Calculus C: Review of the Midterm

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Sandwich Theorem

Recall that if $a_n \leq b_n \leq c_n$ and $a_n, c_n \to x$, then $b_n \to x$. For limit of functions, if $f(x) \leq g(x) \leq h(x)$ and

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = A,$$

then $\lim_{x\to x_0} g(x) = A$.

Example

Assume that $0 < a_1 \leq a_2 \leq ... \leq a_k$. Calculate

$$\lim_{n\to+\infty} (a_1^n + ... + a_k^n)^{\frac{1}{n}}.$$

Proof.

We see that

$$a_k \leq (a_1^n + ... + a_k^n)^{\frac{1}{n}} \leq k^{\frac{1}{n}} a_k.$$

Then $\lim_{n\to+\infty}(a_1^n + ... + a_k^n)^{\frac{1}{n}} = a_k$. □

Sandwich Theorem

Exercise

Assume that $0 < a_1 \leq a_2 \leq ... \leq a_k$. Calculate

$$\lim_{n \to +\infty} (a_1^{\frac{1}{n}} + ... + a_k^{\frac{1}{n}})^n.$$

Exercise

Assume that $a_k \to a > 0$. Calculate

$$\lim_{n \to +\infty} (a_1^n + ... + a_n^n)^{\frac{1}{n}}.$$

Cauchy's Proposition

Proposition

If $\lim_{n\to+\infty} x_n = x$*, then* $\lim_{n\to+\infty} \frac{\sum_{i=1}^n x_i}{n} = x$*.*

Proof.

For any $\varepsilon > 0$, we can choose $N \in \mathbb{Z}_+$ such that $|x_n - x| < \varepsilon$ for any $n > N$. This implies that if $n > N$, there holds

$$\left|\frac{\sum_{i=1}^n x_i}{n} - x\right| \leq \frac{\sum_{i=1}^N |x_i|}{n} + \frac{n-N}{n}\varepsilon.$$

Choosing lager N without changing the notation, we can assume that if $n > N$, then

$$\left|\frac{\sum_{i=1}^n x_i}{n} - x\right| < 2\varepsilon,$$

completing the proof. $\square$

Comparing the Order

Let $a > 0$ and $b > 1$, we have

$$\ln n \ll n^a \ll b^n \ll n! \ll n^n, \tag{1}$$

where we call $f(n) \ll g(n)$ if

$$\lim_{n \to +\infty} \frac{f(n)}{g(n)} = 0.$$

Now, let us show (1). We note that

$$0 < \frac{n!}{n^n} \leq \frac{1}{n},$$

which implies that $n! \ll n^n$. Let $x_n = b^n/(n!)$. We have

$$\frac{b_{n+1}}{b_n} = \frac{b}{n+1}.$$

Comparing the order

As a result, $b^n \ll n!$ follows from the following lemma, whose proof is left for the reader.

Lemma

Let $\{x_n\}$ *be a sequence, if* $\lim_{n\to+\infty}\left|\frac{x_{n+1}}{x_n}\right| < 1$, *then* $\lim_{n\to+\infty} x_n = 0$.

Denote $y_n = \frac{n^a}{b^n}$. By using this lemma, we obtain that $n^a \ll b^n$. Noting that

$$\frac{\ln n}{n^a} = \frac{1}{a}\cdot\frac{\ln n^a}{n^a},$$

we only show that $\frac{\ln n}{n} \to 0^+$.

Exercise

$\lim_{n\to+\infty} n^{\frac{1}{n}} = 1$.

Comparing the Order

Exercise (2024 Fall Mid)

Calculate $\lim_{n\to+\infty} \sqrt[n]{100+\frac{1}{n}}$.

Exercise (2023 Fall Mid)

Calculate $\lim_{n\to+\infty} \sqrt[n]{\ln n}$.

Exercise (2023 Fall Mid)

Calculate $\lim_{n\to+\infty} \frac{3^n}{n!}$.

Exercise (2022 Fall Mid)

Calculate $\lim_{n\to+\infty} \sqrt[n]{2022+\sin n}$.

