NOTES ON EXERCISE COURSE OF CALCULUS A

WEI WANG

Contents

Some rules

Submission rules. The exercise course is on every Tuesday. In this course, you have to submit the homework assigned last Tuesday and the Thursday before the last Thursday. For example, suppose that you need to submit the homework on September 10, you have to submit those assigned on September 3 and August 29.

An exception is that on October 8, you must submit the homework assignment given before September 24 (including this day). After that, you should follow the above rules.

You can send your homework to the email:2201110024@stu.pku.edu.cn, or write on a paper to submit it on the exercise course. If you want to send it by email, you should name the title of this email with "name+student ID+the number of the homework". For example, "Zhang San 0000000000 Homework 1" in Chinese.

Grade of homework. We only care about submitting or not and complete or not. If you submit all the homework and complete all the problems assigned to it, then you will get full marks. Regular grades only come from the submission situation of homework.

Midterm. The midterm exam will take place in Early November and the precise date has not been determined.

Some useful links. We present some useful links associated to calculus.
Lecture notes by Yantong Xie:https://darkoxie.github.io
Mathstackexchange: https://math.stackexchange.com

1. Preliminaries of Calculus

1.1. **Trigonometry.** We introduce some results in trigonometry.

1.1.1. *Some basic formulae.*

$$\sin a + \sin b = 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2},$$
$$\sin a \cos b = \frac{1}{2}(\sin(a+b) + \sin(a-b)).$$

Exercise 1.1. What about $\sin a - \sin b$, $\cos a \pm \cos b$, $\sin a \sin b$, $\cos a \cos b$?

1.1.2. *Reverse functions.* Consider $f(x) = \sin x$. We see that it is increasing in $[-\pi/2, \pi/2]$ and we can define the reverse function of it in this interval by $f^{-1}(x) = \arcsin x$, where $x \in [-1, 1]$.

Exercise 1.2. Consider reverse functions of $\cos x$, $\tan x$.

1.1.3. *Euler formula.*

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

Exercise 1.3. Calculate $\sum_{k=1}^n \cos kx$.

1.2. **Some useful inequalities.**

1.2.1. *Cauchy's inequality.*

Theorem 1.4. *For $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \subset \mathbb{R}$, we have*

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right).$$

Proof. Let $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$. Consider $|a + tb|^2$ with $t \in \mathbb{R}$. $\square$

Question 1.5. What is the condition such that the above inequality satisfies "="?

Proposition 1.6. *Let $a, b, c > 0$ be such that $a + b + c = 1$. We have*

$$a^3 + b^3 + c^3 \geq \frac{a^2 + b^2 + c^2}{3}.$$

Proof. Applying Theorem 1.4, we have

$$(a^2 + b^2 + c^2)^2 \leq (a^3 + b^3 + c^3)(a + b + c) = a^3 + b^3 + c^3.$$

Additionally,

$$a^2 + b^2 + c^2 \geq ab + ac + bc.$$

This implies that $3(a^2 + b^2 + c^2) = (a + b + c)^2 = 1$. Combining all above, the result follows directly. $\square$

1.2.2. *Hölder's inequality.*

Theorem 1.7. *For $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \subset \mathbb{R}_+$, we have*

$$\left(\sum_{i=1}^n a_i b_i\right) \leq \left(\sum_{i=1}^n a_i^p\right)^{1/p}\left(\sum_{i=1}^n b_i^q\right)^{1/q},$$

where $1/p + 1/q = 1$ with $p, q > 0$.

Proof. It is a corollary of Yong's inequality, i.e. for $a, b > 0$, $p, q > 0$ with $1/p + 1/q = 1$, we have $ab \leq a^p/p + b^q/q$. $\square$

1.2.3. *Bernoulli's inequality.*

Theorem 1.8. *Let $n \geq 2$. Assume that $x_1, x_2, ..., x_n > -1$ and have the same sign. Then*

$$\prod_{i=1}^{n}(1+x_i) \geq 1 + \sum_{i=1}^{n} x_i.$$

Proof. The proof is by using induction. For $n = 1$, there is nothing to prove. Assume that the result is true for n. By simple calculations, we have

$$\begin{aligned}
\prod_{i=1}^{n}(1+x_i) &= (1+x_1)\left(\prod_{i=2}^{n}(1+x_i)\right) \\
&\geq (1+x_1)\left(1+\sum_{i=2}^{n} x_i\right) \\
&= 1 + \sum_{i=1}^{n} x_i + x_1\left(\sum_{i=2}^{n} x_i\right) \\
&\geq 1 + \sum_{i=1}^{n} x_i,
\end{aligned}$$

where for the last inequality, we have used the property that x_i have the same sign. Now we complete the proof. $\square$

Exercise 1.9. Let $m \in \mathbb{R}$ and $x > -1$. Show that if $m \in [0, 1]$, then $(1 + x)^m \leq 1 + mx$ and if $m < 0$ or $m > 1$, then $(1 + x)^m \geq 1 + mx$.

