Lecture Notes on Problem-Solving Class: Advanced Mathematics B(II)

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Contents

Preface: Some Important Information

0.1 Course Assessment Information

- **Usual Performance:** Hand in assignments. It accounts for 20 points of the overall evaluation.
- **Mid-term Examination:** On the weekend of the 8th or 9th week. It accounts for 30 points of the overall evaluation.
- **Final Examination:** On Monday evening, June 9th. It accounts for 50 points of the overall evaluation.

0.2 Information of the Class and the Teaching Assistant

- **Teacher:** Wei Wang.
  - **E-mail:** 2201110024@stu.pku.edu.cn
  - **Personal Website:** https://luisyanka.github.io/weiwang.github.io/
- **Location:** Room 313, Teaching Building 2.
- **Students:** Students whose student ID numbers are greater than 2400011822 and less than or equal to 2400015443 should submit their assignments to Teacher Wei Wang's class.

0.3 Some Useful Links

We present some useful links associated with calculus.

- Lecture notes by Yantong Xie: https://darkoxie.github.io
- Mathstackexchange: https://math.stackexchange.com

0.4 Topics of the Class

In this problem-solving class, we will present some classical exercises related to the topics which are delivered by the lecturer in the main course. We will mainly refer to the lecture notes written by Yantong Xie, who was a very good teaching assistant of the course Advanced mathematics (B). We also refer to the book "Guide to Solving

Problems in Advanced Mathematics" by Jianying Zhou and Zhengyuan Li. If you have any advices for this class, then you can contact me with the e-mail. The lecture notes of this class will be updated before the next one on my personal website in the content of "teaching".

Chapter 1

Double integral

1.1 Calculating by the Definition

Example 1.1. There are three points P_0, P_1, P_2 on the plane, which are given by $\{(x_i, y_i)\}_{i=0}^2$. We assume that $x_2 > x_1 > x_0$ and $y_2 > y_1 > y_0$. Please calculate the area of triangle $\Delta P_0P_1P_2$.

Solution: Denote the triangle $\Delta P_0P_1P_2$ by D. By simple calculations, we can determine $P_0P_1 : y = k_1x + b_1$, $P_1P_2 : y = k_2x + b_2$, and $P_1P_3 : y = k_3x + b_3$. WLOG, we assume that $y_1 < k_3x_1 + b_3$. As a result, we have

$$\begin{aligned}\int_D 1\mathrm{d}x\mathrm{d}y &= \int_{x_0}^{x_1} \mathrm{d}x \int_{k_1x+b_1}^{k_3x+b_3} 1\mathrm{d}y + \int_{x_0}^{x_1} \mathrm{d}x \int_{k_2x+b_2}^{k_3x+b_3} 1\mathrm{d}y \\ &= \frac{1}{2}((y_2 - y_0)(x_1 - x_0) - y_1(x_2 - x_0)).\end{aligned}$$

Combined with the case that $y_1 \geq k_3x_1 + b_3$, we obtain

$$A(D) = \frac{1}{2}|x_1y_2 - x_1y_0 - x_0y_2 + x_0y_0 - x_2y_1 + x_0y_1|.$$

Exercise 1.2. Let $A = [0, 1] \times [0, 1]$, find

$$I = \iint_A \frac{y\mathrm{d}x\mathrm{d}y}{(1 + x^2 + y^2)^{\frac{3}{2}}}.$$

Solution: Integrating with respect to y first and then with respect to x, we get

$$\begin{aligned}I &= \int_0^1 \mathrm{d}x \int_0^1 \frac{y\mathrm{d}y}{(1 + x^2 + y^2)^{\frac{3}{2}}} \\ &= \int_0^1 \left(\frac{1}{\sqrt{x^2 + 1}} - \frac{1}{\sqrt{x^2 + 2}} \right) \mathrm{d}x = \ln \frac{2 + \sqrt{2}}{1 + \sqrt{3}}.\end{aligned}$$

1.2 Iterated Integrals

Example 1.3. Calculate $\int_0^1 \int_y^1 \frac{y}{\sqrt{1+x^3}}\mathrm{d}x$.

Solution: Let

$$D := \{(x, y) \in \mathbb{R}^2 : y \leq x,\ x \in [0, 1],\ y \in [0, 1]\}.$$

We have

$$\int_0^1 \mathrm{d}y \int_y^1 \frac{y}{\sqrt{1+x^3}}\mathrm{d}x = \int_D \frac{y}{\sqrt{1+x^3}}\mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^x \frac{y}{\sqrt{1+x^3}}\mathrm{d}y$$
$$= \frac{1}{6}\int_0^1 \frac{\mathrm{d}t}{\sqrt{1+t}} = \frac{1}{3}(\sqrt{2}-1),$$

where for the third inequality, we have used $t = x^3$.

Exercise 1.4. Calculate $\int_0^1 \frac{x-1}{\ln x}\mathrm{d}x$.

Solution: We note

$$\int_0^1 \frac{x-1}{\ln x}\mathrm{d}x = \int_0^1 \mathrm{d}x \int_0^1 x^y \mathrm{d}y = \int_{[0,1]^2} x^y \mathrm{d}x\mathrm{d}y$$
$$= \int_0^1 \mathrm{d}y \int_0^1 x^y \mathrm{d}x = \int_0^1 \frac{1}{y+1}\mathrm{d}y = \ln 2.$$

Exercise 1.5. Suppose that f is continuous on $[0,1]$. Prove that:

$$\int_0^1 \mathrm{d}x \int_x^1 f(t)\mathrm{d}t = \int_0^1 tf(t)\mathrm{d}t.$$

1.3 Change of Variables

Example 1.6 (Observing the region)**.** The region $D \subset \mathbb{R}^2$ is surrounded by the curves $xy = a$, $xy = b$, $y = px$, and $y = qx$, where $0 < a < b$ and $0 < p < q$. Please calculate

$$I = \iint_D xy^3 \mathrm{d}x\mathrm{d}y.$$

Solution: Consider a change of variables as

$$\begin{cases} x' = \dfrac{y}{x}, \\ y' = xy. \end{cases}$$

We can calculate that

$$\left|\frac{\partial(x,y)}{\partial(x',y')}\right| = -\frac{1}{2x'}.$$

As a result,

$$I = s\int_p^q \left(\int_a^b x'(y')^2 \cdot \frac{1}{2x'}\mathrm{d}x'\right)\mathrm{d}y' = \frac{2(b^3-a^3)(q-p)}{3}$$

Example 1.7 (Rotation)**.** Calculate $\int_D |3x+4y|\mathrm{d}x\mathrm{d}y$, where

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}.$$

Solution: Consider a change of variables as

$$\begin{cases} x' = \dfrac{4}{5}x - \dfrac{3}{5}y, \\ y' = \dfrac{3}{5}x + \dfrac{4}{5}y. \end{cases}$$

Indeed, formula above give a rotation, which preserve D and change the line $3x + 4y = t$ to $y' = t$ for any $t \in \mathbb{R}$. As a result, we have

$$\int_D |3x+4y|\mathrm{d}x\mathrm{d}y = 5\int_D |y'|\mathrm{d}x'\mathrm{d}y' = \frac{20}{3}.$$

Question 1.8. Can you give the intuition behind this change of variables?

Example 1.9 (Polar coordinate)**.** Make a polar coordinate transformation to convert the double integral

$$\iint_D f(\sqrt{x^2+y^2})\mathrm{d}x\mathrm{d}y$$

into a definite integral, where $D = \{(x,y) : 0 \le y \le x \le 1\}$.

Solution: Let $x = r\cos\varphi$ and $y = r\sin\varphi$. Then

$$\begin{aligned} \iint_D f(\sqrt{x^2+y^2})\mathrm{d}x\mathrm{d}y &= \iint_D f(r)r\mathrm{d}r\mathrm{d}\varphi \\ &= \int_0^1 \mathrm{d}r \int_0^{\pi/4} f(r)r\mathrm{d}\varphi + \int_1^{\sqrt{2}} \mathrm{d}r \int_{\arccos(1/r)}^{\pi/4} f(r)r\mathrm{d}\varphi \\ &= \frac{\pi}{4}\int_0^1 f(r)r\mathrm{d}r + \int_1^{\sqrt{2}} \left(\frac{\pi}{4} - \arccos\frac{1}{r}\right) f(r)r\mathrm{d}r \\ &= \frac{\pi}{4}\int_0^{\sqrt{2}} f(r)r\mathrm{d}r - \int_1^{\sqrt{2}} \arccos\frac{1}{r} f(r)r\mathrm{d}r. \end{aligned}$$

Remark 1.10. Generally speaking, the generalized polar coordinate transformation

$$x = \frac{1}{a}\left(c + r^{\frac{1}{p}}\cos^{\frac{2}{p}}\theta\right), \quad y = \frac{1}{b}\left(d + r^{\frac{1}{p}}\sin^{\frac{2}{p}}\theta\right),$$

can transform $(ax-c)^p + (by-d)^p$ into r. However, in general, r and θ no longer have the meanings of the usual polar radius and polar angle.

Exercise 1.11. Find

$$\iint_D \left(\sqrt{\frac{x-c}{a}}+\sqrt{\frac{y-c}{b}}\right)\mathrm{d}x\mathrm{d}y,$$

where D is the region bounded by the curve $\sqrt{\frac{x-c}{a}}+\sqrt{\frac{y-c}{b}}=1$, $x=c$, and $y=c$, and $a,b,c>0$.

