Direct Formulation and Minimum Total PE Formulation

Review: Basic FEM Steps

- Preprocessing Phase
  - Step 1: *Discretization* $\rightarrow$ subdivide the problem into nodes and elements
  - Step 2: *Shape Function* $\rightarrow$ assume a function to represent the physical behavior of the element
  - Step 3: *Element Equations* $\rightarrow$ develop the mathematical equation for each element
  - Step 4: *Assembly* $\rightarrow$ element equations throughout the FEM mesh are assembled into a global matrix, modeling the system properties
  - Step 5: *System Constraints* $\rightarrow$ apply boundary conditions, initial conditions, and loading
- Solution Phase
  - Step 6: *Solve for Primary Unknowns* $\rightarrow$ the global equation matrix is solved for the results at each node
- Postprocessing Phase
  - Step 7: *Calculate Derived Variables* $\rightarrow$ determine other information using the nodal values of the primary variables

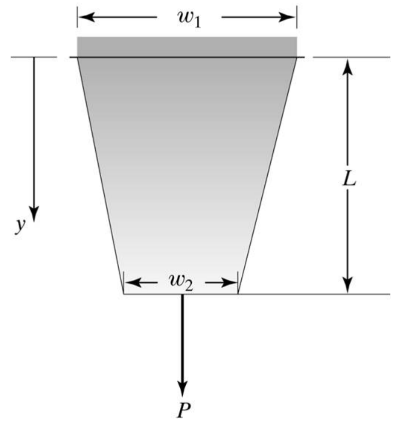
Approaches to Formulating FE Problems

- Direct Formulation
- Minimum Total Potential Energy Formulation
- Weighted Residual Formulation
  - Collocation Method
  - Subdomain Method
  - Galerkin's Method
  - Least Squares Method
  - Variational Method

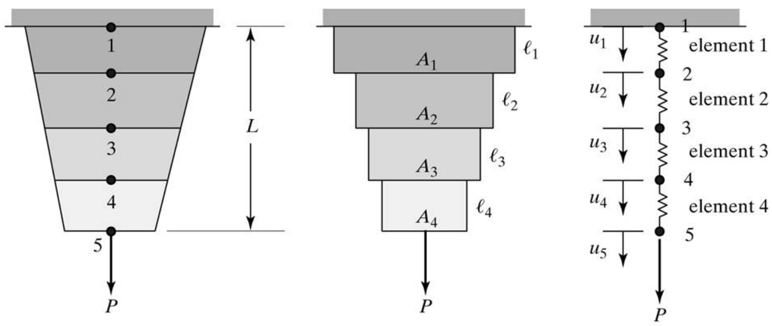
Now which to use? It depends on the application. Remember these are being used on much more complicated models that may have 1000's or 10000's (or more!) of nodes. The best approach is the easiest, the fastest, and most accurate (i.e., forces the error to zero at one point of interest, average error, etc.) but usually it is a trade-off between all these criteria. This determination goes beyond this course, but it is a valuable consideration to keep in mind.

1 Direct Formulation: Ex. 1.1

- Bar with variable cross section
  - Fixed at upper end
  - Supports load P = 1000 lb
  - $E = 10.4 \times 10^6$ lb/in^2
- Dimensions
  - $w_1 = 2$ in
  - $w_2 = 1$ in
  - t = 0.125 in
  - L = 10 in
- Desired information (Weight is negligible)
  - Deflection (u)
  - Stress (σ)



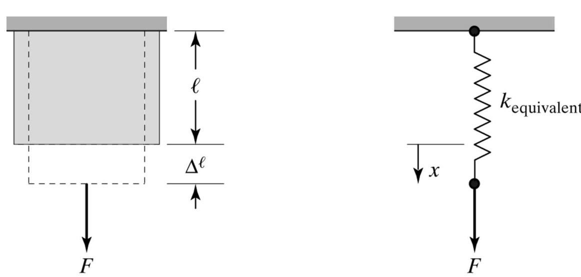
Step 1: Discretize the Solution Domain



- Subdivide the problem
  - Four elements
  - Five nodes
  - Accuracy increases with more nodes and elements
- Element area
  - Constant cross section
  - Based on average area of nodes defining given element
- u_i : displacement distance of node i