Exercise 1.10. Show that if $b > a > 0$, and $n \in \mathbb{Z}_+$, then $a^{n+1} > b^n((n+1)a-nb)$, and $b^{n+1} > a^n((n + 1)b - na)$.

Proposition 1.11. *The sequence $\{(1 + 1/n)^n\}$ is increasing.*

Proof. Let $b = 1 + 1/n$ and $a = 1 + 1/(n + 1)$, the result follows from Exercise 1.10. $\square$

Exercise 1.12. Show that the sequence $\{(1 + 1/n)^{n+1}\}$ is decreasing.

1.3. **Real numbers.**

1.3.1. *Density of real numbers.* One of the most remarkable property of the real numbers is that $\mathbb{R}$ is dense. We can see from the following proposition.

Proposition 1.13. *Let $a, b \in \mathbb{R}$. Show that there exists $r_1 \in \mathbb{Q}$ and $r_2 \in \mathbb{R}\backslash\mathbb{Q}$ such that $a < r_1 < r_2 < b$.*

Proof. Assume that $a \in \mathbb{Q}$. Let $N \gg 1$ be such that $1/N < \pi/N < b-a$, we choose $r_1 = a+1/N$ and $r_2 = a+\pi/N$. Assume that $a \in \mathbb{R}\backslash\mathbb{Q}$, we choose $N \gg 1$ such that $N(b - a) > 10$. As a result, there must be some $n \in \mathbb{Z}_+$ such that $n \in (Na, Nb)$, and then $n/N \in (a, b)$. Taking n/N as new a and applying the previous arguments, we are done. $\square$

1.3.2. *Closeness of calculations for real numbers.*

Proposition 1.14. *There exist $a, b \in \mathbb{R}\backslash\mathbb{Q}$ such that $a^b \in \mathbf{Q}$.*

Proof. Consider $a = b = \sqrt{2}$. If $a^b \in \mathbb{Q}$, then we are done. If not, we see that $(a^b)^{\sqrt{2}} = 2$. □

Exercise 1.15. Give the corresponding examples.

- $a, b \in \mathbb{Q}$, $a^b \in \mathbb{Q}$.
- $a, b \in \mathbb{Q}$, $a^b \in \mathbb{R}\backslash\mathbb{Q}$.
- $a, b \in \mathbb{R}\backslash\mathbb{Q}$, $a + b \in \mathbb{Q}$.

2. Limits of sequences

2.1. **Comparing the order.** Let $a > 0$ and $b > 1$, we have

$$\ln n \ll n^a \ll b^n \ll n! \ll n^n, \tag{2.1}$$

where we call $f(n) \ll g(n)$ if

$$\lim_{n\to+\infty} \frac{f(n)}{g(n)} = 0.$$

Now, let us show (2.1). We note that

$$0 < \frac{n!}{n^n} \leq \frac{1}{n},$$

which implies that $n! \ll n^n$. Let $x_n = b^n/(n!)$. We have

$$\frac{b_{n+1}}{b_n} = \frac{b}{n+1}.$$

As a result $b^n \ll n!$ follows from the following lemma, whose proof is left for the reader.

Lemma 2.1. *Let $\{x_n\}$ be a sequence, if*

$$\lim_{n\to+\infty} \left|\frac{x_{n+1}}{x_n}\right| < 1,$$

then $\lim_{n\to+\infty} x_n = 0$.

Denote $y_n = n^a/b^n$. By using this lemma, we obtain that $n^a \ll b^n$. Finally, we turn to the proof of $\ln n \ll n^a$. Noting that

$$\frac{\ln n}{n^a} = \frac{1}{a} \cdot \frac{\ln n^a}{n^a},$$

we only show that $\ln n/n \to 0^+$. This is a direct consequence of the following exercise.

Exercise 2.2. $\lim_{n\to+\infty} n^{1/n} = 1$.

Hint: Let $n^{1/n} = 1 + x_n$. We have

$$n = (1 + x_n)^n \geq 1 + nx_n + \frac{n(n-1)}{2} x_n.$$

By using Lemma 2.1, we can show $\lim_{n\to+\infty} n^2 q^n = 0$ (in Homework).

