3. *There exists a positive right eigenvector* $\boldsymbol{v}$ *associated with* λ^*, *and a positive left eigenvector* $\boldsymbol{w}$ *associated with* λ^*. *Furthermore,*

$$\lim_{n\to\infty} \frac{1}{(\lambda^*)^n}\boldsymbol{M}^n = \boldsymbol{v}\boldsymbol{w}^T,$$

where the eigenvectors are normalized so that $\boldsymbol{w}^T\boldsymbol{v} = 1$.

The proof of the Perron–Frobenius theorem can be found in many advanced linear algebra textbooks, including Horn and Johnson (1990).

For an ergodic Markov chain, the transition matrix $\boldsymbol{P}$ is regular and $\boldsymbol{P}^N$ is a positive matrix for some integer N. The Perron–Frobenius theorem applies. The Perron–Frobenius eigenvalue of $\boldsymbol{P}^n$ is $\lambda^* = 1$, with associated right eigenvector $\boldsymbol{v} = \mathbf{1}$, and associated left eigenvector $\boldsymbol{w}$.

If $\lambda^* = 1$ is an eigenvalue of $\boldsymbol{P}^N$, then $(\lambda^*)^{1/N} = 1$ is an eigenvalue of $\boldsymbol{P}$, with associated right and left eigenvectors $\boldsymbol{v}$ and $\boldsymbol{w}$, respectively. Normalizing $\boldsymbol{w}$ so that its components sum to 1 gives the unique, positive stationary distribution $\boldsymbol{\pi}$, which is the limiting distribution of the chain. The limiting matrix $\boldsymbol{v}\boldsymbol{w}^T$ is a stochastic matrix all of whose rows are equal to $\boldsymbol{w}^T$. ■

EXERCISES

3.1 Consider a Markov chain with transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 1/2 & 1/4 & 0 & 1/4 \\ 0 & 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 & 0 \\ 0 & 1/4 & 1/2 & 1/4 \end{pmatrix}.$$

Find the stationary distribution. Do not use technology.

3.2 A stochastic matrix is called *doubly stochastic* if its rows and columns sum to 1. Show that a Markov chain whose transition matrix is doubly stochastic has a stationary distribution, which is uniform on the state space.

3.3 Determine which of the following matrices are regular.

$$\boldsymbol{P} = \begin{pmatrix} 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{Q} = \begin{pmatrix} 0 & 1 \\ p & 1-p \end{pmatrix}, \quad \boldsymbol{R} = \begin{pmatrix} 0 & 1 & 0 \\ 0.25 & 0.5 & 0.25 \\ 1 & 0 & 0 \end{pmatrix}.$$

3.4 Consider a Markov chain with transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 1-a & a & 0 \\ 0 & 1-b & b \\ c & 0 & 1-c \end{pmatrix},$$

where $0 < a, b, c < 1$. Find the stationary distribution.

3.5 A Markov chain has transition matrix

$$P = \begin{pmatrix} 0 & 1/4 & 0 & 0 & 3/4 \\ 3/4 & 0 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Describe the set of stationary distributions for the chain.

(b) Use technology to find $\lim_{n\to\infty} P^n$. Explain the long-term behavior of the chain.

(c) Explain why the chain does not have a limiting distribution, and why this does not contradict the existence of a limiting matrix as shown in (b).

3.6 Consider a Markov chain with transition matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \cdots \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & \cdots \\ 2/3 & 0 & 1/3 & 0 & 0 & \cdots \\ 3/4 & 0 & 0 & 1/4 & 0 & \cdots \\ 4/5 & 0 & 0 & 0 & 1/5 & \cdots \\ 5/6 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array},$$

defined by

$$P_{ij} = \begin{cases} i/(i+1), & \text{if } j = 1, \\ 1/(i+1), & \text{if } j = i+1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Does the chain have a stationary distribution? If yes, exhibit the distribution. If no, explain why.

(b) Classify the states of the chain.

