NOTES ON EXERCISE COURSE OF CALCULUS A

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ABSTRACT. This is the lecture note of the exercise course of Calculus A in Peking university.

CONTENTS

SOME RULES

Submission rules. The exercise course is on every Tuesday. In this course, you have to submit the homework assigned last Tuesday and the Thursday before the last Thursday. For example, suppose that you need to submit the homework on September 10, you have to submit those assigned on September 3 and August 29.

An exception is that on October 8, you must submit the homework assignment given before September 24 (including this day). After that, you should follow the above rules.

You can send your homework to the email:2201110024@stu.pku.edu.cn, or write on a paper to submit it on the exercise course. If you want to send it by email, you should name the title of this email with "name+student ID+the number of the homework". For example, "Zhang San 0000000000 Homework 1" in Chinese.

Grade of homework. We only care about submitting or not and complete or not. If you submit all the homework and complete all the problems assigned to it, then you will get full marks. Regular grades only come from the submission situation of homework.

Midterm. The midterm exam will take place in Early November and the precise date has not been determined.

Some useful links. We present some useful links associated to calculus.

Lecture notes by Yantong Xie:https://darkoxie.github.io

Mathstackexchange: https://math.stackexchange.com

1. Preliminaries of Calculus

1.1. **Trigonometry.** We introduce some results in trigonometry.

1.1.1. *Some basic formulae.*

$$\sin a + \sin b = 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2},$$
$$\sin a \cos b = \frac{1}{2}(\sin(a+b) + \sin(a-b)).$$

Exercise 1.1. What about $\sin a - \sin b$, $\cos a \pm \cos b$, $\sin a \sin b$, $\cos a \cos b$?

1.1.2. *Reverse functions.* Consider $f(x) = \sin x$. We see that it is increasing in $[-\pi/2, \pi/2]$ and we can define the reverse function of it in this interval by $f^{-1}(x) = \arcsin x$, where $x \in [-1, 1]$.

Exercise 1.2. Consider reverse functions of $\cos x$, $\tan x$.

1.1.3. *Euler formula.*

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

Exercise 1.3. Calculate $\sum_{k=1}^{n} \cos kx$.

1.2. **Some useful inequalities.**

1.2.1. *Cauchy's inequality.*

Theorem 1.4. *For $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \subset \mathbb{R}$, we have*

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right).$$

Proof. Let $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$. Consider $|a + tb|^2$ with $t \in \mathbb{R}$. □

Question 1.5. What is the condition such that the above inequality satisfies "="?

Proposition 1.6. *Let $a, b, c > 0$ be such that $a + b + c = 1$. We have*

$$a^3 + b^3 + c^3 \geq \frac{a^2 + b^2 + c^2}{3}.$$

Proof. Applying Theorem 1.4, we have

$$(a^2 + b^2 + c^2)^2 \leq (a^3 + b^3 + c^3)(a + b + c) = a^3 + b^3 + c^3.$$

Additionally,

$$a^2 + b^2 + c^2 \geq ab + ac + bc.$$

This implies that $3(a^2 + b^2 + c^2) = (a + b + c)^2 = 1$. Combining all above, the result follows directly. □

1.2.2. *Hölder's inequality.*

Theorem 1.7. *For $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \subset \mathbb{R}_+$, we have*

$$\left(\sum_{i=1}^n a_i b_i\right) \leq \left(\sum_{i=1}^n a_i^p\right)^{1/p} \left(\sum_{i=1}^n b_i^q\right)^{1/q},$$

where $1/p + 1/q = 1$ with $p, q > 0$.

Proof. It is a corollary of Yong's inequality, i.e. for $a, b > 0$, $p, q > 0$ with $1/p + 1/q = 1$, we have $ab \leq a^p/p + b^q/q$. □

1.2.3. *Bernoulli's inequality.*

Theorem 1.8. *Let $n \geq 2$. Assume that $x_1, x_2, ..., x_n > -1$ and have the same sign. Then*

$$\prod_{i=1}^{n}(1+x_i) \geq 1 + \sum_{i=1}^{n} x_i.$$

Proof. The proof is by using induction. For $n = 1$, there is nothing to prove. Assume that the result is true for n. By simple calculations, we have

$$\begin{aligned}\prod_{i=1}^{n}(1+x_i) &= (1+x_1)\left(\prod_{i=2}^{n}(1+x_i)\right)\\ &\geq (1+x_1)\left(1+\sum_{i=2}^{n} x_i\right)\\ &= 1+\sum_{i=1}^{n} x_i + x_1\left(\sum_{i=2}^{n} x_i\right)\\ &\geq 1+\sum_{i=1}^{n} x_i,\end{aligned}$$

where for the last inequality, we have used the property that x_i have the same sign. Now we complete the proof. □

Exercise 1.9. Let $m \in \mathbb{R}$ and $x > -1$. Show that if $m \in [0, 1]$, then $(1 + x)^m \leq 1 + mx$ and if $m < 0$ or $m > 1$, then $(1 + x)^m \geq 1 + mx$.