2.2. **Basic properties on limits.**

2.3. **Nonexistence of limits.**

Proposition 2.3. $\lim_{n\to+\infty} x_n = x$ *if and only if for any subsequence* x_{n_k}*, we have* $\lim_{k\to+\infty} x_{n_k} = x$.

Proof. It follows from the definition the limit. $\square$

We can use this proposition to show that some sequence does not have a limit.

Exercise 2.4. $(-1)^n$ diverges.

Exercise 2.5. For $\{x_n\}$, we have $x_{2k} \to a$ and $x_{2k-1} \to b$, where $a, b \in [-\infty, +\infty]$. Show that x_n is convergent if and only if $a = b \in (-\infty, +\infty)$.

2.4. **Monotone sequence.**

Example 2.6. Let $n \in \mathbb{Z}_+$. Assume that $x_n \in [0, 1]$ is the root of $\sum_{i=1}^n x^i = 1$. Show that $\lim_{n\to+\infty} x_n = \frac{1}{2}$.

Proof. It is easy to see that x_n is decreasing. On the other hand, $x_n - x_n^{n+1} = 1 - x_n$. Letting $n \to +\infty$, we see that $x_n \to \frac{1}{2}$. $\square$

Exercise 2.7. Assume that $0 < b < a$. We let $\{a_n\}, \{b_n\}$ be defined as follows.

$$a_{n+1} = \frac{a_n + b_n}{2},\ b_{n+1} = \sqrt{a_n b_n},\ n \in \mathbb{Z}_+,$$

where $a_1 = a$, $b_1 = b$. Show that $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n$ and exist.

Hint: Show that a_n and b_n are monotone.

Exercise 2.8. $\{x_n\}$ is monotone. Show that $\{x_n\}$ is monotone if and only if it has a convergent subsequence.

2.5. **Boundedness.** Any convergent sequence is bounded.

Example 2.9. Assume that $x_n \to x > 0$. Then x_n has a lower bound but may not has a minimal point.

Proof. For n sufficiently large, we have $x_n > \frac{x}{2}$. Considering $x_n = 1 + \frac{1}{n}$, we see that it does not have a minimal point. $\square$

Exercise 2.10. Assume that $x_n \to +\infty$, then there must be some $n_0 \in \mathbb{Z}_+$ such that $x_{n_0} = \min_{n\in\mathbb{Z}_+}\{x_n\}$.

2.6. **Some important sequence and limits.**

Example 2.11. Show that

$$S_n = \sum_{i=1}^n \frac{1}{i^p}$$

is convergent if and only if $p > 1$.

Proof. Taking $p = 1$ and $p = 2$ for example. For $p = 1$, since

$$\ln(n+1) - \ln(n) < \frac{1}{n},$$

S_n diverges. For $p = 2$, we see that

$$\frac{1}{n^2} < \frac{1}{n-1} - \frac{1}{n},$$

and then S_n is convergent. $\square$

Example 2.12. Consider

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}.$$

Show that $x_n \to \ln 2$.

Proof. Let

$$\gamma_n = \sum_{i=1}^{n} \frac{1}{i^p} - \ln n.$$

We can show that $\gamma_n \to \gamma$ (Euler constant). Then the result follows directly. $\square$

2.7. **Sandwich theorem.** Recall that if $a_n \le b_n \le c_n$ and $a_n, c_n \to x$, then $b_n \to x$.

Example 2.13. Assume that $0 < a_1 \le a_2 \le ... \le a_k$. Calculate $\lim_{n\to+\infty}(a_1^n + ... + a_k^n)^{1/n}$.

Proof. We see that

$$a_k \le (a_1^n + ... + a_k^n)^{1/n} \le k^{1/n} a_k.$$

Then $\lim_{n\to+\infty}(a_1^n + ... + a_k^n)^{1/n} = a_k$. $\square$

Exercise 2.14. Assume that $0 < a_1 \le a_2 \le ... \le a_k$. Calculate $\lim_{n\to+\infty}(a_1^{1/n} + ... + a_k^{1/n})^n$.

Exercise 2.15. Assume that $a_k \to a > 0$. Calculate $\lim_{n\to+\infty}(a_1^n + ... + a_n^n)^{1/n}$.

2.8. **Cauchy's proposition.** It is about the limit of average for a sequence having a limit.

