NOTES ON EXERCISE COURSE OF CALCULUS A

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ABSTRACT. This is the lecture note of the exercise course of Calculus A at Peking University.

CONTENTS

SOME RULES

Submission rules. The exercise course is on every Tuesday. In this course, you have to submit the homework assigned last Tuesday and the Thursday before the last Thursday. For example, suppose that you need to submit the homework on September 10, then you have to submit those assigned on September 3 and August 29.

An exception is that on October 8, you must submit the homework assignment before September 24 (including this day). After that, you should follow the above rules.

You can send your homework to the email:2201110024@stu.pku.edu.cn, or write on paper to submit it to me on the exercise course. If you want to send it by email, you should name the title of this email with "name+student ID+the number of the homework." For example, "Zhang San 0000000000 Homework 1" in Chinese.

Grade of homework. We only care about whether you submit and complete the homework or not. In particular, as long as you submit all the homework and complete all the problems assigned, you will get full marks. Regular grades only come from the submission of homework.

Some useful links. We present some useful links associated with calculus.

Lecture notes by Yantong Xie:https://darkoxie.github.io

Mathstackexchange: https://math.stackexchange.com

1. Preliminaries of calculus

1.1. **Trigonometry.** We introduce some results in trigonometry.

1.1.1. *Some basic formulae.*

$$\sin a + \sin b = 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2},$$
$$\sin a \cos b = \frac{1}{2}(\sin(a+b) + \sin(a-b)).$$

Exercise 1.1. What about $\sin a - \sin b$, $\cos a \pm \cos b$, $\sin a \sin b$, $\cos a \cos b$?

1.1.2. *Reverse functions.* Consider $f(x) = \sin x$. We see that it is increasing in $[-\pi/2, \pi/2]$ and we can define the reverse function of it in this interval by $f^{-1}(x) = \arcsin x$, where $x \in [-1, 1]$.

Exercise 1.2. Consider reverse functions of $\cos x$, $\tan x$.

1.1.3. *Euler formula.*

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

Exercise 1.3. Calculate $\sum_{k=1}^{n} \cos kx$.

1.2. **Some useful inequalities.**

1.2.1. *Cauchy's inequality.*

Theorem 1.4. *For $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \subset \mathbb{R}$, we have*

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right).$$

Proof. Let $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$. Consider $|a + tb|^2$ with $t \in \mathbb{R}$. □

Question 1.5. What is the condition such that the above inequality satisfies "="?

Proposition 1.6. *Let $a, b, c > 0$ be such that $a + b + c = 1$. We have*

$$a^3 + b^3 + c^3 \geq \frac{a^2 + b^2 + c^2}{3}.$$

Proof. Applying Theorem 1.4, we have

$$(a^2 + b^2 + c^2)^2 \leq (a^3 + b^3 + c^3)(a + b + c) = a^3 + b^3 + c^3.$$

Additionally,

$$a^2 + b^2 + c^2 \geq ab + ac + bc.$$

This implies that $3(a^2 + b^2 + c^2) = (a + b + c)^2 = 1$. Combining all the above properties, the result follows directly. □

1.2.2. *Hölder's inequality.*

Theorem 1.7. *For $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \subset \mathbb{R}_+$, we have*

$$\left(\sum_{i=1}^n a_i b_i\right) \leq \left(\sum_{i=1}^n a_i^p\right)^{1/p}\left(\sum_{i=1}^n b_i^q\right)^{1/q},$$

where $1/p + 1/q = 1$ with $p, q > 0$.

Proof. It is a corollary of Yong's inequality, that is, for $a, b > 0$, $p, q > 0$ with $1/p + 1/q = 1$, we have $ab \leq a^p/p + b^q/q$. □

1.2.3. *Bernoulli's inequality.*

Theorem 1.8. *Let $n \geq 2$. Assume that $x_1, x_2, ..., x_n > -1$ and have the same sign. Then*

$$\prod_{i=1}^{n}(1+x_i) \geq 1 + \sum_{i=1}^{n} x_i.$$

Proof. The proof is by using induction. For $n = 1$, there is nothing to prove. Assume that the result is true for n. By simple calculations, we have

$$\begin{aligned}
\prod_{i=1}^{n}(1+x_i) &= (1+x_1)\left(\prod_{i=2}^{n}(1+x_i)\right) \\
&\geq (1+x_1)\left(1+\sum_{i=2}^{n} x_i\right) \\
&= 1 + \sum_{i=1}^{n} x_i + x_1\left(\sum_{i=2}^{n} x_i\right) \\
&\geq 1 + \sum_{i=1}^{n} x_i,
\end{aligned}$$

where for the last inequality, we have used the property that x_i has the same sign. Now, we complete the proof. □

Exercise 1.9. Let $m \in \mathbb{R}$ and $x > -1$. Show that if $m \in [0, 1]$, then $(1 + x)^m \leq 1 + mx$ and if $m < 0$ or $m > 1$, then $(1 + x)^m \geq 1 + mx$.

Exercise 1.10. Show that if $b > a > 0$, and $n \in \mathbb{Z}_+$, then $a^{n+1} > b^n((n + 1)a - nb)$, and $b^{n+1} > a^n((n + 1)b - na)$.

Proposition 1.11. *The sequence $\{(1 + 1/n)^n\}$ is increasing.*

Proof. Let $b = 1 + 1/n$ and $a = 1 + 1/(n + 1)$, the result follows from Exercise 1.10. □

Exercise 1.12. Show that the sequence $\{(1 + 1/n)^{n+1}\}$ is decreasing.

1.3. **Real numbers.**

1.3.1. *Density of real numbers.* One of the most remarkable properties of the real numbers is that $\mathbb{R}$ is dense. We can see from the following proposition.

Proposition 1.13. *Let $a, b \in \mathbb{R}$. Show that there exists $r_1 \in \mathbb{Q}$ and $r_2 \in \mathbb{R}\backslash\mathbb{Q}$ such that $a < r_1 < r_2 < b$.*

Proof. Assume that $a \in \mathbb{Q}$. Let $N \gg 1$ be such that $1/N < \pi/N < b - a$, we choose $r_1 = a + 1/N$ and $r_2 = a + \pi/N$. Assume that $a \in \mathbb{R}\backslash\mathbb{Q}$, we choose $N \gg 1$ such that $N(b - a) > 10$. As a result, there must be some $n \in \mathbb{Z}_+$ such that $n \in (Na, Nb)$, and then $n/N \in (a, b)$. Taking n/N as new a and applying the previous arguments, we complete the proof. □

1.3.2. *Closeness of calculations for real numbers.*

Proposition 1.14. *There exist $a, b \in \mathbb{R}\backslash\mathbb{Q}$ such that $a^b \in \mathbf{Q}$.*

Proof. Consider $a = b = \sqrt{2}$. If $a^b \in \mathbb{Q}$, then we are done. If not, we see that $(a^b)^{\sqrt{2}} = 2$. □

Exercise 1.15. Give the corresponding examples.

- $a, b \in \mathbb{Q}$, $a^b \in \mathbb{Q}$.
- $a, b \in \mathbb{Q}$, $a^b \in \mathbb{R}\backslash\mathbb{Q}$.
- $a, b \in \mathbb{R}\backslash\mathbb{Q}$, $a + b \in \mathbb{Q}$.

2. Limits of sequences

2.1. Comparing the order.

Let $a > 0$ and $b > 1$, we have

$$\ln n \ll n^a \ll b^n \ll n! \ll n^n, \tag{2.1}$$

where we call $f(n) \ll g(n)$ if

$$\lim_{n\to+\infty} \frac{f(n)}{g(n)} = 0.$$

Now, let us show (2.1). We note that

$$0 < \frac{n!}{n^n} \le \frac{1}{n},$$

which implies that $n! \ll n^n$. Let $x_n = b^n/(n!)$. We have

$$\frac{b_{n+1}}{b_n} = \frac{b}{n+1}.$$

As a result, $b^n \ll n!$ follows from the following lemma, whose proof is left for the reader.

Lemma 2.1. *Let $\{x_n\}$ be a sequence, if*

$$\lim_{n\to+\infty} \left|\frac{x_{n+1}}{x_n}\right| < 1,$$

then $\lim_{n\to+\infty} x_n = 0$.

Denote $y_n = n^a/b^n$. By using this lemma, we obtain that $n^a \ll b^n$. Finally, we turn to the proof of $\ln n \ll n^a$. Noting that

$$\frac{\ln n}{n^a} = \frac{1}{a} \cdot \frac{\ln n^a}{n^a},$$

we only show that $\ln n/n \to 0^+$. It is a direct consequence of the following exercise.

Exercise 2.2. $\lim_{n\to+\infty} n^{1/n} = 1$.

Hint: Let $n^{1/n} = 1 + x_n$. We have

$$n = (1 + x_n)^n \ge 1 + nx_n + \frac{n(n-1)}{2} x_n.$$

By using Lemma 2.1, we can show $\lim_{n\to+\infty} n^2 q^n = 0$ (in Homework).

2.2. Basic properties on limits.

2.3. Nonexistence of limits.

Proposition 2.3. $\lim_{n\to+\infty} x_n = x$ *if and only if for any subsequence x_{n_k}, we have* $\lim_{k\to+\infty} x_{n_k} = x$.

Proof. It follows from the definition of the limit. □

We can use this proposition to show that some sequences diverge.

Exercise 2.4. $(-1)^n$ diverges.

Exercise 2.5. For $\{x_n\}$, we have $x_{2k} \to a$ and $x_{2k-1} \to b$, where $a, b \in [-\infty, +\infty]$. Show that x_n is convergent if and only if $a = b \in (-\infty, +\infty)$.

2.4. **Monotone sequence.**

Example 2.6. Let $n \in \mathbb{Z}_+$. Assume that $x_n \in [0,1]$ is the root of $\sum_{i=1}^n x^i = 1$. Show that $\lim_{n\to+\infty} x_n = \frac{1}{2}$.

Proof. It is easy to see that x_n is decreasing. On the other hand, $x_n - x_n^{n+1} = 1 - x_n$. Letting $n \to +\infty$, we see that $x_n \to \frac{1}{2}$. □

Exercise 2.7. Assume that $0 < b < a$. We let $\{a_n\}, \{b_n\}$ be defined as follows.

$$a_{n+1} = \frac{a_n + b_n}{2},\ b_{n+1} = \sqrt{a_n b_n},\ n \in \mathbb{Z}_+,$$

where $a_1 = a$, $b_1 = b$. Show that $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n$ and exist.