Comparing the Order

We also have the following equivalences:

1. $e^x - 1 \sim x \ (x \to 0)$;
2. $\sin x \sim x \ (x \to 0)$;
3. $\ln(1+x) \sim x \ (x \to 0)$;
4. $1 - \cos x \sim \frac{1}{2}x^2 \ (x \to 0)$;
5. $(1+x)^\alpha - 1 \sim \alpha x \ (x \to 0)$;
6. $\arctan x \sim x \ (x \to 0)$.

Remark

The equivalences above can be only used in the calculations of limits with the fractional forms.

Comparing the order

Exercise (2023 Fall Mid)

Calculate
$$\lim_{x \to 1} (1 - x) \tan \frac{\pi x}{2}.$$

Exercise (2022 Fall Mid)

Calculate
$$\lim_{x \to 1} \tan \left(\frac{\sin \pi x}{4(x-1)} \right).$$

Limits with respect to e

We know that

$$e = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n. \tag{2}$$

By this, we can consider related limits.

Exercise (2024 Fall Mid)

Calculate $\lim_{x \to 0}(1 + 3x)^{\frac{1}{\sin x}}$.

Exercise (2024 Fall Mid)

Calculate $\lim_{x \to 1}(\sqrt{x})^{\frac{1}{\sqrt{x}-1}}$.

Exercise (2023 Fall Mid)

Calculate $\lim_{x \to +\infty} \left(\frac{x^3}{(x-1)(x-2)(x-3)}\right)^x$.

Limits with Respect to e

Exercise (2022 Fall Mid)

Calculate

$$\lim_{x\to\infty}\left(\frac{x^2+1}{x^2-1}\right)^{x^2}.$$

Exercise (2022 Fall Mid)

Calculate

$$\lim_{n\to\infty}\frac{\sqrt[n]{n}-1}{\ln\sqrt[n]{n}}.$$

L'Hôpital's Rule

Theorem (L'Hôpital's Rule for $\frac{0}{0}$ Form)

Suppose that functions $f(x)$ and $g(x)$ satisfy the following conditions:

1. $\lim_{x \to x_0} f(x) = 0$ *and* $\lim_{x \to x_0} g(x) = 0$;
2. $f(x)$ *and* $g(x)$ *are differentiable in a deleted neighborhood of* x_0, *and* $g'(x) \neq 0$ *in this neighborhood;*
3. $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = A$ *(where A can be a finite number, $+\infty$ or $-\infty$).*

Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = A.$$

The rule also holds when $x \to x_0^+$, $x \to x_0^-$, $x \to +\infty$ *or* $x \to -\infty$.

L'Hôpital's Rule

Theorem (L'Hôpital's Rule for $\frac{\infty}{\infty}$ Form)

Suppose that functions $f(x)$ and $g(x)$ satisfy the following conditions:

1. $\lim_{x\to x_0} f(x) = \infty$ *and* $\lim_{x\to x_0} g(x) = \infty$;
2. $f(x)$ *and* $g(x)$ *are differentiable in a deleted neighborhood of* x_0, *and* $g'(x) \neq 0$ *in this neighborhood;*
3. $\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = A$ *(where A can be a finite number, $+\infty$ or $-\infty$).*

Then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = A.$$

The rule also holds when $x \to x_0^+$, $x \to x_0^-$, $x \to +\infty$ *or* $x \to -\infty$.

L'Hôpital's Rule

Remark

L'Hôpital's Rule only applies to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ indeterminate forms. For other forms (e.g., $0 \cdot \infty$, $\infty - \infty$, 1^{∞}, 0^0, ∞^0), convert them to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ first.
If $\lim \frac{f'(x)}{g'(x)}$ does not exist (and is not ∞), L'Hôpital's Rule cannot be used, but the original limit may still exist.
Repeated application is allowed if the resulting limit still satisfies the conditions of the rule.