Solution: Let

$$x=c+a\rho\cos^4\theta,\quad y=c+b\rho\sin^4\theta$$

Then

$$J=\left|\frac{\partial(x,y)}{\partial(\rho,\theta)}\right|=4ab\rho\cos^3\theta\sin^3\theta$$

And the integration region becomes $\left\{0\le\theta\le\frac{\pi}{2},0\le\rho\le1\right\}$. Thus

$$\iint_D \left(\sqrt{\frac{x-c}{a}}+\sqrt{\frac{y-c}{b}}\right)\mathrm{d}x\mathrm{d}y=\int_0^{\pi/2}\mathrm{d}\theta\int_0^1 4ab\rho\cos^3\theta\sin^3\theta\sqrt{\rho}\mathrm{d}\rho=\frac{2ab}{15}$$

Exercise 1.12. Find

$$\lim_{R\to+\infty}\iint_{|x|\le R,\ |y|\le R}(x^2+y^2)\mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y.$$

Solution: Let

$$I_R=\iint_{|x|\le R,\ |y|\le R}\left(x^2+y^2\right)\mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y,$$
$$C_R=\iint_{x^2+y^2\le R^2}(x^2+y^2)\mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y$$

Then $C_R\le I_R\le C_{2R}$, and

$$C_R=\int_0^{2\pi}\mathrm{d}\theta\int_0^R r^3\mathrm{e}^{-r^2}\mathrm{d}r=\pi\int_0^{R^2}t\mathrm{e}^{-t}\mathrm{d}t=\pi(1-\mathrm{e}^{-R^2}-R^2\mathrm{e}^{-R^2})\to\pi\quad(R\to+\infty)$$

Similarly, we can prove that $C_{2R}\to\pi$ as $R\to+\infty$. Thus,

$$\lim_{R\to+\infty}I_R=\pi$$

Exercise 1.13. Assume that $f\in C[-1,1]$, show that

$$\int_{|x|+|y|\le1}f(x+y)\mathrm{d}x\mathrm{d}y=\int_{-1}^1 f(z)\mathrm{d}z.$$

Hint: Consider a change of variables as

$$\begin{cases} x' = x - y, \\ y' = x + y. \end{cases}$$

Exercise 1.14. Given the integral

$$I = \iint_D \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right] \mathrm{d}x\mathrm{d}y.$$

Define a transformation $x = x(u, v)$, $y = y(u, v)$, and the region D is transformed into Ω. Assume that the transformation satisfies

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}, \quad \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}.$$

Prove that:

$$I = \iint_\Omega \left[\left(\frac{\partial f}{\partial u} \right)^2 + \left(\frac{\partial f}{\partial v} \right)^2 \right] \mathrm{d}u\mathrm{d}v.$$

1.4 Symmetry

The parity of a function and the symmetry of the integration region can often be used to simplify the calculation of integrals. For example:

1. If the integration region D is symmetric about the x-axis:

   - If $f(x, y) = -f(x, -y)$, then
   $$\iint_D f(x, y)\mathrm{d}x\mathrm{d}y = 0.$$
   - If $f(x, y) = f(x, -y)$, then
   $$\iint_D f(x, y)\mathrm{d}x\mathrm{d}y = 2 \iint_{D\cap\{y\geq 0\}} f(x, y)\mathrm{d}x\mathrm{d}y.$$

2. If the integration region D is symmetric about the y-axis:

   - If $f(x, y) = -f(-x, y)$, then
   $$\iint_D f(x, y)\mathrm{d}x\mathrm{d}y = 0.$$
   - If $f(x, y) = f(-x, y)$, then
   $$\iint_D f(x, y)\mathrm{d}x\mathrm{d}y = 2 \iint_{D\cap\{x\geq 0\}} f(x, y)\mathrm{d}x\mathrm{d}y.$$

3. If D is symmetric about the origin:
   - If $f(x,y) = -f(-x,-y)$, then
   $$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = 0.$$
   - If $f(x,y) = f(-x,-y)$, then
   $$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = 2\iint_{D_1} f(x,y)\mathrm{d}x\mathrm{d}y,$$
   where D_1 is half of the region D.

Example 1.15. Show that:

$$\iint_{|x|+|y|\le 1} (\sqrt{|xy|} + |xy|)\mathrm{d}x\mathrm{d}y \le \frac{3}{2}.$$

Solution: By the symmetric property, we have

$$\iint_{|x|+|y|\le 1} (\sqrt{|xy|} + |xy|)\mathrm{d}x\mathrm{d}y = 4\iint_{x+y\le 1,\ x\ge 0,\ y\ge 0} (\sqrt{xy} + xy)\mathrm{d}x\mathrm{d}y.$$

By direct calculations, the property holds.

1.5 Applications to Proving Integral Inequalities

Example 1.16. Assume that $a < b$ and $f, g \in C[a,b]$, show that

$$\left(\int_a^b fg\right)^2 \le \left(\int_a^b f^2\right)\left(\int_a^b g^2\right).$$

Solution: Note that

$$\int_{[a,b]^2} (f(x)g(y) - f(y)g(x))^2\mathrm{d}x\mathrm{d}y \ge 0.$$

By expanding those in the bracket, the inequality follows directly.

Remark 1.17. The key point in the proof above is to note that the integral variables x and y have the same status.

Exercise 1.18. Let $f \in C[0,1]$ be a positive and non-increasing function. Show that

$$\frac{\int_0^1 xf^2(x)\mathrm{d}x}{\int_0^1 xf(x)\mathrm{d}x} \le \frac{\int_0^1 f^2(x)\mathrm{d}x}{\int_0^1 f(x)\mathrm{d}x}.$$

<u>Solution:</u> It is a direct result from the claim

$$\int_{[0,1]^2} yf(x)f(y)(f(x)-f(y))\mathrm{d}x\mathrm{d}y \geq 0.$$

Consider a change of variables as

$$\begin{cases} x' = x+y, \\ y' = x-y. \end{cases}$$

Define

$$D := \{(x',y') \in \mathbb{R}^2 : 0 \leq x'+y' \leq 2,\ 0 \leq x'-y' \leq 2\}.$$

We see that D is symmetric about the y-axis. Consequently, we have

$$\int_{[0,1]^2} yf(x)f(y)(f(x)-f(y))\mathrm{d}x\mathrm{d}y = \int_D (x'-y')g(x',y')\mathrm{d}x'\mathrm{d}y',$$

where

$$g(x',y') = \frac{1}{2}f\left(\frac{x'+y'}{2}\right)f\left(\frac{x'-y'}{2}\right)\left[f\left(\frac{x'+y'}{2}\right)-f\left(\frac{x'-y'}{2}\right)\right].$$

Obviously, there holds

$$g(x',y') = -g(x',-y') \geq 0$$

if $(x',y') \in D$ and $y' \leq 0$. This implies that

$$\begin{aligned}\int_{D\cap\{y'<0\}} (x'-y')g(x',y')\mathrm{d}x'\mathrm{d}y' &= \int_{D\cap\{y'\geq 0\}} (x'+y')g(x',-y')\mathrm{d}x'\mathrm{d}y' \\ &= -\int_{D\cap\{y'\geq 0\}} (x'+y')g(x',y')\mathrm{d}x'\mathrm{d}y'.\end{aligned}$$

Here, for the first inequality, we have used the change of varaible that sends y' to $-y'$. Moreover, we obtain

$$\begin{aligned}\int_D (x'-y')g(x',y')\mathrm{d}x'\mathrm{d}y' &= \left(\int_{D\cap\{y'\geq 0\}} + \int_{D\cap\{y'<0\}}\right)(x'-y')g(x',y')\mathrm{d}x'\mathrm{d}y' \\ &= \int_{D\cap\{y'\geq 0\}} ((x'-y')-(x'+y'))g(x',y')\mathrm{d}x'\mathrm{d}y' \\ &= -\int_{D\cap\{y'\geq 0\}} 2y'g(x',y')\mathrm{d}x'\mathrm{d}y'.\end{aligned}$$

We see that if $y' \geq 0$, then it follows from the property that f is non-increasing that $g(x',y') \leq 0$. As a result,

$$-\int_{D\cap\{y'\geq 0\}} 2y'g(x',y')\mathrm{d}x'\mathrm{d}y' \geq 0,$$

and the proof is completed.

Exercise 1.19. Assume that $f \in C[0,1]$ and $f > 0$. Show that

$$\left(\int_0^1 \frac{1}{f}\right)\left(\int_0^1 f\right) \geq 1.$$

<u>Solution:</u> We have

$$\begin{aligned}\left(\int_0^1 \frac{1}{f(x)}\mathrm{d}x\right)\left(\int_0^1 f(x)\mathrm{d}x\right) &= \left(\int_0^1 \frac{1}{f(x)}\mathrm{d}x\right)\left(\int_0^1 f(y)\mathrm{d}y\right)\\ &\geq \frac{1}{2}\int_{[0,1]^2}\left(\frac{f(x)}{f(y)}+\frac{f(y)}{f(x)}\right)\mathrm{d}x\mathrm{d}y \geq 1.\end{aligned}$$

1.6 Calculating the Area of the surfaces

1. Parametric representation of a surface: Let a surface S be given parametrically by $\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$, where (u,v) varies over a region D in the $uv-$plane.

   The partial derivatives of $\vec{r}$ with respect to u and v are $\vec{r}_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}$ and $\vec{r}_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}$.

   The cross product $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$

   The surface area $A(S)$ of the surface S is given by the double integral:

   $$A(S) = \iint_D \|\vec{r}_u \times \vec{r}_v\| \, dA = \iint_D \sqrt{\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2} dudv$$

   where $\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$, $\frac{\partial(y,z)}{\partial(u,v)} = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}$, and $\frac{\partial(z,x)}{\partial(u,v)} = \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}$.

2. Explicit Representation of a Surface If the surface S is given explicitly as $z = f(x,y)$, where (x,y) varies over a region D in the $xy-$plane. We can consider the parametric representation $\vec{r}(x,y) = x\vec{i} + y\vec{j} + f(x,y)\vec{k}$.

   The partial derivatives are $\vec{r}_x = \vec{i} + \frac{\partial f}{\partial x}\vec{k}$ and $\vec{r}_y = \vec{j} + \frac{\partial f}{\partial y}\vec{k}$.

   The cross product $\vec{r}_x \times \vec{r}_y = -\frac{\partial f}{\partial x}\vec{i} - \frac{\partial f}{\partial y}\vec{j} + \vec{k}$.

   The magnitude $\|\vec{r}_x \times \vec{r}_y\| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$.

   The surface area $A(S)$ of the surface S is then given by the double integral:

   $$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \mathrm{d}x\mathrm{d}y$$

3. Key Points and Considerations

   (a) Choice of representation: The choice between parametric and explicit representation depends on the nature of the surface. For surfaces like spheres, tori, etc., parametric representation is often more convenient. For surfaces that can be easily written as $z = f(x, y)$ (e.g., graphs of functions), the explicit form is straightforward to use.

