Final Exam of Calculus A: Answers

Wei Wang

Warming: This is not the official form! Since the poor calculation ability of mine, there maybe some typos and mistakes. You can contact me if you find errors in it.

Fall 2022

I. (This question is worth $2 \times 8 = 16$ points)

1. Prove that the line ℓ given by the equations

$$
\begin{aligned}
x - 2y + z &= 0 \\
5x + 2y - 5z &= -6
\end{aligned}
$$

passes through the point $(1, 2, 3)$, and convert this general equation into standard form.

2. Find the equation of the normal plane at the point P corresponding to the parameter $t = 1$ for the curve defined by

$$x = 7t - 14, \quad y = 4t^2, \quad z = 3t^3.$$

Solution:

1. It is easy to verify that $(1, 2, 3)$ is on this line. Note that

$$(1, -2, 1) \times (5, 2, -5) = (8, 10, 12).$$

Then, we have that the standard form of this line is

$$\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-3}{6}.$$

2. We have

$$
\begin{aligned}
\left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right)|_{t=1} &= (7, 8t, 9t^2)|_{t=1} = (7, 8, 9), \\
(x, y, z)|_{t=1} &= (7t - 14, 4t^2, 3t^3)|_{t=1} = (-7, 4, 3).
\end{aligned}
$$

As a result, the normal plane is

$$7x + 8y + 9z = 10.$$

II. (This question is worth $2 \times 10 = 20$ points)

Calculate the following problems:

1. Given $z = \arctan \frac{(x-3)y+(x^2+x-1)y^2}{(x-2)y+(x-3)^2y^4}$, find $\left.\frac{\partial z}{\partial y}\right|_{(3,0)}$.

2. Given $z = z(x, y)$ defined by the equation

$$m\left(x + \frac{z}{y}\right)^n + n\left(y + \frac{z}{x}\right)^m = 1,$$

where m and n are natural numbers. Calculate and simplify: $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + xy$.

Solution:

1. Let

$$f(x,y) = \frac{(x-3)+(x^2+x-1)y}{(x-2)+(x-3)^2y^3}.$$

Note that

$$f(3,0)=1, \quad f_y(3,0)=11.$$

Thus, we have

$$\frac{\partial z}{\partial y}|_{(3,0)} = \frac{1}{1+f^2}\frac{\partial f}{\partial y}|_{(3,0)} = 11.$$

2. Taking derivatives wrt to x and y for both sides of the given equality, we obtain

$$\left(x+\frac{z}{y}\right)^{n-1}\left(1+\frac{z_x}{y}\right)+\left(y+\frac{z}{x}\right)^{m-1}\left(\frac{z_x}{x}-\frac{z}{x^2}\right)=0,$$
$$\left(x+\frac{z}{y}\right)^{n-1}\left(\frac{z_y}{y}-\frac{z}{y^2}\right)+\left(y+\frac{z}{x}\right)^{m-1}\left(1+\frac{z_y}{x}\right)=0.$$

Dividing the first one by the second one, it follows that

$$1+\frac{z_x}{x}+\frac{z_y}{y}=\frac{z^2}{x^2y^2}-\frac{z}{xy^2}z_x-\frac{z}{x^2y}z_y,$$

which implies that

$$xy+xz_x+yz_y=z.$$

III. (This question is worth $3\times 8 = 24$ points)

Consider the following limits. If they exist, find their values; if they do not exist, explain why.

1. $\lim_{x\to 0}\frac{\int_0^{x^3}\sin^3 2t\mathrm{d}t}{\int_0^{x^2}\tan t^5\mathrm{d}t}$;
2. $\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^4+y^2}$;
3. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2}$.

