Sistemas de Ecuaciones con Laplace

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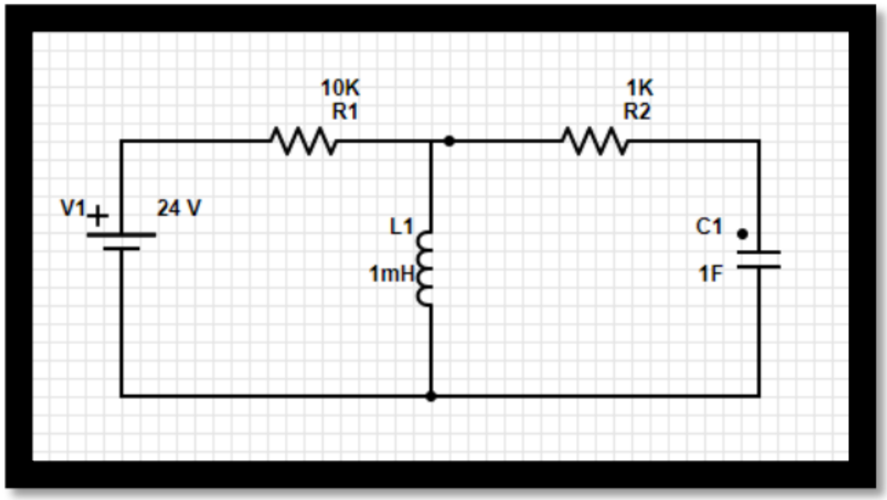
Problemas

1. *Modele y resuelva los siguientes sistemas de ecuaciones diferenciales con los datos proporcionados.*



Condiciones iniciales = 0.

Planteamiento del sistema

$$I_1$$

$$V_{R_1} + V_L = V_T$$
$$I_1R_1 + L\frac{dI}{dt} = V_T$$
$$I_1R_1 + L\left(\frac{dI_1}{dt} - \frac{dI_2}{dt}\right) = V_T$$

$$I_2$$

$$V_{R_2} + V_C + V_L = 0$$
$$I_2R_2 + \frac{\int Idt}{C} + L\left(\frac{dI}{dt}\right) = 0$$
$$I_2R_2 + \frac{\int I_2dt}{C} + L\left(\frac{dI_1}{dt} - \frac{dI_2}{dt}\right) = 0$$

Transformada de Laplace

$$I_1$$

$$I_1(s)R_1 + L\left(sI_1(s) - I_1(0) - \left(sI_2(s) - I_2(0)\right)\right) = V_T$$
$$I_1(s)R_1 + LsI_1(s) - LI_1(0) - LsI_2(s) + LI_2(0) = V_T$$
$$I_1(s)R_1 + LsI_1(s) - LsI_2(s) = V_T$$
$$I_1(s)(R_1 + Ls) + I_2(s)(-Ls) = V_T$$

I_2

$$I_2(s)R_2 + \frac{I_2(s)}{Cs} + L\left(sI_1(s) - I_1(0) - \left(sI_2(s) - I_2(0)\right)\right) = V_T$$
$$I_2(s)R_2 + \frac{I_2(s)}{Cs} + LsI_1(s) - LI_1(0) - LsI_2(s) + LI_2(0) = V_T$$
$$I_2(s)R_2 + \frac{I_2(s)}{Cs} + LsI_1(s) - LsI_2(s) = V_T$$
$$I_1(s)Ls + I_2(s)\left(R_2 + \frac{1}{Cs} - Ls\right) = V_T$$

Sistema de ecuaciones

$$\begin{bmatrix} (R_1 + Ls) & (-Ls) & V_T \\ (Ls) & \left(R_2 + \frac{1}{Cs}\right) & 0 \end{bmatrix}$$
$$Wronksiano = R_1R_2 + \frac{R_1}{Cs} - LsR_1 + LsR_2 + \frac{Ls}{Cs} - L^2S^2 + L^2S^2$$

Se utiliza Cramer

I_1

$$I_1(s) = \frac{\left(24(1Ms + 1k - 1s^2)\right)}{10Gs + 10M - 9ks^2 + s}$$
$$Multiplicando\ por\ 1000 \to \boldsymbol{I_1(s)} = \frac{\mathbf{24(s^2 - 1000000s - 1000)}}{\mathbf{9000s^2 - 10000000001s - 10000000}}$$

I_2

$$\boldsymbol{I_2(s)} = -\frac{24s^2}{10Gs + 10M - 9ks^2} = \frac{\mathbf{24s^2}}{\mathbf{9000s^2 - 10000000001s - 10000000}}$$

