Lecture Notes on Problem-Solving Class: Advanced Mathematics B(I)

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Contents

Preface: Some Important Information

0.1 Course Assessment Information

- **Usual Performance:** Hand in assignments. It accounts for 20 points of the overall evaluation.
- **Mid-term Examination:** On the weekend of the 8th or 9th week. It accounts for 30 points of the overall evaluation.
- **Final Examination:** On Monday evening, June 9th. It accounts for 50 points of the overall evaluation.

0.2 Information of the Class and the Teaching Assistant

- **Teacher:** Wei Wang.
  - **E-mail:** 2201110024@stu.pku.edu.cn
  - **Personal Website:** https://luisyanka.github.io/weiwang.github.io/
- **Location:** Room 313, Teaching Building 2.
- **Students:** Students whose student ID numbers are greater than 2400011822 and less than or equal to 2400015443 should submit their assignments to Teacher Wei Wang's class.

0.3 Some Useful Links

We present some useful links associated with calculus.

- Lecture notes by Yantong Xie: https://darkoxie.github.io
- Mathstackexchange: https://math.stackexchange.com

0.4 Topics of the Class

In this problem-solving class, we will present some classical exercises related to the topics which are delivered by the lecturer in the main course. We will mainly refer to the lecture notes written by Yantong Xie, who was a very good teaching assistant of the course Advanced mathematics (B). We also refer to the book "Guide to Solving Problems in Advanced Mathematics" by Jianying Zhou and Zhengyuan Li. If you have any advices for this class, then you can contact me with the e-mail. The lecture notes of this class will be updated before the next one on my personal website in the content of "teaching".

Chapter 1

Double integral

1.1 Calculating by the Definition

Example 1.1. There are three points P_0, P_1, P_2 on the plane, which are given by $\{(x_i, y_i)\}_{i=0}^2$. We assume that $x_2 > x_1 > x_0$ and $y_2 > y_1 > y_0$. Please calculate the area of triangle $\Delta P_0P_1P_2$.

Solution: Denote the triangle $\Delta P_0P_1P_2$ by D. By simple calculations, we can determine P_0P_1 : $y = k_1x + b_1$, $P_1P_2 : y = k_2x + b_2$, and $P_1P_3 : y = k_3x + b_3$. WLOG, we assume that $y_1 < k_3x_1 + b_3$. As a result, we have

$$\begin{aligned}\int_D 1\mathrm{d}x\mathrm{d}y &= \int_{x_0}^{x_1} \mathrm{d}x \int_{k_1x+b_1}^{k_3x+b_3} 1\mathrm{d}y + \int_{x_0}^{x_1} \mathrm{d}x \int_{k_2x+b_2}^{k_3x+b_3} 1\mathrm{d}y \\ &= \frac{1}{2}((y_2 - y_0)(x_1 - x_0) - y_1(x_2 - x_0)).\end{aligned}$$

Combined with the case that $y_1 \geq k_3x_1 + b_3$, we obtain

$$A(D) = \frac{1}{2}|x_1y_2 - x_1y_0 - x_0y_2 + x_0y_0 - x_2y_1 + x_0y_1|.$$

Exercise 1.2. Let $A = [0, 1] \times [0, 1]$, find

$$I = \iint_A \frac{y\mathrm{d}x\mathrm{d}y}{(1 + x^2 + y^2)^{\frac{3}{2}}}.$$

Solution: Integrating with respect to y first and then with respect to x, we get

$$\begin{aligned}I &= \int_0^1 \mathrm{d}x \int_0^1 \frac{y\mathrm{d}y}{(1 + x^2 + y^2)^{\frac{3}{2}}} \\ &= \int_0^1 \left(\frac{1}{\sqrt{x^2+1}} - \frac{1}{\sqrt{x^2+2}}\right) \mathrm{d}x = \ln\frac{2+\sqrt{2}}{1+\sqrt{3}}.\end{aligned}$$

1.2 Iterated Integrals

Example 1.3. Calculate $\int_0^1 \int_y^1 \frac{y}{\sqrt{1+x^3}}\mathrm{d}x$.

Solution: Let

$$D := \{(x, y) \in \mathbb{R}^2 : y \leq x,\ x \in [0, 1],\ y \in [0, 1]\}.$$

We have

$$\int_0^1 \mathrm{d}y \int_y^1 \frac{y}{\sqrt{1+x^3}}\mathrm{d}x = \int_D \frac{y}{\sqrt{1+x^3}}\mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^x \frac{y}{\sqrt{1+x^3}}\mathrm{d}y$$
$$= \frac{1}{6}\int_0^1 \frac{\mathrm{d}t}{\sqrt{1+t}} = \frac{1}{3}(\sqrt{2}-1),$$

where for the third inequality, we have used $t = x^3$.