Step 2: Develop a Shape Function
Model the Physical Behavior of each Element

For each element:



Average Normal Stress

$$\sigma = \frac{F}{A}$$

Average Normal Strain

$$\varepsilon = \frac{\Delta\ell}{\ell}$$

1.1 Elastic Equation Developement

If linearly elastic, then Hooke's Law applies: $\sigma = E\varepsilon$

Substituting:

$$\frac{F}{A} = E\left(\frac{\Delta\ell}{\ell}\right)$$

$$F = \left(\frac{AE}{\ell}\right)\Delta\ell$$

Since, for springs... $F = kx$

...we can see that elements are elastic springs with an equivalent stiffness or constant of...

$$k_{eq} = \frac{AE}{\ell}$$

Element: Spring Model

Modeling each element as a spring: $f = k_{eq}(u_{i+1} - u_i)$

$f \equiv$ Element Force

$u_i \equiv$ Displacement of node i

$$f = \frac{A_{avg}E}{\ell}(u_{i+1} - u_i)$$

$$f = \left(\frac{A_{i+1} + A_i}{2}\right)\frac{E}{\ell}(u_{i+1} - u_i)$$

finally we get...

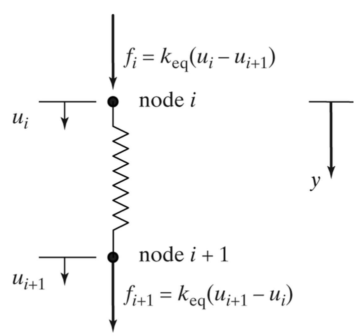
$$k_{eq} = \frac{(A_{i+1} + A_i)E}{2\ell}$$

where:

- $k_{eq} \equiv$ Element stiffness
- $\ell \equiv$ Length of each element
- $A_i, A_{i+1} \equiv$ Cross-sectional areas at nodes i and $i+1$

Step 3: Develop Element Equations

Free Body Diagram of each Element:



where... $f_i, f_{i+1} \equiv$ Forces on elements at nodes i and $i+1$

Initial Matrix Assembly

Knowing:

$$f_i = k_{eq}(u_i - u_{i+1}) \rightarrow f_i = k_{eq}u_i - k_{eq}u_{i+1}$$

$$f_{i+1} = k_{eq}(u_{i+1} - u_i) \rightarrow f_{i+1} = -k_{eq}u_i + k_{eq}u_{i+1}$$

Assemble in matrix form for each element:

$$\begin{Bmatrix} f_i \\ f_{i+1} \end{Bmatrix} = \begin{bmatrix} k_{eq} & -k_{eq} \\ -k_{eq} & k_{eq} \end{bmatrix} \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix}$$

Note the form:

$$\{\text{Load Matrix}\} = [\text{Stiffness Matrix}]\{\text{Displacement Matrix}\}$$

Step 4: Assemble Global Stiffness Matrix
Element Stiffness Matrix:
For Element 1:

$$[\mathbf{K}]^1 = \begin{array}{c} \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ u_2 \end{array}$$

For Element 2:

$$[\mathbf{K}]^2 = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{array}{c} u_2 \\ u_3 \end{array}$$

...etc.
We can now assemble our Global Matrix Equation in either of two ways:

$$\{\text{Load Matrix}\} = [\text{Stiffness Matrix}]\{\text{Displacement Matrix}\}$$

OR

$$[\text{Stiffness Matrix}]\{\text{Displacement Matrix}\} = \{\text{Load Matrix}\}$$

Global Form of Element Matrix

$$\{\textbf{Displacement Matrix}\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}$$

Assemble pieces of Global Stiffness Matrix for each element:

$$[\mathbf{K}]^{1G} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array}$$

$$[\mathbf{K}]^{2G} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array}$$

...etc.