Exercise 1.10. Show that if $b > a > 0$, and $n \in \mathbb{Z}_+$, then $a^{n+1} > b^n((n+1)a-nb)$, and $b^{n+1} > a^n((n + 1)b - na)$.

Proposition 1.11. *The sequence $\{(1 + 1/n)^n\}$ is increasing.*

Proof. Let $b = 1 + 1/n$ and $a = 1 + 1/(n + 1)$, the result follows from Exercise 1.10. □

Exercise 1.12. Show that the sequence $\{(1 + 1/n)^{n+1}\}$ is decreasing.

1.3. **Real numbers.**

1.3.1. *Density of real numbers.* One of the most remarkable property of the real numbers is that $\mathbb{R}$ is dense. We can see from the following proposition.

Proposition 1.13. *Let $a, b \in \mathbb{R}$. Show that there exists $r_1 \in \mathbb{Q}$ and $r_2 \in \mathbb{R}\backslash\mathbb{Q}$ such that $a < r_1 < r_2 < b$.*

Proof. Assume that $a \in \mathbb{Q}$. Let $N \gg 1$ be such that $1/N < \pi/N < b-a$, we choose $r_1 = a+1/N$ and $r_2 = a+\pi/N$. Assume that $a \in \mathbb{R}\backslash\mathbb{Q}$, we choose $N \gg 1$ such that $N(b - a) > 10$. As a result, there must be some $n \in \mathbb{Z}_+$ such that $n \in (Na, Nb)$, and then $n/N \in (a, b)$. Taking n/N as new a and applying the previous arguments, we are done. □

1.3.2. *Closeness of calculations for real numbers.*

Proposition 1.14. *There exist $a, b \in \mathbb{R} \backslash \mathbb{Q}$ such that $a^b \in \mathbf{Q}$.*

Proof. Consider $a = b = \sqrt{2}$. If $a^b \in \mathbb{Q}$, then we are done. If not, we see that $(a^b)^{\sqrt{2}} = 2$. □

Exercise 1.15. Give the corresponding examples.

- $a, b \in \mathbb{Q}$, $a^b \in \mathbb{Q}$.
- $a, b \in \mathbb{Q}$, $a^b \in \mathbb{R} \backslash \mathbb{Q}$.
- $a, b \in \mathbb{R} \backslash \mathbb{Q}$, $a + b \in \mathbb{Q}$.

2. Limits of sequences

2.1. **Comparing the order.** Let $a > 0$ and $b > 1$, we have

$$\ln n \ll n^a \ll b^n \ll n! \ll n^n, \tag{2.1}$$

where we call $f(n) \ll g(n)$ if

$$\lim_{n \to +\infty} \frac{f(n)}{g(n)} = 0.$$

Now, let us show (2.1). We note that

$$0 < \frac{n!}{n^n} \leq \frac{1}{n},$$

which implies that $n! \ll n^n$. Let $x_n = b^n/(n!)$. We have

$$\frac{b_{n+1}}{b_n} = \frac{b}{n+1}.$$

As a result $b^n \ll n!$ follows from the following lemma, whose proof is left for the reader.

Lemma 2.1. *Let $\{x_n\}$ be a sequence, if*

$$\lim_{n \to +\infty} \left| \frac{x_{n+1}}{x_n} \right| < 1,$$

then $\lim_{n \to +\infty} x_n = 0$.

Denote $y_n = n^a/b^n$. By using this lemma, we obtain that $n^a \ll b^n$. Finally, we turn to the proof of $\ln n \ll n^a$. Noting that

$$\frac{\ln n}{n^a} = \frac{1}{a} \cdot \frac{\ln n^a}{n^a},$$

we only show that $\ln n/n \to 0^+$. This is a direct consequence of the following exercise.

Exercise 2.2. $\lim_{n \to +\infty} n^{1/n} = 1$.

Hint: Let $n^{1/n} = 1 + x_n$. We have

$$n = (1 + x_n)^n \geq 1 + nx_n + \frac{n(n-1)}{2} x_n.$$

By using Lemma 2.1, we can show $\lim_{n \to +\infty} n^2 q^n = 0$ (in Homework).

2.2. **Basic properties on limits.**

2.3. **Nonexistence of limits.**

Proposition 2.3. *$\lim_{n\to+\infty} x_n = x$ if and only if for any subsequence x_{n_k}, we have $\lim_{k\to+\infty} x_{n_k} = x$.*

Proof. It follows from the definition the limit. □

We can use this proposition to show that some sequence does not have a limit.

Exercise 2.4. $(-1)^n$ diverges.

Exercise 2.5. For $\{x_n\}$, we have $x_{2k} \to a$ and $x_{2k-1} \to b$, where $a, b \in [-\infty, +\infty]$. Show that x_n is convergent if and only if $a = b \in (-\infty, +\infty)$.

