NOTES ON ELLIPTIC FREE BOUNDARY PROBLEMS

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1. Preface

Elliptic free boundary problems is or great importance in the study of mathematics and physics. There is a famous lecture note by Professor Fanghua Lin discussing this topics named "Lectures on Elliptic Free Boundary Problems". This lecture note is very rich in content, and is an introductory reading material for learning the free boundary theory of elliptic equations. Because this note covers many aspects of theory and is limited in length, there are many omissions in details. In the 2022 winter semester, the author discussed this handout with Yuxuan Chen in a seminar at Peking University. In the process of discussion and learning, we have supplemented the proof details of most of the note, which makes the proof easier to understand.

In the seminar, Yuxuan Chen and I also gave some proofs of different methods of theorems in the handouts. Finally, after the seminar, I sorted it out and wrote a note on Professor Fanghua Lin's lecture note. Due to the limited time, my notes did not disturb the order of the original handouts, and our supplement was added to the specific proof of knowledge to make reading easier.

In general the added supplement mainly comes from the following aspects

- For some unproved conclusions mentioned in the lecture note, we give proofs to make the overall logic more complete.
- For the case of Laplacian operator, we try our best to extend it to the elliptic equation with variable coefficients.
- We will correct some clerical errors in the original text.

In the process of writing notes on this handout, I have gained a lot. I learned theory and exercised my writing ability at the same time. This note can not be produced without the guidance of teachers and the help of classmates. While thanking Yuxuan Chen, the co-organizer of the seminar, I would also like to thank Professor Zhifei Zhang for his important suggestions for the seminar. In addition, I would like to thank Haotong Fu, Feng Shao and Wenzhi Wu for their questions and corrections in the discussion class. Without them, the learning of theory may be more difficult.

Due to my limited level, there will inevitably be some mistakes in this note. I hope readers can point out if they find them when reading. I will be very happy about such behavior.

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In Peking Univeristy
Winter of 2022

Abstract. This series of lectures is devoted to the study of some classical elliptic free boundary problems. The understanding of free boundary problems has been a fascinating and beautiful topic in the analysis and partial differential equations. Here we discuss some general existence and regularity theorems for solutions, as well as the results concerning the regularity of free boundaries. In particular, the classical obstacle problem are carefully examined. The study of singular sets of free boundaries was a very difficult and challenging task for many years. Here the important break through in late 1990 s by Caffarelli is also presented in detail. One believes these ideas and techniques may be applied to a much wide class of problems.

2. Introduction

This write-up covers a set of ten lectures (of total 20 hours) selected from a course given by the author at the Courant Institute of Mathematical Sciences over the years under the title "Elliptic Free Boundary Problems." The author had given similar lectures in the summer schools at Tsinghua University in 2009 and at the East China Normal University in 2010. The literature on this subject is large and many beautiful ideas and techniques have been developed over the last half century. The author has no intention here that this set of lectures would be fully comprehensive or relatively complete description of the whole theory. Rather, one would select a few important and elegant papers that illustrate well some of the key ideas involved in handling a couple of basic questions such as the existence and regularity of classical obstacle problems and the regularity of free boundaries arising in the solutions. The basic references for these lectures are the book by D. Kinderlehrer and G. Stampacchia [16], the book by A. Friedman [7] and the book by L. Caffarelli and S. Salsa [4]. Otherwise, in each chapter I shall provide more references that the lectures are based on.

Examples of Free Boundary Problems

Let us first describe a few classical free boundary value problems.

Example 2.1 (A potential problem)**.** Consider the following PDE in a bounded domain $\Omega \subset \mathbb{R}^n$

$$\begin{cases} -\Delta u = 1 \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial\Omega} = \text{constant}. \end{cases} \tag{2.1}$$

In general, this equation is overdetermined if Ω is given, but we shall ask what is (u, Ω) if they are both unknowns. The answer is that Ω is a ball and u is a radial quadratic function. The proof for this result was given by J. Serrin and H. Weinberger separately in 1972. Serrin's proof adopted the idea of moving plane. Weinberger's proof simply used integration by parts in a tricky way, which does not rely on the maximum principle. Hence, it could be generalized to some cases of elliptic systems.

Example 2.2 (Schiffer's conjecture)**.** Consider the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u \text{ in } \Omega, \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, \\ u|_{\partial\Omega} = \text{constant} \neq 0, \end{cases} \tag{2.2}$$

where λ is positive and is not necessarily the first eigenvalue for $-\Delta$ in Ω. Treating (u, Ω) as unknowns, Schiffer's conjecture asks: is Ω a ball?

Schiffer's conjecture is closely related to the Pompeiu problem, in the sense that the failure of the Pompeiu property is equivalent to the existence of a nontrivial solution of (2.2). A domain Ω is said to satisfy Pompeiu property, if and only if $f = 0$ is the only continuous function in $\mathbb{R}^n$ such that, for any isometry R of $\mathbb{R}^n$, $\int_{R(\Omega)} f(x)dx = 0$. It is known that if a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ has a connected boundary $\partial\Omega$ and if Ω fails to have the Pompeiu property, then $\partial\Omega$ is real analytic.

Example 2.3 (Minimizing the k-th Dirichlet eigenvalue of $-\Delta$ on a domain; Polya-Szegöproblem)**.** For $k \geq 1$, find out $\inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^n, |\Omega| = 1$ and Ω is bounded, connected and Lipschitz$\}$, where λ_k is the k-th Dirichlet eigenvalue of $-\Delta$ on Ω. For $k = 1$, this problem is solved by Faber-Krahn inequality. It is well-known that the first Dirichlet eigenvalue of $-\Delta$ on Ω is given by

$$\lambda_1(\Omega) = \inf\left\{\int_\Omega |\nabla u|^2 : u \in H_0^1(\Omega) \text{ with } \int_\Omega u^2 = 1\right\}.$$

Faber-Krahn inequality states that for $n \geq 2$,

$$\lambda_1(B) \leq \lambda_1(\Omega), \quad \text{ for } \forall\Omega \subset \mathbb{R}^n \text{ with } |\Omega| = |B|,$$

where B is a ball in $\mathbb{R}^n$. The equality holds if and only if Ω is a ball. To make it more clearly, we will state the Faber-Krahn inequality as follows and give a proof for it.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let B be the ball centered at the origin with $|\Omega| = |B|$. Then $\lambda_1(\Omega) \geq \lambda_1(B)$, with equality if and only if $\Omega = B$ almost everywhere.*

Here $\lambda_1(\Omega)$ is the first eigenvalue of the Laplacian, with Dirichlet boundary conditions.

Proof. Recall variational characterization of the first eigenvalue:

$$\lambda_1(\Omega) = \inf\left\{\frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx} : u \in C_0^2(\Omega)\right\}. \tag{2.3}$$

By the Courant nodal domain theorem, we can take a test function for the Rayleigh quotient be non-negative. Let u be a test function and for $0 \leq t \leq \widehat{u} = \max(u)$ let $\Omega_t = \{u > t\}$. Now we define a comparison function $u_* : B \to [0, \infty)$ as follows. First let B_t be the ball centered at the origin with $|B_t| = |\Omega_t|$. Then let u_* be the radially symmetric function such that $B_t = \{u_* > t\}$. By the co-area formula,

$$\int_t^{\widehat{u}} \int_{\partial\Omega_\tau} \frac{dS}{|\nabla u|} d\tau = |\Omega_t| = |B_t| = \int_t^{\widehat{u}} \int_{\partial B_\tau} \frac{dS}{|\nabla u_*|} d\tau$$

Differentiating with respect to t gives us

$$\int_{\partial\Omega_t} \frac{dS}{|\nabla u|} = \int_{\partial B_t} \frac{dS}{|\nabla u_*|} \tag{2.4}$$

for all t. Then

$$\begin{aligned} \int_\Omega u^2 dx &= \int_0^{\widehat{u}} \int_{\partial\Omega_t} \frac{u^2 dS}{|\nabla u|} dt = \int_0^{\widehat{u}} t^2 \int_{\partial\Omega_t} \frac{dS}{|\nabla u|} dt \\ &= \int_0^{\widehat{u}} t^2 \int_{\partial B_t} \frac{dS}{|\nabla u_*|} dt = \int_B u_*^2 dx. \end{aligned} \tag{2.5}$$

Now, for $0 \leq t \leq \widehat{u}$ let

$$\psi(t) = \int_{\Omega_t} |\nabla u|^2 dx, \quad \psi_*(t) = \int_{B_t} |\nabla u_*|^2 dx.$$

By the co-area formula

$$\psi' = -\int_{\partial\Omega_t} |\nabla u| dS, \quad \psi_*' = -\int_{\partial B_t} |\nabla u_*| dS.$$

We use the Cauchy-Schwarz inequality, the isoperimetric inequality and the fact that the normal derivative of u_* is constant on ∂B_t to see

$$\left(\int_{\partial\Omega_t} |\nabla u| dS\right)\left(\int_{\partial D_t} \frac{dS}{|\nabla u|}\right) \geq \left(\int_{\partial\Omega_t} dS\right)^2 = \left(\mathrm{Area}\,(\partial\Omega_t)\right)^2$$
$$\geq \left(\mathrm{Area}\,(\partial B_t)\right)^2 = \left(\int_{\partial B_t} |\nabla u_*| dS\right)\left(\int_{\partial B_t} \frac{dS}{|\nabla u_*|}\right).$$

We use equation (2.4) to cancel the common factor of

$$\int_{\partial\Omega_t} \frac{dS}{|\nabla u|} = \int_{\partial B_t} \frac{dS}{|\nabla u_*|}$$

and so

$$-\psi' = \int_{\partial\Omega_t} |\nabla u| dS \geq \int_{\partial B_t} |\nabla u_*| dS = -\psi_*'.$$

Integrating this last differential inequality and using $\psi(\widehat{u}) = 0 = \psi_*(\widehat{u})$ we see

$$\int_\Omega |\nabla u|^2 dx = \psi(0) \geq \psi_*(0) = \int_B |\nabla u_*|^2 dx.$$

Combine this inequality with (2.5) and (2.3) to give the desired inequality on the eigenvalues:

$$\lambda_1(\Omega) \geq \lambda_1(B).$$

Moreover, equality of the eigenvalues forces the level sets $\partial\Omega_t$ to all be spheres centered at the origin. Also, the equality case of the Cauchy-Schwarz inequality forces $|\nabla u|$ to be constant on the level set $\partial\Omega_t$. Thus u must be radially symmetric and so in this case $u = u_*$. □

For $k > 1$, the problem is still open. We remark that choosing the right conditions on domains is critical. The infimum may not be obtained over the class of bounded connected Lipschitz domains. In fact, we consider $\lambda_2(\Omega)$, where Ω is a bounded connected Lipschitz domain. Since the second eigenfunction u_2 must change sign in Ω, we define Ω_+ (respectively Ω_-) to be the subset of Ω on which $u_2 > 0$ (respectively $u_2 < 0$). Then $u|_{\Omega_+}$ and $u|_{\Omega_-}$ are the first eigenfunctions on Ω_+ and Ω_- respectively. By Faber-Krahn inequality, the corresponding eigenvalue would decrease if $\Omega_\pm$ are replaced by two identical balls $B_\pm$ whose total volume is equal to $|\Omega|$. We denote $\lambda_* = \lambda_1(B_+) = \lambda_1(B_-)$. Clearly, λ_* can not be achieved as $\lambda_2(\Omega)$ for some bounded connected Lipschitz domain Ω. We can further show that there exists a sequence of bounded connected Lipschitz domains Ω_k such that $\lambda_2(\Omega_k) \to \lambda_*$ as $k \to \infty$. Therefore, the infimum is not obtained among bounded connected Lipschitz domains.

An important class of free boundary problems is the obstacle problem, about which we shall discuss a lot.

Example 2.5 (Obstacle problem)**.** Consider a domain Ω whose boundary is decomposed into two parts $\partial\Omega = \Gamma_s \cup \Gamma_f$ with $\Gamma_s \cap \Gamma_f$ intersects only on their boundaries (of Hausdorff dimension less than that of $\partial\Omega$). We fix Γ_s and leave Γ_f as the free boundary. We shall solve for (u, Γ_f) in the following equation

$$\begin{cases} -\Delta u = f \quad \text{in } \Omega, \\ u|_{\Gamma_s} = g \geq 0, \\ u|_{\Gamma_f} = 0, \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\Gamma_f} = 0. \end{cases} \tag{2.6}$$

The equation (2.6) comes from the following model problem in $\mathbb{R}$. Given $\psi \in C^0[p,q]$, $\psi(p) < 0$, $\psi(q) < 0$ and ψ is positive somewhere on (p,q), let $\Omega = \{(x,y) : x \in [p,q], y \geq \psi(x)\}$. Solve

$$\min\{l(\gamma) : \gamma(0) = p, \gamma(1) = q, \gamma \subset \Omega\}, \tag{2.7}$$

where γ is a curve connecting p and q and $l(\gamma)$ is its length. ψ is called the obstacle function.

Suppose u is the solution. We would find $u(x) \geq \psi(x)$, $u(p) = u(q) = 0$ and $u_{xx} = 0$ if $u > \psi$. Denote $q' = \sup\{x : u(x) = \psi(x)\}$. It is easy to show $p < q' < q$ and

$$\begin{cases} u_{xx} = 0 \quad \text{in } (q', q), \\ u(q') = \psi(q'), \\ u_x(q') = \psi_x(q'), \\ u(q) = 0. \end{cases} \tag{2.8}$$

Note that q' is not a priori known here. Let $\omega = u - \psi$. Then ω solves

$$\begin{cases} \omega_{xx} = -\psi_{xx} \quad \text{in } (q', q), \\ \omega > 0 \quad \text{in } (q', q), \\ \omega(q') = \omega_x(q') = 0, \\ \omega(q) = -\psi(q), \end{cases} \tag{2.9}$$

where q' is to be determined. This recovers the form of (2.6) with $\Gamma_f = \{q'\}$ and $\Gamma_s = \{q\}$. To fully solve (2.7), we need to solve (2.8) or (2.9) in a similar manner on many intervals with free boundaries. The landscape could be rather complicated.

Example 2.6 (Harmonic module)**.** Consider a homeomorphism from $B_1 \backslash B_R$ to $B_1 \backslash B_r$, where R, $r < 1$. In 2D, in order to make the map conformal, it is required that $R = r$ and the map is in the form of $az + bz^{-1}$ with a and b to be determined.

For higher dimensional case, we consider $(u_1, u_2, ..., u_n) \triangleq u : B_1^n \backslash B_R^n \to B_1^n \backslash B_r^n$, where u is a homeomorphism and satisfies $\Delta u = 0$. For example, a specific form of u is $ax + \frac{bx}{|x|^n}$. Then the question is whether such condition provides any constraints between R and r. The answer is that, given R, $\exists r(n,R) < 1$, s.t. $0 < r \leq r(n,R)$, i.e. r can be arbitrarily small but can not get too close to 1. The upper bound can be proved easily by Harnack's inequality. Consider $1 - u_1$ on $\Gamma_R \triangleq \partial B^n_{(1+R)/2} \subset\subset B_1^n \backslash B_R^n$. By assumption it is a nonnegative function on $B_1^n \backslash B_R^n$. We can show through topological argument that $1 - u_1 \geq 1$ at some point on Γ_R. By Harnack's inequality,

$$1 \leq \sup_{\Gamma_R}(1 - u_1) \leq C(n,R) \inf_{\Gamma_R}(1 - u_1),$$

which implies $r < \sup_{\Gamma_R} u_1 \leq 1 - C(n,R)^{-1} \triangleq r(n,R) < 1$. This problem is related to the obstacle problem in the following sense. Consider

$$\min_u \left\{ \int_{B_1^n(0)} |\nabla u|^2 dx : u(x) = x \text{ on } \partial B_1^n(0), |u(x)| \geq r \right\}. \tag{2.10}$$

Suppose u is the solution. It is easy to see that $\Delta u = 0$ at x if $|u(x)| > r$. If $\Delta u = 0$ everywhere in B_1^n, u is smooth. Hence, $P \circ u$ is a continuous function from B_n^1 onto S_1^{n-1}, where $P(x) = \frac{x}{|x|}$ which is well-defined and continuous in $B_1^n(0) \backslash B_r^n(0)$. This leads to a contradiction. Therefore, there is a domain $\Omega \subset B_1^n$ to be determined, in which $|u(x)| = r$; $\Delta u = 0$ outside Ω and $|u(x)| > r$.

A natural guess of Ω would be a ball with radius R. It is better to choose R as small as possible since u is harmonic outside Ω and harmonic function obtains minimum energy of the functional in (2.10). A lower bound for possible R is given by the estimate we have just obtained in harmonic modulo.

Example 2.7 (Mullins-Sekerka problem)**.** The following equation is used to formulate solidification and liquidification of materials near zeros specific heat.

$$\begin{cases} \partial_t u - \Delta u = 0 \quad \text{ in } \Omega \backslash \Gamma_t, t \in [0,T], \\ \partial_\nu u = 0 \quad \text{ on } \partial\Omega \times [0,T], \\ u = -\alpha H \quad \text{ on } \Gamma_t, t \in [0,T], \\ [\partial_n u] = v \quad \text{ on } \Gamma_t, t \in [0,T], \\ \Gamma_t|_{t=0} = \Gamma_0. \end{cases} \tag{2.11}$$

In the above equation, Γ_t is the free boundary with v being its velocity in normal direction. $\partial_\nu u$ is the outer unit normal vector along $\partial\Omega$ and $[\partial_n u] = \partial_n^+ u - \partial_n^- u$ is the jump of normal derivatives between two sides of Γ_t. We note that n here is the outward unit normal vector with respect to the domain enclosed by Γ_t. H is the mean curvature of Γ_t s.t. it is positive for a sphere.

Let Ω_t^- be the region enclosed by Γ_t and $\Omega_t^+ = \Omega\backslash(\Gamma_t \cup \Omega_t^-)$. Define $Q^\pm = \cup_{0\le t\le T}\Omega_t^\pm \times \{t\}$ and $u^\pm = u|_{\overline{Q^\pm}}$. If we replace u^+ (or u^-) by a constant then (2.11) is often referred as Hele-Shaw problem, which studies pressure of immiscible fluid in air. That u^+ (respectively u^-) is constant means that the pressure in the air is a constant everywhere and $u^- = -\alpha H$ (respectively $u^+ = -\alpha H$) is the surface tension of the fluid in air (respectively of an air bubble in the fluid). Equations for Hele-Shaw problem write as follows:

$$\begin{cases} \partial_t u - \Delta u = 0 \text{ in } \Omega, \\ u = -\alpha H \text{ on } \partial\Omega, \\ \dfrac{\partial u}{\partial n} = v_n \text{ on } \partial\Omega. \end{cases}$$

Another version for the static case is that

$$\begin{cases} -\Delta u = \delta_0 \text{ in } \Omega, \\ u = -\alpha H \text{ on } \partial\Omega, \\ \dfrac{\partial u}{\partial n} = v_n \text{ on } \partial\Omega. \end{cases}$$

Example 2.8 (Stefan problem)**.**

$$\begin{cases} u_t - u_{xx} = 0 \quad \text{for } 0 < x < s(t), t > 0, \\ u(0,t) = f(t) \ge 0 \quad \text{or} \quad u_x(0,t) = g(t) \quad \text{for } t > 0, \\ u(x,0) = \varphi(x) \ge 0 \quad \text{for } 0 < x < b, \varphi(b) = 0, \\ u(s(t),t) = 0 \quad \text{for } t > 0, \\ \dfrac{ds(t)}{dt} = -u_x(s(t),t) \quad \text{and} \quad s(0) = b. \end{cases}$$

A typical situation Stefan problem models is a 1D system in which water and ice coexist. $u(x)$ characterizes the temperature distribution of the system. $s(t)$ is the interface between water and ice and $[0, s(t)]$ is the water domain. The propagation of the interface is governed by the heat flux at the interface. On the left boundary at $x = 0$, we assign the temperature by setting the Dirichlet boundary condition or assign the heat flux by setting the Neumann boundary condition. In higher dimensional case, the Stefan problem is as follows:

$$\begin{cases} u_t - \Delta u = 0 \quad \text{in} \bigcup\limits_{0\le t\le T} \Omega_t \times \{t\}, \Omega_t \text{ is enclosed by } \Gamma_t, \\ u(x,0) = \varphi(x) \ge 0 \quad \text{for } x \in \Omega_0, \\ u = -H \quad \text{on } \Gamma_t \times [0,T], \\ \dfrac{\partial u}{\partial n} = v_n, \end{cases}$$

where H is again the mean curvature along Γ_t and v_n is the normal propagation velocity of Γ_t.

Example 2.9 (Two fluids with interfaces)**.** The following equation characterizes the dynamics of two incompressible Newtonian fluids with interface Γ_t.

$$\begin{cases} \rho(v_t + v\cdot\nabla v) = -\nabla p \quad \text{in } \mathbb{R}^n\backslash\Gamma_t, \\ \operatorname{div} v = 0 \quad \text{in } \mathbb{R}^n\backslash\Gamma_t, \\ (\partial_t + v\cdot\nabla v) \text{ is tangential to } \Gamma_t, t>0, \\ p_+(t,x) - p_-(t,x) = \varepsilon H_+(t,x). \end{cases}$$

Again H_+ is the mean curvature along the interface, v is the velocity field of the fluid and p is the pressure. That $(\partial_t + v\cdot\nabla v)$ is tangential to Γ_t means that the fluids do not penetrate the interface. The weak formulation of two fluids with interfaces is that

$$\begin{cases} (\rho v)_t + v\cdot\nabla(\rho v) = -\nabla p + \varepsilon H(t,x)\delta_{\Gamma_t}, \\ \operatorname{div} v = 0, \end{cases}$$

Example 2.10 (3D gravity water waves)**.** The water wave equation characterizes the dynamics of the water flow and the water-air interface in the presence of gravity. Let $h(t,x)$ denote the height of water. $\Omega_t = \{(x,z) : x \in \mathbb{R}^2, z \le h(t,x)\}$ is the water domain and $\Gamma_t = \{(x,h(t,x)) : x \in \mathbb{R}^2\}$ is the water-air interface

$$\begin{cases} v_t + v\cdot\nabla v = -\nabla p - g\mathbf{e}_3 \text{ in } \Omega_t, \\ \operatorname{div} v = 0 \quad \text{in } \Omega_t, \\ \partial_t h = v\cdot\nabla_{(x,z)}(h-z) = 0 \text{ for } x\in\mathbb{R}^2, \\ p|_{\Gamma_t} = 0. \end{cases}$$

Example 2.11 (Immersed boundary)**.** The immersed boundary method basically formulates the interaction between the fluid and the elastic object immersed in it. Let Γ_t be the immersed elastic object. The governing equation is

$$\begin{cases} v_t + v\cdot\nabla v + \nabla p = \mu\Delta v + \varepsilon H_{\Gamma_t}\delta_{\Gamma_t}, \\ \operatorname{div} v = 0, \\ \dfrac{dx}{dt} = v(t,x) \quad \text{on } x\in\Gamma_t. \end{cases}$$

where H_{Γ_t} is the mean curvature of Γ_t. Although the immersed boundary method is popular and successful as a numerical method in biological and physical simulation, there are not so many theoretical results concerning this equation.

Example 2.12 (Water-dam problem)**.** In the water-dam problem, the dam is modeled as a porous media that water can go through. Because of the high pressure of upstream water, the water will permeate through the dam slowly and form wet region at the lower part inside the dam. The flow in the dam follows the Darcy's law. Let $\Omega \subset \mathbb{R}^2$ be the wet region inside the dam and let Γ be the interface between dry region and wet region, i.e. the upper boundary of the wet region. Denote the bottom of the dam by Γ_b; denote the underwater parts on the upstream and downstream surface of the dam by Γ_1 and Γ_2 respectively. In general, the height of water inside the dam at the downstream surface is higher than the downstream water level. Denote the wet wall above the water level on the downstream side by Γ_s, where water keeps seeping out. Then the equation with unknowns (u, Ω) for water-dam problem is as follows

$$\begin{cases} -\Delta u = 0 \quad \text{in } \Omega, \\ u = g_i \quad \text{on } \Gamma_i, \\ \dfrac{\partial u}{\partial \nu} \geq 0 \quad \text{on } \Gamma_s, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_b, \\ u(x,y) = y \quad \text{on } \Gamma, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma, \end{cases} \tag{2.12}$$

where ν is the outer normal vector with respect to Ω.

More examples and applications of free boundary problems can be found in [7]. Many of these examples described here are famous and one can find huge volumes of literature addressing these problems by simple searching on internet a key words.

3. General theory of existence and uniqueness

3.1. Riesz lemma and theory of existence in Hilbert space.

Lemma 3.1 (Riesz). *Let K be a closed convex set in a real Hilbert space H and let $y_0 \in H$. Then there is a unique $x_0 \in K$ (denoted by $x_0 = Proj_K y_0$) which is the nearest to y_0, i.e.*

$$\|x_0 - y_0\| = \inf_{x \in K} \|x - y_0\|. \tag{3.1}$$

Moreover, it is equivalent to the following variational inequality:

$$(x_0 - y_0, x - x_0) \geq 0, \quad \forall x \in K. \tag{3.2}$$

Proof. We first prove the equivalence between (3.1) and (3.2). If (3.2) is true, for any $x \in K$, we can write $x = (x - x_0) + x_0$. Since $(x - x_0, x_0 - y_0) \geq 0$, then

$$\|x - y_0\|^2 = \|x - x_0\|^2 + \|x_0 - y_0\|^2 + 2(x - x_0, x_0 - y_0) \geq \|x - x_0\|^2.$$

On the other hand, if (3.1) is true, $\forall x \in K$, $\varepsilon \in [0,1]$, let $x_\varepsilon = x_0 + \varepsilon(x - x_0) = \varepsilon x + (1-\varepsilon)x_0 \in K$. By using the assumption, $\|x_\varepsilon - y_0\|^2 \geq \|x_0 - y_0\|^2$. This gives

$$0 \leq \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \|x_\varepsilon - y_0\|^2 = 2(x_0 - y_0, x - x_0),$$

which proves (3.2). To show the uniqueness, we can assume that there are $x_1, x_2 \in K$ such that (3.1). We can find that for any $x \in K$,

$$(x - x_1, x_1 - y_0) \geq 0, \quad \text{and} \quad (x - x_2, x_2 - y_0) \geq 0.$$

Hence

$$(x_2 - x_1, x_1 - y_0) \geq 0, \quad \text{and} \quad (x_1 - x_2, x_2 - y_0) \geq 0.$$

This implies that $(x_2 - x_1, x_1 - x_2) \geq 0$ and thus $x_1 = x_2$. To prove the existence of such x_0, we take a minimizing sequence $\{x_n\} \subset K$, s.t. $\|x_n - y_0\| \to \inf_{x \in K} \|x - y_0\|$. Note that due to the parallelogram identity

$$\begin{aligned}2\|x_n - y_0\|^2 + 2\|x_m - y_0\|^2 &= \|x_n - x_m\|^2 + 4\left\|\frac{x_n + x_m}{2} - y_0\right\|^2 \\ &\geq \|x_n - x_m\|^2 + 4\inf_{x\in K}\|x - y_0\|^2.\end{aligned}$$

This gives that $\|x_n - x_m\| \to 0$ when $n, m \to \infty$. Then $\{x_n\}$ is a Cauchy sequence. Hence we can define $x_0 \in K$ as its limit in view of the fact that K is closed. x_0 is the point we need. □

Let Ω be a bounded and smooth domain. The Sobolev space $H^1(\Omega)$ is defined by $H^1(\Omega) = \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)\}$, associated with the inner product

$$(u, v) = \int_\Omega \nabla u \nabla v + \int_\Omega uv. \tag{3.3}$$

$H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$ with (3.3)*.*

Definition 3.2. Let $v \in H^1(\Omega)$. We say $v(x_0) \succ 0$ if there $\exists \rho > 0$ and $\phi \in C^1(B_\rho(x_0))$, with $\phi > 0$ in $B_\rho(x_0)$, s.t. $v(x) \geq \phi(x)$ a.e. in $B_\rho(x_0)$.

It is due to the fact that if $v(x_0) > 0$, then there $\exists \rho > 0$ and $\varepsilon_0 > 0$ such that $v(x) \geq \varepsilon_0$ for any $x \in B_\rho(x_0)$. Also, we see that in the Definition 3.2, $\phi \in C^1(\overline{B_{\rho/2}(x_0)})$. Then we can choose $c = \inf_{x\in\overline{B_{\rho/2}(x_0)}} \phi(x)$ and find that $v(x) \geq c$ a.e. $x \in B_{\rho/2}(x_0)$. This means that we can change functions ϕ to positive constants.

We consider functions $a_{ij} \in L^\infty(\Omega)$ for $i, j = 1, 2, ..., n$, s.t.

$$a_{ij} = a_{ji} \in L^\infty(\Omega) \quad \text{and} \quad \lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \forall x \in \Omega. \tag{3.4}$$

with $0 < \lambda < \Lambda < \infty$. We define an elliptic operator L and a bilinear form $a : H_0^1 \times H_0^1 \to \mathbb{R}$,

$$L = -\operatorname{div}(a(x)\nabla) = -\partial_{x_i}(a_{ij}(x)\partial_{x_j}), \tag{3.5}$$

$$a(u, v) = \int_\Omega a_{ij}(x)\partial_{x_i}u(x)\partial_{x_j}v(x)dx, \quad u, v \in H^1(\Omega). \tag{3.6}$$

Note that $a(u, v)$ is an inner product on $H_0^1(\Omega)$ which is equivalent to (3.3) due to uniform ellipticity and Poincaré inequality.

Consider $\psi \in H^1(\Omega)$ with $\psi \leq 0$ on $\partial\Omega$. Define

$$K_\psi = \{v \in H_0^1(\Omega) : v \geq \psi \text{ a.e. in } \Omega\}. \tag{3.7}$$

K is closed and convex set in $H = H_0^1(\Omega)$. Obviously, the strictly convexity given in the original notes is not correct, we can easily construct conterexamples.

Theorem 3.3. *Assume that $a(x)$ satisfies* (3.4)*. For $v \in H_0^1(\Omega)$, let*

$$I(v) = a(v, v) - 2\langle f, v\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)},$$

where $f \in H^{-1}(\Omega)$. Then there exists $u \in K$, s.t. $I(u) = \min_{v\in K_\psi} I(v)$. Moreover, this is equivalent to the following variational inequality:

$$a(u, v - u) \geq \langle f, v - u\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}, \quad \forall v \in K_\psi. \tag{3.8}$$

A typical example of the above variational problem is

$$\min\left\{\int_\Omega |\nabla u|^2 dx : u \in H_0^1(\Omega), u \geq \psi \text{ a.e. in } \Omega\right\}.$$

Proof. By Riesz representation theorem, there exists $g \in H_0^1(\Omega)$ s.t. $\langle f, v\rangle = a(g, v)$, for any $v \in H_0^1(\Omega)$. Hence,

$$I(v) = a(v, v) - 2\langle f, v\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = a(v - g, v - g) = a(g, g).$$

Then

$$u \text{ minimizes } I(\cdot) \text{ over } K \Leftrightarrow u \text{ minimizes } a(v - g, v - g) \text{ among } v \in K_\psi.$$

Recall that $a(u, v)$ defines an equivalent inner product with (3.3) in $H_0^1(\Omega)$. We denote $H_0^1(\Omega)$ equipped with the inner product $a(u, v)$ by $\widetilde{H}$. Hence, u minimizes $\|v - g\|_{\widetilde{H}}$ in K. By Lemma 3.1, there exists unique $u \in K$ s.t. $I(u) = \min_{v\in K_\psi} I(v)$; moreover, it is equivalent to solve

$$a(u - g, v - u) \geq 0, \quad \forall v \in K_\psi,$$

or

$$a(u, v - u) \geq a(g, v - u) = \langle f, v - u\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}, \quad \forall v \in K_\psi.$$

This gives the equivalence between problems of minimizing and variational problems. □

3.2. **Some properties of the solution.**

Definition 3.4. $g \in H_0^1(\Omega)$ is a supersolution of $L(\cdot) - f$ if

$$\langle Lg - f, \xi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = a(g, \xi) - \langle f, \xi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \geq 0, \quad \forall 0 \leq \xi \in H_0^1(\Omega),$$

where L is defined in (3.5).

Proposition 3.5. *u is a supersolution of $L(\cdot) - f$.*

Proof. Let $0 \leq \xi \in H_0^1(\Omega)$. Consider $u_\varepsilon = u + \varepsilon\xi \in K$, $\forall \varepsilon \in (0, 1)$. Then

$$I(u) \leq I(u_\varepsilon) \Rightarrow \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} I(u_\varepsilon) \geq 0 \Rightarrow a(u, \xi) \geq \langle f, \xi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)},$$

which completes the proof. □

Proposition 3.6. *u is a minimum supersolution in the sense that, for any supersolution y of $L(\cdot) - f$ s.t. $y \geq \psi$ and $y \geq 0$ on $y \geq 0$ on $\partial\Omega$, we have $u \leq y$.*

Proof. Let $\xi = \min(u, y) \in K$. By (3.8),

$$a(u, \xi - u) \geq \langle f, \xi - u\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}.$$

Since $\xi - u \leq 0$ in Ω and $\xi - u = 0$ on $\partial\Omega$, one has (as y is a supersolution)

$$a(y, \xi - u) \leq \langle f, \xi - u\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}.$$

Thus

$$\begin{aligned} 0 \geq a(y - u, \xi - u) &= \int_\Omega a_{ij}(x)\partial_{x_i}(y - u)\partial_{x_j}(\xi - u)dx \\ &= \int_{\Omega\cap\{x:y(x)>u(x)\}} a_{ij}(x)\partial_{x_i}(y - u)\partial_{x_j}(\xi - u)dx \\ &\quad + \int_{\Omega\cap\{x:y(x)\leq u(x)\}} a_{ij}(x)\partial_{x_i}(y - u)\partial_{x_j}(\xi - u)dx. \end{aligned}$$

The first term on the right hand side vanishes since $\xi = u$ on $\Omega \cap \{x : y(x) > u(x)\}$; the second term has to be nonnegative as $\xi = y$ on $\Omega \cap \{x : y(x) \leq u(x)\}$ and a_{ij} is uniformly elliptic. Therefore, $u \leq y$ in Ω. □

Corollary 3.7. *Let u be a solution of* (3.8) *with $f = 0$. Suppose $\varphi \leq M$. Then $\psi \leq u \leq M$.*

This corollary is usefull since if we have the boundedness of the obstacle ψ, we can also derive the boundedness of the solution u.

Definition 3.8. For given $\psi \in H^1(\Omega)$ on $\partial\Omega$, let $u \in K$ be solution of (3.8), We define

$$N = \{x \in \Omega : u(x) \succ \psi(x)\} \tag{3.9}$$

as the non-coincidence set, which is open by Definition 3.2. We also define the coincidence set

$$\Lambda = \Omega \backslash N, \tag{3.10}$$

and the free boundary $\Gamma = \partial\Lambda$.

Remark 3.9. In N, one can obtain that $Lu = f$. This is because that N is open and for any $\xi \in C_0^\infty(N)$, there exists $\delta > 0$ such that for any $\varepsilon \in [-\delta, \delta]$, $u + \varepsilon\xi \in K_\psi$. Then

$$I(u) \le I(u_\varepsilon) \Rightarrow \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} I(u_\varepsilon) = 0 \Rightarrow a(u, \xi) = \langle f, \xi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)},$$

which means that $Lu = f$ in N. Since there are still some differences between $\succ$ and $\ge$, then the coinsidence set λ cannot be easily characterized by $\{x \in \Omega : u(x) = \psi(x)\}$.

We note that in the definition of non-coincidence set, (3.9), we have used Definition 3.2. That is

$$\begin{aligned} x \in N &\Leftrightarrow u(x) \succ \psi(x) \text{ in the sense of Deinition 3.2} \\ &\Leftrightarrow \exists c_x > 0,\ \rho_x > 0,\ u(z) \ge \psi(x) + c_x \text{ a.e. in } B_{\rho_x}(x). \end{aligned}$$

Obviously, N is an open set. A remarkable question is that when

$$N = \{x \in \Omega : u(x) \succ \psi(x)\} = \{x \in \Omega : u(x) > \psi(x)\} \tag{3.11}$$

is true? By simple observations, we can see that if u is a lower semi-continuous function, then (3.11) is satisfied. Now, we consider a special case that $a_{ij} = \delta_{ij}$, that is $L = -\Delta$. For this case, we will show that the minimizer u is actually a lower semi-continuous function. In Ω, since u is supersolution, then $-\Delta u \ge 0$ in weak sense. We claim that for any $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$, the function

$$\sigma_x(r) = \fint_{B_r(x)} u(y)dy$$

is non-increasing. The proof of this claim is not hard, since

$$\sigma_x'(r) = \int_{B_1(0)} \nabla u(x + r\xi) \cdot \xi d\xi = -\int_{B_r(x)} \nabla u(y) \cdot \nabla(r^2 - |y - x|^2)dy,$$

and $r^2 - |y - x|^2 \ge 0$ in $B_r(x)$, we can obtain that

$$\sigma_x'(r) = -\int_{B_r(x)} \nabla u(y) \cdot \nabla(r^2 - |y - x|^2)dy \le 0,$$

which implies that the claim is true. If we define $w(x) = \lim_{r\downarrow 0} \fint_{B_r(x)} u$ which is well defined, since the average is monotone non-increasing. Then $w(x) = u(x)$ if x is a Lebesgue points and $w = u$ almost everywhere in Ω. Let us now consider $x_0 \in \Omega$ and let $x_k \to x_0$ as $k \to \infty$. Then, Then, by dominated convergence theorem (actually $u \in H_0^1$ is integrable in Ω, which makes the result be true) we have that

$$\fint_{B_r(x_0)} u = \lim_{k\to\infty} \fint_{B_r(x_k)} u \le \liminf_{k\to\infty} w(x_k)$$

for any $0 < r < \frac{1}{2}\operatorname{dist}(x_0, \partial\Omega)$. For the inequality above, we have used the fact that $\sigma_x(r)$ is non-increasing. Now, by using $r \downarrow 0$ on the left-hand side we reach that

$$w(x_0) \le \liminf_{k\to\infty} w(x_k).$$

that is w is lower semi-continuous. Then we show that, up to changing u in a set of measure 0, u is lower semi-continuous. By using this fact, we can obtain that $\{u > \psi\}$ is an open set if ψ supper semi-cotinuous and $u = \psi$ in Λ at the same time. That is, we can obtain the following results.

Theorem 3.10. *If $u \in K_\psi$ is solution for the variational problem:*

$$\int_\Omega \nabla u \nabla (v - u) dx \geq 0, \quad \forall v \in K_\psi.$$

Then up to a set with lebesgue measure 0, u is lower semi-continuous. If ψ is a supper semi-cotinuous function, then $N = \{x \in \Omega : u(x) > \psi(x)\}$ and $\Lambda = \{x \in \Omega : u(x) = \psi(x)\}$.

Corollary 3.11. *Let $f \in C^{0,\alpha}(\overline{\Omega})$. If $u \in K_\psi$ is solution for the variational problem:*

$$\int_\Omega \nabla u \nabla (v - u) dx \geq \int_\Omega f(v - u) dx, \quad \forall v \in K_\psi.$$

Then up to a set with lebesgue measure 0, u is lower semi-continuous. If ψ is a supper semi-cotinuous function, then $N = \{x \in \Omega : u(x) > \psi(x)\}$ and $\Lambda = \{x \in \Omega : u(x) = \psi(x)\}$.

Proof. We can consider the equation

$$-\Delta w = f \text{ in } \Omega \quad \text{and} \quad w = 0 \text{ in } \partial\Omega.$$

Then we can consider variational problems as follows

$$(V_1) : \min \left\{ \int_\Omega |\nabla u|^2 - 2 \int_\Omega f u : u \in H_0^1(\Omega), u \geq \psi \text{ a.e. in}\Omega \right\},$$
$$(V_2) : \min \left\{ \int_\Omega |\nabla u|^2 : u \in H_0^1(\Omega), u \geq \psi - w \text{ a.e. in } \Omega \right\}.$$

If u is the minimizer of the variational problem (V_1), we want to see that $\widetilde{u} = u - w$ is the minimizer of the variational problem (V_2). For any $v \in H_0^1(\Omega)$ and $v \geq \psi - w$ a.e. in Ω. Then $v + w \geq \psi$ a.e. in Ω and $v + w \in H_0^1(\Omega)$. This implies that

$$\int_\Omega |\nabla (v + w)|^2 dx - 2 \int_\Omega f(v + w) dx \geq \int_\Omega |\nabla u|^2 dx - 2 \int_\Omega f u dx.$$

By simple calcuations, we can find that

$$\int_\Omega |\nabla v|^2 dx \geq \int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla w|^2 dx - 2 \int_\Omega f u dx = \int_\Omega |\nabla (u - w)|^2 dx.$$

This actually completes the proof. Indeed, for u being the solution of the variational problem of (V_1), $u - w$ is the solution of the variational problem of (V_2), by using (3.10), $u - w$ is lower semi-continuous and then u is lower semi-continuous. The rest of the arguments are standard. $\square$

Lemma 3.12. *Let u be the solution of* (3.8). *Assume $B_R(0) = B_R \subset \Omega$. Let $l(x)$ be a linear function and u_0 be the solution of*

$$\begin{cases} \Delta u_0 = 0 \text{ in } B_R, \\ u_0 = u - \psi \geq 0 \text{ on } \partial B_R. \end{cases} \tag{3.12}$$

Then

$$u_0(x) + l(x) - \omega(R) \leq u(x) \leq u_0(x) + l(x) + \omega(R), \quad \text{a.e. in } B_R(0),$$

where

$$\omega(r) = \sup_{B_r(0)} |\psi(x) - l(x)|, \quad 0 \leq r \leq R_0 = \text{diam}(\Omega).$$

In particular, for $l(x) = \psi(0)$, we have

$$|u_0(x) + \psi(x) - u(x)| \leq \sup_{B_R(0)} |\psi(x) - \psi(0)|.$$

Proof of Lemma 3.12. It is know that $0 = -\Delta(u_0 + l - \omega(R)) \le -\Delta u$ in $D'(B_R)$, where we have used facts that $-\Delta(l-\omega(R)) = 0$ and u is a supersolution of $-\Delta(\cdot) - 0$. Moreover, one can derive by definition that $u_0 + l - \omega(R) \le u$ on ∂B_R. By maximum principle, $u_0 + l - \omega(R) \le u$ in B_R. On the other hand, since $u(x) \le u_0(x) + l(x) + \omega(R)$ on ∂B_R and $\mathrm{supp}(\Delta(u - u_0 - l - \omega(R))) \subset \Lambda \cap B_R$ (in view of the fact that $-\Delta u = 0$ in N), we have

$$\begin{aligned}\sup_{B_R}(u - u_0 - l - \omega(R)) &\le \max\left\{\sup_{N\cap B_R}(u - u_0 - l - \omega(R)), \sup_{\Lambda\cap B_R}(u - u_0 - l - \omega(R))\right\}\\ &\le \max\left\{\sup_{\partial N\cup\partial B_R}(u - u_0 - l - \omega(R)), \sup_{\Lambda\cap B_R}(u - u_0 - l - \omega(R))\right\}\\ &\le \max\left\{0, \sup_{\Lambda\cap B_R}(u - u_0 - l - \omega(R))\right\} \le 0,\end{aligned}$$

where for the second inequality, we have used the maximum principle for $u - u_0 - l - \omega(R)$ in $N \cap B_R$, the fact that $\partial(N \cap B_R) \subset N \cup B_R$ and for the third inequality, we have used the fact that for any $x \in \Lambda \cap B_R$, then $u_0(x) \ge \min_{\partial B_R}(u_0) \ge 0$ and

$$u(x) - u_0(x) - l(x) - \omega(R) \le \psi(x) - l(x) - u_0(x) - \omega(R) \le 0.$$

□

Lemma 3.13. *Assume that $0 \in \Lambda$ and let u, l and ω be defined in Lemma 3.12. Then we have*

$$\sup_{x\in B_{\frac{R}{2}}} |u(x) - l(x)| \le C(n)\omega(R),$$

where $C(n)$ is a positive constant only depending on n.