Exercise (2022 Fall Mid)

Calculate
$$\lim_{x \to 0} \frac{x - \arcsin x}{(\tan x)^2 \sin x}.$$

Taylor's Expansion

Theorem (Taylor's Expansion with Peano's remainder)

Assume that f is n-differentiable at x_0. Then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + o\left((x - x_0)^n\right) \quad (x \to x_0)\,.$$

Theorem (Taylor's Expansion with Lagrange's remainder)

Assume that f is $(n+1)$-differentiable in $(x_0 - \delta, x_0 + \delta)$. Then for any $x \in (x_0 - \delta, x_0 + \delta)$ with $x \neq x_0$, there is ξ in the interval of x_0 and x such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}\,(x - x_0)^{n+1}\,.$$

Taylor's Expansion

Remark

Common Taylor expansions at $x_0 = 0$ with Peano's remainder:

1. $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + o(x^n),\ x \to 0;$
2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2k+1}),\ x \to 0;$
3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2k}),\ x \to 0;$
4. $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + o(x^n),\ x \to 0;$
5. $(1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n + o(x^n),$ $x \to 0$ (for any real α).

Exercise (2024 Fall Mid)

Calculate $\lim_{x \to 0} \frac{\cos x - \cos x^2}{x^2}$.

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Applying the Definition

For $y = f(x)$, $f'(x_0) = \lim\limits_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$.

Right derivative: $f'(x_0+) = \lim\limits_{\Delta x \to 0^+} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ (left derivative similarly for $\Delta x \to 0^-$).

Differentiability: Exists iff left and right derivatives exist and are equal.

Exercise (2024 Fall Mid)

Assume that $f(x) = \begin{cases} \ln\left(1 + x^2 \sin \frac{1}{x}\right), & 0 < x < 1, \\ 0, & x = 0. \end{cases}$ Consider the existences of $f(0+)$, right derivative $f'(0+)$ and $\lim_{x \to 0+} f'(x)$.

Exercise (2023 Fall Mid)

Let $f(x)$ be differentiable at 0, and when $|x| < 1$, there is $|f(x)| \leq \ln(1 + |\arcsin x|)$. Prove that $|f'(0)| \leq 1$.

Applying Basic Formulae

- **Basic formulae**: Derivatives of elementary functions (power, exponential, logarithmic, trigonometric, inverse trigonometric).
- **Rules**: Sum, product, exponent, quotient, and chain rule (for composites: $(f(g(x)))' = f'(g(x))g'(x)$).

Exercise (2024 Fall Mid)

Calculate $f'(x)$, where

$$f(x) = \arctan(1 + \sin x) + (1 + x^2)^{x^2}.$$

Implicit Function

Implicit function is Defined by $F(x, y) = 0$ with y as a function of x.

Differentiation: Differentiate both sides w.r.t. x, treat y as $y(x)$, solve for y'.

Exercise (2023 Fall Mid)

Calculate the derivative of function that is determined by

$$y \arctan(x + y) + e^y \ln(1 + (x + y)^2) = 0.$$

Parameterization

We have the parametric equations:

$$\begin{cases} x = \varphi(t) \\ y = \psi(t). \end{cases}$$

First derivative: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\psi'(t)}{\varphi'(t)}$ $(\varphi'(t) \neq 0)$.

Second derivative: $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{\varphi'(t)} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\psi'(t)}{\varphi'(t)} \right)$.

Exercise (2024 Fall Mid)

Assume that $x = 2t - \sin t$ and $y = \cos t$. Calculate $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Exercise (2022 Fall Mid)

Assume that $x = \varphi(t)$ and $y = \psi(t)$. Calculate $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$.

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Calculating the Maximum and Minimum

- **Extreme value**: A point where the function's derivative is zero (critical point) or undefined, and the function changes from increasing to decreasing (maximum) or vice versa (minimum).
- **Global maximum/minimum**: The largest/smallest value of the function over a given domain, found by comparing values at critical points and boundary points.
- **Optimization steps**:
  1. Determine the objective function and constraints.
  2. Find critical points by solving $f'(x) = 0$ or identifying points where $f'(x)$ is undefined.
  3. Analyze the sign change of $f'(x)$ (first derivative test) or the sign of $f''(x)$ (second derivative test) to classify critical points.
  4. Evaluate the function at critical points and boundaries to find global extrema.