The Global Stiffness Matrix
Now combine (i.e., [K] = $[K]^{1G} + [K]^{2G} + [K]^{3G} + [K]^{4G}$)

$$[\mathbf{K}]^G = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix}$$

Step 5: Apply B.C. and Loads
Note that: $u_1 = 0, \quad P = 1000$. Therefore:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 10^3 \end{Bmatrix}$$

(Note: Top row of Stiffness Matrix was changed so that $u_1 = 0$)

Step 6: Solve Algebraic Equations

$$k_{eq} = \frac{(A_{i+1} + A_i)E}{2\ell}$$

$$k_1 = \frac{(0.21875\ in^2 + 0.25\ in^2)(10.4 \cdot 10^6\ lb/in^2)}{2(2.5\ in)} = 975 \cdot 10^3\ lb/in$$

$$k_2 = \frac{(0.1875\ in^2 + 0.21875\ in^2)(10.4 \cdot 10^6\ lb/in^2)}{2(2.5\ in)} = 845 \cdot 10^3\ lb/in$$

$$k_3 = \frac{(0.15625\ in^2 + 0.1875\ in^2)(10.4 \cdot 10^6\ lb/in^2)}{2(2.5\ in)} = 715 \cdot 10^3\ lb/in$$

$$k_4 = \frac{(0.125\ in^2 + 0.15625\ in^2)(10.4 \cdot 10^6\ lb/in^2)}{2(2.5\ in)} = 585 \cdot 10^3\ lb/in$$

Solving for u_i

Therefore $[K]^G$:

$$[K]^G = 10^3 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -975 & 975+845 & -845 & 0 & 0 \\ 0 & -845 & 845+715 & -715 & 0 \\ 0 & 0 & -715 & 715+585 & -585 \\ 0 & 0 & 0 & -585 & 585 \end{bmatrix} \frac{lb}{in}$$

Final matrix equation:

$$[K]^G \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1000 \end{Bmatrix}$$

Note: UNITS must be consistent!!!
Solving...

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.001026 \\ 0.002210 \\ 0.003608 \\ 0.005317 \end{Bmatrix} in$$

Step 7: Obtain Other Information
Stress in each element:

$$\sigma = \frac{f}{A_{avg}}$$

$$\sigma = \frac{k_{eq}(u_{i+1} - u_i)}{A_{avg}}$$

$$\sigma = \frac{\left(\frac{A_{avg}E}{\ell}\right)(u_{i+1} - u_i)}{A_{avg}}$$

$$\sigma = E\left(\frac{u_{i+1} - u_i}{\ell}\right)$$

where:

- $f \equiv$ Element Force
- $u_i \equiv$ Displacement of node

Note relationship to Hooke's Law:

$$\sigma = E\varepsilon \quad \therefore \varepsilon = \left(\frac{u_{i+1} - u_i}{\ell}\right)$$

Calculating Normal Stress

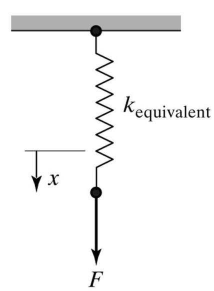
$$\sigma^{(1)} = \frac{(10.4 \cdot 10^6\ lb/in^2)(0.001026\ in - 0\ in)}{2.5\ in} = 4268\ lb/in^2$$

$$\sigma^{(2)} = \frac{(10.4 \cdot 10^6\ lb/in^2)(0.002210\ in - 0.001026\ in)}{2.5\ in} = 4925\ lb/in^2$$

$$\sigma^{(3)} = \frac{(10.4 \cdot 10^6\ lb/in^2)(0.003608\ in - 0.002210\ in)}{2.5\ in} = 5816\ lb/in^2$$

$$\sigma^{(4)} = \frac{(10.4 \cdot 10^6\ lb/in^2)(0.005317\ in - 0.003608\ in)}{2.5\ in} = 7109\ lb/in^2$$

Reaction Force: Method 1
FBD Element 1:



$$F = k_1(u_2 - u_1)$$

$$\sum F_Y = 0$$

$$\sum F_Y = F + R = 0$$

$$\sum F_Y = k_1(u_2 - u_1) + R = 0$$

$$\sum F_Y = k_1(u_2 - u_1) + R = 0$$

$$\mathbf{R = \text{-}1000\ lb}$$

Reaction Force: Method 2

- **Draw FBD of each node**
- **Write equilibrium equations for each node (i.e., $\sum F = 0$)**
- **Assemble into matrix equation of form...**

$$\{\text{React. Matrix}\} = [\text{Stiff. Matrix}]\{\text{Displace. Matrix}\} - \{\text{Ext. Load Matrix}\}$$