2.4. **Monotone sequence.**

Example 2.6. Let $n \in \mathbb{Z}_+$. Assume that $x_n \in [0, 1]$ is the root of $\sum_{i=1}^n x^i = 1$. Show that $\lim_{n\to+\infty} x_n = \frac{1}{2}$.

Proof. It is easy to see that x_n is decreasing. On the other hand, $x_n - x_n^{n+1} = 1 - x_n$. Letting $n \to +\infty$, we see that $x_n \to \frac{1}{2}$. □

Exercise 2.7. Assume that $0 < b < a$. We let $\{a_n\}, \{b_n\}$ be defined as follows.

$$a_{n+1} = \frac{a_n + b_n}{2},\ b_{n+1} = \sqrt{a_n b_n},\ n \in \mathbb{Z}_+,$$

where $a_1 = a$, $b_1 = b$. Show that $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n$ and exist.

Hint: Show that a_n and b_n are monotone.

Exercise 2.8. $\{x_n\}$ is monotone. Show that $\{x_n\}$ is monotone if and only if it has a convergent subsequence.

2.5. **Boundedness.** Any convergent sequence is bounded.

Example 2.9. Assume that $x_n \to x > 0$. Then x_n has a lower bound but may not has a minimal point.

Proof. For n sufficiently large, we have $x_n > \frac{x}{2}$. Considering $x_n = 1 + \frac{1}{n}$, we see that it does not have a minimal point. □

Exercise 2.10. Assume that $x_n \to +\infty$, then there must be some $n_0 \in \mathbb{Z}_+$ such that $x_{n_0} = \min_{n\in\mathbb{Z}_+}\{x_n\}$.

2.6. **Some important sequence and limits.**

Example 2.11. Show that

$$S_n = \sum_{i=1}^n \frac{1}{i^p}$$

is convergent if and only if $p > 1$.

Proof. Taking $p = 1$ and $p = 2$ for example. For $p = 1$, since

$$\ln(n+1) - \ln(n) < \frac{1}{n},$$

S_n diverges. For $p = 2$, we see that

$$\frac{1}{n^2} < \frac{1}{n-1} - \frac{1}{n},$$

and then S_n is convergent. □

Example 2.12. Consider

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}.$$

Show that $x_n \to \ln 2$.

Proof. Let

$$\gamma_n = \sum_{i=1}^{n} \frac{1}{i} - \ln n.$$

We can show that $\gamma_n \to \gamma$ (Euler constant). Then the result follows directly. □

2.7. **Sandwich theorem.** Recall that if $a_n \leq b_n \leq c_n$ and $a_n, c_n \to x$, then $b_n \to x$.

Example 2.13. Assume that $0 < a_1 \leq a_2 \leq ... \leq a_k$. Calculate $\lim_{n\to+\infty}(a_1^n + ... + a_k^n)^{1/n}$.

Proof. We see that

$$a_k \leq (a_1^n + ... + a_k^n)^{1/n} \leq k^{1/n} a_k.$$

Then $\lim_{n\to+\infty}(a_1^n + ... + a_k^n)^{1/n} = a_k$. □

Exercise 2.14. Assume that $0 < a_1 \leq a_2 \leq ... \leq a_k$. Calculate $\lim_{n\to+\infty}(a_1^{1/n} + ... + a_k^{1/n})^n$.

Exercise 2.15. Assume that $a_k \to a > 0$. Calculate $\lim_{n\to+\infty}(a_1^n + ... + a_n^n)^{1/n}$.

2.8. **Cauchy's proposition.** It is about the limit of average for a sequence having a limit.

Proposition 2.16. *If* $\lim_{n\to+\infty} x_n = x$, *then*

$$\lim_{n\to+\infty} \frac{\sum_{i=1}^n x_i}{n} = x. \tag{2.2}$$

Proof. For any $\varepsilon > 0$, we can choose $N \in \mathbb{Z}_+$ such that $|x_n - x| < \varepsilon$ for any $n > N$. This implies that if $n > N$, there holds

$$\left| \frac{\sum_{i=1}^N x_i}{n} - x \right| \leq \frac{\sum_{i=1}^N |x_i|}{n} + \frac{n-N}{n}\varepsilon.$$

Choosing lager N without changing the notation, we can assume that if $n > N$, then

$$\left| \frac{\sum_{i=1}^N x_i}{n} - x \right| < 2\varepsilon.$$

As a result, we have (2.2). □

Exercise 2.17. Assume that $a_n \to a$ and $b_n \to b$. Show that

$$\lim_{n\to+\infty} \frac{\sum_{i=1}^{n-1} a_i b_{n-i}}{n} = ab.$$

Exercise 2.18. Assume that $x_n \to x$. Show that

$$\lim_{n\to+\infty} \frac{\sum_{k=0}^n C_n^k x_k}{2^n} = x$$

Exercise 2.19 (Stolz $\frac{0}{\infty}$)**.** Let $\{a_n\}$ and $\{b_n\}$ be two sequences. Assume that $\{b_n\}$ is a strictly monotone and divergent sequence (i.e. strictly increasing and approaching $+\infty$, or strictly decreasing and approaching $-\infty$) and the following limit exists

$$\lim_{n\to+\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = l.$$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l.$$

Exercise 2.20 (Stolz $\frac{0}{0}$)**.** Let $\{a_n\}$ and $\{b_n\}$ be two sequences. Assume now that $a_n, b_n \to 0$ and $\{b_n\}$ is strictly decreasing. If

$$\lim_{n\to+\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = l.$$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l.$$

2.9. **Limits with respect to the definition of e.** We know that

$$e = \lim_{n\to+\infty} (1+1/n)^n. \tag{2.3}$$

By this we can consider related limits.