Proof. Since $u \ge \psi$ in B_R, we find $u - l \ge -\omega(R)$ in B_R. This proves one side of the estimate. By Lemma 3.12, we can obtain that

$$\sup_{x\in B_{\frac{R}{2}}} (u(x) - l(x)) \le \sup_{B_{\frac{R}{2}}} u_0(x) + \omega(R).$$

Note that u_0 is a nonnegative harmonic function defined by (3.12) in B_R. By Harnack's inequality, we have

$$\sup_{x\in B_{\frac{R}{2}}} u_0(x) \le C(n)u_0(0) \le C(n)(u(0) - l(0) + \omega(R)).$$

Since $u(0) = \psi(0)$, one can derive that

$$\sup_{x\in B_{\frac{R}{2}}} u_0(x) \le C(n)u_0(0) \le C(n)(\psi(0) - l(0) + \omega(R)) \le C(n)\omega(R).$$

Then

$$\sup_{x\in B_{\frac{R}{2}}} (u(x) - l(x)) \le \sup_{x\in B_R} u_0(x) + \omega(R) \le C(n)\omega(R),$$

which complete the proof. □

Corollary 3.14. *If $\psi \in C(\Omega)$ in the definition of K_ψ (see (3.7)) and u is the solution of (3.8) with $a_{ij} = \delta_{ij}$ and $f \equiv 0$, then $u \in C(\Omega)$.*

Proof. For $x_0 \in N$, u is locally a harmonic function and is thus continuous. For $x_0 \in \Lambda$, we apply Lemma 3.13 with $l(x) \equiv \psi(x_0)$ and $B_R(x_0) \subset \Omega$ and find that

$$\sup_{x\in B_{\frac{R}{2}}(x_0)} |u(x) - u(x_0)| = \sup_{x\in B_{\frac{R}{2}}(x_0)} |u(x) - \psi(x_0)| \le C(n) \sup_{x\in B_R(x_0)} |\psi(x) - \psi(x_0)|,$$

where we have used the fact that $u(x_0) = \psi(x_0)$ in Λ. Since $\psi \in C(\Omega)$, we can obtain that if $x \to x_0$, then $u(x) \to u(x_0)$. This shows that u is continuous at x_0. Then we can complete the proof. □

3.3. **Monotone operators and theory in reflexive Banach space.** Next we we shall extend the theory on Hilbert space to the case of reflexive Banach space.

Definition 3.15. Let X be a real reflexive Banach space with dual X^*. $\langle \cdot, \cdot \rangle$ denotes the pairing between X^* and X. A mapping A from $D(A) \subset X$ to X^* is called monotone if

$$\langle Au - Av, u - v \rangle \geq 0, \quad \forall u, v \in D(A).$$

Example 3.16. Let $X = H_0^1(\Omega)$ and $X^* = H^{-1}(\Omega)$. Define $Au = -\partial_{x_i}(a_{ij}\partial_{x_j} u) - f$ with $a_{ij}'s$ satisfying (3.4). Then A is monotone.

Definition 3.17. When $D(A)$ is convex, A is called hemicontinuous if $u, v \in D(A)$, the map

$$t \mapsto \langle A(tu + (1-t)v, u - v) \rangle$$

is continuous for $t \in [0, 1]$.

Definition 3.18. Let $M \subset X$ be a finite dimensional subspace. Let $j : M \hookrightarrow X$ be the embedding and $j^* : X \to M^*$ be the dual map, i.e. $j^* f$ is the restriction of $f \in X^*$ to M. If $j^* A j$ is continuous on $M \cap D(A)$ for any such M, then we say A is continuous on nite dimensional subspace of $D(A)$.

Theorem 3.19. *Suppose that $A : D(A) \subset X \to X^*$ is monotone and hemicontinuous. Then for any bounded closed convex subset K of X, there exists a solution $u_0 \in K$ to the variational inequality*

$$\langle Au_0, v - u_0 \rangle \geq 0, \quad \forall v \in K. \tag{3.13}$$

Theorem 3.20. *Let K be a bounded closed convex subset of X and suppose that $S : D(A) \to X^*$ is monotone, $K \subset D(A)$ and A is continuous on finite dimensional subspace of $D(A)$. Then there exists a $u_0 \in K$, s.t.* (3.13) *holds.*

Lemma 3.21 (Minty)**.** *. Assume K to be any closed convex subset of X. Let $A : K \to X^*$ be monotone and hemicontinuous. Then u_0 satisfying* (3.13) *iff*

$$\langle Av, v - u_0 \rangle \geq 0, \quad \forall v \in K. \tag{3.14}$$

Proof. By monotonicity of A,

$$0 \leq \langle Av - Au_0, v - u_0 \rangle = \langle Av, v - u_0 \rangle - \langle Au_0, v - u_0 \rangle.$$

Hence (3.13) implies (3.14). To show the converse, note that for $\forall w \in K, t \in (0, 1)$,

$$v = tw + (1-t)u_0 = u_0 + t(w - u_0) \in K.$$

By (3.14) one has

$$\langle A(u_0 + t(w - u_0)), w - u_0 \rangle \geq 0.$$

Let $t \to 0^+$ and (3.13) holds for $\forall v = w \in K$ by hemicontinuity. □

Proof of Theorem 3.20. We first consider the case where X is a finite dimensional space, i.e. $X = \mathbb{R}^m$. Let $K \subset \mathbb{R}^m$ be bounded, closed and convex and $A : K \to \mathbb{R}^m$. We assume that A is continuous (but we do not require that it is monotone). The variational inequality (3.13) can be written in the form

$$\langle u_0, v - u_0 \rangle \geq \langle u_0 - Au_0, v - u_0 \rangle, \quad \forall v \in K. \tag{3.15}$$

For $w \in K$, let $Tw = w - Aw$. It is clear that in the finite dimensional space, there is a unique $u \in K$, s.t.

$$u = \mathrm{Proj}_K(w - Aw) = \mathrm{Proj}_K Tw$$
$$\langle u, v - u\rangle \geq \langle Tw, v - u\rangle, \quad \forall v \in K.$$

It is also easy to show that Proj_K is continuous on $\mathbb{R}^m$. Since $\mathrm{Proj}_K \circ (I - A) : K \to K$ is continuous and K is bounded, closed and convex, by Brouwer's fixed-point theorem, there is $u_0 \in K$, s.t. $\mathrm{Proj}_K \circ (I - A)u_0 = u_0$. This proves (3.15).

Now for any finite-dimensional subspace $M \subset X$, by assumption, $j^*Aj : K \cap M \to M^*$ is continuous. Hence, there is a $u_M \in K \cap M$ solving

$$\langle j^*Aju_M, v - u_M\rangle \geq 0, \quad \forall v \in K \cap M.$$

Since $\langle j^*Aju_M, \xi\rangle = \langle Au_M, \xi\rangle$, for $\forall \xi \in K \cap M$, we have

$$\langle Au_M, v - u_M\rangle \geq 0, \quad \forall v \in K \cap M.$$

By Minty's lemma,

$$\langle Av, v - u_M\rangle \geq 0, \quad \forall v \in K \cap M.$$

For $\forall v \in K$, define $S(v) = \{u \in K : \langle Av, v - u\rangle \geq 0\}$. Since $S(v)$ is weakly closed subset of K and since K is bounded and X is reflexive, $S(v)$ is weakly compact by Banach-Alaoglu theorem. We claim that $\forall m$ and $\forall v_1, v_2, ..., v_m \in K$, $\cap_{i=1}^m S(v_i) \neq \emptyset$. In fact, given $M = \mathrm{span}\{v_1, ..., v_m\}$, there exists $u_M \in K \cap M$ satisfying

$$\langle Av, v - u_M\rangle \geq 0, \quad \forall v \in K \cap M,$$

which implies $u_M \in \cap_{i=1}^m S(v_i)$. Hence, $\cap_{v\in K} S(v) \neq \emptyset$ by the definition of compactness. Indeed, $S(v_0)\backslash S(v)$ is relatively open set for fixed $v_0 \in K$ and $\forall v \in K$, where the topology on $S(v_0)$ is induced by weak topology on X. If $\cap_{v\in K} S(v) = \emptyset$, $\{S(v_0)\backslash S(v)\}_{v\in K}$ forms an open covering of $S(v_0)$. Since $S(v_0)$ is compact, there exists a finite subcovering, denoted by $\{S(v_0)\backslash S(v_i)\}_{i=1}^n$. Hence, $\cap_{i=0}^n S(v_i) = \emptyset$, which is a contradiction. Then $u_0 \in \cap_{v\in K} S(v)$ solves (3.14) and thus (3.13). $\square$

Proof of Theorem 3.19. We shall prove that if A is monotone and hemicontinuous, A is continuous on finite dimensional space. Let $M \subset X$ be a finite dimensional subspace and let j and j^* be defined as before.

First we will show that j^*Aj maps bounded sets of M into bounded sets of M^*. Suppose that $\exists \{v_n\} \subset M$, s.t. $\|v_n\| \leq C_0$ and

$$\|j^*Ajv_n\| = \|j^*Av_n\| \to +\infty.$$

By monotonicity, one has

$$\langle j^*Av_n - j^*Au, v_n - u\rangle \geq 0, \quad \forall u \in M.$$

Hence

$$\left\langle y_n - \frac{j^*Au}{\|j^*Av_n\|}, v_n - u\right\rangle \geq 0,$$

where $y_n = \frac{j^*Av_n}{\|j^*Av_n\|}$. Since M and M^* are of finite dimension, there exist subsequences, also denoted by $\{v_n\} \subset M$ and $\{y_n\} \subset M^*$, such that

$$v_n \to v \in M, \quad y_n \to y \in M^*.$$

Note that $\|y\| = 1$. By taking $n \to +\infty$, one find

$$\langle y, v - u\rangle \geq 0, \quad \forall u \in M.$$

Here we use $\|j^*Av_n\| \to +\infty$. This implies $y = 0$, which contradicts with $\|y\| = 1$. Next we shall prove j^*Aj is continuous on M. Suppose not, there is a sequence $\{v_n\} \subset M$, s.t.

$$v_n \to v, \quad j^*Av_n \to w, \quad \text{ and } \quad w \neq j^*Av.$$

Note that $\{v_n\}$ and thus $\{j^*Av_n\}$ are bounded. This gives the existence of a limiting point w and the convergence of $\{j^*Av_n\}$ up to a subsequence. Since $\langle j^*Av_n - j^*Au, v_n - u\rangle \geq 0$,

$$\langle w - j^*Au, v - u\rangle \geq 0, \quad \forall u \in M.$$

Following the proof of Minty's lemma, we can show that this implies

$$\langle w - j^*Av, v - u\rangle \geq 0, \quad \forall u \in M.$$

Hence, we get $w - j^*Av = 0$, which is a contradiction. □

Then Theorem 3.19 is a direct corollary of Theorem 3.20 and the result we just proved.

Remark 3.22. Theorem 3.19 and 3.20 can be generalized to the case where K is unbounded. We shall assume that A is coercive: there is $\phi_0 \in K$, s.t.

$$\frac{1}{\|v\|}\langle Av - A\phi_0, v - \phi_0\rangle \to \infty, \quad \text{ if } v \in K, \|v\| \to +\infty.$$

Theorem 3.23. *Let K be unbounded closed conver set and let A be as in Theorem 2.4 or 2.3. If A is also coercive, then there is a solution of the variational inequality* (3.13).

Remark 3.24. An observation is that $\|u\|$ should be bounded. Suppose $u \in K$ is a solution of (3.13), i.e.

$$\langle Au, v - u\rangle \geq 0, \ \forall v \in K.$$

This gives

$$\frac{1}{\|u\|}\langle Au - Av, u - v\rangle \leq \left\langle Av, \frac{v-u}{\|u\|}\right\rangle, \quad \forall v \in K.$$

In particular, for $v = \phi_0$, one has

$$\frac{1}{\|u\|}\langle Au - A\phi_0, u - \phi_0\rangle \leq \left\langle A\phi_0, \frac{\phi_0 - u}{\|u\|}\right\rangle \leq C_0.$$

This means $\|u\|$ has an a priori upper bound.

Proof. For $\forall R > 0$, let $K_R = K \cap \{\|u\| \leq R\}$ and let $u_R \in K_R$ be the solution of (3.13) corresponding to the convex set K_R.

If $\|u_R\| < R$ for some $R > 0$ then we are done. Indeed, $\forall v \in K$, there $\exists \varepsilon > 0$ depending on v such that $w = u_R + \varepsilon(v - u_Rt) \in K_R$. Hence,

$$0 \leq \langle Au_R, w - u_R\rangle = \varepsilon\langle Au_R, v - u_R\rangle,$$

which implies u_R is a solution of (3.13) in K. If $\|u_R\| = R$ for $\forall R > 0$, let $R > \|\phi_0\|$ and we find $\langle Au_R, u_R - \phi_0\rangle \leq 0$. On the other hand, by the definition of coerciveness

$$\langle Au_R - A\phi_0, u_R - \phi_0\rangle \geq C_R\|u_R\|,$$

where $C_R \to +\infty$ as $R \to +\infty$. Hence

$$\langle Au_R, u_R - \phi_0\rangle \geq (C_R - \|A\phi_0\|)\|u_R - \phi_0\|.$$

The right hand side is positive if R is sufficiently large, which is a contradiction. □

To obtain uniqueness of the solution, we need the following definition.

Definition 3.25. A is called strictly monotone if A is monotone and if

$$\langle Au - Av, u - v\rangle = 0 \Leftrightarrow u = v.$$

Theorem 3.26. *If A is strictly monotone, then there is at most one solution of* (3.13).

Proof. Suppose there are two solutions u_1 and u_2. Then

$$\langle Au_i, v - u_1\rangle \geq 0, \quad \forall v \in K, \text{ for } i = 1, 2.$$

Then

$$\begin{aligned} &\langle Au_1, u_2 - u_1\rangle \geq 0 \leq \langle Au_2, u_1 - u_2\rangle \\ \Rightarrow &\langle Au_2 - Au_1, u_2 - u_1\rangle \leq 0 \\ \Rightarrow &u_1 = u_2. \end{aligned}$$

If we assume

$$\langle Au - Av, u - v\rangle \geq \alpha\|u - v\|^{1+\delta} \tag{3.16}$$

for some $\alpha, \delta > 0$, we can obtain stability estimate. □

Theorem 3.27. *Let A be as in Theorem 3.19 or 3.20 and let* (3.16) *hold. Let $u_i \in K$ $(i = 1, 2)$ be such that*

$$\langle Au_i, v - u_i\rangle \geq \langle f_i, v - u_i\rangle, \quad \forall v \in K,$$

where $f_i \in X^*$. Then

$$\|u_2 - u_1\| \leq \left(\frac{1}{\alpha}\|f_1 - f_2\|\right)^{\frac{1}{s}}.$$

Proof.

$$\alpha\|u_2 - u_1\|^{1+\delta} \leq \langle Au_2 - Au_1, u_2 - u_1\rangle \leq |\langle f_2 - f_1, u_2 - u_1\rangle| \leq \|u_2 - u_1\| \cdot \|f_2 - f_1\|.$$

□

3.4. **A special class of problems.** Assume H to be a Hilbert space and let $K \subset H$ be a closed, convex set. Let $f \in H^* \simeq H$. Define $a(u, v)$ to be a bilinear form on $H \times H$, s.t.

$$\|a(u, v)\| \leq C\|u\|\|v\|, \quad \forall u, v \in H; \qquad \text{(boundedness)} \tag{3.17}$$

$$\|a(u, u)\| \geq \alpha\|u\|^2, \quad \forall u \in H. \qquad \text{(coerciveness)} \tag{3.18}$$

A typical example is that $H = H_0^1(\Omega)$ and

$$a(u, v) = \int_\Omega a_{ij}(x)u_i(x)v_j(x)dx,$$

where

$$a_{ij} \in L^\infty(\Omega), \text{ and } a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2, \quad \forall \xi \in \mathbb{R}^n.$$

Note that there is no symmetry assumption as in (2.4) so that $a(u, v)$ is not necessarily an inner product on H.

Theorem 3.28. *There exists a unique solution u of the variational inequality*

$$a(u, v - u) \geq \langle f, v - u\rangle, \quad \forall v \in K,$$

where $(\cdot, \cdot)$ is the inner product on H. Furthermore, the map $f \mapsto u$ is continuous and

$$\|u_2 - u_1\| \leq \frac{1}{\alpha}\|f_2 - f_1\|, \tag{3.19}$$

where $u_i (i = 1, 2)$ is the solution corresponding to f_i.

Proof. If $a(u,v)$ is symmetric, define $(u,v)=a(u,v)$ to be a new inner product on H. Let $\overline{H}$ be the Hilbert space H equipped with $(\cdot,\cdot)$. By (3.17) and (3.18), $H \simeq \overline{H}$. $\forall f \in H$, by Riesz representation theorem, there $\exists \overline{f} \in \overline{H}$, s.t.

$$(\overline{f}, v) = \langle f, v\rangle, \quad \forall v \in H.$$

Then $u = \mathrm{Proj}_K \overline{f}$ in the metric given by $(\cdot,\cdot)$. For general case, we shall use the continuity method. Define

$$\begin{aligned} &S(u,v) = \frac{1}{2}(a(u,v)+a(v,u)), \quad \sigma(u,v) = \frac{1}{2}(a(u,v)-a(v,u)), \\ &a_t(u,v) = S(u,v) + t\sigma(u,v), \quad 0 \le t \le 1. \end{aligned}$$

Let

$$\begin{aligned} E = \{t \in [0,1] : \ &\text{The variational inequality } a_t(u, v-u) \ge \langle f, v-u\rangle, \\ &\forall v \in K \text{ has a unique solution and (3.19) holds}\}. \end{aligned}$$

It is known that $0 \in E$. We assume $t_0 \in E$ and take $t \in [0,1]$ close to t_0, with $\delta = t_0 - t$. Since $\sigma(u,v)$ is a bounded linear functional in v, by Riesz representation theorem, one has $Tu \in H$, s.t.

$$\langle Tu, v\rangle = \sigma(u,v), \quad \forall v \in H.$$

It is easy to show the map $T : u \mapsto Tu$ is linear and bounded. Denote $f_u^\delta = f + \delta Tu$. We notice that

$$\begin{aligned} a_t(u, v-u) \ge \langle f, v-u\rangle \Leftrightarrow \quad & a_{t_0}(u, v-u) \ge \langle f, v-u\rangle + (t_0 - t)\sigma(u, v-u) \\ \Leftrightarrow \quad & a_{t_0}(u, v-u) \ge \langle f_u^\delta, v-u\rangle. \end{aligned}$$

For $\forall w \in K$, let Lw be the unique solution $z \in K$ of the variational inequality

$$a_{t_0}(z, v-z) \ge \langle f_w^\delta, v-z\rangle, \quad \forall v \in K.$$

We have

$$\|Lw_1 - Lw_2\| \le \frac{1}{\alpha}\|f_{w_1}^\delta - f_{w_2}^\delta\| \le C|\delta|\|w_1 - w_2\| < \frac{1}{2}\|w_1 - w_2\|.$$

The last inequality is true if we take $|\delta| < \delta_* = \frac{1}{2C}$. Note that δ_* is independent of t_0. Hence, by contraction mapping theorem, there $\exists! u \in K$, s.t. $Lu = u$. This u solves the variational inequality associated with $a_t(\cdot,\cdot)$. If we can further show that for $\forall t$, $|t - t_0| < \delta_*$ one has (3.19), then we can start from $0 \in E$ and conclude that $E = [0,1]$, which proves the theorem. To show (3.19), we let t s.t. $|t_0 - t| < \delta_*$. Let $u_i \in K$ $(i = 1, 2)$ be the solution of

$$a_t(u_i, v - u_i) \ge \langle f_i, v - u_i\rangle, \quad \forall v \in K.$$

One immediately has

$$\begin{aligned} &a_t(u_1, u_2 - u_1) \ge \langle f_1, u_2 - u_1\rangle, \\ &a_t(-u_2, u_2 - u_1) \ge \langle -f_2, u_2 - u_1\rangle, \end{aligned}$$

which gives

$$\alpha\|u_1 - u_2\|^2 \le a_t(u_1 - u_2, u_1 - u_2) \le \langle f_1 - f_2, u_1 - u_2\rangle \le \|u_1 - u_2\|\|f_1 - f_2\|.$$

This proves (3.19). □

Much of discussions in this section can be found in the references A. Friedman [7], D. Kinderlehrer and G. Stampacchia [16] and H. Brezis [2].

4. Regularity of solutions

4.1. **Schauder estimates.** In this subsection, we will show that if the obstacle ψ is Hölder continuous, then u is Hölder continuous. We will prove the interior Schauder estimate. These estimates are also priori estimates and we may assume that u is continuous in some cases to ensure the forms of the coincidence set. The results of this subsection can be stated as follows.

Theorem 4.1 (Interior estimates for homogeneous case)**.** *Let u be the solution of the problem:*

$$\min\left\{\int_{\Omega}|\nabla u|^2dx : u\in H_0^1(\Omega), u\geq\psi \ a.e.\ in\ \Omega\right\}$$

where $\psi\leq 0$ on $\partial\Omega$.
(1)If $\psi\in C^{0,\alpha}(\Omega)$, then $u\in C^{0,\alpha}(\Omega)$. (2)If $\psi\in C^{1,\alpha}(\Omega)$, then $u\in C^{1,\alpha}(\Omega)$.

Remark 4.2. Since ψ is continuous here, then by using Corollary 3.14, one can deduce that u is actually continuous locally. The above result can be seen as the fact that if we enhance the regularity of ψ locally, the local regularity of u is also enhanced.

Remark 4.3. In the lecture notes written by Fanghua Lin, there is no explicit estimates in the statements of this theorem. This is to much ambiguous for readers. We will show by using more precise calculations that actually, for any $B_R(x_0)\subset\Omega$, one can obtain if $\psi\in C^{0,\alpha}(\Omega)$, then

$$[u]_{C^{0,\alpha}(B_{\frac{R}{2}}(x_0))}\leq C\left\{\frac{1}{R^\alpha}\left(\fint_{B_R(x_0)}|u|^2\right)^{\frac{1}{2}}+[\psi]_{C^{0,\alpha}(B_R(x_0))}\right\} \tag{4.1}$$

and if $\psi\in C^{1,\alpha}(\Omega)$, then

$$[\nabla u]_{C^{0,\alpha}(B_{\frac{R}{2}}(x_0))}\leq C\left\{\frac{1}{R^{1+\alpha}}\left(\fint_{B_R(x_0)}|u|^2\right)^{\frac{1}{2}}+[\nabla\psi]_{C^{0,\alpha}(B_R(x_0))}\right\}, \tag{4.2}$$

where C depends only on α, n and Ω.

Proof of Theorem 4.1. Part (1). We can suppose that $x_0=0$ since for general cases, the results follow from translations. Assume that $p\in\Lambda\cap B_{\frac{R}{16}}(0)$, the intersection of coincidence set and $B_{\frac{R}{16}}(0)$. If $x\in B_{\frac{R}{16}}(0)$, then $x\in B_{\frac{R}{8}}(p)$. Set $r=|x-p|$, $l(x)=u(p)$, then we can obtain from Lemma 3.13 that

$$|u(x)-u(p)|\leq\sup_{B_r(p)}|u(\cdot)-u(0)|\leq C\omega(2r)=C\omega(2|x-p|).$$

Moreover, by using the definition of $\omega(t)$, one can derive that

$$\omega(2r)=\sup_{x\in B_{2r}(p)}|\psi(x)-u(p)|=\sup_{x\in B_{2r}(p)}|\psi(x)-\psi(p)|\leq C|x-p|^\alpha[\psi]_{C^{0,\alpha}(B_R(0))},$$

where we have used the fact that $\psi(p)=u(p)$ since $p\in\Lambda$. Then,

$$|u(x)-u(p)|\leq C[\psi]_{C^{0,\alpha}(B_R(0))}|x-p|^\alpha, \tag{4.3}$$

for any $x\in B_{\frac{R}{16}}(0)$. We need to show that for any $x,y\in B_{\frac{R}{32}}(0)$,

$$|u(x)-u(y)|\leq C|x-y|^\alpha\left\{\frac{1}{R^\alpha}\left(\fint_{B_R(x_0)}|u|^2\right)^{\frac{1}{2}}+[\psi]_{C^{0,\alpha}(B_R(x_0))}\right\}.$$

This, together with some covering arguments, gives the results. For the proof, we can assume that $\Lambda \cap B_{\frac{R}{16}}(0) \neq \emptyset$, for otherwise, it is not difficult to derive that

$$|u(x)-u(y)| \leq \|\nabla u\|_{L^\infty(B_{\frac{R}{32}}(0))}|x-y| \leq C|x-y|^\alpha R^{1-\alpha} \cdot \frac{C}{R}\left(\fint_{B_{\frac{R}{16}}(0)} |u|^2\right)^{\frac{1}{2}},$$

where we have used the fact that $\Delta u = 0$ in $B_{\frac{R}{16}}(0)$ and Lipschitz estimates of u. This implies that

$$|u(x)-u(y)| \leq C\left(\frac{|x-y|}{R}\right)^\alpha \left(\fint_{B_{\frac{R}{16}}(0)} |u|^2\right)^{\frac{1}{2}}. \tag{4.4}$$

Case (a). Either x or y in $\Lambda \cap B_{\frac{R}{16}}(0)$. Then it is obvious since we can use almost the same arguments by choosing $x = p$ or $y = p$.
Case (b). $x, y \in B_{\frac{R}{16}}(0)\backslash\Lambda$ and

$$|x-y| \geq \frac{1}{4}\min\{\mathrm{dist}(x, \Lambda \cap B_{\frac{R}{16}}(0)), \mathrm{dist}(y, \Lambda \cap B_{\frac{R}{16}}(0))\}.$$

Without the loss of generality, we can assume that $\mathrm{dist}(x, \Lambda \cap B_{\frac{R}{16}}(0)) \leq \mathrm{dist}(y, \Lambda \cap B_{\frac{R}{16}}(0))$. Then we can choose $x^* \in \Lambda \cap B_{\frac{R}{16}}(0)$ s.t.

$$|x-x^*| = \mathrm{dist}(x, \Lambda \cap B_{\frac{R}{16}}(0)) = \min\{\mathrm{dist}(x, \Lambda \cap B_{\frac{R}{16}}(0)), \mathrm{dist}(y, \Lambda \cap B_{\frac{R}{16}}(0))\}.$$

Then by using the fact that $|y-x^*|^\alpha \leq C(|x-y|^\alpha + |x-x^*|^\alpha)$,

$$\begin{aligned}
|u(x)-u(y)| &\leq |u(x)-u(x^*)| + |u(y)-u(x^*)| \\
&\leq C[\psi]_{C^{0,\alpha}(B_R(0))}(|x-x^*|^\alpha + |y-x^*|^\alpha) \\
&\leq C[\psi]_{C^{0,\alpha}(B_R(0))}(|x-y|^\alpha + |x-x^*|^\alpha) \\
&\leq C[\psi]_{C^{0,\alpha}(B_R(0))}|x-y|^\alpha.
\end{aligned}$$

Case (c). $x, y \in B_{\frac{R}{16}}(0)\backslash\Lambda$ and $|x-y| < \frac{1}{4}\min\{\mathrm{dist}(x, \Lambda \cap B_{\frac{R}{16}}(0)), \mathrm{dist}(y, \Lambda \cap B_{\frac{R}{16}}(0))\}$. By using the same methods above, we can still choose x^* and set $r^* = |x-x^*|$. Note that u is harmonic in $B_{\frac{r^*}{3}}(x) \subset B_{\frac{R}{16}}(0)$ and so is $v(\cdot) = u(\cdot) - u(x^*)$. Here we have used the fact that $r^* \leq \frac{R}{32} + \frac{R}{16} \leq \frac{3R}{32}$. Then, we can use the Lipschitz estimates of v and obtain that

$$\begin{aligned}
\|\nabla u\|_{L^\infty(B_{\frac{7r^*}{24}}(x))} &= \|\nabla v\|_{L^\infty(B_{\frac{7r^*}{24}}(x))} \\
&\leq \frac{C}{r^*}\left(\fint_{B_{\frac{r^*}{3}}(x)} |u(\cdot)-u(x^*)|^2\right)^{\frac{1}{2}} \\
&\leq \frac{C}{r^*}\left(\fint_{B_{r^*}(x)} |u(\cdot)-u(x^*)|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Meanwhile, since $|y-x| < \frac{1}{4}r^*$ by definition of r^*, one can show that

$$\begin{aligned}
|u(x)-u(y)| &\leq \|\nabla u\|_{L^\infty(B_{\frac{r^*}{4}}(x))}|x-y| \leq \|\nabla u\|_{L^\infty(B_{\frac{7r^*}{24}}(x))}|x-y| \\
&\leq C(r^*)^{1-\alpha}|x-y|^\alpha \cdot \frac{C}{r^*}\left(\fint_{B_{r^*}(x)} |u(\cdot)-u(x^*)|^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.5}$$

Owing to (4.3), it can be get without difficulty that $|u(z)-u(x^*)| \leq C[\psi]_{C^{0,\alpha}(B_R(0))}|z-x^*|^\alpha$ for $z \in B_{r^*}(x)$. Then

$$\left(\fint_{B_{r^*}(x)} |u(\cdot)-u(x^*)|^2\right)^{\frac{1}{2}} \leq C[\psi]_{C^{0,\alpha}(B_R(0))}(r^*)^\alpha.$$

This, together with (4.5), gives that

$$|u(x)-u(y)| \leq C[\psi]_{C^{0,\alpha}(B_R(0))}|x-y|^\alpha.$$

Combined with (4.4), we can complete the proof of (4.1).

Part (2). One difference between proofs of $C^{1,\alpha}$ and $C^{0,\alpha}$ is that we need to show that ∇u exists at any point in Ω at first. Assume that $p \in \Lambda \cap B_{\frac{R}{16}}(0)$. Then for any $x \in B_{\frac{R}{16}}(0) \subset B_{\frac{R}{8}}(p)$, we can choose $r = |x-p|$ and $l(x) = \psi(p) + \nabla\psi(p)(x-p)$. In view of Lemma 3.13, then for any $x \in B_{\frac{R}{8}}(p)$, we can obtain that

$$|u(x)-\psi(p)-\nabla\psi(p)(x-p)| \leq \sup_{B_r(0)} |(u-l)(\cdot)| \leq C\omega(2r) = C\omega(2|x-p|).$$

Moreover, since $\psi \in C^{1,\alpha}(\Omega)$, it can be shown that

$$\begin{aligned}\omega(2r) &= \sup_{x\in B_r(0)} |\psi(x)-\psi(p)-\nabla\psi(p)(x-p)| \\ &\leq Cr^{1+\alpha}[\nabla\psi]_{C^{0,\alpha}(B_R(0))} \leq C|x-p|^{1+\alpha}[\nabla\psi]_{C^{0,\alpha}(B_R(0))},\end{aligned}$$

Combining these two formula, we have, for any $x \in B_{\frac{R}{16}}(0)$,

$$|u(x)-\psi(p)-\nabla\psi(p)(x-p)| \leq C|x-p|^{1+\alpha}[\nabla\psi]_{C^{0,\alpha}(B_R(0))}. \tag{4.6}$$

Then $\nabla u(p)$ is well defined. Similarly, one can show that $\nabla u(x)$ can be defined pointwise on $B_{\frac{R}{16}}(0)$. We need to check that for any $x, y \in B_{\frac{R}{32}}(0)$,

$$|\nabla u(x)-\nabla u(y)| \leq C|x-y|^\alpha \left\{\frac{1}{R^{1+\alpha}}\left(\fint_{B_R(0)} |u|^2\right)^{\frac{1}{2}} + [\nabla\psi]_{C^{0,\alpha}(B_R(0))}\right\}.$$

Also, if $\Lambda \cap B_{\frac{R}{16}}(0) = \emptyset$, then by using C^2 estimates of harmonic functions, we have

$$|\nabla u(x)-\nabla u(y)| \leq \frac{C}{R}\left(\frac{|x-y|}{R}\right)^\alpha \left(\fint_{B_{\frac{R}{16}}(0)} |u|^2\right)^{\frac{1}{2}}$$

for any $x, y \in B_{\frac{R}{32}}(0)$. Therefore, we can assume that $\Lambda \cap B_{\frac{R}{16}}(0) \neq \emptyset$.

Case (a). Either x or y in $\Lambda \cap B_{\frac{R}{16}}(0)$. We can assume that $y \in \Lambda \cap B_{\frac{R}{16}}(0)$. By (4.6), $\nabla u(y) = \nabla\psi(y)$. Let $x^* \in \Lambda \cap B_{\frac{R}{16}}(0)$ s.t. $r^* = |x-x^*| = \mathrm{dist}(x, \Lambda \cap B_{\frac{R}{16}}(0))$. Set $v(\cdot) = u(\cdot) - \psi(x^*) - \nabla\psi(x^*)(\cdot - x^*)$. Note that v is harmonic in $B_{\frac{r^*}{3}}(x)$, then by using Lipschitz estimates of v, we can obtain that

$$\begin{aligned}|\nabla v(x)| &\leq \frac{C}{r^*}\left(\fint_{B_{\frac{r^*}{3}}} |u(\cdot)-\psi(x^*)-\nabla\psi(x^*)(\cdot - x^*)|^2\right)^{\frac{1}{2}} \\ &\leq \frac{C(r^*)^{1+\alpha}}{r^*}[\nabla\psi]_{C^{0,\alpha}(B_R(0))} \leq C[\nabla\psi]_{C^{0,\alpha}(B_R(0))}(r^*)^\alpha,\end{aligned}$$

where for the second inequality, we have used (4.6). This means that

$$|\nabla u(x)-\nabla\psi(x^*)| \leq C[\nabla\psi]_{C^{0,\alpha}(B_R(0))}(r^*)^\alpha.$$

Then

$$\begin{aligned}|\nabla u(x)-\nabla u(y)| &\le |\nabla u(x)-\nabla\psi(x^*)|+|\nabla\psi(x^*)-\nabla u(y)|\\ &\le |\nabla\psi(x^*)-\nabla\psi(y)|+C[\nabla\psi]_{B_R(0)}(r^*)^\alpha\\ &\le C[\nabla\psi]_{C^{0,\alpha}(B_R(0))}|y-x^*|^\alpha+C[\nabla\psi]_{C^{0,\alpha}(B_R(0))}(r^*)^\alpha\\ &\le C[\nabla\psi]_{C^{0,\alpha}(B_R(0))}(|x-y|^\alpha+|x-x^*|^\alpha)\\ &\le C[\nabla\psi]_{C^{0,\alpha}(B_R(0))}|x-y|^\alpha,\end{aligned}$$

where we have used the fact that $|x-y|\ge|x-x^*|$.
Case (b). $x,y\in B_{\frac{R}{16}}(0)\backslash\Lambda$ and

$$|x-y|\ge\frac{1}{4}\min\{\mathrm{dist}(x,\Lambda\cap B_{\frac{R}{16}}(0)),\mathrm{dist}(y,\Lambda\cap B_{\frac{R}{16}}(0))\}.$$

Without the the loss of generality, we can assume that $\mathrm{dist}(x,\Lambda\cap B_{\frac{R}{16}}(0))\le\mathrm{dist}(y,\Lambda\cap B_{\frac{R}{16}}(0))$. Then we can choose $x^*\in\Lambda\cap B_{\frac{R}{16}}(0)$ s.t.

$$|x-x^*|=\mathrm{dist}(x,\Lambda\cap B_{\frac{R}{16}}(0))=\min\{\mathrm{dist}(x,\Lambda\cap B_{\frac{R}{16}}(0)),\mathrm{dist}(y,\Lambda\cap B_{\frac{R}{16}}(0))\}.$$

Then by using triangular inequality,

$$\begin{aligned}|\nabla u(x)-\nabla u(y)| &\le |\nabla u(x)-\nabla\psi(x^*)|+|\nabla u(y)-\nabla\psi(x^*)|\\ &\le C[\nabla\psi]_{C^{0,\alpha}(B_R(0))}(|x-x^*|^\alpha+|y-x^*|^\alpha)\\ &\le C[\nabla\psi]_{C^{0,\alpha}(B_R(0))}(|x-y|^\alpha+|x-x^*|^\alpha)\\ &\le C[\nabla\psi]_{C^{0,\alpha}(B_R(0))}|x-y|^\alpha.\end{aligned}$$

Case (c). $x,y\in B_{\frac{R}{16}}(0)\backslash\Lambda$ and $|x-y|<\frac{1}{4}\min\{\mathrm{dist}(x,\Lambda\cap B_{\frac{R}{16}}(0)),\mathrm{dist}(y,\Lambda\cap B_{\frac{R}{16}}(0))\}$. By using the same methods above, we can still choose x^* and set $r^*=|x-x^*|$. Note that u is harmonic in $B_{\frac{r^*}{3}}(x)$ and so is $v(\cdot)=u(\cdot)-\psi(x^*)-\nabla\psi(x^*)(\cdot-x^*)$. We can use the ∇^2 estimates of v and obtain that

$$\|\nabla^2 v\|_{L^\infty(B_{\frac{7r^*}{24}}(x))}\le\frac{C}{(r^*)^2}\left(\fint_{B_{\frac{r^*}{3}}(x)}|v|^2\right)^{\frac{1}{2}}\le\frac{C}{(r^*)^2}\left(\fint_{B_{r^*}(x)}|v|^2\right)^{\frac{1}{2}}.$$

Then, since $|y-x|<\frac{1}{4}r^*$, one can show that

$$\begin{aligned}|\nabla u(x)-\nabla u(y)| &\le \|\nabla^2 u\|_{L^\infty(B_{\frac{7r^*}{24}}(x))}|x-y|\\ &\le C(r^*)^{1-\alpha}|x-y|^\alpha\cdot\frac{C}{(r^*)^2}\left(\fint_{B_{r^*}(x)}|v|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Owing to the observation that $|v(z)|\le C[\nabla\psi]_{C^{0,\alpha}(B_R(0))}|z-x^*|^{1+\alpha}$ for any $z\in B_{r^*}(x)$, it can be shown that

$$\left(\fint_{B_{r^*}(x)}|v|^2\right)^{\frac{1}{2}}\le C[\nabla\psi]_{C^{0,\alpha}(B_R(0))}(r^*)^{1+\alpha}.$$

Then

$$|\nabla u(x)-\nabla u(y)|\le C[\nabla\psi]_{C^{0,\alpha}(B_R(0))}|x-y|^\alpha.$$

Now we can complete the proof. □

Remark 4.4. There is a natural generalization of the above Schauder estimates to the variational inequality $\langle-\Delta u,v-u\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\ge\langle f,v-u\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}$, where $f\in C^{0,\alpha}(\Omega)$.

To give the generalization, we will first introduce maximum principle and Harnack's inequality for inhomogeneous case.

Lemma 4.5. *Let $u \in H^1(B_R)$ be the weak solution satisfying $-\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$. Then*

$$\|u\|_{L^\infty(B_R)} \leq 2\left\{R^2\|f\|_{L^\infty(B_R)} + \|g\|_{L^\infty(B_R)}\right\}, \tag{4.7}$$

Proof. Let us consider the function $v = \frac{u}{R^2\|f\|_{L^\infty(B_R)}+\|g\|_{L^\infty(B_R)}}$. We want to show that $\|v\|_{L^\infty(B_R)} \leq 2$. Note that the function v satisfies $-\Delta v = \widetilde{f}$ in Ω and $v = \widetilde{g}$ on $\partial\Omega$ with $\|\widetilde{f}\|_\infty \leq \frac{1}{R^2}$ and $\|\widetilde{g}\|_\infty \leq 1$. Let us consider the function

$$w(x) = -\frac{1}{2nR^2}\sum_{i=1}^n x_i^2 + \frac{1}{2n} + 1.$$

Such function w satisfies the equation $-\Delta w = 1$ in Ω and $w = 1$ on $\partial\Omega$. Then $-\Delta(w - v) \geq 0$ in Ω and $w - v \geq 0$ on $\partial\Omega$. This implies that $v \leq w$ in B_R by using maximum principle. Then, we can complete the proof by using $\frac{1}{2n} + 1 < 2$ for any $n \in \mathbb{N}_+$ □

Lemma 4.6. *Let $B_r(0) = B_r$ with $r > 0$ and $u \in H^1(B_{2R})$ be a nonnegative weak solution of $-\Delta u = f$ in B_{2R}. Then,*

$$\sup_{x\in B_R} u(x) \leq C\left\{\inf_{x\in B_R} u(x) + R^2\|f\|_{L^\infty(B_R)}\right\}, \tag{4.8}$$

where C depends only on n.

Proof. We express u as $u = v + w$ with

$$\begin{cases} -\Delta v = 0 \text{ in } \Omega \\ \quad v = u \text{ on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta w = f \text{ in } \Omega \\ \quad w = 0 \text{ on } \partial\Omega. \end{cases}$$

Then by using Harnack's inequality for homogeneous case and (4.7), one can deduce that

$$\sup_{x\in B_R} v(x) \leq \inf_{x\in B_R} v(x) \quad \text{and} \quad \sup_{x\in B_R} w(x) \leq CR^2\|f\|_{L^\infty(B_{2R})}.$$

Thus it is natural to see that

$$\sup_{x\in B_R} u(x) \leq \sup_{x\in B_R} v(x) + \sup_{x\in B_R} w(x) \leq C\left\{\inf_{x\in B_R} u(x) + R^2\|f\|_{L^\infty(B_R)}\right\}.$$

Notice that we have also used here that $v \leq u$ in B_R. □

Then, we will establish similar results as Lemma 3.12 and Lemma 3.13 by using maximum principle and Harnack's inequality given above for elliptic equation in divergence form. Also, here we assume that u is continuous. This is because we cannot use the same method before to derive that u is lower semi-continuous. To make things simpler, we give the assumptions and state our results.

Lemma 4.7. *Let u be the solution of problem (3.8). Assume that $B_R(0) = B_R \subset \Omega$. let $l(x)$ be a linear function and $u_0 \in H^1(B_R(0))$ be the weak solution of*

$$\begin{cases} -\Delta u_0 = f \in L^\infty(B_R) \text{ in } B_R(0), \\ u_0 = u - \psi \geq 0 \text{ on } \partial B_R(0). \end{cases} \tag{4.9}$$

Then

$$u_0(x) + l(x) - \omega(R) \leq u(x) \leq u_0(x) + l(x) + \omega(R) + CR^2\|f\|_{L^\infty(B_R)},$$

a.e. in $B_R(0)$, where C depends only on n and

$$\omega(r) = \sup_{B_r(0)} |\psi(x) - l(x)|, \quad 0 \leq r \leq R_0 = \operatorname{diam}(\Omega).$$

In particular, for $l(x) = \psi(0)$, we have

$$|u_0(x) + \psi(x) - u(x)| \leq \sup_{B_R(0)} |\psi(x) - \psi(0)|.$$

Proof. The proof of this lemma is similar to Lemma 3.12, we will give it for the sake of completeness. Let $v_0 \in H^1(B_R)$ be the weak solution of the Dirichlet problem: $-\Delta v_0 = f$ in B_R and $v_0 = 0$ on ∂B_R. Note that $f = -\Delta(u_0 + l - \omega(R)) \le -\Delta u$ in $D'(B_R)$, where we have used the fact that u is a supersolution of $-\Delta(\cdot) - f$. Also $u_0 + l - \omega(R) \le u$ on ∂B_R. Then by maximum principle, $u_0 + l - \omega(R) \le u$ in B_R. On the other hand, since $u(x) \le u_0(x) + l(x) + \omega(R)$ on ∂B_R and $\operatorname{supp}(\Delta(u - u_0 - l - \omega(R))) \subset \Lambda \cap B_R$ (in view of the fact that $-\Delta u = f$ in N), we have

$$\sup_{B_R}(u - u_0 - l - \omega(R)) \le \max\left\{0, \sup_{\Lambda\cap B_R}(u - u_0 - l - \omega(R))\right\} \le CR^2\|f\|_{L^\infty(B_R)}.$$

Here, we have used the fact that for any $x \in \Lambda \cap B_R$,

$$-u_0(x) = -(u_0(x) - v_0(x)) - v_0(x) \le CR^2\|f\|_{L^\infty(B_R)}.$$

This completes the proof. □

Lemma 4.8. *Assume that $0 \in \Lambda$ and let u, l and ω be defined in Lemma 4.7. Then we have*

$$\sup_{x\in B_{\frac{R}{2}}} |u(x) - l(x)| \le C(n)\left\{\omega(R) + R^2\|f\|_{L^\infty(B_R)}\right\},$$

where $C(n)$ is a positive constant only depending on n.