Exercise 1.4. Calculate $\int_0^1 \frac{x-1}{\ln x}\mathrm{d}x$.

Solution: We note

$$\int_0^1 \frac{x-1}{\ln x}\mathrm{d}x = \int_0^1 \mathrm{d}x \int_0^1 x^y \mathrm{d}y = \int_{[0,1]^2} x^y \mathrm{d}x\mathrm{d}y$$
$$= \int_0^1 \mathrm{d}y \int_0^1 x^y \mathrm{d}x = \int_0^1 \frac{1}{y+1}\mathrm{d}y = \ln 2.$$

Exercise 1.5. Suppose that f is continuous on $[0,1]$. Prove that:

$$\int_0^1 \mathrm{d}x \int_x^1 f(t)\mathrm{d}t = \int_0^1 tf(t)\mathrm{d}t.$$

1.3 Change of Variables

Example 1.6 (Observing the region)**.** The region $D \subset \mathbb{R}^2$ is surrounded by the curves $xy = a$, $xy = b$, $y = px$, and $y = qx$, where $0 < a < b$ and $0 < p < q$. Please calculate

$$I = \iint_D xy^3 \mathrm{d}x\mathrm{d}y.$$

Solution: Consider a change of variables as

$$\begin{cases} x' = \dfrac{y}{x}, \\ y' = xy. \end{cases}$$

We can calculate that

$$\left|\frac{\partial(x,y)}{\partial(x',y')}\right| = -\frac{1}{2x'}.$$

As a result,

$$I = \int_p^q \left(\int_a^b x'(y')^2 \cdot \frac{1}{2x'}\mathrm{d}x' \right) \mathrm{d}y' = \frac{(b^3-a^3)(q-p)}{3}$$

Example 1.7 (Rotation)**.** Calculate $\int_D |3x+4y|\mathrm{d}x\mathrm{d}y$, where

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Solution: Consider a change of variables as

$$\begin{cases} x' = \dfrac{4}{5}x - \dfrac{3}{5}y, \\ y' = \dfrac{3}{5}x + \dfrac{4}{5}y. \end{cases}$$

Indeed, formula above give a rotation, which preserve D and change the line $3x+4y=t$ to $y'=t$ for any $t\in\mathbb{R}$. As a result, we have

$$\int_D |3x+4y|\mathrm{d}x\mathrm{d}y = 5\int_D |y'|\mathrm{d}x'\mathrm{d}y' = \frac{20}{3}.$$

Question 1.8. Can you give the intuition behind this change of variables?

Example 1.9 (Polar coordinate)**.** Make a polar coordinate transformation to convert the double integral

$$\iint_D f(\sqrt{x^2+y^2})\mathrm{d}x\mathrm{d}y$$

into a definite integral, where $D=\{(x,y)\mid 0\le y\le x\le 1\}$.

Solution: Let $x=r\cos\varphi$ and $y=r\sin\varphi$. Then

$$\begin{aligned}
\iint_D f(\sqrt{x^2+y^2})\mathrm{d}x\mathrm{d}y &= \iint_D f(r)r\mathrm{d}r\mathrm{d}\varphi \\
&= \int_0^1 \mathrm{d}r\int_0^{\pi/4} f(r)r\mathrm{d}\varphi + \int_1^{\sqrt{2}}\mathrm{d}r\int_{\arccos(1/r)}^{\pi/4} f(r)r\mathrm{d}\varphi \\
&= \frac{\pi}{4}\int_0^1 f(r)r\mathrm{d}r + \int_1^{\sqrt{2}}\left(\frac{\pi}{4}-\arccos\frac{1}{r}\right)f(r)r\mathrm{d}r \\
&= \frac{\pi}{4}\int_0^{\sqrt{2}} f(r)r\mathrm{d}r - \int_1^{\sqrt{2}}\arccos\frac{1}{r}f(r)r\mathrm{d}r.
\end{aligned}$$

Remark 1.10. Generally speaking, the generalized polar coordinate transformation

$$x=\frac{1}{a}\left(c+r^{\frac{1}{p}}\cos^{\frac{2}{p}}\theta\right),\quad y=\frac{1}{b}\left(d+r^{\frac{1}{p}}\sin^{\frac{2}{p}}\theta\right),$$

can transform $(ax-c)^p+(by-d)^p$ into r. However, in general, r and θ no longer have the meanings of the usual polar radius and polar angle.