Question 2.21. What about the limit $\lim_{n\to-\infty}(1+1/n)^n$.

Example 2.22. Show that

$$\lim_{n\to+\infty} \left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n = e$$

Proof. It follows from (2.3) that

$$\lim_{n\to+\infty} \left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n = \lim_{n\to+\infty} \left(1+\frac{n+1}{n^2}\right)^{\frac{n^2}{n+1}\cdot\frac{n+1}{n}} = e.$$

□

Example 2.23. Calculate the following limits.

$$\lim_{n\to+\infty} \left(1-\frac{1}{n}\right)^{n^2}, \quad \lim_{n\to+\infty} \left(1-\frac{1}{n^2}\right)^n.$$

Proof. Noting that

$$\left(1-\frac{1}{n}\right)^{n^2} = \left[\left(1-\frac{1}{n}\right)^n\right]^n$$

and $\lim_{n\to+\infty}(1-1/n)^n = 1/e$, we see that the first limit is 0. On the other hand, we obtain that

$$\left(1-\frac{1}{n^2}\right)^n = \left[\left(1-\frac{1}{n^2}\right)^{n^2}\right]^{1/n}.$$

The term in $[\cdot]$ is bounded and then the limit is 1. □

Exercise 2.24. Calculate $\lim_{n\to+\infty}(1+1/n)^{n^2}$ and $\lim_{n\to+\infty}(1+1/n^2)^n$.

Exercise 2.25. If there exists $\{x_n\}$ such that $\lim_{n\to+\infty} x_n = 1$ and $\lim_{n\to+\infty} x_n^n = 1.001$.

Hint: Consider $x_n = 1 + \frac{a}{n}$.

Exercise 2.26. Find $\{x_n\}$ such that the following properties hold.

(1) $\lim_{n\to+\infty}(x_n - e^n) = 0$.
(2) $\lim_{n\to+\infty}(f(x_n) - f(e^n)) \neq 0$, where $f(x) = x\ln x$.

Hint: Consider $y_n = x_n - e^n$.

2.10. **Problems associated with fixed points.** Consider a sequence $\{x_n\}$ such that $x_{n+1} = f(x_n)$. If x_n is convergent with $\lim_{n\to+\infty} x_n = x$ and f is continuous, then $f(x) = x$.

Example 2.27. Let $x_{n+1} = \sqrt{2 + x_n}$ and $x_1 = 1$. Show that $\lim_{n\to+\infty} x_n = 2$.

Proof. 2 is the fix point. We have

$$\frac{x_{n+1} - 2}{x_n - 2} = \frac{1}{2 + \sqrt{x_n + 2}}.$$

Then

$$\left|\frac{x_{n+1} - 2}{x_n - 2}\right| < \frac{1}{2}.$$

□

Exercise 2.28. Let $x_{n+1} = \frac{1}{x_n^{1/2}}$ and $x_1 = 2$. Show that $\lim_{n\to+\infty} x_n = 1$.

Exercise 2.29. Let $x_{n+1} = \frac{1}{x_n^2}$ and $x_1 = 2$. Show that x_n is divergent.

Exercise 2.30. Let $x_{n+1} = \frac{1}{x_n}$ and $x_1 = 2$. Show that x_n is divergent.

Question 2.31. Can you find something interesting in the above three exercises?

Exercise 2.32. $x_1 = \sqrt{2}$. $x_{n+1} = \sqrt{2}^{x_n}$. Consider $\lim_{n\to+\infty}$.

Hint: Draw the picture.

3. Limits of functions

3.1. Heine criterion.

Proposition 3.1. *Assume that $f : [a, b] \to \mathbb{R}$ is a function. For $c \in (a, b)$, $\lim_{x\to c} f(x) = A$, if and only if for any $x_n \to c$, there holds $f(x_n) \to A$ as $n \to +\infty$.*

Proof. The proof is due to the definition. □

Exercise 3.2. Use this Heine criterion and Cauchy criterion of sequences to show the Cauchy criterion of functions.