Sacando transformada inversa

I_1

$$I_1(s) = \frac{24\,(s^2 - 1000000\,s - 1000)}{9000\,s^2 - 10000000001\,s - 10000000}$$
$$= \frac{1000000001\,s + 1000000}{375\,(9000\,s^2 - 10000000001\,s - 10000000)} + \frac{1}{375}$$
$$= \frac{1000000001\,s + 1000000}{375 * 9000\left(s^2 - \frac{10000000001\,s}{9000} - \frac{10000}{9}\right)} + \frac{1}{375}$$
$$= \frac{1000000001\,s + 1000000}{375 * 9000\left(\left(s^2 - \frac{10000000001\,s}{9000} + \left(\frac{10000000001}{18000}\right)^2\right) - \left(\frac{10000000001}{18000}\right)^2 - \frac{10000}{9}\right)} + \frac{1}{375}$$
$$= \frac{1000000001\,s + 1000000}{375 * 9000\left(\left(s - \frac{10000000001}{18000}\right)^2 - \frac{100000000380000000001}{324000000}\right)} + \frac{1}{375}$$

Renombrando constantes

$$I_1(s) = \frac{as + b}{375 * 9000((s-c)^2 - d)} + \frac{1}{375} = \frac{as + b + ac - ac}{375 * 9000((s-c)^2 - d)} + \frac{1}{375}$$
$$= \frac{1}{375} + \frac{1}{375 * 9000}\left(\frac{a(s-c)}{((s-c)^2 - d)} + \frac{b + ac}{((s-c)^2 - d)}\right)$$
$$= \frac{1}{375} + \frac{1}{375 * 9000}\left(\frac{a(s-c)}{((s-c)^2 - d)} + \frac{(b + ac)\sqrt{d}}{\left((s-c)^2 - \left(\sqrt{d}\right)^2\right)\sqrt{d}}\right)$$
$$\mathcal{L}^{-1}\{I_1(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{375} + \frac{1}{375 * 9000}\left(\frac{a(s-c)}{((s-c)^2 - d)} + \frac{(b + ac)\sqrt{d}}{\left((s-c)^2 - \left(\sqrt{d}\right)^2\right)\sqrt{d}}\right)\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{375}\right\} + \frac{1}{375 * 9000}\left(\mathcal{L}^{-1}\left\{\frac{a(s-c)}{((s-c)^2 - d)}\right\} + \mathcal{L}^{-1}\left\{\frac{(b + ac)\sqrt{d}}{\left((s-c)^2 - \left(\sqrt{d}\right)^2\right)\sqrt{d}}\right\}\right)$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{375}\right\} + \frac{1}{375 * 9000}\left(a\mathcal{L}^{-1}\left\{\frac{(s-c)}{((s-c)^2 - d)}\right\} + \frac{b + ac}{\sqrt{d}}\mathcal{L}^{-1}\left\{\frac{\sqrt{d}}{\left((s-c)^2 - \left(\sqrt{d}\right)^2\right)}\right\}\right)$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{375}\right\} + \frac{1}{375 * 9000}\left(ae^{ct}\mathcal{L}^{-1}\left\{\frac{s}{(s^2 - d)}\right\} + \frac{b + ac}{\sqrt{d}}e^{ct}\mathcal{L}^{-1}\left\{\frac{\sqrt{d}}{\left(s^2 - \left(\sqrt{d}\right)^2\right)}\right\}\right)$$
$$= \frac{\delta(t)}{375} + \frac{e^{ct}}{375 * 9000}\left(a\cosh\left(\sqrt{d}t\right) + \frac{b + ac}{\sqrt{d}}\sinh\left(\sqrt{d}t\right)\right)$$

Se deshace la sustitución

$$I_1(t) = \frac{\delta(t)}{375} + \frac{e^{\frac{10000000001}{18000}t}}{3375000}\left(1000000001\cosh\left(\sqrt{\frac{100000000380000000001}{324000000}}t\right) + \frac{1000000 + 1000000001 * \frac{10000000001}{18000}}{\sqrt{\frac{100000000380000000001}{324000000}}}\sinh\left(\sqrt{\frac{100000000380000000001}{324000000}}t\right)\right)$$

$$= \frac{\delta(t)}{375} + \frac{e^{\frac{10000000001}{18000}t}}{3375000}\left(1000000001\cosh\left(\frac{\sqrt{100000000380000000001}}{18000}t\right) + \frac{\frac{10000000029000000001}{18000}}{\frac{\sqrt{100000000380000000001}}{18000}}\sinh\left(\frac{\sqrt{100000000380000000001}}{18000}t\right)\right)$$

$$I_1(t) = \frac{\delta(t)}{375} + \frac{e^{\frac{10000000001}{18000}t}}{3375000}\left(1000000001\cosh\left(\frac{\sqrt{100000000380000000001}}{18000}t\right) + \frac{10000000029000000001}{\sqrt{100000000380000000001}}\sinh\left(\frac{\sqrt{100000000380000000001}}{18000}t\right)\right)$$