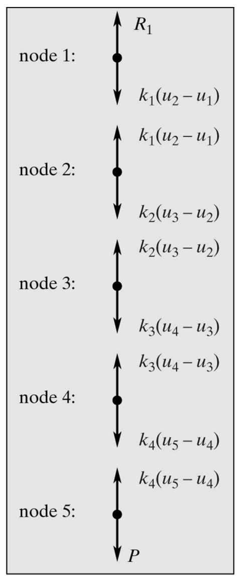
$$\{R\} = [K]\{u\} - \{F\}$$

- **Solve for External Reactions:**

$$\{R\} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ \dots \\ R_i \end{Bmatrix}$$

Method 2 Example: FBD

<u>FBD's of nodes:</u>



Method 2 Example: Equilibrium Equ.

- **Node 1:** $R_1 - k_1(u_2 - u_1) = 0$
- **Node 2:** $k_1(u_2 - u_1) - k_2(u_3 - u_2) = 0$
- **Node 3:** $k_2(u_3 - u_2) - k_3(u_4 - u_3) = 0$
- **Node 4:** $k_3(u_4 - u_3) - k_4(u_5 - u_4) = 0$
- **Node 5:** $k_4(u_5 - u_4) - P = 0$

Method 2 Example: Matrix Form

Rearranging:

$$k_1u_1 - k_1u_2 = -R$$

$$-k_1u_1 + k_1u_2 + k_2u_2 - k_2u_3 = 0$$

$$-k_2u_2 + k_2u_3 + k_3u_3 - k_3u_4 = 0$$

$$-k_3u_3 + k_3u_4 + k_4u_4 - k_4u_5 = 0$$

$$-k_4u_4 + k_4u_5 = -1000$$

In matrix form:

$$\begin{bmatrix} -k_1 & k_1 & & & \\ -k_1 & k_1+k_2 & -k_2 & & \\ & -k_2 & k_2+k_3 & -k_3 & \\ & & -k_3 & k_3+k_4 & -k_4 \\ & & & -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} R \\ 0 \\ 0 \\ 0 \\ -10^3 \end{Bmatrix}$$

Method 2 Example: Separate Forces
Note the form:

$$[\text{Stiffness Matrix}]\{\text{Displace. Matrix}\} = \{\text{Loads \& Reactions}\}$$

Important to distinguish loads from reactionsloads are known,reactions are not!
Therefore, separate loads and reactions:

$$[K]\{u\} = \{F\} + \{R\}$$

OR

$$\{R\} = [K]\{u\} - \{F\}$$

In this case solve for R:

$$\{R\} = \begin{Bmatrix} -P \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

2 Total Potential Energy Formulation

Potential Energy

$$\Pi = \sum_{e=1}^{n} \Lambda^{(e)} - \sum_{i=1}^{m} F_i u_i$$

where:

- $\Pi \equiv$ Total Potential Energy for a body or system with n elements and m nodes
- $\Lambda^{(e)} \equiv$ Strain energy in each element (Lambda)
- $F_i \equiv$ External Force @ Node i
- $u_i \equiv$ Displacement @ Node i

Therefore, the Potential Energy is the difference between the *Total Strain Energy* and the *Work* done by the External Forces.

$$\text{Work} = Fu$$

Not 1/2Fu, since force is a constant—like a weight moving over a distance

Strain Energy

$$\Lambda^{(e)} = \int_V \frac{1}{2}\sigma\varepsilon dV$$

AND

$$\Lambda^{(e)} = \int_V \frac{1}{2}E\varepsilon^2 dV$$

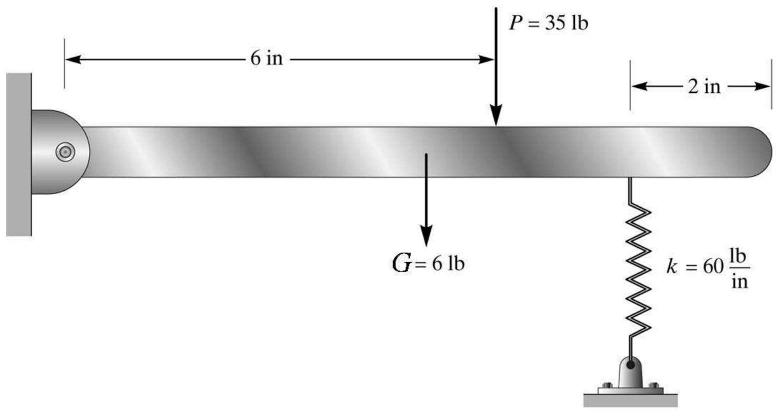
where $V \equiv$ Volume of member
Therefore, the *Strain Energy* is the area under the $\sigma - \varepsilon$ curve multiplied by the material *Volume*