Example 3.3. Show that the Dirichlet function does not have limits at any points in $[0, 1]$. Here Dirichlet function $D(x) : [0, 1] \to \{0, 1\}$ is defined by

$$D(x) = 1 \text{ if } x \in \mathbb{Q}, \quad D(x) = 0 \text{ if } x \notin \mathbb{Q}$$

Exercise 3.4. Show that the Dirichlet function can be represented by

$$D(x) = \lim_{m\to\infty} \left\{ \lim_{n\to\infty} [\cos(\pi m! x)]^{2n} \right\}.$$

Exercise 3.5. The limit $\lim_{x\to+\infty}(a\cos x + b\sin x)$ exists. Show that $a = b = 0$.

Exercise 3.6. Show that $\lim_{x\to 0^+} \sin(\frac{1}{x})$ does not exist.

3.2. **Riemannian function.** Riemannian function is defined as follows.

$$R(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q}, \text{ where } p, q \in \mathbb{Z} \text{ are relatively prime}, q > 0, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Example 3.7. Consider the limit $\lim_{x \to x_0} R(x)$, where $x_0 \notin \mathbb{Q}$.

Proof. Assume that $x_0 \notin \mathbb{Q}$, we claim that $\lim_{x \to x_0} R(x) = 0$. For any $\varepsilon > 0$, we can choose $N \in \mathbb{Z}_+$ such that $\frac{1}{N} < \varepsilon$. Then there are only finite numbers in $\mathbb{Q}$ such that the representation $\frac{p}{q}$ with $p, q \in \mathbb{Z}$, $(p, q) = 1$, and $q > 0$ satisfy $q < N$. Then the result follows. □

3.3. **Some basic calculations.**

Example 3.8. Calculate

$$\lim_{x \to +\infty} \left(\frac{x^2 - 1}{x^2 + 1} \right)^{\frac{x-1}{x+2}}.$$

Proof. Simple calculations imply that it is 1. □

Exercise 3.9. Calculate the following limits.

(1) $\lim_{x \to +\infty} \frac{x^k}{a^x} = 0$, $a > 1, k > 0$.
(2) $\lim_{x \to +\infty} \frac{\ln x}{x^k} = 0$, $k > 0$.
(3) $\lim_{x \to +\infty} x^{1/x} = 1$.

Hint: when $x \notin \mathbb{Z}_+$, consider $[x]$.

3.4. **Two important limits.** We know that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} (1 + x)^{\frac{1}{x}} = 0. \tag{3.1}$$

Example 3.10. Calculate

$$\lim_{x \to \infty} \frac{\sin x}{x}, \quad \lim_{x \to +\infty} (1 + x)^{\frac{1}{x}}.$$

Exercise 3.11. By using (3.1) to calculate the following limits.

(1) $\lim_{x \to +\infty} \left(\frac{2}{\pi} \arctan x \right)^x$;
(2) $\lim_{x \to \frac{pi}{2}-} (\sin x)^{\tan x}$;
(3) $\lim_{x \to \infty} \left(\frac{x^2-1}{x^2+1} \right)^{x^2}$;
(4) $\lim_{x \to \frac{\pi}{2}-} (\cos x)^{\frac{\pi}{2} - x}$;
(5) $\lim_{x \to 0} \frac{\sin 2x - 2\sin x}{x^3}$;
(6) $\lim_{x \to 1} (1 - x) \tan \left(\frac{\pi}{2} x \right)$.

Hint: You have to change the variables.

3.5. **Order analysis.** Assume that $f, g : (a - \delta, a) \cup (a + \delta)$. We have the following definitions.

$f(x) = o(g(x))$ as $x \to a$ if and only if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

$f(x) = O(g(x))$ as $x \to a$ if and only if there exists $M > 0$ such that

$$\left| \frac{f(x)}{g(x)} \right| \leq M.$$

Example 3.12. what is the meaning of $o(1)$ and $O(1)$?

Exercise 3.13. Consider the following limits.

(1) $\lim_{x\to 0} x\sin\frac{1}{x}$;
(2) $\lim_{x\to\infty} x\sin\frac{1}{x}$;
(3) $\lim_{x\to 0} x\sin x$;
(4) $\lim_{x\to\infty} x\sin x$;
(5) $\lim_{x\to 0} \frac{1}{x}\sin\frac{1}{x}$
(6) $\lim_{x\to\infty} \frac{1}{x}\sin\frac{1}{x}$.

Question 3.14. How to use the notation o, O to illustrate the above limits?

4. Continuous function

4.1. Discontinuity.

Example 4.1. Consider the following three functions and classify the continuity and discontinuity of them.

$$f(x)=\begin{cases} e^{\frac{1}{x}}, & x\in(-\infty,0),\\ x^2, & x\in[0,+\infty); \end{cases}$$

$$f(x)=\begin{cases} e^{\frac{1}{x}}, & x\in\mathbb{R}\backslash\{0\}\\ 0, & x=0 \end{cases}$$

$$f(x)=\begin{cases} \frac{\ln(1+x^2)}{x}, & x\in(-1,0)\cup(0,1),\\ 2, & x=0. \end{cases}$$

Question 4.2. If the range of the function f on the interval $[a,b]$ is a closed interval, is f necessarily continuous on $[a,b]$?

