Lecture Notes on Problem-Solving Class: Advanced Mathematics B(II)

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Contents

Preface: Some Important Information

0.1 Course Assessment Information

- **Usual Performance:** Hand in assignments. It accounts for 20 points of the overall evaluation.
- **Mid-term Examination:** On the weekend of the 8th or 9th week. It accounts for 30 points of the overall evaluation.
- **Final Examination:** On Monday evening, June 9th. It accounts for 50 points of the overall evaluation.

0.2 Information of the Class and the Teaching Assistant

- **Teacher:** Wei Wang.
  - **E-mail:** 2201110024@stu.pku.edu.cn
  - **Personal Website:** https://luisyanka.github.io/weiwang.github.io/
- **Location:** Room 313, Teaching Building 2.
- **Students:** Students whose student ID numbers are greater than 2400011822 and less than or equal to 2400015443 should submit their assignments to Teacher Wei Wang's class.

0.3 Some Useful Links

We present some useful links associated with calculus.

- Lecture notes by Yantong Xie: https://darkoxie.github.io
- Mathstackexchange: https://math.stackexchange.com

0.4 Topics of the Class

In this problem-solving class, we will present some classical exercises related to the topics which are delivered by the lecturer in the main course. We will mainly refer to the lecture notes written by Yantong Xie, who was a very good teaching assistant of the course Advanced mathematics (B). We also refer to the book "Guide to Solving

Problems in Advanced Mathematics" by Jianying Zhou and Zhengyuan Li. If you have any advices for this class, then you can contact me with the e-mail. The lecture notes of this class will be updated before the next one on my personal website in the content of "teaching".

Chapter 1

Double integral

1.1 Calculating by the Definition

Example 1.1. There are three points P_0, P_1, P_2 on the plane, which are given by $\{(x_i, y_i)\}_{i=0}^2$. We assume that $x_2 > x_1 > x_0$ and $y_2 > y_1 > y_0$. Please calculate the area of triangle $\Delta P_0 P_1 P_2$.

Solution: Denote the triangle $\Delta P_0 P_1 P_2$ by D. By simple calculations, we can determine $P_0 P_1 : y = k_1 x + b_1$, $P_1 P_2 : y = k_2 x + b_2$, and $P_1 P_3 : y = k_3 x + b_3$. WLOG, we assume that $y_1 < k_3 x_1 + b_3$. As a result, we have

$$\begin{aligned}\int_D 1 \mathrm{d}x\mathrm{d}y &= \int_{x_0}^{x_1} \mathrm{d}x \int_{k_1 x + b_1}^{k_3 x + b_3} 1 \mathrm{d}y + \int_{x_0}^{x_1} \mathrm{d}x \int_{k_2 x + b_2}^{k_3 x + b_3} 1 \mathrm{d}y \\ &= \frac{1}{2}((y_2 - y_0)(x_1 - x_0) - y_1(x_2 - x_0)).\end{aligned}$$

Combined with the case that $y_1 \geq k_3 x_1 + b_3$, we obtain

$$A(D) = \frac{1}{2}|x_1 y_2 - x_1 y_0 - x_0 y_2 + x_0 y_0 - x_2 y_1 + x_0 y_1|.$$

Exercise 1.2. Let $A = [0, 1] \times [0, 1]$, find

$$I = \iint_A \frac{y \mathrm{d}x\mathrm{d}y}{(1 + x^2 + y^2)^{\frac{3}{2}}}.$$

Solution: Integrating with respect to y first and then with respect to x, we get

$$\begin{aligned}I &= \int_0^1 \mathrm{d}x \int_0^1 \frac{y \mathrm{d}y}{(1 + x^2 + y^2)^{\frac{3}{2}}} \\ &= \int_0^1 \left(\frac{1}{\sqrt{x^2 + 1}} - \frac{1}{\sqrt{x^2 + 2}} \right) \mathrm{d}x = \ln \frac{2 + \sqrt{2}}{1 + \sqrt{3}}.\end{aligned}$$

1.2 Iterated Integrals

Example 1.3. Calculate $\int_0^1 \int_y^1 \frac{y}{\sqrt{1+x^3}} \mathrm{d}x$.

Solution: Let

$$D := \{(x, y) \in \mathbb{R}^2 : y \leq x,\ x \in [0, 1],\ y \in [0, 1]\}.$$

We have

$$\int_0^1 \mathrm{d}y \int_y^1 \frac{y}{\sqrt{1+x^3}}\mathrm{d}x = \int_D \frac{y}{\sqrt{1+x^3}}\mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^x \frac{y}{\sqrt{1+x^3}}\mathrm{d}y$$
$$= \frac{1}{6}\int_0^1 \frac{\mathrm{d}t}{\sqrt{1+t}} = \frac{1}{3}(\sqrt{2}-1),$$

where for the third inequality, we have used $t = x^3$.

Exercise 1.4. Calculate $\int_0^1 \frac{x-1}{\ln x}\mathrm{d}x$.

Solution: We note

$$\int_0^1 \frac{x-1}{\ln x}\mathrm{d}x = \int_0^1 \mathrm{d}x \int_0^1 x^y \mathrm{d}y = \int_{[0,1]^2} x^y \mathrm{d}x\mathrm{d}y$$
$$= \int_0^1 \mathrm{d}y \int_0^1 x^y \mathrm{d}x = \int_0^1 \frac{1}{y+1}\mathrm{d}y = \ln 2.$$

Exercise 1.5. Suppose that f is continuous on $[0,1]$. Prove that:

$$\int_0^1 \mathrm{d}x \int_x^1 f(t)\mathrm{d}t = \int_0^1 tf(t)\mathrm{d}t.$$

1.3 Change of Variables

Example 1.6 (Observing the region)**.** The region $D \subset \mathbb{R}^2$ is surrounded by the curves $xy = a$, $xy = b$, $y = px$, and $y = qx$, where $0 < a < b$ and $0 < p < q$. Please calculate

$$I = \iint_D xy^3 \mathrm{d}x\mathrm{d}y.$$

Solution: Consider a change of variables as

$$\begin{cases} x' = \dfrac{y}{x}, \\ y' = xy. \end{cases}$$

We can calculate that

$$\left|\frac{\partial(x,y)}{\partial(x',y')}\right| = -\frac{1}{2x'}.$$

As a result,

$$I = s\int_p^q \left(\int_a^b x'(y')^2 \cdot \frac{1}{2x'}\mathrm{d}x'\right)\mathrm{d}y' = \frac{2(b^3-a^3)(q-p)}{3}$$

Example 1.7 (Rotation)**.** Calculate $\int_D |3x+4y|\mathrm{d}x\mathrm{d}y$, where

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Solution: Consider a change of variables as

$$\begin{cases} x' = \dfrac{4}{5}x - \dfrac{3}{5}y, \\ y' = \dfrac{3}{5}x + \dfrac{4}{5}y. \end{cases}$$

Indeed, formula above give a rotation, which preserve D and change the line $3x + 4y = t$ to $y' = t$ for any $t \in \mathbb{R}$. As a result, we have

$$\int_D |3x+4y|\mathrm{d}x\mathrm{d}y = 5\int_D |y'|\mathrm{d}x'\mathrm{d}y' = \frac{20}{3}.$$

Question 1.8. Can you give the intuition behind this change of variables?

Example 1.9 (Polar coordinate)**.** Make a polar coordinate transformation to convert the double integral

$$\iint_D f(\sqrt{x^2+y^2})\mathrm{d}x\mathrm{d}y$$

into a definite integral, where $D = \{(x,y) : 0 \leq y \leq x \leq 1\}$.

Solution: Let $x = r\cos\varphi$ and $y = r\sin\varphi$. Then

$$\begin{aligned} \iint_D f(\sqrt{x^2+y^2})\mathrm{d}x\mathrm{d}y = &\iint_D f(r)r\mathrm{d}r\mathrm{d}\varphi \\ =&\int_0^1 \mathrm{d}r \int_0^{\pi/4} f(r)r\mathrm{d}\varphi + \int_1^{\sqrt{2}} \mathrm{d}r \int_{\arccos(1/r)}^{\pi/4} f(r)r\mathrm{d}\varphi \\ =&\frac{\pi}{4}\int_0^1 f(r)r\mathrm{d}r + \int_1^{\sqrt{2}} \left(\frac{\pi}{4} - \arccos\frac{1}{r}\right) f(r)r\mathrm{d}r \\ =&\frac{\pi}{4}\int_0^{\sqrt{2}} f(r)r\mathrm{d}r - \int_1^{\sqrt{2}} \arccos\frac{1}{r} f(r)r\mathrm{d}r. \end{aligned}$$

Remark 1.10. Generally speaking, the generalized polar coordinate transformation

$$x = \frac{1}{a}\left(c + r^{\frac{1}{p}}\cos^{\frac{2}{p}}\theta\right), \quad y = \frac{1}{b}\left(d + r^{\frac{1}{p}}\sin^{\frac{2}{p}}\theta\right),$$

can transform $(ax-c)^p + (by-d)^p$ into r. However, in general, r and θ no longer have the meanings of the usual polar radius and polar angle.

Exercise 1.11. Find

$$\iint_D \left(\sqrt{\frac{x-c}{a}} + \sqrt{\frac{y-c}{b}}\right) \mathrm{d}x\mathrm{d}y,$$

where D is the region bounded by the curve $\sqrt{\frac{x-c}{a}} + \sqrt{\frac{y-c}{b}} = 1$, $x = c$, and $y = c$, and $a, b, c > 0$.

Solution: Let

$$x = c + a\rho\cos^4\theta, \quad y = c + b\rho\sin^4\theta$$

Then

$$J = \left|\frac{\partial(x,y)}{\partial(\rho,\theta)}\right| = 4ab\rho\cos^3\theta\sin^3\theta$$

And the integration region becomes $\left\{0 \le \theta \le \frac{\pi}{2}, 0 \le \rho \le 1\right\}$. Thus

$$\iint_D \left(\sqrt{\frac{x-c}{a}} + \sqrt{\frac{y-c}{b}}\right) \mathrm{d}x\mathrm{d}y = \int_0^{\pi/2} \mathrm{d}\theta \int_0^1 4ab\rho\cos^3\theta\sin^3\theta\sqrt{\rho}\mathrm{d}\rho = \frac{2ab}{15}$$

Exercise 1.12. Find

$$\lim_{R\to+\infty} \iint_{|x|\le R,\ |y|\le R} (x^2+y^2)\mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y.$$

Solution: Let

$$I_R = \iint_{|x|\le R,\ |y|\le R} \left(x^2+y^2\right)\mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y,$$
$$C_R = \iint_{x^2+y^2\le R^2} (x^2+y^2)\mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y$$

Then $C_R \le I_R \le C_{2R}$, and

$$C_R = \int_0^{2\pi} \mathrm{d}\theta \int_0^R r^3\mathrm{e}^{-r^2}\mathrm{d}r = \pi\int_0^{R^2} t\mathrm{e}^{-t}\mathrm{d}t = \pi(1-\mathrm{e}^{-R^2} - R^2\mathrm{e}^{-R^2}) \to \pi \quad (R\to+\infty)$$

Similarly, we can prove that $C_{2R} \to \pi$ as $R \to +\infty$. Thus,

$$\lim_{R\to+\infty} I_R = \pi$$

Exercise 1.13. Assume that $f \in C[-1,1]$, show that

$$\int_{|x|+|y|\le 1} f(x+y)\mathrm{d}x\mathrm{d}y = \int_{-1}^1 f(z)\mathrm{d}z.$$

Hint: Consider a change of variables as

$$\begin{cases} x' = x - y, \\ y' = x + y. \end{cases}$$

Exercise 1.14. Given the integral

$$I = \iint_D \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right] \mathrm{d}x \mathrm{d}y.$$

Define a transformation $x = x(u, v)$, $y = y(u, v)$, and the region D is transformed into Ω. Assume that the transformation satisfies

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}, \quad \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}.$$

Prove that:

$$I = \iint_\Omega \left[\left(\frac{\partial f}{\partial u} \right)^2 + \left(\frac{\partial f}{\partial v} \right)^2 \right] \mathrm{d}u \mathrm{d}v.$$

1.4 Symmetry

The parity of a function and the symmetry of the integration region can often be used to simplify the calculation of integrals. For example:

1. If the integration region D is symmetric about the x-axis:
   - If $f(x, y) = -f(x, -y)$, then
   $$\iint_D f(x, y) \mathrm{d}x \mathrm{d}y = 0.$$
   - If $f(x, y) = f(x, -y)$, then
   $$\iint_D f(x, y) \mathrm{d}x \mathrm{d}y = 2 \iint_{D \cap \{y \geq 0\}} f(x, y) \mathrm{d}x \mathrm{d}y.$$
2. If the integration region D is symmetric about the y-axis:
   - If $f(x, y) = -f(-x, y)$, then
   $$\iint_D f(x, y) \mathrm{d}x \mathrm{d}y = 0.$$
   - If $f(x, y) = f(-x, y)$, then
   $$\iint_D f(x, y) \mathrm{d}x \mathrm{d}y = 2 \iint_{D \cap \{x \geq 0\}} f(x, y) \mathrm{d}x \mathrm{d}y.$$

3. If D is symmetric about the origin:
   - If $f(x,y) = -f(-x,-y)$, then
   $$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = 0.$$
   - If $f(x,y) = f(-x,-y)$, then
   $$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = 2\iint_{D_1} f(x,y)\mathrm{d}x\mathrm{d}y,$$
   where D_1 is half of the region D.

Example 1.15. Show that:

$$\iint_{|x|+|y|\leq 1}(\sqrt{|xy|}+|xy|)\mathrm{d}x\mathrm{d}y \leq \frac{3}{2}.$$

Solution: By the symmetric property, we have

$$\iint_{|x|+|y|\leq 1}(\sqrt{|xy|}+|xy|)\mathrm{d}x\mathrm{d}y = 4\iint_{x+y\leq 1,\ x\geq 0,\ y\geq 0}(\sqrt{xy}+xy)\mathrm{d}x\mathrm{d}y.$$

By direct calculations, the property holds.

1.5 Applications to Proving Integral Inequalities

Example 1.16. Assume that $a<b$ and $f,g\in C[a,b]$, show that

$$\left(\int_a^b fg\right)^2 \leq \left(\int_a^b f^2\right)\left(\int_a^b g^2\right).$$

Solution: Note that

$$\int_{[a,b]^2}(f(x)g(y)-f(y)g(x))^2\mathrm{d}x\mathrm{d}y \geq 0.$$

By expanding those in the bracket, the inequality follows directly.

Remark 1.17. The key point in the proof above is to note that the integral variables x and y have the same status.

Exercise 1.18. Let $f\in C[0,1]$ be a positive and non-increasing function. Show that

$$\frac{\int_0^1 xf^2(x)\mathrm{d}x}{\int_0^1 xf(x)\mathrm{d}x} \leq \frac{\int_0^1 f^2(x)\mathrm{d}x}{\int_0^1 f(x)\mathrm{d}x}.$$

Solution: It is a direct result from the claim

$$\int_{[0,1]^2} yf(x)f(y)(f(x)-f(y))\mathrm{d}x\mathrm{d}y \geq 0.$$

Consider a change of variables as

$$\begin{cases} x' = x+y, \\ y' = x-y. \end{cases}$$

Define

$$D := \{(x',y') \in \mathbb{R}^2 : 0 \leq x'+y' \leq 2,\ 0 \leq x'-y' \leq 2\}.$$

We see that D is symmetric about the y-axis. Consequently, we have

$$\int_{[0,1]^2} yf(x)f(y)(f(x)-f(y))\mathrm{d}x\mathrm{d}y = \int_D (x'-y')g(x',y')\mathrm{d}x'\mathrm{d}y',$$

where

$$g(x',y') = \frac{1}{2}f\left(\frac{x'+y'}{2}\right)f\left(\frac{x'-y'}{2}\right)\left[f\left(\frac{x'+y'}{2}\right) - f\left(\frac{x'-y'}{2}\right)\right].$$

Obviously, there holds

$$g(x',y') = -g(x',-y') \geq 0$$

if $(x',y') \in D$ and $y' \leq 0$. This implies that

$$\begin{aligned}\int_{D\cap\{y'<0\}} (x'-y')g(x',y')\mathrm{d}x'\mathrm{d}y' &= \int_{D\cap\{y'\geq 0\}} (x'+y')g(x',-y')\mathrm{d}x'\mathrm{d}y' \\ &= -\int_{D\cap\{y'\geq 0\}} (x'+y')g(x',y')\mathrm{d}x'\mathrm{d}y'.\end{aligned}$$

Here, for the first inequality, we have used the change of varaible that sends y' to $-y'$. Moreover, we obtain

$$\begin{aligned}\int_D (x'-y')g(x',y')\mathrm{d}x'\mathrm{d}y' &= \left(\int_{D\cap\{y'\geq 0\}} + \int_{D\cap\{y'<0\}}\right)(x'-y')g(x',y')\mathrm{d}x'\mathrm{d}y' \\ &= \int_{D\cap\{y'\geq 0\}} ((x'-y')-(x'+y'))g(x',y')\mathrm{d}x'\mathrm{d}y' \\ &= -\int_{D\cap\{y'\geq 0\}} 2y'g(x',y')\mathrm{d}x'\mathrm{d}y'.\end{aligned}$$

We see that if $y' \geq 0$, then it follows from the property that f is non-increasing that $g(x',y') \leq 0$. As a result,

$$-\int_{D\cap\{y'\geq 0\}} 2y'g(x',y')\mathrm{d}x'\mathrm{d}y' \geq 0,$$

and the proof is completed.

Exercise 1.19. Assume that $f \in C[0,1]$ and $f > 0$. Show that

$$\left(\int_0^1 \frac{1}{f}\right)\left(\int_0^1 f\right) \geq 1.$$

Solution: We have

$$\begin{aligned}\left(\int_0^1 \frac{1}{f(x)}\mathrm{d}x\right)\left(\int_0^1 f(x)\mathrm{d}x\right) &= \left(\int_0^1 \frac{1}{f(x)}\mathrm{d}x\right)\left(\int_0^1 f(y)\mathrm{d}y\right)\\ &\geq \frac{1}{2}\int_{[0,1]^2}\left(\frac{f(x)}{f(y)}+\frac{f(y)}{f(x)}\right)\mathrm{d}x\mathrm{d}y \geq 1.\end{aligned}$$

1.6 Calculating the Area of the surfaces

1. Parametric representation of a surface: Let a surface S be given parametrically by $\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$, where (u,v) varies over a region D in the $uv-$plane.

   The partial derivatives of $\vec{r}$ with respect to u and v are $\vec{r}_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}$ and $\vec{r}_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}$.

   The cross product $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$

   The surface area $A(S)$ of the surface S is given by the double integral:

   $$A(S) = \iint_D \|\vec{r}_u \times \vec{r}_v\|\, dA = \iint_D \sqrt{\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2}dudv$$

   where $\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$, $\frac{\partial(y,z)}{\partial(u,v)} = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}$, and $\frac{\partial(z,x)}{\partial(u,v)} = \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}$.

2. Explicit Representation of a Surface If the surface S is given explicitly as $z = f(x,y)$, where (x,y) varies over a region D in the $xy-$plane. We can consider the parametric representation $\vec{r}(x,y) = x\vec{i} + y\vec{j} + f(x,y)\vec{k}$.

   The partial derivatives are $\vec{r}_x = \vec{i} + \frac{\partial f}{\partial x}\vec{k}$ and $\vec{r}_y = \vec{j} + \frac{\partial f}{\partial y}\vec{k}$.

   The cross product $\vec{r}_x \times \vec{r}_y = -\frac{\partial f}{\partial x}\vec{i} - \frac{\partial f}{\partial y}\vec{j} + \vec{k}$.

   The magnitude $\|\vec{r}_x \times \vec{r}_y\| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$.

   The surface area $A(S)$ of the surface S is then given by the double integral:

   $$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\mathrm{d}x\mathrm{d}y$$

3. Key Points and Considerations

   (a) Choice of representation: The choice between parametric and explicit representation depends on the nature of the surface. For surfaces like spheres, tori, etc., parametric representation is often more convenient. For surfaces that can be easily written as $z = f(x, y)$ (e.g., graphs of functions), the explicit form is straightforward to use.

   (b) Region of integration: Identifying the correct region D in the appropriate parameter plane (e.g., uv-plane for parametric surfaces or $xy-$plane for explicit surfaces) is crucial. The limits of integration for the double integral are determined by the boundaries of D.

   (c) Calculation of partial derivatives: Accurately computing the partial derivatives of the functions involved in the representation of the surface is essential. Any error in calculating the partial derivatives will lead to an incorrect result for the surface area.

   (d) Evaluation of the double integral: Once the integrand and the region of integration are determined, the double integral needs to be evaluated. This may involve techniques such as changing the order of integration, using polar coordinates (in the xy-plane for surfaces given as $z = f(x, y)$) or other coordinate transformations depending on the shape of the region D and the integrand.