Hint: Show that a_n and b_n are monotone.

Exercise 2.8. $\{x_n\}$ is monotone. Show that $\{x_n\}$ is monotone if and only if it has a convergent subsequence.

2.5. **Boundedness.** Any convergent sequence is bounded.

Example 2.9. Assume that $x_n \to x > 0$. Then x_n has a lower bound but may not have a minimal point.

Proof. For n sufficiently large, we have $x_n > \frac{x}{2}$. Considering $x_n = 1 + \frac{1}{n}$, we see that it does not have a minimal point. □

Exercise 2.10. Assume that $x_n \to +\infty$, then there must be some $n_0 \in \mathbb{Z}_+$ such that $x_{n_0} = \min_{n\in\mathbb{Z}_+}\{x_n\}$.

2.6. **Some important sequence and limits.**

Example 2.11. Show that

$$S_n = \sum_{i=1}^n \frac{1}{i^p}$$

Is convergent if and only if $p > 1$.

Proof. Taking $p = 1$ and $p = 2$ for example. For $p = 1$, since

$$\ln(n+1) - \ln(n) < \frac{1}{n},$$

S_n diverges. For $p = 2$, we see that

$$\frac{1}{n^2} < \frac{1}{n-1} - \frac{1}{n},$$

and then S_n is convergent. □

Example 2.12. Consider

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}.$$

Show that $x_n \to \ln 2$.

Proof. Let

$$\gamma_n = \sum_{i=1}^n \frac{1}{i} - \ln n.$$

We can show that $\gamma_n \to \gamma$ (Euler constant). Then, the result follows directly. □

2.7. **Sandwich theorem.** Recall that if $a_n \le b_n \le c_n$ and $a_n, c_n \to x$, then $b_n \to x$.

Example 2.13. Assume that $0 < a_1 \le a_2 \le ... \le a_k$. Calculate $\lim_{n\to+\infty}(a_1^n + ... + a_k^n)^{1/n}$.

Proof. We see that

$$a_k \le (a_1^n + ... + a_k^n)^{1/n} \le k^{1/n} a_k.$$

Then $\lim_{n\to+\infty}(a_1^n + ... + a_k^n)^{1/n} = a_k$. □

Exercise 2.14. Assume that $0 < a_1 \le a_2 \le ... \le a_k$. Calculate $\lim_{n\to+\infty}(a_1^{1/n} + ... + a_k^{1/n})^n$.

Exercise 2.15. Assume that $a_k \to a > 0$. Calculate $\lim_{n\to+\infty}(a_1^n + ... + a_n^n)^{1/n}$.

2.8. **Cauchy's proposition.** It is about the limit of average for a sequence having a limit.

Proposition 2.16. *If* $\lim_{n\to+\infty} x_n = x$, *then*

$$\lim_{n\to+\infty} \frac{\sum_{i=1}^n x_i}{n} = x. \tag{2.2}$$

Proof. For any $\varepsilon > 0$, we can choose $N \in \mathbb{Z}_+$ such that $|x_n - x| < \varepsilon$ for any $n > N$. This implies that if $n > N$, there holds

$$\left|\frac{\sum_{i=1}^N x_i}{n} - x\right| \le \frac{\sum_{i=1}^N |x_i|}{n} + \frac{n-N}{n}\varepsilon.$$

Choosing lager N without changing the notation, we can assume that if $n > N$, then

$$\left|\frac{\sum_{i=1}^N x_i}{n} - x\right| < 2\varepsilon.$$

As a result, we have (2.2). □

Exercise 2.17. Assume that $a_n \to a$ and $b_n \to b$. Show that

$$\lim_{n\to+\infty} \frac{\sum_{i=1}^{n-1} a_i b_{n-i}}{n} = ab.$$

Exercise 2.18. Assume that $x_n \to x$. Show that

$$\lim_{n\to+\infty} \frac{\sum_{k=0}^n C_n^k x_k}{2^n} = x$$

Exercise 2.19 (Stolz $\frac{0}{\infty}$)**.** Let $\{a_n\}$ and $\{b_n\}$ be two sequences. Assume that $\{b_n\}$ is a strictly monotone and divergent sequence (namely, strictly increasing and approaching $+\infty$, or strictly decreasing and approaching $-\infty$) and the following limit exists

$$\lim_{n\to+\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = l.$$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l.$$

Exercise 2.20 (Stolz $\frac{0}{0}$)**.** Let $\{a_n\}$ and $\{b_n\}$ be two sequences. Assume now that $a_n, b_n \to 0$ and $\{b_n\}$ is strictly decreasing. If

$$\lim_{n\to+\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = l.$$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l.$$

2.9. **Limits with respect to the definition of e.** We know that

$$e = \lim_{n\to+\infty}(1+1/n)^n. \tag{2.3}$$

By this, we can consider related limits.

Question 2.21. What about the limit $\lim_{n\to-\infty}(1+1/n)^n$.

Example 2.22. Show that

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n = e$$

Proof. It follows from (2.3) that

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n = \lim_{n\to+\infty}\left(1+\frac{n+1}{n^2}\right)^{\frac{n^2}{n+1}\cdot\frac{n+1}{n}} = e.$$

□

Example 2.23. Calculate the following limits.

$$\lim_{n\to+\infty}\left(1-\frac{1}{n}\right)^{n^2}, \quad \lim_{n\to+\infty}\left(1-\frac{1}{n^2}\right)^n.$$

Proof. Noting that

$$\left(1-\frac{1}{n}\right)^{n^2} = \left[\left(1-\frac{1}{n}\right)^n\right]^n$$

and $\lim_{n\to+\infty}(1-1/n)^n = 1/e$, we see that the first limit is 0. On the other hand, we obtain that

$$\left(1-\frac{1}{n^2}\right)^n = \left[\left(1-\frac{1}{n^2}\right)^{n^2}\right]^{1/n}.$$

The term in $[\cdot]$ is bounded, and the limit is 1. □

Exercise 2.24. Calculate $\lim_{n\to+\infty}(1+1/n)^{n^2}$ and $\lim_{n\to+\infty}(1+1/n^2)^n$.

Exercise 2.25. If there exists $\{x_n\}$ such that $\lim_{n\to+\infty}x_n = 1$ and $\lim_{n\to+\infty}x_n^n = 1.001$.

Hint: Consider $x_n = 1+\frac{a}{n}$.

Exercise 2.26. Find $\{x_n\}$ such that the following properties hold.

(1) $\lim_{n\to+\infty}(x_n - e^n) = 0$.
(2) $\lim_{n\to+\infty}(f(x_n) - f(e^n)) \neq 0$, where $f(x) = x\ln x$.

Hint: Consider $y_n = x_n - e^n$.

2.10. **Problems associated with fixed points.** Consider a sequence $\{x_n\}$ such that $x_{n+1} = f(x_n)$. If x_n is convergent with $\lim_{n\to+\infty}x_n = x$ and f is continuous, then $f(x) = x$.

Example 2.27. Let $x_{n+1} = \sqrt{2+x_n}$ and $x_1 = 1$. Show that $\lim_{n\to+\infty}x_n = 2$.

Proof. 2 is the fixed point. We have

$$\frac{x_{n+1}-2}{x_n-2} = \frac{1}{2+\sqrt{x_n+2}}.$$

Then

$$\left|\frac{x_{n+1}-2}{x_n-2}\right| < \frac{1}{2}.$$

□

Exercise 2.28. Let $x_{n+1} = \frac{1}{x_n^{1/2}}$ and $x_1 = 2$. Show that $\lim_{n\to+\infty} x_n = 1$.

Exercise 2.29. Let $x_{n+1} = \frac{1}{x_n^2}$ and $x_1 = 2$. Show that x_n is divergent.

Exercise 2.30. Let $x_{n+1} = \frac{1}{x_n}$ and $x_1 = 2$. Show that x_n is divergent.

Question 2.31. Can you find something interesting in the above three exercises?

Exercise 2.32. $x_1 = \sqrt{2}$. $x_{n+1} = \sqrt{2}^{x_n}$. Consider $\lim_{n\to+\infty}$.

Hint: Draw the picture.

3. Limits of functions

3.1. Heine criterion.

Proposition 3.1. *Assume that $f : [a, b] \to \mathbb{R}$ is a function. For $c \in (a, b)$, $\lim_{x\to c} f(x) = A$, if and only if for any $x_n \to c$, there holds $f(x_n) \to A$ as $n \to +\infty$.*

Proof. The proof is due to the definition. □

Exercise 3.2. Use this Heine criterion and Cauchy criterion of sequences to show the Cauchy criterion of functions.

Example 3.3. Show that the Dirichlet function does not have limits at any points in $[0, 1]$. Here Dirichlet function $D(x) : [0, 1] \to \{0, 1\}$ is defined by

$$D(x) = 1 \text{ if } x \in \mathbb{Q}, \quad D(x) = 0 \text{ if } x \notin \mathbb{Q}$$

Exercise 3.4. Show that the Dirichlet function can be represented by

$$D(x) = \lim_{m\to\infty} \left\{ \lim_{n\to\infty} [\cos(\pi m! x)]^{2n} \right\}.$$

Exercise 3.5. The limit $\lim_{x\to+\infty}(a \cos x + b \sin x)$ exists. Show that $a = b = 0$.

Exercise 3.6. Show that $\lim_{x\to0^+} \sin(\frac{1}{x})$ does not exist.

3.2. Riemannian function.

Riemannian function is defined as follows.

$$R(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q}, \text{ where } p, q \in \mathbb{Z} \text{ are relatively prime}, q > 0, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Example 3.7. Consider the limit $\lim_{x\to x_0} R(x)$, where $x_0 \notin \mathbb{Q}$.

Proof. Assume that $x_0 \notin \mathbb{Q}$, we claim that $\lim_{x\to x_0} R(x) = 0$. For any $\varepsilon > 0$, we can choose $N \in \mathbb{Z}_+$ such that $\frac{1}{N} < \varepsilon$. Then there are only finite numbers in $\mathbb{Q}$ such that the representation $\frac{p}{q}$ with $p, q \in \mathbb{Z}$, $(p, q) = 1$, and $q > 0$ satisfy $q < N$. Then, the result follows. □

3.3. Some basic calculations.

Example 3.8. Calculate

$$\lim_{x\to+\infty} \left(\frac{x^2 - 1}{x^2 + 1} \right)^{\frac{x-1}{x+2}}.$$

Proof. Simple calculations imply that it is 1. □

Exercise 3.9. Calculate the following limits.