I_2

$$I_2(s) = \frac{24s^2}{9000s^2 - 10000000001s - 10000000}$$

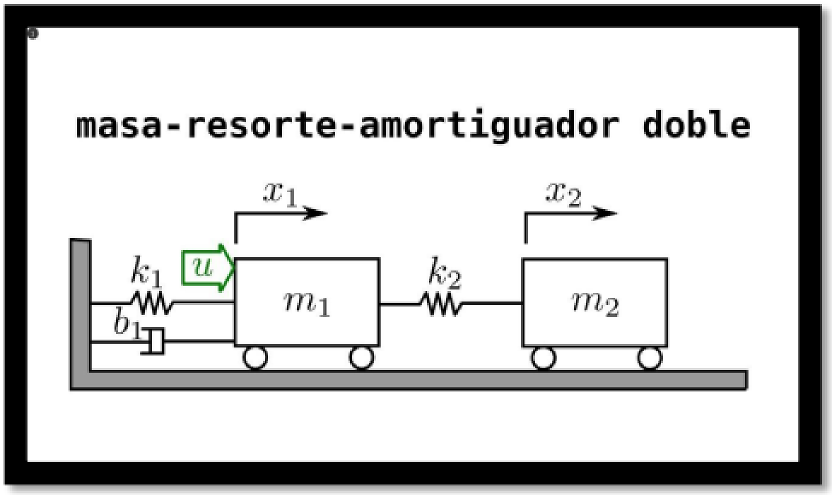
$$= \frac{10000000001\,s + 10000000}{375\,(9000\,s^2 - 10000000001\,s - 10000000)} + \frac{1}{375}$$

$$\mathcal{L}^{-1}\{I_2(s)\} = I_2(t) =$$

$$= \frac{\delta(t)}{375}$$

$$+ \frac{e^{\frac{10000000001}{18000}t}}{3375000}\left(1000000001\cosh\left(\frac{\sqrt{100000000380000000001}}{18000}t\right)\right.$$

$$\left.+ \frac{10000000029000000001}{\sqrt{100000000380000000001}}\sinh\left(\frac{\sqrt{100000000380000000001}}{18000}t\right)\right)$$



Datos Iniciales. $x_1(0) = 0.1,\ x_2(0) = 0.2,\ x_1'(0) = x_2'(0) = 0$

Valores. $k_1 = 5\frac{N}{m},\ \ k_2 = 3\frac{N}{m},\ \ b_1 = 10\frac{Nm}{s},\ m_1 = m_2 = 2\ kg$

Planteamiento del problema

X_1

$$\sum F = 0 = -k_1(x_1) - b_1(x_1') + m_1(x_1'')$$
$$2x_1'' - 10x_1' - 5x_1 = 0$$

X_2

$$-k_2(x_2 - x_1) + m_2(x_2'') = 0$$
$$-3x_2 + 3x_1 + 2x_2'' = 0$$

Transformada de Laplace

X_1

$$2\big(s^2x_1(s) - sx_1(0) - x_1'(0)\big) - 10\big(sx_1(s) - x_1(0)\big) - 5x_1(s) = 0$$
$$2s^2x_1(s) - 2sx_1(0) - 2x_1'(0) - 10sx_1(s) + 10x_1(0) - 5x_1(s) = 0$$
$$2s^2x_1(s) - 0.2s - 10sx_1(s) + 1 - 5x_1(s) = 0$$
$$x_1(s)(2s^2 - 10s - 5) = 0.2s - 1$$
$$x_1(s) = \frac{0.2s - 1}{2s^2 - 10s - 5}$$

X_2

$$-3x_2(s) + 3x_1(s) + 2\big(s^2x_2(s) - sx_2(0) - x_2'(0)\big) = 0$$
$$-3x_2(s) + 3x_1(s) + 2s^2x_2(s) - 2sx_2(0) - 2x_2'(0) = 0$$
$$-3x_2(s) + 3x_1(s) + 2s^2x_2(s) - 0.4s = 0$$
$$x_1(s)(3) + x_2(s)(2s^2 - 3) = 0.4s$$

