

Lecture Notes on Problem-Solving Class:  
Advanced Mathematics B(II)

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# Preface: Some Important Information

## 0.1 Course Assessment Information

- **Usual Performance:** Hand in assignments. It accounts for 20 points of the overall evaluation.
- **Mid-term Examination:** On the weekend of the 8th or 9th week. It accounts for 30 points of the overall evaluation.
- **Final Examination:** On Monday evening, June 9th. It accounts for 50 points of the overall evaluation.

## 0.2 Information of the Class and the Teaching Assistant

- **Teacher:** Wei Wang.
  - **E-mail:** 2201110024@stu.pku.edu.cn
  - **Personal Website:** <https://luisyanka.github.io/weiwang.github.io/>
- **Location:** Room 313, Teaching Building 2.
- **Students:** Students whose student ID numbers are greater than 2400011822 and less than or equal to 2400015443 should submit their assignments to Teacher Wei Wang's class.

## 0.3 Some Useful Links

We present some useful links associated with calculus.

- Lecture notes by Yantong Xie: <https://darkoxie.github.io>
- Mathstackexchange: <https://math.stackexchange.com>

## 0.4 Topics of the Class

In this problem-solving class, we will present some classical exercises related to the topics which are delivered by the lecturer in the main course. We will mainly refer to the lecture notes written by Yantong Xie, who was a very good teaching assistant of the course Advanced mathematics (B). We also refer to the book “Guide to Solving

Problems in Advanced Mathematics” by Jianying Zhou and Zhengyuan Li. If you have any advices for this class, then you can contact me with the e-mail. The lecture notes of this class will be updated before the next one on my personal website in the content of “teaching”.

# Chapter 1

## Double integral

### 1.1 Calculating by the Definition

**Example 1.1.** There are three points  $P_0, P_1, P_2$  on the plane, which are given by  $\{(x_i, y_i)\}_{i=0}^2$ . We assume that  $x_2 > x_1 > x_0$  and  $y_2 > y_1 > y_0$ . Please calculate the area of triangle  $\Delta P_0 P_1 P_2$ .

Solution: Denote the triangle  $\Delta P_0 P_1 P_2$  by  $D$ . By simple calculations, we can determine  $P_0 P_1 : y = k_1 x + b_1$ ,  $P_1 P_2 : y = k_2 x + b_2$ , and  $P_0 P_2 : y = k_3 x + b_3$ . WLOG, we assume that  $y_1 < k_3 x_1 + b_3$ . As a result, we have

$$\begin{aligned}\int_D 1 dx dy &= \int_{x_0}^{x_1} dx \int_{k_1 x + b_1}^{k_3 x + b_3} 1 dy + \int_{x_0}^{x_1} dx \int_{k_2 x + b_2}^{k_3 x + b_3} 1 dy \\ &= \frac{1}{2}((y_2 - y_0)(x_1 - x_0) - y_1(x_2 - x_0)).\end{aligned}$$

Combined with the case that  $y_1 \geq k_3 x_1 + b_3$ , we obtain

$$A(D) = \frac{1}{2} |x_1 y_2 - x_1 y_0 - x_0 y_2 + x_0 y_0 - x_2 y_1 + x_0 y_1|.$$

**Exercise 1.2.** Let  $A = [0, 1] \times [0, 1]$ , find

$$I = \iint_A \frac{y dx dy}{(1 + x^2 + y^2)^{\frac{3}{2}}}.$$

Solution: Integrating with respect to  $y$  first and then with respect to  $x$ , we get

$$\begin{aligned}I &= \int_0^1 dx \int_0^1 \frac{y dy}{(1 + x^2 + y^2)^{\frac{3}{2}}} \\ &= \int_0^1 \left( \frac{1}{\sqrt{x^2 + 1}} - \frac{1}{\sqrt{x^2 + 2}} \right) dx = \ln \frac{2 + \sqrt{2}}{1 + \sqrt{3}}.\end{aligned}$$

### 1.2 Iterated Integrals

**Example 1.3.** Calculate  $\int_0^1 \int_y^1 \frac{y}{\sqrt{1+x^3}} dx$ .

Solution: Let

$$D := \{(x, y) \in \mathbb{R}^2 : y \leq x, x \in [0, 1], y \in [0, 1]\}.$$

We have

$$\begin{aligned}\int_0^1 dy \int_y^1 \frac{y}{\sqrt{1+x^3}} dx &= \int_D \frac{y}{\sqrt{1+x^3}} dx dy = \int_0^1 dx \int_0^x \frac{y}{\sqrt{1+x^3}} dy \\ &= \frac{1}{6} \int_0^1 \frac{dt}{\sqrt{1+t}} = \frac{1}{3}(\sqrt{2}-1),\end{aligned}$$

where for the third inequality, we have used  $t = x^3$ .

**Exercise 1.4.** Calculate  $\int_0^1 \frac{x-1}{\ln x} dx$ .

Solution: We note

$$\begin{aligned}\int_0^1 \frac{x-1}{\ln x} dx &= \int_0^1 dx \int_0^1 x^y dy = \int_{[0,1]^2} x^y dx dy \\ &= \int_0^1 dy \int_0^1 x^y dx = \int_0^1 \frac{1}{y+1} dy = \ln 2.\end{aligned}$$

**Exercise 1.5.** Suppose that  $f$  is continuous on  $[0, 1]$ . Prove that:

$$\int_0^1 dx \int_x^1 f(t) dt = \int_0^1 t f(t) dt.$$

### 1.3 Change of Variables

**Example 1.6** (Observing the region). The region  $D \subset \mathbb{R}^2$  is surrounded by the curves  $xy = a$ ,  $xy = b$ ,  $y = px$ , and  $y = qx$ , where  $0 < a < b$  and  $0 < p < q$ . Please calculate

$$I = \iint_D xy^3 dx dy.$$

Solution: Consider a change of variables as

$$\begin{cases} x' = \frac{y}{x}, \\ y' = xy. \end{cases}$$

We can calculate that

$$\left| \frac{\partial(x, y)}{\partial(x', y')} \right| = -\frac{1}{2x'}.$$

As a result,

$$I = \int_p^q \left( \int_a^b x'(y')^2 \cdot \frac{1}{2x'} dx' \right) dy' = \frac{(b^3 - a^3)(q - p)}{3}$$