(1) $\lim_{x\to+\infty} \frac{x^k}{a^x} = 0$, $a > 1, k > 0$.
(2) $\lim_{x\to+\infty} \frac{\ln x}{x^k} = 0$, $k > 0$.
(3) $\lim_{x\to+\infty} x^{1/x} = 1$.

Hint: when $x \notin \mathbb{Z}_+$, consider $[x]$.

3.4. **Two important limits.** We know that

$$\lim_{x\to 0}\frac{\sin x}{x}=1,\quad \lim_{x\to 0}(1+x)^{\frac{1}{x}}=0. \tag{3.1}$$

Example 3.10. Calculate

$$\lim_{x\to\infty}\frac{\sin x}{x},\qquad \lim_{x\to+\infty}(1+x)^{\frac{1}{x}}.$$

Exercise 3.11. By using (3.1) to calculate the following limits.

(1) $\lim_{x\to+\infty}\left(\frac{2}{\pi}\arctan x\right)^x$;
(2) $\lim_{x\to\frac{pi}{2}-}(\sin x)^{\tan x}$;
(3) $\lim_{x\to\infty}\left(\frac{x^2-1}{x^2+1}\right)^{x^2}$;
(4) $\lim_{x\to\frac{\pi}{2}-}(\cos x)^{\frac{\pi}{2}-x}$;
(5) $\lim_{x\to 0}\frac{\sin 2x-2\sin x}{x^3}$;
(6) $\lim_{x\to 1}(1-x)\tan\left(\frac{\pi}{2}x\right)$.

Hint: You have to change the variables.

3.5. **Order analysis.** Assume that $f,g:(a-\delta,a)\cup(a+\delta)$. We have the following definitions.
$f(x)=o(g(x))$ as $x\to a$ if and only if

$$\lim_{x\to a}\frac{f(x)}{g(x)}=0.$$

$f(x)=O(g(x))$ as $x\to a$ if and only if there exists $M>0$ such that

$$\left|\frac{f(x)}{g(x)}\right|\le M.$$

Example 3.12. what is the meaning of $o(1)$ and $O(1)$?

Exercise 3.13. Consider the following limits.

(1) $\lim_{x\to 0}x\sin\frac{1}{x}$;
(2) $\lim_{x\to\infty}x\sin\frac{1}{x}$;
(3) $\lim_{x\to 0}x\sin x$;
(4) $\lim_{x\to\infty}x\sin x$;
(5) $\lim_{x\to 0}\frac{1}{x}\sin\frac{1}{x}$
(6) $\lim_{x\to\infty}\frac{1}{x}\sin\frac{1}{x}$.

Question 3.14. Please use the notation o,O to illustrate the above limits.

4. Continuous function

4.1. **Discontinuity.**

Example 4.1. Consider the following three functions and classify their continuity and discontinuity of them.

$$f(x)=\begin{cases} e^{\frac{1}{x}}, & x\in(-\infty,0),\\ x^2, & x\in[0,+\infty);\end{cases}$$

$$f(x)=\begin{cases} e^{\frac{1}{x}}, & x\in\mathbb{R}\backslash\{0\}\\ 0, & x=0\end{cases}$$

$$f(x)=\begin{cases} \frac{\ln(1+x^2)}{x}, & x\in(-1,0)\cup(0,1),\\ 2, & x=0.\end{cases}$$

Question 4.2. If the range of the function f on the interval $[a, b]$ is a closed interval, is f necessarily continuous on $[a, b]$?

4.2. **Uniqueness.**

Example 4.3. Assume that $f, g \in C(\mathbb{R})$. If $f(x) = g(x)$ for any $x \in \mathbb{Q}$, then $f \equiv g$.

Proof. By using the continuity of f and g. □

Exercise 4.4. Assume that $f \in C(\mathbb{R})$ and $f(x+y) = f(x) + f(y)$. Show that $f(x) = f(1)x$.

4.3. **Intermediate value theorem.**

Example 4.5. Assume that $0 \leq f \in C[0, 1]$ with $f(0) = f(1) = 0$. Show that for any $a \in (0, 1)$, there exists $x_0 \in [0, 1]$ such that $x_0 + a \in [0, 1]$, and

$$f(x_0) = f(x_0 + a).$$

Proof. Let $g(x) = f(x+a) - f(x)$. The result follows directly from the intermediate value theorem. □

Question 4.6. What will happen? If f does not have the property of nonnegativity.

Example 4.7. Assume that $f \in C[0, 1]$ with $f(0) = f(1)$. Show that for any $n \in \mathbb{Z}_+$, there exists ξ such that

$$f\left(\xi + \frac{1}{n}\right) = f(\xi).$$

Proof. Let $g(x) = f(x + \frac{1}{n}) - f(x)$. Consider $g(0)$, $g(\frac{1}{n})$,... $f(1)$. WLOG, assume that $g(0) > 0$. If there exists $1 \leq k \leq n$ such that $g(\frac{k}{n}) < 1$, then the result follows. If not, it contradicts $f(0) = f(1)$. □

Exercise 4.8. Assume that $f \in C[0, 1]$, and $f([0, 1]) \subset [0, 1]$. Show that the figure of $y = f(x)$ has an intersection with both $y = x$ and $y = 1 - x$.

Hint: Draw the picture, and then you will find the answer.

Exercise 4.9. Let $f_n(x) = x^n + x$ with $n \in \mathbb{Z}_+$. Show that the following properties hold.

(1) For any $n \in \mathbb{Z}_+$, the equation $f_n(x) = 1$ has a unique root in $(\frac{1}{2}, 1)$.
(2) If $c_n \in (\frac{1}{2}, 1)$ is a root of $f_n(x) = 1$ in $(\frac{1}{2}, 1)$, then

$$\lim_{n \to \infty} c_n = 1.$$

Hint: Firstly, show that c_n is increasing. If $c_n < \alpha$ for any $n \in \mathbb{Z}_+$, there is a contradiction.

4.4. **Contraction maps.**

Example 4.10. Let $f \in C(-\infty, +\infty)$. For any $x, y \in \mathbb{R}$, there holds $|f(x) - f(y)| \leq k|x - y|$ with $0 < k < 1$. Show the following properties.

(1) The function $kx - f(x)$ is increasing.
(2) There exists unique $\xi \in \mathbb{R}$, such that $f(\xi) = \xi$.

Hint: Consider the sequence $x_{n+1} = f(x_n)$. Show that x_n is a Cauchy sequence.

Exercise 4.11. Let the function f be defined on the domain $[a, b]$, with the image set $f([a, b]) \subset [a, b]$, and satisfying $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in [a, b]$. For any $x_0 \in [a, b]$, define the sequence recursively by

$$x_{n+1} = \frac{1}{2}(x_n + f(x_n)).$$

Show that the sequence $\{x_n\}$ is convergent.

Hint: Use the MCT.

Exercise 4.12. Let $f \in C[0, +\infty)$. Assume that $f(0) = 0$ and $0 < f(x) < x$ when $x > 0$. For $x_{n+1} = f(x_n)$, show that $\lim_{n\to+\infty} x_n = 0$.

Hint: Draw the picture.

4.5. Boundedness, maximum and minimum.

Example 4.13. Let the function f have only first-type discontinuities on the interval $[a, b]$. Prove that f is bounded on $[a, b]$.

Proof. Prove by the definition. □

Exercise 4.14. If $f \in C[a, +\infty)$ and there exists a finite limit $\lim_{x\to+\infty} f(x)$, prove that f is bounded on $[a, +\infty)$.

Exercise 4.15. If $f \in C[a, +\infty)$ and there exists a finite limit $\lim_{x\to+\infty} f(x)$, prove that f can attain at least either a maximum or a minimum value on $[a, +\infty)$.

Hint: First consider the behavior of f on $[a, a + N]$, $N > 1$.

Exercise 4.16. If $f \in C(a, b)$ and $f(a^+) = f(b^-) = +\infty$, show that f has a minimum value on (a, b).

Exercise 4.17. If $f \in C(a, b)$ and $f(a^+) = f(b^-)$, prove that f can attain at least either a maximum or a minimum value on (a, b).

Exercise 4.18. If $f \in C(-\infty, +\infty)$, $\lim_{x\to\pm\infty} f(x) = +\infty$, and the minimum value of $f(x)$ is $f(a) < a$, prove that the composite function $f(f(x))$ attains its minimum value at least at two points.

Hint: Draw the picture.

4.6. Monotone functions.

Exercise 4.19. Let $f \in C[a, b]$, and for any two rational numbers r_1, r_2 in $[a, b]$ with $r_1 < r_2$, it holds that $f(r_1) \le f(r_2)$. Prove that the function f is monotonically increasing on $[a, b]$.

Exercise 4.20. Let f be a monotonically increasing function on $(-\infty, +\infty)$, and define $g(x) = f(x^+)$ at each point. Prove that g is a function that is right-continuous everywhere on $(-\infty, +\infty)$.

Exercise 4.21. Let f be a monotonically increasing function on $(\infty, +\infty)$, and for all $x, y \in \mathbb{R}$, it satisfies $f(x + y) = f(x) + f(y)$. Prove that $f(x) = f(1)x$.

Exercise 4.22. Let $f \in C[a, b]$. Prove that a necessary and sufficient condition for f to have no extreme points in (a, b) is that f is strictly monotonically increasing on $[a, b]$.

5. Differentiation

5.1. Definitions.

Example 5.1. Given that the even function $f(x)$ is differentiable at the point $x = 0$, find $f'(0)$.

Proof. Considering $f'(0-)$ and $f'(0+)$, it is easy to see that $f'(0) = 0$. □

Exercise 5.2. Let $p(x)$ be an n-degree polynomial with n roots, and denote its distinct roots as $x_1, \cdots, x_k$, where the multiplicity of root x_i is n_i for $i = 1, 2, \cdots, k$, and $n_1 + n_2 + \cdots + n_k = n$. Prove that the following holds.

$$p'(x) = p(x)\left(\sum_{i=1}^{k} \frac{n_i}{x - x_i}\right)$$

Example 5.3. Let the function be defined as

$$f(x) = \begin{cases} \frac{x}{1+e^{1/x}}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Study the differentiability of $f(x)$.

Proof. Consider $\frac{f(x)-f(0)}{x}$. □

Exercise 5.4. Let n be a natural number. Under what conditions is the function

$$f(x) = \begin{cases} x^n \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Study the differentiability of $f(x)$.