(c) Repeat part (a) with the row entries of P switched. That is, let

$$P_{ij} = \begin{cases} 1/(i+1), & \text{if } j = 1, \\ i/(i+1), & \text{if } j = i+1, \\ 0, & \text{otherwise.} \end{cases}$$

3.7 A Markov chain has n states. If the chain is at state k, a coin is flipped, whose heads probability is p. If the coin lands heads, the chain stays at k. If the coin lands tails, the chain moves to a different state uniformly at random. Exhibit the transition matrix and find the stationary distribution.

3.8 Let

$$\boldsymbol{P}_1 = \begin{pmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{P}_2 = \begin{pmatrix} 1/5 & 4/5 \\ 4/5 & 1/5 \end{pmatrix}.$$

Consider a Markov chain on four states whose transition matrix is given by the block matrix

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{P}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}_2 \end{pmatrix}.$$

(a) Does the Markov chain have a unique stationary distribution? If so, find it.

(b) Does $\lim_{n\to\infty} \boldsymbol{P}^n$ exist? If so, find it.

(c) Does the Markov chain have a limiting distribution? If so, find it.

3.9 Let $\boldsymbol{P}$ be a stochastic matrix.

(a) If $\boldsymbol{P}$ is regular, is $\boldsymbol{P}^2$ regular?

(b) If $\boldsymbol{P}$ is the transition matrix of an irreducible Markov chain, is $\boldsymbol{P}^2$ the transition matrix of an irreducible Markov chain?

3.10 A Markov chain has transition matrix $\boldsymbol{P}$ and limiting distribution $\boldsymbol{\pi}$. Further assume that $\boldsymbol{\pi}$ is the initial distribution of the chain. That is, the chain is in stationarity. Find the following:

(a) $\lim_{n\to\infty} P(X_n = j | X_{n-1} = i)$

(b) $\lim_{n\to\infty} P(X_n = j | X_0 = i)$

(c) $\lim_{n\to\infty} P(X_{n+1} = k, X_n = j | X_0 = i)$

(d) $\lim_{n\to\infty} P(X_0 = j | X_n = i)$

3.11 Consider a simple symmetric random walk on $\{0, 1, \ldots, k\}$ with reflecting boundaries. If the walk is at state 0, it moves to 1 on the next step. If the walk is at k, it moves to $k - 1$ on the next step. Otherwise, the walk moves left or right, with probability 1/2.

(a) Find the stationary distribution.

(b) For $k = 1,000$, if the walk starts at 0, how many steps will it take, on average, for the walk to return to 0?

3.12 A Markov chain has transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1/4 & 0 & 3/4 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 3/4 & 0 & 1/4 \end{pmatrix}.$$

Find the set of all stationary distributions.

$\boxed{\textbf{3.13}}$ Find the communication classes of a Markov chain with transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 1/6 & 0 & 0 \\ 0 & 1/4 & 0 & 1/2 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Rewrite the transition matrix in canonical form.

3.14 The California Air Resources Board warns the public when smog levels are above certain thresholds. Days when the board issues warnings are called *episode* days. Lin (1981) models the daily sequence of episode and nonepisode days as a Markov chain with transition matrix

$$P = \begin{array}{c c} & \begin{array}{cc} \text{Nonepisode} & \text{Episode} \end{array} \\ \begin{array}{c} \text{Nonepisode} \\ \text{Episode} \end{array} & \begin{pmatrix} 0.77 & 0.23 \\ 0.24 & 0.76 \end{pmatrix} \end{array}.$$

(a) What is the long-term probability that a given day will be an episode day?

(b) Over a year's time about how many days are expected to be episode days?

(c) In the long-term, what is the average number of days that will transpire between episode days?

3.15 On a chessboard a single random knight performs a simple random walk. From any square, the knight chooses from among its permissible moves with equal probability. If the knight starts on a corner, how long, on average, will it take to return to that corner?

3.16 As in the previous exercise, find the expected return time from a corner square for the following chess pieces: (i) queen, (ii) rook, (iii) king, (iv) bishop. Order the pieces by which pieces return quickest.