4.2. Uniqueness.

Example 4.3. Assume that $f,g\in C(\mathbb{R})$. If $f(x)=g(x)$ for any $x\in\mathbb{Q}$, then $f\equiv g$.

Proof. By using the continuity of f and g. □

Exercise 4.4. Assume that $f\in C(\mathbb{R})$ and $f(x+y)=f(x)+f(y)$. Show that $f(x)=f(1)x$.

4.3. Intermediate value theorem.

Example 4.5. Assume that $0\le f\in C[0,1]$ with $f(0)=f(1)=0$. Show that for any $a\in(0,1)$, there exists $x_0\in[0,1]$ such that $x_0+a\in[0,1]$, and

$$f(x_0)=f(x_0+a).$$

Proof. Let $g(x)=f(x+a)-f(x)$. The result follows directly from intermediate value theorem. □

Question 4.6. If f is not nonnegative, what will happen?

Example 4.7. Assume that $f\in C[0,1]$ with $f(0)=f(1)$. Show that for any $n\in\mathbb{Z}_+$, there exists ξ such that

$$f\left(\xi+\frac{1}{n}\right)=f(\xi).$$

Proof. Let $g(x) = f(x + \frac{1}{n}) - f(x)$. Consider $g(0)$, $g(\frac{1}{n})$,... $f(1)$. WLOG, assume that $g(0) > 0$. If there exists $1 \le k \le n$ such that $g(\frac{k}{n}) < 1$, then the result follows. If not, it is a contradiction to $f(0) = f(1)$. □

Exercise 4.8. Assume that $f \in C[0,1]$, and $f([0,1]) \subset [0,1]$. Show that the figure of $y = f(x)$ has intersection with both $y = x$ and $y = 1 - x$.

Hint: Draw the picture and then you will find the answer.

Exercise 4.9. Let $f_n(x) = x^n + x$ with $n \in \mathbb{Z}_+$. Show that the following properties hold.

(1) For any $n \in \mathbb{Z}_+$, the equation $f_n(x) = 1$ has a unique root in $(\frac{1}{2}, 1)$.
(2) If $c_n \in (\frac{1}{2}, 1)$ is a root of $f_n(x) = 1$ in $(\frac{1}{2}, 1)$, then
$$\lim_{n\to\infty} c_n = 1.$$

Hint: Firstly, show that c_n is increasing. If $c_n < \alpha$ for any $n \in \mathbb{Z}_+$, there is a contradiction.

4.4. **Contraction maps.**

Example 4.10. Let $f \in C(-\infty, +\infty)$. For any $x, y \in \mathbb{R}$, there holds $|f(x)-f(y)| \le k|x - y|$ with $0 < k < 1$. Show the following properties.

(1) The function $kx - f(x)$ is increasing.
(2) There exists unique $\xi \in \mathbb{R}$, such that $f(\xi) = \xi$.

Hint: Consider the sequence $x_{n+1} = f(x_n)$. Show that x_n is a Cauchy sequence.

Exercise 4.11. Let the function f be defined on the domain $[a, b]$, with the image set $f([a, b]) \subset [a, b]$, and satisfying $|f(x) - f(y)| \le |x - y|$ for all $x, y \in [a, b]$. For any $x_0 \in [a, b]$, define the sequence recursively by
$$x_{n+1} = \frac{1}{2}(x_n + f(x_n)).$$
Show that the sequence $\{x_n\}$ is convergent.

Hint: Use the MCT.

Exercise 4.12. Let $f \in C[0, +\infty)$. Assume that $f(0) = 0$ and $0 < f(x) < x$ when $x > 0$. For $x_{n+1} = f(x_n)$, show that $\lim_{n\to+\infty} x_n = 0$.

Hint: Draw the picture.

4.5. **Boundedness, maximum and minimum.**

Example 4.13. Let the function f have only first-type discontinuities on the interval $[a, b]$. Prove that f is bounded on $[a, b]$.

Proof. Prove by the definition. □

Exercise 4.14. If $f \in C[a, +\infty)$ and there exists a finite limit $\lim_{x\to+\infty} f(x)$, prove that f is bounded on $[a, +\infty)$.

Exercise 4.15. If $f \in C[a, +\infty)$ and there exists a finite limit $\lim_{x\to+\infty} f(x)$, prove that f can attain at least either a maximum or a minimum value on $[a, +\infty)$.

Hint: First consider the behavior of f on $[a, a + N]$, $N > 1$.

Exercise 4.16. If $f \in C(a,b)$ and $f(a^+) = f(b^-) = +\infty$, show that f has a minimum value on (a,b).

Exercise 4.17. If $f \in C(a,b)$ and $f(a^+) = f(b^-)$, prove that f can attain at least either a maximum or a minimum value on (a,b).

Exercise 4.18. If $f \in C(-\infty,+\infty)$, $\lim_{x\to\pm\infty} f(x) = +\infty$, and the minimum value of $f(x)$ is $f(a) < a$, prove that the composite function $f(f(x))$ attains its minimum value at least at two points.