Proof. Since $u \ge \psi$ in B_R, we find $u - l \ge -\omega(R)$ in B_R. This proves one side of the estimate. By Lemma 3.12, we can obtain that

$$\sup_{x\in B_{\frac{R}{2}}}(u(x) - l(x)) \le \sup_{x\in B_{\frac{R}{2}}} u_0(x) + \omega(R).$$

Choose $v_0 \in H^1(B_R)$ be the weak solution of the Dirichlet problem: $-\Delta v_0 = f$ in B_R and $v_0 = 0$ on ∂B_R, then $-\Delta(u_0 - v_0) = 0$ in Ω and $u_0 - v_0 = u - \psi \ge 0$. Since $u_0 - v_0$ is a nonnegative harmonic function in B_R. By Harnack's inequality, we have

$$\begin{aligned}\sup_{x\in B_{\frac{R}{2}}}((u_0 - v_0)(x)) &\le C(n)((u_0 - v_0)(0))\\ &\le C(n)(u(0) - l(0) + \omega(R)) + C(n)|v_0(0)|\\ &\le C(n)(u(0) - l(0) + \omega(R)) + C(n)R^2\|f\|_{L^\infty(B_R)}.\end{aligned}$$

Since $u(0) = \psi(0)$, one can derive that

$$\begin{aligned}\sup_{x\in B_{\frac{R}{2}}} u_0(x) &\le \sup_{x\in B_{\frac{R}{2}}}(u_0(x) - v_0(x)) + \sup_{x\in B_{\frac{R}{2}}} v_0(x)\\ &\le C(n)(\psi(0) - l(0) + \omega(R)) + C(n)R^2\|f\|_{L^\infty(B_R)}\\ &\le C(n)\omega(R) + C(n)R^2\|f\|_{L^\infty(B_R)}.\end{aligned}$$

Then

$$\sup_{B_{\frac{R}{2}}}(u(x) - l(x)) \le C(n)\omega(R) + C(n)R^2\|f\|_{L^\infty(B_R)},$$

which complete the proof. □

By using these two lemmas, we can the following results.

Theorem 4.9 (Interior estimates for nonhomogeneous case)**.** *Let u be the solution of problem:*

$$\min\left\{\int_\Omega |\nabla u|^2dx - 2\int_\Omega fudx : u \in H_0^1(\Omega), u \ge \psi \ a.e.\ in\ \Omega\right\}$$

where $\psi \leq 0$ *on* $\partial\Omega$ *and* $f \in C^{0,\alpha}(\Omega)$.
(1)If $\psi \in C^{0,\alpha}(\Omega)$*, then* $u \in C^{0,\alpha}(\Omega)$ *and for any* $B_R(x_0) \subset \Omega$,

$$[u]_{C^{0,\alpha}(B_{\frac{R}{2}})} \leq C\left\{\frac{1}{R^\alpha}\left(\fint_{B_R}|u|^2\right)^{\frac{1}{2}} + [\psi]_{C^{0,\alpha}(B_R)} + R\mathcal{M}(\alpha, x_0, R, f)\right\}, \tag{4.10}$$

(2)If $\psi \in C^{1,\alpha}(\Omega)$*, then* $u \in C^{1,\alpha}(\Omega)$ *and for any* $B_R(x_0) \subset \Omega$,

$$[\nabla u]_{C^{0,\alpha}(B_{\frac{R}{2}})} \leq C\left\{\frac{1}{R^{1+\alpha}}\left(\fint_{B_R}|u|^2\right)^{\frac{1}{2}} + [\nabla\psi]_{C^{0,\alpha}(B_R)} + \mathcal{M}(\alpha, x_0, R, f)\right\}. \tag{4.11}$$

Here

$$\mathcal{M}(\alpha, x_0, R, f) = R^{1-\alpha}\|f\|_{L^\infty(B_R(x_0))} + R[f]_{C^{0,\alpha}(B_R(x_0))}$$

C *depends only on* α, n *and* Ω.

Proof of Theorem 4.9. The proof is almost the same as given before for homogeneous case, here we give the proof here for the sake of completeness. Part (1). We can suppose that $x_0 = 0$ since for general cases, the results follow from translations. Assume that $p \in \Lambda \cap B_{\frac{R}{16}}(0)$, the intersection of coincidence set and $B_{\frac{R}{16}}(0)$. If $x \in B_{\frac{R}{16}}(0)$, then $x \in B_{\frac{R}{8}}(p)$. Set $r = |x-p|$, $l(x) = u(p)$, then we can obtain from Lemma 4.8 that

$$\begin{aligned}
|u(x) - u(p)| &\leq \sup_{z \in B_r(p)} |u(z) - u(0)| \\
&\leq C\omega(2r) + Cr^2\|f\|_{L^\infty(B_{2r}(p))} \\
&= C\omega(2|x-p|) + C|x-p|^2\|f\|_{L^\infty(B_R(0))}.
\end{aligned}$$

Moreover, by using the definition of $\omega(t)$, one can derive that

$$\omega(2r) = \sup_{x \in B_{2r}(p)} |\psi(x) - u(p)| = \sup_{x \in B_{2r}(p)} |\psi(x) - \psi(p)| \leq C|x-p|^\alpha[\psi]_{C^{0,\alpha}(B_R(0))},$$

where we have used the fact that $\psi(p) = u(p)$ since $p \in \Lambda$. Then,

$$|u(x) - u(p)| \leq C\left\{[\psi]_{C^{0,\alpha}(B_R(0))} + R^{2-\alpha}\|f\|_{L^\infty(B_R(0))}\right\}|x-p|^\alpha, \tag{4.12}$$

for any $x \in B_{\frac{R}{16}}(0)$. We need to show that for any $x, y \in B_{\frac{R}{32}}(0)$,

$$|u(x) - u(y)| \leq C|x-y|^\alpha\left\{\frac{1}{R^\alpha}\left(\fint_{B_R(0)}|u|^2\right)^{\frac{1}{2}} + [\psi]_{C^{0,\alpha}(B_R(0))} + R\mathcal{M}(\alpha, 0, R, f)\right\}.$$

This, together with some covering arguments, gives the results. For the proof, we can assume that $\Lambda \cap B_{\frac{R}{16}}(0) \neq \emptyset$, for otherwise, it is not difficult to derive that

$$\begin{aligned}
|u(x) - u(y)| &\leq \|\nabla u\|_{L^\infty(B_{\frac{R}{32}}(0))}|x-y| \\
&\leq C|x-y|\left\{\frac{1}{R}\left(\fint_{B_{\frac{R}{16}}(0)}|u|^2\right)^{\frac{1}{2}} + R^\alpha\mathcal{M}\left(\alpha, 0, \frac{R}{16}, f\right)\right\} \\
&\leq C|x-y|^\alpha\left\{\frac{1}{R^\alpha}\left(\fint_{B_R(0)}|u|^2\right)^{\frac{1}{2}} + R\mathcal{M}(\alpha, 0, R, f)\right\},
\end{aligned} \tag{4.13}$$

where we have used the fact that $-\Delta u = f$ in $B_{\frac{R}{16}}(0)$ and Lipschitz estimates of u when $f \in C^{0,\alpha}$. (Here, actually, we can obtain the $C^{1,\alpha}$ estimate for u and we only need to L^∞ estimates for ∇u.)
Case (a). Either x or y in $\Lambda \cap B_{\frac{R}{16}}(0)$. Then it is obvious since we can use almost the same arguments

by choosing $x = p$ or $y = p$.
Case (b). $x, y \in B_{\frac{R}{16}}(0)\backslash\Lambda$ and

$$|x-y| \geq \frac{1}{4}\min\{\mathrm{dist}(x, \Lambda\cap B_{\frac{R}{16}}(0)), \mathrm{dist}(y, \Lambda\cap B_{\frac{R}{16}}(0))\}.$$

Without the loss of generality, we can assume that

$$\mathrm{dist}(x, \Lambda\cap B_{\frac{R}{16}}(0)) \leq \mathrm{dist}(y, \Lambda\cap B_{\frac{R}{16}}(0)).$$

Then we can choose $x^* \in \Lambda\cap B_{\frac{R}{16}}(0)$ s.t.

$$|x-x^*| = \mathrm{dist}(x, \Lambda\cap B_{\frac{R}{16}}(0)) = \min\{\mathrm{dist}(x, \Lambda\cap B_{\frac{R}{16}}(0)), \mathrm{dist}(y, \Lambda\cap B_{\frac{R}{16}}(0))\}.$$

Then by using the fact that $|y-x^*|^\alpha \leq C(|x-y|^\alpha + |x-x^*|^\alpha)$ and (4.12),

$$\begin{aligned}|u(x)-u(y)| &\leq |u(x)-u(x^*)| + |u(y)-u(x^*)| \\ &\leq C\left\{[\psi]_{C^{0,\alpha}(B_R(0))} + R^{2-\alpha}\|f\|_{L^\infty(B_R(0))}\right\}(|x-x^*|^\alpha + |y-x^*|^\alpha) \\ &\leq C\left\{[\psi]_{C^{0,\alpha}(B_R(0))} + R^{2-\alpha}\|f\|_{L^\infty(B_R(0))}\right\}(|x-y|^\alpha + |x-x^*|^\alpha) \\ &\leq C\left\{[\psi]_{C^{0,\alpha}(B_R(0))} + R^{2-\alpha}\|f\|_{L^\infty(B_R(0))}\right\}|x-y|^\alpha.\end{aligned}$$

Case (c). $x, y \in B_{\frac{R}{16}}(0)\backslash\Lambda$ and

$$|x-y| < \frac{1}{4}\min\{\mathrm{dist}(x, \Lambda\cap B_{\frac{R}{16}}(0)), \mathrm{dist}(y, \Lambda\cap B_{\frac{R}{16}}(0))\}.$$

By using the same methods above, we can still choose x^* and set $r^* = |x-x^*|$. Note that u satisfies $-\Delta u = f$ in $B_{\frac{r^*}{3}}(x) \subset B_{\frac{R}{16}}(0)$ and so is $v(\cdot) = u(\cdot) - u(x^*)$ $(-\Delta v = f)$. Here we have used the fact that $r^* \leq \frac{R}{32} + \frac{R}{16} \leq \frac{3R}{32}$. Then, we can use the Lipschitz estimates of v and obtain that

$$\begin{aligned}\|\nabla u\|_{L^\infty(B_{\frac{7r^*}{24}}(x))} &= \|\nabla v\|_{L^\infty(B_{\frac{7r^*}{24}}(x))} \\ &\leq \frac{C}{r^*}\left(\fint_{B_{\frac{r^*}{3}}(x)} |u(\cdot)-u(x^*)|^2\right)^{\frac{1}{2}} + C(r^*)^\alpha\mathcal{M}\left(\alpha, x, \frac{r^*}{3}, f\right) \\ &\leq \frac{C}{r^*}\left(\fint_{B_{r^*}(x)} |u(\cdot)-u(x^*)|^2\right)^{\frac{1}{2}} + C(r^*)^\alpha\mathcal{M}(\alpha, 0, R, f).\end{aligned}$$

Meanwhile, since $|y-x| < \frac{1}{4}r^*$ by definition of r^*, one can show that

$$\begin{aligned}|u(x)-u(y)| &\leq \|\nabla u\|_{L^\infty(B_{\frac{r^*}{4}}(x))}|x-y| \leq \|\nabla u\|_{L^\infty(B_{\frac{7r^*}{24}}(x))}|x-y| \\ &\leq \frac{C|x-y|^\alpha}{(r^*)^{\alpha-1}}\left\{\frac{1}{r^*}\left(\fint_{B_{r^*}(x)} |u(\cdot)-u(x^*)|^2\right)^{\frac{1}{2}} + (r^*)^\alpha\mathcal{M}(\alpha, 0, R, f)\right\}.\end{aligned} \tag{4.14}$$

Owing to (4.12), it can be get without difficulty that

$$|u(z)-u(x^*)| \leq C\left\{[\psi]_{C^{0,\alpha}(B_R(0))} + R^{2-\alpha}\|f\|_{L^\infty(B_R(0))}\right\}|z-x^*|^\alpha$$

for $z \in B_{r^*}(x)$. Then

$$\left(\fint_{B_{r^*}(x)} |u(\cdot)-u(x^*)|^2\right)^{\frac{1}{2}} \leq C\left\{[\psi]_{C^{0,\alpha}(B_R(0))} + R^{2-\alpha}\|f\|_{L^\infty(B_R(0))}\right\}(r^*)^\alpha.$$

This, together with (4.14), gives that

$$|u(x)-u(y)| \leq C\left\{[\psi]_{C^{0,\alpha}(B_R(0))} + R\mathcal{M}(\alpha, 0, R, f)\right\}|x-y|^\alpha.$$

Combined with (4.13), we can complete the proof of (4.10).
Part (2). One difference between proofs of $C^{1,\alpha}$ and $C^{0,\alpha}$ is that we need to show that ∇u exists at any point in Ω at first. Assume that $p \in \Lambda \cap B_{\frac{R}{16}}(0)$. Then for any $x \in B_{\frac{R}{16}}(0) \subset B_{\frac{R}{8}}(p)$, we can choose $r = |x-p|$ and $l(x) = \psi(p) + \nabla\psi(p)(x-p)$. In view of Lemma 3.13, then for any $x \in B_{\frac{R}{8}}(p)$, we can obtain that

$$\begin{aligned}|u(x) - \psi(p) - \nabla\psi(p)(x-p)| &\leq \sup_{B_r(0)} |(u-l)(\cdot)| \\ &\leq C\omega(2r) + Cr^2 \|f\|_{L^\infty(B_{2r}(p))} \\ &= C\omega(2|x-p|) + C|x-p|^2 \|f\|_{L^\infty(B_R(0))}.\end{aligned}$$

Moreover, since $\psi \in C^{1,\alpha}(\Omega)$, it can be shown that

$$\begin{aligned}\omega(2r) &= \sup_{x \in B_r(0)} |\psi(x) - \psi(p) - \nabla\psi(p)(x-p)| \\ &\leq Cr^{1+\alpha}[\nabla\psi]_{C^{0,\alpha}(B_R(0))} \leq C|x-p|^{1+\alpha}[\nabla\psi]_{C^{0,\alpha}(B_R(0))},\end{aligned}$$

Combining these two formula, we have, for any $x \in B_{\frac{R}{16}}(0)$,

$$|u(x) - \psi(p) - \nabla\psi(p)(x-p)| \leq C|x-p|^{1+\alpha} \left\{[\nabla\psi]_{C^{0,\alpha}(B_R(0))} + R^{1-\alpha}\|f\|_{L^\infty(B_R(0))}\right\}. \tag{4.15}$$

Then $\nabla u(p)$ is well defined. Similarly, one can show that $\nabla u(x)$ can be defined pointwise on $B_{\frac{R}{16}}(0)$. We need to check that for any $x, y \in B_{\frac{R}{32}}(0)$,

$$|\nabla u(x) - \nabla u(y)| \leq C|x-y|^\alpha \left\{\frac{1}{R^{1+\alpha}} \left(⨍_{B_R(0)} |u|^2\right)^{\frac{1}{2}} + [\nabla\psi]_{C^{0,\alpha}(B_R(0))} + \mathcal{M}(\alpha, 0, R, f)\right\}.$$

Also, if $\Lambda \cap B_{\frac{R}{16}}(0) = \emptyset$, then by using $C^{1,\alpha}$ estimates of solutions of $-\Delta u = f$, we have

$$|\nabla u(x) - \nabla u(y)| \leq \frac{C}{R}\left(\frac{|x-y|}{R}\right)^\alpha \left\{\left(⨍_{B_{\frac{R}{16}}(0)} |u|^2\right)^{\frac{1}{2}} + R^{1+\alpha}\mathcal{M}(\alpha, 0, R, f)\right\}$$

for any $x, y \in B_{\frac{R}{32}}(0)$. Therefore, we can assume that $\Lambda \cap B_{\frac{R}{16}}(0) \neq \emptyset$.
Case (a). Either x or y in $\Lambda \cap B_{\frac{R}{16}}(0)$. We can assume that $y \in \Lambda \cap B_{\frac{R}{16}}(0)$. By (4.6), $\nabla u(y) = \nabla\psi(y)$. Let $x^* \in \Lambda \cap B_{\frac{R}{16}}(0)$ s.t. $r^* = |x - x^*| = \mathrm{dist}(x, \Lambda \cap B_{\frac{R}{16}}(0))$. Set $v(\cdot) = u(\cdot) - \psi(x^*) - \nabla\psi(x^*)(\cdot - x^*)$. Note that v satisfies $-\Delta v = f$ in $B_{\frac{r^*}{3}}(x)$, then by using Lipschitz estimates for this equation, we can obtain that

$$\begin{aligned}|\nabla v(x)| &\leq \frac{C}{r^*}\left(⨍_{B_{\frac{r^*}{3}}(x)} |u(\cdot) - \psi(x^*) - \nabla\psi(x^*)(\cdot - x^*)|^2\right)^{\frac{1}{2}} + C(r^*)^\alpha \mathcal{M}\left(\alpha, x, \frac{r^*}{3}, f\right) \\ &\leq \frac{C(r^*)^{1+\alpha}}{r^*}\left\{[\nabla\psi]_{C^{0,\alpha}(B_R(0))} + R^{1-\alpha}\|f\|_{L^\infty(B_R(0))}\right\} + C(r^*)^\alpha \mathcal{M}(\alpha, 0, R, f) \\ &\leq C(r^*)^\alpha \left\{[\nabla\psi]_{C^{0,\alpha}(B_R(0))} + \mathcal{M}(\alpha, 0, R, f)\right\},\end{aligned}$$

where for the second inequality, we have used (4.15). This means that

$$|\nabla u(x) - \nabla\psi(x^*)| \leq C(r^*)^\alpha \left\{[\nabla\psi]_{C^{0,\alpha}(B_R(0))} + \mathcal{M}(\alpha, 0, R, f)\right\}. \tag{4.16}$$

By using the definition of v. Then

$$\begin{aligned}|\nabla u(x)-\nabla u(y)| &\le |\nabla u(x)-\nabla\psi(x^*)|+|\nabla\psi(x^*)-\nabla u(y)|\\ &\le |\nabla\psi(x^*)-\nabla\psi(y)|+C(r^*)^\alpha\left\{[\nabla\psi]_{C^{0,\alpha}(B_R(0))}+\mathcal{M}(\alpha,0,R,f)\right\}\\ &\le C\left\{[\nabla\psi]_{C^{0,\alpha}(B_R(0))}+\mathcal{M}(\alpha,0,R,f)\right\}(|y-x^*|^\alpha+(r^*)^\alpha)\\ &\le C\left\{[\nabla\psi]_{C^{0,\alpha}(B_R(0))}+\mathcal{M}(\alpha,0,R,f)\right\}(|x-y|^\alpha+|x-x^*|^\alpha)\\ &\le C\left\{[\nabla\psi]_{C^{0,\alpha}(B_R(0))}+\mathcal{M}(\alpha,0,R,f)\right\}|x-y|^\alpha,\end{aligned}$$

where we have used the fact that $|x-y|\ge|x-x^*|$.
Case (b). $x,y\in B_{\frac{R}{16}}(0)\backslash\Lambda$ and

$$|x-y|\ge\frac{1}{4}\min\{\mathrm{dist}(x,\Lambda\cap B_{\frac{R}{16}}(0)),\mathrm{dist}(y,\Lambda\cap B_{\frac{R}{16}}(0))\}.$$

Without the the loss of generality, we can assume that

$$\mathrm{dist}(x,\Lambda\cap B_{\frac{R}{16}}(0))\le\mathrm{dist}(y,\Lambda\cap B_{\frac{R}{16}}(0)).$$

Then we can choose $x^*\in\Lambda\cap B_{\frac{R}{16}}(0)$ s.t.

$$|x-x^*|=\mathrm{dist}(x,\Lambda\cap B_{\frac{R}{16}}(0))=\min\{\mathrm{dist}(x,\Lambda\cap B_{\frac{R}{16}}(0)),\mathrm{dist}(y,\Lambda\cap B_{\frac{R}{16}}(0))\}.$$

Then by using triangular inequality and results from Case (a),

$$\begin{aligned}|\nabla u(x)-\nabla u(y)| &\le |\nabla u(x)-\nabla\psi(x^*)|+|\nabla u(y)-\nabla\psi(x^*)|\\ &\le C\left\{[\nabla\psi]_{C^{0,\alpha}(B_R(0))}+\mathcal{M}(\alpha,0,R,f)\right\}(|x-x^*|^\alpha+|y-x^*|^\alpha)\\ &\le C\left\{[\nabla\psi]_{C^{0,\alpha}(B_R(0))}+\mathcal{M}(\alpha,0,R,f)\right\}(|x-y|^\alpha+|x-x^*|^\alpha)\\ &\le C\left\{[\nabla\psi]_{C^{0,\alpha}(B_R(0))}+\mathcal{M}(\alpha,0,R,f)\right\}|x-y|^\alpha.\end{aligned}$$

Case (c). $x,y\in B_{\frac{R}{16}}(0)\backslash\Lambda$ and

$$|x-y|<\frac{1}{4}\min\{\mathrm{dist}(x,\Lambda\cap B_{\frac{R}{16}}(0)),\mathrm{dist}(y,\Lambda\cap B_{\frac{R}{16}}(0))\}.$$

By using the same methods above, we can still choose x^* and set $r^*=|x-x^*|$. Note that $-\Delta u=f$ in $B_{\frac{r^*}{3}}(x)$ and so is $v(\cdot)=u(\cdot)-\psi(x^*)-\nabla\psi(x^*)(\cdot-x^*)$. We can use the $C^{1,\alpha}$ estimates of v and obtain that

$$\begin{aligned}[\nabla v]_{C^{0,\alpha}(B_{\frac{7r^*}{24}}(x))} &\le \frac{C}{(r^*)^{1+\alpha}}\left(\fint_{B_{\frac{r^*}{3}}(x)}|v|^2\right)^{\frac{1}{2}}+C\mathcal{M}\left(\alpha,x,\frac{r^*}{3},f\right)\\ &\le \frac{C}{(r^*)^{1+\alpha}}\left(\fint_{B_{r^*}(x)}|v|^2\right)^{\frac{1}{2}}+C\mathcal{M}(\alpha,0,R,f).\end{aligned}$$

Then, since $|y-x|<\frac{1}{4}r^*$, one can show that

$$\begin{aligned}|\nabla u(x)-\nabla u(y)| &\le [\nabla u]_{C^{0,\alpha}(B_{\frac{7r^*}{24}}(x))}|x-y|^\alpha=[\nabla v]_{C^{0,\alpha}(B_{\frac{7r^*}{24}}(x))}|x-y|^\alpha\\ &\le C|x-y|^\alpha\left\{\frac{1}{(r^*)^{1+\alpha}}\left(\fint_{B_{r^*}(x)}|v|^2\right)^{\frac{1}{2}}+\mathcal{M}(\alpha,0,R,f)\right\}.\end{aligned}\tag{4.17}$$

Owing to the observation from (4.15) that

$$|v(z)|\le C\left\{[\nabla\psi]_{C^{0,\alpha}(B_R(0))}+R^{1-\alpha}\|f\|_{L^\infty(B_R(0))}\right\}|z-x^*|^{1+\alpha}$$

for any $z \in B_{r^*}(x)$, it can be shown that

$$\left(\fint_{B_{r^*}(x)} |v|^2\right)^{\frac{1}{2}} \leq C\left\{[\nabla\psi]_{C^{0,\alpha}(B_R(0))} + R^{1-\alpha}\|f\|_{L^\infty(B_R(0))}\right\}(r^*)^{1+\alpha}.$$

This, together with (4.17), complete the proof for Case (c). □

4.2. $W^{2,p}$ **estimates.** In this section, we shall discuss $W^{2,p}$ estimates of solutions to the obstacle problems. Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain. Define

$$Au = -\sum_{i,j=1}^n a_{ij}(x)\partial_{ij}^2 u + \sum_{i=1}^n b_i(x)\partial_{x_i}u + c(x)u,$$

where there exist $\lambda, \Lambda > 0$, s.t.

$$\begin{aligned}
&\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^n,\\
&\sum_{i,j}\|a_{ij}\|_{C^{0,\alpha}(\overline{\Omega})} + \sum_i \|b_i\|_{C^{0,\alpha}(\overline{\Omega})} + \|c\|_{C^{0,\alpha}(\overline{\Omega})} \leq \Lambda, \qquad (4.18)\\
&c(x) \geq 0,\ \forall x \in \Omega.
\end{aligned}$$

Consider the solution u to the variational inequality (In fact, in this note we cannot give an explicit definition of such variational problem, later in the discussion, we will see what it means.)

$$\langle Au, v-u\rangle \geq \langle f, v-u\rangle, \quad \forall v \in K_\psi, \qquad (4.19)$$

where

$$K_{\psi,\phi} = \{v : v \geq \psi \text{ a.e. in } \Omega, v|_{\partial\Omega} = \phi\} \text{ is a subset of } W^{2,p}(\Omega).$$

We also assume that for some $\alpha \in (0,1)$,

$$\|f\|_{C^{0,\alpha}(\overline{\Omega})} < \infty, \quad \|\psi\|_{C^{2,\alpha}(\overline{\Omega})} < \infty, \quad \|\phi\|_{C^{2,\alpha}(\overline{\Omega})} < \infty, \quad \psi|_{\partial\Omega} \leq \phi. \qquad (4.20)$$

Before we start our statements, we will first introduce some basic properties about the operator A.

Theorem 4.10 (Boundary $C^{2,\alpha}$-estimates of elliptic operators with non divergence form)**.** *Suppose that Ω is a bounded $C^{2,\alpha}$ domain in $\mathbb{R}^n$, $f \in C^{0,\alpha}(\overline{\Omega})$ and $\phi \in C^{2,\alpha}(\overline{\Omega})$. Assume that the coefficients of the operator A satisfies*

$$\begin{aligned}
&\sum_{i,j}\|a_{ij}\|_{C^{0,\alpha}(\overline{\Omega})} + \sum_i \|b_i\|_{C^{0,\alpha}(\overline{\Omega})} + \|c\|_{C^{0,\alpha}(\overline{\Omega})} \leq \Lambda,\\
&\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^n\ (\lambda > 0).
\end{aligned}$$

If $u \in C^2(\overline{\Omega})$ and

$$Au = f \text{ in } \Omega \quad \text{and} \quad u = \phi \text{ in } \partial\Omega,$$

then

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\left\{\|f\|_{C^{0,\alpha}(\overline{\Omega})} + \|\phi\|_{C^{2,\alpha}(\overline{\Omega})}\right\},$$

where C is a constant depending only on λ, Λ and Ω.

Theorem 4.11 (Interior $C^{2,\alpha}$-estimates of elliptic operators with non divergence form)**.** *Suppose that $B_1(0)$ is ball with center 0 and radius 1, $f \in C^{0,\alpha}(\overline{B_1(0)})$, $\Phi \in C^{2,\alpha}(\overline{B_1(0)})$ and*

$$\begin{aligned}
&\sum_{i,j}\|a_{ij}\|_{C^{0,\alpha}(\overline{\Omega})} + \sum_i \|b_i\|_{C^{0,\alpha}(\overline{\Omega})} + \|c\|_{C^{0,\alpha}(\overline{\Omega})} \leq \Lambda,\\
&\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall x \in B_1(0),\ \xi \in \mathbb{R}^n\ (\lambda > 0).
\end{aligned}$$

If $u \in C^2(\overline{\Omega})$ and

$$Au = f \ \text{in} \ B_1(0),$$

then

$$\|u\|_{C^{2,\alpha}(B_{\frac{1}{2}})} \le C\left\{\|u\|_{L^\infty(B_1(0))} + \|f\|_{C^{0,\alpha}(B_1(0))}\right\},$$

where C is a constant depending only on λ, Λ.

Theorem 4.12 (Boundary $W^{2,p}$-estimates of elliptic operators with non divergence form)**.** *For $1 < p < \infty$, assume that*

$$|a_{ij}(x) - a_{ij}(y)| \le \omega(|x-y|) \quad [\omega(t) \to 0 \ \text{if} \ t \to 0], \tag{4.21}$$

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2, \quad \forall x \in \Omega, \ \xi \in \mathbb{R}^n \ (\lambda > 0) \tag{4.22}$$

$$\sum_{i,j} \|a_{ij}\|_{L^\infty(\Omega)} + \sum_i \|b_i\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} \le \Lambda, \tag{4.23}$$

$\partial\Omega$ is C^2 locally, $f \in L^p(\Omega)$ and $\phi \in W^{2,p}(\Omega)$. Then for any $u \in W^{2,p}(\Omega)$ such that

$$Au = f \ \text{in} \ \Omega \quad \text{and} \quad u - \phi \in H_0^1(\Omega),$$

one has

$$\|u\|_{W^{2,p}(\Omega)} \le C\left\{\|f\|_{L^p(\Omega)} + \|\phi\|_{W^{2,p}(\Omega)}\right\}, \tag{4.24}$$

where C is a constant depending only on λ, Λ, ω and Ω.

Theorem 4.13 (Maximum principle)**.** *Assume that (4.22), (4.23) hold and that $c(x) \ge 0$. Let u be a function in $H^2(\Omega) \cap C(\Omega)$ satisfying $Au \le 0$ a.e. in Ω. If u assumes a positive maximum at some point x_0 in Ω, then $u \equiv const$ in Ω (and then $c = 0$ a.e. in Ω).*

Remark 4.14. This result extends also to u, which is not necessarily continuous in Ω; maximum of u is replaced by essential supremum of u. This implies that

$$\text{If } u \in H^2(\Omega) \cap H_0^1(\Omega), \quad Au \le 0 \text{ a.e. in } \Omega, \text{ then } u \le 0 \text{ a.e. in } \Omega. \tag{4.25}$$

The technique to obtain $W^{2,p}$-estimates of u is called the penalty method. Consider the penalized problem

$$Au + \beta_\varepsilon(u - \psi) = f \text{ in } \Omega \quad \text{and} \quad u = \phi \text{ on } \partial\Omega. \tag{4.26}$$

Here $\{\beta_\varepsilon(t)\}_{\varepsilon\in(0,1)}$ are non-decreasing smooth function in t, s.t.

$$\begin{aligned} &\lim_{\varepsilon\to0^+} \beta_\varepsilon(t) \to -\infty \text{ if } t < 0, \ \lim_{\varepsilon\to0^+} \beta_\varepsilon(t) \to 0 \text{ if } t > 0, \\ &\beta_\varepsilon(0) = 0, \ \beta_\varepsilon(t) \le B, \ \forall t \in \mathbb{R}, \ \forall \varepsilon \in (0,1) \end{aligned} \tag{4.27}$$

Lemma 4.15. *There is a solution of (4.26) in $C^{2,\alpha}(\overline{\Omega})$ if $c \ge c_0 > 0$ or $f \equiv 0$, denoted by u_ε.*

Proof. By using the definition of $\beta_\varepsilon(t)$, we can choose a constant $\overline{u} \gg 1$, such that

$$A(\overline{u}) + \beta_\varepsilon(\overline{u} - \psi) - f = c\overline{u} + \beta_\varepsilon(\overline{u} - \psi) - f \ge 0 \text{ in } \Omega \quad \text{and} \quad \overline{u} \ge \phi \text{ on } \partial\Omega.$$

Similarly, we can choose $\underline{u} \ll -1$, such that

$$A(\underline{u}) + \beta_\varepsilon(\underline{u} - \psi) - f = c\underline{u} + \beta_\varepsilon(\underline{u} - \psi) - f \le 0 \text{ in } \Omega \quad \text{and} \quad \underline{u} \le \phi \text{ on } \partial\Omega.$$

Now we write $u_0 = \underline{u}$. Then, for any u_k, $k = 1, 2, ...$, we suppose $u_{k+1} \in C^{2,\alpha}(\overline{\Omega})$ solve

$$A(u_{k+1}) + \mu u_{k+1} + \beta_\varepsilon(u_k - \psi) = f + \mu u_k \text{ in } \Omega \quad \text{and} \quad u_{k+1} = \phi \text{ on } \partial\Omega, \tag{4.28}$$

where $\mu > 0$ is a constant which will be chosen later. We first prove that $\underline{u} \leq u_k \leq \overline{u}$ in Ω. This is obviously true for $k = 0$. Suppose that it holds for some $k \geq 0$. We now consider u_{k+1}. First, we note that

$$\begin{aligned} A(u_{k+1} - \underline{u}) + \mu(u_{k+1} - \underline{u}) &\geq \beta_\varepsilon(\underline{u} - \psi) - \beta_\varepsilon(u_k - \psi) + \mu(u_k - \underline{u}) \\ &= (\mu - \beta_\varepsilon'(\theta))(u_k - \underline{u}), \end{aligned}$$

where θ is between $u_k(x)$ and $\underline{u}$. Then $\theta \in [\underline{u}, \overline{u}]$. Since $\beta_\varepsilon'(t)$ is smooth, we can choose $\mu \gg 1$ such that

$$A(u_{k+1} - \underline{u}) + \mu(u_{k+1} - \underline{u}) \geq 0 \text{ in } \Omega \quad \text{and} \quad u_{k+1} - \underline{u} \geq 0 \text{ on } \partial\Omega.$$

By using the maximum principle, we have $u_{k+1} \geq \underline{u}$ in Ω. Similarly, we have $u_{k+1} \leq \overline{u}$ in Ω. Also, by using the same arguments, we can prove that

$$\underline{u} \leq u_1 \leq u_2 \leq \ldots \leq \overline{u}.$$

Therefore, there exists a function u in Ω such that $u_k(x) \to u(x)$ as $k \to \infty$ for each $x \in \Omega$. Next, the right hand side expression in (4.28) is uniformly bounded independentof k. By the global $C^{1,\alpha}$-estimate, we have

$$\|u_k\|_{C^{1,\alpha}(\overline{\Omega})} \leq C,$$

where C depends only on $n, \mu, \underline{u}, \overline{u}, \Lambda, \lambda$ and Ω. In particular the right hand side expression in (4.28) is uniformly bounded in $C^{1,\alpha}$ norms independent of k. By the global $C^{2,\alpha}$-estimate, we have

$$\|u_k\|_{C^{2,\alpha}(\overline{\Omega})} \leq C,$$

where C depends only on $n, \mu, \underline{u}, \overline{u}, K, \lambda$ and Ω. Therefore, $u \in C^2(\Omega)$ and

$$u_k \to u \text{ in the } C^2\text{-norm in } \Omega.$$

Hence u is the desired solution. □

Remark 4.16. Here, this lemma is very different from what have been given in Lin's note. Actually, in Lin's note, we cannot not use the methods given to complete the proof of existence for the equation. Indeed, we even cannot obtain the supersolution and subsolution of the equation. In order to go on, we assume that $c(x)$ has a positive lower boundedness of $f \equiv 0$.

We shall use however a different approach with desired estimates. For all $N > 0$, we define

$$\beta_{\varepsilon,N}(t) = \max\{\min\{\beta_\varepsilon(t), N\}, -N\}.$$

Consider

$$Au + \beta_{\varepsilon,N}(u - \psi) = f \text{ in } \Omega \quad \text{and} \quad u = \phi \text{ on } \partial\Omega. \tag{4.29}$$

It is known that for each $v \in L^p$, $1 < p < \infty$, there is a unique $w = Tv \in W^{2,p}$ solving

$$Aw = -\beta_{\varepsilon,N}(v - \psi) + f \text{ in } \Omega \quad \text{and} \quad w = \phi \text{ on } \partial\Omega,$$

with

$$\|w\|_{W^{2,p}(\Omega)} \leq C(\varepsilon, N, f, \phi, B) \equiv R_0.$$

In fact, by using (4.24), we can deduce that

$$\|w\|_{W^{2,p}(\Omega)} \leq C\left\{\|\beta_{\varepsilon,N}(v - \psi)\|_{L^p(\Omega)} + \|\phi\|_{W^{2,p}(\Omega)}\right\}.$$

Since $|\beta_{\varepsilon,N}(t)| \leq N$ for any $t \in \mathbb{R}$, we can obtain that $\|\beta_{\varepsilon,N}(v - \psi)\|_{L^p(\Omega)}$ is independent of v and ψ. Hence, if we define

$$T : B_R(0) \subset L^p(\Omega) \to B_R(0),$$

for $R > R_0$, then T is a compact map ($W^{2,p}(\Omega)$ can be compactly embedded into $L^p(\Omega)$) from a bounded closed convex set $B_R(0) \subset L^p(\Omega)$ into itself. By Schauder's fixed-point theorem (A continuous mapping

T from a closed convex set S of Banach space into a compact subset of S has a fixed point), there exists $u_{\varepsilon,N}$, s.t. $Tu_{\varepsilon,N} = u_{\varepsilon,N} \in W^{2,p}(\Omega)$, $1 < p < \infty$, which solves (4.29). In fact, since β_ε is smooth, we can see that $u \in C^{2,\alpha}(\overline{\Omega})$. Now we want to find a lower bound for $\beta_{\varepsilon,N}(u_{\varepsilon,N} - \psi)$ that is independent of N and ε. Let

$$\mu_{\varepsilon,N} = \min_{\Omega} \beta_{\varepsilon,N}(u_{\varepsilon,N} - \psi).$$

We only consider the case when $\mu_{\varepsilon,N} < 0 = \beta_{\varepsilon,N}(0)$; other wise $\mu_{\varepsilon,N}$ is alreasy bounded from below. We may assume that $\mu_{\varepsilon,N}$ is obtained at $x_0 \in \Omega$, i.e. $x_0 \notin \partial\Omega$. Indeed, for $x \in \partial\Omega$, $\beta_{\varepsilon,N}(u_{\varepsilon,N}(x) - \psi(x)) = \beta_{\varepsilon,N}(\phi(x) - \psi(x)) \geq 0$. Since $\beta_{\varepsilon,N}$ is non-decreasing, the minimum of $u_\varepsilon - \psi$ is also achieved at x_0. Then the Hessain matrix (a_{ij}) is semi-positive definite. Then

$$-\sum_{i,j} a_{ij}(x_0)\partial_{ij}^2(u-\psi)(x_0) + \sum_i b_i(x_0)\partial_i(u-\psi)(x_0) + c(x_0)(u-\psi)(x_0) \leq 0.$$

That is $A(u-\psi)(x_0) \leq 0$. Hence,

$$\begin{aligned}\mu_{\varepsilon,N} &= \beta_{\varepsilon,N}(u_{\varepsilon,N}(x_0) - \psi(x_0)) = -Au_{\varepsilon,N}(x_0) + f(x_0)\\ &\geq -A\psi(x_0) + f(x_0) \geq -C(\lambda, \Lambda, \psi, f).\end{aligned}$$

In particular, the constant C does not depend on ε and N. By this, together with the assumptions on β_ε, we conclude that

$$\|Au_{\varepsilon,N}\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)} + \|\beta_{\varepsilon,N}(u_{\varepsilon,N} - \psi)\|_{L^\infty(\Omega)} \leq C(\lambda, \Lambda, \psi, f, B).$$

This implies

$$\|u_{\varepsilon,N}\|_{W^{2,p}(\Omega)} \leq C(n, p, \lambda, \Lambda, \psi, \phi, f, B), \quad \forall p \in (0,1).$$

Note that C is independent of ε and N. Now taking $N \to \infty$, there exists $u_\varepsilon \in W^{2,p}(\Omega)$, s.t. $u_{\varepsilon,N} \to u_\varepsilon$ weakly in $W^{2,p}(\Omega)$ as $N \to \infty$ and

$$\|u_\varepsilon\|_{W^{2,p}(\Omega)} \leq C(n, p, \lambda, \Lambda, \psi, \phi, f, B), \tag{4.30}$$

where C does not depend on ε. Then taking $\varepsilon \to 0^+$, there exists $u \in W^{2,p}(\Omega)$, s.t. $u_\varepsilon \to u$ weakly in $W^{2,p}(\Omega)$ as $\varepsilon \to 0^+$ and $\|u\|_{W^{2,p}(\Omega)} \leq C(n, p, \lambda, \Lambda, \psi, \phi, f, B)$. Note that this holds for any $p \in (1, \infty)$. Hence this implies that $u_\varepsilon \to u$ in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ (here we have used the Ascoli lemma since $W^{2,p}$ can be embedded into $C^{1,\alpha}$ for any $\alpha \in (0,1)$). Now we shall prove that $u \in W^{2,p}(\Omega)$ is a desired solution. It is easy to show from the above calculations that $|\beta_\varepsilon(u_\varepsilon - \psi)| \leq C(\lambda, \Lambda, \psi, f, B)$. This mmediately implies that $u \geq \psi$ by (4.27). Also, $\beta_\varepsilon(u_\varepsilon - \psi) \to 0$ on the set of $\{x : u(x) - \psi(x) > 0\}$ since u_ε is unformly bounded. Therefore,

$$\limsup_{t \to 0^+} \beta_\varepsilon(u_\varepsilon - \psi) \leq 0.$$

It implies that

$$Au = f \text{ in } \{u > \psi\} \quad \text{and} \quad Au \geq f \ a.e. \text{ in } \Omega.$$

Thus we have proved

Theorem 4.17. *Assume that* (4.18) *and* (4.20) *hold. Then there exists a solution u of the variational inequality* (4.19)*, i.e.*

$$\left.\begin{aligned} Au - f &\geq 0\\ u &\geq \psi\\ (Au-f)(u-\psi) &= 0 \end{aligned}\right\} \quad a.e.\ in\ \Omega. \tag{4.31}$$

$$u = \phi \ on\ \partial\Omega,$$

Moreover, $u \in W^{2,p}(\Omega)$*, for any* $p \in (1,\infty)$ *and*

$$\|u\|_{W^{2,p}(\Omega)} \le C(n,p,\lambda,\Lambda,\psi,\phi,f).$$

Note that we omit the dependence on B *since it can be chosen arbitrarily in* (4.27)*.*

Theorem 4.18. *Let* u_1 *and* u_2 *be solutions in* $C(\overline{\Omega}) \cap H^2(\Omega)$ *of* (4.19) *corresponding to* f_1 *and* f_2 *respectively. If* $f_1 \ge f_2$*, then* $u_1 \ge u_2$*.*

Proof. Suppose that $O = \{x \in \Omega : u_2(x) > u_1(x)\}$ is nonempty; O is open. We know that $u_2 > u_1 \ge \psi$ on O and thus $Au_1 \ge f_1$ and $Au_2 = f_2$. Hence, $A(u_1 - u_2) \ge f_1 - f_2 \ge 0$. This together with $u_1 = u_2$ on $\partial\Omega$ implies that $u_1 \ge u_2$ in O. Contradiction! □

Corollary 4.19. *Under the assumptions* (4.18) *and* (4.20)*, the solution of* (4.19) *in* $C(\overline{\Omega}) \cap H^2(\Omega)$ *is unique.*

Let $\rho(x)$ be a function in $C^\infty(\mathbb{R}^n)$ with supported in the unit ball, such that

$$\rho(x) \ge 0,\ \forall x \in \mathbb{R}^n \quad \text{and} \quad \int_{\mathbb{R}^n} \rho(x)dx = 1.$$

Set $\rho_\delta(x) = \delta^{-n}\rho(x/\delta)$ for any $\delta > 0$ and cnsider the mollifier

$$(J_\delta u)(x) = \int_\Omega \rho_\delta(x-y)u(y)dy, \quad u \in L^p(\Omega),\ 1 < p < \infty.$$

By using Minkowski inequality, we can obtain that $J_\delta u$ is in $C^\infty(\mathbb{R}^n)$ and

$$\|J_\delta u - u\|_{L^p(K)} \to 0 \text{ when } \delta \to 0$$

for any compact subset K of Ω. A continuous function $v(x)$ in an open set $\Omega_0 \subset \mathbb{R}^n$ is said to satisfy

$$\frac{\partial^2}{\partial\tau^2} v \ge 0 \text{ in the sense of distributions } D'(\Omega_0),$$

if for any $\zeta \in C_0^\infty(\Omega_0)$, $\zeta \ge 0$, there holds

$$\int_{\Omega_0} v(x)\frac{\partial}{\partial\tau^2}\zeta(x)dx \ge 0,$$

where $\partial/\partial\tau$ is any directional derivative. Taking $\zeta = \rho_\delta$, we conclude that

$$\frac{\partial^2}{\partial\tau^2}(J_\delta v)(x) = \int_{\Omega_0} \frac{\partial^2}{\partial\tau^2}\{\rho_\delta(x-y)\}v(y)dy \ge 0$$

in the usual sense in Ω, provided that $\overline{\Omega} \subset \Omega_0$ and $\delta < \operatorname{dist}(\Omega, \partial\Omega_0)$. We now replace the condition (4.20) by

$$\|f\|_{C^{0,\alpha}(\overline{\Omega})} < \infty, \quad \|\phi\|_{C^{2,\alpha}(\overline{\Omega})} < \infty, \quad \psi|_{\partial\Omega} \le \phi. \tag{4.32}$$

and

$$\begin{cases} \psi \in C^{0,1}(\Omega_0) \\ \dfrac{\partial^2}{\partial\tau^2}\psi \ge -C \text{ in } D'(\Omega_0), \text{ for any direction } \tau. \end{cases} \tag{4.33}$$

The last condition means that

$$\frac{\partial^2}{\partial\tau^2}\left(\psi + \frac{1}{2}C|x|^2\right) \ge 0 \text{ in } D'(\Omega_0) \text{ for any direction } \tau.$$

Setting

$$\psi_\delta = J_\delta\left(\psi + \frac{1}{2}C|x|^2\right) - \frac{1}{2}C|x|^2, \tag{4.34}$$

we easily find that for any $x \in \Omega$,

$$|\nabla \psi_\delta(x)| \le \int_{\Omega_0} \left| \nabla_x \left(\psi(x-y) + \frac{1}{2}|x-y|^2 \right) \right| \rho_\delta(y) dy + C \le C(\psi, \Omega_0),$$

where we have used the fact that $\psi \in C^{0,1}(\Omega_0)$. Also, we have

$$\frac{\partial^2}{\partial \tau^2}\psi_\delta = \int_{\Omega_0} \frac{\partial^2}{\partial \tau^2}\{\rho_\delta(x-y)\} \left(\psi(y) + \frac{1}{2}|y|^2 \right) dy - C \ge -C(\psi).$$

Moreover,

$$\psi_\delta(x) \to \psi(x) \text{ uniformly in } \Omega \text{ as } \delta \to 0.$$

Theorem 4.20. *Let* (4.18)*,* (4.32) *and* (4.33) *hold. Then the assertion of Theorem 4.17 is valid.*

Proof. We repeat the proof of Theorem 4.17, replacing ψ by ψ_δ in the penalized problem and choosing, for instance, $\delta = \varepsilon$. We see that what we need to do is to obtain the lower bound of $\mu_{\varepsilon,N}$, which is independent of ε. That is, if we show that

$$A\psi_\delta \le C, \quad C \text{ is independent of } \delta,$$

then all the estimates remain valid. Now we will show this claim. For any $x_0 \in \Omega$, we can perform an orthogonal transformation such that at $x = x_0$,

$$\sum_{i,j=1}^n a_{ij}(x_0) \frac{\partial^2}{\partial x_i \partial x_j} \text{ becomes } \Delta.$$

Recalling the fact that $\frac{\partial^2}{\partial \tau^2}\psi_\delta \ge -C(\psi)$ in Ω for any direction τ, we can complete the proof. Also, we note that one order derivatives can be estimated by using the property that $|\nabla \psi_\delta| \le C(\Omega_0, \psi)$. □

4.3. **Interior $C^{1,1}$-estimates.**

Example 4.21. Let $\Omega = (-1, 1)$ and $\psi(x) = \frac{3}{4} - x^2$. We consider the following obstacle problem

$$\langle -\Delta u, v - u \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \ge 0, \quad u \in K_\psi = \left\{ w \in H_0^1(\Omega) : w \ge \psi \right\}, \forall v \in K_\psi.$$

It is easy to see that

$$u(x) = \begin{cases} x+1, & x \in (-\frac{1}{2}, \frac{1}{2}), \\ \psi(x), & x \in [-\frac{1}{2}, \frac{1}{2}], \\ 1-x, & x \in (\frac{1}{2}, 1) \end{cases}$$

is the unique solution. However, it is not C^2 at $-\frac{1}{2}$ and $\frac{1}{2}$.