**Example 1.7** (Rotation). Calculate  $\int_D |3x + 4y| dx dy$ , where

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Solution: Consider a change of variables as

$$\begin{cases} x' = \frac{4}{5}x - \frac{3}{5}y, \\ y' = \frac{3}{5}x + \frac{4}{5}y. \end{cases}$$

Indeed, formula above give a rotation, which preserve  $D$  and change the line  $3x + 4y = t$  to  $y' = t$  for any  $t \in \mathbb{R}$ . As a result, we have

$$\int_D |3x + 4y| dx dy = 5 \int_D |y'| dx' dy' = \frac{20}{3}.$$

**Question 1.8.** Can you give the intuition behind this change of variables?

**Example 1.9** (Polar coordinate). Make a polar coordinate transformation to convert the double integral

$$\iint_D f(\sqrt{x^2 + y^2}) dx dy$$

into a definite integral, where  $D = \{(x, y) \mid 0 \leq y \leq x \leq 1\}$ .

Solution: Let  $x = r \cos \varphi$  and  $y = r \sin \varphi$ . Then

$$\begin{aligned} \iint_D f(\sqrt{x^2 + y^2}) dx dy &= \iint_D f(r) r dr d\varphi \\ &= \int_0^1 dr \int_0^{\pi/4} f(r) r d\varphi + \int_1^{\sqrt{2}} dr \int_{\arccos(1/r)}^{\pi/4} f(r) r d\varphi \\ &= \frac{\pi}{4} \int_0^1 f(r) r dr + \int_1^{\sqrt{2}} \left( \frac{\pi}{4} - \arccos \frac{1}{r} \right) f(r) r dr \\ &= \frac{\pi}{4} \int_0^{\sqrt{2}} f(r) r dr - \int_1^{\sqrt{2}} \arccos \frac{1}{r} f(r) r dr. \end{aligned}$$

**Remark 1.10.** Generally speaking, the generalized polar coordinate transformation

$$x = \frac{1}{a} \left( c + r^{\frac{1}{p}} \cos^{\frac{2}{p}} \theta \right), \quad y = \frac{1}{b} \left( d + r^{\frac{1}{p}} \sin^{\frac{2}{p}} \theta \right),$$

can transform  $(ax - c)^p + (by - d)^p$  into  $r$ . However, in general,  $r$  and  $\theta$  no longer have the meanings of the usual polar radius and polar angle.

**Exercise 1.11.** Find

$$\iint_D \left( \sqrt{\frac{x-c}{a}} + \sqrt{\frac{y-c}{b}} \right) dx dy,$$

where  $D$  is the region bounded by the curve  $\sqrt{\frac{x-c}{a}} + \sqrt{\frac{y-c}{b}} = 1$ ,  $x = c$ , and  $y = c$ , and  $a, b, c > 0$ .

Solution: Let

$$x = c + a\rho \cos^4 \theta, \quad y = c + b\rho \sin^4 \theta$$

Then

$$J = \left| \frac{\partial(x, y)}{\partial(\rho, \theta)} \right| = 4ab\rho \cos^3 \theta \sin^3 \theta$$

And the integration region becomes  $\{0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \rho \leq 1\}$ . Thus

$$\iint_D \left( \sqrt{\frac{x-c}{a}} + \sqrt{\frac{y-c}{b}} \right) dx dy = \int_0^{\pi/2} d\theta \int_0^1 4ab\rho \cos^3 \theta \sin^3 \theta \sqrt{\rho} d\rho = \frac{2ab}{15}$$

**Exercise 1.12.** Find

$$\lim_{R \rightarrow +\infty} \iint_{|x| \leq R, |y| \leq R} (x^2 + y^2) e^{-(x^2+y^2)} dx dy.$$

Solution: Let

$$I_R = \iint_{|x| \leq R, |y| \leq R} (x^2 + y^2) e^{-(x^2+y^2)} dx dy, \quad C_R = \iint_{x^2+y^2 \leq R^2} (x^2 + y^2) e^{-(x^2+y^2)} dx dy$$

Then  $C_R \leq I_R \leq C_{2R}$ , and

$$C_R = \int_0^{2\pi} d\theta \int_0^R r^3 e^{-r^2} dr = \pi \int_0^{R^2} t e^{-t} dt = \pi(1 - e^{-R^2} - R^2 e^{-R^2}) \rightarrow \pi \quad (R \rightarrow +\infty)$$

Similarly, we can prove that  $C_{2R} \rightarrow \pi$  as  $R \rightarrow +\infty$ . Thus,

$$\lim_{R \rightarrow +\infty} I_R = \pi$$

**Exercise 1.13.** Assume that  $f \in C[-1, 1]$ , show that

$$\int_{|x|+|y| \leq 1} f(x+y) dx dy = \int_{-1}^1 f(z) dz.$$

Hint: Consider a change of variables as

$$\begin{cases} x' = x - y, \\ y' = x + y. \end{cases}$$



**Exercise 1.14.** Given the integral

$$I = \iint_D \left[ \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 \right] dx dy.$$

Define a transformation  $x = x(u, v)$ ,  $y = y(u, v)$ , and the region  $D$  is transformed into  $\Omega$ . Assume that the transformation satisfies

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}, \quad \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}.$$

Prove that:

$$I = \iint_{\Omega} \left[ \left( \frac{\partial f}{\partial u} \right)^2 + \left( \frac{\partial f}{\partial v} \right)^2 \right] du dv.$$