Hint: Draw the picture.

4.6. **Monotone functions.**

Exercise 4.19. Let $f \in C[a,b]$, and for any two rational numbers r_1, r_2 in $[a,b]$ with $r_1 < r_2$, it holds that $f(r_1) \le f(r_2)$. Prove that the function f is monotonically increasing on $[a,b]$.

Exercise 4.20. Let f be a monotonically increasing function on $(-\infty,+\infty)$, and define $g(x) = f(x^+)$ at each point. Prove that g is a function that is right-continuous everywhere on $(-\infty,+\infty)$.

Exercise 4.21. Let f be a monotonically increasing function on $(\infty,+\infty)$, and for all $x, y \in \mathbb{R}$, it satisfies $f(x+y) = f(x) + f(y)$. Prove that $f(x) = f(1)x$.

Exercise 4.22. Let $f \in C[a,b]$. Prove that a necessary and sufficient condition for f to have no extreme points in (a,b) is that f is strictly monotonically increasing on $[a,b]$.

5. Differentiation

5.1. **Definitions.**

Example 5.1. Given that the even function $f(x)$ is differentiable at the point $x = 0$, find $f'(0)$.

Proof. Considering $f'(0-)$ and $f'(0+)$, it is easy to see that $f'(0) = 0$. □

Exercise 5.2. Let $p(x)$ be an n-degree polynomial with n roots, and denote its distinct roots as $x_1, \cdots, x_k$, where the multiplicity of root x_i is n_i for $i = 1, 2, \cdots, k$, and $n_1 + n_2 + \cdots + n_k = n$. Prove that the following holds.

$$p'(x) = p(x)\left(\sum_{i=1}^{k} \frac{n_i}{x - x_i}\right)$$

Example 5.3. Let the function be defined as

$$f(x) = \begin{cases} \frac{x}{1+e^{1/x}}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Study the differentiability of $f(x)$.

Proof. Consider $\frac{f(x)-f(0)}{x}$. □

Exercise 5.4. Let n be a natural number. Under what conditions is the function

$$f(x) = \begin{cases} x^n \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Study the differentiability of $f(x)$.

Example 5.5. Let the function $f(x)$ satisfy the functional equation $f(x+y) = f(x)f(y)$, and it is known that $f'(0) = 1$. Prove that $f(x)$ is differentiable everywhere and that $f'(x) = f(x)$ holds.

Proof. The assumption that $f(x+y) = f(x)f(y)$ implies that the differentiability can be spreaded. □

Exercise 5.6. Suppose in a neighborhood U of the origin, we have $|f(x)| \leq |g(x)|$, and $g(0) = g'(0) = 0$. Find $f'(0)$.

Hint: Draw the picture.

5.2. **Some tricks.**

Example 5.7. Let $f(0) = 0$ and $f'(0)$ exist. Define the sequence

$$x_n = f\left(\frac{1}{n^2}\right) + f\left(\frac{2}{n^2}\right) + \cdots + f\left(\frac{n}{n^2}\right), \quad n \in \mathbf{N}_+$$

Try to find $\lim_{n\to\infty} x_n$.

Proof. It is easy to find that

$$f\left(\frac{k}{n^2}\right) = f'(0)\frac{k}{n^2} + o(\frac{k}{n^2}).$$

Then we can sum k up from 1 to n, and find that $x_n \to \frac{1}{2}f'(0)$. □

Exercise 5.8. Find the limits of the following sequences.

(1) $\lim_{n\to\infty}\left(\sin\frac{1}{n^2} + \sin\frac{2}{n^2} + \cdots + \sin\frac{n}{n^2}\right)$.
(2) $\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\cdots\left(1+\frac{n}{n^2}\right)\right]$.

Hint: Use the previous limit.

Exercise 5.9. Let $y = \frac{1+x}{\sqrt{1-x}}$, calculate $y^{(n)}(x)$, where $n \in \mathbb{N}_+$.

Hint: Take ln.

Exercise 5.10. Let f have derivatives of all orders on $\mathbf{R}$. Prove that for each natural number n the following holds.

$$\frac{1}{x^{n+1}} f^{(n)}\left(\frac{1}{x}\right) = (-1)^n \left[x^{n-1} f\left(\frac{1}{x}\right)\right]^{(n)}.$$

Exercise 5.11. Using the sum of the series $1 + x + x^2 + \cdots + x^n$, find the sums of the following.

(1) $1 + 2x + 3x^2 + \cdots + nx^{n-1}$.
(2) $1^2 + 2^2x + 3^2x^2 + \cdots + n^2x^{n-1}$.

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6. Mean value theorem

Exercise 6.1. Let f be continuous on $[a, b]$, differentiable on (a, b), and with $0 < a < b$. Prove that there exists $\xi \in (a, b)$ such that

$$f(b) - f(a) = \ln\frac{b}{a} \cdot \xi f'(\xi).$$

Hint: Consider $f(e^x)$.

