READING REPORT OF THE COURSE NONLINEAR ANALYSIS

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Abstract. In the reading report, we provide a concise review of the blow-up result for the generalized static Ginzburg-Landau model discussed in the renowned paper [LW99]. We employ a straightforward argument from [Lin99] to modify the proof of the main theorem presented in [LW99]. The generalized Ginzburg-Landau model is a nonlinear variational model with several specialized structure. These properties enable us to conduct more explicit analysis and apply geometric measure theory, which differs from the primarily topological methods introduced in the“Nonlinear Analysis" course this semester.

Contents

1. Problem Settings and Main Theorems

1.1. **Ginzburg Landau Model.** Assume that (M, g) and (N, h) are two smooth, compact manifolds without boundary, where $m = \dim M$. By the Nash embedding theorem, we can isometrically embed N into $\mathbb{R}^k$ for some $k \in \mathbb{Z}_+$. Let $\delta_N > 0$ be such that the nearest point projection $\Pi_N : N_{2\delta_N} \to N$ is smooth, and $\mathrm{dist}^2(p, N)$ is also smooth for $p \in N_{2\delta_N}$. We consider the Generalized Ginzburg-Landau functional I_ε, defined as

$$I_\varepsilon(u) := \int_M e_\varepsilon(u)\mathrm{d}x = \int_M \left(\frac{|\nabla u|^2}{2} + \frac{F(u)}{\varepsilon^2} \right) \mathrm{d}x,$$

where F is a smooth function on $\mathbb{R}^k$ such that $F(p) = \chi(\mathrm{dist}^2(p, N))$, where $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a smooth, non decreasing function with $\chi(s) = s$ if $s \leq \delta_N$, and $\chi(s) = s \geq 4\delta^2$. For the functional $I_\varepsilon(\cdot)$, the related Euler-Lagrange equation and heat flow are given by

$$-\Delta_g u_\varepsilon = \frac{f(u_\varepsilon)}{\varepsilon^2} \text{ in } M, \tag{1.1}$$

where $f(u) = -D_p F(u)$. Since we only consider the local configuration of the solutions, we can assume that $M = B_1^m$, the Euclidean ball in $\mathbb{R}^m$. If there is no ambiguity, we will omit the superscript m in B_1^m for simplicity.

1.2. **Main theorems.**

Theorem 1.1 (Blow up theorem for static case, Theorem A of [LW99]). *Let $\varepsilon \downarrow 0$ and $\{u_\varepsilon\}_{\varepsilon>0}$ is a sequence of solutions of (1.1) with $I_\varepsilon(u_\varepsilon) \le K < +\infty$ for some positive constant $K > 0$. Assume that $u_\varepsilon \rightharpoonup u \in H^1(B_1)$ and there is no harmonic S^2 in N. There holds $e_\varepsilon(u_\varepsilon)\mathrm{d}x \rightharpoonup^* \frac{1}{2}|\nabla u|^2\mathrm{d}x$, as weak* convergence of Radon measure and $u_\varepsilon \to u$ strongly on $H^1_{\mathrm{loc}}(B_1)$ with $\int_{B_1}\frac{1}{\varepsilon^2}F(u_\varepsilon)\mathrm{d}x \to 0$ as $\varepsilon \downarrow 0$.*

Remark 1.2. Here since we use the truncation in the definition of the potential F, F is always bounded. This implies that the solutions u_ε in Theorem 1.1 are all smooth and we will not worry about taking derivatives and using integral by parts in the proof.

1.3. **Preliminaries.**

Lemma 1.3 (Monotonicity I, [CS89]). *If u_ε satisfies the equation (1.1), then*

$$\begin{aligned}R^{2-m}\int_{B_R(x)} e_\varepsilon(u_\varepsilon) - r^{2-m}\int_{B_r(x)} e_\varepsilon(u_\varepsilon)\\ = \int_r^R t^{2-m}\int_{\partial B_t(x)}\left|\frac{\partial u_\varepsilon}{\partial t}\right|^2 + 2\int_r^R t^{1-m}\int_{B_t(x)}\frac{F(u_\varepsilon)}{\varepsilon^2},\end{aligned}$$

for $x \in B_1$ and $0 < r \le R < d(x, \partial B_1)$. In particular, $r^{2-m}\int_{B_r(x)} e_\varepsilon(u_\varepsilon)$ is monotonically non-decreasing with respect to r.

Lemma 1.4 (ε_0-Regularity theorem, [CS89]). *If u_ε satisfies the equation (1.1), then there exist $\varepsilon_0, K_0 > 0$, depending only on m, N, such that*

$$R^{2-m}\int_{B_R(x)} e_\varepsilon(u_\varepsilon) \le \varepsilon_0^2 \Rightarrow \sup_{B_{R/4}(y)} e_\varepsilon(u_\varepsilon) \le K_0R^{-m}\int_{B_R(x)} e_\varepsilon(u_\varepsilon),$$

for any $y \in B_{R/2}(x)$.

2. Proof of Theorem 1.1

2.1. **The Case of $m = 2$.** Define the concentration set Σ by

$$\Sigma := \bigcap_{r>0}\left\{x \in B_1 : \liminf_{\varepsilon\downarrow 0}\int_{B_r(x)} e_\varepsilon(u_\varepsilon) \ge \varepsilon_0^2\right\},$$

where ε_0 is related to Lemma 1.4. It is easy to see that Σ is closed and locally finite in B_1. For any $x \in B_1\backslash\Sigma$, there exists $r_x > 0$ such that $r_x^{2-m}\int_{B_r(x)} e_\varepsilon(u_\varepsilon) < \varepsilon_0^2$. By Lemma 1.4, $e_\varepsilon(u_\varepsilon)$ is uniformly bounded in $B_{r/4}(x)$. This implies that $u_\varepsilon \to u$ in $C^1(B_1\backslash\Sigma)$ (up to a subsequence if necessary). Moreover, u is a weakly harmonic map in B_1 and $u \in C^\infty(B_1\backslash\Sigma, N)$. Using removable singularity theorem in [SU81], we have $u \in C^\infty(B_1, N)$.

Claim. $\Sigma = \emptyset$. If the claim is true, the result follows directly. If not, by the fact that Σ is locally finite, we can choose $x_0 \in \Sigma$ and $r_0 > 0$ such that $B_{r_0}(x_0) \cap \Sigma = \{x_0\}$. Define

$$Q_\varepsilon(t) = \sup_{y \in B_{r_0}(x_0)}\int_{B_t(y)} e_\varepsilon(u_\varepsilon).$$

For this, the following properties hold.

(1) For fixed ε, $Q_\varepsilon(t) \to 0$ as $t \downarrow 0$.
(2) For fixed $t > 0$, $\liminf_{\varepsilon\downarrow 0} Q_\varepsilon(t) \ge \varepsilon_0^2$.

Indeed, the first property follows from the fact that u_ε is smooth (see Remark 1.2). To get the second one, we note that if there exists some $t_0 > 0$, such that $\liminf_{\varepsilon\downarrow 0} Q_\varepsilon(t_0) < \varepsilon_0^2$, then $\liminf_{\varepsilon\downarrow 0}\int_{B_{t_0}(x_0)} e_\varepsilon(u_\varepsilon) < \varepsilon_0^2$. By Lemma 1.4, up to a subsequence, this implies that $|\nabla u_\varepsilon| \le C$ in $B_{t_0/2}(y_0)$ for some constant C, independent of ε. This is a contradiction to the assumption that $x_0 \in \Sigma$.