3.17 Consider a Markov chain with transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}.$$

Obtain a closed form expression for $\boldsymbol{P}^n$. Exhibit the matrix $\sum_{n=0}^{\infty} \boldsymbol{P}^n$ (some entries may be $+\infty$). Explain what this shows about the recurrence and transience of the states.

3.18 Use first-step analysis to find the expected return time to state b for the Markov chain with transition matrix

$$\boldsymbol{P} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 0 & 3/4 \\ 1/2 & 1/2 & 0 \end{pmatrix} \end{array}.$$

3.19 Consider random walk on the graph in Figure 3.15. Use first-step analysis to find the expected time to hit d for the walk started in a. (*Hint*: By exploiting symmetries in the graph, the solution can be found by solving a 3×3 linear system.)

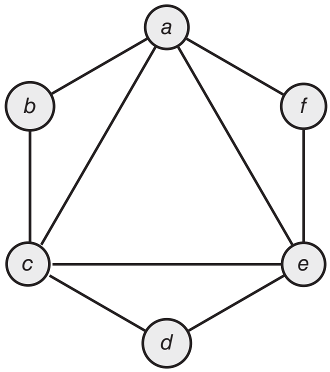

Figure 3.15

3.20 Show that simple symmetric random walk on $\mathbb{Z}^2$, that is, on the integer points in the plane, is recurrent. As in the one-dimensional case, consider the origin.

3.21 Show that simple symmetric random walk on $\mathbb{Z}^3$ is transient. As in the one-dimensional case, consider the origin and show

$$P_{00}^{2n} = \frac{1}{6^{2n}} \sum_{0 \le j+k \le n} \frac{(2n)!}{j!j!k!k!(n-j-k)!(n-j-k)!},$$
$$\le \frac{1}{2^{2n}} \binom{2n}{n} \left(\frac{1}{3^n} \frac{n!}{(n/3)!(n/3)!(n/3)!} \right).$$

Then, use Stirling's approximation.

3.22 Consider the general two-state chain

$$\boldsymbol{P} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \end{array},$$

where p and q are not both 0. Let T be the first return time to state 1, for the chain started in 1.

(a) Show that $P(T \geq n) = p(1-q)^{n-2}$, for $\quad n \geq 2$.

(b) Find $E(T)$ and verify that $E(T) = 1/\pi_1$, where $\boldsymbol{\pi}$ is the stationary distribution of the chain.

3.23 Consider a k-state Markov chain with transition matrix

$$\boldsymbol{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ k-2 \\ k-1 \\ k \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & \cdots & k-2 & k-1 & k \end{array} \\ \begin{pmatrix} 1/k & 1/k & 1/k & \cdots & 1/k & 1/k & 1/k \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

Show that the chain is ergodic and find the limiting distribution.

3.24 Show that the stationary distribution for the modified Ehrenfest chain of Example 3.19 is binomial with parameters N and 1/2.

3.25 Read about the Bernoulli–Laplace model of diffusion in Exercise 2.12.

(a) Find the stationary distribution for the cases $k = 2$ and $k = 3$.

(b) For general k, show that $\pi_j = \binom{k}{j}^2 / \binom{2k}{k}$, for $j = 0, 1, \ldots, k$, satisfies the equations for the stationary distribution and is thus the unique limiting distribution of the chain.

3.26 Assume that $(p_1, \ldots, p_k)$ is a probability vector. Let $\boldsymbol{P}$ be a $k \times k$ transition matrix defined by

$$P_{ij} = \begin{cases} p_j, & \text{if } i = 1, \ldots, k-1, \\ 0, & \text{if } i = k, j < k, \\ 1, & \text{if } i = k, j = k. \end{cases}$$

Describe all the stationary distributions for $\boldsymbol{P}$.

3.27 Sinclair (2005). Consider the infinite Markov chain on the non-negative integers described by Figure 3.16.

(a) Show that the chain is irreducible and aperiodic.

(b) Show that the chain is recurrent by computing the first return time to 0 for the chain started at 0.

(c) Show that the chain is null recurrent.