Calculating the Maximum and Minimum

Exercise (2024 Fall Mid)

Consider the ellipse $L : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Calculate the maximum value of the area of triangular ABC such that $A, B, C \in L$, $AB = AC$, and BC is paralleled to the x-axis.

Exercise (2023 Fall Mid)

Find the extreme points and extreme values of the function $f(x) = x^{\frac{1}{3}}(1 - x)^{\frac{2}{3}}$.

Proving Inequalities

- **Key method using derivatives**: Construct an auxiliary function $f(x)$, then analyze its monotonicity, extreme values, or convexity to prove the inequality.

- **Common strategies**:
  1. **Monotonicity**: Show $f'(x) \geq 0$ (or ≤ 0) to prove $f(x) \geq f(a)$ (or $\leq f(a)$) for $x \geq a$.
  2. **Extreme values**: Prove the minimum (maximum) of $f(x)$ is greater (less) than or equal to 0.
  3. **Convexity**: Use properties of convex/concave functions (second derivative sign) to derive inequalities.

Exercise (2024 Fall Mid)

Show that $\left(1 + \frac{1}{x}\right)^{x+1} > e$, where $x > 0$.

Proving Inequalities

Exercise (2023 Fall Mid)

Let $a > \ln 2 - 1$, prove that $x^2 - 2ax + 1 < e^x$ for $x > 0$.

Exercise (2022 Fall Mid)

Prove that

$$(b+a+1)\ln(b+a+1)-(1+b)\ln(1+b) > (2a+1)\ln(2a+1)-(1+a)\ln(1+a),$$

where $b > a > 0$.

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Discontinuity

Definition (Types of Discontinuities)

Let $f(x)$ be defined in a deleted neighborhood of x_0. We classify discontinuities of $f(x)$ at x_0 as follows:

1. **Removable Discontinuity**: $\lim_{x\to x_0} f(x)$ exists, but either $f(x_0)$ is not defined, or $\lim_{x\to x_0} f(x) \neq f(x_0)$.
2. **Jump Discontinuity**: $\lim_{x\to x_0^+} f(x)$ and $\lim_{x\to x_0^-} f(x)$ both exist but are not equal.
3. **Infinite Discontinuity**: $\lim_{x\to x_0} f(x) = \infty$ (or $+\infty$, $-\infty$), or one-sided limits are infinite.
4. **Oscillatory Discontinuity**: $\lim_{x\to x_0} f(x)$ does not exist and is not infinite (e.g., $f(x) = \sin\frac{1}{x}$ at $x = 0$).

Removable and jump discontinuities are collectively called **first-class discontinuities**; the other two types are **second-class discontinuities**.

Discontinuity

Exercise (2023 Fall Mid)

Let

$$f(x) = \begin{cases} x^2 \ln\left(2 + \cos\left(\frac{1}{x}\right)\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Find the derivative function $f'(x)$, and ask at which points $f'(x)$ is discontinuous and what type of discontinuity they are.

Intermediate theorem

Theorem (Intermediate value theorem)

Let $f(x)$ be continuous on the closed interval $[a, b]$, and let $f(a) \neq f(b)$. For any real number C between $f(a)$ and $f(b)$, there exists at least one point $\xi \in (a, b)$ such that $f(\xi) = C$.

Corollary (Zero point theorem)

Let $f(x)$ be continuous on $[a, b]$, and $f(a) \cdot f(b) < 0$ (i.e., $f(a)$ and $f(b)$ have opposite signs). Then there exists at least one point $\xi \in (a, b)$ such that $f(\xi) = 0$.

Exercise (2024 Fall Mid)

Show that the equation $\ln(2 + \cos x) - \frac{1}{x} = 0$ has infinite numbers of positive roots.

Intermediate Theorem

Exercise (2023 Fall Mid)

Prove that the equation $\left(\frac{1}{2}\right)^x + \left(\frac{3}{4}\right)^x = 1$ has only one real root.

Exercise (2022 Fall Mid)

Show that $f(x) = \frac{\cos x}{x^2} - (\sin x)^5$ has infinite numbers of positive roots.