Now, we can choose a sequence t_ε, related to ε and $x_\varepsilon \to x_0$, such that

$$Q_\varepsilon(t_\varepsilon) = \int_{B_{t_\varepsilon}(x_\varepsilon)} e_\varepsilon(u_\varepsilon) = \frac{\varepsilon_0^2}{2}.$$

Define $v_\varepsilon : \Omega_\varepsilon \to \mathbb{R}^k$ by $v_\varepsilon(x) := u_\varepsilon(x_\varepsilon + t_\varepsilon x)$. Obviously, v_ε satisfies the equation

$$\Delta v_\varepsilon + \frac{t_\varepsilon^2}{\varepsilon^2} f(v_\varepsilon) = 0 \text{ in } \Omega_\varepsilon.$$

In view of the assumption that $I_\varepsilon(u_\varepsilon) \le K$ and the choice of $x_\varepsilon, t_\varepsilon$, we also have

$$\int_{\Omega_\varepsilon} \left(\frac{|\nabla v_\varepsilon|^2}{2} + \frac{t_\varepsilon^2}{\varepsilon^2} F(v_\varepsilon) \right) \le K < +\infty, \tag{2.1}$$

and

$$\int_{B_1(y)} \left(\frac{|\nabla v_\varepsilon|^2}{2} + \frac{t_\varepsilon^2}{\varepsilon^2} F(v_\varepsilon) \right) \le \frac{\varepsilon_0^2}{2}, \tag{2.2}$$

for any $y \in \Omega_\varepsilon$, with equality if $y = 0$ and $\Omega_\varepsilon = t_\varepsilon^{-1}(B_{r_0}(x_0 - x_\varepsilon))$. By (2.1), and the property that $\Omega_\varepsilon \to \mathbb{R}^2$, up to a subsequence, there is $v \in H_1(\mathbb{R}^2)$ such that $v_\varepsilon \rightharpoonup v$ in $H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^k)$. Using (2.2) and Lemma 1.4, $\sup_{B_{1/4}(y)} |\nabla v_\varepsilon| \le C$ for any $y \in \mathbb{R}^2$ and then $v_\varepsilon \to v$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$. Since the equality in (2.2) holds true if $y = 0$, we can apply the C^1 convergence of v_ε to obtain that there are three cases as follows.

(1) If $\frac{\varepsilon}{t_\varepsilon} \to 0$, using standard arguments (see the proof in the case that $m \ge 3$ for more details) we can obtain that $v \in C^\infty(\mathbb{R}^2, N)$, $\frac{t_\varepsilon^2}{\varepsilon^2} F(v_\varepsilon) \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and then

$$\varepsilon_0^2 \le \int_{\mathbb{R}^2} |\nabla v|^2 < +\infty. \tag{2.3}$$

Moreover, v satisfies $\Delta v + A(v)(\nabla v, \nabla v) = 0$. For this case, through conformal map from $\mathbb{R}^2$ to $\mathbb{S}^2$, we can construct a smooth non-trivial harmonic map from $\mathbb{S}^2$ to N, this is a contradiction to the assumption of the theorem.

(2) If $\varepsilon/t_\varepsilon \to +\infty$, we have $\Delta v = 0$ with (2.3). Since the entire function with bounded energy is 0 (see Chapter 4.4 of [GM05] for example), it is a contradiction.

(3) If $\varepsilon/t_\varepsilon \to c > 0$ for some constant $c > 0$, we have

$$\frac{\varepsilon_0^2}{2} \le \int_{\mathbb{R}^2} \left(\frac{|\nabla v|^2}{2} + \frac{F(v)}{c^2} \right) < +\infty.$$

Moreover, v satifies the equation $\Delta v + \frac{1}{c^2} f(v) = 0$. Choose $\phi \in C_0^\infty(B_2)$ be such that $\phi \equiv 1$ in B_1 and $\phi_n(x) := \phi(\frac{x}{n})$. Testing the equation of v by the function $\phi_n x \cdot \nabla v$, we have

$$0 = \int_{\mathbb{R}^2} \partial_\alpha v \partial_\alpha (\phi_n x_\beta \partial_\beta v) + \frac{1}{c^2} \int_{\mathbb{R}^2} \partial_\beta (F(v)) \phi_n x_\beta.$$

This implies that

$$\int_{\mathbb{R}^2} F(v)\phi_n \leq C \int_{B_{2n}\setminus B_n} |\nabla v|^2 \mathrm{d}x \to 0$$

as $n \to +\infty$. As a result, we have $\Delta v = 0$ and this case is reduced to the second case.

2.2. **The Case of $m \geq 3$.** Now assume that $u_\varepsilon \to u$ weakly in $H^1(B_1)$ and $\mu_\varepsilon := e_\varepsilon(u_\varepsilon)\mathrm{d}x \rightharpoonup^* \mu = \frac{1}{2}|\nabla u|^2 \mathrm{d}x + \nu$ as Radon measures for some Radon measure $\nu \geq 0$. By Lemma 1.3, we have

$$\Theta^{m-2}(\mu, x, R) \geq \Theta^{m-2}(\mu, x, r) \text{ for any } 0 < r \leq R < d(x, \partial B_1) \text{ and } x \in B_1, \tag{2.4}$$

where $\Theta^{m-2}(\mu, x, \rho) = \rho^{2-m}\mu(\overline{B}_\rho(x))$. By this, we can define the $(m-2)$-density of μ at $x \in B_1$, denoted by $\Theta^{m-2}(\mu, x)$ as

$$\Theta^{m-2}(\mu, x) = \lim_{\rho \downarrow 0} \Theta(\mu, \rho, x).$$

We note that the existence of $\Theta^{m-2}(\mu, x)$ only depends on (2.4). To be more precise, we define

$$\mathcal{M}_{mo}(\overline{B}_1) = \big\{\lambda \in \mathcal{M}(\overline{B}_1) : \Theta^{m-2}(\lambda, x, r) \text{ is monotonically non increasing for } \\ 0 < r < \operatorname{dist}(x, \partial B_1) \text{ and } x \in B_1\big\}.$$

where $\mathcal{M}(\overline{B}_1)$ is the set of Radon measures on $\overline{B}_1$. For any $\lambda \in \mathcal{M}_{mo}(\overline{B}_1)$, we can define the $m-2$ density $\Theta^{m-2}(\lambda, \cdot)$ in B_1. For $\Theta^{m-2}(\cdot, \cdot)$ and $\Theta^{m-2}(\cdot, \cdot, \cdot)$, we have the following lemma.