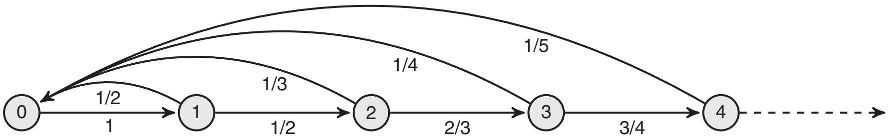

Figure 3.16

3.28 Consider a Markov chain with transition matrix

$$
P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{pmatrix}
1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\
2/3 & 0 & 1/3 & 0 & 0 & 0 & 0 \\
0 & 2/3 & 1/3 & 0 & 0 & 0 & 0 \\
0 & 1/4 & 0 & 1/2 & 0 & 1/4 & 0 \\
0 & 0 & 1/4 & 0 & 1/4 & 0 & 1/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1/8 & 1/8 & 1/8 & 1/8 & 1/4 & 1/4
\end{pmatrix}
\end{array}.
$$

Identify the communication classes. Classify the states as recurrent or transient. For all i and j, determine $\lim_{n\to\infty} P^n_{ij}$ without using technology.

3.29 Consider a Markov chain with transition matrix

$$
P = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array}
\begin{array}{c}
\begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\
\begin{pmatrix}
0.6 & 0.2 & 0 & 0 & 0.2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\
0 & 0 & 0.3 & 0.7 & 0 & 0 & 0 \\
0 & 0.3 & 0.4 & 0.3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0.1 & 0 & 0 & 0 & 0 & 0.9 \\
0 & 0.2 & 0 & 0 & 0 & 0.8 & 0
\end{pmatrix}
\end{array}.
$$

Identify the communication classes. Classify the states as recurrent or transient, and determine the period of each state.

3.30 A graph is *bipartite* if the vertex set can be colored with two colors black and white such that every edge in the graph joins a black vertex and a white vertex. See Figure 3.7(a) for an example of a bipartite graph. Show that for simple random walk on a connected graph, the walk is periodic if and only if the graph is bipartite.

3.31 For the network graph in Figure 3.17, find the PageRank for the nodes of the network using a damping factor of $p = 0.90$. See Example 3.21.

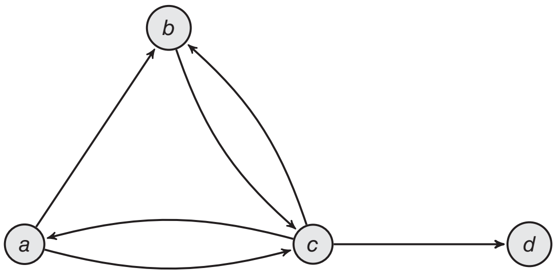

Figure 3.17

3.32 Let $X_0, X_1, \ldots$ be an ergodic Markov chain with transition matrix $\boldsymbol{P}$ and stationary distribution $\boldsymbol{\pi}$. Define the bivariate process $Z_n = (X_n, X_{n-1})$, for $n \geq 1$, with $Z_0 = (X_0, X_0)$.

(a) Give an intuitive explanation for why $Z_0, Z_1, \ldots$ is a Markov chain.

(b) Determine the transition probabilities in terms of $\boldsymbol{P}$. That is, find

$$P(Z_n = (i, j) | Z_{n-1} = (s, t)).$$

(c) Find the limiting distribution.

3.33 Assume that $\boldsymbol{P}$ is a stochastic matrix. Show that if $\boldsymbol{P}^N$ is positive, then $\boldsymbol{P}^{N+m}$ is positive for all $m \geq 0$.

3.34 Let $\boldsymbol{P}$ be the transition matrix of an irreducible, but not necessarily ergodic, Markov chain. For $0 < p < 1$, let

$$\widetilde{\boldsymbol{P}} = p\boldsymbol{P} + (1 - p)\boldsymbol{I},$$

where $\boldsymbol{I}$ is the identity matrix. Show that $\widetilde{\boldsymbol{P}}$ is a stochastic matrix for an ergodic Markov chain with the same stationary distribution as $\boldsymbol{P}$. Give an intuitive description for how the $\widetilde{\boldsymbol{P}}$ chain evolves compared to the $\boldsymbol{P}$-chain.