   (b) Region of integration: Identifying the correct region D in the appropriate parameter plane (e.g., uv-plane for parametric surfaces or $xy-$plane for explicit surfaces) is crucial. The limits of integration for the double integral are determined by the boundaries of D.

   (c) Calculation of partial derivatives: Accurately computing the partial derivatives of the functions involved in the representation of the surface is essential. Any error in calculating the partial derivatives will lead to an incorrect result for the surface area.

   (d) Evaluation of the double integral: Once the integrand and the region of integration are determined, the double integral needs to be evaluated. This may involve techniques such as changing the order of integration, using polar coordinates (in the xy-plane for surfaces given as $z = f(x, y)$) or other coordinate transformations depending on the shape of the region D and the integrand.

In conclusion, double integrals provide a powerful tool for calculating the surface area of a wide variety of surfaces. By carefully choosing the appropriate representation of the surface, correctly identifying the region of integration, and accurately evaluating the resulting double integral, we can obtain the surface area of the given surface.

Example 1.20. Calculating the area of the set

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = a^2,\ x^2 + y^2 \leq ax\}.$$

Solution: First, rewrite the equations and find the relevant partial derivatives: The equation of the sphere is $x^2+y^2+z^2 = a^2$, so $z = \pm\sqrt{a^2 - x^2 - y^2}$. The partial derivatives are $\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{a^2-x^2-y^2}}$ and $\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{a^2-x^2-y^2}}$, and $\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} = \frac{a}{\sqrt{a^2-x^2-y^2}}$. The inequality $x^2+y^2 \leq ax$ in polar coordinates ($x = r\cos\theta, y = r\sin\theta$) becomes $r^2 \leq ar\cos\theta$, i.e., $r \leq a\cos\theta$ with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ (because $r \geq 0$).

Then, use the surface area formula for a surface $z = f(x, y)$: The surface area A of the surface $z = f(x, y)$ over a region D in the xy - plane is $A = 2\iint_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}\mathrm{d}x\mathrm{d}y$ (for the upper and lower hemispheres of the sphere).

Substituting into polar coordinates , we have:

$$\begin{aligned} A &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{a\cos\theta} \frac{a}{\sqrt{a^2-r^2}} r\mathrm{d}r = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\left[-\sqrt{a^2-r^2}\right]_0^{a\cos\theta} d\theta \\ &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\left(a-a|\sin\theta|\right)\mathrm{d}\theta = 2a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-|\sin\theta|)\mathrm{d}\theta = 2a^2\left(\pi-2\right) \end{aligned}$$

Chapter 2

Triple Integrals and n-multiple Integrals

2.1 Calculation of Triple Integrals in Rectangular Coordinate

Example 2.1 (Consider the area of the section)**.** Find the integral

$$I = \iiint_{\Omega} z^2 \mathrm{d}x \mathrm{d}y \mathrm{d}z,$$

where Ω is the common part of the two spheres $x^2+y^2+z^2 \leq R^2$ and $x^2+y^2+z^2 \leq 2Rz$.

Solution: Considering both the integrand and the integration region, the integral can be regarded as the sum of a series of small slices weighted by z^2 for $z \in [0, R]$. According to the composition of the integration region Ω, it can be divided into two sub-regions Ω_1 and Ω_2.

$$\Omega_1 : \begin{cases} x^2+y^2+z^2 \leq 2Rz, \\ 0 \leq z \leq \frac{R}{2}, \end{cases}$$
$$\Omega_2 : \begin{cases} x^2+y^2+z^2 \leq R^2, \\ \frac{R}{2} \leq z \leq R. \end{cases}$$

When $z \in [0, \frac{R}{2}]$, from $x^2+y^2+z^2 \leq 2Rz$, we see that the area of the slice can be obtained as $\pi(2Rz - z^2)$. On the other hand, when $z \in [\frac{R}{2}, R]$, for $x^2+y^2+z^2 \leq R^2$, the area of the slice can be obtained as $\pi(R^2 - z^2)$. As a result,

$$\begin{aligned} I &= \int_0^{R/2} \pi z^2 (2Rz - z^2) \mathrm{d}z + \int_{R/2}^{R} \pi z^2 (R^2 - z^2) \mathrm{d}z \\ &= \left(\frac{1}{2}\pi R z^4 - \frac{1}{5}\pi z^5 \right) \Big|_0^{R/2} + \left(\frac{1}{3}\pi R^2 z^3 - \frac{1}{5}\pi z^5 \right) \Big|_{R/2}^{R} = \frac{59}{480}\pi R^5 \end{aligned}$$

Exercise 2.2. Calculate the volume of the intersection of cylinders $x^2+y^2 \leq a^2$ and $x^2+z^2 \leq a^2$.

Solution:

1. **Cylinder** $x^2+y^2=a^2$: In the xy-plane, the equation $x^2+y^2=a^2$ represents a circle centered at the origin with radius a. When considering this equation in three-dimensional space, it represents a cylinder that extends infinitely along the z-axis.

2. **Cylinder** $x^2+z^2=a^2$: In the xz-plane, the equation $x^2+z^2=a^2$ represents a circle centered at the origin with radius a. In three-dimensional space, it represents a cylinder that extends infinitely along the y-axis.

3. **Intersection of the two cylinders**: The intersection of the cylinders $x^2+y^2\le a^2$ and $x^2+z^2\le a^2$ is a symmetric solid. The solid is symmetric about the x-axis, y-axis, and z-axis. For a fixed $x\in[-a,a]$, the cross-section of the intersection perpendicular to the x-axis is a square. The side length of the square s is given by $s=2\sqrt{a^2-x^2}$ (since from $x^2+y^2=a^2$, we have $y=\pm\sqrt{a^2-x^2}$ and from $x^2+z^2=a^2$, we have $z=\pm\sqrt{a^2-x^2}$).

The formula for the volume V of a solid with cross-sectional area $A(x)$ from $x=c$ to $x=d$ is $V=\int_c^d A(x)\mathrm{d}x$. Here, $c=-a$, $d=a$, and the cross-sectional area $A(x)$ of the intersection of the two cylinders perpendicular to the x-axis is $A(x)=(2\sqrt{a^2-x^2})\times(2\sqrt{a^2-x^2})=4(a^2-x^2)$ (because the cross-section is a square with side length $2\sqrt{a^2-x^2}$).

Then, we calculate the integral:

$$V=\int_{-a}^{a}4(a^2-x^2)\mathrm{d}x=\frac{16}{3}a^3.$$

Example 2.3 (Projection to the plane)**.** Find

$$I=\iiint_\Omega(y^2+z^2)\mathrm{d}V,$$

where Ω represents the region $0\le z\le x^2+y^2\le 1$.

Solution: In cylindrical coordinates $x=r\cos\theta$, $y=r\sin\theta$, $z=z$ and $dV=r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta$, and $y^2+z^2=r^2\sin^2\theta+z^2$. The region $\Omega: 0\le z\le x^2+y^2\le 1$ becomes $0\le z\le r^2$, $0\le r\le 1$, $0\le\theta\le 2\pi$. Then,

$$\begin{aligned}
I&=\int_0^{2\pi}\int_0^1\int_0^{r^2}(r^2\sin^2\theta+z^2)r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta=\int_0^{2\pi}\int_0^1 r\left[\int_0^{r^2}(r^2\sin^2\theta+z^2)\mathrm{d}z\right]\mathrm{d}r\mathrm{d}\theta\\
&=\int_0^{2\pi}\int_0^1 r\left(r^2\sin^2\theta z+\frac{z^3}{3}\Big|_0^{r^2}\right)\mathrm{d}r\mathrm{d}\theta=\int_0^{2\pi}\int_0^1 r\left(r^4\sin^2\theta+\frac{r^6}{3}\right)\mathrm{d}r\mathrm{d}\theta\\
&=\int_0^{2\pi}\left(\sin^2\theta\int_0^1 r^5dr+\frac{1}{3}\int_0^1 r^7dr\right)\mathrm{d}\theta=\int_0^{2\pi}\left(\frac{1}{6}\sin^2\theta+\frac{1}{24}\right)\mathrm{d}\theta=\frac{\pi}{4}.
\end{aligned}$$

Indeed, we also have

$$I=\int_{x^2+y^2\le 1}\left(\int_0^{x^2+y^2}(y^2+z^2)\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y=\frac{\pi}{4}.$$

2.2 Change of Variables for Triple Integrals

Similar to the change of variables for double integrals, there is the following theorem for variable substitution of triple integrals. Let

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w), \quad (u, v, w) \in \Omega'$$

This change of variables satisfies the following conditions:

- It establishes a one-to-one correspondence between Ω and Ω'.
- x, y, and z have continuous partial derivatives with respect to each variable in Ω', and the inverse transformations $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$ also have continuous partial derivatives with respect to each variable in Ω.
- The Jacobian determinant $J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$ has no zeros in Ω'. Then

$$\begin{aligned}&\iiint_{\Omega} f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &\quad = \iiint_{\Omega'} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \mathrm{d}u\mathrm{d}v\mathrm{d}w\end{aligned}$$

There are two commonly used transformations for triple integrals:

1. Cylindrical Coordinate Transformation

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad z = z$$
$$0 \leq \rho < +\infty, \quad 0 \leq \theta < 2\pi, \quad -\infty < z < +\infty$$

The relationship between the triple integral in the rectangular coordinate system and the triple integral in the cylindrical coordinate system is

$$\iiint_{\Omega} f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{\Omega'} f(\rho\cos\theta, \rho\sin\theta, z)\rho\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}z.$$

2. Spherical Coordinate Transformation

$$x = \rho\sin\varphi\cos\theta, \quad y = \rho\sin\varphi\sin\theta, \quad z = \rho\cos\varphi,$$
$$0 \leq \rho < +\infty, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \varphi \leq \pi.$$

The relationship between the triple integral in the rectangular coordinate system and the triple integral in the spherical coordinate system is

$$\iiint_{\Omega} f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{\Omega'} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\varphi.$$