Lemma 2.1. *For any fixed $\lambda_0 \in \mathcal{M}_{mo}(\overline{B}_1)$, $\Theta^{m-2}(\lambda_0, x, r)$ is upper semicontinuous, w.r.t x and r. $\Theta^{m-2}(\lambda, x)$ is upper semicontinuous, w.r.t λ and x.*

Proof. Assume that $x_j \to x$ and $r_j \to r$. For $\delta > 0$ sufficiently small, there exists $j_0 \in \mathbb{Z}_+$ if $j > j_0$, then $B_{r_j}(x_j) \subset B_{r+\delta}(x)$. Now we have

$$\Theta^{m-2}(\lambda_0, x_j, r_j) = r_j^{2-m}\lambda_0(B_{r_j}(x_j)) = \frac{r_j^{2-m}}{(r+\delta)^{2-m}}\Theta(\lambda_0, x, r+\delta).$$

Taking the limsup for both sides, this implies that

$$\limsup_{j \to +\infty} \Theta^{m-2}(\lambda_0, x_j, r_j) = \left(\frac{r+\delta}{r}\right)^{m-2} \Theta(\lambda_0, x, r+\delta).$$

Since $\overline{B}_r(x) = \cap_{\delta > 0}\overline{B}_{r+\delta}(x)$, we have $\lim_{\delta \downarrow 0} \Theta(\lambda_0, x, r+\delta) = \Theta(\lambda_0, x, r)$, which implies that $\Theta^{m-2}(\lambda_0, \cdot, \cdot)$ is semicontinuous. For $\Theta^{m-2}(\cdot, \cdot)$, we assume that $\lambda_j \rightharpoonup *\lambda$ in $\mathcal{M}(\overline{B}_1)$ with $\{\lambda_j\} \cup \{\lambda\} \subset \mathcal{M}_{mo}(\overline{B}_1)$ and $x_j \to x$. For fixed $r, \delta > 0$, there exists $j_0 > 0$ such that if $j > j_0$, $B_r(x_j) \subset B_{r+\delta}(x)$ and then

$$\Theta^{m-2}(\lambda_j, x_j) \leq \Theta^{m-2}(\lambda_j, x_j, r) \leq \frac{r^{2-m}}{(r+\delta)^{2-m}}\Theta^{m-2}(\lambda_j, x, r+\delta).$$

Taking limsup for both sides, we have

$$\limsup_{j \to +\infty} \Theta^{m-2}(\lambda_j, x_j) \leq \frac{r^{2-m}}{(r+\delta)^{2-m}}\Theta^{m-2}(\lambda, x, r+\delta).$$

Choosing $\delta > 0$ and $r > 0$ such that $r + \delta \to 0$ and $\frac{r^{2-m}}{(r+\delta)^{2-m}} \to 1$, it follows that $\Theta^{m-2}(\lambda, x) \geq \limsup_{j \to +\infty} \Theta^{m-2}(\lambda_j, x_j)$, which completes the proof. $\square$

Define the concentration set

$$\Sigma := \bigcap_{r>0} \left\{ x \in B_1 : \liminf_{\varepsilon \downarrow 0} r^{2-m} \int_{B_r(x)} e_\varepsilon(u_\varepsilon) \geq \varepsilon_0^2 \right\},$$

Next, we give some basic results for the set Σ.

Lemma 2.2. *For the set Σ, the following properties hold.*

1. *Σ is relatively closed in B_1.*
2. *$\mathcal{H}^{m-2}(\Sigma \cap B_R)$ is finite for any $0 < R < 1$.*
3. *$u_\varepsilon \to u$ in $C^1_{\mathrm{loc}}(B_1 \backslash \Sigma) \cap H^1(B_1 \backslash \Sigma)$.*
4. *u is a weakly harmonic map in B_1, which is smooth in $B_1 \backslash \Sigma$.*

Proof. To show the first property, we assume that $x_j \in \Sigma$ and $x_j \to x \in B_1$. By the definition of Σ, there holds,

$$\liminf_{j \to +\infty} \liminf_{\varepsilon \downarrow 0} r^{2-m} \int_{B_r(x_j)} e_\varepsilon(u_\varepsilon) \geq \varepsilon_0^2.$$

By the monotonicity formula Lemma 1.3, for sufficiently large j and sufficiently small $r > 0$,

$$r^{2-m} \int_{B_r(x_j)} e_\varepsilon(u_\varepsilon) \leq 2^{m-2} (2r)^{2-m} \int_{B_{2r}(x)} e_\varepsilon(u_\varepsilon) \leq 2^{m-2} C(x, K).$$

This implies that we can interchange the limit and obtain that $x \in \Sigma$. For the second property, we firstly fix $\delta > 0$. For any $x \in \Sigma \cap \overline{B}_R$, there exists $r_x < \delta$ and a subsequence of u_ε, denoted by $u_{\varepsilon_i, x}$ such that $r_x^{2-m} \int_{B_{r_x}(x)} e_{\varepsilon_i}(u_{\varepsilon_i, x}) \geq \varepsilon_0^2$. Now $\{B_{r_x}(x)\}_{x \in \Sigma \cap \overline{B}_R}$ forms a covering of $\Sigma \cap \overline{B}_R$. Since Σ is relatively closed in B_1, $\Sigma \cap \overline{B}_R$ is compact, and we can choose a subsequence of u_ε, still denoted by itself and a finite sub class of the covering, given by $\{B_{r_i}(x_i)\}_{i=1}^\ell$ such that $B_{r_i}(x_i)$ are disjoint, $\Sigma \cap \overline{B}_R \subset \cup_{i=1}^\ell B_{3r_i}(x_i)$, and $\int_{B_{r_i}(x_i)} e_\varepsilon(u_\varepsilon) \geq \varepsilon_0 r_i^{m-2}$. By the construction, we have

$$\mathcal{H}^{m-2}_{3\delta}(\Sigma \cap \overline{B}_R) \leq C \sum_{i=1}^{\ell} (3r_i)^{m-2} \leq \frac{C}{\varepsilon_0^2} \int_{B_{r_i}(x_i)} e_\varepsilon(u_\varepsilon) \leq C(\varepsilon_0, K).$$

To show the third and third properties, we note that for any $x_0 \in B_1 \backslash \Sigma$, there exists $r_0 > 0$ such that $r_0^{2-m} \int_{B_{r_0}(x_0)} e_\varepsilon(u_\varepsilon) < \varepsilon_0^2$. Applying Lemma 1.4, we can obtain that

$$\sup_{B_{r_0/4(x_0)}} \left(|\nabla u_\varepsilon|^2 + \frac{F(u_\varepsilon)}{\varepsilon^2} \right) \leq C. \tag{2.5}$$

This implies that $u_\varepsilon \to u$ in $C^1_{\mathrm{loc}}(B_{r_0/4}(x_0))$, this directly implies the third property. On the other hand, there also holds

$$-\Delta e_\varepsilon(u_\varepsilon) + \frac{\operatorname{dist}^2(u_\varepsilon, N)}{\varepsilon^4} \leq C \tag{2.6}$$

in $B_{r_0/4}(x_0)$. Indeed, through simple calculations, it follows that

$$-\Delta e_\varepsilon(u_\varepsilon) = \frac{2\nabla(f(u_\varepsilon)) \cdot \nabla u_\varepsilon}{\varepsilon^2} - |D^2 u| - \frac{|f(u)|^2}{\varepsilon^4}, \tag{2.7}$$

By (2.5) and the definition of F, if $\varepsilon > 0$ is sufficiently small, we obtain that $u_\varepsilon(x) \to N$ uniformly in $B_{r_0/4}(x_0)$ as $\varepsilon \downarrow 0$. This, together with (2.5), implies that

$$\begin{aligned}|\nabla(f(u_\varepsilon)) \cdot \nabla u_\varepsilon| &\le C(|\nabla_p \operatorname{dist}(u_\varepsilon, N) \cdot \nabla u_\varepsilon|^2 + |D_p^2 \operatorname{dist}(u_\varepsilon, N) \cdot \nabla u_\varepsilon| \operatorname{dist}(u_\varepsilon, N))\\ &\le C(|\nabla u_\varepsilon|^2 + |\nabla u_\varepsilon| \operatorname{dist}(u_\varepsilon, N),\end{aligned}$$

and

$$|f(u_\varepsilon)|^2 \le C|\operatorname{dist}(u_\varepsilon, N)\nabla_p \operatorname{dist}(u_\varepsilon, N)|^2 \le C_1 \operatorname{dist}^2(u_\varepsilon, N).$$