Min. Total Potential Energy

$$\frac{\partial \Pi}{\partial u_i} = \frac{\partial}{\partial u_i}\sum_{e=1}^{n}\Lambda^{(e)} - \frac{\partial}{\partial u_i}\sum_{i=1}^{m}F_i u_i = 0 \qquad i = 1, 2, \ldots n$$

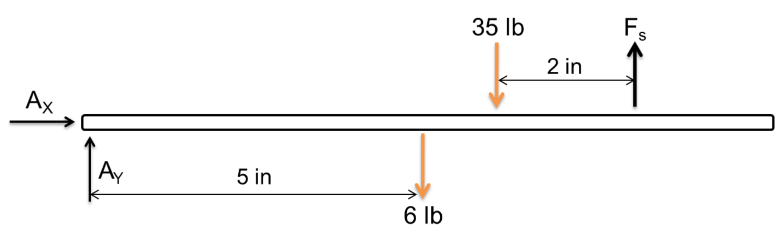
- For a *stable system*, the displacement at the equilibrium position occurs such that the value of the system's total potential energy is a minimum.
- This equation finds the *displacement* when the potential energy is at its minimum.

Min. Tot. Pot. Energy: Example



Find: Deflection of spring using:
(a) FBD and Statics
(b) Minimum Total Potential Energy

Example: Solution to (a)
<u>FBD</u>



$$\sum M_A = 0$$

$$\sum M_A = -(6\ lb)(5\ in) - (35\ lb)(6\ in) + F_s(8\ in) = 0$$

$$F_s = 30\ lb$$

$$F_s = kx$$

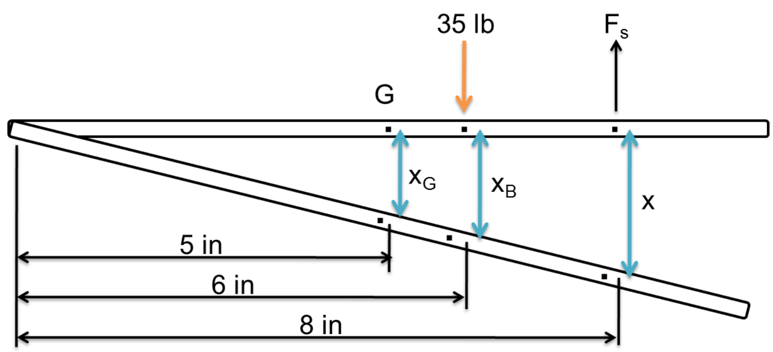
$$x = \frac{F_s}{k} = \frac{30\ lb}{60\ lb/in} = 0.500\ in$$

Example: (b) Strain Energy
Elastic Energy for the spring is:

$$\Lambda = \frac{1}{2}kx^2 = \frac{1}{2}(60\ lb/in)x^2$$

$$\Lambda = (30\ lb/in)x^2$$

Beam displacement:



Example: (b) Work
Similar Triangles:

$$\frac{x}{8} = \frac{x_G}{5}$$

$$x_G = \frac{5}{8}x$$

$$\frac{x}{8} = \frac{x_B}{6}$$

$$x_B = \frac{3}{4}x$$

Work of External Forces:

$$\sum F_i u_i = (6\ lb)x_G + (35\ lb)x_B = (6\ lb)\left(\frac{5}{8}x\right) + (35\ lb)\left(\frac{3}{4}x\right)$$