Exercise 1.11. Find

$$\iint_D\left(\sqrt{\frac{x-c}{a}}+\sqrt{\frac{y-c}{b}}\right)\mathrm{d}x\mathrm{d}y,$$

where D is the region bounded by the curve $\sqrt{\frac{x-c}{a}}+\sqrt{\frac{y-c}{b}}=1$, $x=c$, and $y=c$, and $a,b,c>0$.

Solution: Let

$$x=c+a\rho\cos^4\theta,\quad y=c+b\rho\sin^4\theta$$

Then

$$J=\left|\frac{\partial(x,y)}{\partial(\rho,\theta)}\right|=4ab\rho\cos^3\theta\sin^3\theta$$

And the integration region becomes $\left\{0\le\theta\le\frac{\pi}{2},0\le\rho\le1\right\}$. Thus

$$\iint_D\left(\sqrt{\frac{x-c}{a}}+\sqrt{\frac{y-c}{b}}\right)\mathrm{d}x\mathrm{d}y=\int_0^{\pi/2}\mathrm{d}\theta\int_0^1 4ab\rho\cos^3\theta\sin^3\theta\sqrt{\rho}\mathrm{d}\rho=\frac{2ab}{15}$$

Exercise 1.12. Find

$$\lim_{R\to+\infty}\iint_{|x|\le R,\ |y|\le R}(x^2+y^2)\mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y.$$

<u>Solution:</u> Let

$$I_R=\iint_{|x|\le R,\ |y|\le R}\left(x^2+y^2\right)\mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y,\quad C_R=\iint_{x^2+y^2\le R^2}(x^2+y^2)\mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y$$

Then $C_R\le I_R\le C_{2R}$, and

$$C_R=\int_0^{2\pi}\mathrm{d}\theta\int_0^R r^3\mathrm{e}^{-r^2}\mathrm{d}r=\pi\int_0^{R^2}t\mathrm{e}^{-t}\mathrm{d}t=\pi(1-\mathrm{e}^{-R^2}-R^2\mathrm{e}^{-R^2})\to\pi\quad(R\to+\infty)$$

Similarly, we can prove that $C_{2R}\to\pi$ as $R\to+\infty$. Thus,

$$\lim_{R\to+\infty}I_R=\pi$$

Exercise 1.13. Assume that $f\in C[-1,1]$, show that

$$\int_{|x|+|y|\le1}f(x+y)\mathrm{d}x\mathrm{d}y=\int_{-1}^1 f(z)\mathrm{d}z.$$

<u>Hint:</u> Consider a change of variables as

$$\begin{cases}x'=x-y,\\ y'=x+y.\end{cases}$$

Exercise 1.14. Given the integral

$$I=\iint_D\left[\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right]\mathrm{d}x\mathrm{d}y.$$

Define a transformation $x=x(u,v)$, $y=y(u,v)$, and the region D is transformed into Ω. Assume that the transformation satisfies

$$\frac{\partial x}{\partial u}=\frac{\partial y}{\partial v},\quad \frac{\partial x}{\partial v}=-\frac{\partial y}{\partial u}.$$

Prove that:

$$I=\iint_\Omega\left[\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2\right]\mathrm{d}u\mathrm{d}v.$$

1.4 Symmetry

The parity of a function and the symmetry of the integration region can often be used to simplify the calculation of integrals. For example:

1. If the integration region D is symmetric about the x-axis:

   - If $f(x,y)=-f(x,-y)$, then

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y=0.$$

- If $f(x,y) = f(x,-y)$, then

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = 2\iint_{D\cap\{y\geq 0\}} f(x,y)\mathrm{d}x\mathrm{d}y.$$

2. If the integration region D is symmetric about the y-axis:

- If $f(x,y) = -f(-x,y)$, then

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = 0.$$

- If $f(x,y) = f(-x,y)$, then

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = 2\iint_{D\cap\{x\geq 0\}} f(x,y)\mathrm{d}x\mathrm{d}y.$$

3. If D is symmetric about the origin:

- If $f(x,y) = -f(-x,-y)$, then

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = 0.$$

- If $f(x,y) = f(-x,-y)$, then

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = 2\iint_{D_1} f(x,y)\mathrm{d}x\mathrm{d}y,$$

where D_1 is half of the region D.