Applying Cauchy inequality $ab \le a^2/(4\delta) + \delta b^2$ for any $a, b, \delta > 0$, we have

$$\begin{aligned}-\Delta e_\varepsilon(u_\varepsilon) &\le C|\nabla u_\varepsilon|^4 + \frac{C_1 \operatorname{dist}^2(u_\varepsilon, N)}{2\varepsilon^4} - \frac{C_1 \operatorname{dist}^2(u_\varepsilon, N)}{2\varepsilon^4}\\ &\le C - \frac{C \operatorname{dist}^2(u_\varepsilon, N)}{2\varepsilon^4},\end{aligned}$$

which directly implies (2.7). Choosing a nonnegative function $\phi \in C_0^\infty(B_{r_0/4}(x_0))$ and testing (2.7) by ϕ, it follows from integration by parts that

$$\int_{B_{r_0/4}(x_0)} \frac{\operatorname{dist}^2(u_\varepsilon, N)}{\varepsilon^4}\phi \le \int_{B_{r_0/4}(x_0)} |\Delta\phi| e_\varepsilon(u_\varepsilon) \le C(\phi). \tag{2.8}$$

Using (2.5), we have $\frac{\operatorname{dist}(u_\varepsilon,N)}{\varepsilon^2}$ is uniformly bounded in $L^2(B_{r_0/8}(x_0))$. Since u_ε satisfies (1.1), $D^2 u_\varepsilon$ is also uniformly bounded in $L^2(B_{r_0/8}(x_0))$. Therefore, up to a subsequence, $\Delta u_\varepsilon \rightharpoonup \Delta u$ in $L^2(B_{r_0/8}(x_0))$. Now, we have some function $\lambda \in L^2(B_{r_0/8}(x_0))$, such that $\frac{\operatorname{dist}(u_\varepsilon,N)}{\varepsilon^2} \rightharpoonup \lambda$ weakly in $L^2(B_{r_0/8}(x_0))$. Note also that there exists a unit vector field $\nu_N^\varepsilon \perp T_{\Pi_N(u_\varepsilon)}N$ such that $f(u_\varepsilon) = 2\operatorname{dist}(u_\varepsilon, N)\nu_N^\varepsilon$. As a result, for any vector field $\varphi \in L^2(B_{r_0/8}(x_0))$ with $\varphi(x) \in T_{u(x)}N$ for a.e. $x \in B_{r_0/8}(x_0)$, we have

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} \int_{B_{r_0/8}(x_0)} \frac{f(u_\varepsilon)}{\varepsilon^2}\varphi &= \lim_{\varepsilon\downarrow 0} \int_{B_{r_0/8}(x_0)} \frac{\operatorname{dist}(u_\varepsilon, N)}{\varepsilon^2}\nu_N^\varepsilon \varphi\\ &= \lim_{\varepsilon\downarrow 0} \int_{B_{r_0/8}(x_0)} \lambda \nu_N^\varepsilon \varphi = 0.\end{aligned}$$

This implies that $-\Delta u = \lambda\nu_N(u)$ a.e. in $B_{r_0/8}(x_0)$, where the $\nu_N(u)$is the normal vector field of N at u. This is equivalent to say $-\Delta u \perp T_u N$ and then $-\Delta u = A(u)(\nabla u, \nabla u)$ in $B_{r_0/8}(x_0)$ (by using the higher regularity of elliptic equations. Consequently, $u \in C^\infty(B_1\backslash\Sigma)$ is a harmonic map. Moreover, since Σ has locally finite $\mathcal{H}^{m-2}$ measure, we can use the truncation function to show that $u \in H^1(B_1)$ is a weakly harmonic map. Indeed, for $B_r(x_0) \subset B_1$ and $\delta > 0$, let $T_\delta(\Sigma \cap B_r(x_0)) = \cap_{x\in\Sigma\cap B_r(x_0)} B_\delta(x)$ be the δ-neighborhood of $\Sigma \cap B_r(x_0)$. Choose $t > 0$, $\eta_t \in C^\infty(B_1)$ such that $\eta_t \equiv 1$ in $B_1\backslash T_{2t}(\Sigma \cap B_r(x_0))$, $\eta_t \equiv 0$ in $T_t(\Sigma \cap B_r(x_0))$, $0 \le \eta_t \le 1$ and $|\nabla\eta_t| \le \frac{C}{t}$. For any $\zeta \in C_0^\infty(B_r(x_0), \mathbb{R}^k)$, since $u_\varepsilon \to u$ uniformly in $C^j(B_r(x_0)\backslash T_{2t}(\Sigma \cap B_r(x_0)))$, we have

$$\int_{B_r(x_0)} (\nabla u \cdot \nabla(\eta_t\zeta) - \eta_t\zeta A(u)(\nabla u, \nabla u)) = 0.$$

We claim that $\int_{B_r(x_0)} \nabla u\zeta \cdot \nabla \eta_t \to 0$ as $\varepsilon \downarrow 0$. If this claim is true, we can apply dominant convergence theorem to complete the proof. Indeed,

$$\begin{aligned}\left|\int_{B_r(x_0)} \nabla u\zeta \cdot \nabla \eta_t\right| &\le \frac{C}{t}\left(\int_{(T_{2t}\setminus T_t)(\Sigma\cap B_r(x_0))} |\nabla u|^2\right)^{1/2} |T_t(\Sigma\cap B_r(x_0))|^{1/2} \\ &\le Cr^{m-2}\left(\int_{(T_{2t}\setminus T_t)(\Sigma\cap B_r(x_0))} |\nabla u|^2\right)^{1/2} \to 0,\end{aligned}$$

as $\varepsilon \downarrow 0$. Now we can complete the proof. □

Now we can have a characterization of Σ as follows. Compared to the proof in the convergence of harmonic maps (cf [Lin99]), the difference is the contribution of the term $\frac{1}{\varepsilon^2}F(u_\varepsilon)\mathrm{d}x$.

Lemma 2.3. $\Sigma = \mathrm{sing}(u) \cup \mathrm{supp}(\nu)$, *where*

$$\mathrm{sing}(u) = \{x \in B_1 \times (0,1) : \textit{for each neighborhood of } x,\ u \textit{ is not smooth in it}\}.$$

In particular, $e_\varepsilon(u_\varepsilon)\mathrm{d}x \rightharpoonup^* \frac{1}{2}|\nabla u|^2\mathrm{d}x$ *in* $B_1\setminus\Sigma$.