Example 5.5. Let the function $f(x)$ satisfy the functional equation $f(x+y) = f(x)f(y)$, and it is known that $f'(0) = 1$. Prove that $f(x)$ is differentiable everywhere and that $f'(x) = f(x)$ holds.

Proof. The assumption that $f(x+y) = f(x)f(y)$ implies that the differentiability can be spread. □

Exercise 5.6. Suppose in a neighborhood U of the origin, we have $|f(x)| \leq |g(x)|$, and $g(0) = g'(0) = 0$. Find $f'(0)$.

Hint: Draw the picture.

Exercise 5.7. Assume that $f \in C[a, b]$. Show that if there is some $\alpha, M > 1$ such that for any $x, y \in [a, b]$, $|f(x) - f(y)| \leq M|x-y|^\alpha$, then $f \equiv \text{const}$.

Hint: Show that $f' \equiv 0$.

5.2. **Some tricks.**

Example 5.8. Let $f(0) = 0$ and $f'(0)$ exist. Define the sequence

$$x_n = f\left(\frac{1}{n^2}\right) + f\left(\frac{2}{n^2}\right) + \cdots + f\left(\frac{n}{n^2}\right), \quad n \in \mathbf{N}_+$$

Try to find $\lim_{n\to\infty} x_n$.

Proof. It is easy to find that

$$f\left(\frac{k}{n^2}\right) = f'(0)\frac{k}{n^2} + o\left(\frac{k}{n^2}\right).$$

Then we can sum k up from 1 to n, and find that $x_n \to \frac{1}{2}f'(0)$. □

Exercise 5.9. Find the limits of the following sequences.

(1) $\lim_{n\to\infty} \left(\sin\frac{1}{n^2} + \sin\frac{2}{n^2} + \cdots + \sin\frac{n}{n^2}\right)$.
(2) $\lim_{n\to\infty} \left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\cdots\left(1+\frac{n}{n^2}\right)\right]$.

Hint: Use the previous limit.

Exercise 5.10. Let $y = \frac{1+x}{\sqrt{1-x}}$, calculate $y^{(n)}(x)$, where $n \in \mathbb{N}_+$.

Hint: Take ln.

Exercise 5.11. Let f have derivatives of all orders on $\mathbf{R}$. Prove that for each natural number n the following holds.

$$\frac{1}{x^{n+1}} f^{(n)}\left(\frac{1}{x}\right) = (-1)^n \left[x^{n-1} f\left(\frac{1}{x}\right)\right]^{(n)}.$$

Exercise 5.12. Using the sum of the series $1 + x + x^2 + \cdots + x^n$, find the sums of the following.

(1) $1 + 2x + 3x^2 + \cdots + nx^{n-1}$.
(2) $1^2 + 2^2x + 3^2x^2 + \cdots + n^2x^{n-1}$.

,

6. Indefinite integral

6.1. Pairing method.

Example 6.1. Calculate $\int \frac{\mathrm{d}x}{1+x^4}$.

Consider $\int \frac{x^2\mathrm{d}x}{1+x^4}$.

Exercise 6.2. Calculate the following indefinite integrals.

(1) $\int \frac{\mathrm{d}x}{1+x^2+x^4}$.
(2) $\int \frac{e^x\mathrm{d}x}{e^x+e^{-x}}$.
(3) $\int \frac{b\sin x+a\cos x}{a\sin x+b\cos x}\mathrm{d}x \quad (a \neq b)$.
(4) $\int \frac{\mathrm{d}x}{1+x^3}$.

Hint: (1) Consider $\int \frac{x^2}{1+x^2+x^4}\mathrm{d}x$. (2) Consider $\int \frac{e^{-x}}{e^x+e^{-x}}\mathrm{d}x$. (3) Consider $\int \frac{\sin x}{a\sin x+b\cos x}\mathrm{d}x$ and $\int \frac{\cos x}{a\sin x+b\cos x}\mathrm{d}x$. Consider $\int \frac{x}{1+x^3}\mathrm{d}x$.

Exercise 6.3. Calculate

$$\int \left(\cos^4 x - \sin^4 x\right)\mathrm{d}x \qquad \int \left(\cos^4 x + \sin^4 x\right)\mathrm{d}x$$

and

$$\int \cos^4 x\mathrm{d}x \qquad \int \sin^4 x\mathrm{d}x.$$

6.2. By iterations.

Example 6.4. Calculate $\int \frac{\mathrm{d}x}{(1+x^2)^n}$

Using integration by parts, we have

$$\int \frac{\mathrm{d}x}{(1+x^2)^n} = \frac{x}{(1+x^2)^n} + \int \frac{2nx^2\mathrm{d}x}{(1+x^2)^{n+1}}.$$

Exercise 6.5. Calculate

$$\int \sin^m x\cos^n x\mathrm{d}x \qquad \int \tan^n x\mathrm{d}x,$$

where $m, n \in \mathbb{Z}_+$.

6.3. Some special integrals.

Exercise 6.6 (Rational functions)**.** Calculate the following indefinite integrals.

(1) $\int \frac{x^5-x}{1+x^8}\mathrm{d}x$.
(2) $\int \frac{\mathrm{d}x}{x(x^n+a)} (a \neq 0)$.
(3) $\int \frac{x\mathrm{d}x}{(x+1)(x^2+3)}$.
(4) $\int \frac{x-1}{(x^2+2x+3)^2}\mathrm{d}x$.
(5) $\int \frac{1+x+x^2}{(x-2)^{10}}\mathrm{d}x$.
(6) $\int \frac{x^9+1}{x^3-5x^2+6x}\mathrm{d}x$.
(7) $\int \frac{\mathrm{d}x}{1+x^6}$.
(8) $\int \frac{x^2-x+3}{(x^2+x+1)(x-1)^2}\mathrm{d}x$.

Exercise 6.7 (Triangular functions)**.** Calculate the following indefinite integrals.

(1) $\int \frac{\mathrm{d}x}{1+\cos x}$.
(2) $\int \frac{\mathrm{d}x}{\sin x \cos^4 x}$.
(3) $\int \frac{\mathrm{d}x}{2+\tan^2 x}$.
(4) $\int \tan^5 x \sec^3 x \mathrm{d}x$.
(5) $\int \frac{\sec x}{(1+\sec x)^2} \mathrm{d}x$.
(6) $\int \frac{\sin^2 x \cos x}{\sin x+\cos x} \mathrm{d}x$.
(7) $\int \frac{\sin x \cos x}{1+\sin^4 x} \mathrm{d}x$.
(8) $\int \frac{\mathrm{d}x}{\sin^4 x \cos^2 x}$.

Example 6.8. Calculate the Poisson integral $\int \frac{1-r^2}{1-2r\cos x+r^2}\mathrm{d}x \quad (-1<r<1)$.

Proof. We have

$$\begin{aligned}\int \frac{1-r^2}{1-2r\cos x+r^2}\mathrm{d}x &= \int \frac{1-r^2}{(1+r)^2-4r\cos^2\frac{x}{2}}\mathrm{d}x \\ &= \int \frac{1-r^2}{(1-r)^2\cos^2\frac{x}{2}+(1+r)^2\sin^2\frac{x}{2}}\mathrm{d}x \\ &= \int \frac{2(1-r^2)\mathrm{d}(\tan\frac{x}{2})}{(1-r)^2+(1+r)^2\tan^2\frac{x}{2}}.\end{aligned}$$

□

Exercise 6.9 (Other types)**.** Calculate the following indefinite integrals.

(1) $\int \frac{dx}{\sqrt{\sin x \cos 7x}}$
(2) $\int \frac{\mathrm{d}x}{x\sqrt{2x^2+3}}$
(3) $\int \sqrt{\tan^2 x+2}\mathrm{d}x$.
(4) $\int \frac{x\mathrm{e}^{\mathrm{x}}\mathrm{d}x}{\sqrt{1+\mathrm{e}^x}}$.
(5) $\int \frac{\mathrm{d}x}{\sqrt{(x-1)^3(x-2)}}$.
(6) $\int \frac{\sqrt{x}-1}{\sqrt[3]{x}+1}\mathrm{d}x$.

7. Definite integral

7.1. Calculate limits.

Exercise 7.1. Calculate

$$\lim_{n\to+\infty}\left(\frac{1}{n+1}+...+\frac{1}{2n}\right).$$

Exercise 7.2. Calculate

$$\lim_{n\to+\infty}\left(\frac{1}{n^2}\sum_{k=1}^{n}k\ln(n+k)-\frac{n+1}{2n}\ln n\right).$$

Hint: Note that $\sum_{k=1}^n k = \frac{n(n+1)}{2}$.

Exercise 7.3. Assume that $f \in R[a,b]$. Calculate

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}(-1)^{k-1}f\left(\frac{k}{n}\right).$$

Hint: WLOG, assume that $n = 2m$.

$$\frac{1}{2m}\sum_{k=1}^{2m}(-1)^{k-1}f\left(\frac{k}{2m}\right) = \frac{1}{2m}\sum_{k=1}^{2m}f\left(\frac{k}{2m}\right) - \frac{1}{2m}\sum_{k=1}^{m}f\left(\frac{k}{m}\right).$$

7.2. **Divide into two parts.**

Example 7.4. Show that

$$\lim_{n\to+\infty}\int_0^{\frac{\pi}{2}}\sin^n x\mathrm{d}x = 0.$$

Proof. Consider $(0, \frac{\pi}{2}-\delta)$ and $(\frac{\pi}{2}-\delta, \frac{\pi}{2})$. □

Exercise 7.5. Let $f \in C[-1, 1]$. Show that

$$\lim_{h\to 0^+}\int_{-1}^{1}\frac{h}{h^2+x^2}f(x)\mathrm{d}x = \pi f(0).$$

Hint: Consider intervals $(-1, -\delta)$, $(-\delta, \delta)$, and $(\delta, 1)$.

7.3. **Approximation and Riemann-Lebesgue Lemma.**

Example 7.6. Show that integrals can be approximated by step functions and continuous functions.

Draw the picture.

Exercise 7.7. Assume that $f \in R[0, \pi]$. Show that

$$\lim_{n\to+\infty}\int_0^{\pi}f(x)\sin nx\mathrm{d}x = 0.$$

7.4. **Calculations by symmetry.**

Exercise 7.8. Using symmetry, calculate the following problems.