Solution:

1. By L'Hopital's rule, we have

$$\lim_{x\to 0}\frac{\int_0^{x^3}\sin^3 2t\mathrm{d}t}{\int_0^{x^2}\tan t^5\mathrm{d}t} = \lim_{x\to 0}\frac{3x^2\sin^3(2x^3)}{2x\tan(x^{10})}=12.$$

2. For $(x(t),y(t))=(t,t)$ and (t,t^2), the limit for $t\to 0$ are note the same. Thus, the limit does not exist.

3. We see that

$$\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^4+y^2}=\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{1}{1+\left(\frac{k}{n}\right)^2}=\int_0^1\frac{1}{1+x^2}\mathrm{d}x=\frac{\pi}{4}.$$

IV. (This question is worth $3 \times 8 = 24$ points)

Calculate the following integrals:

1. Let $P_1(a_1, b_1, c_1)$ and $P_2(a_2, b_2, c_2)$ be two different points on the unit sphere defined by $x^2 + y^2 + z^2 = 1$, and let $O(0,0,0)$ be the origin. Calculate

$$\left(\overrightarrow{OP_1} \times \overrightarrow{OP_2}\right)^2 + \left(\overrightarrow{OP_1} \cdot \overrightarrow{OP_2}\right)^2,$$

where $(\vec{r})^2$ denotes the inner product $\overrightarrow{r} \cdot \overrightarrow{r}$.

2. Calculate

$$\int_{-1}^{1} \left(\frac{\sin^2 x}{1+e^x} + \frac{\cos^2 x}{1+e^{-x}} \right) \mathrm{d}x.$$

.

3. Calculate

$$\int_0^{\frac{\pi}{4}} \ln \frac{\sin\left(x + \frac{\pi}{4}\right)}{\cos x} \mathrm{d}x$$

.

Solution:

1. By the definition, we have

$$\left(\overrightarrow{OP_1} \times \overrightarrow{OP_2}\right)^2 = |\overrightarrow{OP_1}|^2 |\overrightarrow{OP_2}|^2 \sin^2\theta,$$
$$\left(\overrightarrow{OP_1} \cdot \overrightarrow{OP_2}\right)^2 = |\overrightarrow{OP_1}|^2 |\overrightarrow{OP_2}|^2 \cos^2\theta.$$

As a result,

$$\left(\overrightarrow{OP_1} \times \overrightarrow{OP_2}\right)^2 + \left(\overrightarrow{OP_1} \cdot \overrightarrow{OP_2}\right)^2 = |\overrightarrow{OP_1}|^2 |\overrightarrow{OP_2}|^2 = 1.$$

2. Let

$$I_1 = \int_{-1}^{1} \frac{\sin^2 x}{1+e^x} \mathrm{d}x, \quad I_2 = \frac{\cos^2 x}{1+e^{-x}} \mathrm{d}x.$$

By the change of variables with $x = -t$, it follows that

$$I_1 = \int_{-1}^{1} \frac{\sin^2 x}{1+e^x} \mathrm{d}x = \int_{-1}^{1} \frac{\sin^2 t}{1+e^{-t}} \mathrm{d}t.$$

Then

$$2I_1 = \int_{-1}^{1} \sin^2 x \mathrm{d}x.$$

Similarly,

$$2I_2 = \int_{-1}^{1} \cos^2 x \mathrm{d}x.$$

Consequently, $I_1 + I_2 = 1$.

3. It follows from the change of variables with $x = \frac{\pi}{4} - t$ that

$$\int_0^{\frac{\pi}{4}} \ln \frac{\sin\left(x + \frac{\pi}{4}\right)}{\cos x} \mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln \frac{\cos x}{\sin\left(x + \frac{\pi}{4}\right)} \mathrm{d}x = 0.$$

V. (This question is worth 8 points)

Let $f(x)$ be twice differentiable on $[a, b]$ with $f(a) = f(b) = 0$ and $f\left(\frac{a+b}{2}\right) > 0$. Prove that there exists a point $\xi \in (a, b)$ such that $f''(\xi) < 0$.