Proof. If $x_0 \in B_1\Sigma$, by the first property of Lemma 2.2 and the definition of Σ, there exists $r_0 > 0$ such that $B_{r_0}(x_0) \subset\subset B_1\setminus\Sigma$ and $r_0^{2-m}\int_{B_{r_0}(x_0)} e_\varepsilon(u_\varepsilon) < \varepsilon_0^2$. Applying the third property of Lemma 2.2, we have that $x_0 \notin \mathrm{sing}(u)$, $\frac{1}{2}|\nabla u_\varepsilon|^2\mathrm{d}x \rightharpoonup^* \frac{1}{2}|\nabla u|^2\mathrm{d}x$ as $\varepsilon \downarrow 0$ in $B_{r_0}(x_0)$ and $F(u_\varepsilon) = \mathrm{dist}^2(u_\varepsilon, N)$ for sufficiently small $\varepsilon > 0$. This implies that $\frac{1}{\varepsilon^2}F(u_\varepsilon)\mathrm{d}x \rightharpoonup^* \nu$ in $B_{r_0}(x_0)$. Using almost the same arguments in the proof of the forth property of Lemma 2.2, or precisely (2.8), we can obtain that

$$0 \le \int_{B_{r_0/8}(x_0)} \frac{F(u_\varepsilon)}{\varepsilon^2} = \varepsilon^2\left(\int_{B_{r_0/8}(x_0)} \frac{\mathrm{dist}^2(u_\varepsilon, N)}{\varepsilon^4}\right) \le C\varepsilon^2 \to 0$$

as $\varepsilon \downarrow 0$. As a result, we have $\nu(B_{r_0/8}(x_0)) = 0$ and then $x_0 \notin \mathrm{supp}(\nu)$. On the other hand, if $x_0 \notin \mathrm{sing}(u)\,\mathrm{supp}(\nu)$, then there exists $r_0 > 0$ such that u is smooth in $B_{r_0}(x_0)$ and $\nu(B_{r_0}(x_0)) = 0$. For some sufficiently small $0 < r < r_0$, there holds $r^{2-m}\int_{B_r(x_0)}(\frac{1}{2}|\nabla u|^2\mathrm{d}x + \mathrm{d}\nu) \le \frac{\varepsilon_0^2}{4}$. Consequently, $r^{2-m}\int_{B_r(x_0)} e_\varepsilon(u_\varepsilon) < \frac{\varepsilon_0^2}{2}$ if $\varepsilon > 0$ is sufficiently small. This implies that $x_0 \notin \Sigma$. Now we can complete the proof. □

Remark 2.4. For any $x_0 \in B_1$ such that $\Theta^{m-2}(\mu, x_0) > 0$, if $x_0 \notin \Sigma$, Lemma 2.3 implies there exists $r_0 > 0$ such that u is smooth in $B_{r_0}(x_0)$ and $\nu(B_{r_0}(x_0)) = 0$. For $0 < r < r_0$ sufficiently small,

$$\begin{aligned}\Theta^{m-2}(\mu, x_0) &\le \Theta^{m-2}(\mu, x_0, r) \\ &= r^{2-m}\int_{B_r(x_0)}\left(\frac{1}{2}|\nabla u|^2\mathrm{d}x + \mathrm{d}\nu\right) \\ &< \frac{1}{2}\Theta^{m-2}(\mu, x_0),\end{aligned}$$

which is a contradiction. This, together with the definition of Σ, implies that

$$\Sigma = \{x \in B_1 : \Theta^{m-2}(\mu, x) \ge \varepsilon_0^2\} = \{x \in B_1 : \Theta^{m-2}(\mu, x) > 0\}.$$

Remark 2.5. Since $u \in H^1(B_1)$, we have $\lim_{r\downarrow 0} r^{2-m}\int_{B_r(x)}|\nabla u|^2 = 0$ (cf. [FZ72]) for $\mathcal{H}^{m-2}$-a.e. $x \in \Sigma$. This implies that $\Theta^{m-2}(\mu, x) = \Theta^{m-2}(\nu, x)$, for $\mathcal{H}^{m-2}$-a.e. $x \in B_1$. Now we have $\nu(x) = \Theta^{m-2}(\mu, x)\mathcal{H}^{m-2}\llcorner\Sigma$ for $x \in B_1$.

In view of (2.4) and Remark 2.4, we have $\varepsilon_0^2 \leq \Theta^{m-2}(\mu, x) \leq C(r, K)$ for any $0 < r < 1$ with $x \in \Sigma \cap B_r$. Next we consider the blow up and stratification of the measure $\mu = \frac{1}{2}|\nabla u|^2\mathrm{d}x + \nu$. For $x_0 \in B_1$ and $\rho > 0$, we define the scaling of this measure by $\mu_{x_0,\rho}$, such that $\mu_{x_0,\rho}(A) = \rho^{2-m}\mu(x_0 + \rho A)$, for any measurable set $A \subset \mathbb{R}^m$ being bounded. In view of Lemma 2.4, $\mu_{x_0,\rho}(\overline{B}_R)$ is uniformly bound for any fixed R, with respect to $\rho > 0$. Now, we can choose a subsequence ρ_i such that $\mu_{x_0,\rho_i} \rightharpoonup^* \eta$ for some nonnegative Radon measure η on $\mathbb{R}^m$. We denote $T_{x_0}\mu$ as the set of all tangent measure of μ at the point x_0. Here we see that $\eta \in T_{x_0}\mu$ Moreover, by diagonal process, we can choose a subsequence ε_i, such that $\mathbf{e}_{\overline{\varepsilon}_i}(u_{\overline{\varepsilon}_i})\mathrm{d}x \rightharpoonup^* \eta$, where $\overline{\varepsilon}_i = \varepsilon_i/\rho_i$. Now we have the following result.

Lemma 2.6 (Properties of the tangent measure)**.** *The limit measure η satisfies the following properties.*

(1) *η is a cone measure on $\mathbb{R}^m$, that is, for any $s > 0$, $\eta_{0,s} = \eta$.*
(2) $\Theta^{m-2}(\eta, 0) = \Theta^{m-2}(\mu, x_0) = \max\{\Theta^{m-2}(\eta, x) : x \in \mathbb{R}^m\}$.
(3) *The set $L_\eta = \{x \in \mathbb{R}^m : \Theta^{m-2}(\eta, x) = \Theta^{m-2}(\eta, 0)\}$ is a linear subspace of $\mathbb{R}^m$ with $\dim(L_\eta) \leq m - 2$.*

Through this lemma, we can define the stratification of Σ by

$$\Sigma_j = \{x \in \Sigma : \dim(L_\eta) \leq j \text{ for any tangent measure } \eta \in T_{x_0}\mu\}.$$

Obviously, we can directly obtain from the definition above that $\Sigma_0 \subset \Sigma_1 \subset ... \subset \Sigma_{m-2} = \Sigma$. Moreover, we can use the famous Federer reduction of dimension method, which is similar in the stratification of harmonic maps (cf. [SU82]) to obtain the following results on the stratification of Σ.

Lemma 2.7 (Stratification)**.** *$\Sigma = \cup_{j=1}^{m-2}\Sigma_j$ and $\dim_{\mathcal{H}}(\Sigma_j) \leq j$ for $j = 0, 1, \dots, m-2$, where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension.*