3.35 Let $\boldsymbol{Q}$ be a $k \times k$ stochastic matrix. Let $\boldsymbol{A}$ be a $k \times k$ matrix each of whose entries is $1/k$. For $0 < p < 1$, let

$$\boldsymbol{P} = p\boldsymbol{Q} + (1 - p)\boldsymbol{A}.$$

Show that $\boldsymbol{P}$ is the transition matrix for an ergodic Markov chain.

3.36 Let $X_0, X_1, \ldots$ be an ergodic Markov chain on $\{1, \ldots, k\}$ with stationary distribution $\boldsymbol{\pi}$. Assume that the chain is in stationarity.

(a) Find $\text{Cov}(X_m, X_{m+n})$.

(b) Find $\lim_{n\to\infty} \text{Cov}(X_m, X_{m+n})$.

3.37 Show that all two-state Markov chains, except for the trivial chain whose transition matrix is the identity matrix, are time reversible.

3.38 You throw five dice and set aside those dice that are sixes. Throw the remaining dice and again set aside the sixes. Continue until you get all sixes.

(a) Exhibit the transition matrix for the associated Markov chain, where X_n is the number of sixes after n throws. See also Exercise 2.11.

(b) How many turns does it take, on average, before you get all sixes?

3.39 Show that if $X_0, X_1, \ldots$ is reversible, then for the chain in stationarity

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = P(X_n = i_0, X_{n-1} = i_1, \ldots, X_0 = i_n),$$

for all $i_0, i_1, \ldots, i_n$.

3.40 Consider a *biased random walk* on the n-cycle, which moves one direction with probability p and the other direction with probability $1 - p$. Determine whether the walk is time reversible.

$\boxed{\textbf{3.41}}$ Show that the Markov chain with transition matrix

$$\boldsymbol{P} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1/6 & 1/6 & 0 & 2/3 \\ 1/5 & 2/5 & 2/5 & 0 \\ 0 & 1/3 & 1/6 & 1/2 \\ 4/9 & 0 & 1/3 & 2/9 \end{pmatrix} \end{array}$$

is reversible. The chain can be described by a random walk on a weighted graph. Exhibit the graph such that all the weights are integers.

$\boxed{\textbf{3.42}}$ Consider random walk on $\{0, 1, 2, \ldots\}$ with one reflecting boundary. If the walk is at 0, it moves to 1 on the next step. Otherwise, it moves left, with probability p, or right, with probability $1 - p$. For what values of p is the chain reversible? For such p, find the stationary distribution.

$\boxed{\textbf{3.43}}$ A Markov chain has transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 1/2 & 1/4 & 0 & 1/4 \\ p & 0 & 1-p & 0 \\ 0 & 1/4 & 1/2 & 1/4 \\ q & 0 & 1-q & 0 \end{pmatrix}.$$

(a) For what values of p and q is the chain ergodic?

(b) For what values of p and q is the chain reversible?

3.44 Markov chains are used to model nucleotide substitutions and mutations in DNA sequences. Kimura gives the following transition matrix for such a model.

$$\boldsymbol{P} = \begin{array}{c} \\ a \\ g \\ c \\ t \end{array} \begin{array}{c} \begin{array}{cccc} a & g & c & t \end{array} \\ \begin{pmatrix} 1-p-2r & p & r & r \\ p & 1-p-2r & r & r \\ q & q & 1-p-2q & p \\ q & q & p & 1-p-2q \end{pmatrix} \end{array}.$$

Find a vector $\boldsymbol{x}$ that satisfies the detailed-balance equations. Show that the chain is reversible and find the stationary distribution. Confirm your result for the case $p = 0.1$, $q = 0.2$, and $r = 0.3$.

3.45 If $\boldsymbol{P}$ is the transition matrix of a reversible Markov chain, show that $\boldsymbol{P}^2$ is, too. Conclude that $\boldsymbol{P}^n$ is the transition matrix of a reversible Markov chain for all $n \geq 1$.