(1) $I_1 = \int_0^{\pi}\frac{x\,\mathrm{d}x}{1+\cos^2 x}$
(2) $I_2 = \int_0^{1}\frac{x}{\mathrm{e}^x+\mathrm{e}^{1-x}}\,\mathrm{d}x$
(3) $I_3 = \int_{-2}^{2}x\ln(1+\mathrm{e}^x)\,\mathrm{d}x$
(4) $I_4 = \int_0^{\pi/4}\ln(1+\tan x)\,\mathrm{d}x$
(5) $I_5 = \int_0^{\pi/2}\frac{\sin^n x}{\sin^n x+\cos^n x}\,\mathrm{d}x$
(6) $I_6 = \int_0^{\pi}\frac{a^n\sin^2 x+b^n\cos^2 x}{a^{2n}\sin^2 x+b^{2n}\cos^2 x}\,\mathrm{d}x$

Exercise 7.9. Let $f \in C[0, a], a > 0$.

(1) If $f(x) + f(a-x) \neq 0$ on $[0, a]$, calculate $I = \int_0^a\frac{f(x)}{f(x)+f(a-x)}\,\mathrm{d}x$;
(2) If $f(x)f(a-x) \equiv 1$ on $[0, a]$, calculate $I = \int_0^a\frac{\mathrm{d}x}{1+f(x)}$.

Exercise 7.10. Let f be a continuous function and a, b be real numbers. Prove the following equalities.

(1) $\int_0^{\pi}xf(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi}f(\sin x)\,\mathrm{d}x$;
(2) $\int_1^a f\left(x^2+\frac{a^2}{x^2}\right)\frac{\mathrm{d}x}{x} = \int_1^a f\left(x+\frac{a^2}{x}\right)\frac{\mathrm{d}x}{x}$;
(3) $\int_0^{2\pi}f(a\cos x+b\sin x)\,\mathrm{d}x = 2\int_0^{\pi}f\left(\sqrt{a^2+b^2}\cos x\right)\,\mathrm{d}x$.

7.5. **Mean value theorem.**

Example 7.11. Let f be a non-negative strictly increasing function that is continuous on the interval $[a,b]$. By the Mean Value Theorem for integrals, for each $p>0$, there exists a unique $x_p\in(a,b)$ such that

$$f^p\left(x_p\right)=\frac{1}{b-a}\int_a^b f^p(t)\mathrm{d}t$$

Find $\lim_{p\to\infty}x_p$.

Proof. WLOG, we assume that $a=0$ and $b=1$. Then

$$f(x_p)=\left(\int_0^1 f^p(t)\mathrm{d}t\right)^{\frac{1}{p}}.$$

Letting $p\to+\infty$, the RHS goes to $f(1^-)$. Thus, $x_p\to 1$. □

Exercise 7.12. Let $f\in C[a,b]$, and let $g(x)$ be a periodic function with period T that is nonnegative integrable on $[0,T]$. Prove that:

$$\lim_{n\to\infty}\int_a^b f(x)g(nx)\mathrm{d}x=\frac{1}{T}\int_a^b f(x)\mathrm{d}x\int_0^T g(x)\mathrm{d}x.$$

Proof. Assume that $b-a=N\frac{T}{n}+c_n(a,b,T)$, where $N\in\mathbb{Z}_+$ and $0\le c_n(a,b,T)<\frac{T}{n}$. We have

$$\begin{aligned}\int_a^b f(x)g(nx)\mathrm{d}x&=\sum_{i=1}^N\int_{a+\frac{kT}{n}}^{a+\frac{(k+1)T}{n}}f(x)g(nx)\mathrm{d}x+\int_{b-c_n(a,b,T)}^b f(x)g(nx)\mathrm{d}x\\&=f(\xi_k)\int_{a+\frac{k(b-1)}{n}}^{a+\frac{(k+1)T}{n}}g(nx)\mathrm{d}x+\int_{b-c_n(a,b,T)}^b f(x)g(nx)\mathrm{d}x,\end{aligned}$$

where for the last equality, we have used the mean value theorem of integration. Then, we are done. □

7.6. **Some inequalities.**

Proposition 7.13 (Cauchy-Schwartz)**.** *Let $f,g\in R[a,b]$. Then*

$$\left(\int_a^b f(x)g(x)\mathrm{d}x\right)^2\le\int_a^b f^2(x)\mathrm{d}x\int_a^b g^2(x)\mathrm{d}x.$$

Hint: Consider

$$\int_a^b(f(x)-tg(x))^2\mathrm{d}x.$$

Example 7.14. $f\in C^1[a,b]$ and let $f(a)=0$. Prove that

$$\int_a^b f^2(x)\mathrm{d}x\le\frac{(b-a)^2}{2}\int_a^b\left(f'(x)\right)^2\mathrm{d}x.$$

Exercise 7.15. It is given that the function $f\in C[a,b]$ and $f(x)>0$ for all x. Prove that

$$\int_a^b f(x)\mathrm{d}x\int_a^b\frac{1}{f(x)}\mathrm{d}x\ge(b-a)^2.$$

Exercise 7.16. Given a non-negative function $f\in R[a,b]$ with $\int_a^b f(x)\mathrm{d}x=1$ and k being a real number, prove that

$$\left(\int_a^b f(x)\cos kx\mathrm{d}x\right)^2+\left(\int_a^b f(x)\sin kx\mathrm{d}x\right)^2\le 1.$$

Proposition 7.17 (Young)**.** *Let f be continuous and differentiable on $[0,\infty)$ and strictly increasing, with $f(0)=0$ and $a,b>0$. Then*

$$ab \le \int_0^a f(x)\mathrm{d}x + \int_0^b g(y)\mathrm{d}y.$$

where $g(y)$ is the inverse function of $f(x)$, and equality holds if and only if $b=f(a)$.

Hint: Draw the picture.

7.7. **Combination of previous topics.**

Exercise 7.18. Calculate the following asymptotic equality

$$\int_0^1 \frac{x^{n-1}}{1+x}\mathrm{d}x = \frac{a}{n} + \frac{b}{n^2} + o\left(\frac{1}{n^2}\right) \quad (n\to\infty)$$

for the undetermined constants a and b.

Hint: By using integration by parts.

Exercise 7.19. Prove that for every natural number n,

$$\frac{2}{3}n\sqrt{n} < 1+\sqrt{2}+\cdots+\sqrt{n} < \frac{4n+3}{6}\sqrt{n}.$$

Exercise 7.20. Let f be a non-increasing function on $[1,+\infty)$. Define

$$a_n = \sum_{k=1}^{n} f(k) - \int_1^n f(x)\mathrm{d}x, \quad n\in\mathbb{N}_+.$$

Prove that the sequence $\{a_n\}$ converges.

Exercise 7.21. Calculate

$$\lim_{n\to\infty}\left(\frac{\sin\pi/n}{n+1} + \frac{\sin 2\pi/n}{n+\frac{1}{2}} + \cdots + \frac{\sin\pi}{n+\frac{1}{n}}\right).$$

Hint: First consider the case

$$\lim_{n\to\infty}\left(\frac{\sin\pi/n}{n} + \frac{\sin 2\pi/n}{n} + \cdots + \frac{\sin\pi}{n}\right).$$

Exercise 7.22. Assume that $f\in C^1[0,a]$, where $a>0$. Prove that

$$|f(0)| \le \frac{1}{a}\int_0^a |f(x)|\mathrm{d}x + \int_0^a |f'(x)|\mathrm{d}x.$$

Hint: WLOG, we assume that $a=1$. We have

$$f(0) - \int_0^1 f = \int_0^1 (f-f(0)) = \int_0^1 \left(\int_0^x f'(t)\mathrm{d}t\right)\mathrm{d}x.$$

Exercise 7.23. Assume that $f\in C^1[0,1]$. Prove that

$$\int_0^1 |f(x)|\mathrm{d}x \le \max\left\{\int_0^1 |f'(x)|\mathrm{d}x, \left|\int_0^1 f(x)\mathrm{d}x\right|\right\}.$$

Hint: If $f\ge 0$ or $f\le 0$ in $[0,1]$, there is nothing to prove. Otherwise, we can apply the N-L formula.

8. Review of the midterm exam

Exercise 8.1. Let

$$x_n = (1+q)(1+q^2)\cdots(1+q^n),$$

where the constant $q \in [0,1)$. Prove that the sequence $\{x_n\}$ has a limit.

Exercise 8.2. Let

$$x_1 > 0, \quad x_{n+1} = \frac{3(1+x_n)}{3+x_n} \quad (n = 1, 2, \cdots).$$

Find

$$\lim_{n\to+\infty} x_n.$$

Exercise 8.3. Find the limit

$$\lim_{x\to+\infty} \frac{1}{x}\int_0^x (t-[t])\mathrm{d}x.$$

Hint: For x, consider $[x]$.

Exercise 8.4. Calculate

$$\int_0^\pi \sqrt{1-\sin x}\mathrm{d}x.$$

Hint: Note that $1-\sin x = (\sin\frac{x}{2} - \cos\frac{x}{2})^2$.

Exercise 8.5. Calculate

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^4 x}{1+e^{-x}}\mathrm{d}x.$$

Hint: Consider the change of variables $y = -x$.

Exercise 8.6. Let the function $y = f(x)$ be differentiable on the interval $[a, b]$ and strictly increasing. Let $g(y)$ be the inverse function of $f(x)$. Prove the following formula:

$$\int_a^b f(x)\mathrm{d}x = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\mathrm{d}y.$$

Hint: By the change of variables and the relation of derivatives for functions and their reverse functions.

Exercise 8.7. Let $f(x)$ be continuous on $(-\infty, +\infty)$. Find

$$\frac{d}{dx}\int_0^x t^2 f\left(x^3 - t^3\right)\mathrm{d}t.$$

Hint: By change of variables.

Exercise 8.8. 33. Let the function $f(x)$ be continuous on $[0, +\infty)$, monotonically increasing, and differentiable on $(0, +\infty)$.

(1) Prove that for any $x > 0$,

$$f(x) \geq \frac{1}{x}\int_0^x f(t)\mathrm{d}t.$$

(2) Define $F(x) = \frac{1}{x}\int_0^x f(t)\mathrm{d}t$. Prove that $F(x)$ is monotonically increasing on $(0, +\infty)$.
(3) Find $\lim_{x\to 0^+} F(x)$.

Exercise 8.9. Let $f(x)$ and $g(x)$ be continuous on $[a, b]$.