## 1.4 Symmetry

The parity of a function and the symmetry of the integration region can often be used to simplify the calculation of integrals. For example:

1. If the integration region  $D$  is symmetric about the  $x$ -axis:

- If  $f(x, y) = -f(x, -y)$ , then

$$\iint_D f(x, y) dx dy = 0.$$

- If  $f(x, y) = f(x, -y)$ , then

$$\iint_D f(x, y) dx dy = 2 \iint_{D \cap \{y \geq 0\}} f(x, y) dx dy.$$

2. If the integration region  $D$  is symmetric about the  $y$ -axis:

- If  $f(x, y) = -f(-x, y)$ , then

$$\iint_D f(x, y) dx dy = 0.$$

- If  $f(x, y) = f(-x, y)$ , then

$$\iint_D f(x, y) dx dy = 2 \iint_{D \cap \{x \geq 0\}} f(x, y) dx dy.$$

3. If  $D$  is symmetric about the origin:

- If  $f(x, y) = -f(-x, -y)$ , then

$$\iint_D f(x, y) dx dy = 0.$$

- If  $f(x, y) = f(-x, -y)$ , then

$$\iint_D f(x, y) dx dy = 2 \iint_{D_1} f(x, y) dx dy,$$

where  $D_1$  is half of the region  $D$ .

**Example 1.15.** Show that:

$$\iint_{|x|+|y|\leq 1} (\sqrt{|xy|} + |xy|) dx dy \leq \frac{3}{2}.$$

Solution: By the symmetric property, we have

$$\iint_{|x|+|y|\leq 1} (\sqrt{|xy|} + |xy|) dx dy = 4 \iint_{x+y\leq 1, x\geq 0, y\geq 0} (\sqrt{xy} + xy) dx dy.$$

By direct calculations, the property holds.

## 1.5 Applications to Proving Integral Inequalities

**Example 1.16.** Assume that  $a < b$  and  $f, g \in C[a, b]$ , show that

$$\left( \int_a^b fg \right)^2 \leq \left( \int_a^b f^2 \right) \left( \int_a^b g^2 \right).$$

Solution: Note that

$$\int_{[a,b]^2} (f(x)g(y) - f(y)g(x))^2 dx dy \geq 0.$$

By expanding those in the bracket, the inequality follows directly.

**Remark 1.17.** The key point in the proof above is to note that the integral variables  $x$  and  $y$  have the same status.

**Exercise 1.18.** Let  $f \in C[0, 1]$  be a positive and non-increasing function. Show that

$$\frac{\int_0^1 x f^2(x) dx}{\int_0^1 x f(x) dx} \leq \frac{\int_0^1 f^2(x) dx}{\int_0^1 f(x) dx}.$$

Solution: It is a direct result from the claim

$$\int_{[0,1]^2} yf(x)f(y)(f(x) - f(y))dxdy \geq 0.$$

Consider a change of variables as

$$\begin{cases} x' = x + y, \\ y' = x - y. \end{cases}$$

Define

$$D := \{(x', y') \in \mathbb{R}^2 : 0 \leq x' + y' \leq 2, 0 \leq x' - y' \leq 2\}.$$

We see that  $D$  is symmetric about the  $y$ -axis. Consequently, we have

$$\int_{[0,1]^2} yf(x)f(y)(f(x) - f(y))dxdy = \int_D (x' - y')g(x', y')dx'dy',$$

where

$$g(x', y') = \frac{1}{2}f\left(\frac{x' + y'}{2}\right)f\left(\frac{x' - y'}{2}\right)\left[f\left(\frac{x' + y'}{2}\right) - f\left(\frac{x' - y'}{2}\right)\right].$$

Obviously, there holds

$$g(x', y') = -g(x', -y') \geq 0$$

if  $(x', y') \in D$  and  $y' \leq 0$ . This implies that

$$\begin{aligned} \int_{D \cap \{y' < 0\}} (x' - y')g(x', y')dx'dy' &= \int_{D \cap \{y' \geq 0\}} (x' + y')g(x', -y')dx'dy' \\ &= - \int_{D \cap \{y' \geq 0\}} (x' + y')g(x', y')dx'dy'. \end{aligned}$$

Here, for the first inequality, we have used the change of variable that sends  $y'$  to  $-y'$ . Moreover, we obtain

$$\begin{aligned} \int_D (x' - y')g(x', y')dx'dy' &= \left( \int_{D \cap \{y' \geq 0\}} + \int_{D \cap \{y' < 0\}} \right) (x' - y')g(x', y')dx'dy' \\ &= \int_{D \cap \{y' \geq 0\}} ((x' - y') - (x' + y'))g(x', y')dx'dy' \\ &= - \int_{D \cap \{y' \geq 0\}} 2y'g(x', y')dx'dy'. \end{aligned}$$

We see that if  $y' \geq 0$ , then it follows from the property that  $f$  is non-increasing that  $g(x', y') \leq 0$ . As a result,

$$- \int_{D \cap \{y' \geq 0\}} 2y'g(x', y')dx'dy' \geq 0,$$

and the proof is completed.

**Exercise 1.19.** Assume that  $f \in C[0, 1]$  and  $f > 0$ . Show that

$$\left( \int_0^1 \frac{1}{f} \right) \left( \int_0^1 f \right) \geq 1.$$

Solution: We have

$$\begin{aligned} \left( \int_0^1 \frac{1}{f(x)} dx \right) \left( \int_0^1 f(x) dx \right) &= \left( \int_0^1 \frac{1}{f(x)} dx \right) \left( \int_0^1 f(y) dy \right) \\ &\geq \frac{1}{2} \int_{[0,1]^2} \left( \frac{f(x)}{f(y)} + \frac{f(y)}{f(x)} \right) dx dy \geq 1. \end{aligned}$$

## Chapter 2

# Triple Integrals and $n$ -multiple Integrals

### 2.1 Calculation of Triple Integrals in Rectangular Coordinate

**Example 2.1** (Consider the area of the section). Find the integral

$$I = \iiint_{\Omega} z^2 dx dy dz,$$

where  $\Omega$  is the common part of the two spheres  $x^2 + y^2 + z^2 \leq R^2$  and  $x^2 + y^2 + z^2 \leq 2Rz$ .