For simplicity, we shall consider in the sequel the following variational inequality:

$$\langle -\Delta u - f, v - u \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \ge 0, \quad \forall u \in K_\psi = \{ w : w \ge \psi, w|_{\partial\Omega} = \phi \}, \forall v \in K_\psi,$$

where f, ψ and ϕ satisfy (4.20) and we also assume $f \in W^{2,p}(\Omega)$ for some $p \in (n, \infty)$. Note the regularity assumption on f can be further reduced. The penalized problem is

$$-\Delta u_\varepsilon + \beta_\varepsilon(u_\varepsilon - \psi) = f \text{ in } \Omega \quad \text{and} \quad u_\varepsilon = \phi \text{ on } \partial\Omega, \tag{4.35}$$

where $\{\beta_\varepsilon(t)\}_{\varepsilon \in (0,1)}$ are non-decreasing smooth functions of t satisfying (4.27) and

$$\beta_\varepsilon'(t) > 0, \ \beta_\varepsilon''(t) \le 0 \text{ for all } t.$$

It is known that the solution u_ε for (4.35) is in $W^{2,q}(\Omega)$, for any $q \in (1, \infty)$. By $-\Delta u_\varepsilon = f - \beta_\varepsilon(u_\varepsilon - \psi) \in W^{2,p}(\Omega)$, we know that $u_\varepsilon \in W^{4,p}(K')$, for any $K' \subset\subset \Omega$ (by using the interior $W^{2,p}$ estimates of the operator $-\Delta$). Note that p comes from the regularity assumption on f. Then the right hand side becomes of $W^{2,p}(K')$, which implies that $u_\varepsilon \in W^{4,p}(K)$, for any $K \subset\subset K'$. For given $K \subset\subset \Omega$, we define a cut-off function $\eta \in C_0^\infty(\Omega)$, s.t.

$$0 \le \eta \le 1 \text{ in } \Omega \quad \text{and} \quad \eta = 1 \text{ in } K. \tag{4.36}$$

Choose an arbitrary vector $\tau \in \mathbb{S}^{n-1}$ as the direction for taking derivative. Our task now is to obtain L^∞-bound of $u_{\varepsilon,\tau\tau}$ in K which is uniform in ε and τ. (Here we use a basic fact that for any $1 \le i, j \le n$, u_{ij} can be represented as a linear combinations of $u_{\xi\xi}$ with $\xi \in \mathbb{S}^{n-1}$.) Then we take $\tau \to 0$ so that $u_\varepsilon \to u$. By virtue of the weak*-compactness of $C^{1,1}(K)$, we know that $u \in C^{1,1}(K)$ and its norm share the same uniform bound. Take pure second partial derivative of (4.35) in the direction of τ and we find

$$-\Delta u_{\varepsilon,\tau\tau} + \beta_\varepsilon'(u_\varepsilon - \psi)(u_{\varepsilon,\tau\tau} - \psi_{\tau\tau}) + \beta_\varepsilon''(u_\varepsilon - \psi)(u_{\varepsilon,\tau} - \psi_\tau)^2 = f_{\tau\tau}.$$

Since $\beta_\varepsilon'' \le 0$, it follows that

$$\Delta u_{\varepsilon,\tau\tau} \le \beta_\varepsilon'(u_\varepsilon - \psi)(u_{\varepsilon,\tau\tau} - \psi_{\tau\tau}) - f_{\tau\tau}.$$

We turn to look at

$$\begin{aligned}
\Delta(\eta u_{\varepsilon,\tau\tau}) &= \eta\Delta u_{\varepsilon,\tau\tau} + \Delta\eta u_{\varepsilon,\tau\tau} + 2\nabla\eta\nabla u_{\varepsilon,\tau\tau} \\
&\le \eta(-f_{\tau\tau} + \beta_\varepsilon'(u_\varepsilon - \psi)(u_{\varepsilon,\tau\tau} - \psi_{\tau\tau})) + u_{\varepsilon,\tau\tau}\Delta\eta + 2\nabla\eta\nabla u_{\varepsilon,\tau\tau} \\
&\le \eta(-f_{\tau\tau} + \beta_\varepsilon'(u_\varepsilon - \psi)(u_{\varepsilon,\tau\tau} - \psi_{\tau\tau})) - u_{\varepsilon,\tau\tau}\Delta\eta + 2\,\mathrm{div}(u_{\varepsilon,\tau\tau}\nabla\eta).
\end{aligned}$$

Let w_ε solve

$$\Delta w_\varepsilon = -\eta f_{\tau\tau} - u_{\varepsilon,\tau\tau}\Delta\eta + 2\,\mathrm{div}(u_{\varepsilon,\tau\tau}\nabla\eta) \text{ in } \Omega \quad \text{and} \quad w_\varepsilon = 0 \text{ in } \partial\Omega.$$

Since $\eta f_{\tau\tau} \in L^p(\Omega)$, $u_{\varepsilon,\tau\tau}\nabla\eta \in L^p(\Omega)$ and $u_{\varepsilon,\tau\tau}\nabla\eta \in L^p(\Omega)$ and $u_{\varepsilon,\tau\tau}\Delta\eta \in L^p(\Omega)$, we know that

$$\|w_\varepsilon\|_{C^{0,\alpha}(\Omega)} \le C(\|f\|_{W^{2,p}(\Omega)} + \|u_\varepsilon\|_{W^{2,p}(\Omega)}) \le C(n, p, \psi, \phi, f, B, \eta) \tag{4.37}$$

for some $\alpha \in (0, 1)$. In particular, the positive constant C does not depend on ε due to (4.30). Also we find that

$$\Delta(\eta u_{\varepsilon,\tau\tau} - w_\varepsilon) \le \eta\beta_\varepsilon'(u_\varepsilon - \psi)(u_{\varepsilon,\tau\tau} - \psi_{\tau\tau}). \tag{4.38}$$

Consider the minimum of $\eta u_{\varepsilon,\tau\tau} - w_\varepsilon$ in $\overline{\Omega}$. We assume that it is obtained at $x_0 \in \overline{\Omega}$. If $\eta(x_0) = 0$, $\eta u_{\varepsilon,\tau\tau}(x) - w_\varepsilon(x) \ge \eta(x_0)u_{\varepsilon,\tau\tau}(x_0) - w_\varepsilon(x_0) \ge -w_\varepsilon(x_0)$ in Ω. Hence, combined with (4.37)

$$u_\varepsilon(x) = \eta(x)u_\varepsilon(x) \ge w_\varepsilon(x) - w_\varepsilon(x_0) \ge -C,$$

in K. If $\eta(x_0) > 0$, $x_0 \in \Omega$. By virtue of (4.38),

$$\Delta(\eta u_{\varepsilon,\tau\tau} - w_\varepsilon)(x_0) \ge 0 \Rightarrow \eta\beta_\varepsilon'(u_\varepsilon - \psi)(x_0)(u_{\varepsilon,\tau\tau} - \psi_{\tau\tau})(x_0) \ge 0 \Rightarrow (\eta u_{\varepsilon,\tau\tau} - \psi_{\tau\tau})(x_0) \ge 0.$$

Here we use the fact that at the minimum point of $(\eta u_{\varepsilon,\tau\tau} - w_\varepsilon)$ the value of $\Delta(\eta u_{\varepsilon,\tau\tau} - w_\varepsilon)$ is greater than zero. Hence

$$\eta u_{\varepsilon,\tau\tau} - w_\varepsilon \ge \eta(x_0) - w_\varepsilon(x_0) \ge \eta(x_0)\psi_{\tau\tau}(x_0) - w_\varepsilon(x_0)$$

in Ω, which implies that $u_{\varepsilon,\tau\tau} \ge -C$ in K. Therefore, in either case, we find for $\tau \in \mathbb{S}^{n-1}$ and $\varepsilon \in (0, 1)$, $u_{\varepsilon,\tau\tau} \ge -C(n, p, \psi, \phi, f, B, \eta)$ in K. On the other hand,

$$\Delta u_\varepsilon = \beta_\varepsilon(u_\varepsilon - \psi) - f \le B + \|f\|_{L^\infty(\Omega)}. \tag{4.39}$$

For $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$ and $y = (y_1, y_2, ..., y_n)^T = Qx$ with Q being an orthogonal matrix, we have

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\right)^T = Q\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, ..., \frac{\partial}{\partial y_n}\right)^T.$$

Then, we have

$$\Delta_x = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\right)\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\right)^T$$
$$= \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, ..., \frac{\partial}{\partial y_n}\right) Q^T Q \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, ..., \frac{\partial}{\partial y_n}\right)^T = \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2} = \Delta_y.$$

To estimate the upper bound of $u_{\varepsilon,\tau\tau}$, in view of the calculations above, we can assume that $\tau = \partial_i$ with $1 \le i \le n$. Then it follows from (4.39) that

$$u_{\varepsilon,ii} \le -\sum_{j=1, j\neq i}^{n} u_{\varepsilon,jj} + B + \|f\|_{L^\infty(\Omega)} \le C(n,p,\psi,\phi,f,B,\eta).$$

Hence we have proved

$$\|u_{\varepsilon,\tau\tau}\|_{L^\infty(K)} \le C(n,p,\psi,\phi,f,B,\eta),$$

for any $\varepsilon \in (0,1)$ and $\tau \in \mathbb{S}^{n-1}$. Next, we can take the limit $\varepsilon \to 0$ and complete the proof. To make it more precise, we see that $\|u_\varepsilon\|_{C^{1,1}(K)} \le C$. This shows that u_ε is uniformly bounded and equicontinuous. Then we can choose a sequence of ε, also denoted by ε and $u \in C^{1,1}(K)$, such that $u_\varepsilon \to u$ in $C^{1,1}(K)$. Moreover u here is the same as what we have obtained in the proof of $W^{2,p}$ estimates.

Remark 4.22. The method above can be extended to general second-order linear elliptic operators of non-divergence type with suitably smooth coefficients.

This remark is not proved in the note given by Lin, we will give an explicit statement for it and prove such result.

Theorem 4.23. *Let the assumptions* (4.18), (4.32) *and* (4.33) *hold. Suppose that* a_{ij} *belongs to* $C^{2,\alpha}(\overline{\Omega})$. *Then the solution of the variational inequality* (4.31) *satisfies* $u \in C^{1,1}(\Omega)$.

Proof. Without loss of generality we may assume that $b = c = 0$; otherwise, we replace f by $f - \sum_i b_i(\partial u/\partial_{x_i}) - cu$. Precisely speaking, in Theorem 4.17, we have shown that $u \in W^{2,p}(\Omega)$ for any $1 < p < \infty$, then, it can be derived that $u \in C^{1,\alpha}$ for any $0 < \alpha < 1$. Then $F = f - \sum_i b_i(\partial u/\partial x_i) - cu \in C^{0,\alpha}(\overline{\Omega})$. Moreover, $\|F\|_{C^{0,\alpha}(\overline{\Omega})} \le C(n,\lambda,\Lambda,\psi,\phi,f)$. Recalling that $u \in W^{2,p}(\Omega)$, we introduce ψ_δ as in (4.34) and consider the penalized problem

$$-\Delta u_{\varepsilon,\delta} + \beta_\varepsilon(u_{\varepsilon,\delta} - \psi_\delta) = f \text{ in } \Omega \quad \text{and} \quad u_{\varepsilon,\delta} = \phi \text{ on } \partial\Omega, \tag{4.40}$$

We have already known that the solution $u_{\varepsilon,\delta}$ satisfies

$$\|u\|_{W^{2,p}(\Omega)} \le C \text{ for any } 1 < p < \infty. \tag{4.41}$$

where C is a constant independent of ε, δ. In the sequel it will be convenient to choose the functions $\beta_\varepsilon(t)$ as before, that is

$$\beta'_\varepsilon(t) > 0, \ \beta''_\varepsilon(t) \le 0 \text{ for all } t.$$

Suppose first that $f \equiv 0$. Note, by elliptic regularity, that $u \in C^{4+\alpha}(\Omega)$. We differential equation (4.40) twice in a direction τ and get

$$Au_{\tau\tau} + \beta'_\varepsilon(u - \psi_\delta)(u_{\tau\tau} - \psi_{\delta,\tau\tau}) + \beta''_\varepsilon(u - \psi_\delta)(u_\tau - \psi_{\delta,\tau})^2 = I_0$$

where $u = u_{\varepsilon,\delta}$ for simplicity and

$$I_0 = 2\sum_{i,j} \frac{\partial a_{ij}}{\partial \tau} \frac{\partial^3 u}{\partial x_i \partial x_j \partial \tau} + \sum_{i} \frac{\partial^2 a_{ij}}{\partial \tau^2} \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Since $\beta_\varepsilon'' \le 0$, we obtain

$$Au_{\tau\tau} + \beta_\varepsilon'(u-\psi_\delta)(u_{\tau\tau} - \psi_{\delta,\tau\tau}) \ge I_0. \tag{4.42}$$

Let Ω_1 be any compact subdomain of Ω such that $\Omega_1 \subset\subset \Omega$ and let $\eta \in C_0^\infty(\Omega)$, $\eta \equiv 1$ in Ω_1, $0 \le \eta \le 1$. Then

$$A(\eta u_{\tau\tau}) - \eta A u_{\tau\tau} = -2\sum_{i,j} a_{ij}\frac{\partial \eta}{\partial x_i}\frac{\partial^3 u}{\partial x_j \partial \tau^2} - \sum_{i,j} a_{ij}\frac{\partial^2 \eta}{\partial x_i \partial x_j} u_{\tau\tau} = I_1.$$

Substituting $Au_{\tau\tau}$ from (4.42), we get

$$A(\eta u_{\tau\tau}) + \eta\beta_\varepsilon'(u-\psi_\delta)(u_{\tau\tau} - \psi_{\delta,\tau\tau}) \ge I_1 + \eta I_0. \tag{4.43}$$

Each term in $I_1 + \eta I_0$ of the form $a\nabla^3 u$ can be written in the form $\nabla(a\nabla^2 u) - \nabla a \nabla^2 u$. Recalling (4.41) and the assumption that the coefficients a_{ij} are in $C^{2,\alpha}(\overline{\Omega})$, we see that

$$I_1 + \eta I_0 = g_0 + \sum_{j=1}^n \partial_j g_j, \tag{4.44}$$

where

$$\sum_{j=1}^n \|g_j\|_{L^p(\Omega)} \le C, \quad C \text{ independent of } \varepsilon, \delta. \tag{4.45}$$

We now need a lemma of Stampacchia as follows.

Lemma 4.24. *Assume that* (4.22), (4.23) *and* $\|\nabla a\|_{L^\infty(\Omega)} \le C_0$ *hold in a bounded domain* Ω *and that* $a(u,v)$ *defined by*

$$a(u,v) = \int_\Omega \sum_{i,j}\left\{ a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_i \left(b_i + \sum_j \frac{\partial a_{ij}}{\partial x_j}\right)\frac{\partial u}{\partial x_i} v + cuv \right\} dx$$

is coercive in $H^1(\Omega)$; *that is*

$$a(u,u) \ge c_0 \|u\|^2_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega)\ (c_0 > 0)$$

Assume also that for any $x_0 \in \partial\Omega$,

$$\liminf_{R\to 0} |B_R(x_0)\backslash\Omega| > 0,\ B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\}.$$

Let $g_1, g_2, ..., g_n$ *be functions in* $L^p(\Omega)$ *and* g_0 *a function in* $L^{p/2}(\Omega)$ *for some* $p > n$. *Let* ϕ *be a continuous function on* $\partial\Omega$. *Then there exists a unique function* w *in* $C(\overline{\Omega}) \cap H^1(\Omega)$ *such that*

$$Aw = g_0 + \sum_{j=1}^n \partial_j g_j \text{ in } \Omega \quad \text{and} \quad w = \phi \text{ on } \partial\Omega.$$

Furthermore,

$$\|w\|_{L^\infty(\Omega)} \le C\left(\sum_{j=1}^n \|g_j\|_{L^p(\Omega)} + \|g_0\|_{L^{p/2}(\Omega)} + \max_{\partial\Omega}|\phi| \right),$$

where C *is a constant depending only on* $C_0, c_0, \lambda, \Lambda, \Omega$.

Without loss of generality we may assume in our case that A is coercive (that is, $a(u,v)$ is coercive). Indeed, otherwise we simply replace A by a coercive operator $A + kI$ $(k > 0)$ and add $k\eta u_{\tau\tau}$ to the righthand side of (4.43). Thus, by using this lemma, there exists a solution w in $C(\overline{\Omega}) \cap H_0^1(\Omega)$ of the equation

$$Aw = I_1 + \eta I_0, \quad w \in H_0^1(\Omega),$$

and

$$\|w\|_{L^\infty(\Omega)} \le C, \quad C \text{ independent of } \varepsilon, \delta. \tag{4.46}$$

Since $I_1 + \eta I_0$ is in $C^{0,\alpha}(\overline{\Omega})$, w is actually in $C^{2,\alpha}(\overline{\Omega})$. Then the function $V = \eta u_{\tau\tau} - w$ satisfies

$$AV + \eta\beta'_\varepsilon(u - \psi_\delta)(u_{\tau\tau} - \psi_{\delta,\tau\tau}) \geq 0 \text{ in } \Omega. \tag{4.47}$$

We shall now estimate V from below. Suppose that V takes a negative minimum at some point $x_0 \in \overline{\Omega}$ (Here we have used the fact that V is actually continous in $\overline{\Omega}$. Moreover, the impact of η is important to get this point.) Since $V = 0$ on $\partial\Omega$, x_0 must belong to Ω. But then

$$AV(x_0) = \sum_{i,j} a_{ij}(x_0)\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) \leq 0,$$

so that by (4.47) and the fact that $\beta'_\varepsilon(t) > 0$ in $\mathbb{R}$, we can obtain

$$\eta(x_0)(u_{\tau\tau}(x_0) - \psi_{\delta,\tau\tau}) \geq 0 \Rightarrow u_{\tau\tau}(x_0) \geq \psi_{\delta,\tau\tau}(x_0) \text{ or } \eta(x_0) = 0.$$

Hence

$$V(x_0) = \eta(x_0)u_{\tau\tau}(x_0) - w(x_0) \geq \eta(x_0)\psi_{\delta,\tau\tau}(x_0) - w(x_0) \geq -C.$$

We have thus proved that $V \geq -C$ and conequently, by (4.46), $\eta u_{\tau\tau} \geq -C$; that is

$$u_{\tau\tau} \geq -C \text{ in every compact subdomain of } \Omega. \tag{4.48}$$

For any point $y \in \Omega$, we can make an orthogonal transformation of the variable x so that in the new variable $z = Qx$,

$$\sum_{ij} a_{ij}(y)\frac{\partial^2 u}{\partial x_i \partial x_j} = -\Delta_z u(y) + 2\sum_j \widetilde{b}_j(y)\frac{\partial u}{\partial z_j} \text{ at } y. \tag{4.49}$$

Since $\beta_\varepsilon \leq C$, we can deduce that $Au(y) \leq C$ and, recalling (4.41) and (4.49), we find that

$$-\Delta_z u(y) \leq C, \quad C \text{ is independent of } \varepsilon, \delta.$$

From this relation and (4.48), it is obvious that $0 \leq u_{\tau\tau} + C \leq C$. Thus $|u_{\tau\tau}| \leq C$, C independent of ε, δ. The rest of the proof is almost the same as what have been given before. We have assumed that $f \equiv 0$. If $f \equiv 0$, then we define u_0 by $Au_0 = f$ in Ω and $u_0 = g$ on $\partial\Omega$ and work with $U = u - u_0$ and with the obstacle $\psi - u_0$. □

4.4. BV-estimates.

Definition 4.25. For $u \in L^1(\Omega)$, define

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + \sup\left\{\int_\Omega u \operatorname{div}\xi dx : \xi \in (C_0^1(\Omega))^n, \|\xi\|_{L^\infty(\Omega)} \leq 1\right\}.$$

We shall say $u \in BV(\Omega)$ iff $\|u\|_{BV(\Omega)} < \infty$.

Example 4.26. It is easy to show that for $v \in W^{1,1}(\Omega)$, $\|v\|_{BV(\Omega)} \leq \|v\|_{W^{1,1}(\Omega)}$.

Example 4.27. Let $E \subset\subset \Omega$ be a bounded smooth open set and $v = \chi_E$. Then

$$\left|\int_\Omega v \operatorname{div}(\xi)dx\right| = \left|\int_E \operatorname{div}(\xi)dx\right| = \left|\int_{\partial E} \xi \cdot n d\mathcal{H}^{n-1}\right| \leq \|\xi\|_{L^\infty(\Omega)}\mathcal{H}^{n-1}(\partial E),$$

where the equality holds for $\xi|_{\partial E} = n$. Hence we have

$$\|v\|_{BV(\Omega)} = |E| + \mathcal{H}^{n-1}(\partial E).$$

Remark 4.28. If one can show $\chi_\Lambda \in BV(\Omega)$, then by a theorem of De Giorgi, the reduced boundary $\partial^*\Lambda$ is rectifiable, i.e. it is contained in a countable union of C^1-hypersurfaces. In this way, we are able to discuss the regularity of coincidence set later in this notes.

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Define

$$Au = -\partial_i(a_{ij}(x)\partial_j u),$$

where $a_{ij} \in W^{2,\infty}(\Omega)$ and

$$\begin{aligned} &\sum_{i,j=1}^{n} \|a_{ij}\|_{W^{2,\infty}(\Omega)} \le \Lambda, \\ &\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \lambda^{-1}|\xi|^2, \forall \xi \in \mathbb{R}^n \end{aligned} \tag{4.50}$$

for some positive Λ and λ. Consider the variational inequality

$$\langle Au - f, v-u\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \ge 0, \quad u \in K_\psi = \left\{w \in H_0^1(\Omega) : w \ge \psi\right\}, \forall v \in K_\psi, \tag{4.51}$$

where $f \in C^{0,1}(\overline{\Omega})$ and $\psi \in W^{3,1}(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$ with $\psi|_{\partial\Omega} \le 0$. Define penalty function $\{\beta_\varepsilon(t)\}_{\varepsilon\in(0,1)}$ as before. Namely they are non-decreasing smooth functions in t satisfying (4.27). Let u_ε solve the penalized problem

$$Au_\varepsilon + \beta_\varepsilon(u_\varepsilon - \psi) = f \text{ in } \Omega \quad \text{and} \quad u_\varepsilon = 0 \text{ on } \partial\Omega.$$

We shall prove that $Au_\varepsilon \in BV_{loc}(\Omega)$. $\forall K \subset\subset \Omega$, as before, we define cut-off function $\eta \in C_0^\infty(\Omega)$ associated with K satisfying (4.36) . We also define $\{\gamma_\delta(t)\}_{\delta\in(0,1)}$ as smooth approximations of $\operatorname{sgn}(t)$, s.t.

$$|\gamma_\delta(t)| \le 1,\ \gamma_\delta'(t) \ge 0,\ \gamma_\delta(0) = 0 \text{ and } \lim_{\delta\to 0^+} \gamma_\delta(t) = \operatorname{sgn}(t).$$

The last convergence is in the pointwise sense. Let $\tau \in \mathbb{S}^{n-1}$ be an arbitrary unit vector. We take derivative in the direction of τ and obtain that

$$\partial_\tau Au_\varepsilon + \beta_\varepsilon'(u_\varepsilon - \psi)\partial_\tau(u_\varepsilon - \psi) = f_\tau. \tag{4.52}$$

Define $\gamma = \gamma_\delta(u_{\varepsilon,\tau} - \psi_\tau)$. We calculate

$$\begin{aligned} \int_\Omega \partial_\tau(Au_\varepsilon)\eta\gamma &= \int_\Omega \partial_\tau(a_{ij}(x)u_{\varepsilon,j})(\gamma_i\eta + \eta_i\gamma) \\ &= \int_\Omega a_{ij}(x)(u_{\varepsilon,\tau} - \psi_\tau)_j\gamma_i\eta + \int_\Omega a_{ij}(x)\psi_{j\tau}\gamma_i\eta \\ &\quad + \int_\Omega \partial_\tau(a_{ij}(x)u_{\varepsilon,j})\eta_i\gamma + \int_\Omega \partial_\tau a_{ij}(x)u_{\varepsilon,j}\gamma_i\eta \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}\,. \end{aligned}$$

Notice that $\gamma_\delta' \ge 0$,

$$\mathrm{I} = \int_\Omega [a_{ij}(u_{\varepsilon,\tau} - \psi_\tau)_j(u_{\varepsilon,\tau} - \psi_\tau)_i] \cdot \gamma_\delta'(u_{\varepsilon,\tau} - \psi_\tau)\eta \ge 0.$$

We also have

$$\begin{aligned} |\,\mathrm{II}\,| &= \left| -\int_\Omega [a_{ij}(x)\psi_{j\tau}\eta]_i\gamma \right| \le C(\Lambda,\eta)\|\psi\|_{W^{3,1}(\Omega)}, \\ |\,\mathrm{III}\,| &= \left| \int_\Omega a_{ij,\tau}u_{\varepsilon,j}\eta_i\gamma + \int_\Omega a_{ij}u_{\varepsilon,j}\eta_i\gamma \right| \le C(\Lambda,\eta)\|u_\varepsilon\|_{W^{2,p}(\Omega)}, \\ |\,\mathrm{IV}\,| &= \left| -\int_\Omega [a_{ij,\tau}u_{\varepsilon,j}\eta]_i\gamma \right| \le C(\Lambda,\eta)\|u_\varepsilon\|_{W^{2,p}(\Omega)}, \end{aligned}$$

In sum

$$-\int_\Omega \partial_\tau(Au_\varepsilon)\eta\gamma \le C(\Lambda,\eta)(\|\psi\|_{W^{3,1}(\Omega)} + \|u_\varepsilon\|_{W^{2,p}(\Omega)}) \le C(n,p,\lambda,\Lambda,\eta,\psi,f).$$

The last inequality comes from the $W^{2,p}$-estimate. By (4.52), we obtain

$$\begin{aligned}\int_\Omega \beta_\varepsilon(u_\varepsilon-\psi)(u_{\varepsilon,\tau}-\psi_\tau)\gamma_\delta(u_{\varepsilon,\tau}-\psi_\tau)\eta &= \int_\Omega f_\tau\gamma\eta - \int_\Omega \partial_\tau(Au_\varepsilon)\eta\gamma \\ &\le C(n,p,\lambda,\Lambda,\eta,\psi,f).\end{aligned}$$

Note that

$$\lim_{\delta\to 0^+} \beta_\varepsilon'(u_{\varepsilon,\tau}-\psi_\tau)\gamma_\delta(u_{\varepsilon,\tau}-\psi_\tau) = |\partial_\tau\beta_\varepsilon(u_\varepsilon-\psi)| \text{ pointwise.}$$

For fixed ε, by bounded convergence theorem

$$\int_\Omega \eta\,|\partial_\tau\beta_\varepsilon(u_\varepsilon-\psi)| \le C(n,p,\lambda,\Lambda,\eta,\psi,f).$$

Here, the we use the $W^{2,p}(\Omega)$ estimates to obtain the boundedness. Again, by (4.52),

$$\int_\Omega \eta|\partial_\tau(Au_\varepsilon)| \le \int_\Omega \eta|\partial_\tau\beta(u_\varepsilon-\psi)| + \int_\Omega \eta|f_\tau| \le C(n,p,\lambda,\Lambda,\eta,\psi,f).$$

Then

$$\int_\Omega \eta|\nabla(Au_\varepsilon)| \le C(n,p,\lambda,\Lambda,\eta,\psi,f),$$

and

$$\int_K |\nabla(Au_\varepsilon)| \le C(n,p,\lambda,\Lambda,\eta,\psi,f).$$

Note that the last inquality, together with

$$\int_K |Au_\varepsilon| \le \int_K |f| + \int_K |\beta_\varepsilon(u_\varepsilon-\psi)| \le C(n,p,\lambda,\Lambda,\eta,\psi,f).$$

implies that $Au_\varepsilon \in BV_{loc}(\Omega)$, in fact in $W^{1,1}_{loc}(\Omega)$. Moreover $\|Au_\varepsilon\|_{BV(K)}$ is independent of ε. By taking $\varepsilon\to 0^+$ and using the compactness of $BV_{loc}(\Omega)$, we know that there exists $v\in BV_{loc}(\Omega)$ and a subsequence of Au_ε, denoted by $\{Au_{\varepsilon_k}\}$, s.t.

$$Au_{\varepsilon_k} \to v \text{ in } L^1_{loc}(\Omega).$$

In fact $Au_{\varepsilon_k} \to v$ weakly in $BV_{loc}(\Omega)$. By using the fact that $BV_{loc}(\Omega)$ is embedded compactly into $L^1_{loc}(\Omega)$, we can obtain the result above. Moreover

$$\|v\|_{BV(K)} \le \liminf_{k\to\infty} \|Au_{\varepsilon_k}\|_{BV(K)}, \quad \forall K \subset\subset \Omega.$$

Recall that u is the weak limit of u_ε in $W^{2,p}(\Omega)$, for any $p\in(1,\infty)$. In particular

$$Au_\varepsilon \to Au \text{ in } L^1_{loc}(\Omega).$$

Therefore, $Au = v\in BV_{loc}(\Omega)$ and

$$\|Au\|_{BV(K)} \le \|Au_{\varepsilon_k}\|_{BV(K)} \le C(K), \quad \forall K\subset\subset\Omega.$$

Thus we have proved

Theorem 4.29. *Suppose that $\psi\in W^{3,1}(\Omega)\cap C^{2,\alpha}(\overline{\Omega})$, $f\in C^{0,1}(\overline{\Omega})$. Let $u\in K_\psi$ be the solution of* (4.51)*, where $A=-\partial_i(a_{ij}(x)\partial_j)$ satisfies* (4.50)*. Then $Au\in BV_{loc}(\Omega)$.*

Here, since $Au\in BV_{loc}(\Omega)$, we can see that u is continuous locally. Then the conincidence set and non-coincidence set are defined well.

Corollary 4.30. *Suppose, in addition, that $A\psi - f\neq 0$. Then the coincidence set $\Lambda=\{x\in\Omega : u(x)=\psi(x)\}$ is of locally finite perimeter, i.e. $\chi_\Lambda\in BV_{loc}(\Omega)$.*

Remark 4.31. The condition $A\psi - f\neq 0$ rules out the case that the landscape of coincidence set could be rather complicated.

Proof. By assumption, $A\psi - f \in W^{1,1}(\Omega)$ and $A\psi - f \neq 0$. We claim that

$$\chi_\Lambda = \frac{Au - f}{A\psi - f} \text{ a.e. in } \Omega.$$

Hence, $\chi_\Lambda \in BV_{loc}(\Omega)$. □

5. Regularity of free boundary

5.1. Examples of free boundary with singularities.

Example 5.1. Consider an analytic curve $\Gamma \subset \mathbb{R}^2$ with a cusp at one point, say 0, having the figure ∞. This means that the figure of this curve looks like ∞. Let Λ be the closed set enclosed by Γ, V an open neighborhood of Γ and $N = V \backslash \overline{\Lambda}$. We consider the boundary value problem:

$$\Delta \widetilde{u} = 1 \text{ in } N, \quad \text{and} \quad \widetilde{u} = 0,\ \frac{\partial \widetilde{u}}{\partial \nu} = 0 \text{ on } \Gamma. \tag{5.1}$$

Set $\widetilde{u} = 0$ in Λ. By the Cauchy-Kowalevski Theorem, (5.1) has a unique analytic solution in N, provided that the neighborhood V is small enough. Since $\frac{\partial^2 \widetilde{u}}{\partial \nu^2} = 1$ on Γ, shrink V further if necessary, we have $\widetilde{u} > 0$ in N. Here we have used the fact that $\nabla_{\tan}\left(\frac{\partial u}{\partial \nu}\right) = 0$, which is a consequence of $\frac{\partial u}{\partial \nu} = 0$ on Γ and $\Delta \widetilde{u} = 1$. Let $u = \widetilde{u} - \frac{|x|^2}{4}$, $\psi = -\frac{|x|^2}{4}$. Then the function u is a solution in $\Omega = \overline{\Lambda} \cup N$ of the obstacle problem:

$$\Delta u \leq 0,\ u \geq \psi,\ (u - \psi)\Delta u = 0 \text{ in } \Omega,$$

with strictly concave obstacle ψ. The free boundary Γ has a cusp at 0.

Example 5.2. Let Ω be a convex C^2 domain which is symmetric with respect to the hypersurface (actually a plane) $P = \{x : x_n = 0\}$. Let $E \subset F \subset \Omega \cap P$ be such that:
(i) E is open and F is relatively closed in $\Omega \cap P$,
(ii) $E = \bigsqcup_i E_i$, $F = \bigsqcup_i F_i$, i.e. disjoint unions of their components E_i and F_i respectively, with $E_i \subset F_i$ for every i. Let $\psi \in C^2(\overline{\Omega})$, $\psi < 0$ on $\partial\Omega$ and u be the solution of

$$\min \int_\Omega |\nabla v|^2 dx, \quad v \in K_\psi.$$

We set $\Lambda = \{x : u(x) = \psi(x)\}$ as the coincidence set and write:

$$\Lambda = \bigsqcup_j \Lambda_j,$$

$$\mathrm{Int}(\Lambda) = \bigsqcup_j \mathrm{Int}(\Lambda_j),$$

the disjoint unions of closed and open components respectively.

Theorem 5.3. *There is a C^∞ superharmonic function ψ in $\overline{\Omega}$ with $\psi \leq 0$ such that $E_j = \mathrm{Int}(\Lambda_j) \cap P$ and $F_j = \Lambda_j \cap P$.*

Remark 5.4. If $(x_0', 0)$ is an accumulation point of components F_j, then in any neighborhood of $(x_0', 0)$, there are infinitely many components of the free boundary. If $n = 2$ and ψ is concave, then N is topologically an annulus. If, in addition ψ is analytic, then Γ is a Jordan curve with analytic parametrization. This latter statements were due to H.Lewy, see [16] and references therein.

Proof of Theorem 5.4. Firstly note that for any open set $O \subset \mathbb{R}^n$, there is a nonnegative, smooth function $\alpha(x)$ such that $O = \{x \in \mathbb{R}^n : \alpha(x) > 0\}$. Indeed, let $\{B_{r_1}(x_i)\}_i$ be a covering of O by balls contained in O, with $r_j < \frac{1}{2}$. Let $\xi \in C^\infty$ be such that $\xi \equiv 0$, if $|x| \geq 1$; and $\xi(x) > 0$, if $|x| < 1$. Then let

$$\alpha(x) = \sum_j \alpha_j \xi\left(\frac{x - x_j}{r_j}\right), \text{ where } \alpha_j = \frac{(r_j)^j}{j!}.$$

We let $O = E \subset P$ and construct $\alpha(x')$ as above. Let

$$\begin{aligned}
&\Gamma_+ = \{(x', \alpha(x')) : x' \in \Omega \cap \mathbb{R}^{n-1}\}, \quad \Gamma_- = \{(x', -\alpha(x')) : x' \in \Omega \cap \mathbb{R}^{n-1}\} \\
&\Omega_+ = \{x : x_n > \alpha(x')\}, \quad \Omega_- = \{x : x_n < -\alpha(x')\} \\
&\Lambda = \{x : -\alpha(x') \le x_n \le \alpha(x')\} \\
&\Omega = \Omega_+ \cup \Omega_- \cup \Lambda.
\end{aligned}$$

Notice that $E \subset \mathrm{Int} \cap P$. Consider the boundary value problem:

$$\begin{cases} \Delta v = 1 + h & \text{in } \Omega_+, \\ v = 0, \quad \dfrac{\partial v}{\partial x_n} = 0 & \text{on } \Gamma_+. \end{cases} \tag{5.2}$$

Lemma 5.5. *There exist functions v, h in $C^\infty(\overline{\Omega}_+)$, for which* (5.2) *hold and h vanishes on Γ_+ as well as its derivatives; furthermore, $v > 0$, $\Delta v > 0$ in Ω_+.*

We extend the definition of v into Ω as:

$$v(x) = \begin{cases} v(x', -x_n) & \text{in } \Omega_-, \\ 0 & \text{in } \Lambda. \end{cases}$$

Letting

$$f = \begin{cases} \Delta v = 1 + h & \text{in } \Omega_+ \cup \Omega_-, \\ 1 & \text{in } \Lambda, \end{cases}$$

we have $f \in C^\infty(\overline{\Omega})$ and $f > 0$. Consider

$$\begin{cases} \Delta g = -f & \text{in } \Omega, \\ g = -v & \text{on } \partial\Omega. \end{cases}$$

Then $u = v + g$ satisfies

$$\text{(i)} \begin{cases} \Delta u = 0 & \text{in } \Omega \backslash \Lambda, \\ u = 0 & \text{on } \partial\Omega; \end{cases}$$

$$\text{(ii)} \begin{cases} u = g, \quad \nabla u = \nabla g & \\ u > \psi, \quad \text{in } \Omega \backslash \Lambda, & \end{cases} \text{on } \partial\Lambda,$$

i.e. u is a solution to the obstacle problem with obstacle ψ. Moreover, for every j we have

$$\begin{cases} F_j = \Lambda_j \cap P, \\ E_j = \mathrm{Int}(\Lambda_j) \cap P. \end{cases}$$

□

5.2. **Free boundary regularity I: basic estimates.** Let u_0 be the unique solution of

$$\min_{K_{\psi,\phi}} \int_\Omega |\nabla v|^2 dx, \quad K_{\psi,\phi} = \left\{ v \in H_0^1(\Omega) : v \ge \psi \text{ in } \Omega, v = \phi \text{ on } \partial\Omega \right\}.$$

Then

$$\Delta u_0 \le 0, \ u_0 \ge \psi \quad \text{and} \quad (u_0 - \psi)\Delta u_0 = 0.$$

We consider $u = u_0 - \psi$ and $f = -\Delta\psi$. Hence we have

$$-\Delta u + f \ge 0, \quad u \ge 0 \quad \text{and} \quad u(-\Delta u + f) = 0.$$

Then, we will focus on the free boundary problem

$$\begin{cases} -\Delta u + f \ge 0 \text{ in } \Omega, \ u \ge 0 \text{ in } \Omega, \\ u(-\Delta u + f) = 0 \text{ in } \Omega, \ u = \phi - \psi \text{ on } \partial\Omega. \end{cases} \tag{5.3}$$

We assume $\Omega \subset \mathbb{R}^n$ to be a bounded domain $\phi \in C^{2,\alpha}(\overline{\Omega})$ and $f \in C^{0,\alpha}(\overline{\Omega})$, $0 < \alpha \le 1$. In view of Theorem 4.23, we have $u \in C^{1,1}(\Omega)$. We define

$$N = \{x \in \Omega : u(x) > 0\},\ \Lambda = \{x \in \Omega : u(x) = 0\} \text{ and } \Gamma = \partial N \cap \Omega.$$

Then (5.3) becomes a more precise form as

$$\begin{cases} -\Delta u + f \ge 0 \text{ in } \Omega,\ u \ge 0 \text{ in } \Omega, \\ u > 0 \text{ in } N(u),\ u = 0 \text{ in } \Lambda(u), \\ u(-\Delta u + f) = 0 \text{ in } \Omega,\ u = \phi - \psi \text{ on } \partial\Omega \end{cases} \tag{5.4}$$
$$\psi, \phi \in C^{2,\alpha}(\overline{\Omega}),\ f \in C^{0,\alpha}(\overline{\Omega}).$$

We first want to show:
(i) Γ has Lebesgue measure zero;
(ii) If $y \in \Gamma$, then $\liminf_{x\to y, x\in N} \partial^2_{ii}(u(x)) \ge 0$ for any i derection.