(1) If $f(x) \geqslant 0$ for $x \in [a,b]$, prove that $F(x) = \int_a^x f(t)\mathrm{d}t$ is monotonically increasing on $[a,b]$. Furthermore, if $F(b) = \int_a^b f(t)\mathrm{d}t = 0$, prove that $F(x) = 0$ and $f(x) = 0$ for $x \in [a,b]$.
(2) If $f(x) \geqslant g(x)$ and $f(x) \not\equiv g(x)$ for $x \in [a,b]$, prove that

$$\int_a^b f(x)\mathrm{d}x > \int_a^b g(x)\mathrm{d}x.$$

Exercise 8.10. Calculate

$$\int \frac{\mathrm{d}x}{x\sqrt{3x^2-2x-1}}.$$

Hint: Let $x = \frac{1}{t}$.

9. Mean value theorem of derivatives

Exercise 9.1. Let f be continuous on $[a,b]$, differentiable on (a,b), and with $0 < a < b$. Prove that there exists $\xi \in (a,b)$ such that

$$f(b) - f(a) = \ln\frac{b}{a} \cdot \xi f'(\xi).$$

Hint: Consider $f(e^x)$.

Exercise 9.2. Let f be differentiable on $[a,b]$. Prove that there exists $\xi \in (a,b)$ such that

$$2\xi[f(b) - f(a)] = \left(b^2 - a^2\right) f'(\xi).$$

Hint: Consider $f(x^{1/2})$.

Example 9.3. Let f, g be continuous on $[a,b]$ and differentiable on (a,b), with $g(x)$ having no zeros in the interval (a,b). Prove that there exists $\xi \in (a,b)$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(\xi) - f(a)}{g(b) - g(\xi)}.$$

Proof. Consider the function

$$F(x) = -f(x)g(x) + g(b)f(x) + f(a)g(x).$$

It is easy to see that $F(a) = g(b)f(a) = F(b)$. Then the result follows from Rolle's theorem. ☐

Exercise 9.4. Let f be continuous on $[a,+\infty)$ and differentiable on $(a,+\infty)$, with $\lim_{x\to+\infty} f(x) = f(a)$. Prove that there exists $\xi > a$ such that $f'(\xi) = 0$.

Exercise 9.5. Let $f(x)$ be differentiable on $[0,+\infty)$ and satisfy $0 \le f(x) \le \frac{x}{1+x^2}$. Prove that there exists $\xi > 0$ such that

$$f'(\xi) = \frac{1-\xi^2}{(1+\xi^2)^2}.$$

Hint: Consider $f(x) - \frac{x}{1+x^2}$.

Exercise 9.6. For (1) $f(x) = ax^2 + bx + c(a \neq 0)$ and (2) $f(x) = \frac{1}{x}(x > 0)$, calculate the value of θ in the formula $f(x + \Delta x) - f(x) = f'(x + \theta\Delta x)\Delta x$ and find the limit $\lim_{\Delta x\to 0}\theta$.

Exercise 9.7. Prove that when $x \ge 0$, we have

$$\sqrt{x+1} - \sqrt{x} = \frac{1}{2\sqrt{x+\theta(x)}},$$

where $\frac{1}{4} \le \theta(x) \le \frac{1}{2}$, and the following properties hold.

$$\lim_{x\to 0^+}\theta(x) = \frac{1}{4}, \quad \lim_{x\to+\infty}\theta(x) = \frac{1}{2}.$$

Exercise 9.8. Let f be differentiable on the interval $[a, b]$. Prove that if $f(a)$ is the maximum value of f, then $f'_+(a) \leq 0$; if $f(b)$ is the maximum value of f, then $f'_-(b) \geq 0$.

Exercise 9.9. Prove that for $|x| \leq \frac{1}{\sqrt{2}}$, the following holds.

$$2 \arcsin x \equiv \arcsin \left(2x\sqrt{1-x^2}\right).$$

Exercise 9.10. Let the function f be twice differentiable on the interval I, and suppose that $f''(x) \equiv 0$. What kind of function is f?

Exercise 9.11. Prove that the derivative of a differentiable function that is unbounded on a finite open interval (a, b) is also unbounded.

Exercise 9.12. Let f be differentiable on $(0, a)$ and $f(0^+) = +\infty$. Prove that $f'(x)$ does not have a lower bound on the right side of the point $x = 0$.

Exercise 9.13. Let f be continuous on $[a, b]$, differentiable on (a, b), and satisfy $f(a) = f(b)$, but f is not a constant function. Prove that there exists $\xi \in (a, b)$ such that $f'(\xi) > 0$.

Exercise 9.14. Let f be continuous on $[0, 1]$ and twice differentiable in $(0, 1)$. It is known that the line segment connecting points $A(0, f(0))$ and $B(1, f(1))$ intersects the curve $y = f(x)$ at point $C(c, f(c))$, where $0 < c < 1$. Prove that there exists a point $\xi \in (0, 1)$ such that $f''(\xi) = 0$.

Hint: Draw the picture.

Exercise 9.15. Let f be continuous on $[a, b]$ and differentiable in (a, b), with $f'(x)$ having no zeros. Prove that there exist $\xi, \eta \in (a, b)$ such that

$$\frac{f'(\xi)}{f'(\eta)} = \frac{e^b - e^a}{b - a} e^{-\eta}.$$

Hint: Consider the case that $\xi = \eta$.

Exercise 9.16. Let f be continuous on $[0, 1]$, differentiable in $(0, 1)$, with $f(0) = f(1) = 0$ and $f\left(\frac{1}{2}\right) = 1$. Prove that:

(1) There exists $\eta \in \left(\frac{1}{2}, 1\right)$ such that $f(\eta) = \eta$.
(2) For any real number λ, there exists $\xi \in (0, \eta)$ such that

$$f'(\xi) - \lambda(f(\xi) - \xi) = 1.$$

Hint: (1): IVT. (2) Consider the function $e^{-\lambda x}(f(x) - x)$.

Exercise 9.17. Let f be a differentiable function on the interval I. Prove that the necessary and sufficient condition for f' to be a constant function on I is that f is a linear function.

Proposition 9.18 (Darboux)**.** *The derivative of a function satisfies IVT.*

Hint: Consider the function $F(x) = f(x) - \eta x$.

10. L'Hospital's rule

Exercise 10.1. $\delta_n = e - \left(1 + \frac{1}{n}\right)^n$. Calculate the limit $\lim_{n\to+\infty} n\delta_n$.

Hint: Consider the limit

$$\lim_{x\to 0} \frac{e - (1 + x)^{\frac{1}{x}}}{x}.$$

Exercise 10.2. The following limits of functions should not be evaluated using L'Hospital's Rule. Why?

(1) $\lim_{x\to 1} \frac{x}{x+1}$.

(2) $\lim_{x\to+\infty} \frac{x+\sin x}{x-\sin x}$.
(3) $\lim_{x\to+\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}}$.
(4) $\lim_{x\to+\infty} \frac{x}{\sqrt{1+x^2}}$.

Exercise 10.3. Find the following limits.

(1) $\lim_{x\to 0} \frac{a^{x^2}-b^{x^2}}{(a^x-b^x)^2}$.
(2) $\lim_{x\to+\infty} \left[\left(x^3-x^2+\frac{x}{2}\right)e^{\frac{1}{x}}-\sqrt{1+x^6}\right]$.
(3) $\lim_{x\to 0^+} \left(\ln\frac{1}{x}\right)^x$.
(4) $\lim_{x\to+\infty} \left(\frac{\pi}{2}-\arctan x\right)^{\frac{1}{\ln x}}$.

11. Taylor expansion and further applications of MVT

Example 11.1. Let the function f be defined in the neighborhood (x_0-r, x_0+r) and be n-times differentiable at $x=x_0$. Let $P_n^T(x)$ be the n-th Taylor polynomial of $f(x)$ at $x=x_0$. For any other n-th degree polynomial $P_n(x)$, prove that there exists a number $\delta\in(0,r)$, depending on the choice of $P_n(x)$, such that

$$|f(x)-P_n^T(x)|\le |f(x)-P_n(x)|$$

holds for all $x\in(x_0-\delta, x_0+\delta)$.

Proof. Let $P_n^T(x)=\sum_{i=0}^n a_i x^i$ and $P_n(x)=\sum_{i=0}^n b_i x^i$. WLOG, we assume that $P_n^T(x)\not\equiv P_n(x)$. Choose $i_0\in\{0,1,2,...,n-1\}$ such that $a_i=b_i$ when $i\le i_0$, and $a_i\ne b_i$ when $i>i_0$. By the Peano's remainder of Taylor expansion, we have

$$\lim_{x\to 0}\frac{|f(x)-P_n^T(x)|}{|x|^{n+1}}=0$$

and

$$\lim_{x\to 0}\frac{\left|f(x)-\sum_{i=1}^{i_0} b_i x^i\right|}{|x|^{i_0+1}}=0.$$

By the choice of i_0, we have

$$\lim_{x\to 0}\frac{|f(x)-P_n(x)|}{|x|^{i_0+1}}=|a_{i_0+1}-b_{i_0+1}|.$$

Then we are done. □

Example 11.2. Let the function $f(x)$ be differentiable on $[a,+\infty)$, and suppose that the limit $\lim_{x\to\infty} f(x)$ exists and is finite. Answer the following questions:

(1) Provide a counterexample to show that $\lim_{x\to\infty} f'(x)=0$ does not always hold.
(2) If $f(x)$ is twice differentiable and $f''(x)$ is bounded on $[a,+\infty)$, prove that $\lim_{x\to\infty} f'(x)=0$.