Solution: Considering both the integrand and the integration region, the integral can be regarded as the sum of a series of small slices weighted by  $z^2$  for  $z \in [0, R]$ . According to the composition of the integration region  $\Omega$ , it can be divided into two sub-regions  $\Omega_1$  and  $\Omega_2$ .

$$\begin{aligned}\Omega_1 : & \begin{cases} x^2 + y^2 + z^2 \leq 2Rz, \\ 0 \leq z \leq \frac{R}{2}, \end{cases} \\ \Omega_2 : & \begin{cases} x^2 + y^2 + z^2 \leq R^2, \\ \frac{R}{2} \leq z \leq R. \end{cases}\end{aligned}$$

When  $z \in [0, \frac{R}{2}]$ , from  $x^2 + y^2 + z^2 \leq 2Rz$ , we see that the area of the slice can be obtained as  $\pi(2Rz - z^2)$ . On the other hand, when  $z \in [\frac{R}{2}, R]$ , for  $x^2 + y^2 + z^2 \leq R^2$ , the area of the slice can be obtained as  $\pi(R^2 - z^2)$ . As a result,

$$\begin{aligned}I &= \int_0^{R/2} \pi z^2 (2Rz - z^2) dz + \int_{R/2}^R \pi z^2 (R^2 - z^2) dz \\ &= \left( \frac{1}{2} \pi R z^4 - \frac{1}{5} \pi z^5 \right) \Big|_0^{R/2} + \left( \frac{1}{3} \pi R^2 z^3 - \frac{1}{5} \pi z^5 \right) \Big|_{R/2}^R = \frac{59}{480} \pi R^5\end{aligned}$$

**Exercise 2.2.** Calculate the volume of the intersection of cylinders  $x^2 + y^2 \leq a^2$  and  $x^2 + z^2 \leq a^2$ .

Solution:

1. **Cylinder**  $x^2 + y^2 = a^2$ : In the  $xy$ -plane, the equation  $x^2 + y^2 = a^2$  represents a circle centered at the origin with radius  $a$ . When considering this equation in three-dimensional space, it represents a cylinder that extends infinitely along the  $z$ -axis.
2. **Cylinder**  $x^2 + z^2 = a^2$ : In the  $xz$ -plane, the equation  $x^2 + z^2 = a^2$  represents a circle centered at the origin with radius  $a$ . In three-dimensional space, it represents a cylinder that extends infinitely along the  $y$ -axis.
3. **Intersection of the two cylinders**: The intersection of the cylinders  $x^2 + y^2 \leq a^2$  and  $x^2 + z^2 \leq a^2$  is a symmetric solid. The solid is symmetric about the  $x$ -axis,  $y$ -axis, and  $z$ -axis. For a fixed  $x \in [-a, a]$ , the cross-section of the intersection perpendicular to the  $x$ -axis is a square. The side length of the square  $s$  is given by  $s = 2\sqrt{a^2 - x^2}$  (since from  $x^2 + y^2 = a^2$ , we have  $y = \pm\sqrt{a^2 - x^2}$  and from  $x^2 + z^2 = a^2$ , we have  $z = \pm\sqrt{a^2 - x^2}$ ).

The formula for the volume  $V$  of a solid with cross-sectional area  $A(x)$  from  $x = c$  to  $x = d$  is  $V = \int_c^d A(x)dx$ . Here,  $c = -a$ ,  $d = a$ , and the cross-sectional area  $A(x)$  of the intersection of the two cylinders perpendicular to the  $x$ -axis is  $A(x) = (2\sqrt{a^2 - x^2}) \times (2\sqrt{a^2 - x^2}) = 4(a^2 - x^2)$  (because the cross-section is a square with side length  $2\sqrt{a^2 - x^2}$ ).

Then, we calculate the integral:

$$V = \int_{-a}^a 4(a^2 - x^2)dx = \frac{16}{3}a^3.$$

**Example 2.3** (Projection to the plane). Find

$$I = \iiint_{\Omega} (y^2 + z^2)dV,$$

where  $\Omega$  represents the region  $0 \leq z \leq x^2 + y^2 \leq 1$ .

Solution: In cylindrical coordinates  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $z = z$  and  $dV = r dz dr d\theta$ , and  $y^2 + z^2 = r^2 \sin^2 \theta + z^2$ . The region  $\Omega : 0 \leq z \leq x^2 + y^2 \leq 1$  becomes  $0 \leq z \leq r^2$ ,  $0 \leq r \leq 1$ ,  $0 \leq \theta \leq 2\pi$ . Then,

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^1 \int_0^{r^2} (r^2 \sin^2 \theta + z^2) r dz dr d\theta = \int_0^{2\pi} \int_0^1 r \left[ \int_0^{r^2} (r^2 \sin^2 \theta + z^2) dz \right] dr d\theta \\ &= \int_0^{2\pi} \int_0^1 r \left( r^2 \sin^2 \theta z + \frac{z^3}{3} \Big|_0^{r^2} \right) dr d\theta = \int_0^{2\pi} \int_0^1 r \left( r^4 \sin^2 \theta + \frac{r^6}{3} \right) dr d\theta \\ &= \int_0^{2\pi} \left( \sin^2 \theta \int_0^1 r^5 dr + \frac{1}{3} \int_0^1 r^7 dr \right) d\theta = \int_0^{2\pi} \left( \frac{1}{6} \sin^2 \theta + \frac{1}{24} \right) d\theta = \frac{\pi}{4}. \end{aligned}$$