Exercise 6.2. Let f be differentiable on $[a, b]$. Prove that there exists $\xi \in (a, b)$ such that

$$2\xi[f(b) - f(a)] = \left(b^2 - a^2\right) f'(\xi).$$

Hint: Consider $f(x^{1/2})$.

Example 6.3. Let f, g be continuous on $[a, b]$ and differentiable on (a, b), with $g(x)$ having no zeros in the interval (a, b). Prove that there exists $\xi \in (a, b)$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(\xi) - f(a)}{g(b) - g(\xi)}.$$

Proof. Consider the function

$$F(x) = -f(x)g(x) + g(b)f(x) + f(a)g(x).$$

It is easy to see that $F(a) = g(b)f(a) = F(b)$. Then the result follows from Rolle's theorem. $\square$

Exercise 6.4. Let f be continuous on $[a, +\infty)$ and differentiable on $(a, +\infty)$, with $\lim_{x\to+\infty} f(x) = f(a)$. Prove that there exists $\xi > a$ such that $f'(\xi) = 0$.

Exercise 6.5. Let $f(x)$ be differentiable on $[0, +\infty)$ and satisfy $0 \le f(x) \le \frac{x}{1+x^2}$. Prove that there exists $\xi > 0$ such that

$$f'(\xi) = \frac{1 - \xi^2}{\left(1 + \xi^2\right)^2}.$$

Hint: Consider $f(x) - \frac{x}{1+x^2}$.

Exercise 6.6. For (1) $f(x) = ax^2 + bx + c(a \neq 0)$ and (2) $f(x) = \frac{1}{x}(x > 0)$, calculate the value of θ in the formula $f(x + \Delta x) - f(x) = f'(x + \theta\Delta x)\Delta x$ and find the limit $\lim_{\Delta x\to 0} \theta$.

Exercise 6.7. Prove that when $x \geqslant 0$, we have

$$\sqrt{x+1} - \sqrt{x} = \frac{1}{2\sqrt{x + \theta(x)}},$$

where $\frac{1}{4} \le \theta(x) \le \frac{1}{2}$, and the following properties hold.

$$\lim_{x\to 0^+} \theta(x) = \frac{1}{4}, \quad \lim_{x\to+\infty} \theta(x) = \frac{1}{2}.$$

Exercise 6.8. Let f be differentiable on the interval $[a, b]$. Prove that if $f(a)$ is the maximum value of f, then $f'_+(a) \le 0$; if $f(b)$ is the maximum value of f, then $f'_-(b) \geqslant 0$.

Exercise 6.9. Prove that for $|x| \le \frac{1}{\sqrt{2}}$, the following holds.

$$2 \arcsin x \equiv \arcsin\left(2x\sqrt{1 - x^2}\right).$$

Exercise 6.10. Let the function f be twice differentiable on the interval I, and suppose that $f''(x) \equiv 0$. What kind of function is f?

Exercise 6.11. Prove that the derivative of a differentiable function that is unbounded on a finite open interval (a, b) is also unbounded.

Exercise 6.12. Let f be differentiable on $(0, a)$ and $f(0^+) = +\infty$. Prove that $f'(x)$ does not have a lower bound on the right side of the point $x = 0$.

Exercise 6.13. Let f be continuous on $[a, b]$, differentiable on (a, b), and satisfy $f(a) = f(b)$, but f is not a constant function. Prove that there exists $\xi \in (a, b)$ such that $f'(\xi) > 0$.

Exercise 6.14. Let f be continuous on $[0, 1]$ and twice differentiable in $(0, 1)$. It is known that the line segment connecting points $A(0, f(0))$ and $B(1, f(1))$ intersects the curve $y = f(x)$ at point $C(c, f(c))$, where $0 < c < 1$. Prove that there exists a point $\xi \in (0, 1)$ such that $f''(\xi) = 0$.

Hint: Draw the picture.

Exercise 6.15. Let f be continuous on $[a, b]$ and differentiable in (a, b), with $f'(x)$ having no zeros. Prove that there exist $\xi, \eta \in (a, b)$ such that

$$\frac{f'(\xi)}{f'(\eta)} = \frac{e^b - e^a}{b - a} e^{-\eta}.$$

Hint: Consider the case that $\xi = \eta$.

Exercise 6.16. Let f be continuous on $[0, 1]$, differentiable in $(0, 1)$, with $f(0) = f(1) = 0$ and $f\left(\frac{1}{2}\right) = 1$. Prove that:

(1) There exists $\eta \in \left(\frac{1}{2}, 1\right)$ such that $f(\eta) = \eta$.
(2) For any real number λ, there exists $\xi \in (0, \eta)$ such that

$$f'(\xi) - \lambda(f(\xi) - \xi) = 1.$$

Hint: (1): IVT. (2) Consider the function $e^{-\lambda x}(f(x) - x)$.

Exercise 6.17. Let f be a differentiable function on the interval I. Prove that the necessary and sufficient condition for f' to be a constant function on I is that f is a linear function.