Proof. For the first point, we consider the function

$$f'(x)=\begin{cases}\text{linear function} & x\in[n,n+n^{-100}],\\ 0 & \text{otherwise.}\end{cases}$$

By Taylor expansion with Lagrange's remainder, we have

$$f(x+t)=f(x)+f'(x)t+\frac{t^2}{2}f''(\xi_{x,t}),$$

where $\xi_x\in[x,x+t]$. □

Example 11.3. Let f be twice differentiable on $(0,+\infty)$, and suppose that

$$M_0 = \sup\{|f(x)| \mid x \in (0,+\infty)\}, \quad \text{and} \quad M_2 = \sup\left\{\left|f''(x)\right| \mid x \in (0,+\infty)\right\}$$

are finite numbers. Prove that $M_1 = \sup\{|f'(x)| \mid x \in (0,+\infty)\}$ is also finite and satisfies the inequality

$$M_1 \le 2\sqrt{M_0 M_2}.$$

Proof. Write

$$f(x+t) = f(x) + f'(x)t + \frac{f''(\xi)}{2}t^2,$$

where $x, t > 0$ and $\xi \in (x, x+t)$. From this, we have the estimate

$$\left|tf'(x)\right| \le \left|f(x+t) - f(x) - \frac{t^2}{2}f''(\xi)\right| \le 2M_0 + \frac{t^2}{2}M_2.$$

Thus, we obtain

$$\left|f'(x)\right| \le \frac{2M_0}{t} + \frac{1}{2}tM_2.$$

This holds for every $x \in (0,+\infty)$. Taking the supremum, we get

$$M_1 \le \frac{2M_0}{t} + \frac{t}{2}M_2.$$

Therefore, M_1 is a finite number. Since this holds for every $t > 0$, to get the best estimate, we can take $t = 2\sqrt{M_0/M_2}$ to minimize the expression on the right, yielding

$$M_1 \le 2\sqrt{M_0 M_2}.$$

□

Example 11.4. Let f be twice differentiable on $[a,b]$. Prove that there exists $\xi \in (a,b)$ such that

$$f(a) - 2f\left(\frac{a+b}{2}\right) + f(b) = \frac{1}{4}(b-a)^2 f''(\xi).$$

Proof. Write the Taylor expansions of $f(a)$ and $f(b)$ at the point $\frac{a+b}{2}$:

$$f(a) = f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(\frac{a-b}{2}\right) + \frac{1}{2}f''\left(\eta_1\right)\left(\frac{b-a}{2}\right)^2,$$
$$f(b) = f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right) + \frac{1}{2}f''\left(\eta_2\right)\left(\frac{b-a}{2}\right)^2,$$

Then adding both equations, we have

$$f(a) - 2f\left(\frac{a+b}{2}\right) + f(b) = \frac{1}{8}(b-a)^2\left[f''\left(\eta_1\right) + f''\left(\eta_2\right)\right].$$

By applying the Darboux theorem to f'', there exists $\xi \in \{a,b\}$ such that

$$f''(\xi) = \frac{1}{2}\left[f''\left(\eta_1\right) + f''\left(\eta_2\right)\right].$$

□

Exercise 11.5. Calculate $\lim_{n\to+\infty} \sin(\pi e n!)$.

Hint: Use Taylor's expansion to e.

Example 11.6. If the Taylor polynomial of any order of the function f at a point x_0 is zero, can we conclude that $f(x) \equiv 0$?

Let f have derivatives of all orders on $[-1,1]$, with $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}_+$, and there exists a constant $C \geq 0$ such that for all $n \in \mathbb{N}_+$ and $x \in [-1,1]$ the inequality $|f^{(n)}(x)| \leq n!C^n$ holds. Prove that $f(x) \equiv 0$.

Hint: Consider the function $e^{-\frac{1}{x^2}}$.

Example 11.7. Let f be twice differentiable on $[a,b]$, and let $f'(a) = f'(b) = 0$. Prove that there exists $\xi \in (a,b)$ such that

$$\left|f''(\xi)\right| \geq \frac{4}{(b-a)^2}|f(b) - f(a)|.$$

Proof. WLOG, we assume that $a = 0$, $b = 1$, and $f(1) > f(0)$. If the result is not true. We see that

$$|f''(x)| < 4(f(1) - f(0)).$$

By Taylor's expansion, it follows that

$$f\left(\frac{1}{2}\right) - f(0) = \frac{1}{2}f''(\xi_1)\left(\frac{1}{2}\right)^2,$$
$$f\left(\frac{1}{2}\right) - f(1) = \frac{1}{2}f''(\xi_2)\left(\frac{1}{2}\right)^2.$$

This is a contradiction. □

Exercise 11.8. Let f be twice differentiable on $[a, +\infty)$, and suppose that $f(x) \geq 0$ and $f''(x) \leq 0$. Prove that for $x \geq a$, it holds that $f'(x) \geq 0$.

Hint: Prove by contradiction.

Exercise 11.9. Let f be differentiable on (a,b). Is it true that for every point $t \in (a,b)$, there always exist two points $x_1, x_2 \in (a,b)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(t)?$$

Let f be differentiable on (a,b), and suppose that at some point $\xi \in (a,b)$, we have $f''(\xi) > 0$. Prove that there exist two points $x_1, x_2 \in (a,b)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi)?$$

Hint: Draw the picture.

Exercise 11.10. Let f be $(n+1)$-times differentiable in $(-1,1)$, with $f^{(n+1)}(0) \neq 0$ for $n \in \mathbb{N}_+$. For $0 < |x| < 1$, we have

$$f(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(\theta x)}{n!}x^n,$$

prove that

$$\lim_{x \to 0} \theta = \frac{1}{n+1}.$$

Hint: Note that

$$\frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(0)}{(n+1)!}x^{n+1} + o(x^{n+2}) = \frac{f^{(n)}(\theta x)}{n!}x^n.$$

Exercise 11.11. Let f be n-times differentiable in the neighborhood $O_\delta(x_0)$, and suppose that $f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$. Prove that when $0 < |h| < \delta$, it holds that

$$f(x_0 + h) - f(x_0) = hf'(x_0 + \theta h), \quad 0 < \theta < 1,$$

and that

$$\lim_{h \to 0} \theta = \frac{1}{n^{\frac{1}{n-1}}}.$$

Exercise 11.12. Let f be twice continuously differentiable on $(-\infty, +\infty)$, and suppose that for all $x, h \in \mathbb{R}$ the following holds:

$$f(x+h) - f(x) = hf'\left(x + \frac{h}{2}\right).$$

Prove that $f(x) = ax^2 + bx + c$.

Hint: Take the derivatives wrt to h.

Exercise 11.13. Let f be differentiable on $[0, +\infty)$, with $f(0) = 0$, and there exists a constant $c > 0$ such that

$$\left|f'(x)\right| \leq c|f(x)| \quad \forall x \in [0, +\infty).$$

Prove that $f(x) \equiv 0$.

Hint: Consider the equation $f'(x) = cf(x)$.

Exercise 11.14. Let f have derivatives of all orders on $(-1, 1)$, and for each $n \geq 0$, suppose that

$$\left|f^{(n)}(x)\right| \leq n!|x|.$$

Prove that $f(x) \equiv 0$.

Hint: Taylor expansion.

12. Application of derivatives

Example 12.1. Let

$$P_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}, \quad n \in \mathbb{Z}_+.$$

Prove the following properties.

(1) When n is even, $P_n(x) > 0$ for all $x \in \mathbb{R}$.
(2) When n is odd, $P_n(x)$ has at least one real root.
(3) When $x < 0$, the inequality $P_{2n}(x) > e^x > P_{2n+1}(x)$ holds.
(4) When $x > 0$, the inequality $e^x > P_n(x) \geq \left(1 + \frac{x}{n}\right)^n$ holds.
(5) For all $x \in \mathbb{R}$, it holds that $\lim_{n\to\infty} P_n(x) = e^x$.

Proof. For (1) and (2), you have to use the induction. For (3), we can apply Taylor expansion. As for (4), we note that For $\left(1 + \frac{1}{n}\right)^n$, we consider the binomial expansion and substitute the definition of the binomial coefficient:

$$\begin{aligned}\left(1 + \frac{x}{n}\right)^n &= 1 + \frac{n}{1} \cdot \frac{x}{n} + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{x^2}{n^2} + \cdots + \frac{n(n-1)\cdots 1}{1 \cdot 2 \cdots n} \cdot \frac{x^n}{n^n} \\ &= 1 + \frac{x}{1!} \cdot \frac{n}{n} + \frac{x^2}{2!} \cdot \frac{n}{n} \cdot \frac{n-1}{n} + \cdots + \frac{x^n}{n!} \cdot \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{1}{n}.\end{aligned}$$

If we scale the fractions $\frac{k}{n}$ on the right side to 1, we have:

$$\left(1 + \frac{x}{n}\right)^n \leq 1 + \sum_{k=1}^{n} \frac{x^k}{k!}.$$

For this last one, we can apply Taylor expansion. □

Exercise 12.2. Let $n \in \mathbb{Z}_+$. Prove that the equation

$$x^{n+2} - 2x^n - 1 = 0$$

has a unique positive real root.

Hint: Draw the picture.

Exercise 12.3. Consider the boundary value problem for the second-order linear ordinary differential equation on the interval $[a, b]$:

$$y'' + p(x)y' + q(x)y = r(x), \quad a < x < b, \quad y(a) = A, \quad y(b) = B,$$

where p, q, r are given functions and A, B are given constants. If $q(x) < 0$ holds for all $x \in (a, b)$, then the equation has at most one solution $y(x)$.

Hint: Consider the case that $A = B = 0$.

Exercise 12.4. Let $|f(x) + f'(x)| \leq 1$ and $f(x)$ be bounded on $(-\infty, +\infty)$. Prove that $|f(x)| \leq 1$.

Hint: Consider the function $g(x) = e^x f(x)$.

Exercise 12.5. Let $f(x)$ be twice differentiable in a neighborhood of $x = 0$, and suppose that

$$\lim_{x \to 0} \left(1 + x + \frac{f(x)}{x}\right)^{\frac{1}{x}} = e^3.$$

Find $f(0)$, $f'(0)$, $f''(0)$, and $\lim_{x \to 0} \left(1 + \frac{f(x)}{x}\right)^{\frac{1}{x}}$.

Hint: Taylor expansion

Exercise 12.6. Prove that for every natural number n, the inequality

$$\frac{e}{2n+2} < e - \left(1 + \frac{1}{n}\right)^n < \frac{e}{2n+1}$$

holds.

Hint: Consider some associated functions.

Exercise 12.7. Prove that when $0 < x < \frac{\pi}{2}$, the inequality

$$\left(\frac{\sin x}{x}\right)^3 > \cos x$$

holds.

Exercise 12.8. Prove that when $0 < x < \frac{\pi}{2}$, the inequality

$$2 \sin x + \tan x \geq 3x$$

holds.

Exercise 12.9. Prove that when $0 < x < 1$, the inequality

$$\pi < \frac{\sin \pi x}{\pi(1 - x)} \leq 4$$

holds.

13. Calculus with multi-variables I

Example 13.1. Let $f(x,y)=|x-y|\varphi(x,y)$, where $\varphi(x,y)$ is defined in a neighborhood of the point $(0,0)$. We require appropriate conditions on the function $\varphi(x,y)$ such that:

(1) $f(x,y)$ is continuous at the point $(0,0)$;
(2) $f(x,y)$ has partial derivatives at the point $(0,0)$;
(3) $f(x,y)$ is differentiable at the point $(0,0)$.