Indeed, we also have

$$I = \int_{x^2+y^2 \leq 1} \left( \int_0^{x^2+y^2} (y^2 + z^2) dz \right) dx dy = \frac{\pi}{4}.$$

## 2.2 Change of Variables for Triple Integrals

Similar to the change of variables for double integrals, there is the following theorem for variable substitution of triple integrals. Let

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w), \quad (u, v, w) \in \Omega'$$

This change of variables satisfies the following conditions:

- It establishes a one-to-one correspondence between  $\Omega$  and  $\Omega'$ .
- $x$ ,  $y$ , and  $z$  have continuous partial derivatives with respect to each variable in  $\Omega'$ , and the inverse transformations  $u = u(x, y, z)$ ,  $v = v(x, y, z)$ ,  $w = w(x, y, z)$  also have continuous partial derivatives with respect to each variable in  $\Omega$ .
- The Jacobian determinant  $J = \frac{\partial(x, y, z)}{\partial(u, v, w)}$  has no zeros in  $\Omega'$ . Then

$$\begin{aligned} \iiint_{\Omega} f(x, y, z) dx dy dz \\ = \iiint_{\Omega'} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw \end{aligned}$$

There are two commonly used transformations for triple integrals:

### 1. Cylindrical Coordinate Transformation

$$\begin{aligned} x &= \rho \cos \theta, & y &= \rho \sin \theta, & z &= z \\ 0 &\leq \rho < +\infty, & 0 &\leq \theta < 2\pi, & -\infty &< z < +\infty \end{aligned}$$

The relationship between the triple integral in the rectangular coordinate system and the triple integral in the cylindrical coordinate system is

$$\iiint_{\Omega} f(x, y, z) dx dy dz = \iiint_{\Omega'} f(\rho \cos \theta, \rho \sin \theta, z) \rho d\rho d\theta dz.$$

### 2. Spherical Coordinate Transformation

$$\begin{aligned} x &= \rho \sin \varphi \cos \theta, & y &= \rho \sin \varphi \sin \theta, & z &= \rho \cos \varphi, \\ 0 &\leq \rho < +\infty, & 0 &\leq \theta < 2\pi, & 0 &\leq \varphi \leq \pi. \end{aligned}$$

The relationship between the triple integral in the rectangular coordinate system and the triple integral in the spherical coordinate system is

$$\iiint_{\Omega} f(x, y, z) dx dy dz = \iiint_{\Omega'} f(\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi) \rho^2 \sin \varphi d\rho d\theta d\varphi.$$

**Example 2.4.** Let  $\Omega = \{(x, y, z) \mid 0 \leq x + y - z \leq 1, 0 \leq y + z - x \leq 1, 0 \leq x + z - y \leq 1\}$  be the region formed by the intersection of six planes. Find the triple integral

$$I = \iiint_{\Omega} (x + y - z)(y + z - x)(x + z - y) dx dy dz.$$

Solution: Consider the change of variables as

$$\begin{cases} x' = x + y - z, \\ y' = y + z - x, \\ z' = x + z - y. \end{cases}$$

It is easy to find that the Jacobi determinant is  $\frac{1}{4}$ . As a result, we have

$$I = \int_{0 \leq x' \leq 1} \int_{0 \leq y' \leq 1} \int_{0 \leq z' \leq 1} \frac{1}{4} x' y' z' dx' dy' dz' = \frac{1}{32}.$$

**Example 2.5.** Calculate the integral

$$H = \iiint_{\substack{x, y, z \geq 0 \\ x^2 + y^2 + z^2 \leq R^2}} \frac{xyz dx dy dz}{\sqrt{a^2 x^2 + b^2 y^2 + c^2 z^2}}, \quad \text{where } a > b > c > 0.$$

Solution: In spherical coordinates,

$$H = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^R \frac{r^4 \sin^3 \varphi \cos \varphi \sin \theta \cos \theta dr d\varphi d\theta}{\sqrt{a^2 \sin^2 \varphi \cos^2 \theta + b^2 \sin^2 \varphi \sin^2 \theta + c^2 \cos^2 \varphi}}.$$

Let  $\sin^2 \varphi = u$ ,  $\sin^2 \theta = v$ . Then

$$\begin{aligned} H &= \frac{1}{4} \int_0^1 \int_0^1 \int_0^R r^4 \frac{u dr dv du}{\sqrt{a^2 u(1-v) + b^2 uv + c^2(1-u)}} \\ &= \frac{1}{20} R^5 \int_0^1 u du \int_0^1 \frac{dv}{\sqrt{[c^2 + (a^2 - c^2)u] + (b^2 - a^2)uv}} \\ &= \frac{1}{20} R^5 \int_0^1 \left\{ \frac{2}{(b^2 - a^2)u} \sqrt{[c^2 + (a^2 - c^2)u] + (b^2 - a^2)uv} \right\} \Big|_{v=0}^{v=1} u du \\ &= \frac{R^5}{10(b^2 - a^2)} \int_0^1 \left\{ \sqrt{[c^2 + (a^2 - c^2)u] + (b^2 - a^2)u} - \sqrt{c^2 + (a^2 - c^2)u} \right\} du \\ &= \frac{R^5}{10(b^2 - a^2)} \left\{ \frac{2}{3(b^2 - c^2)} [c^2 + (b^2 - c^2)u]^{\frac{3}{2}} - \frac{2}{3(a^2 - c^2)} [c^2 + (a^2 - c^2)u]^{\frac{3}{2}} \right\} \Big|_0^1 \\ &= \frac{R^5}{10(b^2 - a^2)} \left[ \frac{2}{3(b^2 - c^2)} (b^3 - c^3) - \frac{2}{3(a^2 - c^2)} (a^3 - c^3) \right] \\ &= \frac{R^5}{15} \cdot \frac{1}{b^2 - a^2} \left( \frac{b^2 + bc + c^2}{b + c} - \frac{a^2 + ac + c^2}{a + c} \right) \\ &= \frac{R^5}{15} \cdot \frac{ab + bc + ca}{(a + b)(b + c)(c + a)}. \end{aligned}$$