In conclusion, double integrals provide a powerful tool for calculating the surface area of a wide variety of surfaces. By carefully choosing the appropriate representation of the surface, correctly identifying the region of integration, and accurately evaluating the resulting double integral, we can obtain the surface area of the given surface.

Example 1.20. Calculating the area of the set

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = a^2,\ x^2 + y^2 \leq ax\}.$$

Solution: First, rewrite the equations and find the relevant partial derivatives: The equation of the sphere is $x^2+y^2+z^2 = a^2$, so $z = \pm\sqrt{a^2 - x^2 - y^2}$. The partial derivatives are $\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{a^2-x^2-y^2}}$ and $\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{a^2-x^2-y^2}}$, and $\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} = \frac{a}{\sqrt{a^2-x^2-y^2}}$. The inequality $x^2+y^2 \leq ax$ in polar coordinates ($x = r\cos\theta, y = r\sin\theta$) becomes $r^2 \leq ar\cos\theta$, i.e., $r \leq a\cos\theta$ with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ (because $r \geq 0$).

Then, use the surface area formula for a surface $z = f(x, y)$: The surface area A of the surface $z = f(x, y)$ over a region D in the xy-plane is $A = 2\iint_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}\mathrm{d}x\mathrm{d}y$ (for the upper and lower hemispheres of the sphere). Substituting into polar coordinates , we have:

$$A = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{a\cos\theta} \frac{a}{\sqrt{a^2-r^2}} r\mathrm{d}r = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\left[-\sqrt{a^2-r^2}\right]_0^{a\cos\theta} d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\left(a - a|\sin\theta|\right)\mathrm{d}\theta = 2a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-|\sin\theta|)\mathrm{d}\theta = 2a^2\left(\pi-2\right)$$

Chapter 2

Triple Integrals and n-multiple Integrals

2.1 Calculation of Triple Integrals in Rectangular Coordinate

Example 2.1 (Consider the area of the section)**.** Find the integral

$$I = \iiint_{\Omega} z^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

where Ω is the common part of the two spheres $x^2+y^2+z^2 \leq R^2$ and $x^2+y^2+z^2 \leq 2Rz$.

Solution: Considering both the integrand and the integration region, the integral can be regarded as the sum of a series of small slices weighted by z^2 for $z \in [0, R]$. According to the composition of the integration region Ω, it can be divided into two sub-regions Ω_1 and Ω_2.

$$\Omega_1 : \begin{cases} x^2+y^2+z^2 \leq 2Rz, \\ 0 \leq z \leq \frac{R}{2}, \end{cases}$$
$$\Omega_2 : \begin{cases} x^2+y^2+z^2 \leq R^2, \\ \frac{R}{2} \leq z \leq R. \end{cases}$$

When $z \in [0, \frac{R}{2}]$, from $x^2+y^2+z^2 \leq 2Rz$, we see that the area of the slice can be obtained as $\pi(2Rz - z^2)$. On the other hand, when $z \in [\frac{R}{2}, R]$, for $x^2+y^2+z^2 \leq R^2$, the area of the slice can be obtained as $\pi(R^2 - z^2)$. As a result,

$$\begin{aligned} I &= \int_0^{R/2} \pi z^2(2Rz - z^2)\mathrm{d}z + \int_{R/2}^{R} \pi z^2(R^2 - z^2)\mathrm{d}z \\ &= \left(\frac{1}{2}\pi R z^4 - \frac{1}{5}\pi z^5\right)\Big|_0^{R/2} + \left(\frac{1}{3}\pi R^2 z^3 - \frac{1}{5}\pi z^5\right)\Big|_{R/2}^{R} = \frac{59}{480}\pi R^5 \end{aligned}$$

Exercise 2.2. Calculate the volume of the intersection of cylinders $x^2+y^2 \leq a^2$ and $x^2+z^2 \leq a^2$.

Solution:

1. **Cylinder** $x^2 + y^2 = a^2$: In the xy-plane, the equation $x^2 + y^2 = a^2$ represents a circle centered at the origin with radius a. When considering this equation in three-dimensional space, it represents a cylinder that extends infinitely along the z-axis.

2. **Cylinder** $x^2 + z^2 = a^2$: In the xz-plane, the equation $x^2 + z^2 = a^2$ represents a circle centered at the origin with radius a. In three-dimensional space, it represents a cylinder that extends infinitely along the y-axis.

3. **Intersection of the two cylinders**: The intersection of the cylinders $x^2 + y^2 \leq a^2$ and $x^2 + z^2 \leq a^2$ is a symmetric solid. The solid is symmetric about the x-axis, y-axis, and z-axis. For a fixed $x \in [-a, a]$, the cross-section of the intersection perpendicular to the x-axis is a square. The side length of the square s is given by $s = 2\sqrt{a^2 - x^2}$ (since from $x^2 + y^2 = a^2$, we have $y = \pm\sqrt{a^2 - x^2}$ and from $x^2 + z^2 = a^2$, we have $z = \pm\sqrt{a^2 - x^2}$).

The formula for the volume V of a solid with cross-sectional area $A(x)$ from $x = c$ to $x = d$ is $V = \int_c^d A(x)\mathrm{d}x$. Here, $c = -a$, $d = a$, and the cross-sectional area $A(x)$ of the intersection of the two cylinders perpendicular to the x-axis is $A(x) = (2\sqrt{a^2 - x^2}) \times (2\sqrt{a^2 - x^2}) = 4(a^2 - x^2)$ (because the cross-section is a square with side length $2\sqrt{a^2 - x^2}$).

Then, we calculate the integral:

$$V = \int_{-a}^{a} 4(a^2 - x^2)\mathrm{d}x = \frac{16}{3}a^3.$$

Example 2.3 (Projection to the plane)**.** Find

$$I = \iiint_\Omega (y^2 + z^2)\mathrm{d}V,$$

where Ω represents the region $0 \leq z \leq x^2 + y^2 \leq 1$.

<u>Solution:</u> In cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ and $\mathrm{d}V = r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta$, and $y^2 + z^2 = r^2\sin^2\theta + z^2$. The region $\Omega : 0 \leq z \leq x^2 + y^2 \leq 1$ becomes $0 \leq z \leq r^2$, $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$. Then,

$$\begin{aligned}
I &= \int_0^{2\pi}\int_0^1\int_0^{r^2} (r^2\sin^2\theta + z^2)r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi}\int_0^1 r\left[\int_0^{r^2}(r^2\sin^2\theta + z^2)\mathrm{d}z\right]\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^1 r\left(r^2\sin^2\theta z + \frac{z^3}{3}\Big|_0^{r^2}\right)\mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi}\int_0^1 r\left(r^4\sin^2\theta + \frac{r^6}{3}\right)\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^{2\pi}\left(\sin^2\theta\int_0^1 r^5\mathrm{d}r + \frac{1}{3}\int_0^1 r^7\mathrm{d}r\right)\mathrm{d}\theta = \int_0^{2\pi}\left(\frac{1}{6}\sin^2\theta + \frac{1}{24}\right)\mathrm{d}\theta = \frac{\pi}{4}.
\end{aligned}$$

Indeed, we also have

$$I = \int_{x^2+y^2\leq 1}\left(\int_0^{x^2+y^2}(y^2 + z^2)\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y = \frac{\pi}{4}.$$

2.2 Change of Variables for Triple Integrals

Similar to the change of variables for double integrals, there is the following theorem for variable substitution of triple integrals. Let

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w), \quad (u, v, w) \in \Omega'$$

This change of variables satisfies the following conditions:

- It establishes a one-to-one correspondence between Ω and Ω'.
- x, y, and z have continuous partial derivatives with respect to each variable in Ω', and the inverse transformations $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$ also have continuous partial derivatives with respect to each variable in Ω.
- The Jacobian determinant $J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$ has no zeros in Ω'. Then

$$\begin{aligned}&\iiint_{\Omega} f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &\qquad = \iiint_{\Omega'} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| \mathrm{d}u\mathrm{d}v\mathrm{d}w\end{aligned}$$

There are two commonly used transformations for triple integrals:

1. Cylindrical Coordinate Transformation

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad z = z$$
$$0 \le \rho < +\infty, \quad 0 \le \theta < 2\pi, \quad -\infty < z < +\infty$$

The relationship between the triple integral in the rectangular coordinate system and the triple integral in the cylindrical coordinate system is

$$\iiint_{\Omega} f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{\Omega'} f(\rho\cos\theta, \rho\sin\theta, z)\rho\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}z.$$

2. Spherical Coordinate Transformation

$$x = \rho\sin\varphi\cos\theta, \quad y = \rho\sin\varphi\sin\theta, \quad z = \rho\cos\varphi,$$
$$0 \le \rho < +\infty, \quad 0 \le \theta < 2\pi, \quad 0 \le \varphi \le \pi.$$

The relationship between the triple integral in the rectangular coordinate system and the triple integral in the spherical coordinate system is

$$\iiint_{\Omega} f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{\Omega'} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\varphi.$$

Example 2.4. Let $\Omega = \{(x, y, z) : 0 \leq x + y - z \leq 1, 0 \leq y + z - x \leq 1, 0 \leq x + z - y \leq 1\}$ be the region formed by the intersection of six planes. Find the triple integral

$$I = \iiint_\Omega (x + y - z)(y + z - x)(x + z - y)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Solution: Consider the change of variables as

$$\begin{cases} x' = x + y - z, \\ y' = y + z - x, \\ z' = x + z - y. \end{cases}$$

It is easy to find that the Jacobi determinant is $\frac{1}{4}$. As a result, we have

$$I = \int_{\substack{0\leq x'\leq 1, \\ 0\leq y'\leq 1, \\ 0\leq z'\leq 1}} \frac{1}{4} x'y'z'\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' = \frac{1}{32}.$$

Example 2.5. Calculate the integral

$$H = \iiint_{\substack{x,y,z\geq 0 \\ x^2+y^2+z^2\leq R^2}} \frac{xyz\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}}, \quad \text{where } a > b > c > 0.$$

Solution: In spherical coordinates,

$$H = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^R \frac{r^4 \sin^3 \varphi \cos \varphi \sin \theta \cos \theta \mathrm{d}r\mathrm{d}\varphi\mathrm{d}\theta}{\sqrt{a^2 \sin^2 \varphi \cos^2 \theta + b^2 \sin^2 \varphi \sin^2 \theta + c^2 \cos^2 \varphi}}.$$

Let $\sin^2 \varphi = u$, $\sin^2 \theta = v$. Then

$$\begin{aligned}
H &= \frac{1}{4} \int_0^1 \int_0^1 \int_0^R r^4 \frac{u\mathrm{d}r\mathrm{d}u\mathrm{d}v}{\sqrt{a^2u(1-v) + b^2uv + c^2(1-u)}} \\
&= \frac{1}{20} R^5 \int_0^1 u\mathrm{d}u \int_0^1 \frac{\mathrm{d}v}{\sqrt{[c^2 + (a^2 - c^2)\, u] + (b^2 - a^2)\, uv}} \\
&= \frac{1}{20} R^5 \int_0^1 \left\{ \frac{2}{(b^2 - a^2)\, u} \sqrt{[c^2 + (a^2 - c^2)\, u] + (b^2 - a^2)\, uv} \right\} \Bigg|_{v=0}^{v=1} u\mathrm{d}u \\
&= \frac{R^5}{10\, (b^2 - a^2)} \int_0^1 \left\{ \sqrt{[c^2 + (a^2 - c^2)\, u] + (b^2 - a^2)\, u} - \sqrt{c^2 + (a^2 - c^2)\, u} \right\} \mathrm{d}u \\
&= \frac{R^5}{10\, (b^2 - a^2)} \left\{ \frac{2}{3\, (b^2 - c^2)} \left[c^2 + \left(b^2 - c^2\right) u\right]^{\frac{3}{2}} - \frac{2}{3\, (a^2 - c^2)} \left[c^2 + \left(a^2 - c^2\right) u\right]^{\frac{3}{2}} \right\} \Bigg|_0^1 \\
&= \frac{R^5}{10\, (b^2 - a^2)} \left[\frac{2}{3\, (b^2 - c^2)} \left(b^3 - c^3\right) - \frac{2}{3\, (a^2 - c^2)} \left(a^3 - c^3\right) \right] \\
&= \frac{R^5}{15} \cdot \frac{1}{b^2 - a^2} \left(\frac{b^2 + bc + c^2}{b + c} - \frac{a^2 + ac + c^2}{a + c} \right) \\
&= \frac{R^5}{15} \cdot \frac{ab + bc + ca}{(a + b)(b + c)(c + a)}.
\end{aligned}$$

Example 2.6. Let $H(x) = \sum_{i,j=1}^{3} a_{ij}x_ix_j$, and $\mathbf{A} = (a_{ij})$ be a positive definite symmetric matrix of order 3. Find

$$I = \iiint_{H(x)\le 1} e^{\sqrt{H(x)}}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3.$$

Solution: There exists an orthogonal matrix $\mathbf{P}$ of order 3 such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

where $\lambda_i > 0$, $i = 1, 2, 3$. Make an orthogonal transformation $x = \mathbf{P}y$, where $x, y \in \mathbb{R}^3$. Then

$$H(x) = H(\mathbf{P}y) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2,$$

and the Jacobian determinant of the transformation $\det \mathbf{P} \equiv 1$. Thus

$$I = \iiint_{\lambda_1 y_1^2+\lambda_2 y_2^2+\lambda_3 y_3^2\le 1} \mathrm{e}^{\sqrt{\lambda_1 y_1^2+\lambda_2 y_2^2+\lambda_3 y_3^2}}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3.$$

Let

$$y_1 = \frac{1}{\sqrt{\lambda_1}}r\sin\varphi\cos\theta, \quad y_2 = \frac{1}{\sqrt{\lambda_2}}r\sin\varphi\sin\theta, \quad y_3 = \frac{1}{\sqrt{\lambda_3}}r\cos\varphi.$$

Then

$$I = \frac{1}{\sqrt{\lambda_1\lambda_2\lambda_3}}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\varphi\int_0^1 r^2\mathrm{e}^r\sin\varphi\mathrm{d}r = \frac{4\pi}{\sqrt{\lambda_1\lambda_2\lambda_3}}\int_0^1 r^2\mathrm{e}^r\mathrm{d}r = \frac{4\pi}{\sqrt{\lambda_1\lambda_2\lambda_3}}(\mathrm{e}-2).$$

Since the determinant of $\mathbf{A}$ is $\det\mathbf{A} = \lambda_1\lambda_2\lambda_3$, so

$$I = \frac{4\pi}{\sqrt{\det\mathbf{A}}}(\mathrm{e}-2).$$

Exercise 2.7. Find

$$\iiiint_{\substack{x,y,z,u\ge 0\\ x^2+y^2+z^2+u^2\le 1}} \sqrt{\frac{1-x^2-y^2-z^2-u^2}{1+x^2+y^2+z^2+u^2}}\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}u.$$

Solution: We use four-dimensional spherical coordinates: $x = r\sin\varphi_1\sin\varphi_2\cos\theta$, $y = r\sin\varphi_1\sin\varphi_2\sin\theta$, $z = r\sin\varphi_1\cos\varphi_2$, $u = r\cos\varphi_1$, with $r \ge 0$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le \varphi_1 \le \frac{\pi}{2}$, $0 \le \varphi_2 \le \frac{\pi}{2}$. The Jacobian of the transformation is $J = r^3\sin^2\varphi_1\sin\varphi_2$, and $x^2 + y^2 + z^2 + u^2 = r^2$.

The given integral

$$
\begin{aligned}
&\iiint\int_{\substack{x,y,z,u\geq 0\\ x^2+y^2+z^2+u^2\leq 1}} \sqrt{\frac{1-x^2-y^2-z^2-u^2}{1+x^2+y^2+z^2+u^2}}\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z\ \mathrm{d}u\\
=&\int_0^{\frac{\pi}{2}}\ \mathrm{d}\theta\int_0^{\frac{\pi}{2}}\sin\varphi_2\ \mathrm{d}\varphi_2\int_0^{\frac{\pi}{2}}\sin^2\varphi_1\ \mathrm{d}\varphi_1\int_0^1 r^3\sqrt{\frac{1-r^2}{1+r^2}}\ \mathrm{d}r\\
=&\frac{\pi}{2}\times\left(-\cos\varphi_2|_0^{\frac{\pi}{2}}\right)\times\frac{1}{2}\left[\varphi_1-\frac{\sin(2\varphi_1)}{2}\right]_0^{\frac{\pi}{2}}\times\frac{1}{2}\left(\int_0^1\frac{t}{\sqrt{1-t^2}}\ \mathrm{d}t-\int_0^1\frac{t^2}{\sqrt{1-t^2}}\ \mathrm{d}t\right)\\
=&\frac{\pi}{2}\times 1\times\frac{\pi}{4}\times\frac{1}{2}\left(-\frac{1}{2}\int_1^0 u^{-\frac{1}{2}}\ \mathrm{d}u-\int_0^{\frac{\pi}{2}}\sin^2\alpha\ \mathrm{d}\alpha\right)\\
=&\frac{\pi}{2}\times 1\times\frac{\pi}{4}\times\frac{1}{2}\left(1-\frac{\pi}{4}\right)\\
=&\frac{\pi^2}{16}\left(1-\frac{\pi}{4}\right)
\end{aligned}
$$

Exercise 2.8. Let

$$F(t)=\iiint_{x^2+y^2+z^2\leq t^2} f(x^2+y^2+z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

where f is a continuous function and $f(1)=1$. Prove that $F'(1)=4\pi$.

Solution: We use spherical coordinates to transform the triple-integral. In spherical coordinates, $x=\rho\sin\varphi\cos\theta$, $y=\rho\sin\varphi\sin\theta$, $z=\rho\cos\varphi$, and $\mathrm{d}V=\rho^2\sin\varphi\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta$, and $x^2+y^2+z^2=\rho^2$. The region $x^2+y^2+z^2\leq t^2$ corresponds to $0\leq\rho\leq t$, $0\leq\varphi\leq\pi$, $0\leq\theta\leq 2\pi$. Then

$$F(t)=\int_0^{2\pi} d\theta\int_0^{\pi}\sin\varphi\mathrm{d}\varphi\int_0^t f(\rho^2)\rho^2\mathrm{d}\rho.$$

Since $\int_0^{2\pi}\mathrm{d}\theta=2\pi$ and $\int_0^{\pi}\sin\varphi\mathrm{d}\varphi=-\cos\varphi|_0^{\pi}=2$, we have

$$F(t)=4\pi\int_0^t f(\rho^2)\rho^2\mathrm{d}\rho.$$

By the fundamental theorem of calculus, if $F(t)=4\pi\int_0^t g(\rho)\mathrm{d}\rho$ (where $g(\rho)=f(\rho^2)\rho^2$), then $F'(t)=4\pi f(t^2)t^2$. When $t=1$, since $f(1)=1$, we get $F'(1)=4\pi\times f(1)\times 1^2=4\pi$.

Exercise 2.9. Find

$$\iiint_{\Omega} z(x^2+y^2+z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

where Ω is the sphere $x^2+y^2+z^2\leq 2z$.

Solution: We rewrite the sphere equation $x^2+y^2+z^2 \leq 2z$ in spherical coordinates. With $x=\rho\sin\varphi\cos\theta$, $y=\rho\sin\varphi\sin\theta$, $z=\rho\cos\varphi$ and $x^2+y^2+z^2=\rho^2$, the sphere becomes $\rho^2 \leq 2\rho\cos\varphi$, so $\rho \leq 2\cos\varphi$ ($\rho \geq 0$). The ranges are $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \frac{\pi}{2}$, and $\mathrm{d}V = \rho^2\sin\varphi \mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta$. The integrand $z(x^2+y^2+z^2)$ is $\rho^3\cos\varphi$.

$$\begin{aligned}\iiint_\Omega z(x^2+y^2+z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^{2\cos\varphi}\rho^3\cos\varphi\cdot\rho^2\sin\varphi\mathrm{d}\rho \\ &= 2\pi\int_0^{\frac{\pi}{2}}\cos\varphi\sin\varphi\left[\frac{\rho^6}{6}\right]_0^{2\cos\varphi}\mathrm{d}\varphi \\ &= \frac{2\pi}{6}\int_0^{\frac{\pi}{2}}64\cos^7\varphi\sin\varphi\mathrm{d}\varphi \\ &= \frac{64\pi}{3}\left[-\frac{\cos^8\varphi}{8}\right]_0^{\frac{\pi}{2}} = \frac{8\pi}{3}.\end{aligned}$$

Exercise 2.10. Let $f(x,y,z)=\sqrt{x^2+y^2+z^2}$, and the region $\Omega \subset \mathbb{R}^3$ be determined by $z \geq \sqrt{x^2+y^2}$ and $4 \leq x^2+y^2+z^2 \leq 16$. Try to calculate the integral average value of the function f over Ω

$$\frac{1}{|\Omega|}\iiint_\Omega f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

where $|\Omega|$ is the volume of Ω.

