EXERCISES

2.1 A Markov chain has transition matrix

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \boldsymbol{P} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0.1 & 0.3 & 0.6 \\ 0 & 0.4 & 0.6 \\ 0.3 & 0.2 & 0.5 \end{pmatrix} \end{array}$$

with initial distribution $\boldsymbol{\alpha} = (0.2, 0.3, 0.5)$. Find the following:

(a) $P(X_7 = 3|X_6 = 2)$

(b) $P(X_9 = 2|X_1 = 2, X_5 = 1, X_7 = 3)$

(c) $P(X_0 = 3|X_1 = 1)$

(d) $E(X_2)$

2.2 Let $X_0, X_1, \ldots$ be a Markov chain with transition matrix

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \end{array}$$

and initial distribution $\boldsymbol{\alpha} = (1/2, 0, 1/2)$. Find the following:

(a) $P(X_2 = 1|X_1 = 3)$

(b) $P(X_1 = 3, X_2 = 1)$

(c) $P(X_1 = 3|X_2 = 1)$

(d) $P(X_9 = 1|X_1 = 3, X_4 = 1, X_7 = 2)$

2.3 See Example 2.6. Consider the Wright–Fisher model with a population of $k = 3$ genes. If the population initially has one A allele, find the probability that there are no A alleles in three generations.

2.4 For the general two-state chain with transition matrix

$$\begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ \boldsymbol{P} = \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \end{array}$$

and initial distribution $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, find the following:

(a) the two-step transition matrix

(b) the distribution of X_1

2.5 Consider a random walk on $\{0, \ldots, k\}$, which moves left and right with respective probabilities q and p. If the walk is at 0 it transitions to 1 on the next step. If the walk is at k it transitions to $k-1$ on the next step. This is called *random walk with reflecting boundaries*. Assume that $k = 3$, $q = 1/4$, $p = 3/4$, and the initial distribution is uniform. For the following, use technology if needed.

(a) Exhibit the transition matrix.

(b) Find $P(X_7 = 1 | X_0 = 3, X_2 = 2, X_4 = 2)$.

(c) Find $P(X_3 = 1, X_5 = 3)$.

2.6 A tetrahedron die has four faces labeled 1, 2, 3, and 4. In repeated independent rolls of the die $R_0, R_1, \ldots$, let $X_n = \max\{R_0, \ldots, R_n\}$ be the maximum value after $n + 1$ rolls, for $n \geq 0$.

(a) Give an intuitive argument for why $X_0, X_1, \ldots$, is a Markov chain, and exhibit the transition matrix.

(b) Find $P(X_3 \geq 3)$.

2.7 Let $X_0, X_1, \ldots$ be a Markov chain with transition matrix $\boldsymbol{P}$. Let $Y_n = X_{3n}$, for $n = 0, 1, 2, \ldots$. Show that $Y_0, Y_1, \ldots$ is a Markov chain and exhibit its transition matrix.

2.8 Give the Markov transition matrix for random walk on the weighted graph in Figure 2.10.

2.9 Give the transition matrix for the transition graph in Figure 2.11.

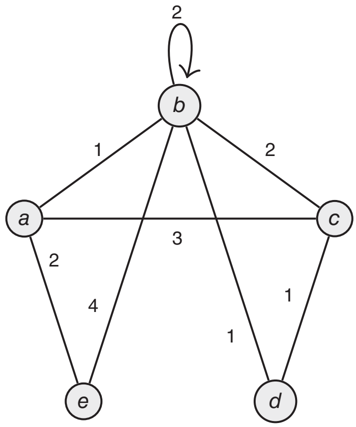

Figure 2.10

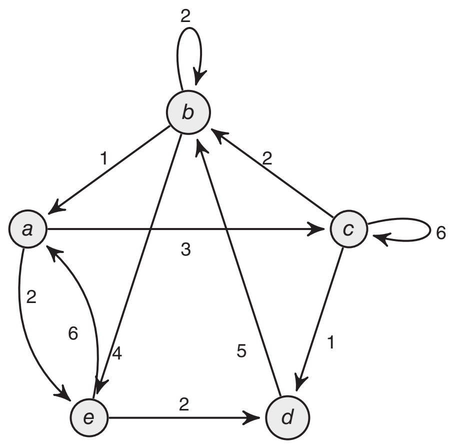

Figure 2.11

2.10 Consider a Markov chain with transition matrix

$$
\boldsymbol{P} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}
\begin{array}{c}
\begin{array}{cccc} a & b & c & d \end{array} \\
\begin{pmatrix} 0 & 3/5 & 1/5 & 1/5 \\ 3/4 & 0 & 1/4 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 1/2 \end{pmatrix}
\end{array}.
$$

(a) Exhibit the directed, weighted transition graph for the chain.