**Example 2.6.** Let  $H(x) = \sum_{i,j=1}^3 a_{ij}x_i x_j$ , and  $\mathbf{A} = (a_{ij})$  be a positive definite symmetric matrix of order 3. Find

$$I = \iiint_{H(x) \leq 1} e^{\sqrt{H(x)}} dx_1 dx_2 dx_3.$$

Solution: There exists an orthogonal matrix  $\mathbf{P}$  of order 3 such that

$$\mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

where  $\lambda_i > 0$ ,  $i = 1, 2, 3$ . Make an orthogonal transformation  $x = \mathbf{P}y$ , where  $x, y \in \mathbb{R}^3$ . Then

$$H(x) = H(\mathbf{P}y) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2,$$

and the Jacobian determinant of the transformation  $\det \mathbf{P} \equiv 1$ . Thus

$$I = \iiint_{\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 \leq 1} e^{\sqrt{\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2}} dy_1 dy_2 dy_3.$$

Let

$$y_1 = \frac{1}{\sqrt{\lambda_1}} r \sin \varphi \cos \theta, \quad y_2 = \frac{1}{\sqrt{\lambda_2}} r \sin \varphi \sin \theta, \quad y_3 = \frac{1}{\sqrt{\lambda_3}} r \cos \varphi.$$

Then

$$I = \frac{1}{\sqrt{\lambda_1 \lambda_2 \lambda_3}} \int_0^{2\pi} d\theta \int_0^\pi d\varphi \int_0^1 r^2 e^r \sin \varphi dr = \frac{4\pi}{\sqrt{\lambda_1 \lambda_2 \lambda_3}} \int_0^1 r^2 e^r dr = \frac{4\pi}{\sqrt{\lambda_1 \lambda_2 \lambda_3}} (e - 2).$$

Since the determinant of  $\mathbf{A}$  is  $\det \mathbf{A} = \lambda_1 \lambda_2 \lambda_3$ , so

$$I = \frac{4\pi}{\sqrt{\det \mathbf{A}}} (e - 2).$$

**Exercise 2.7.** Find

$$\iiint \int_{\substack{x,y,z,u \geq 0 \\ x^2+y^2+z^2+u^2 \leq 1}} \sqrt{\frac{1-x^2-y^2-z^2-u^2}{1+x^2+y^2+z^2+u^2}} dx dy dz du.$$

Solution: We use four-dimensional spherical coordinates:  $x = r \sin \varphi_1 \sin \varphi_2 \cos \theta$ ,  $y = r \sin \varphi_1 \sin \varphi_2 \sin \theta$ ,  $z = r \sin \varphi_1 \cos \varphi_2$ ,  $u = r \cos \varphi_1$ , with  $r \geq 0$ ,  $0 \leq \theta \leq \frac{\pi}{2}$ ,  $0 \leq \varphi_1 \leq \frac{\pi}{2}$ ,  $0 \leq \varphi_2 \leq \frac{\pi}{2}$ . The Jacobian of the transformation is  $J = r^3 \sin^2 \varphi_1 \sin \varphi_2$ , and  $x^2 + y^2 + z^2 + u^2 = r^2$ .

The given integral

$$\begin{aligned}
 & \iiint \int_{\substack{x,y,z,u \geq 0 \\ x^2+y^2+z^2+u^2 \leq 1}} \sqrt{\frac{1-x^2-y^2-z^2-u^2}{1+x^2+y^2+z^2+u^2}} dx dy dz du \\
 &= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} \sin \varphi_2 d\varphi_2 \int_0^{\frac{\pi}{2}} \sin^2 \varphi_1 d\varphi_1 \int_0^1 r^3 \sqrt{\frac{1-r^2}{1+r^2}} dr \\
 &= \frac{\pi}{2} \times \left(-\cos \varphi_2 \Big|_0^{\frac{\pi}{2}}\right) \times \frac{1}{2} \left[\varphi_1 - \frac{\sin(2\varphi_1)}{2}\right]_0^{\frac{\pi}{2}} \times \frac{1}{2} \left(\int_0^1 \frac{t}{\sqrt{1-t^2}} dt - \int_0^1 \frac{t^2}{\sqrt{1-t^2}} dt\right) \\
 &= \frac{\pi}{2} \times 1 \times \frac{\pi}{4} \times \frac{1}{2} \left(-\frac{1}{2} \int_1^0 u^{-\frac{1}{2}} du - \int_0^{\frac{\pi}{2}} \sin^2 \alpha d\alpha\right) \\
 &= \frac{\pi}{2} \times 1 \times \frac{\pi}{4} \times \frac{1}{2} \left(1 - \frac{\pi}{4}\right) \\
 &= \frac{\pi^2}{16} \left(1 - \frac{\pi}{4}\right)
 \end{aligned}$$

**Exercise 2.8.** Let

$$F(t) = \iiint_{x^2+y^2+z^2 \leq t^2} f(x^2+y^2+z^2) dx dy dz.$$

where  $f$  is a continuous function and  $f(1) = 1$ . Prove that  $F'(1) = 4\pi$ .