Solution: In spherical coordinates, $x=\rho\sin\varphi\cos\theta$, $y=\rho\sin\varphi\sin\theta$, $z=\rho\cos\varphi$, and $\mathrm{d}V=\rho^2\sin\varphi\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta$, and $f(x,y,z)=\rho$. The inequality $z \geq \sqrt{x^2+y^2}$ implies $\rho\cos\varphi \geq \rho\sin\varphi$. Since $\rho > 0$ in the non origin part of the region, we have $\tan\varphi \leq 1$, so $0 \leq \varphi \leq \frac{\pi}{4}$. The inequality $4 \leq x^2+y^2+z^2 \leq 16$ implies $2 \leq \rho \leq 4$. And the range of θ is $0 \leq \theta \leq 2\pi$.

$$\begin{aligned}|\Omega| &= \iiint_\Omega \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{4}}\sin\varphi\mathrm{d}\varphi\int_2^4\rho^2\mathrm{d}\rho \\ &= 2\pi\times[-\cos\varphi]_0^{\frac{\pi}{4}}\times\left[\frac{\rho^3}{3}\right]_2^4 = \frac{56\pi(2-\sqrt{2})}{3}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\iiint_\Omega f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \iiint_\Omega \rho\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{4}}\sin\varphi\mathrm{d}\varphi\int_2^4\rho^3\mathrm{d}\rho \\ &= 2\pi\times[-\cos\varphi]_0^{\frac{\pi}{4}}\times\left[\frac{\rho^4}{4}\right]_2^4 = 60\pi(2-\sqrt{2}).\end{aligned}$$

The integral average value of f over Ω is

$$\frac{1}{|\Omega|}\iiint_\Omega f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{60\pi(2-\sqrt{2})}{\frac{56\pi(2-\sqrt{2})}{3}} = \frac{45}{14}.$$

2.3 n-multiple Integrals

Example 2.11. Let $N \in \mathbb{Z}_+$. Denote the volume of the unit ball $\sum_{k=1}^n x_k^2 \leq 1$ in the n-dimensional space as $\alpha(n)$. Calculate $\alpha(4)$ and write out the recurrence formula for the sequence $\alpha(n)$.

Solution:

1. General formula for the volume of an n-dimensional unit ball using integral.

The volume of the n-dimensional unit ball $\Omega_n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : \sum_{k=1}^n x_k^2 \leq 1\}$ is given by the n-fold integral

$$\alpha(n) = \int\!\!\cdots\!\!\int_{\sum_{k=1}^n x_k^2 \leq 1} \mathrm{d}x_1\, \mathrm{d}x_2 \cdots\, \mathrm{d}x_n.$$

We use the following approach to establish a recurrence relation. We can write

$$\alpha(n) = \int_{-1}^{1} \left(\int\!\!\cdots\!\!\int_{\sum_{k=2}^n x_k^2 \leq 1 - x_1^2} \mathrm{d}x_2 \cdots\, \mathrm{d}x_n \right) \mathrm{d}x_1.$$

The inner integral $\int\!\cdots\!\int_{\sum_{k=2}^n x_k^2 \leq 1 - x_1^2} \mathrm{d}x_2 \cdots\, \mathrm{d}x_n$ represents the volume of an $(n-1)$-dimensional ball with radius $r = \sqrt{1 - x_1^2}$. The volume of an $(n-1)$-dimensional ball of radius r is $r^{n-1}\alpha(n-1)$ (by the property of volume scaling in n-dimensions). So

$$\alpha(n) = \alpha(n-1) \int_{-1}^{1} (1 - x_1^2)^{\frac{n-1}{2}}\, \mathrm{d}x_1.$$

Let $x_1 = \sin t$, then $\mathrm{d}x_1 = \cos t\, \mathrm{d}t$. When $x_1 = -1$, $t = -\frac{\pi}{2}$; when $x_1 = 1$, $t = \frac{\pi}{2}$.

$$\int_{-1}^{1} (1 - x_1^2)^{\frac{n-1}{2}}\, \mathrm{d}x_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n t\, \mathrm{d}t = 2 \int_0^{\frac{\pi}{2}} \cos^n t\, \mathrm{d}t.$$

We know that

$$\int_0^{\frac{\pi}{2}} \cos^n t\, \mathrm{d}t = \begin{cases} \frac{(n-1)!!}{n!!} \cdot \frac{\pi}{2}, & n \text{ is even}, \\ \frac{(n-1)!!}{n!!}, & n \text{ is odd}. \end{cases}$$

So the recurrence formula is

$$\alpha(n) = 2\alpha(n-1) \int_0^{\frac{\pi}{2}} \cos^n t\, \mathrm{d}t.$$

2. Initial values.

For $n = 1$, the unit ball is the interval $[-1, 1]$, so $\alpha(1) = 2$.

For $n = 2$, $\alpha(2) = \pi$ (since the unit ball in 2-D is a unit circle $x_1^2 + x_2^2 \leq 1$ and its area is $\pi r^2 = \pi$).

3. Calculate $\alpha(4)$.
We first use the recurrence formula

$$\alpha(n) = 2\alpha(n-1)\int_0^{\frac{\pi}{2}} \cos^n t \ \mathrm{d}t.$$

For $n = 3$, $\int_0^{\frac{\pi}{2}} \cos^3 t \ \mathrm{d}t = \frac{2!!}{3!!} = \frac{2}{3}$, and

$$\alpha(3) = 2\alpha(2)\int_0^{\frac{\pi}{2}} \cos^3 t \ \mathrm{d}t = 2\pi \times \frac{2}{3} = \frac{4\pi}{3}.$$

For $n = 4$, $\int_0^{\frac{\pi}{2}} \cos^4 t \ \mathrm{d}t = \frac{3!!}{4!!} \cdot \frac{\pi}{2} = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{16}$.

$$\alpha(4) = 2\alpha(3)\int_0^{\frac{\pi}{2}} \cos^4 t \ \mathrm{d}t = 2 \times \frac{4\pi}{3} \times \frac{3\pi}{16} = \frac{\pi^2}{2}.$$

2.4 Symmetry

1. Symmetry of the Integration Region and the Integrand. Let D be an n-dimensional integration region in $\mathbb{R}^n$, and let $f(x_1, x_2, \cdots, x_n)$ be an $n-$variable function that is integrable over D.

Symmetry about coordinate hyperplanes:

- Even and odd functions: A function $f(x_1, x_2, \cdots, x_n)$ is said to be odd with respect to the variable x_i if

  $$f(x_1, \cdots, x_{i-1}, -x_i, x_{i+1}, \cdots, x_n) = -f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n).$$

  for all $(x_1, \cdots, x_n) \in \mathbb{R}^n$. It is said to be even with respect to x_i if

  $$f(x_1, \cdots, x_{i-1}, -x_i, x_{i+1}, \cdots, x_n) = f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n).$$

- Integral property: Suppose that the region D is symmetric about the hyperplane $x_i = 0$, i.e., if $(x_1, \cdots, x_n) \in D$, then $(x_1, \cdots, x_{i-1}, -x_i, x_{i+1}, \cdots, x_n) \in D$.

  - If f is odd with respect to x_i, then

    $$\int \cdots \int_D f(x_1, \cdots, x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n = 0.$$

  - If f is even with respect to x_i, then

    $$\int \cdots \int_D f(x_1, \cdots, x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n = 2\int \cdots \int_{D_i} f(x_1, \cdots, x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

    where $D_i = \{(x_1, \cdots, x_n) \in D : x_i \geq 0\}$.

General symmetry of the region.

Let $T:\mathbb{R}^n\to\mathbb{R}^n$ be a linear transformation that preserves the region D, i.e., $T(D)=D$. If $f(T(x_1,\cdots,x_n))=f(x_1,\cdots,x_n)$ for all $(x_1,\cdots,x_n)\in D$, then

$$\int\cdots\int_D f(x_1,\cdots,x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n=\int\cdots\int_D f(T(x_1,\cdots,x_n))|J_T|\mathrm{d}x_1\cdots\mathrm{d}x_n,$$

where J_T is the Jacobian determinant of the transformation T. In the case where T is an orthogonal transformation (e.g., a rotation or a reflection), $|J_T|=1$.

2. Examples in n-dimensional space

- Integration over the n-dimensional unit ball: Let

$$B^n=\left\{(x_1,\cdots,x_n)\in\mathbb{R}^n:\sum_{i=1}^n x_i^2\le 1\right\}$$

be the n-dimensional unit ball. Consider the integral

$$\int\cdots\int_{B^n} x_1^{2k_1}x_2^{2k_2}\cdots x_n^{2k_n}\mathrm{d}x_1\cdots\mathrm{d}x_n,$$

where $k_1,k_2,\cdots,k_n$ are non-negative integers. Since B^n is symmetric about each of the coordinate hyperplanes $x_i=0$, and the function

$$g(x_1,\cdots,x_n)=x_1^{2k_1}x_2^{2k_2}\cdots x_n^{2k_n}$$

is even with respect to each x_i, we can reduce the integral to an integral over the sub-ball

$$B_+^n=\{(x_1,\cdots,x_n)\in B^n:x_i\ge 0,i=1,\cdots,n\}.$$

So

$$\int\cdots\int_{B^n} x_1^{2k_1}x_2^{2k_2}\cdots x_n^{2k_n}\mathrm{d}x_1\cdots\mathrm{d}x_n=2^n\int\cdots\int_{B_+^n} x_1^{2k_1}x_2^{2k_2}\cdots x_n^{2k_n}\mathrm{d}x_1\cdots\mathrm{d}x_n.$$

- Integration over a symmetric polyhedron: Let P be an $n-$dimensional symmetric polyhedron in $\mathbb{R}^n$, symmetric about the origin. Let $f(x_1,\cdots,x_n)$ be a function such that $f(-x_1,\cdots,-x_n)=f(x_1,\cdots,x_n)$ (the function is even with respect to the origin). We can use the symmetry of P to simplify the integral

$$\int\cdots\int_P f(x_1,\cdots,x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n.$$

For example, if P can be divided into 2^n congruent sub-polyhedra $P_1,\cdots,P_{2^n}$ by the coordinate hyperplanes $x_i=0$ $(i=1,\cdots,n)$, then

$$\int\cdots\int_P f(x_1,\cdots,x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n=2^n\int\cdots\int_{P_1} f(x_1,\cdots,x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n,$$

where P_1 is the part of P with $x_i\ge 0$ for $i=1,\cdots,n$.

3. Conclusions In n-dimensional integration, identifying the symmetry of the integration region (such as symmetry about coordinate hyperplanes, symmetry about the origin, or symmetry under a linear transformation) and the parity properties of the integrand (even or odd with respect to variables or under a transformation) is a powerful tool for simplifying multi-integrals. By using these symmetries, we can often reduce the original integral over a large region to an integral over a smaller, more manageable sub-region, or even directly conclude that the integral is zero in the case of odd functions with respect to a symmetric region. This not only simplifies the computational process but also provides a deeper understanding of the geometric and algebraic properties of the functions and regions involved in the integration.

Example 2.12. Calculate

$$\iiint_\Omega (x+y+z)^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

where

$$\Omega := \left\{(x,y,z)\in\mathbb{R}^3 : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\right\}.$$

Solution: First, expand $(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2zx$. Since $\Omega = \left\{(x,y,z)\in\mathbb{R}^3 : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\right\}$ is symmetric about the coordinate planes $x=0$, $y=0$ and $z=0$, we have:

$$\iiint_\Omega xy\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0$$
$$\iiint_\Omega yz\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0$$
$$\iiint_\Omega zx\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0$$

By the transformation $x=au$, $y=bv$, $z=cw$, then $\mathrm{d}x\mathrm{d}y\mathrm{d}z = abc\mathrm{d}u\mathrm{d}v\mathrm{d}w$, and the region Ω is transformed into the unit ball $B=\left\{(u,v,w)\in\mathbb{R}^3 : u^2+v^2+w^2\le 1\right\}$. We use spherical coordinates $u=\rho\sin\varphi\cos\theta$, $v=\rho\sin\varphi\sin\theta$, $w=\rho\cos\varphi$, where $\mathrm{d}u\mathrm{d}v\mathrm{d}w = \rho^2\sin\varphi\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta$ and the limits of integration for the unit ball B are $0\le\rho\le 1$, $0\le\varphi\le\pi$, $0\le\theta\le 2\pi$. We calculate $\iiint_B (u^2+v^2+w^2)\mathrm{d}u\mathrm{d}v\mathrm{d}w$ as follows:

$$\iiint_B (u^2+v^2+w^2)\mathrm{d}u\mathrm{d}v\mathrm{d}w = \int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\mathrm{d}\varphi\int_0^1 \rho^2\cdot\rho^2\mathrm{d}\rho = \frac{4\pi}{5}.$$

Since $\iiint_B u^2\mathrm{d}u\mathrm{d}v\mathrm{d}w = \iiint_B v^2\mathrm{d}u\mathrm{d}v\mathrm{d}w = \iiint_B w^2\mathrm{d}u\mathrm{d}v\mathrm{d}w$ and $\iiint_B (u^2+v^2+w^2)\mathrm{d}u\mathrm{d}v\mathrm{d}w = 3\iiint_B u^2\mathrm{d}u\mathrm{d}v\mathrm{d}w$, we get $\iiint_B u^2\mathrm{d}u\mathrm{d}v\mathrm{d}w = \frac{4\pi}{15}$. Then $\iiint_\Omega x^2\mathrm{d}x\mathrm{d}y\mathrm{d}z = a^3bc\iiint_B u^2\mathrm{d}u\mathrm{d}v\mathrm{d}w = \frac{4\pi a^3bc}{15}$, $\iiint_\Omega y^2\mathrm{d}x\mathrm{d}y\mathrm{d}z = b^3ac\iiint_B v^2\mathrm{d}u\mathrm{d}v\mathrm{d}w = \frac{4\pi ab^3c}{15}$, $\iiint_\Omega z^2\mathrm{d}x\mathrm{d}y\mathrm{d}z = c^3ab\iiint_B w^2\mathrm{d}u\mathrm{d}v\mathrm{d}w =$

$\frac{4\pi abc^3}{15}$. So,

$$\begin{aligned}\iiint_\Omega (x+y+z)^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z &= \iiint_\Omega x^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z + \iiint_\Omega y^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z + \iiint_\Omega z^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \frac{4\pi a^3 bc}{15} + \frac{4\pi ab^3 c}{15} + \frac{4\pi abc^3}{15} \\ &= \frac{4\pi abc(a^2+b^2+c^2)}{15}.\end{aligned}$$

Exercise 2.13. Find

$$I = \iiint_\Omega (x+y+z)^2 \mathrm{d}V,$$

where Ω is the intersection of $x^2+y^2 \le 2z$ and $x^2+y^2+z^2 \le 3$.

Solution: Given $I = \iiint_\Omega (x+y+z)^2 \mathrm{d}V$. Expand $(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2zx$. Because of the symmetry of the region Ω (the intersection of $x^2+y^2 \le 2z$ and $x^2+y^2+z^2 \le 3$) with respect to the coordinate planes, we have $\iiint_\Omega 2xy\mathrm{d}V = 0$, $\iiint_\Omega 2yz\mathrm{d}V = 0$, and $\iiint_\Omega 2zx\mathrm{d}V = 0$. So, $I = \iiint_\Omega (x^2+y^2+z^2)\mathrm{d}V$.

In cylindrical coordinates, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, and $\mathrm{d}V = r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$. The inequalities become $r^2 \le 2z$ and $r^2+z^2 \le 3$. To find the intersection of $r^2 = 2z$ and $r^2+z^2 = 3$, substitute $r^2 = 2z$ into $r^2+z^2 = 3$, getting $z^2+2z-3 = 0$. The non-negative root is $z = 1$, and then $r = \sqrt{2}$. The integral I can be written as:

$$\begin{aligned}I &= \int_0^{2\pi} \mathrm{d}\theta \int_0^{\sqrt{2}} r\mathrm{d}r \int_{\frac{r^2}{2}}^{\sqrt{3-r^2}} (r^2+z^2)\mathrm{d}z = 2\pi \int_0^{\sqrt{2}} r\left(r^2 z + \frac{z^3}{3}\right)\Big|_{\frac{r^2}{2}}^{\sqrt{3-r^2}} \mathrm{d}r \\ &= 2\pi \int_0^{\sqrt{2}} r\left(r^2\sqrt{3-r^2} + \frac{(3-r^2)^{\frac{3}{2}}}{3} - \frac{r^4}{2} - \frac{r^6}{24}\right)\mathrm{d}r \\ &= 2\pi\left(\int_0^{\sqrt{2}} r^3\sqrt{3-r^2}\mathrm{d}r + \frac{1}{3}\int_0^{\sqrt{2}} r(3-r^2)^{\frac{3}{2}}\mathrm{d}r - \frac{1}{2}\int_0^{\sqrt{2}} r^5\mathrm{d}r - \frac{1}{24}\int_0^{\sqrt{2}} r^7\mathrm{d}r\right) \\ &= \frac{\pi(17-2\sqrt{3})}{5}\end{aligned}$$

Chapter 3

Curvilinear Integral

3.1 Relations between type I and type II

The coordinate differential element is given by $\mathrm{d}\boldsymbol{\tau} = \boldsymbol{\tau}\,\mathrm{d}s$, where $\boldsymbol{\tau}$ is the unit tangent vector of the curve in the specified direction. Thus, we have

$$\int_\Gamma \boldsymbol{F}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{r} = \int_\Gamma (\boldsymbol{F}(\boldsymbol{x}) \cdot \boldsymbol{\tau})\,\mathrm{d}s.$$

For a space curve Γ, let $\cos\alpha, \cos\beta, \cos\gamma$ be the direction cosines of the tangent. Then,

$$\int_\Gamma P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z = \int_\Gamma (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,\mathrm{d}s.$$

When Γ is a plane curve, if we take the direction of the normal $\boldsymbol{n}$ and the direction of the tangent $\boldsymbol{\tau}$ such that the angle $\angle(\boldsymbol{n}, \boldsymbol{\tau}) = \frac{\pi}{2}$, we have

$$\angle(x, \boldsymbol{n}) = \angle(x, \boldsymbol{\tau}) + \angle(\boldsymbol{\tau}, \boldsymbol{n}) = \alpha - \frac{\pi}{2}.$$

Thus,

$$\int_\Gamma P\,\mathrm{d}x + Q\,\mathrm{d}y = \int_\Gamma [-P\sin(x, \boldsymbol{n}) + Q\cos(x, \boldsymbol{n})]\,\mathrm{d}s.$$

3.2 Calculations of curvilinear integral: type I

Example 3.1. Calculate

$$\int_L \sqrt{x^2 + y^2}\mathrm{d}s,$$

where $L : x^2 + y^2 = ax$.

Solution: First, convert the curve $x^2 + y^2 = ax$ to polar coordinates: In polar coordinates, $x = r\cos\theta, \quad y = r\sin\theta$, so $r^2 = ar\cos\theta$, then $r = a\cos\theta$ $(r \neq 0)$. Then calculate the curvilinear integral:

$$\int_L \sqrt{x^2 + y^2}\mathrm{d}s = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r \cdot a\mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\cos\theta \cdot a\mathrm{d}\theta = 2a^2$$

Example 3.2. Find $I = \oint_C x^2\mathrm{d}s$, where C is given by

$$\begin{cases} x^2 + y^2 + z^2 = R^2 \\ x + y + z = 0 \end{cases}$$

Solution:

(Conventional method) First, write out the parametric expressions of the curve C. Since C is the intersection line of the spherical surface $x^2+y^2+z^2=R^2$ and the plane $x+y+z=0$ passing through the center of the sphere, it is a circular circumference in space. Its projection on the xOy plane is an ellipse, and the equation of this ellipse can be obtained by eliminating z from the equations of the two surfaces. That is, substitute $z=-(x+y)$ into $x^2+y^2+z^2=R^2$, and we get

$$x^2+xy+y^2=\frac{R^2}{2}.$$

Complete the square for the left side into the sum of squares

$$\left(\frac{\sqrt{3}}{2}x\right)^2+\left(\frac{x}{2}+y\right)^2=\frac{R^2}{2}.$$

Let

$$\frac{\sqrt{3}}{2}x=\frac{R}{\sqrt{2}}\cos t,\quad \frac{x}{2}+y=\frac{R}{\sqrt{2}}\sin t,\quad t\in[0,2\pi].$$

That is, we obtain the parametric representation

$$x=\sqrt{\frac{2}{3}}R\cos t,\quad y=\frac{R}{\sqrt{2}}\sin t-\frac{R}{\sqrt{6}}\cos t,\quad t\in[0,2\pi].$$

Substitute it into $z=-(x+y)$, and we get

$$z=-\frac{R}{\sqrt{6}}\cos t-\frac{R}{\sqrt{2}}\sin t,\quad t\in[0,2\pi].$$

From this, we have

$$\begin{aligned}
\mathrm{d}s&=\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\mathrm{d}t\\
&=R\sqrt{\frac{2}{3}\sin^2 t+\left(\frac{\cos t}{\sqrt{2}}+\frac{\sin t}{\sqrt{6}}\right)^2+\left(\frac{\sin t}{\sqrt{6}}-\frac{\cos t}{\sqrt{2}}\right)^2}\mathrm{d}t\\
&=R\mathrm{d}t.
\end{aligned}$$

Therefore, we have

$$\int_C x^2\mathrm{d}s=\int_0^{2\pi}\frac{2}{3}R^3\cos^2 t\mathrm{d}t=\frac{2}{3}\pi R^3.$$

(Using symmetry) By symmetry, we have

$$\int_C x^2\mathrm{d}s=\int_C y^2\mathrm{d}s=\int_C z^2\mathrm{d}s.$$

Then

$$\int_C x^2\mathrm{d}s=\frac{1}{3}\int_C\left(x^2+y^2+z^2\right)\mathrm{d}s=\frac{1}{3}R^2\int_C\mathrm{d}s=\frac{2}{3}\pi R^3.$$

Exercise 3.3. Calculate the following curvilinear integrals of type I:

1. $\int_C \left(x^{4/3}+y^{4/3}\right) \mathrm{d}s$, where C is the astroid curve $x^{2/3}+y^{2/3}=a^{2/3}$.

