NOTES ON EXERCISE COURSE OF CALCULUS A

Contents

Some rules

Submission rules. The exercise course is on every Tuesday. In this course, you have to submit the homework assigned last Tuesday and the Thursday before the last Thursday. For example, suppose that you need to submit the homework on September 10, you have to submit those assigned on September 3 and August 29.

An exception is that on October 8, you must submit the homework assignment given before September 24 (including this day). After that, you should follow the above rules.

You can send your homework to the email:2201110024@stu.pku.edu.cn, or write on a paper to submit it on the exercise course. If you want to send it by email, you should name the title of this email with "name+student ID+the number of the homework". For example, "Zhang San 0000000000 Homework 1" in Chinese.

Grade of homework. We only care about submitting or not and complete or not. If you submit all the homework and complete all the problems assigned to it, then you will get full marks. Regular grades only come from the submission situation of homework.

Midterm. The midterm exam will take place in Early November and the precise date has not been determined.

Some useful links. We present some useful links associated to calculus.

Lecture notes by Yantong Xie:https://darkoxie.github.io

Mathstackexchange: https://math.stackexchange.com

1. Preliminaries of Calculus

1.1. **Trigonometry.** We introduce some results in trigonometry.

1.1.1. *Some basic formulae.*

$$\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2},$$

$$\sin a\cos b = \frac{1}{2}(\sin(a+b)+\sin(a-b)).$$

Exercise 1.1. What about $\sin a - \sin b$, $\cos a \pm \cos b$, $\sin a \sin b$, $\cos a \cos b$?

1.1.2. *Reverse functions.* Consider $f(x) = \sin x$. We see that it is increasing in $[-\pi/2, \pi/2]$ and we can define the reverse function of it in this interval by $f^{-1}(x) = \arcsin x$, where $x \in [-1, 1]$.

Exercise 1.2. Consider reverse functions of $\cos x$, $\tan x$.

1.1.3. *Euler formula.*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Exercise 1.3. Calculate $\sum_{k=1}^{n} \cos kx$.

1.2. **Some useful inequalities.**

1.2.1. *Cauchy's inequality.*

Theorem 1.4. *For $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \subset \mathbb{R}$, we have*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Proof. Let $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$. Consider $|a + tb|^2$ with $t \in \mathbb{R}$. $\square$

Question 1.5. What is the condition such that the above inequality satisfies "="?

Proposition 1.6. *Let $a, b, c > 0$ be such that $a + b + c = 1$. We have*

$$a^3 + b^3 + c^3 \geq \frac{a^2 + b^2 + c^2}{3}.$$

Proof. Applying Theorem 1.4, we have

$$(a^2 + b^2 + c^2)^2 \leq (a^3 + b^3 + c^3)(a + b + c) = a^3 + b^3 + c^3.$$

Additionally,

$$a^2 + b^2 + c^2 \geq ab + ac + bc.$$

This implies that $3(a^2 + b^2 + c^2) = (a + b + c)^2 = 1$. Combining all above, the result follows directly. $\square$

1.2.2. *Hölder's inequality.*

Theorem 1.7. *For $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \subset \mathbb{R}_+$, we have*

$$\left(\sum_{i=1}^{n} a_i b_i\right) \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p}\left(\sum_{i=1}^{n} b_i^q\right)^{1/q},$$

where $1/p + 1/q = 1$ with $p, q > 0$.

Proof. It is a corollary of Yong's inequality, i.e. for $a, b > 0$, $p, q > 0$ with $1/p + 1/q = 1$, we have $ab \leq a^p/p + b^q/q$. $\square$

1.2.3. *Bernoulli's inequality.*

Theorem 1.8. *Let $n \geq 2$. Assume that $x_1, x_2, ..., x_n > -1$ and have the same sign. Then*

$$\prod_{i=1}^{n}(1+x_i) \geq 1 + \sum_{i=1}^{n} x_i.$$

Proof. The proof is by using induction. For $n = 1$, there is nothing to prove. Assume that the result is true for n. By simple calculations, we have

$$\begin{aligned}\prod_{i=1}^{n}(1+x_i) &= (1+x_1)\left(\prod_{i=2}^{n}(1+x_i)\right)\\ &\geq (1+x_1)\left(1+\sum_{i=2}^{n} x_i\right)\\ &= 1+\sum_{i=1}^{n} x_i + x_1\left(\sum_{i=2}^{n} x_i\right)\\ &\geq 1+\sum_{i=1}^{n} x_i,\end{aligned}$$

where for the last inequality, we have used the property that x_i have the same sign. Now we complete the proof. □

Exercise 1.9. Let $m \in \mathbb{R}$ and $x > -1$. Show that if $m \in [0, 1]$, then $(1 + x)^m \leq 1 + mx$ and if $m < 0$ or $m > 1$, then $(1 + x)^m \geq 1 + mx$.

Exercise 1.10. Show that if $b > a > 0$, and $n \in \mathbb{Z}_+$, then $a^{n+1} > b^n((n+1)a-nb)$, and $b^{n+1} > a^n((n + 1)b - na)$.

Proposition 1.11. *The sequence $\{(1 + 1/n)^n\}$ is increasing.*

Proof. Let $b = 1 + 1/n$ and $a = 1 + 1/(n + 1)$, the result follows from Exercise 1.10. □

Exercise 1.12. Show that the sequence $\{(1 + 1/n)^{n+1}\}$ is decreasing.

1.3. **Real numbers.**

1.3.1. *Density of real numbers.* One of the most remarkable property of the real numbers is that $\mathbb{R}$ is dense. We can see from the following proposition.

Proposition 1.13. *Let $a, b \in \mathbb{R}$. Show that there exists $r_1 \in \mathbb{Q}$ and $r_2 \in \mathbb{R}\backslash\mathbb{Q}$ such that $a < r_1 < r_2 < b$.*

Proof. Assume that $a \in \mathbb{Q}$. Let $N \gg 1$ be such that $1/N < \pi/N < b-a$, we choose $r_1 = a+1/N$ and $r_2 = a+\pi/N$. Assume that $a \in \mathbb{R}\backslash\mathbb{Q}$, we choose $N \gg 1$ such that $N(b - a) > 10$. As a result, there must be some $n \in \mathbb{Z}_+$ such that $n \in (Na, Nb)$, and then $n/N \in (a, b)$. Taking n/N as new a and applying the previous arguments, we are done. □

1.3.2. *Closeness of calculations for real numbers.*

Proposition 1.14. *There exist* $a, b \in \mathbb{R}\backslash\mathbb{Q}$ *such that* $a^b \in \mathbf{Q}$.

Proof. Consider $a = b = \sqrt{2}$. If $a^b \in \mathbb{Q}$, then we are done. If not, we see that $(a^b)^{\sqrt{2}} = 2$. $\square$

Exercise 1.15. Give the corresponding examples.

- $a, b \in \mathbb{Q}$, $a^b \in \mathbb{Q}$.
- $a, b \in \mathbb{Q}$, $a^b \in \mathbb{R}\backslash\mathbb{Q}$.
- $a, b \in \mathbb{R}\backslash\mathbb{Q}$, $a + b \in \mathbb{Q}$.