Proof of Lemma 2.6. For $\alpha, \beta = 1, 2, \dots, m$, define $\mu_{i,1}^{\alpha\beta} := \partial_\alpha u_{\overline{\varepsilon}_i}\partial_\beta u_{\overline{\varepsilon}_i}\mathrm{d}x$ and $\mu_{i,2} := \frac{1}{\overline{\varepsilon}_i}F(u_{\overline{\varepsilon}_1})\mathrm{d}x$. By the compactness result, we can assume that $\mu_{i,1}^{\alpha\beta} \rightharpoonup^* \eta_1^{\alpha\beta}$ and $\mu_{i,2} \rightharpoonup^* \eta_2$ for Radon measures $\eta_1^{\alpha\beta}$ and η_2 on $\mathbb{R}^m$. In view of Lemma 1.3, we can obtain that

$$\begin{aligned} &\Theta^{m-2}(\eta, x, R) - \Theta^{m-2}(\eta, x, r) \\ &\quad= 2\int_{B_R(x)\setminus B_r(x)} |y - x|^{-m}(y - x)_\alpha(y - x)_\beta \mathrm{d}\eta_1^{\alpha\beta} \\ &\qquad + 2\int_r^R t^{1-m}\eta_2(B_t(x))\mathrm{d}t, \end{aligned} \tag{2.9}$$

for any $x \in \mathbb{R}^m$ and a.e. $0 < r \leq R < +\infty$. Now we claim that for any $r > 0$, $\Theta^{m-2}(\eta, 0, r) = \Theta^{m-2}(\eta, 0) = \Theta^{m-2}(\mu, x_0)$. Indeed, for a.e. $r > 0$, $\eta_*(\partial B_r) = 0$ and then

$$\begin{aligned} \Theta^{m-2}(\eta, 0, r) &= \lim_{i\to+\infty}\Theta^{m-2}(\mu_{x_0,\rho_i}, 0, r) \\ &= \lim_{i\to+\infty}\Theta^{m-2}(\mu, x_0, r\rho_i) \\ &= \Theta^{m-2}(\mu, x_0). \end{aligned}$$

This directly implies claim. Choosing $x=0$ in the formula (2.9), we obtain that

$$\int_{B_R\setminus B_r}|y|^{-m}y_\alpha y_\beta \mathrm{d}\eta_1^{\alpha\beta}=0 \text{ and } \int_r^R t^{1-m}\eta_2(B_t)\mathrm{d}t=0,$$

for a.e. $0<r\leq R<+\infty$, where we have also used the property from convergence that $\eta_1^{\alpha\beta}\llcorner x_\alpha x_\beta\geq 0$ and $\eta_2\geq 0$. As a result, $\eta_1^{\alpha\beta}\llcorner x_\alpha x_\beta=0$ and $\eta_2=0$. Note that for any $\phi\in C_0^\infty(\mathbb{R}^m,\mathbb{R})$, since

$$\left(\int_{\mathbb{R}^m}x_\alpha\partial_\beta\phi \mathrm{d}\mu_{i,1}^{\alpha\beta}\right)^2\leq\left(\int_{\mathbb{R}^m}\partial_\alpha\phi\partial_\beta\phi \mathrm{d}\mu_{i,1}^{\alpha\beta}\right)\left(\int_{\operatorname{supp}\phi}x_\alpha x_\beta \mathrm{d}\mu_{i,1}^{\alpha\beta}\right),$$

we have, by taking $i\to+\infty$, there holds $\int_{\mathbb{R}^m}x_\alpha\partial_\beta\phi \mathrm{d}\eta_1^{\alpha\beta}=0$. For any $\psi\in C_0^\infty(\mathbb{R}^m)$ and $s>0$, define $\psi_s(x)=\psi(\frac{x}{s})$. Since $u_{\bar{\varepsilon}_i}$ satisfies the equation $\Delta u_{\bar{\varepsilon}_i}+\frac{1}{\bar{\varepsilon}_i^2}f(u_{\bar{\varepsilon}_i})=0$, we can test it by the funtion $\psi_s x\cdot\nabla u_\varepsilon$ and obtain from integration by parts that

$$\begin{aligned}\int_{\mathbb{R}^m}&\left((2-m)\psi_s-\frac{x}{s}\cdot\nabla\psi\left(\frac{x}{s}\right)\right)|\nabla u_{\bar{\varepsilon}}|^2\mathrm{d}x\\&=-2\int_{\mathbb{R}^m}x_\beta\partial_\alpha\psi_s \mathrm{d}\mu_{i,1}^{\alpha\beta}+\int_{\mathbb{R}^m}\partial_\beta(x_\beta\psi_s)\mathrm{d}\mu_{i,2}.\end{aligned}$$

Taking $i\to+\infty$, we have

$$\int_{\mathbb{R}^m}\left((2-m)\psi_s-\frac{x}{s}\cdot\nabla\psi\left(\frac{x}{s}\right)\right)\mathrm{d}\eta=0.$$

Through simple calculations, it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}(\eta_{0,s}(\psi))&=\frac{\mathrm{d}}{\mathrm{d}s}\left(s^{2-m}\int_{\mathbb{R}^m}\psi_s\mathrm{d}\eta\right)\\&=s^{1-m}\int_{\mathbb{R}^m}\left((2-m)\psi_s-\frac{x}{s}\cdot\nabla\psi\left(\frac{x}{s}\right)\right)d\eta,\end{aligned}$$

we conclude that $\frac{\mathrm{d}}{\mathrm{d}s}(\eta_{0,s}(\psi))=0$ and hence $\eta_{0,s}=\eta$ for any $s>0$. Now, we complete the proof of the first property. To show the second one, since we have already obtained that $\Theta^{m-2}(\eta,0)=\Theta^{m-2}(\mu,x_0)$, we only need to prove that for any $x\in\mathbb{R}^m$, $\Theta^{m-2}(\eta,x)\leq\Theta^{m-2}(\eta,0)$. Indeed, by Lemma 2.4, for any $R\geq r>0$, we have

$$\begin{aligned}\Theta^{m-2}(\eta,x)&\leq\Theta^{m-2}(\eta,x,R)\\&\leq\left(\frac{R+|x|}{R}\right)^{m-2}\Theta^{m-2}(\eta,0,R+|x|)\\&=\left(\frac{R+|x|}{R}\right)^{m-2}\Theta^{m-2}(\eta,0).\end{aligned}\tag{2.10}$$

Choosing $R\to+\infty$, we obtain that $\Theta^{m-2}(\eta,x)\leq\Theta^{m-2}(\eta,0)$. For the last property, we note that for any $x\in L_\eta$, it follows from (2.10) that $\Theta^{m-2}(\eta,x,r)=\Theta(\eta,x)$ for any $r>0$. Using (2.9), we can get that

$$\int_{B_R(x)\setminus B_r(x)}|y-x|^{-m}(y-x)_\alpha(y-x)_\beta \mathrm{d}\eta_1^{\alpha\beta}=0$$

for a.e. $0<r<R<+\infty$. Using almost the same arguments in the proof of the second property, we have $\eta_{x,s}=\eta$ for any $s>0$. Now assume that $y\in L_\eta$, we claim that for any

$x \in \mathbb{R}^m$, $t \in \mathbb{R}$, there holds $\eta_{x,1} = \eta_{x+ty,1}$. In fact, for any measure set $A \subset \mathbb{R}^m$ and $s > 0$, we have

$$\begin{aligned}\eta_{x,1}(A) &= s^{2-m}\eta(sx + sA)\\ &= s^{m-2}\eta\left(y + \frac{sx - y}{s^2} + \frac{A}{s}\right)\\ &= \eta(x + (s - s^{-1})y + A),\end{aligned}$$

where for the first and last inequality, we have used $\eta_{0,s} = \eta$ for any $s > 0$ and for the second inequality, we have used $\eta_{y,s} = \eta$ for any $s > 0$. Since $s - s^{-1}$ can be any real number, we have completed the proof of this claim. Now for any $y_1, y_2 \in L_\eta$ and $t_1, t_2 \in \mathbb{R}$, by using this claim, it can be obtained that $\eta_{t_1y_1+t_2y_2,1} = \eta$ and then obviously $\Theta^{m-2}(\eta, t_1y_1 + t_2y_2) = \Theta^{m-2}(\eta, 0)$, which implies that $t_1y_1 + t_2y_2 \in L_\eta$ and then L_η is linear subspace of $\mathbb{R}^m$. Finally, we prove that $\dim(L_\eta) \le m - 2$. To see this, we can prove by contradiction. Taking $\dim(L_\eta) = m - 1$ as an example (the case $\dim(L_\eta) = m$ is almost the same), Without loss of generality, we assume that $L_\eta = \mathbb{R}^{m-1}$. Since $B_R^{m-1} \times [-1, 1]$ contains number $\sim R^{m-1}$ balls $\{B_1(y_i)\}$ such that $y_i \in \mathbb{R}^{m-1}$. Then by the assumptions on η,

$$\Theta^{m-2}(\eta, 0, R) \ge \sum_{i=1}^{N} R^{2-m}\eta(B_1(y_i)) \sim R\eta(B_1) \to +\infty$$

if $R \to \infty$. This is a contradiction to the fact that $\Theta^{m-2}(\eta, 0)$ is finite. Now we can complete the proof. □