Lemma 5.6 (Nondegeneracy of u)**.** *Suppose $f \ge \lambda > 0$, that is $\Delta\psi \le -\lambda$ in Ω and let $x_0 \in \overline{N}$. Then for any ball $B_r(x_0) \subset \Omega$,*

$$\sup_{x\in B_r(x_0)} (u(x) - u(x_0)) \ge \frac{\lambda r^2}{2n}. \tag{5.5}$$

Proof. Firstlt we assume that $x_0 \in N$ and consider the auxiliary function

$$w(x) = u(x) - u(x_0) - \frac{\lambda}{2n}|x - x_0|^2.$$

Then

$$\Delta w = \Delta u - \lambda = f - \lambda \ge 0$$

in N and $w(x_0) = 0$. Hence by the maximum principle,

$$\sup_{x\in B_r(x_0)\cap N} w(x) \ge 0,\ (\text{since } w(x_0) \ge 0),$$

and it is attained on the boundary $\partial(N \cap B_r(x_0))$. By using the fact that $w \in C^{1,1}(\Omega)$ and $u = 0$ in Λ, we can see that $w \le 0$ on $\partial N \cap B_r(x_0)$. Then there exists an $x_1 \in \partial B_r(x_0) \cap \overline{N}$ such that $w(x_1) \ge 0$. Hence (5.5) follows. If $x_0 \in \overline{N}\backslash N$, then we can take a sequence of $\{x_k\} \subset N$ converging to x_0 apply the above argument. ☐

Remark 5.7. For general elliptic operator

$$A = -\sum_{i,j=1}^{n} a_{ij}(x)\partial^2_{ij} + \sum_{i=1}^{n} b_i(x)\partial_i + c(x), \tag{5.6}$$

with

$$\begin{aligned} &\sum_{i,j} \|a_{ij}\|_{C^{2,\alpha}(\Omega)} + \sum_i \|b_i\|_{C^{0,\alpha}(\Omega)} + \|c\|_{C^{0,\alpha}(\Omega)} \le \mu_1,\ c(x) \ge 0 \\ &\sum_{i,j} a_{ij}(x)\xi_i\xi_j \ge \nu|\xi|^2, \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^n\ (\nu > 0), \end{aligned} \tag{5.7}$$

the free boundary problem (5.4) becomes

$$\begin{cases} Au + f \ge 0 \text{ in } \Omega,\ u \ge 0 \text{ in } \Omega, \\ u > 0 \text{ in } N(u),\ u = 0 \text{ in } \Lambda(u), \\ u(Au + f) = 0 \text{ in } \Omega,\ u = \phi - \psi \text{ on } \partial\Omega, \end{cases} \tag{5.8}$$
$$A \text{ is give by (5.6) and satisfies (5.7)};\ \psi, \phi \in C^{2,\alpha}(\overline{\Omega}),\ f \in C^{0,\alpha}(\overline{\Omega}).$$

Firstly, we define a constant $C_1 = 10(\mu_1+1)(\nu+1)(\mathrm{diam}(\Omega)+1)$. Next, we can choose

$$w(x) = u(x) - u(x_0) - \frac{\lambda}{2nC_1}|x-x_0|^2$$

as the new auxiliary function. Then it can be obtained by simple calculations that

$$\begin{aligned}-Aw &= f - \sum_{i=1}^n \frac{\lambda}{nC_1} a_{ii}(x) + \sum_{i=1}^n \frac{2\lambda}{nC_1} b_i(x)(x_i - x_{0,i}) + \frac{\lambda}{2nC_1} c(x)|x-x_0|^2 \\ &\geq f - \left(\sum_{i=1}^n \frac{\nu}{nC_1} + \sum_{i=1}^n \frac{2\,\mathrm{diam}(\Omega)\mu_1}{nC_1}\right)\lambda \geq f - \lambda \geq 0.\end{aligned}$$

This, together with maximal principle, shows that

$$\sup_{x\in B_r(x_0)} (u(x) - u(x_0)) \geq \frac{\lambda r^2}{2nC_1}. \tag{5.9}$$

To some extent, such estimate is enough to deal with the problems since the what is the exact constant is not essential for the problem.

Lemma 5.8. *If $x \in N$, $\mathrm{dist}(x,\Gamma) < \varepsilon$, $\mathrm{dist}(x,\partial\Omega) \geq 1$, then*

$$u(x) \leq M\varepsilon^2, \ \textit{where } M = \|\nabla^2 u\|_{L^\infty(\Omega)}, \tag{5.10}$$

$$|\nabla u(x)| \leq C_M \varepsilon. \tag{5.11}$$

Proof. Since $u = 0$ in Λ and $u \in C^{1,1}(\Omega)$, we can obtain that $u = 0$ and $|\nabla u| = 0$ on Γ. Another perspective to see the fact that $|\nabla u| = 0$ is that in Γ, u achieves its minimum. These show that (5.10). Indeed, we can see that for any $x \in N$ such that $\mathrm{dist}(x,\Gamma) < \varepsilon$, $\mathrm{dist}(x,\partial\Omega) \geq 1$,

$$u(x) - u(x_0) = \frac{1}{2}\sum_{i,j=1}^n \partial_{ij}^2 u(x_0 + \theta(x-x_0))(x_i - x_{0,i})(x_j - x_{0,j}),$$

where x_0 is choosen as the point in Γ such that $|x - x_0| = \mathrm{dist}(x,\Gamma)$. This shows that $u(x) \leq M\varepsilon^2$. For $x \in N$, we have the expansion

$$0 \leq u(x+se_i) = u(x) + \partial_i u(x)s + \int_0^s \int_0^\tau \partial_{ij}^2 u(x+te_i)\,dt\,d\tau.$$

Choose e_i so that $-\partial_i u(x) = |\nabla u|(x)$. Here e_i need not to be unit direction. Let $s = \varepsilon$ to get that

$$0 \leq M\varepsilon^2 - |\nabla u(x)|\varepsilon + \frac{1}{2}M\varepsilon^2,$$

and rearranging yields the claim. In fact, we can simply use the expansion of ∇u to prove (5.11). In other words, this lemma is only a simple use of fundamental theorem of calculus. □

Theorem 5.9. *If $f \geq \lambda > 0$ in Ω and $x_0 \in \Gamma$, $\mathrm{dist}(x_0,\partial\Omega) \geq \delta_0 > 0$, then for problem* (5.8)*, there exists an ε_0 and r such that*

$$\frac{|B_\varepsilon(x_0)\cap N|}{|B_\varepsilon(x_0)|} \geq r > 0, \quad \forall \varepsilon \in (0,\varepsilon_0), \tag{5.12}$$

where r, ε_0 depend only on λ, δ_0 and $M = \|D^2 u\|_{L^\infty(\Omega_{\delta_0})}$ with

$$\Omega_{\delta_0} = \{x \in \Omega : \mathrm{dist}(x,\partial\Omega) \geq \delta_0\}.$$

Corollary 5.10. *If $f \geq \lambda > 0$, then Γ has Lebesgue measure zero for problem* (5.8).

Proof of Corollary 5.10. This is a standard method to show some set is of zero measure. Briefly speaking, we show that for a.e. x in the set have density in $(0,1)$. If E is a measurable subset of $\mathbb{R}^n$ with positive measure (note that Γ is measurable, which is a result that Γ is relative closed in Ω and Ω is open), then

$$\text{density of } x_0 \text{ in } E = DE(x_0) = \lim_{\varepsilon\to 0^+} \frac{|B_\varepsilon(x_0)\cap E|}{|B_\varepsilon(x_0)|} = 1 \text{ for a.e. } x_0 \in E.$$

Then (5.12) implies for almost every $x_0 \in \Gamma$, $D\Gamma(x_0) < 1$. Thus, we conclude $|\Gamma| = 0$. □

Proof. We will only prove this theorem for problem (5.4) and the case for (5.4) is almost the same since we only need to change the constants in the property of Nondegeneracy of u. Firstly, we set $\varepsilon < \delta_0/2$. By using Lemma 5.6, there is a point $x \in B_{\varepsilon/2}(x_0)$ such that

$$u(x) = u(x) - u(x_0) \geq \frac{\lambda\varepsilon^2}{8n}.$$

Here we have used the fact that the maximum is obtained on the boundary, that is, on $\partial B_{\varepsilon/2}(x_0)$. Consider $y \in B_{\delta\varepsilon}(x)$ with δ small. Since $B_{\delta\varepsilon}(x)$ is in the ε neighborhood of Γ, by using Lemma 5.8 we can see that

$$\|\nabla u\|_{L^\infty(B_{\delta\varepsilon}(x))} \leq C\varepsilon,$$

where C depends on M, δ_0. Then

$$|u(x) - u(y)| \leq \delta\varepsilon \cdot C\varepsilon = C\delta\varepsilon^2 \text{ for any } y \in B_{\delta\varepsilon}(x).$$

Thus,

$$u(y) \geq \frac{\lambda\varepsilon^2}{8n} - C\delta\varepsilon^2 > \frac{\lambda\varepsilon^2}{16n}, \quad \text{for } \delta < \frac{\lambda}{16nC},$$

i.e. $B_{\delta\varepsilon}(x) \subset N$. This means that

$$\frac{|B_\varepsilon(x_0)\cap N|}{|B_\varepsilon(x_0)|} \geq \frac{|B_{\delta\varepsilon}(x)\cap N|}{|B_\varepsilon(x_0)|} \geq \delta^n = r > 0,$$

for any $\varepsilon < \delta_0/2$. □

Theorem 5.11. *Let f be a constant function, u be the solution for the problem* (5.4)*, $y \in \Gamma$,* $\operatorname{dist}(y, \partial\Omega) \geq \delta_0 > 0$*. Then if $x \in N$*

$$\partial^2_{ii} u(x) \geq -\frac{C}{|\log|x-y||^\varepsilon}, \quad \varepsilon = \frac{1}{2(n-1)}, \tag{5.13}$$

where C depends only on δ_0, n and $M = \|u\|_{C^{1,1}(\overline{\Omega_{\delta_0/2}})}$.

Remark 5.12. (a) Note that for any direction e_i and $y \in \Gamma$, we have $|x-y| \geq \operatorname{dist}(x,\Gamma)$ and

$$\partial^2_{ii} u(x) \geq -\frac{C}{|\log(\operatorname{dist}(x,\Gamma))|^\varepsilon}, \quad \varepsilon = \frac{1}{2(n-1)},$$

follows from (5.13) and hence

$$\liminf_{x\in N, x\to y\in\Gamma} \partial^2_{ii} u(x) \geq 0.$$

(b) In general cases, if u is the solution of (5.8), then

$$\liminf_{x\in N, x\to y\in\Gamma} \partial^2_{ii} u(x) \geq 0 \tag{5.14}$$

as well.

First we shall show (5.14) in the case Δ as a corollary of Theorem 5.11.

Proof of (5.14) *with* $A = \Delta$. (Blow-up argument) We will prove this lower bound by contradiction. Suppose (5.14) is not true. Then there exist a positive constant $\varepsilon_0 > 0$ and a sequence $\{x_m\} \subset N$ such that $x_m \to 0 = y \in \Gamma$ (we assume that $y = 0$ for simplicity and for general case, we can obtain the same results by translation) and

$$\partial_{ii}^2 u(x_m) \le -\varepsilon_0 < 0.$$

Meanwhile, we can choose $x_m^* \in \Gamma$ such that $\mathrm{dist}(x_m, \Gamma) = |x_m - x_m^*|$. Now consider

$$v_m(x) = \lambda_m^{-2} u(x_m^* + \lambda_m x),$$

where $x \in B_2(0)$ and $\lambda_m = \frac{1}{\delta}|x_m - x_m^*| \to 0$ with $\delta < 1$ chosen sufficiently small so that

$$\frac{C}{|\log \delta|^{\varepsilon}} < \frac{\varepsilon_0}{2}.$$

Here, the reason why v_m can be defined in $B_2(0)$ is that $\lambda_m \to 0$ no matter how small the δ is chosen. Then as long as m is sufficiently large, v_m can be defined in $B_2(0)$, that is $B(x_m^*, \lambda_m) \subset \Omega$, more precisely, $B(x_m^*, \lambda_m) \subset \Omega_{\delta_0/2}$. Then by using the properties of u, we can find without difficulty that $v_m \ge 0$ and

$$\|\partial^2 v_m\|_{L^\infty(B_2(0))} = \|\partial^2 u\|_{L^\infty(B(x_m^*, \lambda_m))} \le \|\partial^2 u\|_{L^\infty(\Omega_{\delta_0/2})} \le M.$$

Letting $f_m(x) = f(x_m^* + \lambda_m x)$, we have $[f_m]_{C^{0,\alpha}(B_2(0))} \le c_0 \lambda_m^\alpha$, where c_0 depends only on Ω, δ_0, δ and $\|f\|_{C^{0,\alpha}(\Omega_{\delta_0/2})}$. We note that

$$\begin{aligned} x_m \in N(u) \Leftrightarrow u(x_m) > 0 \Leftrightarrow v_m\left(x_m^* + \lambda_m\left(\lambda_m^{-1}(x_m - x_m^*)\right)\right) > 0 \\ \Leftrightarrow \xi_m = \lambda_m^{-1}(x_m - x_m^*) \in N(v_m), \end{aligned}$$

and $|\xi_m| = \delta$. Since $x_m^* \in \Gamma$ is chosen as the point such that $|x_m - x_m^*| = \mathrm{dist}(x_m, \Gamma)$, then ξ_m satisfies $|\xi_m - 0| = \mathrm{dist}(\xi_m, \Gamma(v_m))$ with $0 \in \Gamma(v_m)$. This implies that $B_\delta(\xi_m) \subset N(v_m) \cap B_2(0)$. Now, we may assume, up to a subsequence, $\xi_m \to \xi_* \in B_2(0)$. We see that when m tends to ∞, $\Gamma(v_m)$ become "flatter". Then as their limit, ξ_* satisfies $B_{2\delta/3}(\xi_*) \subset N(v_m)$. By using the definition of $N(v_m)$, we have $\Delta v_m = f_m$ Furthermore, we have the interior estimate $\|v_m\|_{C^{2,\alpha}(B_{\delta/2}(\xi_*))} \le C\|f_m\|_{C^{0,\alpha}(B_{2\delta/3}(\xi_*))}$. Once we could establish $v_m \to v_*$ in $C^{1,1}$, $f_m \to f_* = f(0) =$ constant, $0 \in N(v_*)$ and $B_{\delta/2}(\xi_*) \subset N(v_*)$, we would have

$$\partial_{ii}^2 v_*(\xi_*) \ge -\frac{C}{|\log \delta|^{\varepsilon}} > -\varepsilon_0/2$$

due to Theorem 5.11, which will give the desired contradiction with

$$\partial_{ii}^2 v_m(\xi_*) \le -\varepsilon_0 < 0.$$

To this end, we need the following notion and the compactness result. We define the set of functions $P_1(M, C_0, \lambda_0)$ such that $u \in P_1(M, C_0, \lambda_0)$ if and only if

$$\begin{cases} \Delta u = f \text{ in } N(u) = \{u > 0\} \cap B_1(0) \text{ for some } f \ge \lambda_0 > 0 \text{ with } \|f\|_{C^{0,\alpha}(B_1(0))} \le C_0, \\ u \ge 0 \text{ in } B_1(0), \quad 0 \in \partial N(u) = \Gamma(u), \quad \|u\|_{C^{1,1}(B_1(0))} \le M. \end{cases}$$

Lemma 5.13 (Compactness of $P_1(M, C_0, \lambda_0)$). *For any $\{u_m\} \subset P_1(M, C_0, \lambda_0)$, there is a subsequence re-labeled as $\{u_{m'}\}$ converging to $u \in P_1(M, C_0, \lambda_0)$ in $C^{1,1}$ in the weak-* sense, that is,*

$$\begin{aligned} &u_{m'} \to u \text{ uniformly in } C^0(B_1(0)) \\ &\nabla u_{m'} \to \nabla u \text{ unifromly in } C^0(B_1(0)) \qquad (5.15) \\ &\nabla^2 u_{m'} \to \nabla^2 u \text{ in weak*-topology of } L^\infty(B_1(0)), \end{aligned}$$

and we have

$$\Gamma(u_m') \to \Gamma(u), \text{ i.e. } \overline{\lim_{m' \to \infty}} N(u_{m'}) \subset \overline{N(u)}, \text{ and } \overline{\lim_{m' \to \infty}} \Lambda(u_{m'}) \subset \Lambda(u),$$

where the super limit of a sequence of sets $\{A_m\}$ is give by the definition that

$$x \in \overline{\lim_{m\to\infty}} A_m \Leftrightarrow \exists \{m_k\},\ \{x_{m_k}\} \textit{ such that } x_{n_k} \in A_{m_k} \textit{ and } \lim_{k\to\infty} x_{m_k} = x.$$

Proof of Lemma 5.13. By definitions, we can assume that there exists a sequence of functions $\{u_m\}$ and $\{f_m\}$ such that

$$\begin{cases} \Delta u_m = f_m \text{ in } N(u_m) = \{u_m > 0\} \cap B_1(0),\ f_m \geq \lambda_0 > 0 \text{ with } \|f_m\|_{C^{0,\alpha}(B_1(0))} \leq C_0, \\ u_m \geq 0 \text{ in } B_1(0),\ 0 \in \partial N(u_m) = \Gamma(u_m),\ \|u_m\|_{C^{1,1}(B_1(0))} \leq M. \end{cases}$$

Since $\|u_m\|_{C^{1,1}(B_1(0))}$ is uniformly bounded by M, then u_m and ∇u_m are equicontinuous and unformly bounded. This shows that we can choose a subsequence of $\{u_m\}$ denoted by $\{u_{m'}\}$ and $u \in C^{1,1}(B_1(0))$ such that

$$u_{m'} \to u \text{ in } C^0(B_1(0)),\ \text{i.e. } \|u_{m'} - u\|_{L^\infty(B_1(0))} \to 0$$
$$\nabla u_{m'} \to \nabla u \text{ in } C^0(B_1(0)),\ \text{i.e. } \|\nabla u_{m'} - \nabla u\|_{L^\infty(B_1(0))} \to 0$$

when $m' \to \infty$. Moreover, in view of the fact that $\|\nabla^2 u_{m'}\|_{L^\infty(B_1(0))} \leq M$, by using Alaoglu theorem, we can choose a subsequence of $\{u_{m'}\}$ also denoted by $\{u_{m'}\}$ that $\nabla^2 u_{m'} \to v \in L^\infty(B_1(0))$ in weak-* sense in $L^\infty(B_1(0))$, where $v \in L^\infty(B_1(0))$. Obviously, we can obtain that $v = \nabla^2 u$. Hence, the weak-* convergence in $C^{1,1}(B_1(0))$ follows. As a consequence, we can obtain that $\|u\|_{C^{1,1}(B_1(0))} \leq M$.

The nonnegativity condition in the limit follows by using the equations satisfied by u_m and the uniform interior estimate, that is, $u_{m'} \to u$ uniformly in $B_1(0)$ and $u_{m'} \geq 0$ in $B_1(0)$. Meanwhile, since $\|f_{m'}\|_{C^{0,\alpha}(B_1(0))} \leq C_0$, we can choose a subsequence, which is still denoted by $f_{m'}$ and $f \in C^{0,\alpha}(\overline{B_1(0)})$ such that $f_{m'} \to f$ uniformly in $C^0(B_1(0))$ norm. Next, we will prove that $\Delta u = f$ in $N(u)$. Obviously, since $u_{m'} \in P(M, C_0, \lambda_0)$, then $u_{m'}(\Delta u_{m'} - f) = 0$. By using the fact that $\Delta u_{m'} \to \Delta u$ in weak*-topology of L^∞ and $u_{m'} \to u$ uniformly in $C^0(B_1(0))$, we can obtain that $u(\Delta u - f) = 0$ in $B_1(0)$. This gives the result that $\Delta u = f$ in $N(u)$.

Hence, it remains to prove convergence of the free boundary only. $\overline{\lim}_{m'} \Lambda(u_{m'}) \subset \Lambda(u)$ holds by definition and the uniform convergence of $\{u_{m'}\}$. Precisely speaking, for any $x_0 \in \overline{\lim}_{m'}(u_{m'})$, we have, $\{m'_k\}$ and $\{x_{m'_k}\}$ such that $x_{m'_k} \in \Lambda(u_{m'_k})$ and $x_{m'_k} \to x_0$. Then

$$|u(x_0)| = |u_{m'_k}(x_0) - u(x_0)| + |u_{m'_k}(x_{m'_k}) - u(x_{m'_k})| + |u(x_{m'_k}) - u(x_0)| \to 0 \text{ as } k \to \infty$$

shows that $u(x_0) = 0$.

For the remaining part, let $x_0 \notin \overline{N(u)}$. Then $u(x) \equiv 0$ for $x \in B_\rho(x_0)$ with $\rho > 0$. We want to show that $x_0 \notin \overline{\lim}_{m'} N(u_{m'})$. We prove this by contradiction. Without loss of generality we can assume that for any m', $x_0 \in N(u_{m'})$. Then we can obtain a sequence of points $\{x_{m'}\}$ converging to $x_{m'}$ such that $u_{m'}(x_{m'}) > 0$, then using the nondegeneracy lemma 5.6 we obtain that for sufficiently large m',

$$\sup_{x \in \partial B_{\rho/2}(x_{m'})} u_{m'}(x) \geq \frac{\lambda_0}{2n}(\rho/2)^2.$$

By the uniform convergence $u_m \to u_0$ in $C^0(B_1(0))$, we have,

$$\sup_{B_{2\rho/3}(x_0)} u_0(x) \geq \frac{\lambda_0}{2n}(\rho/2)^2 > 0$$

which is a contradiction with $u(x) \equiv 0$ in $B_\rho(x_0)$. Finally, we will show that $0 \in \Gamma(u)$. Since $0 \in \Gamma(u_m)$ for any m, then we can obtain $x_m \in N(u_m)$ such that $|x_m| < 1/m$. This shows that $0 \in \overline{\lim}_{m'} N(u_{m'})$ and then $0 \in \overline{N(u)}$. On the other hand, we can get that $u(0) = 0$. Combining these two, we can obtain that $0 \in \Gamma(u)$. □

By using Lemma 5.13, we can complete the proof of (5.14). □

Remark 5.14. To treat general case, that is, the operator A is given by (5.6), we can define $P_1(M, C_0, \lambda_0, \mu_0, \nu_0)$ as the set of function u such that

$$\begin{cases} -Au = f \text{ in } N(u) = \{u > 0\} \cap B_1(0) \text{ for some } f \geq \lambda_0 > 0 \text{ with } \|f\|_{C^{0,\alpha}(B_1(0))} \leq C_0, \\ u \geq 0 \text{ in } B_1(0),\ 0 \in \partial N(u) = \Gamma(u),\ \|u\|_{C^{1,1}(B_1(0))} \leq M, \\ \sum_{i,j} \|a_{ij}\|_{C^{2,\alpha}(B_1(0))} + \sum_i \|b_i\|_{C^{0,\alpha}(B_1(0))} + \|c\|_{C^{0,\alpha}(B_1(0))} \leq \mu_0,\ c(x) \geq 0, \\ \sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \nu_0|\xi|^2, \quad \forall x \in B_1(0),\ \xi = (\xi_i)_{i=1}^n \in \mathbb{R}^n\ (\nu_0 > 0). \end{cases}$$

For $P_1(M, C_0, \lambda_0, \mu_0, \nu_0)$, we can also obtain a lemma of compactness similar to Lemma 5.13. For $\{u_m\} \subset P_1(M, C_0, \lambda_0, \mu_0, \nu_0)$, we have

$$\begin{cases} -A_m u_m = f_m \text{ in } N(u_m) = \{u_m > 0\} \cap B_1(0),\ f_m \geq \lambda_0 > 0 \text{ with } \|f_m\|_{C^{0,\alpha}(B_1(0))} \leq C_0, \\ u_m \geq 0 \text{ in } B_1(0),\ 0 \in \partial N(u_m) = \Gamma(u_m),\ \|u_m\|_{C^{1,1}(B_1(0))} \leq M, \\ \sum_{i,j} \|a_{ij}^{(m)}\|_{C^{2,\alpha}(B_1(0))} + \sum_i \|b_i^{(m)}\|_{C^{0,\alpha}(B_1(0))} + \|c^{(m)}\|_{C^{0,\alpha}(B_1(0))} \leq \mu_0,\ c(x) \geq 0, \\ \sum_{i,j} a_{ij}^{(m)}(x)\xi_i\xi_j \geq \nu_0|\xi|^2, \quad \forall x \in B_1(0),\ \xi = (\xi_i)_{i=1}^n \in \mathbb{R}^n\ (\nu_0 > 0). \end{cases}$$

Since $\|u_m\|_{C^{1,1}(B_1(0))}$ is uniformly bounded by M we can also obtain a subsequence of $\{u_m\}$ denoted by $\{u_{m'}\}$ and $u \in C^{1,1}(B_1(0))$ such that

$$\begin{aligned} u_{m'} &\to u \text{ in } C^0(B_1(0)),\ \text{i.e. } \|u_{m'} - u\|_{L^\infty(B_1(0))} \to 0 \\ \nabla u_{m'} &\to \nabla u \text{ in } C^0(B_1(0)),\ \text{i.e. } \|\nabla u_{m'} - \nabla u\|_{L^\infty(B_1(0))} \to 0 \\ \nabla^2 u_{m'} &\to \nabla^2 u \text{ in weak*-topology in } L^\infty(B_1(0)), \end{aligned}$$

when $m' \to \infty$. Moreover, $\|u\|_{C^{1,1}(B_1(0))} \leq M$. Since

$$\begin{aligned} &\sum_{i,j} \|a_{ij}^{(m)}\|_{C^{2,\alpha}(B_1(0))} + \sum_i \|b_i^{(m)}\|_{C^{0,\alpha}(B_1(0))} + \|c^{(m)}\|_{C^{0,\alpha}(B_1(0))} \leq \mu_0,\ c(x) \geq 0, \\ &\sum_{i,j} a_{ij}^{(m)}(x)\xi_i\xi_j \geq \nu_0|\xi|^2, \quad \forall x \in B_1(0),\ \xi = (\xi_i)_{i=1}^n \in \mathbb{R}^n\ (\nu_0 > 0), \end{aligned}$$

we can choose a subsequence of $\{a_{ij}^{(m)}(x)\}$, $\{b_i^{(m)}(x)\}$, $\{c^{(m)}(x)\}$, $\{f_m\}$, denoted as $\{a_{ij}^{(m')}(x)\}$, $\{b_i^{(m')}(x)\}$, $\{c^{(m')}(x)\}$, $\{f_{m'}\}$ and $a_{ij}(x)$, $b_i(x)$, $c(x)$, f such that

$$\begin{aligned} &a_{ij}^{(m')}(x) \to a_{ij}(x),\ \nabla a_{ij}^{(m')}(x) \to \nabla a_{ij}(x),\ \nabla^2 a_{ij}^{(m')}(x) \to \nabla^2 a_{ij}(x) \text{ in } C^0(B_1(0)), \\ &b_i^{(m')}(x) \to b_i(x),\ c^{(m)}(x) \to c(x),\ f_m(x) \to f(x) \text{ in } C^0(B_1(0)), \\ &\sum_{i,j} \|a_{ij}\|_{C^{2,\alpha}(B_1(0))} + \sum_i \|b_i\|_{C^{0,\alpha}(B_1(0))} + \|c\|_{C^{0,\alpha}(B_1(0))} \leq \mu_0,\ c(x) \geq 0, \\ &\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \nu_0|\xi|^2, \quad \forall x \in B_1(0),\ \xi = (\xi_i)_{i=1}^n \in \mathbb{R}^n\ (\nu_0 > 0), \\ &\|f\|_{C^{0,\alpha}(B_1(0))} \leq C_0,\ f(x) \geq \lambda_0 > 0 \text{ in } B_1(0). \end{aligned}$$

Then we can easily get that u satisfies

$$\begin{cases} -Au = f \text{ in } N(u) = \{u > 0\} \cap B_1(0),\ f \geq \lambda_0 > 0 \text{ with } \|f\|_{C^{0,\alpha}(B_1(0))} \leq C_0, \\ u \geq 0 \text{ in } B_1(0),\ 0 \in \partial N(u) = \Gamma(u),\ \|u\|_{C^{1,1}(B_1(0))} \leq M, \\ \sum_{i,j} \|a_{ij}\|_{C^{2,\alpha}(B_1(0))} + \sum_i \|b_i\|_{C^{0,\alpha}(B_1(0))} + \|c\|_{C^{0,\alpha}(B_1(0))} \leq \mu_0,\ c(x) \geq 0, \\ \sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \nu_0|\xi|^2, \quad \forall x \in B_1(0),\ \xi = (\xi_i)_{i=1}^n \in \mathbb{R}^n\ (\nu_0 > 0), \end{cases}$$

i.e. $u \in P_1(M, C_0, \lambda_0, \mu_0, \nu_0)$. This compactness results can be used to show (5.14) with A satisfying (5.7). By simple observation, we can assume that b_i, c are vanishing, for other wise, we can use the fact that $u \in C^{1,\alpha}(\overline{\Omega})$ by $W^{2,p}(\Omega)$ estimates of u to achieve that $b_i\partial_i u + cu \in C^{0,\alpha}(\overline{\Omega})$. This reduce the problem to the special case that b_i and c are 0. We can still use the blow-up arguments. Suppose (5.14) is not true. Then there exist a positive constant $\varepsilon_0 > 0$ and a sequence $\{x_m\} \subset N$ such that $x_m \to 0 = y \in \Gamma$ and

$$\partial_{ii}^2 u(x_m) \leq -\varepsilon_0 < 0.$$

Let $x_m^* \in \Gamma$ such that $\operatorname{dist}(x_m, \Gamma) = |x_m - x_m^*|$. Now consider

$$v_m(x) = \lambda_m^{-2} u(x_m^* + \lambda_m x),\ \lambda_m = \frac{1}{\delta}\,|x_m - x_m^*| \to 0,\ x \in B_2(0),$$

with $\delta \ll 1$ so that $\frac{C}{|\log\delta|^\varepsilon} < \frac{\varepsilon_0}{2}$. For $m \gg 1$, v_m can be defined in $B_2(0)$. Then $v_m \geq 0$ and

$$\|\partial^2 v_m\|_{L^\infty(B_2(0))} = \|\partial^2 u\|_{L^\infty(B(x_m^*, \lambda_m))} \leq \|\partial^2 u\|_{L^\infty(\Omega_{\delta_0/2})} \leq M.$$

Letting $f_m(x) = f\left(x_m^* + \lambda_m x\right)$, we have $[f_m]_{C^{0,\alpha}(B_2(0))} \leq c_0\lambda_m^\alpha$, where c_0 depends only on Ω, δ_0, δ and $\|f\|_{C^{0,\alpha}(\Omega_{\delta_0/2})}$. We note that

$$x_m \in N(u) \Leftrightarrow \xi_m = \lambda_m^{-1}(x_m - x_m^*) \in N\left(v_m\right),$$

and $|\xi_m| = \delta$. Since $|x_m - x_m^*| = \operatorname{dist}(x_m, \Gamma)$, then ξ_m satisfies $|\xi_m| = \operatorname{dist}(\xi_m, \Gamma(v_m))$ and $0 \in \Gamma(v_m)$. This implies that $B_\delta(\xi_m) \subset N(v_m) \cap B_2(0)$. Up to a subsequence, $\xi_m \to \xi_* \in B_2(0)$. Then as their limit, ξ_* satisfies $B_{2\delta/3}(\xi_*) \subset N(v_m)$. In $N(v_m)$, we have $-A_m v_m = f_m$, where

$$A_m = \sum_{i,j=1}^n a_{ij}(x_m^* + \lambda_m x)\partial_{ij}^2 \triangleq \sum_{i,j=1}^n a_{ij}^{(m)}(x)\partial_{ij}^2.$$

By simple calculations, we can find that

$$\begin{aligned} &\|a_{ij}^{(m)}\|_{C^{2,\alpha}(B_{2\delta/3}(\xi_*))} \leq \mu_1,\ [a_{ij}^{(m)}]_{C^{0,\alpha}(B_{2\delta/3}(\xi_*))} \leq \mu_1\lambda_m^\alpha, \\ &\|f_m\|_{C^{0,\alpha}(B_{2\delta/3}(\xi_*))} \leq c_0, [f_m]_{C^{0,\alpha}(B_{2\delta/3}(\xi_*))} \leq c_0\lambda_m^\alpha \end{aligned} \tag{5.16}$$

This shows that ther is a constant $a_{ij}^{(\infty)}$, such that

$$a_{ij}^{(m)}(x) \to a_{ij}^{(\infty)}, f_m(x) \to f(0) \text{ uniformly in } C^0(B_{2\delta/3}(\xi_*)).$$

Furthermore, we have the interior estimate $\|v_m\|_{C^{2,\alpha}(B_{\delta/2}(\xi_*))} \leq C\|f_m\|_{C^{0,\alpha}(B_{2\delta/3}(\xi_*))}$. Finally, we can establish that $v_m \to v_*$ in $C^{1,1}(B_{2\delta/3}(\xi_*))$ and $B_{\delta/2}(\xi_*) \subset N(v_*)$, then

$$\partial_{ii}^2 v_*(\xi_*) \geq -\frac{C}{|\log\delta|^\varepsilon} > -\varepsilon_0/2$$

due to Theorem 5.11, which will give the desired contradiction with

$$\partial_{ii}^2 v_m(\xi_*) \leq -\varepsilon_0 < 0.$$

Proof of Theorem 5.11. In order to complete the proof of the theorem, there remains to prove the lower bound for $\partial_{ii}^2 u$. For simplicity, we can assume that $y = 0$ and for general case, we can show the results by translation. Define

$$-M_k = \inf_{x\in B_{2^{-k}}(0)} \partial_{ii}^2 u(x).$$

We shall estimate M_k inductively for $M_k > 0$. Otherwise, if $M_k \le 0$ for some k, we have nothing to prove. Note that $|M_k| \le M$ uniformly and $\partial_{ii}^2 u \ge -M$.

Let $x \in N$ with $|x| \le 2^{-(k+1)}$ and let $B_s(x)$ be the largest ball in N. Then $s \le 2^{-(k+1)}$ and $B_s(x) \subset B_{2^{-k}}(0)$. Let $y_0 \in \Gamma \cap \partial B_s(x)$. Let y_1 be a point lying in the segment between x and y_0 such that $|y_1 - y_0| = \delta s$, for small enough δ to be chosen later. Then $u(y_1) \le \frac{M_k}{2}(\delta s)^2$ and $|\nabla u(y_1)| \le CM_k\delta s$.

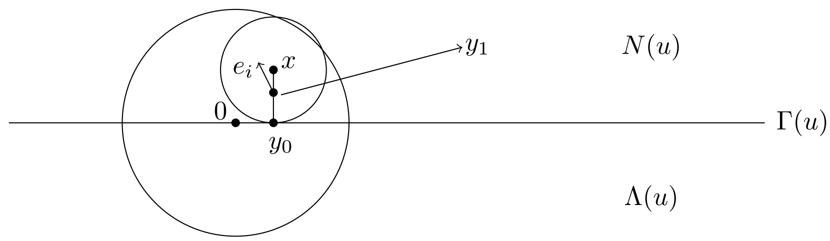

Figure 1.

In each axis direction of the coordinate, we can choose unit vector e_i, s.t. $(e_i, x - y_0) \ge 0$. (Here e_i is given in the assumption, the meaning for this is that e_i has two directions and we choose the one satisfies the condition we set.) Then since u is smooth in $B_s(x)$, we have $\partial_{ii}^2 u = \partial_{-i,-i}^2 u = \partial_{e_i,e_i}^2 u$ in $B_s(x)$ and $y_1 + te_i \in B_s(x) \subset N$ for $0 \le t \le \sqrt{\delta}s$. Here, we use the fact that

$$|x - (y_1 + te_i)|^2 \le (1-\delta)^2 s^2 + \frac{\delta}{4} s^2 \le \left(1 - \frac{\delta}{2}\right)^2 s^2 \le s^2$$

if $\delta \ll 1$. Let $h = \sqrt{\delta}s/2$. Since

$$0 \le u(y_1 + he_i) = u(y_1) + u_i(y_1)h + \int_0^h \int_0^\tau \partial_{ii}^2 u(y_1 + te_i)dtd\tau,$$

we have

$$\int_0^{\sqrt{\delta}s/2} \int_0^\tau \partial_{ii}^2 u(y_1 + te_i)dtd\tau \ge -\frac{M}{2}(\delta s)^2 - CM\delta s \cdot \sqrt{\delta}s \ge -\delta^{3/2}s^2 CM.$$

Hence

$$\sup_{0<t<\frac{\sqrt{\delta}s}{2}} \partial_{ii}^2 u(y_1 + te_i) \ge -C\delta^{\frac{1}{2}}. \tag{5.17}$$

Consider $w = \partial_{ii}^2 u + M_k$. It satisfies (here we assume that f is a constant)

$$\Delta w = 0 \text{ in } B_s(x), \quad w \ge 0.$$

Moreover, in view of (5.17),

$$w(\widetilde{y}) \ge M_k - C\delta^{\frac{1}{2}}$$

for some $\widetilde{y} \in B_{\frac{1}{2}\sqrt{\delta}s - \delta s}(x) \subset B_{s(1-\delta/2)}(x)$. Now we apply Harnack's inequality to w to get

$$w(\widetilde{y}) \le \left(\frac{s}{(s - (1-\delta/2)s)}\right)^{d-2} \left(\frac{s + (1-\delta/2)s}{s - (1-\delta/2)s}\right) w(x) \Rightarrow w(x) \ge w(\widetilde{y}) \cdot C\delta^{n-1},$$

equivalently

$$\partial_{ii}^2 u(x) \ge -M_k + C\delta^{n-1}(M_k - C\delta^{\frac{1}{2}}).$$

Choosing δ such that $c\delta^{\frac{1}{2}} = \varepsilon_0 M_k$ (and $\delta < \frac{1}{2}$ if ε_0 is small) we have

$$\partial_{ii}^2 u(x) \ge -M_k + CM_k^{2n-1}.$$

Since x is arbitrary in $B_{2^{-k-1}}(0)$, this gives the set of inequalities

$$-M_{k+1} \geq -M_k + CM_k^{2n-1} \Leftrightarrow M_{k+1} \leq M_k - CM_k^{2n-1}.$$

Hence, M_k is positive and decreasing in k. It is easy to show

$$0 < M_k \leq M_0 - C\sum_{i=0}^{k-1} M_i^{2n-1} \leq M_0 - CkM_k^{2n-1}.$$

This gives $CkM_k^{2n-1} \leq M_0$ and therefore

$$M_k \leq Ck^{-\varepsilon},$$

where $\varepsilon = \frac{1}{2(n-1)}$. We note that the $\varepsilon = \frac{1}{2(n-1)}$ is note sharp since $\frac{1}{2n-1}$ is obvious a better index. However it is not essential for such problems. For $|x| \approx 2^{-k}$, we conclude that

$$\partial_{ii}^2 u(x) \geq -\frac{C}{(\log|x|)^{\varepsilon}},$$

which complete the proof of the theorem. □

Remark 5.15. For problem (5.8) we can also generate (5.13). By using the fact that $u \in C^{1,1}(\Omega)$, we can still assume that b_i, c are 0. In order to proceed as in the proof of (5.13), we must be able to apply Harnack's inequality to some "small perturbation" of $D_{ii} + M_k$. To accomplish this, we note first that for any direction e_i,

$$A(\partial_{ii}^2 u) = \partial_{ii}^2 f + 2\sum \partial_i a_{jk}\partial_{ijk}^3 u + \sum \partial_{ii}^2 a_{jk}\partial_{jk}^2 u. \tag{5.18}$$

Here $\partial_{ii}^2 f$ is in the sense of general derivatives. Let u^0 be the solution of

$$Au^0 = f \text{ in } B_s(x) \quad \text{and} \quad u^0 = 0 \text{ on } \partial B_s(x).$$

Then $u_0 \in C^{2,\alpha}(\overline{B_s(x)})$ and by writing the Schauder estimates for $\widetilde{u}(y) = s^{-2}u^0(x+sy)$ in $\{|y|<1\}$, we have

$$\sum_{i,j} a_{ij}(x+sy)\partial_{ij}^2\widetilde{u}(y) = f(x+sy) \text{ in } B_1(0) \quad \text{and} \quad \widetilde{u} = 0 \text{ on } \partial B_1(0).$$

Then, we can obtain by standard Schauder estimates that

$$[\nabla^2\widetilde{u}]_{C^{0,\alpha}(B_1(0))} \leq C[f]_{C^{0,\alpha}(B_1(0))}, \quad \|\nabla^2\widetilde{u}\|_{L^\infty(B_1(0))} \leq C\|f\|_{C^{0,\alpha}(B_1(0))}.$$

By changing the variable, it is easy to find

$$[\nabla^2 u^0]_{C^{0,\alpha}(B_s(x))} \leq C\|f\|_{C^{0,\alpha}(B_s(x))}, \quad \|\nabla^2 u\|_{L^\infty(B_s(x))} \leq C\|f\|_{C^{0,\alpha}(B_s(x))}, \tag{5.19}$$

where C is a positive constant independent of $0 < s < 1$. Formally,

$$A(\partial_{ii}^2 u^0) = \partial_{ii}^2 f + 2\sum \partial_i a_{jk}\partial_{ijk}^3 u^0 + \sum \partial_{ii}^2 a_{jk}\partial_{jk}^2 u^0.$$

Let u^1 be the solution of

$$Au^1 = -2\sum \partial_i a_{jk}\partial_{jk}^2 u^0 \text{ in } B_s(x) \quad \text{and} \quad u^1 = 0 \text{ on } \partial B_s(x).$$

In view of (5.19), it can be seen that

$$\begin{aligned}[\nabla^2 u^1]_{C^{0,\alpha}(B_s(x))} &\leq C\left\{[\nabla a]_{C^{0,\alpha}(B_s(x))}\|\nabla^2 u^0\|_{L^\infty(B_s(x))} + \|\nabla a\|_{L^\infty(B_s(x))}[\nabla^2 u^0]_{C^{0,\alpha}(B_s(x))}\right\} \\ &\leq C\|f\|_{C^{0,\alpha}(B_s(x))}\end{aligned}$$

and

$$\|\nabla^2 u^1\|_{L^\infty(B_s(x))} \leq C\left\|2\sum \partial_i a_{jk}\partial_{jk}^2 u^0\right\|_{C^{0,\alpha}(B_s(x))} \leq C\|f\|_{C^{0,\alpha}(B_s(x))}.$$

Similarly, it can get that

$$\|\nabla u^1\|_{L^\infty(B_s(x))} \leq C\|f\|_{C^{0,\alpha}(B_s(x))}, \ [\nabla u^1]_{C^{0,\alpha}(B_s(x))} \leq C\|f\|_{C^{0,\alpha}(B_s(x))}.$$

Moreover we have

$$A(\partial_i u^1) = -2\sum \partial_i a_{jk}\partial^2_{ijk}u^0 - 2\sum \partial^2_{ii}a_{jk}\partial^2_{jk}u^0 + \sum \partial_i a_{jk}\partial^2_{jk}u^1.$$

Next let u^2 be the solution of

$$Au^2 = \sum \partial^2_{ii}a_{jk}\partial^2_{jk}u^0 - \sum \partial_i a_{jk}\partial^2_{jk}u^1 \text{ in } B_s(x) \quad \text{and} \quad u^2 = 0 \text{ on } \partial B_s(x).$$

Then by using (5.19) again, one can infers that

$$\begin{aligned}[\nabla^2 u^2]_{C^{0,\alpha}(B_s(x))} &\le C\left\{[\nabla^2 a]_{C^{0,\alpha}(B_s(x))}\|\nabla^2 u^0\|_{L^\infty(B_s(x))} + \|\nabla^2 a\|_{L^\infty(B_s(x))}[\nabla^2 u^0]_{C^{0,\alpha}(B_s(x))}\right\}\\ &\quad + C\left\{[\nabla a]_{C^{0,\alpha}(B_s(x))}\|\nabla^2 u^1\|_{L^\infty(B_s(x))} + \|\nabla a\|_{L^\infty(B_s(x))}[\nabla^2 u^1]_{C^{0,\alpha}(B_s(x))}\right\}\\ &\le C\|f\|_{C^{0,\alpha}(B_s(x))}\end{aligned}$$

and

$$\|u^2\|_{L^\infty(B_s(x))} \le C\|f\|_{C^{0,\alpha}(B_s(x))},\ [u^2]_{C^{0,\alpha}(B_s(x))} \le C\|f\|_{C^{0,\alpha}(B_s(x))}.$$

Finally, if

$$A\overline{v} = 0 \text{ in } B_s(x) \quad \text{and} \quad \overline{v} = \partial^2_{ii}u^0 + \partial_i u^1 + u^2 \text{ on } \partial B_s(x), \tag{5.20}$$

then the function $v = \partial^2_{ii}u^0 + \partial_i u^1 + u^2 - \overline{v}$ satisfies

$$Av = \partial^2_{ii}f \text{ in } B_s(x) \quad \text{and} \quad \overline{v} = 0 \text{ on } \partial B_s(x), \tag{5.21}$$

and $\|v\|_{C^{0,\beta}(B_s(x))} \le C$ for some $\beta > 0$. Here, we use the fact that $\partial^2_{ii}u^0$, $\partial_i u^1$, u^2 and $\overline{v}$ are all Hölder continuous for some $0 < \beta < 1$ and in $C^{0,\beta}(\overline{B_s(x)})$. The fact that $\overline{v} \in C^{0,\beta}(B_s(x))$ is not trivial. This is because the inverse of the elliptic operator A, denoted by A^{-1} is bounded from $C^{2,\alpha}(\partial B_s(x))$ to $C^{2,\alpha}(B_s(x))$ from Schauder estimates and is also bounded from $C(\partial B_s(x)) \to C(B_s(x))$ due to maximal principle. We can use intepolations to on that it is bounded from $C^\alpha(\partial B_s(x))$ to $C^\alpha(B_s(x))$. From (5.18) we see that

$$\begin{aligned}A(\partial^2_{ii}u) &= \partial^2_{ii}f + 2\sum \partial_i a_{jk}\partial^3_{ijk}u + \sum \partial^2_{ii}a_{jk}\partial^2_{jk}u\\ &= \partial^2_{ii}f + 2\sum \partial_i(\partial_i a_{jk}\partial^2_{jk}u) - \sum \partial^2_{ii}a_{jk}\partial^2_{jk}u\\ &= \partial^2_{ii}f + g_0 + \sum_{j=1}^n \partial_j g_j,\end{aligned}$$

where g_0, g_j are L^∞ functions. Following the preceding methos, we can find functions v_i such that

$$Av_0 = g_0, \quad Av_j = \partial_j g_j\ (1 \le j \le n)$$

in $B_s(x)$ and $v_i = 0$ on $\partial B_s(x)$. Then

$$\|v\|_{C^{0,\gamma}(B_s(x))} \le C \text{ for any } 0 < \gamma < 1.$$

Let $V = v + \sum_{j=0}^n v_j$ and consider the function

$$w = \partial^2_{ii}u - V + M_k + Cs^\beta \text{ in } B_s(x). \tag{5.22}$$

It is a solution of $Aw = 0$. We recall the Harnack inequality:

Lemma 5.16. *Let u satisfy*

$$-\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = 0,$$

in $B_\rho(0)$ and assume that

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge \nu|\xi|^2,\ \sum_{i=1}^n \|b_i\|_{L^\infty(B_\rho(0))} \le M,\ 0 \le c(x) \le M,$$

$$|a_{ij}(x) - a_{ij}(y)| \le A_0|x-y|^{\varepsilon_0}, \ (\nu > 0, M > 0, A_0 > 0, \varepsilon_0 > 0).$$

If u is positive in B_ρ then, for any $|x| < \rho/2$, $|y| < \rho$,

$$\frac{1}{\widetilde{C}} u(y) \frac{\rho}{\rho - |y|} \ge u(x) \ge \widetilde{C} u(y) \frac{(\rho - |y|)^{n-1}}{\rho^{n-1}},$$

where C is a positive constant depending only on ν, M, A_0 and ε_0.