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Rolle's Theorem

Theorem

Let $f(x)$ satisfy:

1. *Continuous on $[a, b]$;*
2. *Differentiable on (a, b);*
3. $f(a) = f(b)$.

Then there exists at least one $\xi \in (a, b)$ such that $f'(\xi) = 0$.

Lagrange's Mean Value Theorem

Theorem

Let $f(x)$ satisfy:

1. *Continuous on $[a, b]$;*
2. *Differentiable on (a, b).*

Then there exists at least one $\xi \in (a, b)$ such that

$$f(b) - f(a) = f'(\xi)(b - a).$$

The formula can also be written as $f(a + h) - f(a) = f'(a + \theta h)h$ for some $\theta \in (0, 1)$ (where $h = b - a$).

Cauchy's Mean Value Theorem

Theorem

Let $f(x)$ and $g(x)$ satisfy:

1. *Continuous on $[a, b]$;*
2. *Differentiable on (a, b);*
3. $g'(x) \neq 0$ *for all* $x \in (a, b)$;
4. $g(a) \neq g(b)$.

Then there exists at least one $\xi \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

Exercises of Mean value theorems

Exercise (2024 Fall Mid)

Assume that $f(x)$ is differentiable in (a, b). If for some $x_0 \in (a, b)$, $\lim_{x \to x_0} f'(x)$ exists. Show that $\lim_{x \to x_0} f'(x) = f'(x_0)$.

Exercise (2023 Fall Mid)

Let $g(x)$ be continuous on $[a, b]$ and differentiable on (a, b), $g(a) = 0$, and satisfy $|g'(x)| \leq \frac{1}{2(b-a)}|g(x)|, x \in (a, b)$. Prove that $g(x) = 0, x \in [a, b]$.

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Key Theorems for Sequence Convergence

Theorem (Monotone bounded theorem)

Every monotone (non-decreasing or non-increasing) and bounded sequence must converge.

- *If $\{x_n\}$ is non-decreasing and bounded above, then* $\lim_{n\to+\infty} x_n = \sup\{x_n \mid n \in \mathbb{N}\}$*;*
- *If $\{x_n\}$ is non-increasing and bounded below, then* $\lim_{n\to+\infty} x_n = \inf\{x_n \mid n \in \mathbb{N}\}$*.*

Theorem (Cauchy Convergence Criterion)

A sequence $\{x_n\}$ converges if and only if for any $\varepsilon > 0$, there exists a positive integer N such that for all $m, n > N$, we have $|x_m - x_n| < \varepsilon$. A sequence that satisfies the above condition is called a ***Cauchy sequence****.*

Remarks and Exercises

Remark

Common strategies for proving sequence convergence:

1. For recursive sequences (e.g., $x_{n+1} = f(x_n)$), use the **Monotone Bounded Theorem**: first prove monotonicity (via induction or $x_{n+1} - x_n$), then prove boundedness (via induction or inequality estimation);
2. For sequences where monotonicity is hard to verify, use the **Cauchy Convergence Criterion** (estimate $|x_m - x_n|$ directly, often using telescoping series or geometric series bounds);

Exercise (2023 Fall Mid)

Let $x_{n+1} = \sin x_n, n = 0, 1, 2, \cdots$, where x_0 is any real number. Prove that x_n has a limit and find the limit.

Exercise (2022 Fall Mid)

Given $0 < b < a$. Let $a_0 = a, b_0 = b$. Assume that

$$a_n = \frac{a_{n-1} + b_{n-1}}{2}, \quad b_n = \sqrt{a_{n-1} b_{n-1}},\ n \in \mathbb{Z}_+.$$

Let $\{a_n\}$ and $\{b_n\}$ be two sequences. Show that these two sequences have the same limit.

Exercise (2021 Fall Mid)

Let $x_1 > 0$. Assume that for any $n \in \mathbb{Z}_+$, $x_{n+1} = \frac{2(1+x_n)}{2+x_n}$. Show that $\lim_{n\to+\infty} x_n$ exists and calculate the limit.

Thank you for listening!