Example 1.15. Show that:

$$\iint_{|x|+|y|\leq 1} (\sqrt{|xy|} + |xy|)\mathrm{d}x\mathrm{d}y \leq \frac{3}{2}.$$

Solution: By the symmetric property, we have

$$\iint_{|x|+|y|\leq 1} (\sqrt{|xy|} + |xy|)\mathrm{d}x\mathrm{d}y = 4\iint_{x+y\leq 1,\ x\geq 0,\ y\geq 0} (\sqrt{xy} + xy)\mathrm{d}x\mathrm{d}y.$$

By direct calculations, the property holds.

1.5 Applications to Proving Integral Inequalities

Example 1.16. Assume that $a < b$ and $f, g \in C[a,b]$, show that

$$\left(\int_a^b fg\right)^2 \leq \left(\int_a^b f^2\right)\left(\int_a^b g^2\right).$$

Solution: Note that

$$\int_{[a,b]^2} (f(x)g(y) - f(y)g(x))^2 \mathrm{d}x\mathrm{d}y \geq 0.$$

By expanding those in the bracket, the inequality follows directly.

Remark 1.17. The key point in the proof above is to note that the integral variables x and y have the same status.

Exercise 1.18. Let $f \in C[0,1]$ be a positive and non-increasing function. Show that

$$\frac{\int_0^1 xf^2(x)\mathrm{d}x}{\int_0^1 xf(x)\mathrm{d}x} \le \frac{\int_0^1 f^2(x)\mathrm{d}x}{\int_0^1 f(x)\mathrm{d}x}.$$

Solution: It is a direct result from the claim

$$\int_{[0,1]^2} yf(x)f(y)(f(x)-f(y))\mathrm{d}x\mathrm{d}y \ge 0.$$

Consider a change of variables as

$$\begin{cases} x' = x-y, \\ y' = x+y. \end{cases}$$

Define

$$D := \{(x',y') \in \mathbb{R}^2 : 0 \le x'+y' \le 2,\ 0 \le x'-y' \le 2\}.$$

Consequently, we have

$$\int_{[0,1]^2} yf(x)f(y)(f(x)-f(y))\mathrm{d}x\mathrm{d}y = \int_D (x'-y')g(x',y')\mathrm{d}x'\mathrm{d}y',$$

where

$$g(x',y') = \frac{1}{2}f\left(\frac{x'+y'}{2}\right)f\left(\frac{x'-y'}{2}\right)\left[f\left(\frac{x'+y'}{2}\right)-f\left(\frac{x'-y'}{2}\right)\right].$$

Obviously, there holds

$$g(x',y') = -g(x',-y') \ge 0$$

if $(x',y') \in D$ and $y' \le 0$. This implies that

$$\begin{aligned}\int_{D\cap\{y'<0\}} (x'-y')g(x',y')\mathrm{d}x'\mathrm{d}y' &= \int_{D\cap\{y'\ge 0\}} (x'+y')g(x',-y')\mathrm{d}x'\mathrm{d}y' \\ &= -\int_{D\cap\{y'\ge 0\}} g(x',y')\mathrm{d}x'\mathrm{d}y'.\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}\int_D (x'-y')g(x',y')\mathrm{d}x'\mathrm{d}y' &= \left(\int_{D\cap\{y'\ge 0\}} + \int_{D\cap\{y'<0\}}\right)(x'-y')g(x',y')\mathrm{d}x'\mathrm{d}y' \\ &= \int_{D\cap\{y'\ge 0\}} ((x'-y')-(x'+y'))g(x',y')\mathrm{d}x'\mathrm{d}y' \ge 0.\end{aligned}$$

Exercise 1.19. Assume that $f \in C[0,1]$ and $f > 0$. Show that

$$\left(\int_0^1 \frac{1}{f}\right)\left(\int_0^1 f\right) \ge 1.$$

Solution: We have

$$\begin{aligned}\left(\int_0^1 \frac{1}{f(x)}\mathrm{d}x\right)\left(\int_0^1 f(x)\mathrm{d}x\right) &= \left(\int_0^1 \frac{1}{f(x)}\mathrm{d}x\right)\left(\int_0^1 f(y)\mathrm{d}y\right) \\ &\ge \frac{1}{2}\int_{[0,1]^2}\left(\frac{f(x)}{f(y)}+\frac{f(y)}{f(x)}\right)\mathrm{d}x\mathrm{d}y \ge 1.\end{aligned}$$