2. $\int_C \mathrm{e}^{\sqrt{x^2+y^2}} \mathrm{d}s$, where C is composed of the curve $r=a$, $\varphi=0$, $\varphi=\frac{\pi}{4}$ (r and φ are polar-coordinates).

3. $\int_C |y| \mathrm{d}s$, where C is the lemniscate $\left(x^2+y^2\right)^2=a^2\left(x^2-y^2\right)$.

4. $\int_C \frac{\mathrm{d}s}{y^2}$, where C is the catenary $y=a \operatorname{ch} \frac{x}{a}$.

5. $\int_C z \mathrm{d}s$, where C is the curve $x^2+y^2=z^2$, $y^2=ax$ from the point $(0,0,0)$ to the point $(a, a, \sqrt{2}a)$.

Solution: 1. First, parameterize the astroid curve $x^{2/3}+y^{2/3}=a^{2/3}$ with $x=a\cos^3 t$, $y=a\sin^3 t$, $t\in[0,2\pi]$. Then

$$
\begin{aligned}
\int_C (x^{4/3}+y^{4/3})\mathrm{d}s &= \int_0^{2\pi} (a^{4/3}\cos^4 t + a^{4/3}\sin^4 t)\cdot 3a|\sin t\cos t|\mathrm{d}t \\
&= 4\int_0^{\frac{\pi}{2}} (a^{4/3}\cos^4 t + a^{4/3}\sin^4 t)\cdot 3a\sin t\cos t\mathrm{d}t = 4a^{7/3}
\end{aligned}
$$

2. $\frac{\pi a e^a}{4}$. 3. $4a^2(1-\frac{\sqrt{2}}{2}) = 2a^2(2-\sqrt{2})$. 4. $\frac{\pi}{a}$

Exercise 3.4. Let $f(x,y)$ be continuous on L, and L be a piecewise smooth simple closed curve. Prove that:

$$
u(x,y) = \oint_L f(\xi,\eta)\ln\sqrt{(x-\xi)^2+(y-\eta)^2}\mathrm{d}s.
$$

A necessary and sufficient condition for $u(x,y)$ to tend to 0 as $x^2+y^2\to+\infty$ is that

$$
\oint_L f(\xi,\eta)\mathrm{d}s = 0.
$$

Solution: Note that

$$
\begin{aligned}
&\left|\oint_L f(\xi,\eta)\ln\sqrt{(x-\xi)^2+(y-\eta)^2}\mathrm{d}s - \oint_L f(\xi,\eta)\ln\sqrt{x^2+y^2}\mathrm{d}s\right| \\
&= \oint_L |f(\xi,\eta)|\ln\frac{\sqrt{(x-\xi)^2+(y-\eta)^2}}{\sqrt{x^2+y^2}}\mathrm{d}s.
\end{aligned}
$$

The right hand side converges to 0. This implies the equivalence of the result in this problem.

3.3 Calculations of curvilinear integral: type II

Example 3.5. Calculate the integral

$$I = \int_C (x^2 + 2xy)\, dy$$

where C represents the upper half of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ oriented counterclockwise.

Solution: Using the parametric expressions for the ellipse,

$$x = a\cos t, \quad y = b\sin t$$

with t varying from 0 to π in the given direction, we substitute x and y into the integral and replace $\mathrm{d}y$ with $b\cos t, \mathrm{d}t$:

$$\begin{aligned} I &= \int_0^\pi \left(a^2\cos^2 t + 2ab\cos t\sin t\right) b\cos t\, dt \\ &= a^2 b\int_0^\pi \cos^3 t\, dt + 2ab^2\int_0^\pi \cos^2 t\sin t\, dt = \frac{4}{3}ab^2. \end{aligned}$$

Example 3.6. Find

$$I = \int_\Gamma y^2\, dx + z^2\, dy + x^2\, dz,$$

where Γ is the curve defined by

$$\begin{cases} x^2 + y^2 + z^2 = a^2 \\ x^2 + y^2 = ax \quad (a > 0) \end{cases}$$

the part of the curve where $z \geq 0$, and viewed from the positive direction of the x-axis, Γ is oriented counterclockwise.

Solution: First, we need to find the parametric expressions for the curve Γ. If we use one of x, y, z as a parameter, square roots will be involved, and branches will appear, making it inconvenient. From the second equation, it is easy to see that we can introduce cylindrical coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

Thus, the parametric equations for Γ become

$$\begin{aligned} &x = a\cos^2\theta, \quad y = a\cos\theta\sin\theta \\ &z = \sqrt{a^2 - r^2} = a|\sin\theta|, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}. \end{aligned}$$

According to the orientation of the curve, we have

$$I = \int_{-\pi/2}^{\pi/2} \big[a^2\cos^2\theta\sin^2\theta\cdot 2a\cos\theta(-\sin\theta) \\ +a^2\sin^2\theta\cdot a(\cos^2\theta-\sin^2\theta)+a^2\cos^4\theta\cdot z'(\theta)\big]\,d\theta.$$

The first term is an odd function, and since $z(\theta)$ is an even function, $z'(\theta)$ is an odd function; thus, the third term is also an odd function. Then

$$\begin{aligned} I &= \int_{-\pi/2}^{\pi/2} a^3\sin^2\theta\left(\cos^2\theta-\sin^2\theta\right)\mathrm{d}\theta \\ &= 2a^3\int_0^{\pi/2}\left(\sin^2\theta-2\sin^4\theta\right)\mathrm{d}\theta = -\frac{\pi}{4}a^3 \end{aligned}$$

Example 3.7. Let the arc length of the curve C be L. Prove that:

$$\left|\int_C P(x,y)dx+Q(x,y)dy\right| \le ML$$

where $M=\max_{(x,y)\in C}\sqrt{P^2(x,y)+Q^2(x,y)}$. Using the conclusion of the previous question, prove that

$$\lim_{R\to 0}\oint_{x^2+y^2=R^2}\frac{y\mathrm{d}x-x\mathrm{d}y}{(x^2+xy+y^2)^2}=0.$$

Solution: We parametrize L as $x(t)$ and $y(t)$, where $t\in[a,b]$. Then

$$\begin{aligned} \left|\int_C P(x,y)dx+Q(x,y)dy\right| &= \left|\int_a^b (P(x(t),y(t))x'(t)+Q(x(t),y(t))y'(t))\mathrm{d}t\right| \\ &\le M\int_a^b\sqrt{(x'(t))^2+(y'(t))^2}\mathrm{d}t \le ML. \end{aligned}$$

For the second one, we see that for corresponding P,Q, there holds that

$$|P|,|Q|\le\frac{1}{(x^2+y^2)^{3/2}}.$$

Then

$$\left|\oint_{x^2+y^2=R^2}\frac{y\mathrm{d}x-x\mathrm{d}y}{(x^2+xy+y^2)^2}\right|\le CR\cdot\frac{1}{R^3}\to 0.$$

Solution:

Exercise 3.8. Calculate $\int_C(x+y)^2\mathrm{d}x+(x^2-y^2)\mathrm{d}y$, where C is the triangle with vertices $A(1,1)$, $B(3,2)$, and $D(3,1)$, and the direction of C is clockwise.

Solution: $-\frac{8}{3}$.

Exercise 3.9. Find

$$\oint_C \left(y^2+z^2\right)\,\mathrm{d}x+\left(z^2+x^2\right)\,\mathrm{d}y+\left(x^2+y^2\right)\,\mathrm{d}z,$$

where C is the intersection curve of the surfaces $x^2+y^2+z^2=2Rx$ and $x^2+y^2=2ax$ (with $0<a<R$ and $z>0$), and is oriented counterclockwise when viewed from the positive direction of the z-axis.

Solution: $2\pi a^2(R-a)$.

Exercise 3.10. Calculate

$$\int_L (x-2xy^2)\mathrm{d}x+(y-2x^2y)\mathrm{d}y,$$

where L is given by $y=x^2$ from $(0,0)$ to $(2,4)$.

3.4 Using Green's formula

Green's formula is an important formula that establishes a connection between the double integral over a certain region in the plane and a specific type II curvilinear integral on the boundary of that region.

Let D be a bounded closed region in $\mathbb{R}^2$, and its boundary ∂D consists of smooth curves or piece-wise smooth curves. Also, assume that the functions P and Q have continuous partial derivatives with respect to the independent variables x and y on D. Then the following Green's formula holds:

$$\iint_D \left(-\frac{\partial P}{\partial y}+\frac{\partial Q}{\partial x}\right)\mathrm{d}x\mathrm{d}y=\oint_{\partial D} P\mathrm{d}x+Q\mathrm{d}y,$$

where the direction of ∂D is the positive direction with respect to D. Similarly,

$$\begin{aligned}\iint_D \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\mathrm{d}x\mathrm{d}y &= \oint_{\partial D} -Q\mathrm{d}x+P\mathrm{d}y\\ &= \oint_{\partial D} [Q\sin(x,\boldsymbol{n})+P\cos(x,\boldsymbol{n})]\mathrm{d}s,\end{aligned}$$

where $\boldsymbol{n}$ is the unit outer normal vector on ∂D. Using $\angle(x,\boldsymbol{n})=\angle(x,y)+\angle(y,\boldsymbol{n})=\frac{\pi}{2}+\angle(y,\boldsymbol{n})$ (see Figure 24.2), then

$$\iint_D \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\mathrm{d}x\mathrm{d}y=\oint_{\partial D}[P\cos(x,\boldsymbol{n})+Q\cos(y,\boldsymbol{n})]\mathrm{d}s$$

Formulas above provide us with an indirect method for calculating curvilinear integrals in the plane. That is, we use Green's formula to convert the curvilinear integral into a double integral. Even if the curve C is not closed, we can use the method of adding "auxiliary lines".

Example 3.11. Let C be the arc of the parabola $2x = \pi y^2$ from $(0,0)$ to $\left(\frac{\pi}{2}, 1\right)$. Calculate

$$I = \int_C \left(2xy^3 - y^2 \cos x\right) \mathrm{d}x + \left(1 - 2y \sin x + 3x^2 y^2\right) \mathrm{d}y.$$

Solution: Let $P(x,y) = 2xy^3 - y^2 \cos x$ and $Q(x,y) = 1 - 2y \sin x + 3x^2 y^2$. Then

$$-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = 0.$$

Set $A = (\frac{\pi}{2}, 0)$. To use Green's formula, we add auxiliary lines $O = (0,0) \to A$ and $A \to B = (\frac{\pi}{2}, 1)$. Then

$$I = \int_C + \int_{\widehat{BA}} + \int_{\widehat{AO}} + \int_{\widehat{AB}} + \int_{\widehat{OA}}$$

By Green's formula, the integral of the first three terms is zero. So

$$I = \int_{\widehat{AB}} + \int_{\widehat{OA}} = \int_0^1 \left[1 - 2y \sin \frac{\pi}{2} + 3 \left(\frac{\pi}{2}\right)^2 y^2\right] \mathrm{d}y = \frac{\pi^2}{4}.$$

Example 3.12. Calculate the integral

$$I = \oint_C \frac{\cos(\boldsymbol{r}, \boldsymbol{n})}{r} \mathrm{d}s,$$

where C is a piecewise smooth simple closed curve, $\boldsymbol{r} = (x,y)$, $r = |\boldsymbol{r}| = \sqrt{x^2 + y^2}$, and $\boldsymbol{n}$ is the unit outer normal vector on C.

Solution: From $\cos(\boldsymbol{r}, \boldsymbol{n}) = \frac{\boldsymbol{r} \cdot \boldsymbol{n}}{r} = \frac{1}{r}(x \cos(\boldsymbol{n}, \boldsymbol{x}) + y \cos(\boldsymbol{n}, y))$, we get

$$I = \oint_C \left(\frac{x}{r^2} \cos(\boldsymbol{n}, x) + \frac{y}{r^2} \cos(\boldsymbol{n}, y)\right) \mathrm{d}s.$$

When $(0,0)$ is outside C, using Green's formula (24.10), we have

$$I = \iint_D \left[\partial_x \left(\frac{x}{r^2}\right) + \partial_y \left(\frac{y}{r^2}\right)\right] \mathrm{d}x\mathrm{d}y = 0,$$

where D is the region bounded by C. When $(0,0)$ is inside C, we draw a circle C_ε with the center $(0,0)$ and a sufficiently small radius ε such that C_ε is inside C. Let the region bounded by C and C_ε be D_ε, then

$$\begin{aligned} I = &\oint_C \left(\frac{x}{r^2} \cos(\boldsymbol{n}, x) + \frac{y}{r^2} \cos(\boldsymbol{n}, y)\right) \mathrm{d}s + \oint_{C_\varepsilon} \left(\frac{x}{r^2} \cos(\boldsymbol{n}, x) + \frac{y}{r^2} \cos(\boldsymbol{n}, y)\right) \mathrm{d}s \\ &- \oint_{C_\varepsilon} \left(\frac{x}{r^2} \cos(\boldsymbol{n}, x) + \frac{y}{r^2} \cos(\boldsymbol{n}, y)\right) \mathrm{d}s, \end{aligned}$$

where the direction of the unit normal vector $\boldsymbol{n}$ on C_ε points to the origin of coordinates. Applying Green's formula to the first two terms, we get

$$I = -\oint_{C_\varepsilon} \left(\frac{x}{r^2} \cos(\boldsymbol{n}, x) + \frac{y}{r^2} \cos(\boldsymbol{n}, y) \right) \mathrm{d}s.$$

On the circle C_ε,

$$\cos(\boldsymbol{n}, x) = -\frac{x}{\varepsilon}, \quad \cos(\boldsymbol{n}, y) = -\frac{y}{\varepsilon}, \quad r = \varepsilon$$

Thus

$$I = \oint_{C_\varepsilon} \frac{1}{\varepsilon} ds = 2\pi$$

When $(0,0) \in C$, we see that the integral is not well defined since the functions have singularities at $(0,0)$. As a result, we define

$$I = \lim_{\varepsilon \to 0^+} \int_{C \setminus B_\varepsilon} \left(\frac{x}{r^2} \cos(\boldsymbol{n}, x) + \frac{y}{r^2} \cos(\boldsymbol{n}, y) \right) \mathrm{d}s,$$

where $B_\varepsilon = \left\{ x^2 + y^2 \leq \varepsilon \right\}$. Using Green's formula, we obtain that

$$I = \lim_{\varepsilon \to 0} \int_{C_\varepsilon} \frac{1}{\varepsilon} \mathrm{d}s$$

We draw the tangents OA and OB of the curve C passing through the origin. Let the included angle between OA and OB be θ. If the curve C is smooth at the origin, then $\theta = \pi$. We draw a circle ∂B_ε with the center $(0,0)$ and radius ε, and denote the part of the circumference of ∂B_ε inside C as C_ε. From the above calculation, we know that

$$I = \lim_{\varepsilon \to 0} \int_{C_\varepsilon} \frac{1}{\varepsilon} \mathrm{d}s = \lim_{\varepsilon \to 0} \theta_\varepsilon = \theta$$

3.5 Path independence

Let Ω be a region in $\mathbb{R}^2$, and $P(x,y), Q(x,y)$ be continuous on Ω. Denote

$$w = P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y.$$

Arbitrarily take points $A, B \in \Omega$. A piecewise smooth simple curve in Ω from A to B is called a path in Ω from A to B. For any path L in Ω from A to B, if the second-type curvilinear integral

$$\int_L w = \int_L P dx + Q dy$$

depends only on A and B, and is independent of the specific choice of L, then the first-order differential form w is said to have a curvilinear integral independent of the path in Ω. If the region enclosed by any simple closed curve in the planar region D is completely contained in D, then D is called a simply connected region. Let D be a simply connected region in the plane, and $w = Pdx + Qdy$, where both P and Q have continuous partial derivatives on D. Then the following conclusions are equivalent:

1. For any closed curve C in D, we have

$$\oint_C w = 0;$$

2. For any path C in D, the integral $\int_C w$ depends only on the starting point and the ending point of C, and is independent of the path taken;

3. In D (everywhere), the following holds

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

4. There exists a function $\varphi(x, y)$ such that in D, the following holds

$$\mathrm{d}\varphi(x, y) = P(x, y)dx + Q(x, y)dy$$

That is, $P\mathrm{d}x + Q\mathrm{d}y$ is the total differential of the function φ. At this time, φ is called the potential function or primitive function of w, and w is also called an exact differential form. At this time,

$$\varphi(x, y) = \int_{x_0}^{x} P(x, y_0)\mathrm{d}x + \int_{y_0}^{y} Q(x, y)\mathrm{d}y + C,$$

where (x_0, y_0) is an arbitrary point in D, C is an arbitrary constant, and

$$\int_{(x_0,y_0)}^{(x,y)} P\mathrm{d}x + Q\mathrm{d}y = \varphi\big|_{(x_0,y_0)}^{(x,y)} = \varphi(x, y) - \varphi(x_0, y_0).$$

Example 3.13. First, prove that the curvilinear integral is independent of the path, and then calculate the value of the integral:

1 $\int_{(1,2)}^{(3,4)} \varphi(x)\mathrm{d}x + \psi(y)\mathrm{d}y$, where $\varphi(x)$ and $\psi(y)$ are continuous functions;

2. $\int_{(1,0)}^{(6,8)} \frac{x\mathrm{d}x+y\mathrm{d}y}{x^2+y^2}$, along a path that does not pass through the origin.

<u>Solution:</u> 1. For the integral $\int_{(1,2)}^{(3,4)} \varphi(x)\mathrm{d}x + \psi(y)\mathrm{d}y$, let $P(x, y) = \varphi(x)$ and $Q(x, y) = \psi(y)$. We calculate the partial derivatives: $\frac{\partial P}{\partial y} = 0$ and $\frac{\partial Q}{\partial x} = 0$. Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ holds for all $(x, y) \in \mathbb{R}^2$ (because $\varphi(x)$ depends only on x and $\psi(y)$ depends

only on y), and the domain $\mathbb{R}^2$ is a simply-connected region, by the theorem of path-independence of line integrals, the line integral $\int_{(1,2)}^{(3,4)} \varphi(x)\mathrm{d}x+\psi(y)\mathrm{d}y$ is independent of the path.

We can choose a path composed of two line segments. First, from the point $(1,2)$ to the point $(3,2)$ and then from the point $(3,2)$ to the point $(3,4)$. So, $\int_{(1,2)}^{(3,4)} \varphi(x)\mathrm{d}x + \psi(y)\mathrm{d}y = \int_1^3 \varphi(x)\mathrm{d}x + \int_2^4 \psi(y)\mathrm{d}y$.

2. Notice that $\frac{x\mathrm{d}x+y\mathrm{d}y}{x^2+y^2} = \frac{1}{2}\frac{\mathrm{d}(x^2+y^2)}{x^2+y^2}$. Let $u = x^2 + y^2$, then the integral $\int \frac{x\mathrm{d}x+y\mathrm{d}y}{x^2+y^2} = \frac{1}{2}\int \frac{\mathrm{d}u}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+y^2) + C$.

Using the fundamental theorem of line integrals, if $F(x,y) = \frac{1}{2}\ln(x^2+y^2)$, then

$$\int_{(1,0)}^{(6,8)} \frac{x\mathrm{d}x + y\mathrm{d}y}{x^2+y^2} = F(6,8) - F(1,0) = \ln 10$$

Exercise 3.14. Let L be the unit circle $x^2 + y^2 = 1$ with the counterclockwise direction. Find the integral

$$\oint_L \frac{(x-y)\mathrm{d}x + (x+4y)\mathrm{d}y}{x^2+4y^2}.$$

<u>Solution:</u> π.