(b) The transition graph for this chain can be given as a weighted graph without directed edges. Exhibit the graph.

2.11 You start with five dice. Roll all the dice and put aside those dice that come up 6. Then, roll the remaining dice, putting aside those dice that come up 6. And so on. Let X_n be the number of dice that are sixes after n rolls.

(a) Describe the transition matrix $\boldsymbol{P}$ for this Markov chain.

(b) Find the probability of getting all sixes by the third play.

(c) What do you expect $\boldsymbol{P}^{100}$ to look like? Use technology to confirm your answer.

2.12 Two urns contain k balls each. Initially, the balls in the left urn are all red and the balls in the right urn are all blue. At each step, pick a ball at random from each urn and exchange them. Let X_n be the number of blue balls in the left urn. (Note that necessarily $X_0 = 0$ and $X_1 = 1$.) Argue that the process is a Markov chain. Find the transition matrix. This model is called the Bernoulli–Laplace model of diffusion and was introduced by Daniel Bernoulli in 1769 as a model for the flow of two incompressible liquids between two containers.

2.13 See the move-to-front process in Example 2.10. Here is another way to organize the bookshelf. When a book is returned it is put back on the library shelf one position forward from where it was originally. If the book at the front of the shelf is returned it is put back at the front of the shelf. Thus, if the order of books is (a, b, c, d, e) and book d is picked, the new order is (a, b, d, c, e). This reorganization method is called the *transposition*, or *move-ahead-1*, scheme. Give the transition matrix for the transposition scheme for a shelf with three books.

2.14 There are k songs on Mary's music player. The player is set to *shuffle* mode, which plays songs uniformly at random, sampling with replacement. Thus, repeats are possible. Let X_n denote the number of *unique* songs that have been heard after the nth play.

(a) Show that $X_0, X_1, \ldots$ is a Markov chain and give the transition matrix.

(b) If Mary has four songs on her music player, find the probability that all songs are heard after six plays.

2.15 Assume that $X_0, X_1, \ldots$ is a two-state Markov chain on $\mathcal{S} = \{0, 1\}$ with transition matrix

$$\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & \quad 1 \end{array} \\ \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \end{array}.$$

The present state of the chain only depends on the previous state. One can model a bivariate process that looks back two time periods by the following construction. Let $Z_n = (X_{n-1}, X_n)$, for $n \geq 1$. The sequence $Z_1, Z_2, \ldots$ is a Markov chain with state space $\mathcal{S} \times \mathcal{S} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Give the transition matrix of the new chain.

2.16 Assume that $\boldsymbol{P}$ is a stochastic matrix with equal rows. Show that $\boldsymbol{P}^n = \boldsymbol{P}$, for all $n \geq 1$.

2.17 Let $\boldsymbol{P}$ be a stochastic matrix. Show that $\lambda = 1$ is an eigenvalue of $\boldsymbol{P}$. What is the associated eigenvector?

2.18 A stochastic matrix is called *doubly stochastic* if its columns sum to 1. Let $X_0, X_1, \ldots$ be a Markov chain on $\{1, \ldots, k\}$ with doubly stochastic transition matrix and initial distribution that is uniform on $\{1, \ldots, k\}$. Show that the distribution of X_n is uniform on $\{1, \ldots, k\}$, for all $n \geq 0$.

2.19 Let $\boldsymbol{P}$ be the transition matrix of a Markov chain on k states. Let $\mathbf{I}$ denote the $k \times k$ identity matrix. Consider the matrix

$$\boldsymbol{Q} = (1 - p)\mathbf{I} + p\boldsymbol{P}, \text{ for } 0 < p < 1.$$

Show that $\boldsymbol{Q}$ is a stochastic matrix. Give a probabilistic interpretation for the dynamics of a Markov chain governed by the $\boldsymbol{Q}$ matrix in terms of the original Markov chain.