Example 2.4. Let $\Omega = \{(x,y,z) : 0 \le x+y-z \le 1, 0 \le y+z-x \le 1, 0 \le x+z-y \le 1\}$ be the region formed by the intersection of six planes. Find the triple integral

$$I = \iiint_{\Omega} (x+y-z)(y+z-x)(x+z-y)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Solution: Consider the change of variables as

$$\begin{cases} x' = x+y-z, \\ y' = y+z-x, \\ z' = x+z-y. \end{cases}$$

It is easy to find that the Jacobi determinant is $\frac{1}{4}$. As a result, we have

$$I = \int_{\substack{0\le x'\le 1,\\ 0\le y'\le 1,\\ 0\le z'\le 1}} \frac{1}{4} x'y'z'\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' = \frac{1}{32}.$$

Example 2.5. Calculate the integral

$$H = \iiint_{\substack{x,y,z\ge 0\\ x^2+y^2+z^2\le R^2}} \frac{xyz\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}, \quad \text{where } a>b>c>0.$$

Solution: In spherical coordinates,

$$H = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^R \frac{r^4\sin^3\varphi\cos\varphi\sin\theta\cos\theta\mathrm{d}r\mathrm{d}\varphi\mathrm{d}\theta}{\sqrt{a^2\sin^2\varphi\cos^2\theta + b^2\sin^2\varphi\sin^2\theta + c^2\cos^2\varphi}}.$$

Let $\sin^2\varphi = u$, $\sin^2\theta = v$. Then

$$\begin{aligned}
H &= \frac{1}{4}\int_0^1\int_0^1\int_0^R r^4 \frac{u\mathrm{d}r\mathrm{d}u\mathrm{d}v}{\sqrt{a^2u(1-v)+b^2uv+c^2(1-u)}} \\
&= \frac{1}{20}R^5\int_0^1 u\mathrm{d}u\int_0^1 \frac{\mathrm{d}v}{\sqrt{[c^2+(a^2-c^2)\,u]+(b^2-a^2)\,uv}} \\
&= \frac{1}{20}R^5\int_0^1 \left\{\frac{2}{(b^2-a^2)\,u}\sqrt{[c^2+(a^2-c^2)\,u]+(b^2-a^2)\,uv}\right\}\Bigg|_{v=0}^{v=1} u\mathrm{d}u \\
&= \frac{R^5}{10\,(b^2-a^2)}\int_0^1 \left\{\sqrt{[c^2+(a^2-c^2)\,u]+(b^2-a^2)\,u} - \sqrt{c^2+(a^2-c^2)\,u}\right\}\mathrm{d}u \\
&= \frac{R^5}{10\,(b^2-a^2)}\left\{\frac{2}{3\,(b^2-c^2)}\left[c^2+\left(b^2-c^2\right)u\right]^{\frac{3}{2}} - \frac{2}{3\,(a^2-c^2)}\left[c^2+\left(a^2-c^2\right)u\right]^{\frac{3}{2}}\right\}\Bigg|_0^1 \\
&= \frac{R^5}{10\,(b^2-a^2)}\left[\frac{2}{3\,(b^2-c^2)}\left(b^3-c^3\right) - \frac{2}{3\,(a^2-c^2)}\left(a^3-c^3\right)\right] \\
&= \frac{R^5}{15}\cdot\frac{1}{b^2-a^2}\left(\frac{b^2+bc+c^2}{b+c} - \frac{a^2+ac+c^2}{a+c}\right) \\
&= \frac{R^5}{15}\cdot\frac{ab+bc+ca}{(a+b)(b+c)(c+a)}.
\end{aligned}$$

Example 2.6. Let $H(x) = \sum_{i,j=1}^{3} a_{ij}x_i x_j$, and $\mathbf{A} = (a_{ij})$ be a positive definite symmetric matrix of order 3. Find

$$I = \iiint_{H(x)\le 1} e^{\sqrt{H(x)}}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3.$$

<u>Solution:</u> There exists an orthogonal matrix $\mathbf{P}$ of order 3 such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

where $\lambda_i > 0$, $i = 1, 2, 3$. Make an orthogonal transformation $x = \mathbf{P}y$, where $x, y \in \mathbb{R}^3$. Then

$$H(x) = H(\mathbf{P}y) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2,$$

and the Jacobian determinant of the transformation $\det \mathbf{P} \equiv 1$. Thus

$$I = \iiint_{\lambda_1 y_1^2+\lambda_2 y_2^2+\lambda_3 y_3^2\le 1} \mathrm{e}^{\sqrt{\lambda_1 y_1^2+\lambda_2 y_2^2+\lambda_3 y_3^2}}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3.$$

Let

$$y_1 = \frac{1}{\sqrt{\lambda_1}}r\sin\varphi\cos\theta, \quad y_2 = \frac{1}{\sqrt{\lambda_2}}r\sin\varphi\sin\theta, \quad y_3 = \frac{1}{\sqrt{\lambda_3}}r\cos\varphi.$$

Then

$$I = \frac{1}{\sqrt{\lambda_1\lambda_2\lambda_3}}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\varphi\int_0^1 r^2\mathrm{e}^r\sin\varphi\mathrm{d}r = \frac{4\pi}{\sqrt{\lambda_1\lambda_2\lambda_3}}\int_0^1 r^2\mathrm{e}^r\mathrm{d}r = \frac{4\pi}{\sqrt{\lambda_1\lambda_2\lambda_3}}(\mathrm{e}-2).$$

Since the determinant of $\mathbf{A}$ is $\det\mathbf{A} = \lambda_1\lambda_2\lambda_3$, so

$$I = \frac{4\pi}{\sqrt{\det\mathbf{A}}}(\mathrm{e}-2).$$

Exercise 2.7. Find

$$\iiint\int_{\substack{x,y,z,u\ge 0\\ x^2+y^2+z^2+u^2\le 1}} \sqrt{\frac{1-x^2-y^2-z^2-u^2}{1+x^2+y^2+z^2+u^2}}\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}u.$$

<u>Solution:</u> We use four-dimensional spherical coordinates: $x = r\sin\varphi_1\sin\varphi_2\cos\theta$, $y = r\sin\varphi_1\sin\varphi_2\sin\theta$, $z = r\sin\varphi_1\cos\varphi_2$, $u = r\cos\varphi_1$, with $r \ge 0$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le \varphi_1 \le \frac{\pi}{2}$, $0 \le \varphi_2 \le \frac{\pi}{2}$. The Jacobian of the transformation is $J = r^3\sin^2\varphi_1\sin\varphi_2$, and $x^2 + y^2 + z^2 + u^2 = r^2$.

The given integral

$$\iiint\int_{\substack{x,y,z,u\geq 0\\ x^2+y^2+z^2+u^2\leq 1}} \sqrt{\frac{1-x^2-y^2-z^2-u^2}{1+x^2+y^2+z^2+u^2}}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}u$$
$$=\int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{\frac{\pi}{2}} \sin\varphi_2\, \mathrm{d}\varphi_2 \int_0^{\frac{\pi}{2}} \sin^2\varphi_1\, \mathrm{d}\varphi_1 \int_0^1 r^3\sqrt{\frac{1-r^2}{1+r^2}}\, \mathrm{d}r$$
$$=\frac{\pi}{2}\times\left(-\cos\varphi_2\big|_0^{\frac{\pi}{2}}\right)\times\frac{1}{2}\left[\varphi_1-\frac{\sin(2\varphi_1)}{2}\right]_0^{\frac{\pi}{2}}\times\frac{1}{2}\left(\int_0^1\frac{t}{\sqrt{1-t^2}}\, \mathrm{d}t-\int_0^1\frac{t^2}{\sqrt{1-t^2}}\, \mathrm{d}t\right)$$
$$=\frac{\pi}{2}\times 1\times\frac{\pi}{4}\times\frac{1}{2}\left(-\frac{1}{2}\int_1^0 u^{-\frac{1}{2}}\, \mathrm{d}u-\int_0^{\frac{\pi}{2}}\sin^2\alpha\, \mathrm{d}\alpha\right)$$
$$=\frac{\pi}{2}\times 1\times\frac{\pi}{4}\times\frac{1}{2}\left(1-\frac{\pi}{4}\right)$$
$$=\frac{\pi^2}{16}\left(1-\frac{\pi}{4}\right)$$

Exercise 2.8. Let

$$F(t)=\iiint_{x^2+y^2+z^2\leq t^2} f(x^2+y^2+z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

where f is a continuous function and $f(1)=1$. Prove that $F'(1)=4\pi$.

Solution: We use spherical coordinates to transform the triple-integral. In spherical coordinates, $x=\rho\sin\varphi\cos\theta$, $y=\rho\sin\varphi\sin\theta$, $z=\rho\cos\varphi$, and $\mathrm{d}V=\rho^2\sin\varphi\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta$, and $x^2+y^2+z^2=\rho^2$. The region $x^2+y^2+z^2\leq t^2$ corresponds to $0\leq\rho\leq t$, $0\leq\varphi\leq\pi$, $0\leq\theta\leq 2\pi$. Then

$$F(t)=\int_0^{2\pi} d\theta\int_0^{\pi}\sin\varphi\mathrm{d}\varphi\int_0^t f(\rho^2)\rho^2\mathrm{d}\rho.$$

Since $\int_0^{2\pi}\mathrm{d}\theta=2\pi$ and $\int_0^{\pi}\sin\varphi\mathrm{d}\varphi=-\cos\varphi|_0^{\pi}=2$, we have

$$F(t)=4\pi\int_0^t f(\rho^2)\rho^2\mathrm{d}\rho.$$

By the fundamental theorem of calculus, if $F(t)=4\pi\int_0^t g(\rho)\mathrm{d}\rho$ (where $g(\rho)=f(\rho^2)\rho^2$), then $F'(t)=4\pi f(t^2)t^2$. When $t=1$, since $f(1)=1$, we get $F'(1)=4\pi\times f(1)\times 1^2=4\pi$.

Exercise 2.9. Find

$$\iiint_{\Omega} z(x^2+y^2+z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

where Ω is the sphere $x^2+y^2+z^2\leq 2z$.