Solution:

By LMT, there is $\xi_1 \in (a, \frac{a+b}{2})$ such that

$$f'(\xi) = \frac{f(\frac{a+b}{2}) - f(a)}{\frac{a+b}{2} - a} > 0.$$

If for any $x \in (a, b)$, $f''(x) \geq 0$, then

$$f'(x) \geq f'(\xi_1) > 0 \text{ for any } x \in [\xi_1, b).$$

As a result, f is increasing on (ξ_1, b) and we obtain that

$$0 = f(b) \geq f\left(\frac{a+b}{2}\right),$$

which is a contradiction.

VI. (This question is worth 8 points)

Let $f \in C^1[0, 2]$ satisfy $f(0) = f(2) = 0$. Let $M = \max_{x\in[0,2]} |f(x)|$. Prove:

1. There exists $\epsilon \in (0, 2)$ such that $|f'(\xi)| \geq M$.
2. If for all $x \in (0, 2)$, $|f'(x)| \leq M$, then $f = 0$.

Solution:

1. WLOG, we assume that there is $x_0 \in [0, 1]$ such that $f(x_0) = M$. Then LMT implies that there exists $\xi \in (0, x_0)$ with

$$f'(\xi) = \frac{f(x_0) - f(0)}{x_0 - 0} \geq M.$$

2. By 1, we see that $x_0 = 1$, since for otherwise $f'(\xi) > M$. Since $|f'(x)| \leq M$ for any $x \in [0, 2]$, we see that

$$f(x) \leq g(x), \quad g(x) = \begin{cases} Mx & x \in [0, 1] \\ -Mx + 2M & x \in [1, 2] \end{cases}.$$

We claim that $f \equiv g$. If such a claim is true, then it is a contradiction to the assume that f is differentiable since g is not differentiable at 1. To show this claim, WLOG, we assume that there is $x_1 \in (0, 1)$ such that $f(x_1) < g(x_1)$. Then LMT implies that there is $\eta \in (x_1, 1)$ such that

$$f'(\eta) = \frac{f(1) - f(x_1)}{1 - x_1} > M,$$

which is a contradiction.

Fall 2023

I. (This question is worth 20 points)

1.1 Find the limit:
$$\lim_{x\to 0}\frac{1}{x}\left((1+x)^{\frac{1}{2}}-e\right)$$
.

1.2 Let $f(x)$ be $(n+1)$-times differentiable at $x=0$, and
$$f(0)=f'(0)=\cdots=f^{(n-1)}(0)=0,\quad f^{(n)}(0)=a.$$
Find the limit:
$$\lim_{x\to 0}\frac{f\left(e^x-1\right)-f(x)}{x^{n+1}}$$
.

Solution:
1.1 By L'Hopital's rule, the limit is $-\frac{e}{2}$.
1.2 By Taylor expansion, we have
$$f(y)=\frac{a}{n!}y^n+\frac{f^{(n+1)}(0)}{(n+1)!}y^{n+1}+o(y^{n+1}).$$
It follows that
$$f(e^x-1)=\frac{a}{n!}(e^x-1)^n+\frac{f^{(n+1)}(0)}{(n+1)!}(e^x-1)^{n+1}+o(x^{n+1}),$$
$$f(x)=\frac{a}{n!}x^n+\frac{f^{(n+1)}(0)}{(n+1)!}x^{n+1}+o(x^{n+1}).$$
Note that
$$(e^x-1)^k-x^k=\left(x+\frac{1}{2}x^2+o(x^2)\right)^k-x^k=\frac{k}{2}x^{k+1}+o(x^{k+1})$$
for any $k\in\mathbb{Z}_+$, we see that
$$f(e^x-1)-f(x)=\frac{an}{2n!}x^{n+1}+o(x^{n+1}),$$
which implies that the limit is $\frac{a}{2(n-1)!}$.

II. (This question is worth 20 points)

2.1 Let the function $F(u,v)$ have continuous second-order partial derivatives, and let $z=z(x,y)$ be an implicit function defined by the equation $F(x-z,y-z)=0$. Calculate and simplify

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y\partial x}+\frac{\partial^2 z}{\partial y^2}.$$

2.2 Given the system of equations

$$(*)\begin{cases} xy+yz^2+4=0\\ x^2y+yz-z^2+5=0.\end{cases}$$

Discuss which implicit functions can be determined near the point $P_0(1,-2,1)$ for the system $(*)$. Also, calculate the derivatives of the implicit functions determined at the point P_0.