$$\sum F_i u_i = \frac{15\ lb}{4}x + \frac{105\ lb}{4}x = \frac{120\ lb}{4}x$$

$$\sum F_i u_i = (30\ lb)x$$

Example: Solution to (b)
Total Potential Energy of the System

$$\Pi = \sum \Lambda - \sum F_i u_i$$

$$\Pi = (30\ lb/in)x^2 - (30\ lb)x$$

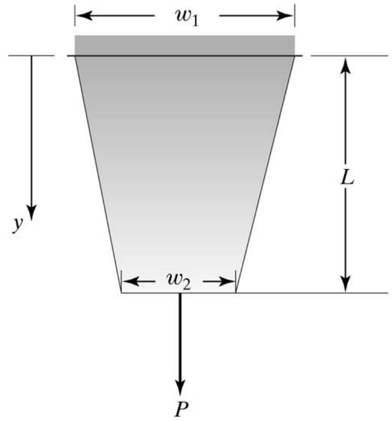
Minimize

$$\frac{\partial \Pi}{\partial x} = \frac{\partial}{\partial x}(30x^2 - 30x) = 0$$

$$60x - 30 = 0$$

$$\mathbf{x = 0.500\ in}$$

Example: Tapered Bar



Find:
Displacement (using the Minimum Total Potential Energy Formulation)

Tapered Bar: Strain Energy
Strain Energy:

$$\Lambda^{(e)} = \int_V \frac{1}{2} E \varepsilon^2 dV$$

$$\Lambda^{(e)} = \frac{1}{2} E \varepsilon^2 \int_V dV$$

$$\Lambda^{(e)} = \frac{1}{2} E \varepsilon^2 V$$

From before, strain for one element:

$$\varepsilon = \frac{(u_{i+1} - u_i)}{\ell}$$

And

$$V = A_{avg} \ell$$

$$\Lambda^{(e)} = \frac{1}{2} E \left(\frac{u_{i+1} - u_i}{\ell} \right)^2 (A_{avg} \ \ell)$$

$$\Lambda^{(e)} = \frac{A_{avg} E}{2\ell} (u_{i+1}^2 + u_i^2 - 2u_{i+1} u_i)$$

Tapered Bar: Min. Strain Energy
Minimizing Strain Energy:

$$\frac{\partial \Lambda^{(e)}}{\partial u_i} = \frac{\partial}{\partial u_i} \left(\frac{A_{avg} E}{2\ell} (u_{i+1}^2 + u_i^2 - 2u_{i+1} u_i) \right)$$

$$\frac{\partial \Lambda^{(e)}}{\partial u_i} = \frac{A_{avg} E}{2\ell} (0 + 2u_i - 2u_{i+1})$$

$$\frac{\partial \Lambda^{(e)}}{\partial u_i} = \frac{A_{avg} E}{\ell} (u_i - u_{i+1})$$

Also...

$$\frac{\partial \Lambda^{(e)}}{\partial u_{i+1}} = \frac{A_{avg} E}{\ell} (u_{i+1} - u_i)$$

Tapered Bar: Matrix Form and Forces

In Matrix Form

$$\begin{Bmatrix} \frac{\partial \Lambda^{(e)}}{\partial u_i} \\ \frac{\partial \Lambda^{(e)}}{\partial u_{i+1}} \end{Bmatrix} = \begin{bmatrix} k_{eq} & -k_{eq} \\ -k_{eq} & k_{eq} \end{bmatrix} \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix}$$

Where:

$$k_{eq} = \frac{A_{avg} E}{\ell}$$

Work of External Forces: $F_i u_i \qquad F_{i+1} u_{i+1}$

Minimizing External Work:

$$\frac{\partial}{\partial u_i}(F_i u_i) = F_i$$

$$\frac{\partial}{\partial u_{i+1}}(F_{i+1} u_{i+1}) = F_{i+1}$$

Tapered Bar: Min. Tot. Pot. Energy

Minimum Total Potential Energy:

$$\frac{\partial \Pi}{\partial u_i} = \frac{\partial}{\partial u_i} \sum_{e=1}^{n} \Lambda^{(e)} - \frac{\partial}{\partial u_i} \sum_{i=1}^{m} F_i u_i = 0 \qquad i = 1, 2, 3, \ldots n$$

$$[K]^{(G)} \{u\} - \{F\} = 0$$

$$[\mathbf{K}]^G = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix}$$

Tapered Bar: Solution

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}$$

$$\{F\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -10^3 \end{Bmatrix}$$

Row 1 of [K] matrix will change to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 10^3 \end{Bmatrix}$$

...due to B.C., $u_1 = 0$ **Same solution as Direct Formulation!!**