Solution: We rewrite the sphere equation $x^2 + y^2 + z^2 \leq 2z$ in spherical coordinates. With $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, $z = \rho \cos\varphi$ and $x^2+y^2+z^2 = \rho^2$, the sphere becomes $\rho^2 \leq 2\rho\cos\varphi$, so $\rho \leq 2\cos\varphi$ ($\rho \geq 0$). The ranges are $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \frac{\pi}{2}$, and $\mathrm{d}V = \rho^2 \sin\varphi \mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta$. The integrand $z(x^2+y^2+z^2)$ is $\rho^3\cos\varphi$.

$$\begin{aligned}\iiint_\Omega z(x^2+y^2+z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^{2\cos\varphi}\rho^3\cos\varphi\cdot\rho^2\sin\varphi\mathrm{d}\rho\\ &= 2\pi\int_0^{\frac{\pi}{2}}\cos\varphi\sin\varphi\left[\frac{\rho^6}{6}\right]_0^{2\cos\varphi}\mathrm{d}\varphi\\ &= \frac{2\pi}{6}\int_0^{\frac{\pi}{2}}64\cos^7\varphi\sin\varphi\mathrm{d}\varphi\\ &= \frac{64\pi}{3}\left[-\frac{\cos^8\varphi}{8}\right]_0^{\frac{\pi}{2}} = \frac{8\pi}{3}.\end{aligned}$$

Exercise 2.10. Let $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, and the region $\Omega \subset \mathbb{R}^3$ be determined by $z \geq \sqrt{x^2+y^2}$ and $4 \leq x^2+y^2+z^2 \leq 16$. Try to calculate the integral average value of the function f over Ω

$$\frac{1}{|\Omega|}\iiint_\Omega f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

where $|\Omega|$ is the volume of Ω.

Solution: In spherical coordinates, $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, $z = \rho\cos\varphi$, and $dV = \rho^2\sin\varphi\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta$, and $f(x,y,z) = \rho$. The inequality $z \geq \sqrt{x^2+y^2}$ implies $\rho\cos\varphi \geq \rho\sin\varphi$. Since $\rho > 0$ in the non origin part of the region, we have $\tan\varphi \leq 1$, so $0 \leq \varphi \leq \frac{\pi}{4}$. The inequality $4 \leq x^2+y^2+z^2 \leq 16$ implies $2 \leq \rho \leq 4$. And the range of θ is $0 \leq \theta \leq 2\pi$.

$$\begin{aligned}|\Omega| = \iiint_\Omega \mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{4}}\sin\varphi\mathrm{d}\varphi\int_2^4\rho^2\mathrm{d}\rho\\ = 2\pi\times[-\cos\varphi]_0^{\frac{\pi}{4}}\times\left[\frac{\rho^3}{3}\right]_2^4 &= \frac{56\pi(2-\sqrt{2})}{3}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\iiint_\Omega f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_\Omega \rho\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{4}}\sin\varphi\mathrm{d}\varphi\int_2^4\rho^3\mathrm{d}\rho\\ &= 2\pi\times[-\cos\varphi]_0^{\frac{\pi}{4}}\times\left[\frac{\rho^4}{4}\right]_2^4 = 60\pi(2-\sqrt{2}).\end{aligned}$$

The integral average value of f over Ω is

$$\frac{1}{|\Omega|}\iiint_\Omega f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{60\pi(2-\sqrt{2})}{\frac{56\pi(2-\sqrt{2})}{3}} = \frac{45}{14}.$$

2.3 n-multiple Integrals

Example 2.11. Let $N \in \mathbb{Z}_+$. Denote the volume of the unit ball $\sum_{k=1}^{n} x_k^2 \leq 1$ in the n-dimensional space as $\alpha(n)$. Calculate $\alpha(4)$ and write out the recurrence formula for the sequence $\alpha(n)$.

Solution:

1. General formula for the volume of an n-dimensional unit ball using integral.

The volume of the n-dimensional unit ball $\Omega_n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : \sum_{k=1}^{n} x_k^2 \leq 1\}$ is given by the n-fold integral

$$\alpha(n) = \int \cdots \int_{\sum_{k=1}^{n} x_k^2 \leq 1} \mathrm{d}x_1 \, \mathrm{d}x_2 \cdots \, \mathrm{d}x_n.$$

We use the following approach to establish a recurrence relation. We can write

$$\alpha(n) = \int_{-1}^{1} \left(\int \cdots \int_{\sum_{k=2}^{n} x_k^2 \leq 1 - x_1^2} \mathrm{d}x_2 \cdots \, \mathrm{d}x_n \right) \mathrm{d}x_1.$$

The inner integral $\int \cdots \int_{\sum_{k=2}^{n} x_k^2 \leq 1-x_1^2} \mathrm{d}x_2 \cdots \, \mathrm{d}x_n$ represents the volume of an $(n-1)$-dimensional ball with radius $r = \sqrt{1 - x_1^2}$. The volume of an $(n-1)$-dimensional ball of radius r is $r^{n-1}\alpha(n-1)$ (by the property of volume scaling in n-dimensions). So

$$\alpha(n) = \alpha(n-1) \int_{-1}^{1} (1 - x_1^2)^{\frac{n-1}{2}} \, \mathrm{d}x_1.$$

Let $x_1 = \sin t$, then $\mathrm{d}x_1 = \cos t \, \mathrm{d}t$. When $x_1 = -1$, $t = -\frac{\pi}{2}$; when $x_1 = 1$, $t = \frac{\pi}{2}$.

$$\int_{-1}^{1} (1 - x_1^2)^{\frac{n-1}{2}} \, \mathrm{d}x_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n t \, \mathrm{d}t = 2 \int_{0}^{\frac{\pi}{2}} \cos^n t \, \mathrm{d}t.$$

We know that

$$\int_{0}^{\frac{\pi}{2}} \cos^n t \, \mathrm{d}t = \begin{cases} \frac{(n-1)!!}{n!!} \cdot \frac{\pi}{2}, & n \text{ is even,} \\ \frac{(n-1)!!}{n!!}, & n \text{ is odd.} \end{cases}$$

So the recurrence formula is

$$\alpha(n) = 2\alpha(n-1) \int_{0}^{\frac{\pi}{2}} \cos^n t \, \mathrm{d}t.$$

2. Initial values.

For $n = 1$, the unit ball is the interval $[-1, 1]$, so $\alpha(1) = 2$.

For $n = 2$, $\alpha(2) = \pi$ (since the unit ball in 2-D is a unit circle $x_1^2 + x_2^2 \leq 1$ and its area is $\pi r^2 = \pi$).

3. Calculate $\alpha(4)$.
We first use the recurrence formula

$$\alpha(n) = 2\alpha(n-1)\int_0^{\frac{\pi}{2}} \cos^n t \text{ d}t.$$

For $n = 3$, $\int_0^{\frac{\pi}{2}} \cos^3 t \text{ d}t = \frac{2!!}{3!!} = \frac{2}{3}$, and

$$\alpha(3) = 2\alpha(2)\int_0^{\frac{\pi}{2}} \cos^3 t \text{ d}t = 2\pi \times \frac{2}{3} = \frac{4\pi}{3}.$$

For $n = 4$, $\int_0^{\frac{\pi}{2}} \cos^4 t \text{ d}t = \frac{3!!}{4!!} \cdot \frac{\pi}{2} = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{16}$.

$$\alpha(4) = 2\alpha(3)\int_0^{\frac{\pi}{2}} \cos^4 t \text{ d}t = 2 \times \frac{4\pi}{3} \times \frac{3\pi}{16} = \frac{\pi^2}{2}.$$

2.4 Symmetry

1. Symmetry of the Integration Region and the Integrand. Let D be an n-dimensional integration region in $\mathbb{R}^n$, and let $f(x_1, x_2, \cdots, x_n)$ be an $n-$variable function that is integrable over D.

Symmetry about coordinate hyperplanes:

- Even and odd functions: A function $f(x_1, x_2, \cdots, x_n)$ is said to be odd with respect to the variable x_i if

  $$f(x_1, \cdots, x_{i-1}, -x_i, x_{i+1}, \cdots, x_n) = -f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n).$$

  for all $(x_1, \cdots, x_n) \in \mathbb{R}^n$. It is said to be even with respect to x_i if

  $$f(x_1, \cdots, x_{i-1}, -x_i, x_{i+1}, \cdots, x_n) = f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n).$$

- Integral property: Suppose that the region D is symmetric about the hyperplane $x_i = 0$, i.e., if $(x_1, \cdots, x_n) \in D$, then $(x_1, \cdots, x_{i-1}, -x_i, x_{i+1}, \cdots, x_n) \in D$.

  - If f is odd with respect to x_i, then

    $$\int \cdots \int_D f(x_1, \cdots, x_n) \text{d}x_1 \cdots \text{d}x_n = 0.$$

  - If f is even with respect to x_i, then

    $$\int \cdots \int_D f(x_1, \cdots, x_n) \text{d}x_1 \cdots \text{d}x_n = 2 \int \cdots \int_{D_i} f(x_1, \cdots, x_n) \text{d}x_1 \cdots \text{d}x_n,$$

    where $D_i = \{(x_1, \cdots, x_n) \in D : x_i \geq 0\}$.

General symmetry of the region.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation that preserves the region D, i.e., $T(D) = D$. If $f(T(x_1, \cdots, x_n)) = f(x_1, \cdots, x_n)$ for all $(x_1, \cdots, x_n) \in D$, then

$$\int \cdots \int_D f(x_1, \cdots, x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n = \int \cdots \int_D f(T(x_1, \cdots, x_n))|J_T|\mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

where J_T is the Jacobian determinant of the transformation T. In the case where T is an orthogonal transformation (e.g., a rotation or a reflection), $|J_T| = 1$.

