ways can we fill a board of size $2 \times n$?

UP-Numbers

An UP-number is a number that doesn't contain two consecutive pairs in its digits. For example: 314385

Conclusion

Conclusions

There is no easy way to learn how to use DP, it more like a habit that must be acquired trough practice and solving new problems, and gradually it becomes easier to identify when is a good idea to implement a DP solution.

Famous DP Problems

Famous DP Problems

- Knapsack problem (Mochila)
- Largest Increasing Sub-sequence
- Largest Common Sub-sequence
- Edit Distance
- Coin change problem
- Optimal matrix multiplication

Largest Increasing Sub-sequence (LIS)

LIS

Given a sequence X of n integers, the objective is to find the longest sub-sequence, $X_{k_1}, X_{k_2}, \ldots, X_{k_m}$, such that $k_i > k_{i-1}$ and $X_{k_i} > X_{k_{i-1}}$. For example, for the following sequence:

$$3, 8, 2, 7, 3, 9, 12, 4, 1, 6, 10,$$

the longest increasing sub-sequence would be:

$$2, 3, 4, 6, 10.$$

The idea of the algorithm is to keep an array L where L_i represents the length of a *LIS* with X_i as its final element.

First start L_i with 1, then for every j from 0 to $i-1$, check if $X_j < X_i$ and $L_j + 1 > L_i$, if that happens then make $L_i = L_j + 1$. What we are doing here is to check if we can add the element X_i to the *LIS* that ends in X_j

LIS Example

For the sequence above, the array L will look like this.

X =	3	8	2	7	3	9	12	4	1	6	10
L =	1	2	1	2	2	3	4	3	1	4	5

The value of L_i represents the length of the *LIS* ending with X_i.

Q & A

1. How the LIS algorithm looks like?
2. What is the time complexity of the algorithm?
3. How can we print a valid LIS?
4. How can we improve the time of the algorithm?

Longest Common Sub-sequence (LCS)

LCS

A sub-sequence is a sequence that can be obtained from another sequence by removing some of its elements and preserving the order of the remaining elements.

The *LCS* of two sequences is a sub-sequence that is common to both sequences and has a maximal length.

LCS - Example

$$X = mexico$$
$$Y = america,$$

their *LCS* is *meic*.

LCS - Algorithm

The algorithm to find the *LCS* of two strings consists on having a matrix C of $n \times m$, where $C_{i,j}$ represents the length of the *LCS* using the first i letters from X and the first j letters from Y. So the result will be stored in position $C_{n,m}$

LCS - Algorithm

For the case where X_i is equal to Y_j that means that adding letter X_i and Y_j increments the length of the *LCS* by one, $C_{i,j} = C_{i-1,j-1} + 1$.

And for the case where X_i and Y_j are different, we only need to keep the greatest value in C so far, $C_{i,j} = \max(C_{i-1,j}, C_{i,j-1})$.

LCS - Algorithm Example

		m	a	c	e	t	a
	0	0	0	0	0	0	0
b	0	0	0	0	0	0	0
o	0	0	0	0	0	0	0
t	0	0	0	0	0	1	1
e	0	0	0	0	1	1	1
l	0	0	0	0	1	1	1
l	0	0	0	0	1	1	1
a	0	0	1	1	1	1	2

The value of C_{ij} represents the length of the *LCS* considering the first *i* characters of the