Solution:

2.1 We denote $F_1=F_u$ and $F_2=F_v$. Then we have

$$F_1(1-z_x)+F_2(-z_x)=0,\quad F_1(-z_y)+F_2(1-z_y)=0,$$

which implies that

$$z_x+z_y=1.$$

Consequently,

$$z_{xx}+z_{yx}+z_{xy}+z_{yy}=0.$$

2.2 Let

$$f(x,y,z)=xy+yz^2+4,$$
$$g(x,y,z)=x^2y+yz-z^2+5.$$

Then we see that

$$\begin{pmatrix} f_x & f_y\\ g_x & g_y\end{pmatrix}|_{(1,-2,1)}=\begin{pmatrix} -2 & 2\\ -4 & 2\end{pmatrix},$$

$$\begin{pmatrix} f_x & f_z\\ g_x & g_z\end{pmatrix}|_{(1,-2,1)}=\begin{pmatrix} -2 & -4\\ -4 & -4\end{pmatrix},$$

$$\begin{pmatrix} f_y & f_z\\ g_y & g_z\end{pmatrix}|_{(1,-2,1)}=\begin{pmatrix} 2 & -4\\ 2 & -4\end{pmatrix}.$$

Consequently, we have $(x(z),y(z))$ and $(x(y),z(y))$ near P_0. In particular, we have

$$\begin{pmatrix} x'(1)\\ y'(1)\end{pmatrix}=-\left(\begin{pmatrix} f_x & f_y\\ g_x & g_y\end{pmatrix}|_{(1,-2,1)}\right)^{-1}\begin{pmatrix} f_z\\ g_z\end{pmatrix}|_{(1,-2,1)}=\begin{pmatrix} 1\\ 3\end{pmatrix},$$

$$\begin{pmatrix} x'(-2)\\ z'(-2)\end{pmatrix}=-\left(\begin{pmatrix} f_x & f_z\\ g_x & g_z\end{pmatrix}|_{(1,-2,1)}\right)^{-1}\begin{pmatrix} f_y\\ g_y\end{pmatrix}|_{(1,-2,1)}=\begin{pmatrix} 0\\ -\frac{1}{2}\end{pmatrix},$$

III. (This question is worth 20 points)

Find the extrema of the function

$$f(x,y) = (y - x^2)(y - x^3)$$

.

Solution:

Consider the solution of

$$f_x = 0, \quad f_y = 0.$$

Then we have

$$\begin{aligned} &x = 0,\ y = 0,\\ &x = \frac{2}{3},\ y = \frac{10}{27},\\ &x = 1,\ y = 1. \end{aligned}$$

We can calculate the Hessian matrix at these points

$$\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} \frac{100}{27} & -\frac{8}{3} \\ -\frac{8}{3} & 2 \end{pmatrix} \quad \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix}.$$

Then we see that $(\frac{2}{3}, \frac{10}{27})$ is a local maximum point since the Hessian matrix at this point is positive definite. $(1,1)$ is not an extrema point since the Hessian matrix at this point is note semi positive definite or semi negative definite. $(0,0)$ is also not an extrema point since $f(0,0) = 0$, $f(t,0) = t^5$, which is negative if $t < 0$ and positive if $t > 0$. As a result, the local minimum is $-\frac{4}{729}$.

IV. (This question is worth 20 points)

4.1 Let the function $f(x,y)$ be defined in a neighborhood of the point $(0,0)$ and continuous at $(0,0)$. If the limit

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{x^2+y^2}$$

exists, prove that $f(x,y)$ is differentiable at the point $(0,0)$.

4.2 The plane $x + y + z = 1$ intersects the cylindrical surface $x^2 + y^2 = 1$ to form an ellipse. Use multi-variable calculus to find the points on this ellipse that are closest to and farthest from the origin.