2. Examples in n-dimensional space

- Integration over the n-dimensional unit ball: Let

  $$B^n = \left\{(x_1, \cdots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 \leq 1\right\}$$

  be the n-dimensional unit ball. Consider the integral

  $$\int \cdots \int_{B^n} x_1^{2k_1} x_2^{2k_2} \cdots x_n^{2k_n} \mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

  where $k_1, k_2, \cdots, k_n$ are non - negative integers. Since B^n is symmetric about each of the coordinate hyperplanes $x_i = 0$, and the function

  $$g(x_1, \cdots, x_n) = x_1^{2k_1} x_2^{2k_2} \cdots x_n^{2k_n}$$

  is even with respect to each x_i, we can reduce the integral to an integral over the sub-ball

  $$B_+^n = \{(x_1, \cdots, x_n) \in B^n : x_i \geq 0, i = 1, \cdots, n\}.$$

  So

  $$\int \cdots \int_{B^n} x_1^{2k_1} x_2^{2k_2} \cdots x_n^{2k_n} \mathrm{d}x_1 \cdots \mathrm{d}x_n = 2^n \int \cdots \int_{B_+^n} x_1^{2k_1} x_2^{2k_2} \cdots x_n^{2k_n} \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

- Integration over a symmetric polyhedron: Let P be an $n-$dimensional symmetric polyhedron in $\mathbb{R}^n$, symmetric about the origin. Let $f(x_1, \cdots, x_n)$ be a function such that $f(-x_1, \cdots, -x_n) = f(x_1, \cdots, x_n)$ (the function is even with respect to the origin). We can use the symmetry of P to simplify the integral

  $$\int \cdots \int_P f(x_1, \cdots, x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

  For example, if P can be divided into 2^n congruent sub - polyhedra $P_1, \cdots, P_{2^n}$ by the coordinate hyperplanes $x_i = 0$ $(i = 1, \cdots, n)$, then

  $$\int \cdots \int_P f(x_1, \cdots, x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n = 2^n \int \cdots \int_{P_1} f(x_1, \cdots, x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

  where P_1 is the part of P with $x_i \geq 0$ for $i = 1, \cdots, n$.

3. Conclusions In n-dimensional integration, identifying the symmetry of the integration region (such as symmetry about coordinate hyperplanes, symmetry about the origin, or symmetry under a linear transformation) and the parity properties of the integrand (even or odd with respect to variables or under a transformation) is a powerful tool for simplifying multi - integrals. By using these symmetries, we can often reduce the original integral over a large region to an integral over a smaller, more manageable sub - region, or even directly conclude that the integral is zero in the case of odd functions with respect to a symmetric region. This not only simplifies the computational process but also provides a deeper understanding of the geometric and algebraic properties of the functions and regions involved in the integration.

Example 2.12. Calculate

$$\iiint_{\Omega}(x+y+z)^2\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

where

$$\Omega := \left\{(x,y,z)\in\mathbb{R}^3 : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\right\}.$$

Solution: First, expand $(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2zx$. Since $\Omega = \left\{(x,y,z)\in\mathbb{R}^3 : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\right\}$ is symmetric about the coordinate planes $x=0$, $y=0$ and $z=0$, we have:

$$\iiint_{\Omega} xy\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0$$
$$\iiint_{\Omega} yz\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0$$
$$\iiint_{\Omega} zx\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0$$

By the transformation $x=au$, $y=bv$, $z=cw$, then $\mathrm{d}x\mathrm{d}y\mathrm{d}z = abc\mathrm{d}u\mathrm{d}v\mathrm{d}w$, and the region Ω is transformed into the unit ball $B=\left\{(u,v,w)\in\mathbb{R}^3 : u^2+v^2+w^2\le 1\right\}$. We use spherical coordinates $u=\rho\sin\varphi\cos\theta$, $v=\rho\sin\varphi\sin\theta$, $w=\rho\cos\varphi$, where $\mathrm{d}u\mathrm{d}v\mathrm{d}w = \rho^2\sin\varphi\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta$ and the limits of integration for the unit ball B are $0\le\rho\le 1$, $0\le\varphi\le\pi$, $0\le\theta\le 2\pi$. We calculate $\iiint_B(u^2+v^2+w^2)\mathrm{d}u\mathrm{d}v\mathrm{d}w$ as follows:

$$\iiint_B(u^2+v^2+w^2)\mathrm{d}u\mathrm{d}v\mathrm{d}w = \int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\mathrm{d}\varphi\int_0^1\rho^2\cdot\rho^2\mathrm{d}\rho = \frac{4\pi}{5}.$$

Since $\iiint_B u^2\mathrm{d}u\mathrm{d}v\mathrm{d}w = \iiint_B v^2\mathrm{d}u\mathrm{d}v\mathrm{d}w = \iiint_B w^2\mathrm{d}u\mathrm{d}v\mathrm{d}w$ and $\iiint_B(u^2+v^2+w^2)\mathrm{d}u\mathrm{d}v\mathrm{d}w = 3\iiint_B u^2\mathrm{d}u\mathrm{d}v\mathrm{d}w$, we get $\iiint_B u^2\mathrm{d}u\mathrm{d}v\mathrm{d}w = \frac{4\pi}{15}$. Then $\iiint_\Omega x^2\mathrm{d}x\mathrm{d}y\mathrm{d}z =$

$a^3bc\iiint_B u^2\mathrm{d}u\mathrm{d}v\mathrm{d}w = \frac{4\pi a^3bc}{15}$, $\iiint_\Omega y^2\mathrm{d}x\mathrm{d}y\mathrm{d}z = b^3ac\iiint_B v^2\mathrm{d}u\mathrm{d}v\mathrm{d}w = \frac{4\pi ab^3c}{15}$, $\iiint_\Omega z^2\mathrm{d}x\mathrm{d}y\mathrm{d}z = c^3ab\iiint_B w^2\mathrm{d}u\mathrm{d}v\mathrm{d}w = \frac{4\pi abc^3}{15}$. So,

$$\begin{aligned}\iiint_\Omega (x+y+z)^2\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \iiint_\Omega x^2\mathrm{d}x\mathrm{d}y\mathrm{d}z + \iiint_\Omega y^2\mathrm{d}x\mathrm{d}y\mathrm{d}z + \iiint_\Omega z^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &= \frac{4\pi a^3bc}{15} + \frac{4\pi ab^3c}{15} + \frac{4\pi abc^3}{15}\\ &= \frac{4\pi abc(a^2+b^2+c^2)}{15}.\end{aligned}$$

Exercise 2.13. Find

$$I = \iiint_\Omega (x+y+z)^2\mathrm{d}V,$$

where Ω is the intersection of $x^2+y^2\le 2z$ and $x^2+y^2+z^2\le 3$.

Solution: Given $I = \iiint_\Omega (x+y+z)^2\mathrm{d}V$. Expand $(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2zx$. Because of the symmetry of the region Ω (the intersection of $x^2+y^2\le 2z$ and $x^2+y^2+z^2\le 3$) with respect to the coordinate planes, we have $\iiint_\Omega 2xy\mathrm{d}V = 0$, $\iiint_\Omega 2yz\mathrm{d}V = 0$, and $\iiint_\Omega 2zx\mathrm{d}V = 0$. So, $I = \iiint_\Omega (x^2+y^2+z^2)\mathrm{d}V$.

In cylindrical coordinates, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, and $\mathrm{d}V = r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$. The inequalities become $r^2\le 2z$ and $r^2+z^2\le 3$. To find the intersection of $r^2 = 2z$ and $r^2+z^2 = 3$, substitute $r^2 = 2z$ into $r^2+z^2 = 3$, getting $z^2+2z-3 = 0$. The non - negative root is $z = 1$, and then $r = \sqrt{2}$. The integral I can be written as:

$$\begin{aligned}I &= \int_0^{2\pi}\mathrm{d}\theta\int_0^{\sqrt{2}} r\mathrm{d}r\int_{\frac{r^2}{2}}^{\sqrt{3-r^2}}(r^2+z^2)\mathrm{d}z = 2\pi\int_0^{\sqrt{2}} r\left(r^2z+\frac{z^3}{3}\right)\Big|_{\frac{r^2}{2}}^{\sqrt{3-r^2}}\mathrm{d}r\\ &= 2\pi\int_0^{\sqrt{2}} r\left(r^2\sqrt{3-r^2}+\frac{(3-r^2)^{\frac{3}{2}}}{3}-\frac{r^4}{2}-\frac{r^6}{24}\right)\mathrm{d}r\\ &= 2\pi\left(\int_0^{\sqrt{2}} r^3\sqrt{3-r^2}\mathrm{d}r+\frac{1}{3}\int_0^{\sqrt{2}} r(3-r^2)^{\frac{3}{2}}\mathrm{d}r-\frac{1}{2}\int_0^{\sqrt{2}} r^5\mathrm{d}r-\frac{1}{24}\int_0^{\sqrt{2}} r^7\mathrm{d}r\right)\\ &= \frac{\pi(17-2\sqrt{3})}{5}\end{aligned}$$

Chapter 3

Curvilinear integral

3.1 Relations between type I and type II

The coordinate differential element is given by $\mathrm{d}\boldsymbol{\tau} = \boldsymbol{\tau}\,\mathrm{d}s$, where $\boldsymbol{\tau}$ is the unit tangent vector of the curve in the specified direction. Thus, we have

$$\int_\Gamma \boldsymbol{F}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{r} = \int_\Gamma (\boldsymbol{F}(\boldsymbol{x}) \cdot \boldsymbol{\tau})\,\mathrm{d}s.$$

For a space curve Γ, let $\cos\alpha, \cos\beta, \cos\gamma$ be the direction cosines of the tangent. Then,

$$\int_\Gamma P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z = \int_\Gamma (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,\mathrm{d}s.$$

When Γ is a plane curve, if we take the direction of the normal $\boldsymbol{n}$ and the direction of the tangent $\boldsymbol{\tau}$ such that the angle $\angle(\boldsymbol{n}, \boldsymbol{\tau}) = \frac{\pi}{2}$, we have

$$\angle(x, \boldsymbol{n}) = \angle(x, \boldsymbol{\tau}) + \angle(\boldsymbol{\tau}, \boldsymbol{n}) = \alpha - \frac{\pi}{2}.$$

Thus,

$$\int_\Gamma P\,\mathrm{d}x + Q\,\mathrm{d}y = \int_\Gamma [-P\sin(x, \boldsymbol{n}) + Q\cos(x, \boldsymbol{n})]\,\mathrm{d}s.$$

3.2 Calculations of curvilinear integral: type I

Example 3.1. Calculate

$$\int_L \sqrt{x^2 + y^2}\mathrm{d}s,$$

where $L : x^2 + y^2 = ax$.