Chapter 4

Surface Integrals

4.1 Surface integrals of type I

Consider a smooth surface S bounded by a piecewise smooth closed curve L. Suppose this surface is divided into m parts $S_1, S_2, \cdots, S_m$ by a piecewise smooth curve network. And for each part S_i, we arbitrarily choose a point M_i inside it. Project the element S_i vertically onto the tangent plane of the surface at the point M_i, obtaining a planar figure T_i in the tangent plane. The limit of the sum of the areas ΔT_i $(i = 1, 2, \cdots, m)$ as the diameters of each element S_i approach zero

$$\lim_{\lambda \to 0} \sum_{i=1}^{m} \Delta T_i$$

is called the area of the surface S, where $\lambda = \max_{1 \le i \le m}\{d(S_i)\}$ is the maximum diameter among S_i $(i = 1, 2, \cdots, m)$. If this limit is a finite number, then the surface S is said to be a surface of measurable area. Let S be a surface of measurable area in $\mathbb{R}^3$, and let the function f be defined on S. For any partition $T = \{S_1, S_2, \cdots, S_m\}$ of S, denote the area of S_i by ΔS_i. Arbitrarily choose a point $(\xi_i, \eta_i, \zeta_i) \in S_i$. If, as $d(T) = \max_{1 \le i \le m}\{d(S_i)\} \to 0$, the sum

$$\sum_{i=1}^{m} f(\xi_i, \eta_i, \zeta_i) \Delta S_i$$

converges, and the limit does not depend on the specific partition of S and the specific choice of the points $(\xi_i, \eta_i, \zeta_i) \in S_i$, then we say that the first kind of surface integral of the function f on S exists. This limit value is called the first kind of surface integral of $f(x, y, z)$ on S, denoted as

$$\iint_S f(x, y, z) \mathrm{d}S = \lim_{d(T) \to 0} \sum_{i=1}^{m} f(\xi_i, \eta_i, \zeta_i) \Delta S_i.$$

If the parametric equations of the surface S are If the parametric equations of the surface S are

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad (u, v) \in D,$$

where D is a region of measurable area in the uOv-plane, and $x(u, v)$, $y(u, v)$, $z(u, v)$ have continuous partial derivatives on D, then

$$\mathrm{d}S = \sqrt{EG - F^2} \mathrm{d}u \mathrm{d}v$$

where

$$\begin{aligned} E &= x_u^2 + y_u^2 + z_u^2 \\ F &= x_u x_v + y_u y_v + z_u z_v \\ G &= x_v^2 + y_v^2 + z_v^2 \end{aligned}$$

and

$$\iint_S f(x,y,z)\mathrm{d}S = \iint_D f(x(u,v), y(u,v), z(u,v))\sqrt{EG - F^2}\mathrm{d}u\mathrm{d}v$$

In particular, if the equation of the surface S is

$$z = z(x,y), \quad (x,y) \in D$$

then

$$\iint_S f(x,y,z)\mathrm{d}S = \iint_D f(x,y,z(x,y))\sqrt{1 + z_x^2 + z_y^2}\mathrm{d}x\mathrm{d}y$$

Example 4.1. Let S be the part of the surface $z = \sqrt{x^2 + y^2}$ that is cut off by the cylinder $x^2 + y^2 = 2ax$. Find

$$I = \iint_S (x^2y^2 + y^2z^2 + z^2x^2)\mathrm{d}S.$$

Solution 1: Calculate it in the rectangular coordinate system.

$$z_x = \frac{x}{z}, \quad z_y = \frac{y}{z},$$

$$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{2}$$

$$I = \iint_{x^2+y^2 \le 2ax} [x^2y^2 + (x^2 + y^2)^2]\sqrt{2}\mathrm{d}x\mathrm{d}y.$$

Use the polar coordinate transformation to evaluate the above double integral. Then

$$\begin{aligned} I &= \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{2a\cos\theta} (r^4\cos^2\theta\sin^2\theta + r^4)r\mathrm{d}r \\ &= \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2\theta\sin^2\theta + 1)\cdot\left(\frac{1}{6}r^6\Big|_0^{2a\cos\theta}\right)\mathrm{d}\theta \\ &= \frac{\sqrt{2}}{6}(2a)^6\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^6\theta(\cos^2\theta\sin^2\theta + 1)\mathrm{d}\theta = \frac{29}{8}\sqrt{2}\pi a^6 \end{aligned}$$

Solution 2: Calculate it using the parametric form. The equation of $z = \sqrt{x^2 + y^2}$ in the spherical coordinate system is $\varphi = \frac{\pi}{4}$. Thus, the parametric equations of the surface S are

$$x = \frac{1}{\sqrt{2}}r\cos\theta, \quad y = \frac{1}{\sqrt{2}}r\sin\theta, \quad z = \frac{1}{\sqrt{2}}r, \quad (r,\theta) \in D$$

Moreover, the boundary curves of the surface S

$$z=\sqrt{x^2+y^2},\quad x^2+y^2=2ax$$

can be expressed in spherical coordinates as

$$\varphi=\frac{\pi}{4},\quad r^2\sin^2\varphi=2ar\sin\varphi\cos\theta$$

Then

$$D=\left\{(r,\theta)\,\middle|\,-\frac{\pi}{2}\le\theta\le\frac{\pi}{2},0\le r\le 2\sqrt{2}a\cos\theta\right\}$$

By calculation, we get

$$E=\frac{r^2}{2},\quad F=0,\quad G=1$$

Finally, we obtain

$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{2\sqrt{2}a\cos\theta}\left(\frac{1}{4}r^4\cos^2\theta\sin^2\theta+\frac{1}{4}r^4\right)\frac{r}{\sqrt{2}}\mathrm{d}r=\frac{29}{8}\sqrt{2}\pi a^6$$

Exercise 4.2. Calculate $\iint_S|xyz|\mathrm{d}S$, where

1. S is the surface given by $|x|+|y|+|z|=1$;
2. S is the part of the paraboloid $z=x^2+y^2$ cut off by $z=1$.

Solution:

1. When S is the surface given by $|x|+|y|+|z|=1$ **Solution**: The surface $S:|x|+|y|+|z|=1$ is symmetric about the $x-y$, $y-z$, and $z-x$ planes. So we can first consider the part of the surface in the first-octant where $x\ge 0,y\ge 0,z\ge 0$ and $z=1-x-y$.

The surface element $\mathrm{d}S=\sqrt{1+z_x^2+z_y^2}\mathrm{d}x\mathrm{d}y$, where $z_x=-1$ and $z_y=-1$, so $\mathrm{d}S=\sqrt{1+(-1)^2+(-1)^2}\mathrm{d}x\mathrm{d}y=\sqrt{3}\mathrm{d}x\mathrm{d}y$. The region D in the xOy plane is bounded by $x\ge 0,y\ge 0,x+y\le 1$. The integral over the part of the surface in the first-octant is $\iint_{D_1}xy(1-x-y)\sqrt{3}\mathrm{d}x\mathrm{d}y$, where $D_1=\{(x,y):x\ge 0,y\ge 0,x+y\le 1\}$.

$$\begin{aligned}\iint_{D_1}xy(1-x-y)\mathrm{d}x\mathrm{d}y&=\int_0^1 x\mathrm{d}x\int_0^{1-x}(y-xy-y^2)\mathrm{d}y\\&=\int_0^1 x(1-x)^2\left(\frac{1}{2}-\frac{1}{2}x\right)\mathrm{d}x-\frac{1}{3}\int_0^1 x(1-x)^3\mathrm{d}x.\end{aligned}$$

Let $t=1-x$, then $x=1-t$ and $\mathrm{d}x=-\mathrm{d}t$.

$$\int_0^1 x(1-x)^2\left(\frac{1}{2}-\frac{1}{2}x\right)\mathrm{d}x=\frac{1}{2}\int_0^1(1-t)t^2(1-(1-t))\mathrm{d}t=\frac{1}{2}\left(\frac{1}{4}-\frac{1}{5}\right),$$

$$\int_0^1 x(1-x)^3\mathrm{d}x=\int_0^1(1-t)t^3\mathrm{d}t=\frac{1}{4}-\frac{1}{5}.$$

The integral over the whole surface S is 8 times the integral over the first-octant part (due to the eight-fold symmetry of the surface $|x|+|y|+|z|=1$)

$$\iint_S |xyz|\mathrm{d}S = 8\sqrt{3}\iint_{D_1} xy(1-x-y)\mathrm{d}x\mathrm{d}y = \frac{\sqrt{3}}{15}$$

2. When S is the part of the paraboloid $z=x^2+y^2$ cut off by $z=1$ We use polar coordinates $x=r\cos\theta, y=r\sin\theta$, then $z=r^2$ and $\mathrm{d}S=\sqrt{1+z_x^2+z_y^2}\mathrm{d}x\mathrm{d}y = \sqrt{1+4x^2+4y^2}\mathrm{d}x\mathrm{d}y=\sqrt{1+4r^2}r\mathrm{d}r\mathrm{d}\theta$. The region D in the $x-y$ plane is $x^2+y^2\le 1$, or in polar coordinates $0\le r\le 1, 0\le\theta\le 2\pi$. $\iint_S |xyz|\mathrm{d}S = \int_0^{2\pi}|\cos\theta\sin\theta|\mathrm{d}\theta\int_0^1 r^4\cdot r\sqrt{1+4r^2}\mathrm{d}r$. Since $\int_0^{2\pi}|\cos\theta\sin\theta|\mathrm{d}\theta = 4\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\mathrm{d}\theta=2$. Let $u=1+4r^2$, then $r^2=\frac{u-1}{4}$ and $\mathrm{d}r=\frac{1}{8r}\mathrm{d}u$. $\int_0^1 r^5\sqrt{1+4r^2}\mathrm{d}r=\frac{1}{128}\int_1^5(u-1)^2\sqrt{u}\mathrm{d}u$.

$$\begin{aligned}\frac{1}{128}\int_1^5(u^2-2u+1)\sqrt{u}\mathrm{d}u &= \frac{1}{128}\int_1^5(u^{\frac{5}{2}}-2u^{\frac{3}{2}}+u^{\frac{1}{2}})\mathrm{d}u\\ &= \frac{1}{128}\left[\frac{2}{7}u^{\frac{7}{2}}-\frac{4}{5}u^{\frac{5}{2}}+\frac{2}{3}u^{\frac{3}{2}}\right]_1^5 = \frac{25}{84}\sqrt{5}-\frac{1}{420}.\end{aligned}$$

Exercise 4.3. Calculate $\iint_S(x^2+y^2+z^2)\mathrm{d}S$, where S is the surface given by $|x|+|y|+|z|=a$.

<u>Solution:</u> First, note the symmetry of the surface $S: |x|+|y|+|z|=a$. The surface S is symmetric about the xOy, yOz, and zOx planes. We know that $\iint_S x^2\mathrm{d}S=\iint_S y^2\mathrm{d}S=\iint_S z^2\mathrm{d}S$ due to the symmetry of the surface. So $\iint_S(x^2+y^2+z^2)\mathrm{d}S=3\iint_S x^2\mathrm{d}S$. We only need to consider the part of the surface in the first-octant, where $x\ge 0, y\ge 0, z\ge 0$ and $x+y+z=a$, or $z=a-x-y$. The surface element $\mathrm{d}S=\sqrt{1+z_x^2+z_y^2}\mathrm{d}x\mathrm{d}y$. Since $z_x=-1$ and $z_y=-1$, then $\mathrm{d}S=\sqrt{1+(-1)^2+(-1)^2}\mathrm{d}x\mathrm{d}y=\sqrt{3}\mathrm{d}x\mathrm{d}y$. The region D in the xOy plane is bounded by $x\ge 0, y\ge 0, x+y\le a$. $\iint_S x^2\mathrm{d}S=8\iint_D x^2\sqrt{3}\mathrm{d}x\mathrm{d}y$ (the factor of 8 comes from the eight-fold symmetry of the surface $|x|+|y|+|z|=a$). We calculate the double integral $\iint_D x^2\mathrm{d}x\mathrm{d}y=\int_0^a x^2\mathrm{d}x\int_0^{a-x}\mathrm{d}y=\int_0^a x^2(a-x)\mathrm{d}x$ as

$$\int_0^a x^2(a-x)\mathrm{d}x = a\int_0^a x^2\mathrm{d}x-\int_0^a x^3\mathrm{d}x=\frac{a^4}{12}.$$

So

$$\iint_S(x^2+y^2+z^2)\mathrm{d}S = 3\times 8\times\sqrt{3}\times\frac{a^4}{12}=2\sqrt{3}a^4.$$

4.2 Surface integrals of type II

Let S be a piece-wise smooth oriented surface, and P,Q,R be functions defined on the surface S. Make a partition T on the specified side of the surface S, which divides the surface S into n small surfaces $S_1, S_2, \cdots, S_n$. The fineness of the partition T is $\|T\| = \max_{1\le i\le n}\{\text{diameter of } S_i\}$. Denote $\Delta S_i^{(1)}, \Delta S_i^{(2)}, \Delta S_i^{(3)}$ as the areas of the projection regions of S_i on the three coordinate planes yOz, zOx, and xOy respectively. Their signs are determined by the direction of S_i. When the positive direction of the normal of S_i forms an acute angle with the positive direction of the z-axis, the area $\Delta S_i^{(3)}$ of the projection region of S_i on the xOy plane is positive; otherwise, it is negative. The signs of $\Delta S_i^{(1)}$ and $\Delta S_i^{(2)}$ can be defined similarly. Arbitrarily take a point (ξ_i, η_i, ζ_i) on each S_i. If

$$\lim_{\|T\|\to 0}\sum_{i=1}^{n} P(\xi_i,\eta_i,\zeta_i)\Delta S_i^{(1)} + \lim_{\|T\|\to 0}\sum_{i=1}^{n} Q(\xi_i,\eta_i,\zeta_i)\Delta S_i^{(2)} + \lim_{\|T\|\to 0}\sum_{i=1}^{n} R(\xi_i,\eta_i,\zeta_i)\Delta S_i^{(3)}$$

exists and is independent of the partition T of the surface S and the choice of the points (ξ_i, η_i, ζ_i) on S_i, then this limit is called the surface integral of the second kind of the functions P, Q, R on the specified side of the surface S, denoted as

$$\iint_S P(x,y,z)\mathrm{d}y\mathrm{d}z + Q(x,y,z)\mathrm{d}z\mathrm{d}x + R(x,y,z)\mathrm{d}x\mathrm{d}y.$$

Let $R(x, y, z)$ be a continuous function defined on the smooth surface

$$S : z = z(x, y), \quad (x, y) \in D_{xy}.$$

Taking the upper side of S as the positive side, then

$$\iint_S R(x,y,z)\mathrm{d}x\mathrm{d}y = \pm\iint_{D_{xy}} R(x,y,z(x,y))\mathrm{d}x\mathrm{d}y.$$

If the integration is carried out on the positive side, take the plus sign in front of the integral sign; if the integration is carried out on the negative side, take the minus sign in front of the integral sign. Similarly, when $P(x, y, z)$ is continuous on the smooth surface

$$S : x = x(y, z), \quad (y, z) \in D_{yz}.$$

There is

$$\iint_S P(x,y,z)\mathrm{d}y\mathrm{d}z = \pm\iint_{D_{yz}} P(x(y,z),y,z)\mathrm{d}y\mathrm{d}z.$$

Here, the positive side of S is the side where the normal direction of S forms an acute angle with the positive direction of the x-axis. When $Q(x, y, z)$ is continuous on the smooth surface

$$S : y = y(z, x), \quad (z, x) \in D_{zx}.$$

There is

$$\iint_S Q(x,y,z)\mathrm{d}z\mathrm{d}x = \pm \iint_{D_{zx}} Q(x,y(z,x),z)\mathrm{d}z\mathrm{d}x.$$

Here, the positive side of S is the side where the normal direction of S forms an acute angle with the positive direction of the y-axis. If the smooth surface S is given by the parametric equations:

$$S: x = x(u,v), \quad y = y(u,v), \quad z = z(u,v), \quad (u,v) \in D.$$

And the determinants

$$A = \frac{\partial(y,z)}{\partial(u,v)}, \quad B = \frac{\partial(z,x)}{\partial(u,v)}, \quad C = \frac{\partial(x,y)}{\partial(u,v)}$$

are not all zero at the same time.

$$\iint_S P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = \pm \iint_D (PA + QB + RC)\mathrm{d}u\mathrm{d}v.$$

The rule for choosing the plus or minus sign in front of the integral sign is as follows: If the angle between the vector (A,B,C) and the direction of the normal vector of the pre-selected side of the surface S is not greater than a right angle, then take the " + " sign; otherwise, take the "-" sign.

Example 4.4. Let Σ be the upper half of the unit sphere $z = \sqrt{1-(x^2+y^2)}$, and take the inner side. Find

$$I = \iint_\Sigma \mathrm{d}y\mathrm{d}z + \mathrm{d}z\mathrm{d}x + \mathrm{d}x\mathrm{d}y.$$

Solution:

1. Calculation in the rectangular coordinate system

$$I = \iint_\Sigma \mathrm{d}y\mathrm{d}z + \iint_\Sigma \mathrm{d}z\mathrm{d}x + \iint_\Sigma \mathrm{d}x\mathrm{d}y = I_1 + I_2 + I_3.$$

Calculate I_1: $\Sigma = \Sigma_1 + \Sigma_2$, where $\Sigma_1 : x = \sqrt{1-y^2-z^2}, \quad (y,z) \in D_{yz} = \{y^2+z^2 \leq 1, z \geq 0\}$, taking the rear side, $\Sigma_2 : x = -\sqrt{1-y^2-z^2}, \quad (y,z) \in D_{yz}$, taking the front side. Then

$$I_1 = \iint_{\Sigma_1} \mathrm{d}y\mathrm{d}z + \iint_{\Sigma_2} \mathrm{d}y\mathrm{d}z = -\iint_{D_{yz}} \mathrm{d}y\mathrm{d}z + \iint_{D_{yz}} \mathrm{d}y\mathrm{d}z = 0.$$

Similarly, we have

$$I_2 = 0, I_3 = -\iint_{D_{xy}} \mathrm{d}x\mathrm{d}y = -\pi$$

Finally, we get

$$I = -\pi$$

2. Using parametric equations

$$\Sigma : x = \sin\varphi\cos\theta, \quad y = \sin\varphi\sin\theta, \quad z = \cos\varphi,$$
$$0 \le \varphi \le \frac{\pi}{2}, 0 \le \theta \le 2\pi$$

Calculate the determinants

$$A = \frac{\partial(y,z)}{\partial(\varphi,\theta)} = \sin^2\varphi\cos\theta, \quad B = \frac{\partial(z,x)}{\partial(\varphi,\theta)} = \sin^2\varphi\sin\theta$$
$$C = \frac{\partial(x,y)}{\partial(\varphi,\theta)} = \sin\varphi\cos\varphi.$$

Since the direction of (A, B, C) is opposite to the direction of the normal vector of the inner side of the upper hemisphere S, we take the "-" sign in front of the integral sign. Then we get

$$\begin{aligned}
I &= -\iint_D \left(\sin^2\varphi\cos\theta + \sin^2\varphi\sin\theta + \sin\varphi\cos\varphi\right) \mathrm{d}\varphi\mathrm{d}\theta \\
&= -\int_0^{\pi/2} \mathrm{d}\varphi \int_0^{2\pi} \left[\sin^2\varphi(\cos\theta + \sin\theta) + \sin\varphi\cos\varphi\right] \mathrm{d}\theta \\
&= -2\pi \int_0^{\pi/2} \sin\varphi\cos\varphi\mathrm{d}\varphi = -\pi
\end{aligned}$$

Exercise 4.5. Calculate the surface integral $I = \iint_S x\mathrm{d}y\mathrm{d}z + y\mathrm{d}z\mathrm{d}x + z\mathrm{d}x\mathrm{d}y$, where S is the finite part of the paraboloid $z = x^2 + y^2$ cut off by the plane $z = 4$, and the direction of integration is the outer side.

Solution: Let $\vec{F} = (x, y, z)$. The divergence of the vector field $\vec{F}$ is calculated as follows:

$$\mathrm{div}\vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

We consider the closed surface $S + S_1$, where S is the paraboloid $z = x^2 + y^2$ and S_1 is the disk $x^2 + y^2 \le 4, z = 4$ with the downward orientation.

By the divergence theorem, we have $\iiint_\Omega \mathrm{div}\vec{F}\mathrm{d}V = \iint_{S+S_1} \vec{F} \cdot d\vec{S}$, where Ω is the region bounded by S and S_1.

We use cylindrical coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, and $\mathrm{d}V = rdzdrd\theta$. The limits of integration are: $0 \le r \le 2$, $r^2 \le z \le 4$, $0 \le \theta \le 2\pi$.