To show Theorem 1.1, we will prove by contradiction. Assume that $e_\varepsilon(u_\varepsilon)\mathrm{d}x \nrightarrow \frac{1}{2}|\nabla u|^2\mathrm{d}x$. This is equivalent to say that $\mathcal{H}^{m-2}(\Sigma) > 0$ or $\nu(B_1) > 0$. We choose $x_0 \in \Sigma$, such that the following properties hold.

(1) $\Theta^{m-2}(\mu, x_0) = \Theta^{m-2}(\nu, x_0) \ge \varepsilon_0^2$.
(2) $\lim_{r\downarrow 0} r^{2-m} \int_{B_r(x_0)} |\nabla u|^2 = 0$.
(3) $x_0 \in \Sigma_{n-2}\backslash\Sigma_{n-3}$.

The first and second one is from the Remark 2.4, 2.5 and the last properties is the consequence of the fact that $\mathcal{H}^{m-2}(\Sigma) > 0$ and Lemma 2.7. Consider the rescaling measures $\mu_{x_0,\rho}$. By using (2.4), and the assumption that $\Theta^{m-2}(\mu, x_0) \ge \varepsilon_0^2$, we have $2^{m-2}\varepsilon_0^2 \le \mu_{x_0,\rho}(\overline{B}_2) \le C(x_0, K) < +\infty$. By the compactness of Radon measures, we can choose a subsequence $\rho_i \downarrow 0$ such that $\mu_i = \mu_{x_0,\rho_i} \rightharpoonup^* \mu_*$, where μ_* is a nonnegative Radon measure. By diagonal method, we can choose a subsequence $\{\varepsilon_i\}_{i=1}^{+\infty}$ such that $u_{\varepsilon_i} \rightharpoonup u_*$ weakly in $H^1(B_1)$ and $e_{\overline{\varepsilon}_i}(u_{\overline{\varepsilon}_i})\mathrm{d}x \rightharpoonup^* \mu_*$, where $\overline{\varepsilon}_i = \varepsilon_i/\rho_i$. By the second property above, we can further assume that u_* is a constant function. This implies that $\mathrm{supp}(\mu_*) = \Sigma_*$, where

$$\begin{aligned}\Sigma_* &= \bigcap_{r>0}\left\{x \in \mathbb{R}^m : \liminf_{i\to+\infty} r^{2-m}\int_{B_r(x)} e_{\overline{\varepsilon}_i}(u_{\overline{\varepsilon}_i}) \ge \varepsilon_0^2\right\}\\ &= \{x \in \mathbb{R}^m : \Theta^{m-2}(\mu_*, x) > 0\}.\end{aligned} \tag{2.11}$$

By the third property, we further assume that $\dim(L_{\mu_*}) = m - 2$. In view of Lemma 2.6, we have $\Theta^{m-2}(\mu_*, x) = \Theta^{m-2}(\mu_*, 0) = \Theta^{m-2}(\mu, x_0) > \varepsilon_0^2$ for any $x \in L_{\mu_*}$. By the definition of Σ_*, there holds that $L_{\mu_*} \subset \Sigma_*$. We claim that $\Sigma_* = L_{\mu_*}$. Firstly we show that $(\Sigma_*)_{x,s} = \Sigma_*$ for any $x \in L_{\mu_*}$ and $s > 0$, where $(\Sigma_*)_{x,s} = \frac{\Sigma_* - x}{s}$. This property directly implies that $\Sigma_* = L_{\mu_*}$. Without loss of generality, we assume that $x = 0$ and for other points, we apply the almost the same arguments. Indeed, we show that for $x \in \Sigma_*$ and $s > 0$, $sx \in \Sigma_*$. Using (2.11), we only need to show that $\Theta^{m-2}(\mu_*, sx) > 0$. This is because

$$\begin{aligned}\Theta^{m-2}(\mu_*, sx) &= \lim_{r\downarrow 0} r^{2-m}\mu_*(\overline{B_r(sx)})\\ &= \lim_{r\downarrow 0}\left(\frac{r}{s}\right)^{2-m}\mu_*(\overline{B_{r/s}(x)})\\ &= \Theta^{m-2}(\mu_*, x) > 0,\end{aligned}$$

where for the second equality, we have used $(\mu_*)_{0,s} = \mu_*$ and for the inequality, we have used $x \in \Sigma_*$. Using Remark 2.5, we deduce that

$$\mu_*(x) = \Theta^{m-2}(\mu, x)\mathcal{H}^{m-2}\llcorner\Sigma_* = \Theta^{m-2}(\mu, x)\mathcal{H}^{m-2}\llcorner(\mathbb{R}^{m-2} \times \{(0,0)\}),$$

where we have identify $L_{\mu_*} = \mathbb{R}^{m-2} \times \{(0,0)\}$. Choose

$$\xi^j_{\pm} = (0, \dots, \pm 4, \dots, 0) \times \{(0,0)\}$$

with $j = 1, 2, \dots, m-2$, where there is only ± 4 in j-th position. Now applying Lemma 1.3 to different $\xi^j_{\pm}$, we can obtain that

$$\lim_{i\to+\infty}\int_{B_1^{m-2}\times B_1^2}\left(\sum_{j=1}^{m-2}\left|\frac{\partial u_{\overline{\varepsilon}_i}}{\partial x_j}\right|^2 + \frac{F(u_{\overline{\varepsilon}_i})}{\overline{\varepsilon}_i^2}\right) = 0. \tag{2.12}$$

Set $X_1 = (x_1, x_2, \dots, x_{m-2})$ and $X_2 = (x_{m-1}, x_m)$. For $i \geq 1$, we define the function $f_i : B^{m-2}_{1/2} \to \mathbb{R}$ by

$$f_i(X_1) := \int_{B_1^2}\left(\sum_{j=1}^{m-2}\left|\frac{\partial u_{\overline{\varepsilon}_i}}{\partial x_j}\right|^2 + \frac{F(u_{\overline{\varepsilon}_i})}{\overline{\varepsilon}_i^2}\right)(X_1, X_2)\mathrm{d}X_2.$$

In view of Fubini theorem and (2.12), we can obtain that

$$\lim_{i\to+\infty}\|f_i\|_{L^1(B^{m-2}_{1/2})} = 0. \tag{2.13}$$

Define the local Hardy-Littlewood maximal function of f_i by

$$M(f_i)(X_1) = \sup_{0<r\leq\frac{1}{2}} r^{2-m}\int_{B_r^{m-2}(X_1)} f_i,\ X_1 \in B^{m-2}_{1/2}.$$