Now, since

$$\|V\|_{C^{0,\beta}(B_s(x))} \le \text{const}, \quad V = 0 \text{ on } \partial B_s(x),$$

we can choose C large enough in (5.22) so that w is positive in $B_s(x)$. Applying the Harnack inequality as in the proof of (5.13), we obtain that

$$\partial_{ii}^2 u(x) \ge -M_k - Cs^\beta + \widetilde{C}\delta^{n-1}(M_k - C\delta^{\frac{1}{2}} + Cs^\beta).$$

Making the same choice of δ as before, we get that

$$-M_{k+1} \ge -M_k + CM_k^{2n-1} - C2^{-\beta k},$$

or equivalently

$$M_{k+1} \le M_k - CM_k^{2n-1} + C2^{-\beta k}$$

Then we can establish as before that $M_k \le Ck^{-\varepsilon}$ with $\varepsilon = \frac{1}{2(n-1)}$. This is achieved by induction. For $k = 0, 1$, the results is true if we set the constant is sufficiently large (depending on M_0 and M_1). If we assume that the result is true for any $l \le k$. Then

$$M_{k+1} \le Ck^{-\varepsilon} + C2^{-\beta k}.$$

By changing the constan, we can complete the proof.

5.3. **Free boundary regularity II: main theorems.** To this end, we are able to study the regularity of the free boundary Γ. For convenience, we introduce the following notation as a special case of previously defined $P_1(M, C_0, \lambda_0)$.

Definition 5.17. We say $u \in P_R(M)$ if and only if $u \in C^{1,1}(B_R(0))$ with $\|\partial^2 u\|_{L^\infty(B_R(0))} \le M$, s.t.

$$\begin{cases} \Delta u = 1, & \text{in } \{x : u > 0\} \cap B_R(0), \\ u \ge 0, & 0 \in \Gamma = \partial\{x : u > 0\}. \end{cases}$$

Moreover, if $u \in P_R(M)$ is also convex, we say $u \in P_R^*(M)$.

Remark 5.18. Comparing this definition with $P_1(M, C_0, \lambda_0)$, we can find that there is some little differences. Firstly, we only assume that $\|\partial^2 u\|_{L^\infty(B_R(0))}$ is bounded, istead of assuming that $\|u\|_{C^{1,1}(B_R(0))}$ is bounded. The reason for it is that in the defintion of $P_R(M)$, we further assume that $0 \in \Gamma$, which means that $u(0) = 0$ and $|\nabla u(0)| = 0$. Then $\|\nabla u\|_{L^\infty(B_R(0))} \le RM$ and $\|u\|_{L^\infty(B_R(0))} \le R^2 M$. By using these estimates, we can obtain that $\|u\|_{C^{1,1}(B_R(0))}$ is also uniformly bounded in $B_R(0)$.

Remark 5.19. (a) (Compactness) If $\{u_m\} \subset P_R(M)$, there is a subsequence of it, denoted by $\{u_{m'}\}$, s.t. $u_m \to u_\infty \in P_R(M)$. The convergence is in the sense of weak-* in $C^{1,1}(B_R)$. We shall omit this in the sequel if no confusion would be made. Aslo, one can deduce that $u_{m'} \to u_\infty$ in C^0 norm, that is $\|u_{m'} - u_\infty\|_{L^\infty(B_R)} \to 0$ when $m' \to \infty$.

(b) Given $\{u_m\} \subset P_R(M)$ and $\lambda_m \to 0^+$, define

$$v_m(x) = \lambda_m^{-2} u_m(\lambda_m x) \in P_R(M).$$

Here, we have used the fact that

$$\|\partial^2 v_m\|_{L^\infty(B_R(0))} \le \|u_m\|_{L^\infty(B_{\lambda_m^{-1}R}(0))} \le \|u_m\|_{L^\infty(B_R(0))} \le M.$$

This, together with the property that $0 \in \Gamma(v_m)$, implies that $\|v_m\|_{C^{1,1}(B_R(0))} \leq C$. Then there exists a subsequence $v_{m'} \to v_0 \in P_R(M)$. Moreover, $v_0 \in P^*_\infty(M)$. First of all, we can see that for any $r > 0$, if $m \gg 1$, v_m have definitions since $v_m \in P_{R/\lambda_m}(M)$. Next, it is easy to get that for any fixed $r > 0$, up to a subsequence $v_m \to v_0 \in P_r(M)$. That is

$$v_m \to v_0, \nabla v_m \to \nabla v_0 \text{ uniformly in } C^0(B_r(0)),$$
$$\nabla v_m \to \nabla^2 v_0 \text{ in weak*-topology of } L^\infty(B_r(0)).$$

Then by letting r to ∞, we can obtain that $v_0 \in P_\infty(M)$. To show that $v_0 \in P^*_R(M)$, we see that for any fixed $r > 0$ and $x \in B_r(0)$, by (5.13), then if $m \gg 1$, we have

$$\partial^2_{ii} v_m(x) = \partial^2_{ii} u_m(\lambda_m x) \geq -\frac{C}{|\log(\lambda_m r)|^\varepsilon},$$

for any direction e_i. This means that for m sufficiently large,

$$v_m(x) + \frac{C}{|\log(\lambda_m r)|^\varepsilon}|x|^2 \text{ is convex in } B_r(0).$$

In view of the fact that $v_m \to v_0$ uniformly in $C^0(B_r(0))$ and

$$\frac{C}{|\log(\lambda_m r)|^\varepsilon}|x|^2 \leq \frac{C}{|\log(\lambda_m r)|^\varepsilon} r^2 \to 0 \text{ when } m \to \infty.$$

Then v_0 is convex.

We also recall the following lemma on the convergence of the free boundary as a special case of Lemma 5.13.

Lemma 5.20. *Suppose $\{u_m\} \subset P_R(M)$ and $u_m \to u_0 \in P_R(M)$. Then*

$$\overline{\lim_{m\to\infty}}\, N(u_m) \subset \overline{N(u_0)}, \quad \overline{\lim_{m\to\infty}}\, \Lambda(u_m) \subset \Lambda(u_0).$$

Lemma 5.21. *Given $u \in P_R(M)$ and $\lambda_m \to 0^+$, let*

$$u_{\lambda_m}(x) = \lambda_m^{-2} u(\lambda_m x) \in P_R(M).$$

Then there exists $u_0 \in P^*_R(M)$, s.t. $u_{\lambda_m} \to u_0$.

Lemma 5.22. *Given $u \in P^*_1(M)$ and $\lambda_m \to 0^+$, then, up to a subsequence, $u_{\lambda_m}(x) \to u_0(x)$, i.e. for any fixed $r > 0$,*

$$\begin{aligned} &u_{\lambda_m} \to u_0,\ \nabla u_{\lambda_m} \to \nabla u \textit{ uniformly in } C^0(B_r(0)),\\ &\nabla^2 u_{\lambda_m} \to \nabla^2 u \textit{ in weak*-topology of } L^\infty(B_r(0)), \end{aligned} \tag{5.23}$$

where in some coordinate systems,

$$u_0(x) = \sum_{i=1}^n \frac{a_i}{2} x_i^2 \textit{ with } a_i \geq 0,\ \sum_{i=1}^n a_i = 1 \quad \textit{or} \quad u_0(x) = \frac{1}{2}(x_n^+)^2,$$

with $x_n^+ = \max\{x_n, 0\}$.

Proof. By using standard arguments, (5.23) is easy to verify. Then, $u_0 \in P_\infty(M)$ is true if we let r tends to ∞ in (5.23). By using (b) of Remark 5.19, we can obtain that u_0 is convex. Then $u_0 \in P^*_\infty(M)$. This gives that $\Lambda(u_0)$ is a convex, closed set in $\mathbb{R}^n$. A well known result shows that for any bounded, convex, closed set in $\mathbb{R}^n$, if the interior of it is empty, then it can be contained in a hypersurface in $\mathbb{R}^n$ and if the interior of it is not empty, there is a homeomorphism from it to a unit ball. Here, we will also use similar methods to study the explicit shape of $\Lambda(u_0)$. If $\mathrm{Int}(\Lambda(u_0)) = \emptyset$, then $\Lambda(u_0)$ is contained in a hyperplane. We know that

$$\begin{cases} \Delta u_0 = 1 \text{ a.e. in } \mathbb{R}^n, \\ u_0 \geq 0,\ \left\|\partial^2 u_0\right\|_{L^\infty(\mathbb{R}^n)} \leq M, \end{cases}$$

which implies that u is the classical solution of the equation. Then u is smooth and $\nabla^2 u_0$ is harmonic. Since $\nabla^2 u_0$ is bounded, it can be inferred from Liouville's theorem that $\nabla^2 u$ is a constant. Then u_0 is a quadratic polynomial. Considering the case that $u_0(0)=0$ and $\nabla u_0(0)=0$, we have

$$u_0(x)=\sum_{i=1}^{n}\frac{a_i}{2}x_i^2 \text{ with } a_i\geq 0,\ \sum_{i=1}^{n}a_i=1.$$

Suppose $\mathrm{Int}(\Lambda(u_0))\neq\emptyset$. To characterize $\Lambda(u_0)$, we define the cone of a set $S\subset\mathbb{R}^n$ by

$$C_0(S)=\{x\in\mathbb{R}^n:\exists y\in S \text{ and } a\in(0,\infty),\ \text{s.t. } x=ay\}.$$

We will show that

$$\Lambda\left(u_0\right)=\overline{C_0(\Lambda(u))}.$$

Note that there $\Lambda(u_0)$ is not necessarily equal to $C_0(\Lambda(u))$. Since

$$C_0(\{(x_1,x_2,...,x_n):x_1^2+x_2^2+...+x_{n-1}^2+(x_n+1)^2\leq 1\})=\{(x_1,x_2,...,x_n):x_n<0\}$$

is not closed, we will use the closure of $C_0(\Lambda(u))$ to substitute $C_0(\Lambda(u))$. First of all, we can see that $\overline{\lim}_{m\to\infty}\Lambda(u_m)=\overline{C_0(\Lambda(u))}$. Then by using the fact that $\mathrm{Int}(\Lambda(u_0))\subset\overline{\lim}_{m\to\infty}\Lambda(u_m)$ and $\Lambda(u_0)$ is closed, we have $\Lambda(u_0)\subset\overline{C_0(\Lambda(u))}$. On the other hand, for any $x_0\in\overline{C_0(\Lambda(u))}$, we can choose $R_0>0$ such that $|x_0|<\frac{R_0}{2}$. It can be seen that since $|u_m(0)|=|\nabla u_m(0)|=0$, then $\|\nabla u_m\|_{L^\infty(B_{R_0}(0))}\leq R_0 M$ and $\|u_m\|_{L^\infty(B_{R_0}(0))}\leq R_0^2 M$. In view of the assumption that $x_0\in\overline{C_0(\Lambda(u))}$, we have

$$\lim_{m\to\infty}\mathrm{dist}(x_0,\Lambda(u_m))=0\Leftarrow\mathrm{Int}(\Lambda(u_0))\subset\overline{\lim_{m\to\infty}}\Lambda(u_m).$$

Then there exists $\overline{x}_m$ such that

$$\overline{x}_m\in\Lambda(u_m) \text{ and } \mathrm{dist}(x_0,\Lambda(u_m))=|x_0-\overline{x}_m|.$$

By using Lagrange mean value theorem,

$$|u_m(x_0)|=|u_m(x_0)-u_m(\overline{x}_m)|\leq|x_0-\overline{x}_m|R_0 M\to 0, \tag{5.24}$$

when $m\to\infty$. If $x_0\notin\Lambda(u_0)$, i.e. $u_0(x_0)>0$. Then since $u_m\to u_0$ uniformly in $B_{R_0}(0)$, we can find that if m is sufficiently large $u_m(x_0)>\frac{1}{2}u(x_0)$, this is a contradiction to (5.24). Hence, either $\Lambda(u_0)=\{x\in\mathbb{R}^n:x_n\leq 0\}\triangleq C_*$ or $\Lambda(u_0)\subsetneqq C_*$. Consider $v=\frac{\partial u_0}{\partial x_n}$. In the first case where $\Lambda(u_0)=C_*$,

$$\begin{cases}\Delta v=0 \text{ in } \{x_n>0\},\\ v=0 \text{ on } \{x_n=0\},\\ v>0 \text{ in } \{x_n>0\} \text{ grows linearly at } \infty.\end{cases}$$

The last property is due to the scaling law $u\left(\lambda_m x\right)/\lambda_m^2$ and Lemma 5.6. Therefore, this implies $v=Cx_n$ in $\{x_n>0\}$ and $u_0=\frac{1}{2}(x_n^+)^2$. In fact, since $v=0$ and $\nabla v=0$ on $\{x_n=0\}$, we can extend v to the whole space and the results follow directly. In the second case where $\Lambda(u_0)\neq C_*$, again we have

$$\begin{cases}\Delta v=0 \text{ in } N(u_0),\\ v=0 \text{ on } \Gamma(u_0),\\ v>0 \text{ in } N(u_0).\end{cases}$$

Recall that u takes off from $\Gamma(u)$ quadratically due to Lemma 5.6. Consider the Dirichlet eigenvalue problem of $-\Delta_{\mathbb{S}^{n-1}}$ on $N\left(u_0\right)\cap\mathbb{S}^{n-1}$, i.e.

$$\begin{cases}\Delta_{\mathbb{S}^{n-1}}\phi+\mu\phi=0 \quad \text{on } N(u_0)\cap\mathbb{S}^{n-1}\\ \phi=0 \quad \text{on } \partial N(u_0)\cap\mathbb{S}^{n-1}\end{cases}$$

From $\mathbb{S}^{n-1}_+ \subsetneq N(u_0)\cap\mathbb{S}^{n-1}$, we know that the first eigenvalue $\mu < n-1$, where $(n-1)$ is the first Dirichlet eigenvalue of $\mathbb{S}^{n-1}_+$. Now we look for a positive solution of

$$\Delta h = 0 \quad \text{in } N(u_0), \quad h = 0 \quad \text{on } \partial N(u_0)$$

in the form of $h(x) = r^\alpha(\omega)$, where ϕ is the eigenfunction corresponding to μ and where ω is the coordinate on $\mathbb{S}^{n-1}$. This would give $\alpha \in (0,1)$. It can be proved by maximum principle that $v \geq \varepsilon$ for some small $\varepsilon > 0$ on $N(u_0) \cap B_\delta(0)$ with sufficiently small δ. Hence v is not Lipschitz, which is a contradiction with $u_0 \in C^{1,1}$. □

Remark 5.23. Here is another way to deal with the case for $\Lambda(u_0) \neq C_*$ without considering the eigenvectors of laplacian operator on a sphere. Let us first consider the case that $n = 2$. Firstly, consider the solution for the problem

$$\begin{cases} \Delta f = 0 \text{ in } \{x = re^{i\theta} : 0 \leq r < \infty,\ 0 \leq \theta \leq \alpha\} \\ \quad f = 0 \text{ on } \{x = re^{i\theta} : 0 \leq r < \infty,\ \theta = 0 \text{ or } \alpha\} \end{cases} \tag{5.25}$$

where $\pi < \alpha < 2\pi$. By uisng basic methods in complex analysis, we can obtain that the solution of or (5.25) can be represented by

$$f(re^{i\theta}) = \mathrm{Im}(z^{\frac{\pi}{\alpha}}) = r^{\frac{\pi}{\alpha}} \sin\left(\frac{\pi}{\alpha}\theta\right).$$

Then, we can see that since $\pi < \alpha < 2\pi$, then ∇f will blow up at 0. Assume that

$$\Lambda(u_0) = \{x = re^{i\theta} : r \geq 0,\ 0 \leq \theta \leq \theta_0\},$$

where $0 < \theta_0 < \pi$. Chooe $\theta_1 \ll 1$ such that $0 < \theta_0 + 2\theta_1 < \pi$. Then we can obtain an auxiliary function v such that

$$\begin{cases} \Delta v = 0 \text{ in } \{x = re^{i\theta} : 0 \leq r < \infty,\ \theta_0 + \theta_1 \leq \theta \leq 2\pi - \theta_1\} \\ \quad v = 0 \text{ on } \{x = re^{i\theta} : 0 \leq r < \infty,\ \theta = \theta_0 + \theta_1 \text{ or } 2\pi - \theta_1\} \end{cases}$$

For $e_i = (\cos(\gamma), \sin(\gamma))$ with $\pi < \gamma < \pi + \theta_0$, we can obtain that $\partial_i u_0$ is increasing due to the convexity of u_0. Then we can choose $\rho > 0$ such that $\partial_i u_0$ is positive on

$$G = \{x = re^{i\theta} : r = \rho,\ \theta_0 + \theta_1 \leq \theta \leq 2\pi - \theta_1\}.$$

Since $\partial_i u_0$ is continuous in $\mathbb{R}^n$, we can obtain that there exists $0 < \varepsilon \ll 1$ such taht $(\partial_i u_0 - \varepsilon v)(x) \geq 0$ when $x \in G$. Obviously, for

$$\widetilde{G} = \{x = re^{i\theta} : 0 \leq r \leq \rho,\ \theta_0 + \theta_1 \leq \theta \leq 2\pi - \theta_1\},$$

we can obtain that

$$\Delta(\partial_i u_0 - \varepsilon v) = 0 \text{ in } \widetilde{G} \quad \text{and} \quad \partial_i u_0 - \varepsilon v \geq 0 \text{ on } \partial\widetilde{G}.$$

By using the maximum principle, it can be got that $\partial_i u_0 \geq \varepsilon v$ in $\widetilde{G}$. Then let $x \to 0$, it is a contradiction since $v \to \infty$ by using the properties of auxiliary function defined by (5.25). For the case that $n \geq 3$, we can restrict the cone in $\mathbb{R}^n$ to a two dimensional plane and use the proofs of the case that $n = 2$.

Remark 5.24. In the proof above, we essentially show that if $u \in P_1^*(M)$ and $\hat{\Lambda}(u_0) \neq \emptyset$, then

$$C_0(\Lambda(u)) = \{x : x_n \leq 0\}.$$

Definition 5.25. The minimum diameter of a set $\Omega \subset \mathbb{R}^n$, denoted by $MD(\Omega)$, is defined by

$$\begin{aligned} MD(\Omega) = \inf\{D : \Omega &\text{ can be put between two parallel hyperplanes} \\ &\Pi_1 \text{ and } \Pi_2 \text{ with distance } D\}. \end{aligned}$$

Lemma 5.26. *Let*

$$\delta_{x_0,r}(\Lambda(u))=\frac{MD(\Lambda(u)\cap B_r(x_0))}{2r}.$$

In particular, for $x_0=0$, we define

$$\delta_r(\Lambda(u))=\frac{MD(\Lambda(u)\cap B_r(0))}{2r}.$$

For $\forall\varepsilon>0$ and $\delta>0$, there is $\lambda=\lambda(\varepsilon,\delta)>0$, s.t. if $u\in P_1^*(M)$ and $\delta_1(\Lambda(u))>\varepsilon$, then in a suitable coordinate system, one has

$$\begin{aligned}&(a)\ \Lambda(u)\supset B_\lambda(0)\cap\left\{x:\alpha(x,-e_n)<\frac{\pi}{2}-\delta\right\},\\&(b)\ N(u)\supset B_\lambda(0)\cap\left\{x:\alpha(x,e_n)<\frac{\pi}{2}-\delta\right\},\end{aligned}\tag{5.26}$$

where $\alpha(x,y)$ is the angle between vectors x and y.

Proof. First we show that (a) implies (b). In fact, if (a) is true and if $\exists p\in\Lambda(u)\cap B_\lambda\cap\{x:\alpha(x,e_n)<\frac{\pi}{2}-\delta\}$, then by the convexity of u (and thus $\Lambda(u)$), we know that there is a neighborhood of 0 contained in $\Lambda(u)$, which contradicts with $0\in\partial\{u>0\}$.

We shall prove (a) by contradiction. Suppose that there exists positive ε_0 and δ_0, a sequence of $\lambda_m\to 0^+$ and a sequence of $\{u_m\}$ satisfying $\delta_1(\Lambda(u_m))>\varepsilon_0$, s.t. (a) is not true for u_m on the scale of λ_m for any choice of e_n. That is

$$\text{for any} e\in\mathbb{S}^{n-1},\ B_{\lambda_m}(0)\cap\left\{x:\alpha(x,e)<\frac{\pi}{2}-\delta_0\right\}\ \text{cannot be contained in}\ \Lambda(u_m).$$

Equivalently, by using the scaling arguments,

$$B_1(0)\cap\left\{x:\alpha(x,e)<\frac{\pi}{2}-\delta_0\right\}\ \text{cannot be contained in}\ \Lambda((u_m)_{\lambda_m}).$$

On the other hand, we know that $u_m\to u\in P_1^*(M)$ and $u_{\lambda_m}\to u_0\in P_\infty^*(M)$ up to a subsequence, with $\delta_1(\Lambda(u))\ge\varepsilon_0>0$. Indeed, if $\delta_1(\Lambda(u))<\varepsilon_0$, we can choose a small neighborhood of $\Lambda(u)$, denoted by $\widetilde{\Lambda}(u)$ such that in $\overline{B_1(0)}\backslash\widetilde{\Lambda}(u)$, $u\ge c_1$ for some constant $c_1>0$. Then for $m\gg 1$,

$$\Lambda(u_m)\cap(\overline{B_1(0)}\backslash\widetilde{\Lambda}(u))\neq\emptyset,$$

which is a controdiction if $\widetilde{\Lambda}(u)$ is sufficiently close to $\Lambda(u)$. Hence, $\mathrm{Int}(\Lambda(u))\neq\emptyset$. This implies $C_0(\Lambda(u))=\{x:x_n\le 0\}$ for a suitable choice of coordinate (cf. Remark 5.24). By using Lemma 5.22, we can obtain that $\Lambda(u_0)=C_0(\Lambda(u))=\{x:x_n\le 0\}$ with some coordinate system. Then for the domain $\{x:\alpha(x,-e_n)<\frac{\pi}{2}-\delta_0\}$, it can not be contained in to $\Lambda((u_m)_{\lambda_m})$. Then since $\Lambda((u_m)_{\lambda_m})$ is convex, there exists $\{y_m\}$ such that

$$\{y_m\}\subset\left\{x:\alpha(x,-e_n)<\frac{\pi}{2}-\delta_0\right\},\quad \|y_m\|=\frac{1}{2}$$

and $y_m\in N((u_m)_{\lambda_m})$. Then we can see tha by choosing a subsequence, we can assume that

$$y_m\to y\in B_{\frac{1}{2}}(0)\cap\left\{x:\alpha(x,-e_n)<\frac{\pi}{2}-\frac{\delta_0}{2}\right\}.$$

Since

$$\overline{\lim_{m\to\infty}}N((u_m)_{\lambda_m})\subset\overline{N(u_0)}=\{x:x_n\ge 0\},$$

then $y\in\overline{N(u_0)}$, which is a contradiction. $\square$

Remark 5.27. Let $u\in P_1^*(M)$. If $a\in\Gamma=\partial\{u>0\}$, s.t.

$$\liminf_{r\to 0^+}\frac{L^n(\Lambda(u)\cap B_r(a))}{r^n}\ge\delta_0>0,\tag{5.27}$$

where L^n is the n-dimensional Lebesgue measure, then
$$\frac{MD(\Lambda(u)\cap B_r(0))}{2r}\geq C(\delta_0)>0$$
for r sufficiently small. Therefore, we can replace the condition $\delta_1(\Lambda(u))\geq\varepsilon_0$ by a density hypothesis in the form of (5.27).

To this end, one might ask whether a similar result of Lemma 5.26 holds at points in a neighborhood of 0. We start from the following observation. Assume $\delta_1(\Lambda(u))>\varepsilon$ and $y\in\Gamma(u)=\partial\Lambda(u)$ with $|y|<\frac{1}{4}$. We notice that
$$\begin{aligned}&\delta_{y,\frac{1}{2}}(\Lambda(u))=MD\left(\Lambda(u)\cap B_{\frac{1}{2}}(y)\right)\geq MD\left(\Lambda(u)\cap B_{\frac{1}{4}}(0)\right)\\ &\Rightarrow\delta_{\frac{1}{2}}\left(\Lambda(u)\cap B_{\frac{1}{2}}(y)\right)\geq\frac{1}{4}\delta_{\frac{1}{2}}(\Lambda(u)).\end{aligned}$$
Then
$$\text{Lemma 5.26 can be used to } y \text{ such that } |y|<\frac{1}{4}. \tag{5.28}$$

Corollary 5.28. *$\forall\varepsilon>0$ and $\delta>0$, there is $\lambda=\lambda(\varepsilon,\delta)>0$, s.t. if $u\in P_1^*(M)$ and $\delta_{\frac{1}{4}}(\Lambda(u))>\varepsilon$, then $\forall y\in\Gamma\cap B_{\frac{1}{4}}(0)$, in a suitable coordinate system (depending on y), one has*
(a) $\Lambda(u)\supset B_\lambda(y)\cap\{x:\alpha(x-y,-e_n(y))<\frac{\pi}{2}-\delta\}$,
(b) $N(u)\supset B_\lambda(y)\cap\{x:\alpha(x-y,e_n(y))<\frac{\pi}{2}-\delta\}$,
where $\alpha(x,y)$ is the angle between vectors x and y.

In the notation of Corollary 5.28, one can show by simple geometric argument that
$$\alpha(e_n(0),e_n(y))\leq C\delta.$$
if $|y|<\frac{\lambda}{2}$. Indeed, for sets
$$\begin{aligned}B_\lambda(0)\cap\left\{x:\alpha(x,e_n(0))<\frac{\pi}{2}-\delta\right\}&=B_\lambda(0)\cap\left\{x:\frac{x}{|x|}\cdot e_n(0)>\lambda\sin\delta\right\},\\ B_\lambda(y)\cap\left\{x:\alpha(x-y,-e_n(y))<\frac{\pi}{2}-\delta\right\}&=B_\lambda(y)\cap\left\{x:-\frac{x-y}{|x-y|}\cdot e_n(y)>\lambda\sin\delta\right\},\end{aligned}$$
we can see that there are disjoint. This gives that
$$\alpha(e_n(0),e_n(y))\leq\frac{\pi}{2}-\left(\pi-\delta+\frac{\pi}{2}-\delta-\pi\right)=2\delta.$$
More precisely, it is true if $n=2$. For $n\geq 3$, we can consider the section of planes. In view of the fact that $\alpha(e_n(0),e_n(y))\leq 2\delta$ if $|y|<\frac{\lambda}{2}$. we can see that
(a) $\Lambda(u)\supset B_\lambda(y)\cap\{x:\alpha(x-y,-e_n(0))<\frac{\pi}{2}-3\delta\}$,
(b) $N(u)\supset B_\lambda(y)\cap\{x:\alpha(x-y,e_n(0))<\frac{\pi}{2}-3\delta\}$,
are true for any $|y|<\frac{\lambda}{2}$. Obviously we need to choose δ sufficiently small. Such results give that Γ is actually Lipschitz. More explicitly, we can obtain the following theorem.

Theorem 5.29 (Lipschitz regularity)**.** *Assume the same hypothesis as in Corollary 5.28. Then $\Gamma\cap B_{\frac{1}{4}}(0)$ is the graph of a Lipschitz function f with $\|\nabla f\|_{L^\infty}\leq C\delta$, i.e. in a suitable coordinate, $\Gamma\cap B_{\frac{1}{4}}(0)$ can be represented by $x_n=f(x')$.*

Moreover we can establish

Proposition 5.30 (C^1 regularity)**.** *If $u\in P_1^*(M)$ and $\delta_{\frac{1}{4}}(\Lambda(u))\geq\varepsilon>0$, then there is an appropriate system of coordinates, a positive number $r>0$ and a C^1 function $g(x_1,x_2,...,x_n)$ such that*
$$\Lambda(u)\cap B_r(0)\subset\{x:x_n\leq g(x_1,x_2,...,x_{n-1})\},$$
where r and the module of continuity of ∇g depend only on M and ε.

Proof. Like what have been discussed before, Lemma 5.26 can be applied with respect to any point $y \in \Gamma$, $|y| < \frac{1}{4}$. By Lemma 5.26 with $y = 0$, for any $\delta = 1/m$ there is a system of coordinates (e_i^m) $(i = 1, 2, ..., n)$ such that (5.26) holds with $e_n = e_n^m$. Since $\Lambda(u)$ is convex, any line $x_0 + te_n^m$ intersects $\Lambda(u)$ in a segment lying in $\{t < t_0\}$ and $N(u)$ in segment lying in $\{t > t_0\}$. If $x = \sum x_i e_i^m$, then we can represent $x_0 + t_0 e_n^m$ by

$$x_n = g^m(x') \quad x' = (x_2, ...x_n) \tag{5.29}$$

and $\Lambda(u)$ by $x_n < g^m(x')$.

It is important to notice that if we use any coordinate system centered at 0 in which the direction e_n is near the direction e_n^m, then the remark above regarding $x_0 + te_n^m$ is valid also with respect to $x_0 + te_n$ provided that $|x_0|$ is small enough.

Since (5.26) holds for (e_i^m) in B_{λ_m} (λ_m deqpends on $\delta = 1/m$ and ε; $\lambda_m \to 0$) it follows that for a suitable choice of the e_i^m, $l < i < n-1$, and for a subsequence

$$e_i^m \to e_i^0,$$

$$(e_i^0) = T^m(e_i^m), \quad T^m \text{ an orthonormal matrix,}$$

and

$$T^m \to I, \quad I \text{ the identity matrix.} \tag{5.30}$$

From (5.26) we get

$$\frac{|g^m(x') - g^m(0)|}{|x'|} < \frac{C}{m}, \quad |x'| < \lambda_m,$$

and using (5.30) we easily deduce that

$$\frac{|g^0(x') - g^0(0)|}{|x'|} < \beta_m \quad \text{ if } |x'| < \lambda_m \text{ where } \beta_m \to 0, \tag{5.31}$$

here $x_n = g^0(x)$ is a representation of $\Gamma(u)$ which exists in the coordinate system (e_i^0) with $x = \sum x_i e_i^0$ by the remark following (5.29) and by (5.30). From (5.31) we see that $g^0(x')$ is differentiable at 0 with zero gradient.

In view of (5.28) we can do the same about, each point $y \in \Lambda(u)$ with $|y|$ small enough. Thus there is a system of coordinates $(e_i^{m,y})$ and a limiting one (as $m \to \infty$) $(e_i^{0,y})$, and $\Lambda(u)$ can be represented in a ρ_0 neighborhaod of y (ρ_0 is itidependent of y) in the form

$$x_n \le g_n^{m,y}(x') \quad \text{or} \quad x_n \le g^{0,y}(x')$$

where $x = \Sigma x_i e_i^{m,y}$ or $x = \sum x_i e_i^{0,y}$, respectively. Furthermore,

$$\frac{|g^{0,y}(x') - g^0(0)|}{|x'|} < \beta_m \quad \text{if} |x'| < \lambda_m, \beta_m \to 0. \tag{5.32}$$

The system of coordinates $e_i^{m,y}$ is related to $e_i^{0,y}$ by

$$e_i^{m,y} = T^{m,y}(e_i^{0,y}),$$

where $T^{m,y} \to I$ uniformly with respect to y. Notice also that $(e_i^{0,y})$ is related to (e_i^0) by

$$(e_i^{0,y}) = T^{0,y}(e_i^0) \tag{5.33}$$

and

$$T^{0,y} \to I \quad \text{if} y \to 0 \tag{5.34}$$

(with appropriate choice of $e_i^{0,y}$, $1 < i < n-1$); this follows, in the same way as (5.30), from the assertion (5.26) for $y = 0$ and for y near 0.

We can rewrite the inequality (5.32) in terms of the systems of coordinates (e_i^0); taking into account the rotation (5.33) and (5.34), we obtain from (5.32)

$$g^0(y+h) - g^0(y) = h \cdot c(y) + h \cdot o(1),$$

where $o(1) \to 0$ as $|h| \to 0$, uniformly with respect to y, and $|c(y)| < C$. This implies that $g^0 \in C^1$. $\square$

In the general case, we assume $u \in P_1(M)$ satisfies $\delta_\rho(\Lambda(u)) \geq \varepsilon_0 > 0$. Note that there is no convexity assumption on u. By a blow-up argument, we can show in a suitable coordinate,

$$\Lambda(u) \supset B_{\frac{\rho\lambda}{2}}(0) \cap \{x : -x_n \geq 2\rho\lambda\delta\}$$

for sufficiently small ρ. One can also show a similar result for $N(u)$ using non-degeneracy.

A critical issue in this case is that, the coordinate on each scale ρ is "scale-dependent". This means when it comes to a finer scale, i.e. when ρ becomes smaller, the coordinate has to (and it has the freedom) rotate to fit the free boundary. We need to control this rotation from scale to scale, so that the set $B_{\frac{\rho\lambda}{2}}(0) \cap \{x : -x_n \geq 2\rho\lambda\delta\}$ can "converge" in a particular sense. Such bounds on the rotation come from the convexity estimates on the solution. Recall that we only have logarithmic decay of pure second derivative when it approaches the free boundary (cf. Theorem 5.11 and Remark 5.12). This would give a logarithmically decaying sequence of the rotation angles when we come to finer and finer scales, which is not convergent. Therefore, we need to improve the convexity to obtain Hölder decay, i.e. the rotation angle behaves like r^α, where r is the size of scale. Now the sequence is summable!

General case: Now we consider the general case $u \in P_1(M)$, hence removing the convexity assumption.

Lemma 5.31. *Given $\epsilon > 0$ and $0 < \delta < \frac{1}{4}$, there is a $\rho_0 = \rho_0(\epsilon, \delta) > 0$ such that for any $u \in P_1(M)$, if $\delta_\rho(\Lambda(u)) \geq \epsilon$ with $0 < \rho < \rho_0$, then in a suitable system of coordinates*

$$\begin{cases} \Lambda(u) \supset B_{\rho\lambda/2}(0) \cap \{x : -x_n \geq 2\rho\lambda\delta\}, \\ N(u) \supset B_{\rho\lambda/2}(0) \cap \{x : x_n \geq 2\rho\lambda\delta\}, \end{cases} \tag{5.35}$$

where $\lambda = \lambda(\epsilon, \delta)$ is as defined in Lemma 5.26.

Proof. If one of the assertions is not true, then for given $\epsilon > 0$ and $0 < \delta < \frac{1}{4}$ there are sequences $\{u^{(m)}\} \subset P_1(M)$ and $\rho_m > 0$, where $\rho_m \to 0^+$ such that (5.35) is not valid for any system of coordinates. That is, for any $m \in \mathbb{N}_+$, there exists $\widetilde{\rho}_m > 0$ such that $0 < \widetilde{\rho}_m < \rho_m$, $\delta_{\widetilde{\rho}_m}(u^{(m)}) \geq \varepsilon$ and for any coordinate system, $B_{\widetilde{\rho}_m\lambda/2}(0) \cap \{x : -x_n \geq 2\widetilde{\rho}_m\lambda\delta\}$ cannot be contained into $\Lambda(u^{(m)})$, or equivalently, for any coordinate system,

$$B_{\lambda/2}(0) \cap \{x : -x_n \geq 2\lambda\delta\} \text{ cannot be contained into } \Lambda(u^{(m)}_{\widetilde{\rho}_m}).$$

For simplicity, we do not nee to distinguish between $\widetilde{\rho}_m$ and ρ_m. Then in the followsing calculations, we will set $\widetilde{\rho}_m$ by ρ_m. By taking a subsequence, if necessary, we have

$$u^{(m)}_{\rho_m} \to u \in P_1^*(M), \quad \Lambda(u) \supset \varlimsup_{m\to\infty} \Lambda(u^{(m)}_{\rho_m}), \quad \overline{N(u)} \supset \varlimsup_{m\to\infty} N(u^{(m)}_{\rho_m}). \tag{5.36}$$

Since λ is chosen as in Lemma 5.26, we can obtain that changing coodinate systems if necessary, $B_\lambda(0) \cap \{x : \alpha(x, -e_n) < \frac{\pi}{2} - \delta\} \subset \Lambda(u)$. By using the fact that $\sin\delta \leq 4\delta$, we get that

$$B_{\lambda/2}(0) \cap \{x : -x_n \geq 2\lambda\delta\} \subset\subset B_\lambda(0) \cap \left\{x : \alpha(x, -e_n) < \frac{\pi}{2} - \delta\right\} \subset \Lambda(u).$$

In view of the assumptions of $u^{(m)}$, it is obvious that $B_{\lambda/2}(0) \cap \{x : -x_n \geq 2\lambda\delta\}$ cannot be contained into $\Lambda(u^{(m)}_{\rho_m})$. Then we can choose $\{y_m\}$ such that

$$\{y_m\} \subset B_{\lambda/2}(0) \cap \{x : -x_n \geq 2\lambda\delta\}$$

and $y_m \in N(u^{(m)}_{\rho_m})$. Owing to the fact that $\{y_m\}$ is bounded, we can assume that $\lim_{m\to\infty} y_m = y$ and

$$y \in B_\lambda(0) \cap \left\{x : \alpha(x, -e_n) < \frac{\pi}{2} - \delta - \delta'\right\} \cap \{x : -x_n \geq 2\lambda\delta\} \subset \mathrm{Int}(\Lambda(u))$$

with $\delta' \ll 1$. Therefore, by using (5.36), it is easy to deduce that $y \in \overline{N(u)}$. However, taking the choice of y into consideration, we can see that $y \in \mathrm{Int}(\Lambda(u))$, which is a contradiction. □

Note that from (5.35) one has

$$\delta_{\rho\lambda/2}(\Lambda(u)) \geq \frac{\frac{\lambda\rho}{2} - 2\rho\lambda\delta}{2 \cdot \frac{\lambda\rho}{2}} = \frac{1}{2} - 2\delta > \frac{1}{4},$$

if $\delta < \frac{1}{8}$. Hence one may apply (5.35) again to replace $\rho\lambda/2$ with ρ, if $\epsilon \leq \frac{1}{8}$ in Lemma 5.31, which may be assumed.

Lemma 5.32. *Let $\epsilon < \frac{1}{8}$, $\delta < \frac{1}{8}$, $\lambda = \lambda(\epsilon, \delta)$, $\rho_0 = \rho_0(\epsilon, \delta)$ as what have been given in Lemma 5.26. If $u \in P_1(M)$, $\delta_\rho(\Lambda(u)) \geq \epsilon$, $\rho < \rho_0$, then there exists a sequence of coordinates $\{e^{(k)}_i : 1 \leq i \leq n\}$ such that*

$$\begin{cases} \Lambda(u) \supset B_{\rho(\frac{\lambda}{2})^k}(0) \cap \{x : -x^{(k)}_n \geq \rho(\lambda/2)^k 4\delta\}, \\ N(u) \supset B_{\rho(\frac{\lambda}{2})^k}(0) \cap \{x : x^{(k)}_n \geq \rho(\lambda/2)^k 4\delta\}, \end{cases} \tag{5.37}$$

$\forall k \in \mathbb{Z}_+$.