Sistema de ecuaciones

$$x_1(s) = \frac{0.2s - 1}{2s^2 - 10s - 5}$$

X_2

$$\left(\frac{0.2s - 1}{2s^2 - 10s - 5}\right)(3) + x_2(s)(2s^2 - 3) = 0.4s$$

$$x_2(2s^2 - 3) = 0.4s - \frac{3(0.2s - 1)}{2s^2 - 10s - 5}$$

$$x_2 = \frac{0.4s - \frac{3(0.2s-1)}{2s^2-10s-5}}{2s^2 - 3} = \frac{0.4s}{2s^2 - 3} - \frac{0.6s - 3}{(2s^2 - 10s - 5)(2s^2 - 3)}$$

$$= \frac{0.2s}{s^2 - \frac{3}{2}} - \frac{0.6s - 3}{4s^4 - 6s^2 - 20s^3 + 30s - 10s^2 + 15}$$

$$x_2 = \frac{0.2s}{s^2 - \frac{3}{2}} - \frac{0.6s - 3}{4s^4 - 20s^3 - 16s^2 + 30s + 15}$$

Sacando transformada inversa

X_1

$$x_1(s) = \frac{0.2s - 1}{2\left(s^2 - 5s - \frac{5}{2}\right)} = \frac{\frac{s}{5} - 1}{2\left(\left(s - \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)\right)} = \frac{\frac{s}{5} - 1}{2\left(\left(s - \frac{5}{2}\right)^2 - \frac{35}{4}\right)}$$

$$= \frac{\frac{s}{5} - \frac{1}{2} - 1 + \frac{1}{2}}{2\left(\left(s - \frac{5}{2}\right)^2 - \frac{35}{4}\right)} = \frac{\frac{1}{5}\left(s - \frac{5}{2}\right) - 1 + \frac{1}{2}}{2\left(\left(s - \frac{5}{2}\right)^2 - \frac{35}{4}\right)} = \frac{s - \frac{5}{2}}{10\left(\left(s - \frac{5}{2}\right)^2 - \frac{35}{4}\right)} - \frac{1}{4\left(\left(s - \frac{5}{2}\right)^2 - \frac{35}{4}\right)}$$

$$\mathcal{L}^{-1}\{x_1] = \frac{e^{\frac{5t}{2}}}{10}\mathcal{L}^{-1}\left\{\frac{s}{s^2 - \frac{35}{4}}\right\} - \frac{e^{\frac{5t}{2}}}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2 - \frac{35}{4}}\right\} =$$

$$= \frac{e^{\frac{5t}{2}}}{10}\mathcal{L}^{-1}\left\{\frac{s}{s^2 - \left(\sqrt{\frac{35}{4}}\right)^2}\right\} - \frac{e^{\frac{5t}{2}}}{4\sqrt{\frac{35}{4}}}\mathcal{L}^{-1}\left\{\frac{\sqrt{\frac{35}{4}}}{s^2 - \left(\sqrt{\frac{35}{4}}\right)^2}\right\}$$

$$\mathcal{L}^{-1}\{x_1\} = \frac{e^{\frac{5t}{2}}\, cosh\left(\frac{\sqrt{35}}{2}t\right)}{10} - \frac{e^{\frac{5t}{2}}\sinh\left(\frac{\sqrt{35}}{2}t\right)}{2\sqrt{35}}$$

X_2

$$x_2 = \frac{0.2s}{s^2 - \frac{3}{2}} - \frac{0.6s - 3}{4s^4 - 20s^3 - 16s^2 + 30s + 15}$$

$$= \frac{0.2s}{s^2 - \frac{3}{2}} - \left(\frac{0.032453}{s\ -\ 1.22474} - \frac{0.10726}{s\ +\ 0.45804} + \frac{0.0743963}{s\ +\ 1.22474} + \frac{0.000410511}{s\ -\ 5.45804}\right)$$

$$\mathcal{L}^{-1}\{x_2\} = \mathcal{L}^{-1}\left\{\frac{0.2s}{s^2 - \left(\sqrt{\frac{3}{2}}\right)^2}\right\}$$
$$- \left(\mathcal{L}^{-1}\left\{\frac{0.032453}{s\ -\ 1.22474}\right\} - \mathcal{L}^{-1}\left\{\frac{0.10726}{s\ +\ 0.45804}\right\} + \mathcal{L}^{-1}\left\{\frac{0.0743963}{s\ +\ 1.22474}\right\}\right.$$
$$\left. + \mathcal{L}^{-1}\left\{\frac{0.000410511}{s\ -\ 5.45804}\right\}\right)$$

$$\mathcal{L}^{-1}\{x_2\} = \frac{\cosh\left(\sqrt{\frac{3}{2}}t\right)}{5} - 0.032453e^{1.22474t} + 0.10726e^{-0.45804t} - 0.0743963e^{-1.22474t} - 0.000410511e^{5.45804t}$$

NOTAS ADICIONALES