Solution: First, convert the curve $x^2 + y^2 = ax$ to polar coordinates: In polar coordinates, $x = r\cos\theta, \quad y = r\sin\theta$, so $r^2 = ar\cos\theta$, then $r = a\cos\theta$ $(r \neq 0)$. Then calculate the line integral:

$$\int_L \sqrt{x^2 + y^2}\mathrm{d}s = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r \cdot a\mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\cos\theta \cdot a\mathrm{d}\theta = 2a^2$$

Example 3.2. Find $I = \oint_C x^2\ \mathrm{d}s$, where C is given by

$$\begin{cases} x^2 + y^2 + z^2 = R^2 \\ x + y + z = 0 \end{cases}$$

Solution:

(Conventional method) First, write out the parametric expressions of the curve C. Since C is the intersection line of the spherical surface $x^2+y^2+z^2=R^2$ and the plane $x+y+z=0$ passing through the center of the sphere, it is a circular circumference in space. Its projection on the xOy plane is an ellipse, and the equation of this ellipse can be obtained by eliminating z from the equations of the two surfaces. That is, substitute $z=-(x+y)$ into $x^2+y^2+z^2=R^2$, and we get

$$x^2+xy+y^2=\frac{R^2}{2}.$$

Complete the square for the left side into the sum of squares

$$\left(\frac{\sqrt{3}}{2}x\right)^2+\left(\frac{x}{2}+y\right)^2=\frac{R^2}{2}.$$

Let

$$\frac{\sqrt{3}}{2}x=\frac{R}{\sqrt{2}}\cos t,\quad \frac{x}{2}+y=\frac{R}{\sqrt{2}}\sin t,\quad t\in[0,2\pi].$$

That is, we obtain the parametric representation

$$x=\sqrt{\frac{2}{3}}R\cos t,\quad y=\frac{R}{\sqrt{2}}\sin t-\frac{R}{\sqrt{6}}\cos t,\quad t\in[0,2\pi].$$

Substitute it into $z=-(x+y)$, and we get

$$z=-\frac{R}{\sqrt{6}}\cos t-\frac{R}{\sqrt{2}}\sin t,\quad t\in[0,2\pi].$$

From this, we have

$$\begin{aligned}
\mathrm{d}s&=\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\ \mathrm{d}t\\
&=R\sqrt{\frac{2}{3}\sin^2 t+\left(\frac{\cos t}{\sqrt{2}}+\frac{\sin t}{\sqrt{6}}\right)^2+\left(\frac{\sin t}{\sqrt{6}}-\frac{\cos t}{\sqrt{2}}\right)^2}\ \mathrm{d}t\\
&=R\ \mathrm{d}t.
\end{aligned}$$

Therefore, we have

$$\int_C x^2\ \mathrm{d}s=\int_0^{2\pi}\frac{2}{3}R^3\cos^2 t\ \mathrm{d}t=\frac{2}{3}\pi R^3.$$

(Using symmetry) By symmetry, we have

$$\int_C x^2\ \mathrm{d}s=\int_C y^2\ \mathrm{d}s=\int_C z^2\ \mathrm{d}s.$$

Then

$$\int_C x^2\ \mathrm{d}s=\frac{1}{3}\int_C\left(x^2+y^2+z^2\right)\mathrm{d}s=\frac{1}{3}R^2\int_C\ \mathrm{d}s=\frac{2}{3}\pi R^3.$$

Exercise 3.3. Calculate the following line integrals of type I:

1. $\int_C \left(x^{4/3} + y^{4/3}\right) \mathrm{d}s$, where C is the astroid curve $x^{2/3} + y^{2/3} = a^{2/3}$.

2. $\int_C \mathrm{e}^{\sqrt{x^2+y^2}} \mathrm{d}s$, where C is composed of the curve $r = a$, $\varphi = 0$, $\varphi = \frac{\pi}{4}$ (r and φ are polar-coordinates).

3. $\int_C |y| \mathrm{d}s$, where C is the lemniscate $\left(x^2 + y^2\right)^2 = a^2 \left(x^2 - y^2\right)$.

4. $\int_C \frac{\mathrm{d}s}{y^2}$, where C is the catenary $y = a \operatorname{ch} \frac{x}{a}$.

5. $\int_C z \mathrm{d}s$, where C is the curve $x^2 + y^2 = z^2$, $y^2 = ax$ from the point $(0, 0, 0)$ to the point $(a, a, \sqrt{2}a)$.

Solution: 1. First, parameterize the astroid curve $x^{2/3} + y^{2/3} = a^{2/3}$ with $x = a \cos^3 t$, $y = a \sin^3 t$, $t \in [0, 2\pi]$. Then

$$\begin{aligned}\int_C (x^{4/3} + y^{4/3}) \mathrm{d}s &= \int_0^{2\pi} (a^{4/3} \cos^4 t + a^{4/3} \sin^4 t) \cdot 3a |\sin t \cos t| \mathrm{d}t \\ &= 4 \int_0^{\frac{\pi}{2}} (a^{4/3} \cos^4 t + a^{4/3} \sin^4 t) \cdot 3a \sin t \cos t \mathrm{d}t = 4a^{7/3}\end{aligned}$$

2. $\frac{\pi a e^a}{4}$. 3. $4a^2(1 - \frac{\sqrt{2}}{2}) = 2a^2(2 - \sqrt{2})$. 4. $\frac{\pi}{a}$

Exercise 3.4. Let $f(x, y)$ be continuous on L, and L be a piecewise smooth simple closed curve. Prove that:

$$u(x, y) = \oint_L f(\xi, \eta) \ln \sqrt{(x - \xi)^2 + (y - \eta)^2} \mathrm{d}s.$$

A necessary and sufficient condition for $u(x, y)$ to tend to 0 as $x^2 + y^2 \to +\infty$ is that

$$\oint_L f(\xi, \eta) \mathrm{d}s = 0.$$

Solution: Note that

$$\begin{aligned}&\left| \oint_L f(\xi, \eta) \ln \sqrt{(x - \xi)^2 + (y - \eta)^2} \mathrm{d}s - \oint_L f(\xi, \eta) \ln \sqrt{x^2 + y^2} \mathrm{d}s \right| \\ &= \oint_L |f(\xi, \eta)| \ln \frac{\sqrt{(x - \xi)^2 + (y - \eta)^2}}{\sqrt{x^2 + y^2}} \mathrm{d}s.\end{aligned}$$

The right hand side converges to 0. This implies the equivalence of the result in this problem.

3.3 Calculations of curvilinear integral: type II

Example 3.5. Calculate the integral

$$I = \int_C (x^2 + 2xy)\,\mathrm{d}y$$

where C represents the upper half of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ oriented counterclockwise.

Solution: Using the parametric expressions for the ellipse,

$$x = a\cos t, \quad y = b\sin t$$

with t varying from 0 to π in the given direction, we substitute x and y into the integral and replace $\mathrm{d}y$ with $b\cos t, \mathrm{d}t$:

$$\begin{aligned} I &= \int_0^{\pi} \left(a^2\cos^2 t + 2ab\cos t\sin t\right) b\cos t\,dt \\ &= a^2 b\int_0^{\pi} \cos^3 t\,dt + 2ab^2\int_0^{\pi} \cos^2 t\sin t\,dt = \frac{4}{3}ab^2. \end{aligned}$$

Example 3.6. Find

$$I = \int_\Gamma y^2\,\mathrm{d}x + z^2\,\mathrm{d}y + x^2\,dz,$$

where Γ is the curve defined by

$$\begin{cases} x^2 + y^2 + z^2 = a^2 \\ x^2 + y^2 = ax \quad (a > 0) \end{cases}$$

the part of the curve where $z \geq 0$, and viewed from the positive direction of the x-axis, Γ is oriented counterclockwise.

Solution: First, we need to find the parametric expressions for the curve Γ. If we use one of x, y, z as a parameter, square roots will be involved, and branches will appear, making it inconvenient. From the second equation, it is easy to see that we can introduce cylindrical coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

Thus, the parametric equations for Γ become

$$\begin{aligned} &x = a\cos^2\theta, \quad y = a\cos\theta\sin\theta \\ &z = \sqrt{a^2 - r^2} = a|\sin\theta|, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}. \end{aligned}$$

According to the orientation of the curve, we have

$$I = \int_{-\pi/2}^{\pi/2} \big[a^2 \cos^2\theta \sin^2\theta \cdot 2a\cos\theta(-\sin\theta) \\ + a^2\sin^2\theta \cdot a(\cos^2\theta - \sin^2\theta) + a^2\cos^4\theta \cdot z'(\theta)\big]\, d\theta.$$

The first term is an odd function, and since $z(\theta)$ is an even function, $z'(\theta)$ is an odd function; thus, the third term is also an odd function. Then

$$\begin{aligned} I &= \int_{-\pi/2}^{\pi/2} a^3 \sin^2\theta \left(\cos^2\theta - \sin^2\theta\right) \mathrm{d}\theta \\ &= 2a^3 \int_0^{\pi/2} \left(\sin^2\theta - 2\sin^4\theta\right) \mathrm{d}\theta = -\frac{\pi}{4}a^3 \end{aligned}$$

Example 3.7. Let the arc length of the curve C be L. Prove that:

$$\left| \int_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y \right| \le ML$$

where $M = \max_{(x,y)\in C} \sqrt{P^2(x,y) + Q^2(x,y)}$. Using the conclusion of the previous question, prove that

$$\lim_{R\to 0} \oint_{x^2+y^2=R^2} \frac{y\mathrm{d}x - x\mathrm{d}y}{(x^2+xy+y^2)^2} = 0.$$

Solution: We parametrize L as $x(t)$ and $y(t)$, where $t \in [a,b]$. Then

$$\begin{aligned} \left| \int_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y \right| &= \left| \int_a^b (P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t))\mathrm{d}t \right| \\ &\le M \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\mathrm{d}t \le ML. \end{aligned}$$

For the second one, we see that for corresponding P, Q, there holds that

$$|P|, |Q| \le \frac{1}{(x^2+y^2)^{3/2}}.$$

Then

$$\left| \oint_{x^2+y^2=R^2} \frac{y\mathrm{d}x - x\mathrm{d}y}{(x^2+xy+y^2)^2} \right| \le CR \cdot \frac{1}{R^3} \to 0.$$

Solution:

Exercise 3.8. Calculate $\int_C (x+y)^2\mathrm{d}x + (x^2-y^2)\mathrm{d}y$, where C is the triangle with vertices $A(1,1)$, $B(3,2)$, and $D(3,1)$, and the direction of C is clockwise.