Proof. Applying Lemma 5.31 iteratively gives the claim. □

Lemma 5.33. *Let $\epsilon < \frac{1}{8}$, $\delta < \frac{1}{8}$, $\lambda = \lambda(\epsilon, \delta)$, $\rho_0 = \rho_0(\epsilon, \delta)$ as what have been given in Lemma 5.26. If $u \in P_1(M)$, then there is $\mu \in (0, 1)$ such that*

$$\partial^2_{ii} u \geq -M\mu^{k-1} \quad \textit{in } N(u) \cap B_{\rho(\lambda/2)^k}(0).$$

Proof. We prove by induction. The base case $k = 1$ follows from the uniform second derivative estimate. Consider $\Delta h = 0$ in B_1 such that

$$h = \begin{cases} 0 & \text{on } \partial B_1(0) \cap \left\{x : x^{(k)}_n < -\dfrac{1}{2}\right\}, \\ -1 & \text{on } \partial B_1(0) \cap \left\{x : x^{(k)}_n \geq -\dfrac{1}{2}\right\}. \end{cases}$$

Let $\mu = -\inf_{B_{\frac{1}{2}}} h$ and note $0 < \mu < 1$ by maximal principle. We have the inductive hypothesis

$$\partial^2_{ii} u(x) \geq -M\mu^{k-1} \quad \text{in } N(u) \cap B_{\rho(\lambda/2)^k}(0).$$

Consider the auxiliary function

$$w(x) = M\mu^{k-1} h\left(\frac{x}{\rho(\lambda/2)^k}\right) \quad \text{in } N(u) \cap B_{\rho(\lambda/2)^k}(0).$$

Then both w and $\partial^2_{ii} u$ are harmonic in $N(u) \cap B_{\rho(\lambda/2)^k}(0)$. For any $x \in N(u) \cap \partial B_{\rho(\lambda/2)^k}(0)$, by using (5.37) we can obtain that

$$x \notin B_{\rho(\lambda/2)^k}(0) \cap \{x : -x^{(k)}_n \geq \rho(\lambda/2)^k 4\delta\}.$$

That is

$$x \in B_{\rho(\lambda/2)^k}(0) \cap \{x : x^{(k)}_n > -\rho(\lambda/2)^k 4\delta\}.$$

Since $\delta < \frac{1}{8}$, we have

$$B_{\rho(\lambda/2)^k}(0) \cap \{x : x^{(k)}_n \geq -\rho(\lambda/2)^k 4\delta\} \subset B_{\rho(\lambda/2)^k}(0) \cap \left\{x : x^{(k)}_n \geq -\frac{1}{2}\rho(\lambda/2)^k\right\}.$$

Then we can obtain that $x \in \partial B_{\rho(\lambda/2)^k}(0) \cap \{x : x_n^{(k)} \geq -\frac{1}{2}\rho(\lambda/2)^k\}$. Then $w(x) = -M\mu^{k-1}$ and $\partial^2_{ii}u(x) \geq w(x)$. For $x \in \partial N(u) \cap B_{\rho(\lambda/2)^k}(0)$, by using (5.14) we have $\partial^2_{ii}u(x) \geq w(x)$. Hence

$$\partial^2_{ii}u \geq w(x) \geq M\mu^{k-1}h\left(\frac{x}{\rho(\lambda/2)^k}\right) \geq -M\mu^k$$

in $B_{\rho(\lambda/2)^{k+1}}(0) \cap N(u)$ by the maximum principle. □

Lemma 5.34. *If $u \in C^{1,1}(B_R(0))$, $u \geq 0$, $u(0) = 0, \partial^2_{ii}u \geq -\tau$ on $\overline{0x_0}$ for any direction $e_i \in \mathbb{S}^{n-1}$, where $|x_0| < R$, $\tau \geq 0$, then for $0 < t < 1$*

$$u(tx_0) \leq \frac{1}{2}|x_0|^2\tau + u(x_0).$$

Proof. Let $u(t_0x_0) = \max_{0\leq t\leq 1} u(t_0)$. We may assume that $0 < t_0 < 1$, as otherwise the claim becomes trivial. So $(\partial_i u)(t_0x_0) = 0$, where $e_i = \frac{x_0}{|x_0|}$. Hence by Taylor expansion

$$\begin{aligned} u(x_0) &= u(t_0x_0) + \int_0^{(1-t_0)|x_0|}\int_0^{\widetilde{t}} \partial^2_{ii}u\left(t_0x_0 + t\frac{x_0}{|x_0|}\right)dtd\widetilde{t} \\ &\geq u(t_0x_0) - \tau(1-t_0)^2|x_0|^2/2 \\ &\geq u(t_0x_0) - \tau|x_0|^2/2, \end{aligned}$$

and rearranging gives the claim. □

Lemma 5.35. *Assume that $\epsilon < \frac{1}{8}$, $\delta < \frac{1}{8}$, $\lambda = \lambda(\epsilon,\delta)$, $\rho_0 = \rho_0(\epsilon,\delta)$ as what have been given in Lemma 5.26 and μ is given by Lemma 5.33. Let $\theta_k = (3Mn)^{\frac{1}{2}}\mu^{(k-2)/2}\rho(\lambda/2)^{k-1}$, $u \in P_1(M)$ and $x \in N(u) \cap (\overline{B}_{\rho(\lambda/2)^k}(0)\backslash B_{\rho(\lambda/2)^{k+1}}(0))$. Then there is a point y such that $|x-y| \leq \theta_k$ and the points sy are in $N(u)$ for all s with $s \geq 1$ and $|sy| \leq \rho(\lambda/2)^{k-1}$.*

By using translations, we can change 0 in Lemma 5.35 to other points. That is, we can obtain the results as follows.

Corollary 5.36. *If $u \in C^{1,1}(B_R(x_0))$, $u \geq 0$, $u(x_0) = 0, \partial^2_{ii}u \geq -\tau$ on $\overline{x_0x_1}$ for any direction $e_i \in \mathbb{S}^{n-1}$, where $|x_1 - x_0| < R$, $\tau \geq 0$, then for $0 < t < 1$*

$$u(x_0 + t(x_1 - x_0)) \leq \frac{1}{2}|x_1 - x_0|^2\tau + u(x_0).$$

Proof. By using Lemma 5.6 and not that the parameter λ in that lemma is actually 1 here, we can obtain that there is y such that $|x-y| \leq \theta_k$ and $u(y) \geq \frac{\theta_k^2}{2n}$. Let s be as above, that is $s \geq 1$ and $|sy| \leq \rho(\lambda/2)^{k-1}$. From the arguments above, we can see that $y \in N(u)$ and by using the fact that $N(u)$ is open, $cy \in N(u)$ if $|c-1| \ll 1$. We can choose

$$c_0 = \inf\{c \leq 1 : ly \in N(u) \text{ for any } c \leq l \leq 1\}$$

and get that $c_0y \in \Gamma(u)$. Indeed if $c_0y \in N(u)$, it is a contradiction to the property that $N(u)$ is open and if $c_0y \in \mathrm{Int}(\Lambda(u))$ the arguments are almost the same. Take $x_0 = sy$, with $t = \frac{1-c_0}{s-c_0}$ in Lemma 5.34. Note that by Corollary 5.36, then $\partial^2_{ii}u \geq -M\mu^{k-2}$ on $\overline{(c_0y)sy}$ since $\overline{(c_0y)sy} \subset N(u) \cap B_{\rho(\lambda/2)^k}(0)$. Then we can deduce from Corollary 5.36 that

$$\begin{aligned} \frac{\theta_k^2}{2n} &\leq u(y) = u\left(c_0y + t(sy - c_0y)\right) \\ &\leq \frac{1}{2}|(s-c_0)y|^2M\mu^{k-2} + u(sy) \leq \frac{1}{2}|sy|^2M\mu^{k-2} + u(sy). \end{aligned}$$

By the definition of θ_k, it can be seen that $u(sy) > 0$, i.e. $sy \in N(u)$. □

Note that in the previous lemma we have

$$x \in N(u) \cap (\overline{B}_{\rho(\lambda/2)^k}(0) \backslash B_{\rho(\lambda/2)^{k+1}}(0)), \quad |x-y| \leq \theta_k = \sqrt{C(n)M}\mu^{\frac{k-2}{2}}\rho(\lambda/2)^{k-1}.$$

Then

$$\alpha(x,y) \leq C(n,M)\frac{\mu^{\frac{k-2}{2}}}{\lambda^2} \to 0, \quad \text{as} \quad k \to \infty.$$

Lemma 5.37. *Under the assumptions of Lemma 5.32 there exists a $k_0 = k_0(\epsilon,\delta)$ and $C = C(\epsilon,\delta) > 0$ such that in a suitable system of coordinates*

$$\begin{cases} \Lambda(u) \supset B_{\rho(\frac{\lambda}{2})^{k_0}}(0) \cap \{x : \alpha(x,-e_n) < \pi/2 - C\delta\}, \\ N(u) \supset B_{\rho(\frac{\lambda}{2})^{k_0}}(0) \cap \{x : \alpha(x,e_n) < \pi/2 - C\delta\}. \end{cases} \tag{5.38}$$

Proof. Take e_n as $e_n^{(k)}$ with $k = k_0 - 1$ as in Lemma 5.32. The logic here is that k_0 is a number to be determined. We assume that the properties given by (5.38) is not choose and obtain some results from it. Next, we choose k_0 to get a contradiction. Suppose that the first statement in (5.38) is not true, then there is a point

$$\overline{x} \in B_{\rho(\lambda/2)^{k_0}}(0) \cap \{x : \alpha(x, -e_n^{(k_0-1)}) < \pi/2 - C\delta\} \cap N(u).$$

More precisely, we can choose $j \geq k_0$ and denote $\overline{x} = x_j$ such that

$$x_j \in N(u) \cap (B_{\rho(\lambda/2)^j}(0) \backslash B_{\rho(\lambda/2)^{j+1}}(0))$$

and

$$\alpha\left(x_j, -e_n^{(k_0-1)}\right) < \pi/2 - C\delta.$$

Here we can assume that $j > k_0$ for otherwise the result is trivial if we take smaller k_0. By Lemma 5.35 there exists a y_j and a segment $s_j y_j$ in $N(u)$, where $s_j \geq 1$, $|s_j y_j| \leq \rho(\lambda/2)^{j-1}$. Taking $x_{j-1} = s_j y_j$ so that

$$|s_j y_j| = \rho(\lambda/2)^{j-1}.$$

Then $x_{j-1} \in N(u) \cap (\overline{B}_{\rho(\lambda/2)^{j-1}}(0) \backslash B_{\rho(\lambda/2)^j}(0))$ and

$$\alpha(x_j, x_{j-1}) = \alpha(x_j, y_j) \leq \frac{C}{\lambda^2}\mu^{\frac{j-2}{2}}.$$

Moreover since the lenghths of vectors do not have influence on the angle of two vectors, we can assume that $x_{j-1} \in N(u) \cap \partial\overline{B}_{\rho(\lambda/2)^{j-1}}(0)$. Repeating this procedure, we find a sequence of points $\{x_i\}$, $k_0 \leq i < j$, such that

$$x_i \in N(u) \cap \partial B_{\rho(\lambda/2)^i}(0) \text{ and } \alpha(x_i, x_{i-1}) \leq \frac{C}{\lambda^2}\mu^{(i-2)/2}.$$

Thus, $x = x_{k_0}$ satisfies

$$x \in N(u) \cap \partial B_{\rho(\lambda/2)^{k_0}}(0) \text{ and } \alpha(x, x_j) \leq \widetilde{C}(\lambda,\mu)\mu^{\frac{k_0-2}{2}},$$

where we have use the summation that

$$\alpha(x_{k_0}, x_j) \leq \sum_{i=j}^{k_0-1} \alpha(x_i, x_{i+1}) \leq \sum_{i=j}^{k_0-1} \frac{C}{\lambda^2}\mu^{(i-2)/2} \leq \widetilde{C}(\lambda,\mu)\mu^{\frac{k_0-2}{2}}.$$

We let $C \geq 5$ in the first statement in (5.38) and k_0 sufficiently large, then

$$\alpha(x, -e_n^{(k_0-1)}) < \frac{\pi}{2} - 4\delta,$$

which is a contradiction to Lemma 5.32. Indeed, Lemma 5.32 implies that

$$\begin{aligned} x &\in \partial B_{\rho(\lambda/2)^{k_0}}(0) \cap \left\{x : \alpha(x, -e_n^{(k_0-1)}) < \frac{\pi}{2} - 4\delta\right\} \\ &\subset B_{\rho(\lambda/2)^{k_0-1}(0)} \cap \left\{x : x_n^{(k_0-1)} \leq -\rho(\lambda/2)^{k_0-1}4\delta\right\} \subset \Lambda(u), \end{aligned}$$

where we have used the fact that $\sin(4\delta) \le 4\delta$ for $\delta \ll 1$. This is a contradiction that $x \in N(u)$. $\square$

Finally, we note that the first statement in (5.38) implies the latter by a blow-up argument.

Remark 5.38. Note that Lemma implies the free boundary regularity for the general case as in the convex case, cf. Theorem 5.29.

To make things more explicity, we will give the definition of regular points.

Definition 5.39. Let $u \in P_1(M)$ and let $N(u)$, $\Lambda(u)$ and $\Gamma(u)$ be defined as before. $x_0 \in \Gamma(u)$ is said to be in the regular set of $\Gamma(u)$, denoted by $x_0 \in \operatorname{reg} u$ if there exists $\varepsilon > 0$, such that for any $r > 0$, there exists $0 < \rho < r$, satisfying $\delta_\rho(\Lambda(u)) \ge \varepsilon$.

A simple observation shows that if $0 \in \Gamma(u) \cap \operatorname{reg} u$ and $\delta_\rho(\Lambda(u)) \ge \varepsilon > 0$ with $\rho > 0$, then for any $|y| < \frac{\rho}{4}$,

$$\begin{aligned} &\delta_{y,\frac{\rho}{2}}(\Lambda(u)) = \frac{MD\left(\Lambda(u) \cap B_{\frac{\rho}{2}}(y)\right)}{\rho} \ge \frac{MD\left(\Lambda(u) \cap B_{\frac{\rho}{4}}(0)\right)}{\rho} \\ &\Rightarrow \delta_{y,\frac{\rho}{2}}(\Lambda(u)) \ge \frac{1}{4}\delta_{\frac{\rho}{2}}(\Lambda(u)) \ge \frac{\varepsilon}{4}. \end{aligned} \tag{5.39}$$

Now we will show that actually $\operatorname{reg} u$ is relatively open in $\Gamma(u)$. In Lemma 5.31, 5.32, 5.33 and 5.38, we establish results for 0. Obviously, if $0 \in \operatorname{reg} u$, we can see that the conditions of Lemma 5.31 are satisfied. For general case, i.e. $0 \ne x \in \Gamma(u)$, we can do the same procedure. However the number ρ_0 is not the same for points $x \ne 0$. It is important to reveal the relations of ρ_0 between different $x \in \Gamma(u)$. For $x_0 \in \Gamma(u)$, there exists $r > 0$ such that $B_r(x_0) \subset B_1(x_0)$. we can consider the function $v(x) = r^{-2}u(x_0 + rx)$. It is easy to see that $v \in P_1(M)$ and $0 \in \Gamma(v)$, then we can see that here if $\varepsilon > 0$ and $\delta > 0$ are given, ρ_0 is the same as Lemma 5.31. It can be see that if $x \in [(\Lambda(u) - x_0)/r] \cap B_\rho(0)$, we have $rx + x_0 \in \Lambda(u) \cap B_{r\rho}(x_0)$ and

$$\begin{aligned} \delta_\rho(\Lambda(v)) &= \frac{MD(\Lambda(v) \cap B_\rho(0))}{2\rho} = \frac{MD((\Lambda(u) - x_0)/r \cap B_\rho(0))}{2\rho} \\ &= \frac{MD([(\Lambda(u) - x_0)/r] \cap B_\rho(0))}{2\rho} = \frac{MD(\Lambda(u) \cap B_{r\rho}(x_0))}{2r\rho} = \delta_{x_0, r\rho}(\Lambda(u)). \end{aligned}$$

Then the corresponding ρ_0 for u at x_0 is $r\rho_0$. This together with (5.39) implies for any y such that $|y| < \frac{\rho}{2}$, $y \in \operatorname{reg} u$.

Remark 5.40. One can see further that we have actually proved that the free boundary is C^1 under the same assumptions. The proof is almost the same as what has been given in Proposition 5.30.

Remark 5.41. One can generalize the results above to the elliptic operator to (5.6) with conditions (5.7). That is

$$A = -\sum_{i,j=1}^n a_{ij}(x)\partial_{ij}^2 + \sum_{i=1}^n b_i(x)\partial_i + c(x),$$

with

$$\begin{aligned} &\sum_{i,j} \|a_{ij}\|_{C^{2,\alpha}(\Omega)} + \sum_i \|b_i\|_{C^{0,\alpha}(\Omega)} + \|c\|_{C^{0,\alpha}(\Omega)} \le \mu_1, \ c(x) \ge 0 \\ &\sum_{i,j} a_{ij}(x)\xi_i\xi_j \ge \nu|\xi|^2, \quad \forall x \in \Omega, \ \xi \in \mathbb{R}^n \ (\nu > 0). \end{aligned}$$

To make things more explicit, we have the following results.

Definition 5.42. We say $u \in P_R(M, C_0, \lambda_0, \mu_0, \nu_0)$ if and only if $u \in C^{1,1}(B_R(0))$ such that

$$\begin{cases} -Au = f \text{ in } N(u) = \{u > 0\} \cap B_R(0) \text{ for some } f \geq \lambda_0 > 0 \text{ with } \|f\|_{C^{0,\alpha}(B_R(0))} \leq C_0, \\ u \geq 0 \text{ in } B_R(0),\ 0 \in \partial N(u) = \Gamma(u),\ \|u\|_{C^{1,1}(B_R(0))} \leq M, \\ \sum_{i,j} \|a_{ij}\|_{C^{2,\alpha}(B_R(0))} + \sum_i \|b_i\|_{C^{0,\alpha}(B_R(0))} + \|c\|_{C^{0,\alpha}(B_R(0))} \leq \mu_0,\ c(x) \geq 0, \\ \sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \nu_0|\xi|^2, \quad \forall x \in B_R(0),\ \xi = (\xi_i)_{i=1}^n \in \mathbb{R}^n\ (\nu_0 > 0). \end{cases}$$

Moreover, if $u \in P_R(M, C_0, \lambda_0, \mu_0, \nu_0)$ is also convex, we say $u \in P_R^*(M, C_0, \lambda_0, \mu_0, \nu_0)$.

Lemma 5.43. *Suppose $\{u_m\} \subset P_R(M, C_0, \lambda_0, \mu_0, \nu_0)$ and $u_m \to u_0 \in P_R(M, C_0, \lambda_0, \mu_0, \nu_0)$. Then*

$$\overline{\lim_{m\to\infty}}\, N(u_m) \subset \overline{N(u_0)}, \quad \overline{\lim_{m\to\infty}}\, \Lambda(u_m) \subset \Lambda(u_0).$$

Lemma 5.44. *Given $u \in P_R(M, C_0, \lambda_0, \mu_0, \nu_0)$ and $\lambda_m \to 0^+$, let*

$$u_{\lambda_m}(x) = \lambda_m^{-2} u(\lambda_m x) \in P_R(M, C_0, \lambda_0, \mu_0, \nu_0).$$

Then there exists $u_0 \in P_R^*(M, C_0, \lambda_0, \mu_0, \nu_0)$, s.t. $u_{\lambda_m} \to u_0$.

Lemma 5.45. *Given $u \in P_R^*(M, C_0, \lambda_0, \mu_0, \nu_0)$ and $\lambda_m \to 0^+$, then, up to a subsequence, $u_{\lambda_m}(x) \to u_0(x)$, i.e. for any fixed $r > 0$,*

$$\begin{aligned} &u_{\lambda_m} \to u_0,\ \nabla u_{\lambda_m} \to \nabla u \textit{ uniformly in } C^0(B_r(0)), \\ &\nabla^2 u_{\lambda_m} \to \nabla^2 u \textit{ in weak*-topology of } L^\infty(B_r(0)), \end{aligned} \tag{5.40}$$

where in some coordinate systems,

$$u_0(x) = \sum_{i=1}^n \frac{a_i}{2} x_i^2 \textit{ with } a_i \geq 0,\ \sum_{i=1}^n a_i = 1 \quad \textit{or} \quad u_0(x) = \frac{1}{2}(x_n^+)^2,$$

with $x_n^+ = \max\{x_n, 0\}$.

Lemma 5.46. *$\forall \varepsilon > 0$ and $\delta > 0$, there is $\lambda = \lambda(\varepsilon, \delta) > 0$, s.t. if $u \in P_1^*(M, C_0, \lambda_0, \mu_0, \nu_0)$ and $\delta_1(\Lambda(u)) > \varepsilon$, then in a suitable coordinate system, one has*

$$\begin{aligned} &(a)\ \Lambda(u) \supset B_\lambda(0) \cap \left\{x : \alpha(x, -e_n) < \frac{\pi}{2} - \delta\right\}, \\ &(b)\ N(u) \supset B_\lambda(0) \cap \left\{x : \alpha(x, e_n) < \frac{\pi}{2} - \delta\right\}, \end{aligned} \tag{5.41}$$

where $\alpha(x, y)$ is the angle between vectors x and y.

Corollary 5.47. *$\forall \varepsilon > 0$ and $\delta > 0$, there is $\lambda = \lambda(\varepsilon, \delta) > 0$, s.t. if $u \in P_1^*(M, C_0, \lambda_0, \mu_0, \nu_0)$ and $\delta_{\frac{1}{4}}(\Lambda(u)) > \varepsilon$, then $\forall y \in \Gamma \cap B_{\frac{1}{4}}(0)$, in a suitable coordinate system (depending on y), one has*
(a) $\Lambda(u) \supset B_\lambda(y) \cap \{x : \alpha(x - y, -e_n(y)) < \frac{\pi}{2} - \delta\}$,
(b) $N(u) \supset B_\lambda(y) \cap \{x : \alpha(x - y, e_n(y)) < \frac{\pi}{2} - \delta\}$,
where $\alpha(x, y)$ is the angle between vectors x and y.

Lemma 5.48. *Given $\epsilon > 0$ and $0 < \delta < \frac{1}{4}$, there is a $\rho_0 = \rho_0(\epsilon, \delta) > 0$ such that for any $u \in P_1(M, C_0, \lambda_0, \mu_0, \nu_0)$, if $\delta_\rho(\Lambda(u)) \geq \epsilon$ with $0 < \rho < \rho_0$, then in a suitable system of coordinates*

$$\begin{cases} \Lambda(u) \supset B_{\rho\lambda/2}(0) \cap \{x : -x_n \geq 2\rho\lambda\delta\}, \\ N(u) \supset B_{\rho\lambda/2}(0) \cap \{x : x_n \geq 2\rho\lambda\delta\}, \end{cases} \tag{5.42}$$

where $\lambda = \lambda(\epsilon, \delta)$ is as defined in Lemma 5.46.

Lemma 5.49. *Let $\epsilon < \frac{1}{8}$, $\delta < \frac{1}{8}$, $\lambda = \lambda(\epsilon, \delta)$, $\rho_0 = \rho_0(\epsilon, \delta)$ as what have been given in Lemma 5.26. If $u \in P_1(M, C_0, \lambda_0, \mu_0, \nu_0)$, $\delta_\rho(\Lambda(u)) \geq \epsilon$, $\rho < \rho_0$, then there exists a sequence of coordinates $\{e_i^{(k)} : 1 \leq i \leq n\}$ such that*

$$\begin{cases} \Lambda(u) \supset B_{\rho(\frac{\lambda}{2})^k}(0) \cap \{x : -x_n^{(k)} \geq \rho(\lambda/2)^k 4\delta\}, \\ N(u) \supset B_{\rho(\frac{\lambda}{2})^k}(0) \cap \{x : x_n^{(k)} \geq \rho(\lambda/2)^k 4\delta\}, \end{cases} \tag{5.43}$$

$\forall k \in \mathbb{Z}_+$.

Lemma 5.50. *Let $\epsilon < \frac{1}{8}$, $\delta < \frac{1}{8}$, $\lambda = \lambda(\epsilon, \delta)$, $\rho_0 = \rho_0(\epsilon, \delta)$ as what have been given in Lemma 5.46. If $u \in P_1(M, C_0, \lambda_0, \mu_0, \nu_0)$, then there is $\mu \in (0,1)$ and a constant $C_1 > 0$ such that*

$$\partial_{ii}^2 u \geq -C_1 \mu^{k-1} \quad \text{in } N(u) \cap B_{\rho(\lambda/2)^k}(0).$$

Lemma 5.51. *Under the assumptions of Lemma 5.49 there exists a $k_0 = k_0(\epsilon, \delta)$ and $C = C(\epsilon, \delta) > 0$ such that in a suitable system of coordinates*

$$\begin{cases} \Lambda(u) \supset B_{\rho(\frac{\lambda}{2})^{k_0}}(0) \cap \{x : \alpha(x, -e_n) < \pi/2 - C\delta\}, \\ N(u) \supset B_{\rho(\frac{\lambda}{2})^{k_0}}(0) \cap \{x : \alpha(x, e_n) < \pi/2 - C\delta\}. \end{cases} \tag{5.44}$$

In fact by some simple observation, we only need to show Lemma 5.50 since others are standard generalization without any difficulty. Indeed, in the procedure of blow-up, the coeffients of L, a, b, c and data f may converge to constants. This observation makes most of the proof above are still valid. Indeed, if

$$-\sum_{i,j=1}^n a_{ij}(x)\partial_{ij}^2 u(x) + \sum_{i=1}^n b_i(x)\partial_i u(x) + c(x)u(x) = f(x) \text{ in } B_1(0),$$

we have, for $u^\lambda(x) = \lambda^{-2}u(\lambda x)$ with $\lambda \to 0$,

$$-\sum_{i,j=1}^n a_{ij}(\lambda x)\partial_{ij}^2 u^\lambda(x) + \sum_{i=1}^n \lambda b_i(\lambda x)\partial_i u^\lambda(x) + \lambda c(\lambda x)u^\lambda(x) = f(\lambda x) \text{ in } B_{1/\lambda}(0).$$

This shows that when we use the compactness methods, we $b^\lambda(x) = \lambda b(\lambda x)$, $c^\lambda(x) = \lambda^2 c(\lambda x)$ will converge to 0, $a^\lambda(x) = a(\lambda x)$ converges to a constant matrix and $f^\lambda(x) = f(\lambda x)$ converge to a constant that is greater than λ_0.

Proof of Lemma 5.50. Fistly, we consider the case that b, c are 0. Let

$$-M_k = \inf_{x \in B_{\rho(\frac{\lambda}{2})^k}(0) \cap N(u)} \partial_{ii}^2 u.$$

For e_i fixed, and $s = \rho(\frac{\lambda}{2})^k$, define u^0, u^1, u^2, $\overline{v}$, v_j, g_j $(0 \leq j \leq n)$, β and V as Remark 5.15 with $x = 0$, it can be obtained that for $w = \partial_{ii}^2 u - V + Cs^\beta$ in $B_s(0)$,

$$Aw = 0 \text{ in } B_s(0) = B_{\rho(\frac{\lambda}{2})^k}(0).$$

Now, consider $-\sum_{i,j=1}^n a_{ij}(\rho(\frac{\lambda}{2})^k x)\partial_{ij}^2 h_k(x) = 0$ in B_1 such that

$$h_k(x) = \begin{cases} 0 & \text{on } \partial B_1(0) \cap \left\{x : x_n^{(k)} < -\dfrac{1}{2}\right\}, \\ -1 & \text{on } \partial B_1(0) \cap \left\{x : x_n^{(k)} \geq -\dfrac{1}{2}\right\}. \end{cases}$$

Let $\mu_k = -\inf_{B_{\frac{1}{2}}} h_k$ and note $0 < \mu_k < 1$ by maximal principle. Since $\rho(\frac{\lambda}{2})^k \to 0$ when $k \to \infty$ and then $a_{ij}(\rho(\frac{\lambda}{2})^k x)$ convergene to constants, we can see that there exists $\mu \in (0,1)$ such that $\mu_k \in (0, \mu)$ for any k. By using the definition of M_k, we have

$$\partial_{ii}^2 u(x) \geq -M_k \quad \text{in } N(u) \cap B_{\rho(\lambda/2)^k}(0).$$

Consider the auxiliary function

$$\widetilde{h}_k(x) = M_k h_k\left(\frac{x}{\rho(\lambda/2)^k}\right) \qquad \text{in } N(u)\cap B_{\rho(\lambda/2)^k}(0).$$

Then $A(w-\widetilde{h}) = 0$ in $N(u)\cap B_{\rho(\lambda/2)^k}(0)$. For any $x\in N(u)\cap \partial B_{\rho(\lambda/2)^k}(0)$, by using (5.43) we can obtain that

$$x\notin B_{\rho(\lambda/2)^k}(0)\cap\{x : -x_n^{(k)} \geq \rho(\lambda/2)^k 4\delta\}.$$

That is

$$x\in B_{\rho(\lambda/2)^k}(0)\cap\{x : x_n^{(k)} > -\rho(\lambda/2)^k 4\delta\}.$$

Since $\delta < \frac{1}{8}$, we have

$$B_{\rho(\lambda/2)^k}(0)\cap\{x : x_n^{(k)} \geq -\rho(\lambda/2)^k 4\delta\} \subset B_{\rho(\lambda/2)^k}(0)\cap\left\{x : x_n^{(k)} \geq -\frac{1}{2}\rho(\lambda/2)^k\right\}.$$

Then we can obtain that

$$x\in \partial B_{\rho(\lambda/2)^k}(0)\cap\left\{x : x_n^{(k)} \geq -\frac{1}{2}\rho(\lambda/2)^k\right\}.$$

Then $\widetilde{h}_k(x) = -M_k$ and $w(x)\geq \partial^2_{ii}u(x)\geq \widetilde{h}_k(x)$. For $x\in \partial N(u)\cap B_{\rho(\lambda/2)^k}(0)$, by using (5.14) we have $w(x)\geq \partial^2_{ii}u(x)\geq \widetilde{h}_k(x)$. Hence

$$\begin{aligned}\partial^2_{ii}u &\geq \widetilde{h}_k(x) - C(\lambda/2)^{\beta k}\\ &\geq M_k h_k\left(\frac{x}{\rho(\lambda/2)^k}\right) - C(\lambda/2)^{\beta k}\\ &\geq -M_k\mu - C(\lambda/2)^{\beta k}\end{aligned}$$

in $B_{\rho(\lambda/2)^{k+1}}(0)\cap N(u)$ by the maximum principle. Then

$$M_{k+1}\leq \mu M_k + C(\lambda/2)^{\beta k}.$$

We will show that $M_k \leq C_1\widetilde{\mu}^{k-1}$ for some $C_1>0$ and $0<\widetilde{\mu}<1$. We can prove this by induction, if for k is true, we need

$$M_{k+1}\leq \mu M_k + C(\lambda/2)^{\beta k} \leq C_1\widetilde{\mu}^{k-1}\mu + C(\lambda/2)^{\beta k}\leq C_1\widetilde{\mu}^{k}.$$

that is

$$C_1\widetilde{\mu}^{k-1}(1-\mu)\geq C(\lambda/2)^{\beta k}.$$

This can be done since we can choose $\lambda < 1$. □

Remark 5.52. Finally, we consider a different approach to the regularity of the free boundary. In particular, we will show that a flat free boundary is Lipschitz. Given

$$\left|u - \frac{1}{2}(x_n^+)^2\right|_{C^{1,\alpha}(B_1(0))} \ll 1,$$

consider

$$h = u_{x_n} - \frac{1}{2}\left(u - \frac{|x-x_0|^2}{2n}\right), \quad x_0\in N\cap B_{\frac{1}{4}}(0).$$

Then

$$\begin{cases}\Delta h = 0 & \text{in } N\cap B_1(0)\\ h\geq 0 & \text{on } \Gamma\cap B_1(0) \text{ and } \partial B_1(0)\cap N.\end{cases}$$

Therefore, $h\geq 0$ in $N\cap B_1(0)$. Hence, $(u_{x_n} - \frac{1}{2}u)(x_0)\geq 0$ and at every point in $B_{\frac{1}{4}}(0)\cap N$. Similarly, $u_e - \frac{1}{2}u\geq 0$ in $B_{\frac{1}{4}}(0)\cap N$, whenever e is close to e_n. We conclude that Γ is a Lipschitz graph in $B_{\frac{1}{4}}(0)$. A more careful argument would lead to that Γ is C^1. Next we shall present yet another approach to show that the Lipschitz regularity of Γ implies that $\Gamma\in C^{1,\alpha}$.

The following two theorems are also well-known results for the regularity of free boundary when there is no convexity assumption on u. Rather one assumes the free boundary is a graph in some direction.

Theorem 5.53 (Alt)**.** *Let $u \in P_1(M)$, $\frac{\partial u}{\partial x_1} \geq 0$ in $B_1(0)$ and $\frac{\partial u}{\partial x_1} > 0$ in $N(u) \cap B_1$. Let*

$$\partial N \triangleq \Gamma = \{x : x_1 = f(x_2, x_3, ..., x_n)\}.$$

Then f is Lipschitz continuous, i.e. Γ is a Lipschitz graph.

Theorem 5.54 (Athanasopoulos-Caffarelli)**.** *Under the assumptions of Theorem 5.53, if f is Lipschitz continuous, then f is locally $C^{1,\alpha}$.*

Proof of Theorem 5.53. First we want to show there is a cone $C \subset \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 > 0\}$ such that $\forall y_0 \in B_{\frac{1}{2}}(0) \cap \Gamma$, whenever $\xi \in C \cap \mathbb{S}^{n-1}$.

$$u_\xi(x) = \nabla u(x) \cdot \xi \geq 0 \quad \text{ for } x \in B_{\frac{1}{4}}(y_0),$$
$$u_\xi(x) > 0 \quad \text{ for } x \in B_{\frac{1}{4}}(y_0) \cap N(u).$$

Let

$$R(x) = \begin{cases} 0 \text{ if } |x - y_0| < \dfrac{1}{4}, \\ \left[|x-y_0|^2 - \dfrac{1}{4^2}\right]^2 \text{ if } |x - y_0| \geq \dfrac{1}{4}, \end{cases}$$

and assume $0 \neq \xi \in \mathbb{R}^n$ and $\varepsilon_0 > 0$ to be chosen later. We consider

$$v(x) = u_\xi(x) - u(x) + \varepsilon_0 R(x) \quad \text{ in } B_{\frac{1}{2}}(y_0).$$

We choose ε_0 sufficiently small so that

$$\Delta v(x) = 0 - 1 + \varepsilon_0 \Delta R(x) \leq -\frac{1}{2} < 0 \quad \text{ in } N(u) \cap B_{\frac{1}{2}}(y_0).$$

We want to choose ξ properly so that $v \geq 0$ on $\partial(N(u) \cap B_{\frac{1}{2}}(y_0))$.

First, one observes that on $\partial N(u) \cap B_{\frac{1}{2}}(y_0) \subset \Lambda(u)$, $u(x) = u_\xi(x) = 0$ and $R(x) \geq 0$. Next we let $\sum_{j=2}^n |\xi_j| \leq 1$. Hence by using the fact that $\|u\|_{C^{1,1}(B_1(0))}$ is bounded, we have

$$\sum_{j=2}^n |\xi_j u_{x_j}(x)| \leq C_0 \text{ and } |u(x)| \leq C_0 \quad \text{ in } B_1(0).$$

It is possible to choose ξ_1 large so that

$$u_\xi(x) - u(x) + \varepsilon_0 R(x) \geq 0 \quad \text{ on } \partial B_{\frac{1}{2}}(y_0) \cap N(u).$$

Indeed, since $|\nabla u| = u = 0$ on Γ, $u_{x_1} > 0$ in $N(u) \cap B_1(0)$ and $u \in C^{1,1}(B_1(0))$,

$$\begin{aligned} u_\xi(x) - u(x) + \varepsilon_0 R(x) &= \sum_{j=2}^n \xi_j u_{x_j}(x) + \xi_i u_{x_1}(x) - u(x) + \varepsilon_0 R(x) \\ &\geq - \sum_{j=2}^n |\xi_j u_{x_j}(x)| - u(x) + \varepsilon_0 R(x) \\ &\geq \varepsilon_0 \left(\frac{3}{16}\right)^2 - \sum_{j=2}^n |\xi_j u_{x_j}| - u(x) \geq \left(\frac{1}{16}\right)^2 \varepsilon_0, \end{aligned}$$

if $\text{dist}(x, \Gamma) < \delta_0$ for some small $\delta_0 > 0$, where we also use the fact that when $x \in \partial B_{\frac{1}{2}}(y_0) \cap N(u)$, $|x - y_0| = \frac{1}{2}$, $R(x) = (\frac{1}{2})^2 - \left(\frac{1}{4}\right)^2 = \frac{3}{16}$ and arguments similar in the proof of (5.11). On the other hand, if $\text{dist}(x, \Gamma) \geq \delta_0$, then $u_{x_1}(x) \geq C(\delta_0) u_{x_1}(\frac{1}{2}, 0)$ by Harnack's inequality. Hence, we choose

$$\xi_1 \geq \frac{3C_0}{C(\delta_0) u_{x_1}(\frac{1}{2}, 0)} \triangleq M_0,$$

which gives

$$u_\xi(x) - u(x) + \varepsilon_0 R(x) \geq -2C_0 + \xi_1 u_{x_1} \geq C_0 > 0.$$

Therefore, by maximal principle, for

$$\forall \xi \in C \cap \mathbb{S}^{n-1} = \{(\xi_1, ..., \xi_n) \in \mathbb{S}^{n-1} : \xi_1 \geq M_0 \sum_{j=2}^{n} |\xi_j|\},$$

we have

$$u_\xi(x) - u(x) + \varepsilon_0 R(x) \geq 0 \quad \text{ for } x \in N(u) \cap B_{\frac{1}{2}}(y_0).$$

In particular, by using the definition of $R_0(x)$,

$$u_\xi(x) \geq u(x) > 0 \quad \text{ in } N(u) \cap B_{\frac{1}{4}}(y_0) \quad \text{ and } u_\xi(x) \geq u(x) \geq 0 \quad \text{ in } B_{\frac{1}{4}}(y_0).$$

The geometric consequence of the above fact is that $\forall y_0 \in \Gamma$ with $B_{\frac{1}{2}}(y_0) \subset B_1$,

$$\Lambda_+(y_0) = \left\{ y \in B_{\frac{1}{4}}(y_0) : \frac{y - y_0}{|y - y_0|} \in C \cap \mathbb{S}^{n-1} \right\} \subset N(u) \cap B_{\frac{1}{4}}(y_0),$$

$$\Lambda_-(y_0) = \left\{ y \in B_{\frac{1}{4}}(y_0) : -\frac{y - y_0}{|y - y_0|} \in C \cap \mathbb{S}^{n-1} \right\} \subset \Lambda(u) \cap B_{\frac{1}{4}}(y_0).$$

Hence $\Gamma = \{x_1 = f(x_2, ..., x_n)\}$ is a Lipschitz graph. □

To prove Theorem 5.54, we need the following lemma.

Lemma 5.55. *Let $L = -\operatorname{div}(A(y)\nabla)$ be a uniformly elliptic operator in $B_1(0)$ with bounded measurable coefficients. If*

$$\begin{aligned} Lv = Lw = 0 &\quad \text{in } B_1^+(0) \\ v > 0,\ w > 0 &\quad \text{in } B_1^+(0) \\ v = w = 0 &\quad \text{on } \{y_1 = 0\} \end{aligned}$$

then $\frac{v}{w} \in C^{0,\alpha}\left(\overline{B^+_{\frac{1}{2}}(0)(0)}\right)$.

Proof of Theorem 5.54. Since $u(f(x_2, ..., x_n), x_2, ..., x_n) \equiv 0$,

$$u_{x_j} + u_{x_1} f_{x_j} = 0, \quad j = 2, ..., n.$$

Thus $f_{x_j} = -\left.\frac{u_{x_j}}{u_{x_1}}\right|_\Gamma$. Here we need to note that f is already Lipschitz, which means that f_{x_j} exists a.e. and when f_{x_j}, the above result is true. Note that $\frac{u_{x_j}}{u_{x_1}}$ is well-defined in $N(u)$.

Next we observe that $\Delta u_{x_j} = 0$ in $N(u)$ for $j = 1, ..., n$. We define $T : x \mapsto y$ by

$$y_1 = x_1 - f(x_2, ..., x_n), \text{ and } y_i = x_i \text{ for } i = 2, ..., n.$$

By change of variable, $w_j(y) \triangleq u_{x_j}(T^{-1}y)$, we find

$$Lw_j = 0 \quad \text{ in } \{y_1 > 0\},\ j = 1, ..., n,$$

for some uniformly elliptic operator $L = \operatorname{div}(A(y)\nabla)$ with bounded measurable coefficients. Since $w_1 > 0$ in $\{y_1 > 0\}$ and $w_j \equiv 0$ on $\{y_1 = 0\}$ for $j = 1, ..., n$. Let $C_0 > 0$ and $v_j(x) = C_0 w_1(x) + w_j(x)$. Then

$$u_{x_j} + C_0 u_{x_1} = (C_0^2 + 1)^{\frac{1}{2}} \left(\frac{1}{(C_0^2 + 1)^{\frac{1}{2}}} u_{x_j} + \frac{C_0}{(C_0^2 + 1)^{\frac{1}{2}}} u_{x_1} \right) = (C_0^2 + 1)^{\frac{1}{2}} \partial_\tau u$$

with $\tau = \frac{1}{(C_0^2+1)} e_j + \frac{C_0}{(C_0^2+1)} e_1$. By using the proof of Theorem 5.53, we see that if $\tau \cdot e_n > c_0$ for some uniform constant c_0, $\partial_\tau u \geq 0$. Then if $C_0 \gg 1$ it can be obtained without difficulty. The implies that $\partial_\tau u(x) \geq 0$ for any $x \in B_r(0)$ with $r \ll 1$. Then $v_j(x) \geq 0$ if $x \in B_\delta(0)$ with $\delta \ll 1$.

Now applying Lemma 5.55, we obtain $\frac{v_j}{w_1}\Big|_{y_1=0}$ is Hölder continuous, i.e. $f_{x_j} = -C_0 - \frac{u_{x_j}}{u_{x_1}}\Big|_{\Gamma}$ is Hölder continuous. $\square$

There remains to prove Lemma 5.55.

Lemma 5.56 (Harnack's principle at the boundary)**.** *Let L be a uniformly elliptic operater with bounded measurable coefficients, i.e.*

$$L = \frac{\partial}{\partial x^i}\left(a^{ij}(x)\frac{\partial}{\partial x^j}\right)$$
$$a^{ij} \in L^\infty, \quad \lambda|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \lambda^{-1}|\xi|^2$$

Suppose $Lv = Lw = 0$ in $B_1^+(0)$, with $v = w = 0$ on $B_1(0)\cap\{x_1 = 0\}$ and $v, w > 0$ in $B_1^+(0)$. Then there is a constant $C(n,\lambda)$, s.t.

$$\sup_{B^+_{\frac{1}{2}}(0)(0)} w \leq Cw\left(\frac{1}{2}, 0\right)$$
$$\sup_{B^+_{\frac{1}{2}}(0)(0)} \frac{v}{w} \leq C\frac{\sup_{B^+_{\frac{1}{2}}(0)(0)} v}{\sup_{B^+_{\frac{1}{2}}(0)(0)} w}$$

For convenience, we recall some classic estimates in elliptic equations (for more details, see Chapter 8).

Lemma 5.57 (Moser's Harnack inequality)**.** *Let L be defined as in Lemma 5.56. For any positive solution of $Lu = 0$ in $B_1(0)$ and for $\forall R \in (0,1)$,*

$$\sup_{B_R(0)} u \leq C(R,n,\lambda)\inf_{B_R(0)} u$$

Lemma 5.58 (De Giorgi-Nash)**.** *Let L be defined as in Lemma 5.56. Let $Lu = 0$ in $B_1(0)$. Then for $\forall R \in (0,1)$, $\forall x, y \in B_R(0)$*

$$|u(x) - u(y)| \leq C(R,n,\lambda,\alpha)\|u\|_{L^2(B_1(0))}|x-y|^\alpha.$$

Define

$$A = \{x = (x_1, x_2, ..., x_n) : 0 < x_n < 8, |x_j| < 4, \forall j = 1, ..., n-1\},$$
$$A_0 = \{x = (x_1, x_2, ..., x_n) : 0 < x_n < 2, |x_j| < 1, \forall j = 1, ..., n-1\},$$

and let z_0 be the center of A_0, i.e. $z_0 = (0, ..., 0, 1)$.