Solution: We use spherical coordinates to transform the triple-integral. In spherical coordinates,  $x = \rho \sin \varphi \cos \theta$ ,  $y = \rho \sin \varphi \sin \theta$ ,  $z = \rho \cos \varphi$ , and  $dV = \rho^2 \sin \varphi d\rho d\varphi d\theta$ , and  $x^2 + y^2 + z^2 = \rho^2$ . The region  $x^2 + y^2 + z^2 \leq t^2$  corresponds to  $0 \leq \rho \leq t$ ,  $0 \leq \varphi \leq \pi$ ,  $0 \leq \theta \leq 2\pi$ . Then

$$F(t) = \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_0^t f(\rho^2) \rho^2 d\rho.$$

Since  $\int_0^{2\pi} d\theta = 2\pi$  and  $\int_0^\pi \sin \varphi d\varphi = -\cos \varphi \Big|_0^\pi = 2$ , we have

$$F(t) = 4\pi \int_0^t f(\rho^2) \rho^2 d\rho.$$

By the fundamental theorem of calculus, if  $F(t) = 4\pi \int_0^t g(\rho) d\rho$  (where  $g(\rho) = f(\rho^2) \rho^2$ ), then  $F'(t) = 4\pi f(t^2) t^2$ . When  $t = 1$ , since  $f(1) = 1$ , we get  $F'(1) = 4\pi \times f(1) \times 1^2 = 4\pi$ .

**Exercise 2.9.** Find

$$\iiint_{\Omega} z(x^2 + y^2 + z^2) dx dy dz,$$

where  $\Omega$  is the sphere  $x^2 + y^2 + z^2 \leq 2z$ .

**Solution:** We rewrite the sphere equation  $x^2 + y^2 + z^2 \leq 2z$  in spherical coordinates. With  $x = \rho \sin \varphi \cos \theta$ ,  $y = \rho \sin \varphi \sin \theta$ ,  $z = \rho \cos \varphi$  and  $x^2 + y^2 + z^2 = \rho^2$ , the sphere becomes  $\rho^2 \leq 2\rho \cos \varphi$ , so  $\rho \leq 2 \cos \varphi$  ( $\rho \geq 0$ ). The ranges are  $0 \leq \theta \leq 2\pi$ ,  $0 \leq \varphi \leq \frac{\pi}{2}$ , and  $dV = \rho^2 \sin \varphi d\rho d\varphi d\theta$ . The integrand  $z(x^2 + y^2 + z^2)$  is  $\rho^3 \cos \varphi$ .

$$\begin{aligned} \iiint_{\Omega} z(x^2 + y^2 + z^2) dx dy dz &= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2 \cos \varphi} \rho^3 \cos \varphi \cdot \rho^2 \sin \varphi d\rho \\ &= 2\pi \int_0^{\frac{\pi}{2}} \cos \varphi \sin \varphi \left[ \frac{\rho^6}{6} \right]_0^{2 \cos \varphi} d\varphi \\ &= \frac{2\pi}{6} \int_0^{\frac{\pi}{2}} 64 \cos^7 \varphi \sin \varphi d\varphi \\ &= \frac{64\pi}{3} \left[ -\frac{\cos^8 \varphi}{8} \right]_0^{\frac{\pi}{2}} = \frac{8\pi}{3}. \end{aligned}$$

**Exercise 2.10.** Let  $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ , and the region  $\Omega \subset \mathbb{R}^3$  be determined by  $z \geq \sqrt{x^2 + y^2}$  and  $4 \leq x^2 + y^2 + z^2 \leq 16$ . Try to calculate the integral average value of the function  $f$  over  $\Omega$

$$\frac{1}{|\Omega|} \iiint_{\Omega} f(x, y, z) dx dy dz$$

where  $|\Omega|$  is the volume of  $\Omega$ .

**Solution:** In spherical coordinates,  $x = \rho \sin \varphi \cos \theta$ ,  $y = \rho \sin \varphi \sin \theta$ ,  $z = \rho \cos \varphi$ , and  $dV = \rho^2 \sin \varphi d\rho d\varphi d\theta$ , and  $f(x, y, z) = \rho$ . The inequality  $z \geq \sqrt{x^2 + y^2}$  implies  $\rho \cos \varphi \geq \rho \sin \varphi$ . Since  $\rho > 0$  in the non origin part of the region, we have  $\tan \varphi \leq 1$ , so  $0 \leq \varphi \leq \frac{\pi}{4}$ . The inequality  $4 \leq x^2 + y^2 + z^2 \leq 16$  implies  $2 \leq \rho \leq 4$ . And the range of  $\theta$  is  $0 \leq \theta \leq 2\pi$ .

$$\begin{aligned} |\Omega| &= \iiint_{\Omega} dx dy dz = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin \varphi d\varphi \int_2^4 \rho^2 d\rho \\ &= 2\pi \times [-\cos \varphi]_0^{\frac{\pi}{4}} \times \left[ \frac{\rho^3}{3} \right]_2^4 = \frac{56\pi(2 - \sqrt{2})}{3}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \iiint_{\Omega} f(x, y, z) dx dy dz &= \iiint_{\Omega} \rho dx dy dz = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin \varphi d\varphi \int_2^4 \rho^3 d\rho \\ &= 2\pi \times [-\cos \varphi]_0^{\frac{\pi}{4}} \times \left[ \frac{\rho^4}{4} \right]_2^4 = 60\pi(2 - \sqrt{2}). \end{aligned}$$

The integral average value of  $f$  over  $\Omega$  is

$$\frac{1}{|\Omega|} \iiint_{\Omega} f(x, y, z) dx dy dz = \frac{60\pi(2 - \sqrt{2})}{\frac{56\pi(2 - \sqrt{2})}{3}} = \frac{45}{14}.$$

### 2.2.1 $n$ -multiple Integrals

**Example 2.11.** Let  $N \in \mathbb{Z}_+$ . Denote the volume of the unit ball  $\sum_{k=1}^n x_k^2 \leq 1$  in the  $n$ -dimensional space as  $\alpha(n)$ . Calculate  $\alpha(4)$  and write out the recurrence formula for the sequence  $\alpha(n)$ .