First, calculate the triple-integral:

$$\iiint_\Omega 3\mathrm{d}V = 3\int_0^{2\pi} d\theta \int_0^2 r\left(\int_{r^2}^4 dz\right)\mathrm{d}r = 6\pi\int_0^2 (4r - r^3)\mathrm{d}r = 24\pi$$

Now, calculate $\iint_{S_1} \vec{F} \cdot d\vec{S}$. On S_1, the normal vector $\boldsymbol{n} = (0, 0, -1)$, and $\vec{F} = (x, y, 4)$. So, $\vec{F} \cdot \boldsymbol{n} = -4$.

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{x^2+y^2\le 4} -4\mathrm{d}x\mathrm{d}y = -4 \cdot \pi \cdot 2^2 = -16\pi$$

Then,

$$\iint_S \vec{F} \cdot d\vec{S} = 24\pi - (-16\pi) = 40\pi$$

Exercise 4.6. Calculate the surface integral $I = \iint_S x^3 \mathrm{d}y\mathrm{d}z + y^3 \mathrm{d}z\mathrm{d}x + z^3 \mathrm{d}x\mathrm{d}y$, where S is the part of the unit sphere in the first quadrant, and the outer side is taken.

Solution: Let $\vec{F} = (x^3, y^3, z^3)$. The divergence of the vector field $\vec{F}$ is:

$$\mathrm{div}\vec{F} = 3(x^2 + y^2 + z^2)$$

By the divergence theorem, $\iint_S \vec{F} \cdot d\vec{S} = \iiint_\Omega \mathrm{div}\vec{F}\mathrm{d}V$, where Ω is the part of the unit ball $x^2 + y^2 + z^2 \leq 1$ in the first quadrant. We use spherical coordinates: $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, $z = \rho \cos\varphi$, and $\mathrm{d}V = \rho^2 \sin\varphi d\rho d\varphi d\theta$. The limits of integration are: $0 \leq \rho \leq 1$, $0 \leq \varphi \leq \frac{\pi}{2}$, $0 \leq \theta \leq \frac{\pi}{2}$. Calculate the triple-integral:

$$\begin{aligned}\iiint_\Omega 3(x^2 + y^2 + z^2)\mathrm{d}V &= 3\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi d\varphi \int_0^1 \rho^2 \cdot \rho^2 d\rho \\ &= 3 \cdot \frac{\pi}{2} \cdot [-\cos\varphi]_0^{\frac{\pi}{2}} \cdot \left[\frac{1}{5}\rho^5\right]_0^1 = 3 \cdot \frac{\pi}{2} \cdot (0+1) \cdot \frac{1}{5} = \frac{3\pi}{10}\end{aligned}$$

Exercise 4.7. Calculate the surface integral $I = \iint_S x^2 \mathrm{d}y\mathrm{d}z + y^2 \mathrm{d}z\mathrm{d}x + xy\mathrm{d}x\mathrm{d}y$, where S is the boundary of the spatial region $\Omega : \frac{(x-1)^2}{4} + \frac{(y-1)^2}{9} \leq z \leq 1$, and the outer side is taken.

Solution: Let $\vec{F} = (x^2, y^2, xy)$. The divergence of the vector field $\vec{F}$ is:

$$\mathrm{div}\vec{F} = 2x + 2y$$

By the divergence theorem, $\iint_S \vec{F} \cdot d\vec{S} = \iiint_\Omega (2x + 2y)\mathrm{d}V$. We make a translation $u = x - 1$, $v = y - 1$, then $x = u + 1$, $y = v + 1$, and the region Ω becomes $\frac{u^2}{4} + \frac{v^2}{9} \leq z \leq 1$.

$$\iiint_\Omega (2x + 2y)\mathrm{d}V = \iiint_\Omega (2(u+1) + 2(v+1))\mathrm{d}V = \iiint_\Omega (2u + 2v + 4)\mathrm{d}V$$

Since $\iiint_\Omega 2u\mathrm{d}V = 0$ and $\iiint_\Omega 2v\mathrm{d}V = 0$ due to the symmetry of the region about the planes $u = 0$ and $v = 0$. We only need to calculate $\iiint_\Omega 4\mathrm{d}V$. The base of the region in the uv-plane is an ellipse $\frac{u^2}{4} + \frac{v^2}{9} \leq 1$. We use the transformation $u = 2r\cos\theta$, $v = 3r\sin\theta$, $dudv = 6r\mathrm{d}rd\theta$, and the region in the $r\theta$-plane is $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$.

$$\begin{aligned}\iiint_\Omega 4\mathrm{d}V &= 4\int_0^{2\pi} d\theta \int_0^1 6r\left(1 - r^2\right)\mathrm{d}r \\ &= 24\int_0^{2\pi} d\theta \int_0^1 (r - r^3)\mathrm{d}r = 24 \cdot 2\pi \left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^1 = 24\pi\end{aligned}$$

4.3 The connections between type I and type II

Let S be an orientable surface, and $\boldsymbol{n}$ be the normal vector of a selected side on S, then

$$\begin{aligned}&\iint_S P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y\\ &= \iint_S [P\cos(\boldsymbol{n},x) + Q\cos(\boldsymbol{n},y) + R\cos(\boldsymbol{n},z)]\mathrm{d}S.\end{aligned}$$

One of the uses of the above relation is that it can simplify the calculation of surface integrals. When the calculation of a certain type of surface integral is relatively complicated, the formula above can be used to transform it into another type of surface integral for calculation.

Example 4.8. Calculate

$$I = \iint_\Sigma (y-z)\mathrm{d}y\mathrm{d}z + (z-x)\mathrm{d}z\mathrm{d}x + (x-y)\mathrm{d}x\mathrm{d}y,$$

where Σ is the part of the spherical surface $x^2+y^2+z^2 = 2Rx$ that is cut off by the cylindrical surface $x^2+y^2 = 2rx(0<r<R)$ and located in the region $z \geq 0$, and the outer side is taken.

Solution: Rewrite the equation of the spherical surface as $(x-R)^2+y^2+z^2 = R^2$. The normal vector of its outer side is

$$\boldsymbol{n} = \left(\frac{x-R}{R}, \frac{y}{R}, \frac{z}{R}\right)$$

We have

$$\begin{aligned}I &= \iint_\Sigma (y-z, z-x, x-y)\cdot\boldsymbol{n}\mathrm{d}S\\ &= \frac{1}{R}\iint_\Sigma [(y-z)(x-R) + (z-x)y + (x-y)z]\mathrm{d}S\\ &= \iint_\Sigma (z-y)\mathrm{d}S.\end{aligned}$$

Since Σ is symmetric about the xOz plane, and the function y is an odd function, then

$$\begin{aligned}I &= \iint_\Sigma z\mathrm{d}S = \iint_{x^2+y^2\leq 2rx} \sqrt{2Rx-x^2-y^2}\frac{R}{\sqrt{2Rx-x^2-y^2}}\mathrm{d}x\mathrm{d}y\\ &= R\cdot\pi r^2 = \pi R r^2.\end{aligned}$$

Exercise 4.9. Let the equation of the surface Σ be $z = z(x,y), (x,y)\in D$, and $z(x,y)$ be continuously differentiable in $\overline{D}$. Prove that:

$$\begin{aligned}&\iint_\Sigma P(x,y,z)\mathrm{d}y\mathrm{d}z + Q(x,y,z)\mathrm{d}z\mathrm{d}x + R(x,y,z)\mathrm{d}x\mathrm{d}y\\ &= \pm\iint_D (-Pz_x - Qz_y + R)\bigg|_{z=z(x,y)}\mathrm{d}x\mathrm{d}y\end{aligned}$$

The positive sign is taken when Σ takes the upper side, and the negative sign is taken when Σ takes the lower side.

Solution: It follows from the formula at the beginning of this section.

Exercise 4.10. If Σ is piecewise smooth and symmetric about the xOy plane, $f(x,y,z)$ is continuous on $\overline{\Sigma}$, and satisfies $f(x,y,z) = -f(x,y,-z)$. Ask: Is $\iint_\Sigma f(x,y,z)\mathrm{d}S = 2\iint_{\Sigma_1} f(x,y,z)\mathrm{d}S$ or equal to 0? (where Σ_1 is the part of Σ above the xOy plane.)

Solution: 0.

Exercise 4.11. If Σ is piecewise smooth and symmetric about the xOy plane, $R(x,y,z)$ is continuous on $\overline{\Sigma}$, and satisfies $R(x,y,z) = -R(x,y,-z)$. Ask: Is $\iint_\Sigma R(x,y,z)\mathrm{d}x\mathrm{d}y = 2\iint_{\Sigma_1} R(x,y,z)\mathrm{d}x\mathrm{d}y$ or equal to 0? (where Σ_1 is the part of Σ above the xOy plane, and the orientation is consistent with that of Σ.)

Solution: $\iint_\Sigma R(x,y,z)\mathrm{d}x\mathrm{d}y = 2\iint_{\Sigma_1} R(x,y,z)\mathrm{d}x\mathrm{d}y$.

Exercise 4.12. Let Σ be a bounded region in the plane Π, and its area is S. The normal vector of Π taking the upper side is $\boldsymbol{n}$, and $\cos(\boldsymbol{n},z) = \mu$. Prove that the area of the projection of Σ on the xOy plane is μS.

Solution: Assume that the projective region is D. On D. We see that

$$\iint_\Sigma \cos(\boldsymbol{n},z)\mathrm{d}S = \mu S = \iint_\Sigma \mathrm{d}x\mathrm{d}y = \iint_D \mathrm{d}x\mathrm{d}y.$$

Exercise 4.13. Find $I_1 = \iint_\Sigma z\mathrm{d}x\mathrm{d}y$, $I_2 = \iint_\Sigma z^2\mathrm{d}x\mathrm{d}y$, where Σ is the spherical surface $x^2+y^2+z^2=a^2$, taking the outer side.

Solution: 1 Calculate $I_1 = \iint_\Sigma z\mathrm{d}x\mathrm{d}y$ The sphere $\Sigma : x^2+y^2+z^2 = a^2$ is symmetric about the xOy-plane. We split the sphere $\Sigma = \Sigma_1 + \Sigma_2$, where Σ_1 is the upper-hemisphere $z = \sqrt{a^2-x^2-y^2}$ (with upward orientation) and Σ_2 is the lower-hemisphere $z = -\sqrt{a^2-x^2-y^2}$ (with downward orientation). The projection of the sphere on the xOy-plane is the disk $D : x^2+y^2 \le a^2$. By the property of surface integrals and orientation: $I_1 = \iint_\Sigma z\mathrm{d}x\mathrm{d}y = 2\iint_{\Sigma_1} z\mathrm{d}x\mathrm{d}y$ Using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $\mathrm{d}x\mathrm{d}y = r\mathrm{d}rd\theta$ and $z = \sqrt{a^2-r^2}$ on Σ_1 $I_1 = 2\int_0^{2\pi} d\theta \int_0^a \sqrt{a^2-r^2}r\mathrm{d}r$ Let $u = a^2-r^2$, $du = -2r\mathrm{d}r$. When $r = 0$, $u = a^2$; when $r = a$, $u = 0$ $I_1 = 2\times 2\pi \times \left(-\frac{1}{2}\right)\int_{a^2}^0 \sqrt{u}du = \frac{4}{3}\pi a^3$.

2. Calculate $I_2 = \iint_\Sigma z^2\mathrm{d}x\mathrm{d}y$. Since the sphere Σ is symmetric about the xOy-plane and the integrand $f(x,y,z) = z^2$ satisfies $f(x,y,z) = f(x,y,-z)$. Also, the orientation of the upper-hemisphere Σ_1 and the lower-hemisphere Σ_2 is opposite with respect to the xOy-plane. For the surface integral $\iint_\Sigma z^2\mathrm{d}x\mathrm{d}y$, the contributions from Σ_1 and Σ_2 cancel each other out. So $I_2 = 0$.

4.4 Gauss' formula

Gauss' formula is an important formula that establishes a connection between the triple-integral over a certain region in $\mathbb{R}^3$ and a specific surface integral over the boundary of this region.

Let D be a bounded region in $\mathbb{R}^3$, and its boundary ∂D consists of smooth surfaces or piece-wise smooth surfaces, with the orientation being the outer side (relative to the region D). Suppose that the functions P, Q, R all have continuous partial derivatives with respect to x, y, z on D. Then the following Gauss' formula holds:

$$\iiint_D \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \oiint_{\partial D} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y.$$

Using the relationship between the two types of surface integrals, Gauss' formula can also be written as

$$\begin{aligned}&\iiint_D \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ =&\oiint_{\partial D}(P\cos(\boldsymbol{n},\boldsymbol{x})+Q\cos(\boldsymbol{n},y)+R\cos(\boldsymbol{n},z))\mathrm{d}\boldsymbol{S},\end{aligned}$$

where $\boldsymbol{n}$ is the outer normal vector to the surface ∂D.

Example 4.14 (Auxiliary surfaces)**.** Calculate

$$I=\iint_\Sigma 4xz\mathrm{d}y\mathrm{d}z - 2yz\mathrm{d}z\mathrm{d}x + \left(1-z^2\right)\mathrm{d}x\mathrm{d}y,$$

where Σ is the surface of revolution generated by rotating the curve $z=\mathrm{e}^y(0\le y\le a)$ about the z-axis, with the lower side orientation.

Solution: Solve the equation of Σ as

$$z=\mathrm{e}^{\sqrt{x^2+y^2}}\left(x^2+y^2\le a^2\right).$$

Direct calculation is rather complicated. Consider using Gauss' formula. Since Σ is not a closed surface, we need to add an auxiliary surface $\Sigma_1: z=\mathrm{e}^a, \left(x^2+y^2\le a^2\right)$, with the upward orientation. Let the region enclosed by Σ and Σ_1 be D. Let

$$P=4xz,\quad Q=-2yz,\quad R=1-z^2.$$

Then

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0.$$

According to Gauss' formula, we have

$$\begin{aligned}I&=\left(\iint_\Sigma+\iint_{\Sigma_1}-\iint_{\Sigma_1}\right)\left(1-z^2\right)\mathrm{d}x\mathrm{d}y=-\iint_{\Sigma_1}\left(1-z^2\right)\mathrm{d}x\mathrm{d}y\\ &=\left(\mathrm{e}^{2a}-1\right)\iint_{x^2+y^2\le a^2}\mathrm{d}x\mathrm{d}y=\left(\mathrm{e}^{2a}-1\right)\pi a^2.\end{aligned}$$

Exercise 4.15. First, add an auxiliary surface and then use Gauss' formula to calculate the following surface integrals.

1. $\iint_\Sigma (x^2 \cos\alpha + y^2 \cos\beta + z^2 \cos\gamma) dS$, where Σ is the part of the cone $z^2 = x^2 + y^2$ for $0 \le z \le h$, and $(\cos\alpha, \cos\beta, \cos\gamma)$ is the unit normal vector on Σ with the downward direction.

2. $\iint_\Sigma x^3 \mathrm{d}y\mathrm{d}z + y^3 \mathrm{d}z\mathrm{d}x + z^3 \mathrm{d}x\mathrm{d}y$, where Σ is the upper-half of the sphere $x^2 + y^2 + z^2 = a^2$.

3. $\iint_\Sigma \left(\frac{x^3}{a^3} + y^3 z^3\right) \mathrm{d}y\mathrm{d}z + \left(\frac{y^3}{b^3} + z^3 x^3\right) \mathrm{d}z\mathrm{d}x + \left(\frac{z^3}{c^3} + x^3 y^3\right) \mathrm{d}x\mathrm{d}y$, where Σ is the part of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ with $x \ge 0$, and the orientation is the back-side.

Solution:

1. Add a plane $\Sigma_1 : z = h, x^2 + y^2 \le h^2$ with the upward orientation. Then the closed surface $\Omega = \Sigma + \Sigma_1$ is a closed-surface enclosing a cone-shaped region V. According to Gauss' formula, $\oiint_\Omega (x^2 \cos\alpha + y^2 \cos\beta + z^2 \cos\gamma) dS = \iiint_V (2x + 2y + 2z) \mathrm{d}V$. Use cylindrical coordinates: $x = r\cos\theta, y = r\sin\theta, z = z$, and $\mathrm{d}V = r\, dz\, \mathrm{d}r\, d\theta$. The limits of integration are $0 \le \theta \le 2\pi, 0 \le r \le h, r \le z \le h$. $\iiint_V (2x + 2y + 2z) \mathrm{d}V = \int_0^{2\pi} \mathrm{d}\theta \int_0^h r\ \mathrm{d}r \int_r^h (2r\cos\theta + 2r\sin\theta + 2z) dz$. Since $\int_0^{2\pi} \cos\theta d\theta = \int_0^{2\pi} \sin\theta d\theta = 0$, we have:

$$\iiint_V (2x + 2y + 2z) \mathrm{d}V = \int_0^{2\pi} d\theta \int_0^h r\left[(2r\cos\theta + 2r\sin\theta)(h - r) + h^2 - r^2\right] \mathrm{d}r.$$

After integrating with respect to θ, the terms with $\cos\theta$ and $\sin\theta$ disappear. So we only need to calculate

$$\int_0^{2\pi} d\theta \int_0^h r(h^2 - r^2) \mathrm{d}r = 2\pi \int_0^h (h^2 r - r^3) \mathrm{d}r = 2\pi \left[\frac{h^2 r^2}{2} - \frac{r^4}{4}\right]_0^h = \frac{\pi h^4}{2}.$$

Calculate the surface integral on Σ_1. On Σ_1, $\cos\alpha = \cos\beta = 0, \cos\gamma = 1$, and $z = h$. So $\iint_{\Sigma_1} (x^2 \cos\alpha + y^2 \cos\beta + z^2 \cos\gamma) dS = \iint_{x^2+y^2 \le h^2} h^2 \mathrm{d}x\mathrm{d}y = \pi h^4$. Then $\iint_\Sigma (x^2 \cos\alpha + y^2 \cos\beta + z^2 \cos\gamma) dS = \frac{\pi h^4}{2} - \pi h^4 = -\frac{\pi h^4}{2}$.

2. Add a plane $\Sigma_1 : z = 0, x^2 + y^2 \le a^2$ with the downward orientation. Then the closed surface $\Omega = \Sigma + \Sigma_1$ encloses the upper-half of the sphere V. According to Gauss' formula, $\oiint_\Omega x^3 \mathrm{d}y\mathrm{d}z + y^3 \mathrm{d}z\mathrm{d}x + z^3 \mathrm{d}x\mathrm{d}y = \iiint_V (3x^2 + 3y^2 + 3z^2) \mathrm{d}V$. Use spherical coordinates: $x = \rho\sin\varphi\cos\theta, y = \rho\sin\varphi\sin\theta, z = \rho\cos\varphi$, and $\mathrm{d}V = \rho^2 \sin\varphi d\rho d\varphi d\theta$. The limits of integration are $0 \le \theta \le 2\pi, 0 \le \varphi \le \frac{\pi}{2}, 0 \le \rho \le a$.

$$\iiint_V (3x^2 + 3y^2 + 3z^2) \mathrm{d}V = 3\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi d\varphi \int_0^a \rho^4 d\rho.$$

$\int_0^{2\pi} d\theta = 2\pi, \int_0^{\frac{\pi}{2}} \sin\varphi d\varphi = 1, \int_0^a \rho^4 d\rho = \frac{a^5}{5}$. So $\iiint_V (3x^2+3y^2+3z^2)\mathrm{d}V = \frac{6\pi a^5}{5}$. Calculate the surface integral on Σ_1. On Σ_1, $z=0$, so $\iint_{\Sigma_1} x^3\mathrm{d}y\mathrm{d}z + y^3\mathrm{d}z\mathrm{d}x + z^3\mathrm{d}x\mathrm{d}y = 0$. Then $\iint_\Sigma x^3\mathrm{d}y\mathrm{d}z + y^3\mathrm{d}z\mathrm{d}x + z^3\mathrm{d}x\mathrm{d}y = \frac{6\pi a^5}{5}$.