By the weak L^1 estimate for Hardy-Littlewood maximal functions and (2.13), we can find $\{X_1^i\} \subset B^{m-2}_{1/2}$ such that

$$\lim_{i\to+\infty} M(f_i)(X_1^i) = 0. \tag{2.14}$$

We claim that for sufficiently large i, there exist $0 < \delta_i \to 0$, and $\{X_2^i\} \subset B^2_{1/4}$ with $X_2^i \to (0,0)$ such that

$$\begin{aligned}&\delta_i^{2-m}\int_{B^{m-2}_{\delta_i}(X_1^i)\times B^2_{\delta_i}(X_2^i)} e_{\overline{\varepsilon}_i}(u_{\overline{\varepsilon}_i})\\ &= \max_{X_2\in B^2_{1/2}}\delta_i^{2-m}\int_{B^{m-2}_{\delta_i}(X_1^i)\times B^2_{\delta_i}(X_2)} e_{\overline{\varepsilon}_i}(u_{\overline{\varepsilon}_i}) = \frac{\varepsilon_0^2}{C(m)},\end{aligned} \tag{2.15}$$

where $C(m) > 0$ is a large constant to be chosen. To prove this claim we define

$$g_i(\delta) = \max_{X_2 \in B^2_{1/2}} \delta^{2-m} \int_{B^{m-2}_\delta(X^i_1)\times B^2_{\delta_i}(X_2)} e_{\bar{\varepsilon}_i}(u_{\bar{\varepsilon}_i}).$$

Firstly, we observe that u_{ε_i} is smooth near $X^i_1 \times B^2_{1/2}$ and $F(u_{\varepsilon_i})$ is bounded. This implies that for fixed $i \geq 0$, there holds $\lim_{\delta\downarrow 0} g_i(\delta) = 0$. On the other hand, for fixed $\delta > 0$, we have $\lim_{i\to+\infty} g_i(\delta) > \varepsilon_0^2$. This is because if there exists $\delta_0 > 0$ such that $g_i(\delta_0) < \varepsilon_0^2$ for i sufficiently large, we can use Lemma 1.4 to obtain that

$$\delta_0^2 \max_{B^{m-2}_{\delta_0/4}\times B^2_{\delta_0/4}} e_{\bar{\varepsilon}_i}(u_{\bar{\varepsilon}_i}) \leq C\varepsilon_0^2.$$

This is implies that $\mu_* \llcorner B^{m-2}_{\delta_0/4} \times B^2_{\delta_0/4} = 0$ and is contradictory to the assumption of μ_*. Now we can choose $\delta_i \to 0$ and $\{X^i_2\} \subset B^2_{1/4}$ such that (2.15) is true. Finally, we only need to show that $X^i_2 \to (0,0)$. If not, we have $X^i_2 \nrightarrow (0,0)$. Without loss of generality, we assume that $X^i_2 \to X^0_2 \neq (0,0)$ with $X^i_2 \in B^2_{1/2}$. Define $r_0 := |X^0_2| > 0$, we can use Lemma 1.3 to obtain

$$\left(\frac{r_0}{2}\right)^{2-m} \int_{B_{r_0/2}(X^i_1, X^0_2)} e_{\bar{\varepsilon}_i}(u_{\bar{\varepsilon}_i}) \geq \frac{\varepsilon_0^2}{C(m)}$$

for i sufficiently large. In particular,

$$\int_{B^{m-2}_1 \times (B^2_{2r_0}\backslash B^2_{r_0/2})} e_{\bar{\varepsilon}_i}(u_{\bar{\varepsilon}_i}) \geq C(\varepsilon_0, r_0) > 0$$

for sufficiently large i. This is contradictory to the property that $u_{\bar{\varepsilon}_i}$ converges to a constant function in $C^1(B^m_1 \backslash (B^{m-2}_1 \times \{(0,0)\}), \mathbb{R}^k)$, and $\frac{1}{\bar{\varepsilon}_i^2}(B^m_1\backslash(B^{m-2}_1 \times \{(0,0)\}) \to 0$ in $L^1_{\mathrm{loc}}((B^m_1\backslash(B^{m-2}_1 \times \{(0,0)\})$, where we have used the third property of Lemma 2.2 and the proof of it. Now we proceed the blow-up scheme as follows. Let $p_i = (X^i_1, X^i_2)$, $R_i = (4\delta_i)^{-1}$, and $\Omega_i = B^{m-2}_{R_i} \times B^2_{R_i}$. Since $R_i \to +\infty$, Ω_i converges to $\mathbb{R}^m$. Define $w_i : \Omega_i \to \mathbb{R}^k$ by $w_i(x) = u_{\bar{\varepsilon}_i}(p_i + \delta_i x)$, $x \in \Omega_i$. Then w_i satisfies $\Delta w_i + (\frac{\delta_i}{\bar{\varepsilon}_i})^2 f(w_i) = 0$ in Ω_i. Moreover, (2.14) and (2.15) imply

$$\lim_{i\to+\infty} \sup_{0<R<R_i} R^{2-m} \left(\int_{B^{m-2}_R \times B^2_{R_i}} \sum_{j=1}^{m-2} \left| \frac{\partial w_i}{\partial x_j} \right|^2 + \frac{F(w_i)}{\widehat{\varepsilon}_i^2} \right) = 0, \ \widehat{\varepsilon}_i = \frac{\bar{\varepsilon}_i}{\delta_i} \tag{2.16}$$

$$\int_{B^{m-2}_1 \times B^2_1} e_{\widehat{\varepsilon}_i}(w_i) = \frac{\varepsilon_0^2}{C(m)} = \max\left\{ \int_{B^{m-2}_1 \times B^2_1(X_2)} e_{\widehat{\varepsilon}_i}(w_i) : X_2 \in B^2_{R_i - 1} \right\}, \tag{2.17}$$

$$\sup_i \left\{ R^{2-m} \int_{B^{m-2}_R \times B^2_R} e_{\widehat{\varepsilon}_i}(w_i) \right\} \leq C, \text{ for any } 0 < R < R_i. \tag{2.18}$$

Therefore, by (2.17) and Lemma 1.4, if $\widehat{\varepsilon}_i \to 0$, we may assume that there exists a smooth harmonic map $w : B^{m-2}_{1/2} \times \mathbb{R}^2 \to N$ such that

$$w_i \to w \text{ in } C^2_{\mathrm{loc}}(B^{m-2}_{1/2} \times \mathbb{R}^2, N) \text{ as } i \to +\infty.$$

It follows from (2.16) that

$$\sum_{j=1}^{m-2} \int_{B_{1/2}^{m-2}\times\mathbb{R}^2} \left|\frac{\partial w}{\partial x_j}\right|^2 = 0$$

so that $w(X_1, X_2) = w(X_2)$ is independent of the first $m-2$ variables. Hence $w : \mathbb{R}^2 \to N$ is a smooth harmonic map. This, (2.18) and (2.16) imply that

$$0 < \int_{\mathbb{R}^2} |\nabla w|^2 \leq C.$$

Therefore w can be lifted to be a nontrivial harmonic map from S^2 to N by the singularity theorem in [SU81]. If $\widehat{\varepsilon}_i \to c > 0$ or $+\infty$, we can use almost the same arguments in the proof of the case that $m = 2$ to get contradictions.

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