Solution:

4.1 We claim that $\mathrm{d}f|_(0,0) = 0\mathrm{d}x + 0\mathrm{d}y$. Assume that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{x^2+y^2} = A \in \mathbb{R}.$$

For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $(x, y) \in \mathbb{R}^2$ with $0 < x^2+y^2 < \delta$, then

$$|f(x, y) - A(x^2 + y^2)| \leq \varepsilon(x^2 + y^2).$$

Then for such (x, y), we have

$$\frac{|f(x, y)|}{\sqrt{x^2 + y^2}} \leq (A + \varepsilon)\sqrt{x^2 + y^2},$$

which implies that

$$\lim_{(x,y)\to(0,0)} \frac{|f(x, y)|}{\sqrt{x^2 + y^2}} = 0.$$

Then the result follows directly.

4.2 Consider

$$f(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(x + y + z - 1) - \mu(x^2 + y^2 - 1).$$

We have

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial \lambda} = 0, \quad \frac{\partial f}{\partial \mu} = 0.$$

As a result, we solve that

$$\begin{aligned}
&x = 0,\ y = 1,\ z = 0,\ \lambda = 0,\ \mu = -1,\\
&x = 1,\ y = 0,\ z = 0,\ \lambda = 0,\ \mu = -1,\\
&x = -\frac{1}{\sqrt{2}},\ y = -\frac{1}{\sqrt{2}},\ z = 1 + \sqrt{2},\ \lambda = 2(-1 - \sqrt{2}),\ \mu = -3 - \sqrt{2},\\
&x = \frac{1}{\sqrt{2}},\ y = \frac{1}{\sqrt{2}},\ z = 1 - \sqrt{2},\ \lambda = 2(-1 + \sqrt{2}),\ \mu = -3 + \sqrt{2}.
\end{aligned}$$

As a result

$$\text{Max} = 4 + 2\sqrt{2}, \quad \text{Min} = 1.$$

V. (This question is worth 20 points)

5.1 Let $f(x)$ be a smooth function defined on $\mathbb{R}$ with a period $T \neq 0$, and let k be any given natural number. Prove that there exists a point $\xi \in \mathbb{R}$ such that $f^{(k)}(\xi) = 0$.

5.2 Let the function $f(u, v)$ have continuous partial derivatives $f_u(u, v)$ and $f_v(u, v)$, and satisfy $f(x, 1-x) = 1$. Prove that the function $f(u, v)$ has at least two distinct points on the unit circle $u^2 + v^2 = 1$ that satisfy the equation:

$$vf_u(u, v) = uf_v(u, v)$$

.

Solution:

5.1 We prove a stronger result, that is, for any $k \in \mathbb{Z}_+$, there exists a sequence of different numbers $\{\xi_i^{(k)}\}_{i\in\mathbb{Z}_+}$ such that $f^{(k)}(\xi_i^{(k)}) = 0$ for any $i \in \mathbb{Z}_+$. We will prove by the induction. For $k = 1$, since f is periodic wrt T, for any $a \in \mathbb{R}$, we have

$$f(a) = f(a + nT), \quad n \in \mathbb{Z}_+.$$

Then LMT implies the result for $k = 1$. Assume that we have already shown the result for k, it is trivial to obtain that for $k+1$ with the help of LMT. Then we complete the proof.

5.2 Let $u = r\cos\theta$ and $v = r\sin\theta$. We see that

$$\frac{\partial}{\partial u} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \quad \frac{\partial}{\partial v} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}.$$

Then we can rewrite $vf_u(u,v) = uf_v(u,v)$ as

$$g_\theta(r,\theta) = 0,$$

where $g(r,\theta) = f(r\cos\theta, r\sin\theta)$. Since $f(x, 1-x) = 1$, we see that $g(1,0) = 1 = g(1,\frac{\pi}{2})$. Then it follows from LMT that there must be some $\theta_0 \in (0,\frac{\pi}{2})$ such that $g_\theta(1,\theta_0) = 0$.