Solution:

**1. General formula for the volume of an  $n$ -dimensional unit ball using integral.**

The volume of the  $n$ -dimensional unit ball  $\Omega_n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \sum_{k=1}^n x_k^2 \leq 1\}$  is given by the  $n$ -fold integral

$$\alpha(n) = \int \cdots \int_{\sum_{k=1}^n x_k^2 \leq 1} dx_1 dx_2 \cdots dx_n.$$

We use the following approach to establish a recurrence relation. We can write

$$\alpha(n) = \int_{-1}^1 \left( \int \cdots \int_{\sum_{k=2}^n x_k^2 \leq 1-x_1^2} dx_2 \cdots dx_n \right) dx_1.$$

The inner integral  $\int \cdots \int_{\sum_{k=2}^n x_k^2 \leq 1-x_1^2} dx_2 \cdots dx_n$  represents the volume of an  $(n-1)$ -dimensional ball with radius  $r = \sqrt{1-x_1^2}$ . The volume of an  $(n-1)$ -dimensional ball of radius  $r$  is  $r^{n-1} \alpha(n-1)$  (by the property of volume scaling in  $n$ -dimensions). So

$$\alpha(n) = \alpha(n-1) \int_{-1}^1 (1-x_1^2)^{\frac{n-1}{2}} dx_1.$$

Let  $x_1 = \sin t$ , then  $dx_1 = \cos t dt$ . When  $x_1 = -1$ ,  $t = -\frac{\pi}{2}$ ; when  $x_1 = 1$ ,  $t = \frac{\pi}{2}$ .

$$\int_{-1}^1 (1-x_1^2)^{\frac{n-1}{2}} dx_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n t dt = 2 \int_0^{\frac{\pi}{2}} \cos^n t dt.$$

We know that

$$\int_0^{\frac{\pi}{2}} \cos^n t dt = \begin{cases} \frac{(n-1)!!}{n!!} \cdot \frac{\pi}{2}, & n \text{ is even,} \\ \frac{(n-1)!!}{n!!}, & n \text{ is odd.} \end{cases}$$

So the recurrence formula is

$$\alpha(n) = 2\alpha(n-1) \int_0^{\frac{\pi}{2}} \cos^n t dt.$$

#### 2. Initial values.

For  $n = 1$ , the unit ball is the interval  $[-1, 1]$ , so  $\alpha(1) = 2$ .

For  $n = 2$ ,  $\alpha(2) = \pi$  (since the unit ball in 2-D is a unit circle  $x_1^2 + x_2^2 \leq 1$  and its area is  $\pi r^2 = \pi$ ).

#### 3. Calculate $\alpha(4)$ .

We first use the recurrence formula

$$\alpha(n) = 2\alpha(n-1) \int_0^{\frac{\pi}{2}} \cos^n t \, dt.$$

For  $n = 3$ ,  $\int_0^{\frac{\pi}{2}} \cos^3 t \, dt = \frac{2!!}{3!!} = \frac{2}{3}$ , and

$$\alpha(3) = 2\alpha(2) \int_0^{\frac{\pi}{2}} \cos^3 t \, dt = 2\pi \times \frac{2}{3} = \frac{4\pi}{3}.$$

For  $n = 4$ ,  $\int_0^{\frac{\pi}{2}} \cos^4 t \, dt = \frac{3!!}{4!!} \cdot \frac{\pi}{2} = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{16}$ .

$$\alpha(4) = 2\alpha(3) \int_0^{\frac{\pi}{2}} \cos^4 t \, dt = 2 \times \frac{4\pi}{3} \times \frac{3\pi}{16} = \frac{\pi^2}{2}.$$

## 2.3 Symmetry

**Example 2.12.** Find

$$I = \iiint_{\Omega} (x+y+z)^2 dV,$$

where  $\Omega$  is the intersection of  $x^2 + y^2 \leq 2z$  and  $x^2 + y^2 + z^2 \leq 3$ .

Solution: Given  $I = \iiint_{\Omega} (x+y+z)^2 dV$ . Expand  $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$ . Because of the symmetry of the region  $\Omega$  (the intersection of  $x^2 + y^2 \leq 2z$  and  $x^2 + y^2 + z^2 \leq 3$ ) with respect to the coordinate planes, we have  $\iiint_{\Omega} 2xy dV = 0$ ,  $\iiint_{\Omega} 2yz dV = 0$ , and  $\iiint_{\Omega} 2zx dV = 0$ . So,  $I = \iiint_{\Omega} (x^2 + y^2 + z^2) dV$ .

In cylindrical coordinates,  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $z = z$ , and  $dV = r dr d\theta dz$ . The inequalities become  $r^2 \leq 2z$  and  $r^2 + z^2 \leq 3$ . To find the intersection of  $r^2 = 2z$  and  $r^2 + z^2 = 3$ , substitute  $r^2 = 2z$  into  $r^2 + z^2 = 3$ , getting  $z^2 + 2z - 3 = 0$ . The non-negative root is  $z = 1$ , and then  $r = \sqrt{2}$ . The integral  $I$  can be written as:

$$\begin{aligned} I &= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} r dr \int_{\frac{r^2}{2}}^{\sqrt{3-r^2}} (r^2 + z^2) dz = 2\pi \int_0^{\sqrt{2}} r \left( r^2 z + \frac{z^3}{3} \right) \Big|_{\frac{r^2}{2}}^{\sqrt{3-r^2}} dr \\ &= 2\pi \int_0^{\sqrt{2}} r \left( r^2 \sqrt{3-r^2} + \frac{(3-r^2)^{\frac{3}{2}}}{3} - \frac{r^4}{2} - \frac{r^6}{24} \right) dr \\ &= 2\pi \left( \int_0^{\sqrt{2}} r^3 \sqrt{3-r^2} dr + \frac{1}{3} \int_0^{\sqrt{2}} r(3-r^2)^{\frac{3}{2}} dr - \frac{1}{2} \int_0^{\sqrt{2}} r^5 dr - \frac{1}{24} \int_0^{\sqrt{2}} r^7 dr \right) \\ &= \frac{\pi(17 - 2\sqrt{3})}{5} \end{aligned}$$