Proposition 2.16. *If* $\lim_{n\to+\infty} x_n = x$, *then*

$$\lim_{n\to+\infty} \frac{\sum_{i=1}^{n} x_i}{n} = x. \tag{2.2}$$

Proof. For any $\varepsilon > 0$, we can choose $N \in \mathbb{Z}_+$ such that $|x_n - x| < \varepsilon$ for any $n > N$. This implies that if $n > N$, there holds

$$\left| \frac{\sum_{i=1}^{N} x_i}{n} - x \right| \le \frac{\sum_{i=1}^{N} |x_i|}{n} + \frac{n-N}{n}\varepsilon.$$

Choosing lager N without changing the notation, we can assume that if $n > N$, then

$$\left| \frac{\sum_{i=1}^{N} x_i}{n} - x \right| < 2\varepsilon.$$

As a result, we have (2.2). $\square$

Exercise 2.17. Assume that $a_n \to a$ and $b_n \to b$. Show that

$$\lim_{n\to+\infty} \frac{\sum_{i=1}^{n-1} a_i b_{n-i}}{n} = ab.$$

Exercise 2.18. Assume that $x_n \to x$. Show that

$$\lim_{n\to+\infty} \frac{\sum_{k=0}^{n} C_n^k x_k}{2^n} = x$$

Exercise 2.19. Calculate $\lim_{n\to+\infty} \frac{n!2^n}{n^n}$.

Hint: Consider a_{n+1}/a_n and use the production form of Cauchy's proposition.

Exercise 2.20 (Stolz $\frac{0}{\infty}$)**.** Let $\{a_n\}$ and be two sequences. Assume that $\{b_n\}$ is a strictly monotone and divergent sequence (i.e. strictly increasing and approaching $+\infty$, or strictly decreasing and approaching $-\infty$) and the following limit exists

$$\lim_{n\to+\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = l.$$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l.$$

Exercise 2.21 (Stolz $\frac{0}{0}$)**.** Let $\{a_n\}$ and be two sequences. Assume now that $a_n, b_n \to 0$ and $\{b_n\}$ is strictly decreasing. If

$$\lim_{n\to+\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = l.$$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l.$$

2.9. **Limits with respect to the definition of e.** We know that

$$e = \lim_{n\to+\infty} (1+1/n)^n. \tag{2.3}$$

By this we can consider related limits.

Question 2.22. What about the limit $\lim_{n\to-\infty}(1+1/n)^n$.

Example 2.23. Show that

$$\lim_{n\to+\infty} \left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n = e$$

Proof. It follows from (2.3) that

$$\lim_{n\to+\infty} \left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n = \lim_{n\to+\infty} \left(1+\frac{n+1}{n^2}\right)^{\frac{n^2}{n+1}\cdot\frac{n+1}{n}} = e.$$

□

Example 2.24. Calculate the following limits.

$$\lim_{n\to+\infty} \left(1-\frac{1}{n}\right)^{n^2}, \lim_{n\to+\infty} \left(1-\frac{1}{n^2}\right)^n.$$

Proof. Noting that

$$\left(1-\frac{1}{n}\right)^{n^2} = \left[\left(1-\frac{1}{n}\right)^n\right]^n$$

and $\lim_{n\to+\infty}(1-1/n)^n = 1/e$, we see that the first limit is 0. On the other hand, we obtain that

$$\left(1-\frac{1}{n^2}\right)^n = \left[\left(1-\frac{1}{n^2}\right)^{n^2}\right]^{1/n}.$$

The term in $[\cdot]$ is bounded and then the limit is 1. □

Exercise 2.25. Calculate $\lim_{n\to+\infty}(1+1/n)^{n^2}$ and $\lim_{n\to+\infty}(1+1/n^2)^n$.

2.10. **Problems associated with fixed points.** Consider a sequence $\{x_n\}$ such that $x_{n+1} = f(x_n)$. If x_n is convergent with $\lim_{n\to+\infty} x_n = x$ and f is continuous, then $f(x) = x$.

Example 2.26. Let $x_{n+1} = \sqrt{2 + x_n}$ and $x_1 = 1$. Show that $\lim_{n\to+\infty} x_n = 2$.

Proof. 2 is the fix point. We have

$$\frac{x_{n+1} - 2}{x_n - 2} = \frac{1}{2 + \sqrt{x_n + 2}}.$$

Then

$$\left|\frac{x_{n+1} - 2}{x_n - 2}\right| < \frac{1}{2}.$$

□

Exercise 2.27. Let $x_{n+1} = \frac{1}{x_n^{1/2}}$ and $x_1 = 2$. Show that $\lim_{n\to+\infty} x_n = 1$.

Exercise 2.28. Let $x_{n+1} = \frac{1}{x_n^2}$ and $x_1 = 2$. Show that x_n is divergent.

Exercise 2.29. Let $x_{n+1} = \frac{1}{x_n}$ and $x_1 = 2$. Show that x_n is divergent.

Question 2.30. Can you find something interesting in the above three exercises?