3.46 Given a Markov chain with transition matrix $\boldsymbol{P}$ and stationary distribution $\boldsymbol{\pi}$, the *time reversal* is a Markov chain with transition matrix $\widetilde{\boldsymbol{P}}$ defined by

$$\widetilde{P}_{ij} = \frac{\pi_j P_{ji}}{\pi_i}, \quad \text{for all } i, j.$$

(a) Show that a Markov chain with transition matrix $\boldsymbol{P}$ is reversible if and only if $\boldsymbol{P} = \widetilde{\boldsymbol{P}}$.

(b) Show that the time reversal Markov chain has the same stationary distribution as the original chain.

3.47 Consider a Markov chain with transition matrix

$$\boldsymbol{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & \quad 2 & \quad 3 \end{array} \\ \begin{pmatrix} 1/3 & 0 & 2/3 \\ 1/2 & 1/2 & 0 \\ 1/6 & 1/3 & 1/2 \end{pmatrix} \end{array}.$$

Find the transition matrix of the time reversal chain (see Exercise 3.46).

3.48 Consider a Markov chain with transition matrix

$$\boldsymbol{P} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & \quad b & \quad c \end{array} \\ \begin{pmatrix} 1-\alpha & \alpha & 0 \\ 0 & 1-\beta & \beta \\ \gamma & 0 & 1-\gamma \end{pmatrix} \end{array},$$

where $0 < \alpha, \beta, \gamma < 1$. Find the transition matrix of the time reversal chain (see Exercise 3.46).

3.49 Consider an absorbing chain with t transient and $k - t$ absorbing states. For transient state i and absorbing state j, let B_{ij} denote the probability starting at i that the chain is absorbed in j. Let $\boldsymbol{B}$ be the resulting $t \times (k - t)$ matrix. By first-step analysis show that $\boldsymbol{B} = (\boldsymbol{I} - \boldsymbol{Q})^{-1}\boldsymbol{R}$.

3.50 Consider the following method for shuffling a deck of cards. Pick two cards from the deck uniformly at random and then switch their positions. If the same two cards are chosen, the deck does not change. This is called the *random transpositions* shuffle.

(a) Argue that the chain is ergodic and the stationary distribution is uniform.

(b) Exhibit the 6×6 transition matrix for a three-card deck.

(c) How many shuffles does it take on average to reverse the original order of the deck of cards?

3.51 A deck of k cards is shuffled by the *top-to-random* method: the top card is placed in a uniformly random position in the deck. (After one shuffle, the top card stays where it is with probability $1/k$.) Assume that the top card of the deck is the ace of hearts. Consider a Markov chain where X_n is the position of the ace of hearts after n top-to-random shuffles, with $X_0 = 1$. The state space is $\{1, \ldots, k\}$. Assume that $k = 6$.

(a) Exhibit the transition matrix and find the expected number of shuffles for the ace of hearts to return to the top of the deck.

(b) Find the expected number of shuffles for the bottom card to reach the top of the deck.

3.52 The board for a modified Snakes and Ladder game is shown in Figure 3.18. The game is played with a tetrahedron (four-faced) die.

(a) Find the expected length of the game.

(b) Assume that the player is on square 6. Find the probability that they will find themselves on square 3 before finishing the game.

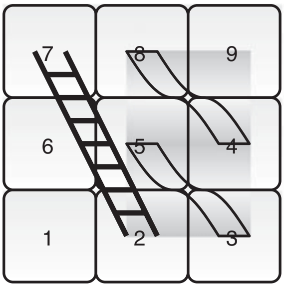

Figure 3.18

3.53 When an NFL football game ends in a tie, under *sudden-death* overtime the two teams play at most 15 extra minutes and the team that scores first wins the game. A Markov chain analysis of sudden-death is given in Jones (2004). Assuming two teams A and B are evenly matched, a four-state absorbing Markov chain is given with states *PA*: team A gains possession, *PB*: team B gains possession, *A*: A wins, and *B*: B wins. The transition matrix is

$$\boldsymbol{P} = \begin{array}{c} \\ PA \\ PB \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cccc} PA & PB & A & B \end{array} \\ \begin{pmatrix} 0 & 1-p & p & 0 \\ 1-p & 0 & 0 & p \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array},$$

where p is the probability that a team scores when it has the ball. Which team first receives the ball in overtime is decided by a coin flip.