3. Add a plane $\Sigma_1 : x = 0, \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$ with the forward orientation. Then the closed surface $\Omega = \Sigma + \Sigma_1$ encloses the right-half of the ellipsoid V. According to Gauss' formula,

$$\oiint_\Omega \left(\frac{x^3}{a^3} + y^3z^3\right)\mathrm{d}y\mathrm{d}z + \left(\frac{y^3}{b^3} + z^3x^3\right)\mathrm{d}z\mathrm{d}x + \left(\frac{z^3}{c^3} + x^3y^3\right)\mathrm{d}x\mathrm{d}y = \iiint_V \left(\frac{3x^2}{a^3} + \frac{3y^2}{b^3} + \frac{3z^2}{c^3}\right)\mathrm{d}V.$$

Use the generalized spherical coordinates: $x = a\rho\sin\varphi\cos\theta, y = b\rho\sin\varphi\sin\theta, z = c\rho\cos\varphi$, and $\mathrm{d}V = abc\rho^2\sin\varphi d\rho d\varphi d\theta$. The limits of integration are $0 \le \theta \le 2\pi, 0 \le \varphi \le \pi, 0 \le \rho \le 1$.

$$\begin{aligned}&\iiint_V \left(\frac{3x^2}{a^3} + \frac{3y^2}{b^3} + \frac{3z^2}{c^3}\right)\mathrm{d}V \\ &= 3abc\int_0^{2\pi} d\theta \int_0^\pi \sin\varphi d\varphi \int_0^1 \left(\frac{a\rho^2\sin^2\varphi\cos^2\theta}{a^3} + \frac{b\rho^2\sin^2\varphi\sin^2\theta}{b^3} + \frac{c\rho^2\cos^2\varphi}{c^3}\right)\rho^2 d\rho.\end{aligned}$$

After a series of integrations and simplifications, we get $\iiint_V \left(\frac{3x^2}{a^3} + \frac{3y^2}{b^3} + \frac{3z^2}{c^3}\right)\mathrm{d}V = \frac{4\pi abc}{5}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$. Calculate the surface integral on Σ_1. On Σ_1, $x = 0$, so

$$\iint_{\Sigma_1} \left(\frac{x^3}{a^3} + y^3z^3\right)\mathrm{d}y\mathrm{d}z + \left(\frac{y^3}{b^3} + z^3x^3\right)\mathrm{d}z\mathrm{d}x + \left(\frac{z^3}{c^3} + x^3y^3\right)\mathrm{d}x\mathrm{d}y = \iint_{\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1} \left(y^3z^3\right)\mathrm{d}y\mathrm{d}z = 0$$

(by symmetry). Since Σ is taken with the back-side orientation,

$$\iint_\Sigma \left(\frac{x^3}{a^3} + y^3z^3\right)\mathrm{d}y\mathrm{d}z + \left(\frac{y^3}{b^3} + z^3x^3\right)\mathrm{d}z\mathrm{d}x + \left(\frac{z^3}{c^3} + x^3y^3\right)\mathrm{d}x\mathrm{d}y = -\frac{4\pi abc}{5}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Example 4.16 (Digging holes)**.** Calculate the surface integral

$$I = \iint_S \frac{x\mathrm{d}y\mathrm{d}z + y\mathrm{d}z\mathrm{d}x + z\mathrm{d}x\mathrm{d}y}{(ax^2+by^2+cz^2)^{3/2}},$$

where S is the sphere $x^2+y^2+z^2 = 1$, with the outer-side orientation ($a > 0, b > 0, c > 0$).

Solution: Denote $P(x,y,z) = \frac{x}{(ax^2+by^2+cz^2)^{3/2}}$, $Q(x,y,z) = \frac{y}{(ax^2+by^2+cz^2)^{3/2}}$, and $R(x,y,z) = \frac{z}{(ax^2+by^2+cz^2)^{3/2}}$. Then, in any region that does not contain the origin,

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0.$$

To use Gauss' formula, for a sufficiently small $\varepsilon > 0$, construct a closed surface

$$S_\varepsilon = \left\{ax^2 + by^2 + cz^2 = \varepsilon^2\right\}$$

with the outer-side orientation. By Gauss' formula,

$$I = \oiint_{S_\varepsilon} \frac{x\mathrm{d}y\mathrm{d}z + y\mathrm{d}z\mathrm{d}x + z\mathrm{d}x\mathrm{d}y}{\left(ax^2 + by^2 + cz^2\right)^{3/2}} = \frac{1}{\varepsilon^3} \oiint_{S_\varepsilon} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}z\mathrm{d}x + z\mathrm{d}x\mathrm{d}y.$$

The above integral is taken over the outer side of S_ε. Using Gauss' formula again, we have

$$I = \frac{3}{\varepsilon^3} \iiint_{ax^2+by^2+cz^2 \le \varepsilon^2} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{3}{\varepsilon^3} \cdot \frac{4\pi}{3} \cdot \frac{\varepsilon^3}{\sqrt{abc}} = \frac{4\pi}{\sqrt{abc}}$$

Example 4.17. Let S be a smooth closed surface that encloses the region Ω. Suppose the functions $u(x, y, z)$ and $v(x, y, z)$ have continuous second-order partial derivatives on $\Omega \cup S$. Prove the following equalities.

1. Let n_1 be the first component of the unit normal vector $\boldsymbol{n}(x, y, z)$. Then $\iiint_\Omega u'_x v \mathrm{d}V = -\iiint_\Omega u v'_x \mathrm{d}V + \iint_S u v n_1 \mathrm{d}S$.

2. Denote by Δu the function obtained by applying the Laplace operator to u, by ∇u the gradient of u, and by $\frac{\partial u}{\partial \boldsymbol{n}}$ the directional derivative of u in the direction of the outer normal vector on the boundary S. Then

$$\iiint_\Omega \nabla u \cdot \nabla v \mathrm{d}V = -\iiint_\Omega u \Delta v \mathrm{d}V + \iint_S u \frac{\partial v}{\partial \boldsymbol{n}} \mathrm{d}S.$$

<u>Solution:</u> By Gauss' formula, we obtain that the divergence formula

$$\int_S \vec{F} \cdot \boldsymbol{n} \mathrm{d}S = \int_\Omega \operatorname{div} \vec{F} \mathrm{d}V.$$

Letting $F = (uv, 0, 0)$, property 1 holds. Letting $v = v_x$ in property 1 and sum with x, y, z, property 2 holds.

4.5 Stokes' formula

Stokes' formula is an important formula that establishes a connection between the second type of surface integral over a surface in space and the second type of curve integral over the boundary of this surface. Let D be a piecewise smooth surface in $\mathbf{R}^3$, and the boundary ∂D of D is composed of piecewise smooth curves. Suppose that P, Q, R have continuous partial derivatives with respect to x, y, z. Then the following Stokes' formula holds:

$$\oint_{\partial D} P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = \iint_D \begin{vmatrix} \mathrm{d}y\mathrm{d}z & \mathrm{d}z\mathrm{d}x & \mathrm{d}x\mathrm{d}y \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R, \end{vmatrix}$$

where the direction of ∂D and the direction of D follow the right-hand rule. From the relationship between the two types of surface integrals, the above formula an be expressed by

$$\oint_{\partial D} P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = \iint_D \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \mathrm{d}S,$$

where $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of the normal vector on the surface D.

Example 4.18. Calculate

$$I = \oint_C \left(y^2 - z^2\right)\mathrm{d}x + \left(z^2 - x^2\right)\mathrm{d}y + \left(x^2 - y^2\right)\mathrm{d}z$$

where C is the intersection curve of the surface of the cube $\{(x, y, z) : 0 \le x \le a, 0 \le y \le a, 0 \le z \le a\}$ and the plane $x + y + z = \frac{3}{2}a$, and the orientation is counterclockwise when viewed from the positive direction of the z-axis.

Solution: Solution Let Σ be the part of the plane $x + y + z = \frac{3}{2}a$ enclosed by C, and take the upper side. Then the orientation of C is consistent with the side orientation of Σ. Applying Stokes' formula, we

$$\begin{aligned} I &= \iint_\Sigma \begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} \mathrm{d}S \\ &= \frac{1}{\sqrt{3}} \iint_\Sigma -4(x + y + z)\mathrm{d}S \\ &= -\frac{4}{\sqrt{3}} \iint_\Sigma \frac{3}{2}a\mathrm{d}S = -2\sqrt{3}a \cdot |\Sigma| \\ &= -2\sqrt{3}a \cdot \frac{3\sqrt{3}}{4}a^2 = -\frac{9}{2}a^3. \end{aligned}$$

Exercise 4.19. Use Stokes' formula to calculate

$$I = \oint_C \left(y^2 + z^2\right)\mathrm{d}x + \left(z^2 + x^2\right)\mathrm{d}y + \left(x^2 + y^2\right)\mathrm{d}z,$$

where C is the intersection curve of $x^2 + y^2 + z^2 = 2Rx$ and $x^2 + y^2 = 2rx$ $(0 < r < R, z > 0)$. The orientation of C is such that the smaller region on the sphere enclosed by C is kept on the left-hand side.

Solution: Let S be the outer side of the part of the sphere $x^2+y^2+z^2=2Rx$ cut by the cylinder $x^2+y^2=2rx$. By Stokes' formula, we have

$$\begin{aligned} I &= \iint_S \begin{vmatrix} \frac{x-R}{R} & \frac{y}{R} & \frac{z}{R} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & z^2+x^2 & x^2+y^2 \end{vmatrix} \mathrm{d}S \\ &= \frac{2}{R}\iint_S [(y-z)(x-R)+(z-x)y+(x-y)z]\mathrm{d}S \\ &= 2\iint_S (z-y)\mathrm{d}S = 2\iint_S z\mathrm{d}S = 2R\iint_{x^2+y^2\leqslant 2rx} \mathrm{d}x\mathrm{d}y = 2\pi r^2 R. \end{aligned}$$

Exercise 4.20. Let C be an arbitrary piece-wise smooth simple closed curve in space, and $f(x)$, $g(x)$, $h(x)$ be arbitrary continuous functions. Prove that

$$\oint_C [f(x)-yz]\mathrm{d}x + [g(y)-xz]\mathrm{d}y + [h(z)-xy]\mathrm{d}z = 0.$$

Solution: We use Stokes' theorem $\oint_C \vec{F}\cdot \mathrm{d}\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\mathrm{d}\vec{S}$, where $\vec{F} = (f(x)-yz, g(y)-xz, h(z)-xy)$. Calculate that $\nabla\times\vec{F} = (0,0,0)$. By Stokes' theorem, $\oint_C [f(x)-yz]\mathrm{d}x + [g(y)-xz]\mathrm{d}y + [h(z)-xy]\mathrm{d}z = \iint_S (\nabla\times\vec{F})\cdot d\vec{S} = 0$

Exercise 4.21. Evaluate the surface integral

$$\iint_\Sigma \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-z & x^3-yz & -3xy^2 \end{vmatrix} \mathrm{d}S,$$

where Σ is the part of the sphere $x^2+y^2+z^2=R^2$ with $z\geq 0$, and $(\cos\alpha,\cos\beta,\cos\gamma)$ is the unit normal vector of the lower side of Σ.

Solution: First, expand the determinant:

$$\begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-z & x^3-yz & -3xy^2 \end{vmatrix} = (-6xy+y)\cos\alpha - (-3y^2+1)\cos\beta + 3x^2\cos\gamma.$$

Let $\vec{F} = (-6xy+y, 3y^2-1, 3x^2)$. Then, use the divergence theorem $\iint_\Sigma \vec{F}\cdot\boldsymbol{n}\mathrm{d}S = \iiint_V \operatorname{div}\vec{F}\mathrm{d}V$, where V is the upper-hemisphere $x^2+y^2+z^2\leq R^2, z\geq 0$ and $\boldsymbol{n}$ is the unit normal vector of the lower side of Σ. Calculate $\operatorname{div}\vec{F} = \frac{\partial(-6xy+y)}{\partial x} + \frac{\partial(3y^2-1)}{\partial y} + \frac{\partial(3x^2)}{\partial z} = -6y+6y+0 = 0$. So

$$\iint_\Sigma \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-z & x^3-yz & -3xy^2 \end{vmatrix} \mathrm{d}S = \iiint_V \mathrm{div}\vec{F}\mathrm{d}V = 0.$$

Exercise 4.22. Evaluate the line integral $\oint_C y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z$, where C is the intersection curve of the sphere $x^2+y^2+z^2=a^2$ and the plane $x+y+z=0$, and the orientation of C is counter-clockwise when viewed from the positive z-axis.

Solution: Use Stokes' theorem $\oint_C \vec{F}\cdot \mathrm{d}\vec{r} = \iint_S (\nabla\times\vec{F})\cdot \mathrm{d}\vec{S}$, where $\vec{F}=(y,z,x)$. Calculate $\nabla\times\vec{F} = \left(\frac{\partial x}{\partial y}-\frac{\partial z}{\partial z}, \frac{\partial y}{\partial z}-\frac{\partial x}{\partial x}, \frac{\partial z}{\partial x}-\frac{\partial y}{\partial y}\right) = (-1,-1,-1)$. The normal vector $\boldsymbol{n}$ of the plane $x+y+z=0$ with the right-hand rule (since C is counter-clockwise when viewed from the positive z-axis) is $\boldsymbol{n}=\frac{(1,1,1)}{\sqrt{3}}$. The surface S bounded by C is a disk with radius a and area $A=\pi a^2$. Calculate $(\nabla\times\vec{F})\cdot\boldsymbol{n} = \frac{-1\times1-1\times1-1\times1}{\sqrt{3}} = -\sqrt{3}$. Then $\oint_C y\mathrm{d}x+z\mathrm{d}y+x\mathrm{d}z = \iint_S(\nabla\times\vec{F})\cdot\mathrm{d}\vec{S} = -\sqrt{3}\iint_S \mathrm{d}S = -\sqrt{3}\pi a^2$

Exercise 4.23. Let Σ be a smooth closed surface, and the region enclosed by it be Ω. Let $\boldsymbol{n}$ be the unit outer normal vector on Σ, and (x_0,y_0,z_0) be a fixed point inside Ω. For $(x,y,z)\in\Sigma$, let $\boldsymbol{r}=(x-x_0,y-y_0,z-z_0)$. Prove that:

$$\oiint_{\Sigma}\cos(\boldsymbol{n},\boldsymbol{r})\mathrm{d}S = 2\iiint_{\Omega}\frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{|\boldsymbol{r}|}$$

Solution: WLOG, we assume that $x_0=y_0=z_0=0$. Then,

$$\oiint_{\Sigma}\cos(\boldsymbol{n},\boldsymbol{r})\mathrm{d}S = \oiint_{\Sigma}\frac{\boldsymbol{n}\cdot\boldsymbol{r}}{|\boldsymbol{r}|}\mathrm{d}S = \iiint_{\Omega}\operatorname{div}\left(\frac{\boldsymbol{r}}{|\boldsymbol{r}|}\right)\mathrm{d}V = 2\iiint_{\Omega}\frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{|\boldsymbol{r}|}.$$

Indeed, the above calculation is not rigorous and you have to use the digging hole arguments.

Exercise 4.24. Given the plane $\Pi: Ax+By+Cz=D$. For any orientation of Π, find $w=P(x,y,z)\mathrm{d}x+Q(x,y,z)\mathrm{d}y+R(x,y,z)\mathrm{d}z$ such that for any piecewise smooth simple closed curve Γ on Π (the orientation of Γ is consistent with the orientation of Π), we always have

$$\oint_{\Gamma} w = S(\Gamma)$$

where $S(\Gamma)$ is the area of the region enclosed by Γ on Π.

Solution: Using Stokes' formula, we have

$$\begin{aligned}\oint_{\Gamma} P\mathrm{d}x+Q\mathrm{d}y+R\mathrm{d}z &= \iint_{\Sigma}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathrm{d}y\mathrm{d}z+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathrm{d}z\mathrm{d}x+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y\\ &= \iiint_{\Sigma}\frac{A\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+B\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)+C\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)}{\sqrt{A^2+B^2+C^2}}\mathrm{d}S\\ &= S(\Gamma).\end{aligned}$$

Chapter 5

ODE theory

5.1 Elementary integral method

Example 5.1 (Separable variable equation)**.** Solve:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(1-x)}{x}.$$

Solution: This is a separable variable equation. Assume $y \neq 0$. Separate the variables and integrate both sides, we have

$$\int \frac{\mathrm{d}y}{y} = \int \frac{1-x}{x}\mathrm{d}x + c_1.$$

We obtain $\ln|y| = \ln|x| - x + c_1$. Remove the logarithmic notation, we get

$$|y| = |x|\mathrm{e}^{-x+c_1}.$$

Then remove the absolute-value notation, and let $c = \pm\mathrm{e}^{c_1}$. Thus, we obtain the general solution

$$y = cx\mathrm{e}^{-x}.$$

In addition, $y = 0$ is also a solution of the original equation, which can be considered to be included in the above expression ($c = 0$).

Example 5.2 (Homogeneous equation)**.** Solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy}{x^2+y^2}.$$

Solution: Denote $f(x,y) = \frac{xy}{x^2+y^2}$. Then

$$f(x,ux) = \frac{x^2u}{x^2+x^2u^2} = \frac{u}{1+u^2},$$

which is independent of x. So, we know that the equation is a homogeneous equation. Let $y = ux$. Then the original equation is transformed into

$$u + x\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u}{1+u^2}.$$

Simplify and rearrange the terms to get

$$-\frac{1+u^2}{u^3}\mathrm{d}u = \frac{\mathrm{d}x}{x}.$$

Integrate both sides to obtain

$$\frac{1}{2u^2} - \ln|u| = \ln|c_1 x|$$

Substitute back the original variables to get the general integral $y = c\mathrm{e}^{\frac{x^2}{2y^2}}$, where $c = \pm\frac{1}{c_1}$ is an arbitrary constant. In addition, $u = 0$, that is $y = 0$, is also a solution of the original equation, which can be considered to be included in $y = c\mathrm{e}^{\frac{x^2}{2y^2}}$ (corresponding to $c = 0$).

Example 5.3 (First-order linear differential equation)**.** Solve the equation $\frac{\mathrm{d}y}{\mathrm{d}x} - y\cot x = 2x\sin x$, with the initial condition $y|_{x=\frac{\pi}{2}} = 0$.

Solution:

Method 1: Method of variation of constants First, solve the corresponding homogeneous linear equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y\cot x = 0$$

Separate the variables: $\frac{\mathrm{d}y}{y} = \frac{\cos x}{\sin x}\mathrm{d}x$. Integrate both sides: $\ln|y| = \ln|\sin x| + \ln|c|$. We get $y = c\sin x$. Vary the constant c. Let $y = u\sin x$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x\frac{\mathrm{d}u}{\mathrm{d}x} + u\cos x.$$

Substitute it into the original equation:

$$\left(\sin x\frac{\mathrm{d}u}{\mathrm{d}x} + u\cos x\right) - u\sin x\cot x = 2x\sin x.$$

So, $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$. Solve for u: $u = x^2 + c$. Substitute u back into $y = u\sin x$, we get the general solution of the original equation:

$$y = (x^2 + c)\sin x.$$

Substitute the initial condition $y|_{x=\frac{\pi}{2}} = 0$ into the general solution. We have $0 = \left(\left(\frac{\pi}{2}\right)^2 + c\right)\sin\frac{\pi}{2}$, which gives $c = -\frac{\pi^2}{4}$. So the required solution is $y = \left(x^2 - \frac{\pi^2}{4}\right)\sin x$.

Method 2: Using the formula for the solution of the initial-value problem of first-order linear differential equations We have

$$y = \mathrm{e}^{\int_{\frac{\pi}{2}}^{x}\cot\xi\mathrm{d}\xi}\left[\int_{\frac{\pi}{2}}^{x} 2\zeta\sin\zeta\cdot\mathrm{e}^{-\int_{\frac{\pi}{2}}^{\zeta}\cot\eta\mathrm{d}\eta}\mathrm{d}\zeta + 0\right] = |\sin x|\left[\int_{\frac{\pi}{2}}^{x}\frac{2\zeta\sin\zeta}{|\sin\zeta|}\mathrm{d}\zeta\right]$$

Since the original equation has the coefficient $\cot x$, then $x \neq n\pi (n = 0, \pm 1, \cdots)$. The initial condition is $y|_{x=\frac{\pi}{2}} = 0$. So the existence interval of the solution should be an interval that contains $x = \frac{\pi}{2}$ but $x \neq n\pi$. Thus, $0 < x < \pi$. Then

$$|\sin\zeta| = \sin\zeta, |\sin x| = \sin x$$

So the particular solution is

$$y = \sin x \cdot \int_{\frac{\pi}{2}}^{x} 2\zeta \mathrm{d}\zeta = \sin x \cdot \left(x^2 - \frac{\pi^2}{4}\right) = \left(x^2 - \frac{\pi^2}{4}\right) \sin x$$

Example 5.4 (Bernoulli's equation)**.** Find the solutions of the equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = a(\ln x)y^2$.

Solution: Divide both sides of the equation by y^2, we get

$$y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y^{-1} = a\ln x.$$

Let $z = y^{-1}$, then $\frac{\mathrm{d}z}{\mathrm{d}x} = -y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x}$. Thus the original equation is transformed into a linear equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{z}{x} = -a\ln x.$$

According to the general solution formula, we have

$$z = x\left[\int (-a\ln x)\frac{1}{x}\mathrm{d}x + c\right] = x\left[c - \frac{a}{2}(\ln x)^2\right].$$

Then the general solution is

$$y = \frac{1}{x\left[c - \frac{a}{2}(\ln x)^2\right]},$$

where c is an arbitrary constant. In addition, there is an obvious solution $y = 0$. This solution is not included in the above-mentioned expression. It can be seen that the general solution of this problem does not include all solutions.

Exercise 5.5. Let the functions $p(x)$ and $f(x)$ be continuous on the interval $[0, +\infty)$, and $\lim_{x\to+\infty} p(x) = a > 0$, $|f(x)| \leq b$, where a and b are constants. Prove that all solutions of the equation $\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = f(x)$ are bounded on $[0, +\infty)$.