Proof. (1) Since $f(0,0)=0$, in the vicinity of the point $(0,0)$,

$$|f(x,y)|\leqslant 2\sqrt{x^2+y^2}|\varphi(x,y)|,$$

thus, when

$$\lim_{(x,y)\to(0,0)}\sqrt{x^2+y^2}\varphi(x,y)=0,$$

$f(x,y)$ is continuous at $(0,0)$. In particular, if $\varphi(x,y)$ is bounded in the vicinity of $(0,0)$, then $f(x,y)$ is continuous at $(0,0)$.

(2) The one-sided derivatives are given by

$$\begin{aligned}\left(\frac{\partial f}{\partial x}\right)_{\pm}(0,0)&=\lim_{x\to0^{\pm}}\frac{f(x,0)-f(0,0)}{x}\\&=\lim_{x\to0^{\pm}}\frac{\pm x\varphi(x,0)}{x}=\pm\lim_{x\to0^{\pm}}\varphi(x,0).\end{aligned}$$

Thus, when $\lim_{x\to0^+}\varphi(x,0)=-\lim_{x\to0^-}\varphi(x,0)$, we have $\frac{\partial f}{\partial x}(0,0)=\lim_{x\to0^+}\varphi(x,0)$. Similarly, when $\lim_{y\to0^+}\varphi(0,y)=-\lim_{y\to0^-}\varphi(0,y)$, we find $\frac{\partial f}{\partial y}(0,0)=\lim_{y\to0^+}\varphi(0,y)$. In particular, when $\lim_{(x,y)\to(0,0)}\varphi(x,y)=0$, we have $\frac{\partial f}{\partial x}(0,0)=\frac{\partial f}{\partial y}(0,0)=0$.

(3) Since

$$\begin{aligned}&f(x,y)-\frac{\partial f}{\partial x}(0,0)x-\frac{\partial f}{\partial y}(0,0)y\\=&|x-y|\varphi(x,y)-\left[\lim_{x\to0^+}\varphi(x,0)\right]x-\left[\lim_{y\to0^+}\varphi(0,y)\right]y\\=&\begin{cases}\left[\varphi(x,y)-\lim_{x\to0^+}\varphi(x,0)\right]x-\left[\varphi(x,y)+\lim_{y\to0^+}\varphi(0,y)\right]y, & \text{if } x\geqslant y,\\-\left[\varphi(x,y)+\lim_{x\to0^+}\varphi(x,0)\right]x+\left[\varphi(x,y)-\lim_{y\to0^+}\varphi(0,y)\right]y, & \text{if } x<y,\end{cases}\end{aligned}$$

Thus, we can conclude that when

$$\lim_{(x,y)\to(0,0)}\varphi(x,y)=0,$$

then $\frac{\partial f}{\partial x}(0,0)=\frac{\partial f}{\partial y}(0,0)=0$ and $f(x,y)$ is differentiable at the point $(0,0)$. □

Exercise 13.2. Why does the existence of $f_x(x_0,y_0)$ guarantee that the one-variable function $f(x,y_0)$ is continuous at the point x_0? Can we further assert that for y_1 sufficiently close to y_0, the one-variable function $f(x,y_1)$ is continuous at the point x_0?

Exercise 13.3. Provide examples to illustrate:

(1) $f(x,y)$ has partial derivatives in a neighborhood of a point but may not be continuous at that point and thus may not be differentiable;
(2) $f(x,y)$ is continuous at a point but the partial derivatives may not exist at that point and thus may not be differentiable;

(3) $f(x, y)$ is differentiable at a point, but the partial derivatives may not be continuous at that point.

Hint: By using the dummy variables.

Exercise 13.4. Prove: If $f_x(x, y)$ exists at the point (x_0, y_0) and $f_y(x, y)$ is continuous at the point (x_0, y_0), then $f(x, y)$ is differentiable at the point (x_0, y_0).

Hint: Consider the function $f(x, y_0)$. Precisely, we have

$$f(x, y) - f(x, y_0) = \int_{y_0}^{y} \partial_1 f(x, t) \mathrm{d}t.$$

Exercise 13.5. Let $z = f(x, y)$ be differentiable in the open set $D = (a, b) \times (c, d)$, and the total differential $\mathrm{d}z$ is always zero. Does this imply that $f(x, y)$ must take a constant value in D? Prove your conclusion.

Hint: Connectness.

Exercise 13.6. Let

$$f(x, y) = \begin{cases} \frac{\sqrt{|x-y|}}{x^2+y^2} \sin\left(x^2 + y^2\right), & x^2 + y^2 \neq 0, \\ 0, & x^2 + y^2 = 0. \end{cases}$$

Discuss:

(1) Is $f(x, y)$ continuous at the point $(0, 0)$?
(2) Is $f(x, y)$ differentiable at the point $(0, 0)$?

Exercise 13.7. Let

$$f(x, y) = \begin{cases} xy \sin \frac{1}{x^2+y^2}, & x^2 + y^2 \neq 0, \\ 0, & x^2 + y^2 = 0. \end{cases}$$

Prove:

(1) Both $f_x(0, 0)$ and $f_y(0, 0)$ exist;
(2) $f_x(0, 0)$ and $f_y(0, 0)$ are not continuous at the point $(0, 0)$;
(3) $f(x, y)$ is differentiable at the point $(0, 0)$.

Exercise 13.8. Let $f(x, y, z)$ be defined in an open set D. If $f_x(x, y, z)$ and $f_y(x, y, z)$ are bounded in D, and for fixed (x, y), $f(x, y, z)$ is a continuous function of z, prove that $f(x, y, z)$ is continuous in D.

Exercise 13.9. Find the total differential of $u = \ln(1 + x^2 + y^2)$ at the point $(x, y) = (1, 2)$.

Exercise 13.10. Let $f(x, y)$ be defined on the rectangle $I = [a, b] \times [c, d]$, and f_y is continuous on I. Prove that $f(x, y)$ satisfies a uniform Lipschitz condition with respect to y, i.e., there exists $L > 0$ such that for all $(x, y_1), (x, y_2) \in I$,

$$|f(x, y_1) - f(x, y_2)| \leq L |y_1 - y_2|,$$

where L is independent of x.

Exercise 13.11. If the partial derivatives f_x and f_y of a function $f(x, y)$ exist in the region D, and for all $(x, y) \in D$, $f_x(x, y) = f_y(x, y) = 0$, prove that $f(x, y)$ is a constant function in D.

Exercise 13.12. Let $\Omega \subset \mathbb{R}^2$ be an open region, and let $u(x, y), v(x, y)$ satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \quad u^2 + v^2 = C,$$

where C is a constant. Please show that $u(x, y)$ and $v(x, y)$ are constant functions in Ω.

Exercise 13.13. Let $f(x, y)$ be defined on $G = \{(x, y) \mid x^2 + y^2 < 1\}$. If $f(x, 0)$ is continuous at the point $x = 0$ and $f_y(x, y)$ is bounded in G, prove that $f(x, y)$ is continuous at the point $(0, 0)$.

14. Calculus with multi-variables I

A function $f(x,y)$ is called an n-th homogeneous function if it satisfies $f(tx,ty)=t^n f(x,y)$ for $t>0$. Homogeneous functions have the following properties:

Exercise 14.1. If f has continuous partial derivatives, then a necessary and sufficient condition for f to be an n-th homogeneous function is

$$x\frac{\partial f}{\partial x}(x,y)+y\frac{\partial f}{\partial y}(x,y)=nf(x,y).$$

Exercise 14.2. If $f(x,y)$ is a twice continuously differentiable n-th homogeneous function, then

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 f(x,y)=n(n-1)f(x,y).$$

Here,

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2=x^2\left(\frac{\partial}{\partial x}\right)^2+2xy\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y}\right)+y^2\left(\frac{\partial}{\partial y}\right)^2$$

is a shorthand notation, where x, y, $\frac{\partial}{\partial x}$, and $\frac{\partial}{\partial y}$ are treated as constants during the binomial expansion, and it does not mean applying the operator $x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ to $f(x,y)$ twice in succession.

Exercise 14.3. If $f(x,y)$ is a twice continuously differentiable n-th homogeneous function, then $f_x(x,y)$ and $f_y(x,y)$ are $(n-1)$-th homogeneous functions.

Exercise 14.4. If $f(x,y)$ is a continuous n-th homogeneous function on $\mathbb{R}^2\setminus\{0\}$, then

$$|f(x,y)|\leq C\left(x^2+y^2\right)^{\frac{n}{2}},$$

where C is a constant.

Exercise 14.5. Suppose $u(x,y)$ and $v(x,y)$ have continuous first partial derivatives in the region $\mathbf{R}^2$, and there exists a constant $C>0$ such that for any two points $(x_i,y_i)\in\mathbf{R}^2$ $(i=1,2)$, the following holds:

$$(u_1-u_2)^2+(v_1-v_2)^2\geq C\left[(x_1-x_2)^2+(y_1-y_2)^2\right],$$

where $u_i=u(x_i,y_i)$ and $v_i=v(x_i,y_i)$ for $i=1,2$. Then for all $(x,y)\in\mathbf{R}^2$, we have

$$\frac{\partial(u,v)}{\partial(x,y)}\neq 0.$$

Exercise 14.6. Proof: A differentiable function $z=f(x,y)$ is a function of $ax+by$ (where $ab\neq 0$) if and only if

$$b\frac{\partial z}{\partial x}=a\frac{\partial z}{\partial y}.$$

Exercise 14.7. Proof: The equation $F(x,y)=1-\mathrm{e}^{-x}+y^3\mathrm{e}^{-y}=0$ has a unique solution $y=y(x)$ for $x>0$ in $\{x>0,y\in\mathbb{R}^1\}$, and $y(x)$ is continuously differentiable.

Exercise 14.8. Let $f(x,y)$ satisfy: f_x exists on $\mathbb{R}^2$, f_y exists and is continuous on $\mathbb{R}^2$, and

$$|f_x|<M\left|f_y\right|,\quad f(x_0,y_0)=0,$$

where M is a constant. Prove that $f(x,y)=0$ uniquely determines a differentiable solution defined on $\mathbb{R}$ as $y=y(x)$, satisfying $y(x_0)=y_0$. Is the condition $|f_x|<M\left|f_y\right|$ necessary? If we remove the condition $f(x_0,y_0)=0$, does the conclusion still hold?