2.20 Let $X_0, X_1, \ldots$ be a Markov chain with transition matrix

$$
\boldsymbol{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}
\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & 1-p & 0 \end{pmatrix} \end{array},
$$

for $0 < p < 1$. Let g be a function defined by

$$
g(x) = \begin{cases} 0, & \text{if } x = 1, \\ 1, & \text{if } x = 2, 3. \end{cases}
$$

Let $Y_n = g(X_n)$, for $n \geq 0$. Show that $Y_0, Y_1, \ldots$ is not a Markov chain.

2.21 Let P and Q be two transition matrices on the same state space. We define two processes, both started in some initial state i.
In process #1, a coin is flipped. If it lands heads, then the process unfolds according to the P matrix. If it lands tails, the process unfolds according to the Q matrix.
In process #2, at each step a coin is flipped. If it lands heads, the next state is chosen according to the P matrix. If it lands tails, the next state is chosen according to the Q matrix.
Thus, in #1, one coin is initially flipped, which governs the entire evolution of the process. And in #2, a coin is flipped at each step to decide the next step of the process.
Decide whether either of these processes is a Markov chain. If not, explain why, if yes, exhibit the transition matrix.

2.22 Prove the following using mathematical induction.

(a) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

(b) $1^2 + 2^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6$.

(c) For all real $x > -1$, $(1 + x)^n \geq 1 + nx$.

2.23 `R` : Simulate the first 20 letters (vowel/consonant) of the Pushkin poem Markov chain of Example 2.2.

2.24 `R` : Simulate 50 steps of the random walk on the graph in Figure 2.1. Repeat the simulation 10 times. How many of your simulations end at vertex c? Compare with the exact long-term probability the walk visits c.

$\boxed{\textbf{2.25}}$ `R` : The behavior of dolphins in the presence of tour boats in Patagonia, Argentina is studied in Dans et al. (2012). A Markov chain model is developed, with state space consisting of five primary dolphin activities (socializing, traveling, milling, feeding, and resting). The following transition matrix is obtained.

$$
\begin{array}{c c}
 & \begin{array}{ccccc} s & t & m & f & r \end{array} \\
P = \begin{array}{c} s \\ t \\ m \\ f \\ r \end{array} & \begin{pmatrix} 0.84 & 0.11 & 0.01 & 0.04 & 0.00 \\ 0.03 & 0.80 & 0.04 & 0.10 & 0.03 \\ 0.01 & 0.15 & 0.70 & 0.07 & 0.07 \\ 0.03 & 0.19 & 0.02 & 0.75 & 0.01 \\ 0.03 & 0.09 & 0.05 & 0.00 & 0.83 \end{pmatrix}
\end{array}.
$$

Use technology to estimate the long-term distribution of dolphin activity.

2.26 R : In computer security applications, a *honeypot* is a trap set on a network to detect and counteract computer hackers. Honeypot data are studied in Kimou et al. (2010) using Markov chains. The authors obtain honeypot data from a central database and observe attacks against four computer ports—80, 135, 139, and 445—over 1 year. The ports are the states of a Markov chain along with a state corresponding to no port is attacked. Weekly data are monitored, and the port most often attacked during the week is recorded. The estimated Markov transition matrix for weekly attacks is

$$
\begin{array}{c c}
 & \begin{array}{ccccc} 80 & 135 & 139 & 445 & \text{No attack} \end{array} \\
P = \begin{array}{c} 80 \\ 135 \\ 139 \\ 445 \\ \text{No} \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 8/13 & 3/13 & 1/13 & 1/13 \\ 1/16 & 3/16 & 3/8 & 1/4 & 1/8 \\ 0 & 1/11 & 4/11 & 5/11 & 1/11 \\ 0 & 1/8 & 1/2 & 1/8 & 1/4 \end{pmatrix}
\end{array},
$$

with initial distribution $\boldsymbol{\alpha} = (0, 0, 0, 0, 1)$.

(a) Which are the least and most likely attacked ports after 2 weeks?

(b) Find the long-term distribution of attacked ports.

2.27 R : See **gamblersruin.R.** Simulate gambler's ruin for a gambler with initial stake \$2, playing a fair game.

(a) Estimate the probability that the gambler is ruined before he wins \$5.

(b) Construct the transition matrix for the associated Markov chain. Estimate the desired probability in (a) by taking high matrix powers.

(c) Compare your results with the exact probability.