(a) If team A receives the ball in overtime, find the probability that A wins.

(b) An alternate overtime procedure is the *first-to-six rule*, where the first time to score six points in overtime wins the game. Consider two evenly matched teams. Let α be the probability that a team scores a touchdown (six points). Let β be the probability that a team scores a field goal (three points). Assume for simplicity that touchdowns and field goals are the only way points can be scored. Develop a 10-state Markov chain model for overtime play.

(c) For the 2002 regular NFL season, there were 6,049 possessions, 1,270 touchdowns, and 737 field goals. Using these data compare the probability that A wins the game for each of the two overtime procedures.

3.54 A mouse is placed in the maze in Figure 3.19 starting in box A. A piece of cheese is put in box I. From each room the mouse moves to an adjacent room through an open door, choosing from the available doors with equal probability.

(a) How many rooms, on average, will the mouse visit before it finds the cheese?

(b) How many times, on average, will the mouse visit room A before it finds the cheese?

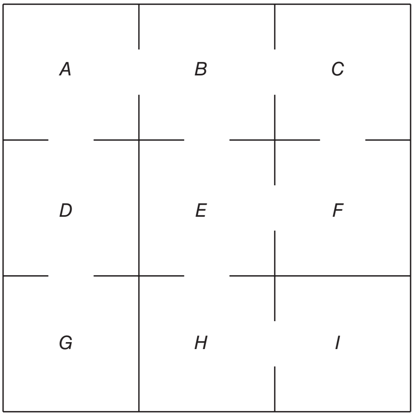

Figure 3.19 Mouse in a maze.

3.55 In a sequence of fair coin flips, how many flips, on average, are required to first see the pattern H-H-T-H?

3.56 A biased coin has heads probability 1/3 and tails probability 2/3. If the coin is tossed repeatedly, find the expected number of flips required until the pattern H-T-T-H-H appears.

3.57 In repeated coin flips, consider the set of all three-element patterns:

$$\{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}.$$

Which patterns take the longest time, on average, to appear in repeated sampling? Which take the shortest?

3.58 A sequence of 0s and 1s is generated by a Markov chain with transition matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 1/4 & 3/4 \\ 3/4 & 1/4 \end{pmatrix} \end{array}.$$

The first element of the sequence is decided by a fair coin flip. On average, how many steps are required for the pattern 0-0-1-1 to first appear?

3.59 Consider random walk on the weighted graph in Figure 3.20.

(a) If the walk starts in a, find the expected number of steps to return to a.

(b) If the walk starts in a, find the expected number of steps to first hit b.

(c) If the walk starts in a, find the probability that the walk hits b before c.

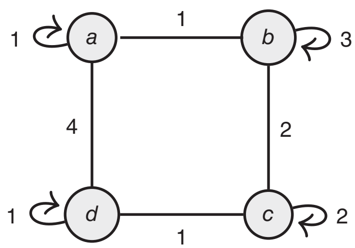

Figure 3.20

3.60 For a Markov chain started in state i, let T denote the *fifth time* the chain visits state i. Is T a stopping time? Explain.

3.61 Consider the weather Markov chain $X_0, X_1, \ldots$ of Example 2.3. Let T be the first time that it rains for 40 days in a row. Is $X_T, X_{T+1}, X_{T+2}, \ldots$ a Markov chain? Explain.

3.62 Let S be a random variable that is constant, with probability 1, where that constant is some positive integer. Show that S is a stopping time. Conclude that the Markov property follows from the strong Markov property.

3.63 `R` : Hourly wind speeds in a northwestern region of Turkey are modeled by a Markov chain in Sahin and Sen (2001). Seven wind speed levels are the states