Solution: We first solve this equation as

$$y(x) = y(0)e^{-\int_0^x p(t)\mathrm{d}t} + \int_0^x f(t)\mathrm{e}^{-\int_t^x p(s)\mathrm{d}s}\mathrm{d}t.$$

Combining the assumption, y is bounded.

Exercise 5.6. Consider the initial-value problem

$$\begin{cases} x\frac{\mathrm{d}y}{\mathrm{d}x} - (2x^2+1)y = x^2, x \geq 1 \\ y(1) = y_1. \end{cases}$$

1. Find the solution of the above initial-value problem (expressed in terms of integrals).
2. Does there exist an appropriate y_1 such that the corresponding solution $y(x)$ has a finite limit as $x \to +\infty$? If so, how many such y_1 are there? Find them and calculate $\lim_{x\to+\infty} y(x)$.

Solution: To solve the initial-value problem

$$\begin{cases} x\frac{\mathrm{d}y}{\mathrm{d}x} - \left(2x^2+1\right) y = x^2, \quad x \geq 1 \\ y(1) = y_1 \end{cases}$$

we first rewrite the differential equation in standard linear form

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \left(2x + \frac{1}{x}\right) y = x.$$

The integrating factor is calculated as:

$$\mu(x) = \exp\left(-\int \left(2x + \frac{1}{x}\right) \mathrm{d}x\right) = \exp\left(-x^2 - \ln x\right) = \frac{1}{x} e^{-x^2}.$$

Using the integrating factor, the solution to the differential equation is

$$y(x) = \frac{1}{\mu(x)} \left(\int_1^x \mu(t) \cdot t\mathrm{d}t + y_1 \mu(1)\right).$$

Substituting $\mu(x)$ and simplifying, we get

$$y(x) = xe^{x^2} \left(\int_1^x e^{-t^2}\ \mathrm{d}t + \frac{y_1}{e}\right).$$

For the second part, we analyze the behavior of $y(x)$ as $x \to \infty$. The integral $\int_1^x e^{-t^2}\ \mathrm{d}t$ approaches $\int_1^\infty e^{-t^2}\ \mathrm{d}t$ as $x \to \infty$. To ensure the solution has a finite limit, the term inside the brackets must approach zero. This requires

$$\int_1^\infty e^{-t^2}\ \mathrm{d}t + \frac{y_1}{e} = 0.$$

Solving for y_1, we find

$$y_1 = -e \int_1^\infty e^{-t^2} \mathrm{d}t.$$

With this y_1, the solution $y(x)$ asymptotically approaches $-\frac{1}{2}$ as $x \to \infty$.

5.2 Linear differential equation

Example 5.7 (Linear differential equation with constant variables). Let $f(x)$ be a continuous function and satisfy

$$f(x) = \mathrm{e}^{-x} + \frac{1}{2}\int_0^x (x-t)^2 f(t)\mathrm{d}t.$$

Find $f(x)$.

Solution: Split the integral into three terms

$$\begin{aligned}\frac{1}{2}\int_0^x (x-t)^2 f(t)\mathrm{d}t &= \frac{1}{2}\int_0^x (x^2-2xt+t^2) f(t)\mathrm{d}t \\ &= \frac{1}{2}x^2\int_0^x f(t)\mathrm{d}t - x\int_0^x tf(t)\mathrm{d}t + \frac{1}{2}\int_0^x t^2 f(t)\mathrm{d}t,\end{aligned}$$

Differentiate both sides of the given integral equation, we get

$$\begin{aligned}f'(x) &= -\mathrm{e}^{-x} + x\int_0^x f(t)\mathrm{d}t + \frac{x^2}{2}f(x) - x^2 f(x) \\ &\quad - \int_0^x tf(t)\mathrm{d}t + \frac{1}{2}x^2 f(x) \\ &= -\mathrm{e}^{-x} + x\int_0^x f(t)\mathrm{d}t - \int_0^x tf(t)\mathrm{d}t \\ f''(t) &= \mathrm{e}^{-x} + \int_0^x f(t)\mathrm{d}t \\ f'''(x) &= -\mathrm{e}^{-x} + f(x).\end{aligned}$$

We have $f(0) = 1,\ f'(0) = -1,\ f''(0) = 1$. Next, solve

$$f'''(x) - f(x) = -\mathrm{e}^{-x}.$$

The characteristic equation is

$$\lambda^3 - 1 = 0, \text{ that is } (\lambda-1)(\lambda^2+\lambda+1) = 0,$$

The characteristic roots are $\lambda_1 = 1$, $\lambda_{2,3} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$. The general solution of the corresponding homogeneous equation is

$$f(x) = c_1\mathrm{e}^x + \mathrm{e}^{-\frac{1}{2}x}(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x).$$

To find a particular solution $f^*(x)$, let

$$f^*(x) = A\mathrm{e}^{-x}.$$

We have $f^{*\prime}(x) = -A\mathrm{e}^{-x}$, $f^{*\prime\prime}(x) = A\mathrm{e}^{-x}$, $f^{*\prime\prime\prime}(x) = -A\mathrm{e}^{-x}$. Then $-2A = -1$, $A = \frac{1}{2}$.

Example 5.8 (Euler equation)**.** Find the general solution of the equation $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} = 6\ln x - \frac{1}{x}$.

Solution: Let $x = \mathrm{e}^t$, then the above equation is transformed into

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 6t - \mathrm{e}^{-t}.$$

Integrate twice successively, and we can obtain its general solution as

$$y = A + Bt + t^3 - \mathrm{e}^{-t}.$$

Substitute back to the variable x, then we get the general solution of the original equation ($x > 0$)

$$y = A + B\ln x + (\ln x)^3 - x^{-1},$$

where A and B are arbitrary constants.

Exercise 5.9. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the homogeneous linear equation corresponding to the second-order non-homogeneous linear equation

$$y'' + p(x)y' + q(x)y = f(x)$$

and their Wronskian determinant is $W(x)$. Prove that the general solution of this non-homogeneous linear equation is

$$\begin{aligned} y &= c_1y_1(x) + c_2y_2(x) \\ &\quad + \int_{x_0}^{x} \frac{1}{W(\xi)}\left[y_1(\xi)y_2(x) - y_2(\xi)y_1(x)\right] f(\xi)\mathrm{d}\xi, \end{aligned}$$

where $p(x), q(x)$ and $f(x)$ are continuous in the interval (a, b), and $x_0 \in (a, b)$.

Solution: We know that the general solution of a second-order non-homogeneous linear equation $y'' + p(x)y' + q(x)y = f(x)$ is composed of the general solution of the corresponding homogeneous equation y_h and a particular solution y_p of the non-homogeneous equation, i.e., $y = y_h + y_p$. Since $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$, the general solution of the homogeneous equation is $y_h = c_1y_1(x) + c_2y_2(x)$, where c_1 and c_2 are arbitrary constants. We will use the method of variation of parameters to find a particular solution y_p of the non-homogeneous equation. Assume that the particular solution has the form

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Differentiating y_p with respect to x, we get $y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$. To simplify the calculation, we impose the condition $u_1'y_1 + u_2'y_2 = 0$. Then $y_p' = u_1y_1' + u_2y_2'$. Differentiating y_p' with respect to x again, we have

$$y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''.$$

Substitute y_p, y_p' and y_p'' into the non-homogeneous equation $y'' + p(x)y' + q(x)y = f(x)$. We get

$$\left(u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''\right) + p(x)\left(u_1y_1' + u_2y_2'\right) + q(x)\left(u_1y_1 + u_2y_2\right) = f(x).$$

Since y_1 and y_2 are solutions of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$, the terms $u_1\left(y_1'' + p(x)y_1' + q(x)y_1\right)$ and $u_2\left(y_2'' + p(x)y_2' + q(x)y_2\right)$ are equal to zero. So we have $u_1'y_1' + u_2'y_2' = f(x)$. Combined with the condition $u_1'y_1 + u_2'y_2 = 0$, we get a system of linear equations for u_1' and u_2':

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = f(x). \end{cases}$$

The coefficient matrix of this system is $\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$, and its determinant is the Wronskian determinant

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1' \neq 0$$

(because y_1 and y_2 are linearly independent). By Cramer's rule, we have: $u_1' = -\frac{y_2 f(x)}{W(x)}$ and $u_2' = \frac{y_1 f(x)}{W(x)}$. Integrating u_1' and u_2' from x_0 to x to get u_1 and u_2:

$$u_1(x) = -\int_{x_0}^{x} \frac{y_2(\xi)f(\xi)}{W(\xi)}\mathrm{d}\xi, \quad u_2(x) = \int_{x_0}^{x} \frac{y_1(\xi)f(\xi)}{W(\xi)}\mathrm{d}\xi.$$

The particular solution $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

$$\begin{aligned} y_p &= y_1(x)\left(-\int_{x_0}^{x} \frac{y_2(\xi)f(\xi)}{W(\xi)}\mathrm{d}\xi\right) + y_2(x)\int_{x_0}^{x} \frac{y_1(\xi)f(\xi)}{W(\xi)}\mathrm{d}\xi \\ &= \int_{x_0}^{x} \frac{1}{W(\xi)}\left[y_1(\xi)y_2(x) - y_2(\xi)y_1(x)\right] f(\xi)\mathrm{d}\xi \end{aligned}$$

The general solution of the non-homogeneous equation $y = y_h + y_p$ is

$$\begin{aligned} y &= c_1y_1(x) + c_2y_2(x) \\ &\quad + \int_{x_0}^{x} \frac{1}{W(\xi)}\left[y_1(\xi)y_2(x) - y_2(\xi)y_1(x)\right] f(\xi)\mathrm{d}\xi \end{aligned}$$

Chapter 6

Number Series

6.1 Definition

Example 6.1. Recall the Cauchy's criterion for the convergence of number series.

Example 6.2. Consider the relations of the following statements.

1. $\sum_{n=1}^{\infty} a_n$ is convergent.
2. $\lim_{n\to+\infty} a_n = 0$.
3. For any p, $\lim_{n\to+\infty} \sum_{k=n+1}^{n+p} a_k = 0$.

Solution: 1 implies 2 and 3, 2 implies 3, but does not imply 1, and 3 does not imply 1.

Exercise 6.3. Let $\{a_n\}$ be a monotonically increasing and bounded sequence. Prove that the series $\sum_{n=1}^{\infty} \left(1 - \frac{a_n}{a_{n+1}}\right)$ is convergent.

Solution: Since $\{a_n\}$ is increasing, we note that for any $p \in \mathbb{Z}_+$ and $n \in \mathbb{Z}_+$,

$$\sum_{i=n+1}^{n+p} \left(\frac{a_{i+1} - a_i}{a_i}\right) \leq \sum_{i=n+1}^{n+p} \left(\frac{a_{i+1} - a_i}{a_{n+1}}\right) = \frac{a_{n+p} - a_{n+1}}{a_{n+1}}.$$

Since $a_n \uparrow M > 0$, we see that the right-hand side of the above formula approximates to 0 as long as n is sufficiently large, proving the result.

6.2 Positive number series

Let the general terms of the positive series $\sum a_n$ and $\sum b_n$ satisfy the condition $a_n \leqslant b_n$ for $n = 1, 2, \cdots$. Then the following conclusions hold:

1. If $\sum b_n$ converges, then $\sum a_n$ converges;
2. If $\sum a_n$ diverges, then $\sum b_n$ diverges. Note that the inequality relationship between the general terms of the two series only needs to hold for sufficiently large values of n.

Using the idea of comparison test, we have some useful methods of test.

Cauchy's root test: If $\overline{\lim}_{n\to\infty}\sqrt[n]{a_n}=c$, the series converges when $c<1$ and diverges when $c>1$.

d'Alembert's test: If $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=d$, the series converges when $d<1$ and diverges when $d>1$.

Raabe's test: If $\lim_{n\to\infty}n\left(\frac{a_n}{a_{n+1}}-1\right)=r$, the series converges when $r>1$ and diverges when $r<1$.

Gauss' test: If $\frac{a_n}{a_{n+1}}=1+\frac{\mu}{n}+O\left(\frac{1}{n^{1+\varepsilon}}\right)$ with $\varepsilon>0$, the series converges when $\mu>1$ and diverges when $\mu\leqslant 1$.

Integral test: Let f be a monotonically decreasing function on the interval $[1,+\infty)$. Then the series $\sum_{n=1}^{\infty}f(n)$ and the improper integral of infinite limit $\int_1^{+\infty}f(x)\mathrm{d}x$ either both converge or both diverge.

Example 6.4. Prove the Raabe's test.

Solution: Taking ln for both sides, we see that

$$\ln a_n-\ln a_{n+1}=\ln\left(1+\frac{r}{n}+o(1)\right)=\frac{r+o_{n\to+\infty}(1)}{n}.$$

Sum n up, we obtain the result.

Exercise 6.5. Assume that $\sum_{n=1}^{\infty}a_n$ is a positive number series and

$$\lim_{n\to+\infty}n\ln\left(\frac{a_n}{a_{n+1}}\right)=\ell.$$

Show the following statements.

1. If $\ell>1$, then $\sum_{n=1}^{\infty}a_n$ is convergent.
2. If $\ell>1$, then $\sum_{n=1}^{\infty}a_n$ is divergent.
3. If $\ell=1$, then the convergence of this series cannot be determined.

Solution: The first and the second results follow from almost the same arguments in the proof of Raabe's test. For the third one, we need to give examples of a_n such that the desired limit is 1, which are convergent and divergent respectively. Let $a_n=\frac{1}{n}$, we see that

$$\lim_{n\to+\infty}n\ln\left(\frac{a_n}{a_{n+1}}\right)=1,$$

and $\sum_{n=1}^{\infty}a_n$ diverges. For $a_n=\frac{1}{n(\ln n)^2}$, $\sum_{n=1}^{\infty}a_n$ converges due to integral test and

$$\lim_{n\to+\infty}n\ln\left(\frac{a_n}{a_{n+1}}\right)=1.$$

Question 6.6. Can you give the comparison test for $\sum_{n=1}^{\infty} a_n$, which may not be positive?

Example 6.7. Discuss the convergence and divergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{1}{3^{\ln n}}$;
2. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$;
3. $\sum_{n=3}^{\infty} \frac{1}{(\ln n)^{\ln \ln n}}$.

Solution: As long as we know that for any two positive numbers $a, b > 0$, the following holds:

$$a^{\ln b} = \left(e^{\ln a}\right)^{\ln b} = e^{\ln a \ln b} = \left(e^{\ln b}\right)^{\ln a} = b^{\ln a}$$

it is not difficult to solve the first two problems. By using $3^{\ln n} = n^{\ln 3}$ and $\ln 3 > 1$, we can see that $\sum_{n=1}^{\infty} \frac{1}{3^{\ln n}}$ converges. For the second one, by using $(\ln n)^{\ln n} = n^{\ln \ln n}$ and when n is large enough, $n^{\ln \ln n} > n^2$. We can use $\sum_{n=1}^{\infty} \frac{1}{n^2}$ as a comparison series, and thus know that it converges. Similarly, for the third problem, we have $(\ln n)^{\ln \ln n} = e^{(\ln \ln n)^2} < e^{\ln n} = n$. Therefore, from the divergence of $\sum_{n=1}^{\infty} \frac{1}{n}$, we can know that it diverges.

Exercise 6.8. Discuss the convergence and divergence of the following series:

1. $\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln \ln n)^p}$
2. $\sum_{n=1}^{+\infty} \frac{a^n}{1+a^{2n}}$, $a > 0$;
3. $\sum_{n=1}^{\infty} \left[e - \left(1 + \frac{1}{n}\right)^n\right]$.

Solution: 1. It converges for $p > 1$, and diverges otherwise. 2. Converges. 3. Diverges.

Exercise 6.9. Discuss the convergence and divergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{1}{3^{\sqrt{n}}}$;
2. $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$;
3. $\sum_{n=1}^{\infty} \left[\frac{1}{n} - \ln\left(1 + \frac{1}{n}\right)\right]$;
4. $\sum_{n=1}^{\infty} \frac{n^{n-1}}{(2n^2+n+1)^{\frac{n-1}{2}}}$;
5. $\sum_{n=2}^{\infty} \frac{n^{\ln n}}{(\ln n)^n}$.

Solution: 1. $3^{\sqrt{n}} > 3^{\ln n} = n^{\ln 3}$. Converges. 2. $n^{\frac{1}{n}} \sim 1$. Diverges. 3. $\frac{1}{n} - \ln\left(1+\frac{1}{n}\right) \sim \frac{1}{n^2}$. Converges. 4. $\frac{n^{n-1}}{(2n^2+n+1)^{\frac{n-1}{2}}} \sim \frac{1}{(2+\frac{1}{n}+\frac{1}{n^2})^{\frac{n-1}{2}}}$. Converges. 5. $\frac{n^{\ln n}}{(\ln n)^{\frac{n}{2}}} \to 0$. Converges. 6. $\mathrm{e}^{-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)} \sim n^{-1}$. $p>0$: converges; $p \leq 0$.

Exercise 6.10. Let the parameter p be a positive real number, and determine the convergence of the positive series $\sum_{n=1}^{\infty}(\sqrt{n+1}-\sqrt{n})^p \ln \frac{n+1}{n-1}$.

Solution: We have that

$$(\sqrt{n+1}-\sqrt{n})^p \sim \frac{1}{n^{\frac{p}{2}}}, \quad \ln\frac{n+1}{n-1} \sim \frac{1}{n}.$$

As a result,

$$(\sqrt{n+1}-\sqrt{n})^p \ln\frac{n+1}{n-1} \sim \frac{1}{n^{\frac{p+2}{2}}}.$$

Then, the series is convergent if and only if $p>2$.

Exercise 6.11. Determine the convergence of the positive series $\sum_{n=1}^{\infty}\left(n^{\frac{1}{n}} - \sin\frac{1}{n}\right)^{n^2}$.

Solution: By Talyor's expansion, we have

$$\left(n^{\frac{1}{n}} - \sin\frac{1}{n}\right)^{n^2} = e^{n^2 \ln\left(1+\frac{1}{n}\ln n - \frac{1}{n} - \frac{(\ln n)^2}{2n^2} + O\left(\frac{1}{n^3}\right)\right)} \to +\infty.$$

Then, it is divergent.

6.3 General number series

A convergent series $\sum a_n$ is called an absolutely convergent series if $\sum |a_n|$ is also convergent; otherwise, it is called a conditionally convergent series.

Example 6.12. Consider the convergence property of

$$\sum_{n=1}^{\infty} \ln\left(1+\frac{(-1)^{n-1}}{n^p}\right).$$

Solution: For $p>1$, we see that

$$\left|\ln\left(1+\frac{(-1)^{n-1}}{n^p}\right)\right| \leq \frac{1}{n^p}.$$

Since $\sum_{n\geq 1}\frac{1}{n^p}$ is convergent, we see that $\sum_{n=1}^{\infty}\ln\left(1+\frac{(-1)^{n-1}}{n^p}\right)$ is absolutely convergent due to comparison test. For $p \in (\frac{1}{2}, 1]$, it follows from Taylor's expansion that

$$\ln\left(1+\frac{(-1)^{n-1}}{n^p}\right) = \frac{(-1)^{n-1}}{n^p} - \frac{(1+o(1))}{n^{2p}}.$$

This implies that the series is conditionally convergent. For $p \in (0, \frac{1}{2}]$, the above formula shows that it diverges.

Dirichlet's test: If the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$ is bounded, and the sequence $\{b_n\}$ is monotonic and converges to 0 , then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

Abel's test: If the series $\sum_{n=1}^{\infty} a_n$ converges, and the sequence $\{b_n\}$ is monotonic and bounded, then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

Example 6.13. Determine the convergence of the general term series

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \frac{\sin n}{n}.$$

Solution: $\sum_{k=1}^{n} \sin k$ is uniformly bounded. $\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \frac{1}{n}$ is decreasing to 0. It follows from Dirichlet test that the series is convergent.

Exercise 6.14. Discuss the absolute convergence and conditional convergence of the following series:

1. $\sum_{n=2}^{\infty} \frac{\sin \frac{n\pi}{12}}{\ln n}$;
2. $\sum_{n=1}^{\infty} \frac{(-1)^n 2^{\frac{1}{n}}}{\sqrt{n}}$;
3. $\sum_{n=1}^{\infty} (-1)^n \frac{\cos^2 n}{n}$;
4. $\sum_{n=1}^{\infty} \frac{\sin n}{n} \left(1 + \frac{1}{n}\right)^n$;
5. $\sum_{n=1}^{\infty} \sin \left(\pi \sqrt{n^2 + a^2}\right)$;
6. $\sum_{n=1}^{\infty} \frac{(-1)^n \sin n}{n}$;
7. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{[n + (-1)^{n-1}]^p}$;
8. $\sum_{n=2}^{\infty} \sin \left(n\pi + \frac{1}{\ln n}\right)$;
9. $\sum_{n=1}^{\infty} (-1)^n \left(n^{\frac{1}{n}} - 1\right)$.