Solution: $-\frac{8}{3}$.

Exercise 3.9. Find

$$\oint_C \left(y^2+z^2\right) \mathrm{d}x + \left(z^2+x^2\right) \mathrm{d}y + \left(x^2+y^2\right) \mathrm{d}z,$$

where C is the intersection curve of the surfaces $x^2+y^2+z^2=2Rx$ and $x^2+y^2=2ax$ (with $0<a<R$ and $z>0$), and is oriented counterclockwise when viewed from the positive direction of the z-axis.

Solution: $2\pi a^2(R-a)$.

Exercise 3.10. Calculate

$$\int_L (x-2xy^2)\mathrm{d}x + (y-2x^2y)\mathrm{d}y,$$

where L is given by $y=x^2$ from $(0,0)$ to $(2,4)$.

3.4 Using Green's formula

Green's formula is an important formula that establishes a connection between the double integral over a certain region in the plane and a specific type II curvilinear integral on the boundary of that region.

Let D be a bounded closed region in $\mathbb{R}^2$, and its boundary ∂D consists of smooth curves or piece - wise smooth curves. Also, assume that the functions P and Q have continuous partial derivatives with respect to the independent variables x and y on D. Then the following Green's formula holds:

$$\iint_D \left(-\frac{\partial P}{\partial y}+\frac{\partial Q}{\partial x}\right) \mathrm{d}x\mathrm{d}y = \oint_{\partial D} P\mathrm{d}x + Q\mathrm{d}y,$$

where the direction of ∂D is the positive direction with respect to D. Similarly,

$$\begin{aligned}\iint_D \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right) \mathrm{d}x\mathrm{d}y &= \oint_{\partial D} -Q\mathrm{d}x + P\mathrm{d}y \\ &= \oint_{\partial D} [Q\sin(x,\boldsymbol{n}) + P\cos(x,\boldsymbol{n})]\mathrm{d}s,\end{aligned}$$

where $\boldsymbol{n}$ is the unit outer normal vector on ∂D. Using $\angle(x,\boldsymbol{n}) = \angle(x,y)+\angle(y,\boldsymbol{n}) = \frac{\pi}{2}+\angle(y,\boldsymbol{n})$ (see Figure 24.2), then

$$\iint_D \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right) \mathrm{d}x\mathrm{d}y = \oint_{\partial D} [P\cos(x,\boldsymbol{n}) + Q\cos(y,\boldsymbol{n})]\mathrm{d}s$$

Formulas above provide us with an indirect method for calculating line integrals in the plane. That is, we use Green's formula to convert the line integral into a double integral. Even if the curve C is not closed, we can use the method of adding "auxiliary lines".

Example 3.11. Let C be the arc of the parabola $2x = \pi y^2$ from $(0,0)$ to $\left(\frac{\pi}{2}, 1\right)$. Calculate

$$I = \int_C \left(2xy^3 - y^2 \cos x\right) \mathrm{d}x + \left(1 - 2y \sin x + 3x^2 y^2\right) \mathrm{d}y.$$

<u>Solution:</u> Let $P(x,y) = 2xy^3 - y^2 \cos x$ and $Q(x,y) = 1 - 2y \sin x + 3x^2y^2$. Then

$$-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = 0.$$

Set $A = (\frac{\pi}{2}, 0)$. To use Green's formula, we add auxiliary lines $O = (0,0) \to A$ and $A \to B = (\frac{\pi}{2}, 1)$. Then

$$I = \int_C + \int_{\widehat{BA}} + \int_{\widehat{AO}} + \int_{\widehat{AB}} + \int_{\widehat{OA}}$$

By Green's formula, the integral of the first three terms is zero. So

$$I = \int_{\widehat{AB}} + \int_{\widehat{OA}} = \int_0^1 \left[1 - 2y \sin\frac{\pi}{2} + 3\left(\frac{\pi}{2}\right)^2 y^2\right] \mathrm{d}y = \frac{\pi^2}{4}.$$

Example 3.12. Calculate the integral

$$I = \oint_C \frac{\cos(\boldsymbol{r}, \boldsymbol{n})}{r} \mathrm{d}s,$$

where C is a piecewise smooth simple closed curve, $\boldsymbol{r} = (x, y)$, $r = |\boldsymbol{r}| = \sqrt{x^2 + y^2}$, and $\boldsymbol{n}$ is the unit outer normal vector on C.

<u>Solution:</u> From $\cos(\boldsymbol{r}, \boldsymbol{n}) = \frac{\boldsymbol{r} \cdot \boldsymbol{n}}{r} = \frac{1}{r}(x \cos(\boldsymbol{n}, \boldsymbol{x}) + y \cos(\boldsymbol{n}, y))$, we get

$$I = \oint_C \left(\frac{x}{r^2} \cos(\boldsymbol{n}, x) + \frac{y}{r^2} \cos(\boldsymbol{n}, y)\right) \mathrm{d}s.$$

When $(0,0)$ is outside C, using Green's formula (24.10), we have

$$I = \iint_D \left[\partial_x \left(\frac{x}{r^2}\right) + \partial_y \left(\frac{y}{r^2}\right)\right] \mathrm{d}x\mathrm{d}y = 0,$$

where D is the region bounded by C. When $(0,0)$ is inside C, we draw a circle C_ε with the center $(0,0)$ and a sufficiently small radius ε such that C_ε is inside C. Let the region bounded by C and C_ε be D_ε, then

$$\begin{aligned} I = &\oint_C \left(\frac{x}{r^2} \cos(\boldsymbol{n}, x) + \frac{y}{r^2} \cos(\boldsymbol{n}, y)\right) \mathrm{d}s + \oint_{C_\varepsilon} \left(\frac{x}{r^2} \cos(\boldsymbol{n}, x) + \frac{y}{r^2} \cos(\boldsymbol{n}, y)\right) \mathrm{d}s \\ &- \oint_{C_\varepsilon} \left(\frac{x}{r^2} \cos(\boldsymbol{n}, x) + \frac{y}{r^2} \cos(\boldsymbol{n}, y)\right) \mathrm{d}s, \end{aligned}$$

where the direction of the unit normal vector $\boldsymbol{n}$ on C_ε points to the origin of coordinates. Applying Green's formula to the first two terms, we get

$$I = -\oint_{C_\varepsilon} \left(\frac{x}{r^2} \cos(\boldsymbol{n}, x) + \frac{y}{r^2} \cos(\boldsymbol{n}, y) \right) \mathrm{d}s.$$

On the circle C_ε,

$$\cos(\boldsymbol{n}, x) = -\frac{x}{\varepsilon}, \quad \cos(\boldsymbol{n}, y) = -\frac{y}{\varepsilon}, \quad r = \varepsilon$$

Thus

$$I = \oint_{C_\varepsilon} \frac{1}{\varepsilon} \mathrm{d}s = 2\pi$$

When $(0,0) \in C$, we draw the tangents OA and OB of the curve C passing through the origin. Let the included angle between OA and OB be θ. If the curve C is smooth at the origin, then $\theta = \pi$. We draw a circle B_ε with the center $(0,0)$ and radius ε, and denote the part of the circumference of B_ε inside C as C_ε. From the above calculation, we know that

$$I = \lim_{\varepsilon \to 0} \int_{C_\varepsilon} \frac{1}{\varepsilon} ds = \lim_{\varepsilon \to 0} \theta_\varepsilon = \theta$$

3.5 Path independence

Let Ω be a region in $\mathbf{R}^2$, and $P(x,y), Q(x,y)$ be continuous on Ω. Denote

$$w = P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y.$$

Arbitrarily take points $A, B \in \Omega$. A piecewise smooth simple curve in Ω from A to B is called a path in Ω from A to B. For any path L in Ω from A to B, if the second - type line integral

$$\int_L w = \int_L P\mathrm{d}x + Q\mathrm{d}y$$

depends only on A and B, and is independent of the specific choice of L, then the first - order differential form w is said to have a line integral independent of the path in Ω. If the region enclosed by any simple closed curve in the planar region D is completely contained in D, then D is called a simply connected region. Let D be a simply connected region in the plane, and $w = P\mathrm{d}x + Q\mathrm{d}y$, where both P and Q have continuous partial derivatives on D. Then the following conclusions are equivalent:

1. For any closed curve C in D, we have

$$\oint_C w = 0;$$

2. For any path C in D, the integral $\int_C w$ depends only on the starting point and the ending point of C, and is independent of the path taken;
3. In D (everywhere), the following holds

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

4. There exists a function $\varphi(x, y)$ such that in D, the following holds

$$\mathrm{d}\varphi(x, y) = P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y$$

That is, $P\mathrm{d}x + Q\mathrm{d}y$ is the total differential of the function φ. At this time, φ is called the potential function or primitive function of w, and w is also called an exact differential form. At this time,

$$\varphi(x, y) = \int_{x_0}^{x} P(x, y_0)\mathrm{d}x + \int_{y_0}^{y} Q(x, y)\mathrm{d}y + C,$$

where (x_0, y_0) is an arbitrary point in D, C is an arbitrary constant, and

$$\int_{(x_0,y_0)}^{(x,y)} P\mathrm{d}x + Q\mathrm{d}y = \varphi|_{(x_0,y_0)}^{(x,y)} = \varphi(x, y) - \varphi(x_0, y_0).$$