Lemma 5.59. *Let L be defined as in Lemma 5.56. There is a constant $C = C(\lambda, n)$, s.t. for $u \in H^1(A)$, $u > 0$ in A solving $Lu = 0$ in A and $u(x', 0) = 0$, we have*

$$\sup_{A_0} u \leq Cu(z_0).$$

Proof. We may assume $u(z_0) = 1$. Also by reflecting u across $\{x_n = 0\}$ as an odd function in x_n, we may assume

$$\widetilde{L}u = 0 \text{ in } \widetilde{A} \triangleq \{x : |x_n| < 8,\ |x_j| < 4,\ j = 1, ..., n-1\}.$$

We note that $\widetilde{L}$ share the same ellipticity constant λ with L. Define $\omega_Q(u) = \operatorname{osc}_{cQ} u$. Then Moser's Harnack inequality (Precisely speacking, we have used (8.5)) gives

$$\omega_Q(u) \geq \rho\,\omega_{\frac{1}{2}Q}(u) \tag{5.45}$$

with constant $\rho = \rho(n, \lambda) > 1$. Also, by Moser's Harnack inequality,

$$u(x', x_n) \le Ku(x', 2x_n) \quad \text{for } |x_j| \le 2,\ j = 1.2, ..., n-1,\ 0 < x_n \le \frac{3}{2},$$
$$u(x', x_n) \le Ku(z_0) \quad \text{for } |x_j| \le 2,\ j = 1.2, ..., n-1,\ 1 \le x_n \le 3,$$

where $K = K(n, \lambda) > 1$.

Suppose there is a point $Y_0 \in A_0$ such that $u(Y_0) \ge K^{k_0+2}$ with k_0 to be determined. Then $\operatorname{dist}(Y_0, \mathbb{R}^{n-1}) \le 2^{-k_0}$. (Otherwise, $u(Y_0) \le K^{k_0+1} u(z_0)$.) Also,

$$\omega_{Q(Y_0, 2^{-k_0+N})}(u) \ge \rho^N \omega_{Q(Y_0, 2^{-k_0})}(u) \ge \rho^N K^{k_0+2},$$

where $Q(x, r)$ is the cube in $\mathbb{R}^n$ centered at x with edge length $2r$. Now we choose N, which will be fixed for all, so that $\rho^N \ge 2K^2$. Hence, $\omega_{Q(Y_0,2)}(u) \ge 2K^{k_0+4}$. Since u is an odd function, there is a $Y_1 \in Q(Y_0, 2^{-k_0+N}) \cap \mathbb{R}^n_+$, s.t. $u(Y_1) \ge K^{k_0+4}$. Thus, $\operatorname{dist}(Y_1, \mathbb{R}^{n-1}) \le 2^{-k_0-2}$ and

$$\omega_{Q(Y_1, 2^{-k_0-2+N})}(u) \ge \rho^N K^{k_0+4} \ge 2K^{k_0+6}.$$

Therefore, $\exists Y_2 \in Q(Y_1, 2^{-k_0-2+N}) \cap \mathbb{R}^n_+$, s.t. $u(Y_2) \ge K^{k_0+6}$. Repeating this process, we will obtain a sequence $\{Y_k\}$ such that $\forall k = 1, 2, ...,$
(i) $\operatorname{dist}(Y_k, \mathbb{R}^{n-1}) \le 2^{-k_0-2k}$; (ii) $Y_k \in Q(Y_{k-1}, 2^{-k_0-2(k-1)+N}) \cap \mathbb{R}^n_+$; (iii) $u(Y_k) \ge K^{k_0+2(k+1)}$. We want to show all Y_k's are strictly contained in A. Indeed, if k_0 is sufficiently large, e.g. taking $k_0 = N + 3$, then for any $i = 1, ..., n-1$,

$$|Y_{k,i}| \le \sum_{j=1}^{k} |Y_{j,i} - Y_{j-1,i}| + |Y_{0,i}| \le \left(\sum_{j=1}^{k} 2^{-2(j-1)} \right) 2^{-k_0+N} + 1 < \frac{3}{2}.$$

This leads to the contradiction since $u(Y_k) \to +\infty$. □

Remark 5.60. There is a simple version of proof for Lemma 5.59. Which is due to (8.6). We may still assume $u(z_0) = 1$. Also by reflecting u across $\{x_n = 0\}$ as an odd function in x_n, then $u > 0$ in $\{x_n > 0\}$, $u = 0$ on $\{x_n = 0\}$ and $u < 0$ in $\{x_n < 0\}$

$$\widetilde{L}u = 0 \text{ in } \widetilde{A} = \{x = (x_1, x_2, ..., x_n) : |x_n| < 8,\ |x_j| < 4,\ j = 1, 2, ..., n-1\}$$

Here $\widetilde{L}$ share the same ellipticity constant λ with L. By using (8.6), we have $\sup_Q u \ge \rho \sup_{\frac{1}{2}Q} u$ for any $2Q \subset \widetilde{A}$ with $\rho > 1$ as long as

$$|Q \cap \{u \le 0\}| \ge \mu |Q| \text{ for some fixed } \mu.$$

By Moser's Harnack inequality, there exists $K > 1$, such that

$$u(x', x_n) \le Ku(x', 2x_n) \quad \text{for } |x_j| \le 2,\ j = 1.2, ..., n-1,\ 0 < x_n \le \frac{3}{2}, \tag{5.46}$$
$$u(x', x_n) \le Ku(z_0) = K \quad \text{for } |x_j| \le 2,\ j = 1.2, ..., n-1,\ 1 \le x_n \le 3, \tag{5.47}$$

Suppose there is a point $Y_0 \in A_0$ such that $u(Y_0) \ge K^{k_0+2}$ with k_0 to be determined. Then $\operatorname{dist}(Y_0, \mathbb{R}^{n-1}) \le 2^{-k_0}$. Otherwise, by using (5.46) and (5.47), we have

$$u(Y_0) = u(Y_0', Y_{0,n}) \le Ku(Y_0', 2Y_{0,n}) \le ... \le K^{k_0} u(Y_0', 2^{k_0} Y_{0,n}) \le K^{k_0+1} u(z_0) \le K^{k_0+1},$$

which is contradiction to the condition $u(Y_0) \ge K^{k_0+2}$. For $N \in \mathbb{N}_+$, we can see that

$$\begin{aligned} |Q(Y_0, 2^{-k_0+N}) \cap \{u \le 0\}| &= |Q(Y_0, 2^{-k_0+N}) \cap \mathbb{R}^n_-| = \frac{2^{-k_0+N} - 2^{-k_0}}{2 \cdot 2^{-k_0+N}} |Q(Y_0, 2^{-k_0+N})| \\ &= \frac{1 - 2^{-N}}{2} |Q(Y_0, 2^{-k_0+N})| \ge \frac{1}{4} |Q(Y_0, 2^{-k_0+N})| \end{aligned}$$

Then by using (8.6), we have

$$\sup_{Q(Y_0,2^{-k_0+N})} u \geq \rho \sup_{Q(Y_0,2^{-k_0+N})} \geq \ldots \geq \rho^N \sup_{Q(Y_0,2^{-k_0})} u \geq \rho^N u(Y_0) \geq \rho^N K^{k_0+2}.$$

We can choose N such that $\rho^N > K^2$ and $k_0 > N$. Then there exists $Y_1 \in Q(Y_0, 2^{-k_0+N}) \cap \mathbb{R}^n_+$, such that $u(Y_1) \geq K^{k_0+4}$. By using almost the same arguments, we can obtain that $\mathrm{dist}(Y_1, \mathbb{R}^{n-1}) \leq 2^{-k_0-2}$ and

$$\sup_{Q(Y_1,2^{-k_0-2+N})} u \geq \rho^N \sup_{Q(Y_1,2^{-k_0-2})} u \geq \rho^N u(Y_1) \geq \rho^N K^{k_0+4} \geq K^{k_0+6}$$

Repeating this process, we will obtain a sequence $\{Y_k\}$ such that $\forall k = 1, 2, \ldots,$
(i) $\mathrm{dist}(Y_k, \mathbb{R}^{n-1}) \leq 2^{-k_0-2k}$; (ii) $Y_k \in Q(Y_{k-1}, 2^{-k_0-2(k-1)+N}) \cap \mathbb{R}^n_+$; (iii) $u(Y_k) \geq K^{k_0+2(k+1)}$.
We want to show all Y_k's are strictly contained in A. Indeed, if k_0 is sufficiently large, e.g. taking $k_0 = N + 3$, then for any $i = 1, \ldots, n-1$,

$$|Y_{k,i}| \leq \sum_{j=1}^{k} |Y_{j,i} - Y_{j-1,i}| + |Y_{0,i}| \leq \left(\sum_{j=1}^{k} 2^{-2(j-1)} \right) 2^{-k_0+N} + 1 < \frac{3}{2}.$$

This leads to the contradiction since $u(Y_k) \to +\infty$.

Lemma 5.61. *Let u and w be positive solutions of $Lu = Lw = 0$ in A with $u(x', 0) = w(x', 0) = 0$. Then*

$$\sup_{A_0} \frac{u}{w} \leq C(n, \lambda) \frac{u(z_0)}{w(z_0)}$$

Proof. Let

$$A_1 = \{x = (x_1, x_2, \ldots, x_n) : 0 < x_n < 4,\ |x_j| < 2,\ \forall j = 1, 2, \ldots, n-1\}.$$

We introduce the notation

$$\alpha = \partial A_1 \cap \{x_n > 1\}, \quad \beta = \partial A_1 \cap \{x_n > 0\}.$$

Consider

$$\begin{cases} Lu_1 = 0 & \text{in } A_1, \\ u_1 = \chi_\alpha & \text{on } \partial A_1, \end{cases} \quad \text{and} \quad \begin{cases} Lu_2 = 0 & \text{in } A_1, \\ u_2 = \chi_\beta & \text{on } \partial A_1. \end{cases}$$

Note that $u(x) \leq Cu(z_0)$ for $x \in A_1$ by the preceding lemma. Hence for $x \in A_1$

$$u(x) \leq Cu(z_0) u_2(x).$$

This is from maximal principle for L, that is

$$\left.\begin{array}{l} L(Cu(z_0)u_2 - u) = 0 \text{ in } A_1, \\ Cu(z_0)u_2(x) - u(x) = Cu(z_0) - u(x) \geq 0 \text{ on } \beta \\ Cu_2(z_0)u_2(x) - u(x) = 0 \text{ on } \partial A_1 \cap \{x_n = 0\} \end{array}\right\} \Rightarrow u(x) \leq Cu(z_0)u_2(x) \text{ in } A_1.$$

Also by using De Giogi-Nash inequality we can see that w is continuous on α. Then we can choose $c \ll 1$ such that $w(x) \geq \inf_{y \in \alpha} w(y) \geq cw(z_0)$ for $x \in \alpha$ and hence by using maximal principle for L again, we have

$$\left.\begin{array}{l} L(w - cw(z_0)u_1) = 0 \text{ in } A_1, \\ w(x) - cw(z_0)u_1(x) = w(x) - c(w(z_0)) \geq 0 \text{ on } \alpha \\ w(x) - cw(z_0)u_1(x) = w(x) \geq 0 \text{ on } \partial A_1 \cap \{0 \leq x_n \leq 1\} \end{array}\right\} \Rightarrow w(x) \geq cw(z_0)u_1(x) \text{ in } A_1.$$

Claim 1: $u_2(x) \leq Cu_1(x)$ for every $x \in A_0$. Hence

$$\sup_{A_0} \frac{u}{w} \leq C(n, \lambda) \frac{u(z_0)}{w(z_0)}.$$

Proof of Claim 1. Let $0 \le \phi \le 1$, $\phi = 1$ on β and $\phi \in C_0^\infty(\mathbb{R}^n \backslash \overline{A}_{\frac{1}{2}})$. For every $x \in A_0$, one has for G the Green function of L on A_1

$$\begin{aligned} u_2(x) &\le \int_{\partial A_1} P(x,y)\phi(y)dy = \int_{\partial A_1} \langle A(y)\nabla_y G(x,y), \nu(y)\rangle \phi(y)dy \\ &\le C\left(\int_{A_1\backslash A_{\frac{1}{2}}} |\nabla_y G(x,y)|^2 dy\right)^{\frac{1}{2}} \le C\left(\int_{A_1\backslash A_{\frac{1}{4}}} G^2(x,y)dy\right)^{\frac{1}{2}} \le CG(x,y_0), \end{aligned}$$

where the last step again follows from Lemma 5.59. To conclude the proof of the claim, we need the following estimate:

Claim 2:

$$|G(x,y_0)| \le \begin{cases} C|x-y_0|^{-(n-2)} & \text{if } n \ge 3 \\ C(1+\ln(|x-y_0|) & \text{if } n = 2 \end{cases} \le C$$

in $A_1 \backslash B_{\frac{1}{2}}(y_0)$.

For more details about green functions of variable coeffients, we refer to [3] and [13]. Note that this second claim implies $G(x,y_0) \le Cu_1(x)$ for every $x \in A_1\backslash B_{\frac{1}{2}}(y_0)$ by Harnack applied to u_1 and the comparison principle. Precisely speacking, by using Harnack inequality u_1, namely Lemma 5.57 and De Giogi-Nash inequality (which implies that u_1 is continuous in A_1), we can obtain that

$$C \inf_{B_{\frac{1}{2}}(y_0)} u_1 = C \inf_{\overline{B_{\frac{1}{2}}(y_0)}} u \ge \sup_{\overline{B_{\frac{1}{2}}(y_0)}} u > 0 \Rightarrow C \inf_{\partial B_{\frac{1}{2}}(y_0)} u_1 > 0.$$

Choosing $C \ll 1$, we can obtain

$$\left.\begin{aligned} &L(G(x,y_0)) = Lu_1 = 0 \text{ in } A_1\backslash B_{\frac{1}{2}}(y_0) \\ &G(x,y_0) \le Cu_1(x) \text{ on } A_1 \cap B_{\frac{1}{2}}(y_0) \\ &G(x,y_0) = 0 \le Cu_1(x) \text{ on } \partial A_1 \end{aligned}\right\} \Rightarrow G(x,y_0) \le Cu_1(x) \text{ in } A_1\backslash B_{\frac{1}{2}}(y_0).$$

Consequently, we can conclude that $u_2(x) \le Cu_1(x)$.

This second claim follows from a simple duality argument using the fact that G is the Green's function associated with L and bounding $\|G(x,\cdot)\|_{L^{\frac{2n}{n-2}}}$ by duality implies the claim. We only consider the case for $n \ge 3$ and the case for $n = 2$ is much more complicated (see [3]). Consider the problem

$$\begin{cases} Lu = f \in L^{\frac{2n}{n-2}}(A_1), \\ u = 0 \quad \text{on } \partial A_1. \end{cases}$$

Then by standard elliptic estimates, $\nabla u \in L^2(A_1)$ and by the Poincaré inequality, $u \in L^2(A_1)$. Assume that the support of f is in $B_{\frac{1}{4}}(y_0)$. We have the representation formula

$$u(\xi) = \int_{A_1} G(\xi,y)f(y)dy = \int_{B_{\frac{1}{4}}(y_0)} G(\xi,y)f(y)dy.$$

Note that $\|u\|_{L^2(A_1)} \le C\|f\|_{L^{\frac{2n}{n-2}}}$ implies

$$u(x) \le \|u\|_{L^\infty(A_1\backslash B_{\frac{1}{2}}(y_0))} \le C\|f\|_{L^{\frac{2n}{n+2}}}.$$

Hence we have

$$u(x) = \int_{B_{\frac{1}{4}}(y_0)} G(\xi,y)f(y)dy \le C\|f\|_{L^{\frac{2n}{n-2}}},$$

which by duality gives the desired bound for the $L^{\frac{2n}{n-2}}$-norm of $G(x,\cdot)$. Thus, the proof of Lemma 5.61 is complete. $\square$

Note that Lemmas 5.59 and 5.61 precisely amount to Lemma 5.56. Now we are ready complete the proof of the main lemma 5.55.

Proof of Lemma 5.55. Let $\alpha_k = \sup_{B^+_{2^{-k}}(0)} \frac{v}{w}$ and $\beta_k = \inf_{B^+_{2^{-k}}(0)} \frac{v}{w}$ for $k = 1, 2, ...$ We want to show that for some $\delta < 1$, we have $\alpha_k - \beta_k \le M_0 \delta^{k-1}$ for $k = 1, 2, ...$ Here

$$M_0 = \sup_{B^+_{\frac{1}{2}}(0)} \frac{v}{w} - \inf_{B^+_{\frac{1}{2}}(0)} \frac{v}{w} \le C \left[\frac{\sup_{B^+_{\frac{1}{2}}(0)} v}{\sup_{B^+_{\frac{1}{2}}(0)} w} - \frac{\sup_{B^+_{\frac{1}{2}}(0)} (-v)}{\sup_{B^+_{\frac{1}{2}}(0)} w} \right] = C \frac{\operatorname{osc}_{B^+_{\frac{1}{2}}(0)} v}{\sup_{B^+_{\frac{1}{2}}(0)} w}.$$

We argue by induction on k. The base case $k = 1$ holds by the preceding lemma. Suppose that the claim holds for k. Then we prove the case for $k+1$. We assume

$$\frac{v(0, 2^{-k-1})}{w(0, 2^{-k-1})} \ge \beta_k + \frac{1}{2}(\alpha_k - \beta_k) = \frac{\alpha_k + \beta_k}{2},$$

and consider $v^*(x) = (v - \beta_k w)(2^{-k}x) \ge 0$ on B^+_1, where $w^*(x) = w(2^{-k}x)$. Then applying Lemma 5.56 to v^* and w^* yields

$$\begin{aligned} \sup_{B^+_{\frac{1}{2}}(0)} \frac{w^*}{v^*} &\le C \frac{\sup_{B^+_{\frac{1}{2}}(0)} w^*}{\sup_{B^+_{\frac{1}{2}}(0)} v^*} \le \widetilde{C} \frac{w^*(0, \frac{1}{2})}{v^*(0, \frac{1}{2})} = \widetilde{C} \frac{w(0, 2^{-k-1})}{[v(0, 2^{-k-1}) - \beta_k w(0, 2^{-k-1})]} \\ &= \widetilde{C} \frac{1}{[\frac{v(0,2^{-k-1})}{w(0,2^{-k-1})} - \beta_k]} \le \widetilde{C} \frac{1}{(\frac{\alpha_k + \beta_k}{2} - \beta_k)} \le \frac{\widetilde{C}}{\alpha_k - \beta_k}. \end{aligned}$$

Now observe that

$$\sup_{B_{2^{-k-1}}} \frac{v}{w} = \alpha_{k+1} \le \alpha_k,$$

and

$$\inf_{B_{2^{-k-1}}} \frac{v}{w} = \beta_{k+1} = \inf_{B^+_{\frac{1}{2}}(0)} \frac{v^*}{w^*} + \beta_k \ge \beta_k + \frac{1}{\widetilde{C}}(\alpha_k - \beta_k).$$

Therefore,

$$\alpha_{k+1} - \beta_{k+1} \le \left(1 - \frac{1}{\widetilde{C}}\right)(\alpha_k - \beta_k) = \delta(\alpha_k - \beta_k).$$

On the other hand, if

$$\frac{v(0, 2^{-k-1})}{w(0, 2^{-k-1})} < \beta_k + \frac{1}{2}(\alpha_k - \beta_k),$$

we can consider $v^*(x) = (\alpha_k v - w)(2^{-k}x) \ge 0$ on B^+_1, where $w^*(x) = w(2^{-k}x)$. Still, by applying Lemma 5.56 to v^* and w^* yields

$$\begin{aligned} \sup_{B^+_{\frac{1}{2}}(0)} \frac{w^*}{v^*} &\le C \frac{\sup_{B^+_{\frac{1}{2}}(0)} w^*}{\sup_{B^+_{\frac{1}{2}}(0)} v^*} \le \widetilde{C} \frac{w^*(0, \frac{1}{2})}{v^*(0, \frac{1}{2})} = \widetilde{C} \frac{w(0, 2^{-k-1})}{[\alpha_k v(0, 2^{-k-1}) - w(0, 2^{-k-1})]} \\ &= \widetilde{C} \frac{1}{[\alpha_k - \frac{v(0,2^{-k-1})}{w(0,2^{-k-1})}]} \le \widetilde{C} \frac{1}{(\alpha_k - \frac{\alpha_k + \beta_k}{2})} \le \frac{\widetilde{C}}{\alpha_k - \beta_k}. \end{aligned}$$

Now we can use the same methods above to complete the proof. Hence, we have proved the claim. $\square$

5.4. **Application to Minimal Surface and Prescribed Mean Curvature Problems.** In this section we will follow [18] to show how the boundary behavior of solutions of the nonpara-metric least area and prescribed mean curvature problem can be tackled by a reduction to obstacle problems, for which the regularity has been established in the preceding section.

We first consider the Dirichlet problem for the minimal surface equation in a bounded C^2-domain Ω of $\mathbb{R}^2$:

$$\begin{cases} \operatorname{div}\left(\dfrac{u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = \varphi. \end{cases} \tag{5.48}$$

When Ω is convex, (5.48) has a unique solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ for every $\varphi \in C^0(\partial\Omega)$. In contrast, in the case of nonconvex φ one can construct a smooth φ for which (5.48) fails to have a solution. See for instance [8] for the issues concerning the solvability of (5.48). However, one can instead study the solution to the following variational problem:

$$\begin{cases} \min I(v) \text{ for } v \in BV(\Omega), \\ I(v) = \int_\Omega \sqrt{1+|\nabla v|^2} + \int_{\theta\Omega} |v-\varphi|, \end{cases} \tag{5.49}$$

which is called the nonparametric least area problem. Here we characterize $BV(\Omega)$ as the space of integrable functions that can be approximated in $L^1(\Omega)$ norm by a sequence of Lipschitz functions v_i that satisfy:

$$\limsup_i \int_\Omega |Dv_i| < \infty.$$

Note that in case the limit v is also Lipschitz, $I(v)$ corresponds to the area of the Lipschitz surface obtained by taking the union of the graph of v over Ω and the part of the boundary cylinder $\partial\Omega \times \mathbb{R}$ enclosed by the curves $\{(x, \varphi(x)) : x \in \partial\Omega\}$ and $\{(x, v(x)) : x \in \partial\Omega\}$.

Giusti showed in [9] the existence of a unique solution $u \in C^2(\Omega) \cap BV(\Omega)$ to the problem (5.49). Furthermore, we can decompose $\partial\Omega$ as:

$$\partial\Omega = \partial_+(\Omega) \cup \partial_-(\Omega) \cup \Gamma,$$

where:

$$\begin{cases} \partial_+(\Omega) = \text{ int } \{x \in \partial\Omega : H_{\partial\Omega}(x) \geq 0\} \\ \partial_-(\Omega) = \{x \in \partial\Omega : H_{\partial\Omega}(x) < 0\}, \\ \Gamma = \partial\Omega \backslash \left(\partial_+(\Omega) \cup \partial_-(\Omega)\right). \end{cases}$$

Here $H_{\partial\Omega}(x)$ denotes the mean curvature of $\partial\Omega$ with respect to inward normal at $x \in \partial\Omega$. Then $u = \varphi$ on $\partial_+(\Omega)$ and u is Hölder continuous at each point of $\partial_+(\Omega)$ provided φ is Lipschitz continuous.

In fact, in [20] L. Simon proved that if $\partial\Omega$ is C^4, then u is Hölder continuous at each point of $\partial_-(\Omega)$ too and the restriction of u to $\partial_-(\Omega)$ is locally Lipschitz continuous, as long as φ is Lipschitz continuous there.

Here we aim to show the higher regularity of the trace of u over the part of $\partial_-(\Omega)$ where $u \neq \varphi$ and obtain the $C^{\frac{1}{2}}$-Hölder estimate of u near such points. Our strategy is to reduce the problem to a free boundary one and make use of the regularity theory for free boundaries we developed in the preceding chapter, as well as the higher regularity results that we will prove in the last chapter of these notes.

For Ω bounded, C^2 domain in $\mathbb{R}^2$ and $\varphi \in C^0(\partial\Omega)$, we study the trace of the unique solution u to the variational problem (5.49). The trace of u over $\partial\Omega$ is defined by the requirement:

$$\lim_{\rho\to 0+} \rho^{-2} \int_{\Omega\cap\{\xi:|\xi-x|<\rho\}} |u(\xi) - u(x)| d\xi = 0$$

for all $x \in \partial\Omega$, where such a value $u(x)$ exists. Let $\gamma = \gamma_- \cup \gamma_+$, where:

$$\begin{cases} \gamma_+ = \{x \in \partial_-(\Omega) : \varphi(x) > u(x)\}, \\ \gamma_- = \{x \in \partial_-(\Omega) : \varphi(x) < u(x)\}. \end{cases}$$

Note that the sets γ_+, γ_- and γ are well-defined up to linear measure zero sets.

Theorem 5.62. *Let Ω be a $C^{2,\alpha}$ domain in $\mathbb{R}^2$ for $0 < \alpha < 1$ and let φ, u, γ_+ be as above. Suppose $x_0 \in \gamma_+$ so that $(\partial\Omega \cap B_{r_0}(x_0)) \backslash \gamma_+$ is a set of one-dimensional Hausdorff measure zero for some $r_0 > 0$. Then u restricted to $\partial\Omega \cap B_{\frac{r_0}{2}}(x_0)$ is a $C^{1,\alpha}$ function. Furthermore, its $C^{1,\alpha}$-norm depends only on the $C^{2,\alpha}$-norm of $\partial\Omega$, $L^\infty(\Omega)$-norm of φ, r_0 and:*

$$\inf\{|H_{\partial\Omega}(x)| : x \in \partial\Omega \cap B_{r_0}(x_0)\}.$$

The analogous statement holds for γ_- in place of γ_+ as well.

Corollary 5.63. *Under the hypothesis of Theorem 5.62, u is Hölder continuous in $B_{\frac{r_0}{2}}(x_0) \cap \overline{\Omega}$ with Hölder exponent exactly equal to $\frac{1}{2}$. If, in addition, $\partial\Omega$ is of class $C^{k,\alpha}$ or analytic, then u restricted to $B_{\frac{r_0}{2}}(x_0) \cap \partial\Omega$ is $C^{k-1,\alpha}$ for $k = 3, 4, ...$ and $0 < \alpha < 1$, or analytic respectively.*

For the proof, we refere to [18]. In order to restrict our attention to the free boundary aspect of the problem, we also state the following measure-theoretic results without proof.

Lemma 5.64. *Let Q be the multiplicity-one integral 2-current in $\mathbb{R}^3$ with $\operatorname{spt}(Q) \subseteq \partial\Omega \times \mathbb{R}$ and $\partial Q = T - Tu$, where:*

$$\begin{cases} T = \{(x, \varphi(x)) : x \in \partial\Omega\}, \\ Tu = \{(x, u(x)) : x \in \partial\Omega\}. \end{cases}$$

Then $\Delta = \operatorname{graph}(u) + Q$ is the unique area minimizing integral whose boundary is the given T and whose support lies in $\overline{\Omega} \times \mathbb{R}$.

Corollary 5.65. *$\Delta \backslash T$ is a $C^{1,1}$ is a $C^{1,1}$ surface with its $C^{1,1}$-character depending only on the $L^\infty(\partial\Omega)$ norm of φ, $C^{1,1}$-character of $\partial\Omega$ and the distance to T.*

Lemma 5.66. *Let φ, u, Ω be as above. We define:*

$$\varphi^* = \begin{cases} \varphi \text{ on } \partial\Omega \backslash B_{r_0}(x_0) \\ \varphi + 2r_0 \text{ on } \partial\Omega \cap B_{r_0}(x_0), \end{cases}$$

and u^* as the solution of (4.15) with boundary data φ^*. Then $u \equiv u^*$.

Proof. Use Corollary 5.65 and the generalized maximum principle of R. Finn, see for example Theorem 13.10 in [8]. □

Proof of Theorem 5.62. Now we will restrict ourselves to a neighborhood of an arbitrary point in

$$\gamma_* = \{(x, u(x)) : x \in \gamma_+ \cap B_{\frac{r_0}{2}}(x_0)\}$$

and study the local behavior of Δ at such an arbitrary point. By Lemma 5.66 $\operatorname{dist}(\gamma_*, \partial\Delta) \geq \frac{r_0}{2}$. Fix a point $p \in \gamma_*$ and introduce a coordinate system $(y, y_3) = (y_1, y_2, y_3)$ such that:
(i) p is the origin of $\mathbb{R}^3$,
(ii) y_1-axis is the same as x_3-axis,
(iii) y_2, y_3-plane is the tangent plane of $\partial\Omega \times \mathbb{R}$ at p,
(iv) e_3 is the unit normal of $\partial\Omega \times \mathbb{R}$ at p, hence normal to Δ at p by Corollary 5.65.

By Lemma 5.66 and Corollary 5.65, Δ can be locally represented as the graph of a $C^{1,1}$-function $u = u(y)$, say over $D_\delta = \left\{y \in \mathbb{R}^2 : |y| < \delta\right\}$, where $\delta = \delta(r_0, \partial\Omega) > 0$. Likewise, over D_δ. $\partial\Omega \times \mathbb{R}$ can be represented as the graph of a $C^{2,\alpha}$-function $w(y) = w(y_2)$. By scaling we can assume $\delta = 1$. Let:

$$K = \{v \in C^{0,1}(D_1) : v \geq w \text{ in } B_1(0) \text{ and } v = u \text{ on } \partial B_1(0)\}.$$

We note that $u \in K$ satisfies the variational inequality:

$$\int_{D_1} a_j(\nabla u)(v-u)_j dy \geq 0, \tag{5.50}$$

for all $v \in K$, where $a_j(\nabla u) = u_j/\sqrt{1+|\nabla u|^2}$. $u_j = u_{y_j}$. Let $h = u - w$ so that $h \in C^{1,1}(B_1(0))$. Define:

$$\begin{aligned}
\Lambda(h) &= \{y \in B_1 : h(y) = 0\},\\
\Omega(h) &= \{y \in B_1 : h(y) > 0\},\\
F(h) &= \partial(\Omega(h)) \cup \partial(\Lambda(h)).
\end{aligned}$$

Clearly: $u(F(h)) = \gamma_*$ in $B_1(0) \times \mathbb{R}$. There remains to show that γ_* is the graph of a $C^{1,\alpha}$-function over $\gamma_+ \cap B_{\frac{r_0}{2}}(x_0)$. In other words, we will first show that γ_* is a $C^{1,\alpha}$-curve by showing that $F(h)$ is $C^{1,\alpha}$ (and hence, so is its image.) Next, we will show that γ_+ is C^1-graph over $\gamma_+ \cap B_{\frac{r_0}{2}}(x_0)$.

Consider $p = 0 \in F(h)$. Either 0 is an isolated point of $F(h)$, or it has positive density with respect to $\Lambda(h)$, or it has zero density with respect to $\Lambda(h)$. Note that by the regularity analysis in the preceding section, if 0 has positive density with respect to $\Lambda(h)$, then it is a regular point of the free boundary and $F(h)$ is a locally C^1 curve. Hence, $C^{1,\alpha}$ regularity would follow by the higher regularity theory we will establish in the ultimate chapter of these notese.

On the other hand. $\gamma_* = u(F(h))$ is the trace of a $BV(\Omega)$ function, hence 0 cannot be an isolated point of $F(h)$. Thus, it suffices to show that 0 cannot be a point of zero density with respect to $\Lambda(h)$. For this purpose, consider the Jacobi-field equation on the minimal surfaces:

$$\Delta_M V + |A_M|^2 V = 0, \tag{5.51}$$

where $|A_M|^2$ is the squared length of the second fundamental form and Δ_M the Laplace-Beltrami operator, both on the surface M. In our coordinate system: $V = -u_{y_1}/\sqrt{1+|du|^2}$, Lipschitz on $\overline{\Omega(h)}$ and satisfying:

$$\begin{cases} \Delta_M V + |A_M|^2 V = 0 \text{ in } \Omega(h), \quad V > 0 \text{ in } \Omega(h),\\ V|_{P(h)} = 0. \end{cases} \tag{5.52}$$

Recall that in the case 0 is a point of zero density with respect to $\Lambda(h)$, by the blow-up analysis of the preceding section:

$$\Lambda(h) \cap B_\eta(0) \subset \mathbb{C} = \left\{y, \theta(y, (\pm, 0)) \leq \Phi^{-1}(h)\right\},$$

and $h(y) = ay_2^2 + o\left(y_1^2 + y_2^2\right)$ for $(y_1, y_2) \in (\Omega(h) \cap B_\eta(0)) \backslash \mathbb{C}$. In particular, $\Lambda(h) \cap B_\eta(0) \subset \mathbb{C}_1$, where $C_1 = \{(y_1, y_2) \in B_\eta(0) : y_1 \geq |y_2|\}$.

On $\mathbb{C}_1$ consider $U_0(y) = r^\alpha \cos(\alpha\theta)$, where $\alpha = 2/3$, $\theta(y_1, y_2)$ is the angle between vectors (y_1, y_2) and $(-1, 0)$. Note that U_0 solves:

$$\begin{cases} \Delta_M U_0 = 0, \quad U_0 > 0 \text{ in } \mathbb{C}_1,\\ U_0|_{\partial\mathbb{C}_1} = 0. \end{cases} \tag{5.53}$$

Hence, by the maximum principle we have: $V \geq \epsilon U_0$ in $\mathbb{C}_1 \cap B_\eta(0)$ for some $\epsilon > 0$. Consequently, V cannot be Lipschitz at 0 , a contradiction. So 0 cannot be a point of zero density with respect to $\Lambda(h)$. So $F(h)$ is C^1 near 0 . By higher regularity theory for free boundaries to be addressed in the ultimate chapter, we have: $\gamma_* - u(F(h))$ is a $C^{1,\alpha}$-curve.

There remains to show that γ_* is the graph pf a $C^{1,\alpha}$-function. We argue by contradiction. Note that $u \in C^2$ unfiromly in $\overline{\Omega(h)} \cap B_{\frac{1}{2}}$. If γ_* is not a uniformly C^1-graph over γ_+, then we would obtain a point, say 0 by another change of coordinates, in $F(h) \cap D_{\frac{1}{2}}$ such that the second order blow-up of h at 0 is of form:

$$h_0(y) = ay_2^2, \quad 2a = -H_{\partial\Omega}(0) > 0, \tag{5.54}$$

i.e. $h(y) = ay_2^2 + o(y_1^2 + y_2^2)$. However, (4.18) and the Hopf boundary-point Lemma give: $\partial_{y_2} V \neq 0$ at 0. Hence: $\partial_{y_1}\partial_{y_2} h(0) \neq 0$, a contradiction with (5.54). Hence, the proof is complete. □

In the remaining part of this section, we consider the equation for surfaces with prescribed mean curvature in a bounded C^2 domain $\Omega \subset \mathbb{R}^2$, that is:

$$\operatorname{div}\left(\frac{u}{\sqrt{1+|\nabla u|^2}}\right) = H(x). \tag{5.55}$$

We begin with stating a few general facts regarding the solvability of (5.55), for which we refer to [11]. A necessary condition for (5.54) to have a solution $u \in C^2(\Omega)$ is:

$$\left|\int_A H dx\right| < \mathcal{H}_1(\partial A), \tag{5.56}$$

for every Caccioppoli set $A \subset \Omega$ such that $\emptyset \neq A \neq \Omega$. Here $\mathcal{H}_1$ denotes the one-dimensional Hausdorff measure.

In [11] the extremal case of equality in (5.56) is studied. In [18] his result is partially recovered, alongside a proof of higher regularity for u restricted to the part of $\partial\Omega$ where $H_{\partial\Omega}(x) < H(x)$ whenever H and $\partial\Omega$ are smooth.

Let Ω be a bounded, C^2 domain in $\mathbb{R}^2$. and let H be a Lipschitz function on $\overline{\Omega}$. Consider the unique solution u (up to an additive constant) of (5.55) with H sayisfying (5.56) with equality. Let $\Omega_1 \subsetneq \Omega$ with $\operatorname{per}(\Omega_1) < \infty$. Then the set $U = \{(x_1, x_2, x_3) \in \Omega_1 \times \mathbb{R} : x_3 < u(x)\}$ is a minimizer of the functional

$$\int |\nabla X_U| dx + \int H X_U dx dx_3 \text{ in } \Omega_1 \times \mathbb{R} \tag{5.57}$$

in the sense that for every set $V \subset \Omega_1 \times \mathbb{R}$, coinciding with U outside some compact set $K \subset \Omega_1 \times \mathbb{R}$, we have:

$$\int_K |\nabla X_U| dx + \int_K H X_U dx dx_3 \leq \int_K |\nabla X_V| dx + \int_K H X_V dx dx_3. \tag{5.58}$$

Moreover, if in a neighborhood of $x_0 \in \partial\Omega$, the inequality $H_{\partial\Omega}(x) < H(x)$ is satisfied, then $u(x)$ is bounded from above on this neighborhood. u is always bounded from below in this case. Hence, $u \geq 0$ can be assumed. See [11] for further discussion.

Suppose there exist positive constants C_0, r_0 such that:

$$H_{\partial\Omega}(x) \leq H(x) - C_0, \quad \forall x \in B_{r_0}(x_0) \cap \partial\Omega. \tag{5.59}$$

for some $x_0 \in \partial\Omega$. We will show that u restricted to $B_{r_0/2}(x_0) \cap \partial\Omega$ is smooth in the following sense:

Theorem 5.67. *Let Ω be a bounded, $C^{2,\alpha}$ domain in $\mathbb{R}^2$ and let H be a Lipschitz function on $\overline{\Omega}$ which satisfies* (5.56) *with equality. Suppose* (5.59) *holds. Then the restriction of u to $B_{\frac{r_0}{2}}(x_0) \cap \partial\Omega$ is uniformly $C^{1,\alpha}$. Moreover, u is $C^{0,\frac{1}{2}}$ in $\overline{\Omega} \cap B_{\frac{r_0}{2}}(x_0)$.*

Once again, using the higher regularity theory to be addressed in the ultimate chapter of these notes, one would deduce from Theorem 5.67:

Corollary 5.68. *In Theorem 5.67:*
(i) if Ω is $C^{k,\alpha}$ and if H is $C^{k-2,\alpha}$, then the restriction of u to $B_{\frac{r_0}{2}}(x_0) \cap \partial\Omega$ is $C^{k-1,\alpha}$ for $k = 3, 4, ...,$ $0 < \alpha < 1$;
(ii) if Ω and H are analytic, then u is also analytic on $B_{\frac{r_0}{2}}(x_0) \cap \partial\Omega$.

Let Q be the multiplicity-one integral current in $\mathbb{R}^3$ with $\text{spt}(Q) \subseteq (\partial\Omega \cap B_{r_0}(x_0)) \times \mathbb{R}$ and $\partial Q = T - Tu$, where:

$$T = \{(x, u^*(x)) : x \in \partial\Omega\}, \quad Tu = \{(x, u(x)) : x \in \partial\Omega\},$$

and

$$u^*(x) = \begin{cases} u(x) & \text{if} \quad x \in \partial\Omega \backslash B_{r_0}(x_0), \\ u(x) + \eta(x) & \text{if} \quad x \in \partial\Omega \cap B_{r_0}(x_0). \end{cases}$$

Above $\eta(x)$ is a smooth function such that $0 \leq \eta \leq M_0$ and

$$\eta(x) \equiv M_0 = \sup\{u(x) : x \in \partial\Omega \cap B_{r_0}(x_0)\}$$

for $x \in \partial\Omega \cap B_{\frac{r_0}{2}}(x_0)$. Then we have the following lemma analogous to Lemma 5.64:

Lemma 5.69. $\Delta = [\text{graph}(u) + Q] \cap (B_{r_0}(x_0) \times \mathbb{R})$ *is the unique integral current which minimizes*

$$M(\mathbb{S}) + \int_{B_{r_0}(x_0) \cap \Omega} H(x) \eta_{\mathbb{S}}(x) dx,$$

where $\eta_{\mathbb{S}}(x)$ is the slice $\langle \mathbb{S}, P, x \rangle$, where $P(x_1, x_2) = x$ and $\partial\mathbb{S} = \partial\Delta$.

For measure-theoretic definitions and facts we refer to [5].

The argument for proving Theorem 5.67 is completely analogous to that for proving Theorem 5.62. Yet this time the Jacobi-field equation is:

$$\Delta_M V + (|A_M|^2 - H^2)V = \delta_3 H, \tag{5.60}$$

where $V = 1/\sqrt{1 + |\nabla u|^2}$ and $|\delta_3 H| \leq |\nabla H| V$. Nevertheless, the comparison argument employed in the case of nonparemetric least area and the resulting estimate are still valid.

In [11], E. Giusti also showed that if $H(x) \equiv H_{\partial\Omega}(x)$ in an open arc $l \subset \partial\Omega$, then:

$$\lim_{x \to x_0} u(x) = +\infty \qquad \text{uniformly in } x_0 \in K \subset\subset l.$$

This can also be deduced from Lemma 5.69 and the regularity theory for obstacle problems. We briefly sketch the idea:

If there is a sequence $x_i \in K$ with:

$$\lim_{x_i \to x_0} u(x_i) = u(x_0) < \infty,$$

then $(x_0, u(x_0))$ will be a point in the support of the minimizer. Choose a ball $\partial\Omega \cap B_{r_0}(x_0, t)$ and move it upward from $t < 0$ towards $(x_0, u(x_0))$. Here we choose r_0 small enough that $B_{r_0}(x, t) \subset K \times \mathbb{R}$, Then it must touch the free boundary at some point p, possibly different from $(x_0, u(x_0))$. Then in a neighborhood of p, the vertical cylinder $\partial\Omega \times \mathbb{R}$ and the graph(u) are graphs over the tangent plane to $\partial\Omega \times \mathbb{R}$ at p. Now we may apply the Hopf boundary-point lemma at p to obtain a contradiction.

For further discussion on the solvability of the Dirichlet problem for (5.55) and the extension of the analysis that leads to Theorems 5.62 and 5.67 to the case of parametric elliptic integrals, we refer to [18].

The regularity theory of free boundaries in higher dimensions were first established by Luis Caffarelli: The regularity of free boundaries in higher dimensions. Acta Math. 139 (1977), no. 3-4, 155-184. The presentation here follows the approach used by Caffarelli: Compactness methods in free boundary problems. Comm. Partial Differential Equations 5 (1980), no. 4, 427-448. Further developments using the boundary Harnack inequality approach may be found in a much detailed exposition [4]. See also [7].

6. Regularity of Singular Sets of Free Boundary

6.1. Singular Set of the Free Boundary.

Definition 6.1. Let $u \in P_1(M)$ and let $N(u)$, $\Lambda(u)$ and $\Gamma(u)$ (or Γ) be defined as before. $x_0 \in \Gamma(u)$ is said to be in the singular set of $\Gamma(u)$, denoted by $x_0 \in \operatorname{sing} u$ if for $\forall r > 0$, $\delta_{x_0,r}(\Lambda(u)) \leq \sigma(r)$, where $\delta_r(\Lambda(u))$ is defined in Lemma 5.26 and $\sigma(r)$ is some monotone increasing function defined on $\mathbb{R}_+$ satisfying $\sigma(0^+) = 0$. That is to say, for $\forall r > 0$, $\Lambda(u) \cap B_r(x_0)$ is contained in a strip of width $r\sigma(r)$.

Remark 6.2. If we fix the point $x_0 \in B_1(0)$ we can see that $\delta_{x_0,r}(\Lambda(u))$ is a function of r with $0 \leq \delta_{x_0,r}(\Lambda(u)) \leq 1$, which is given by the fact that $\frac{MD(\Lambda(u)\cap B_r(x_0))}{2r} \leq \frac{2r}{2r} = 1$. We claim that the existence of $\sigma(r)$ is equivalent to $\lim_{r\to 0} \delta_{x_0,r}(\Lambda(u)) = 0$. If $\lim_{r\to 0} \delta_{x_0,r}(\Lambda(u)) = 0$ is not true, we can choose $0 < \varepsilon_0 \ll 1$ and a sequence $\{r_j\} \subset (0,1)$ such that $r_{j+1} < r_j$ for any $j \in \mathbb{N}_+$, $\lim_{j\to\infty} r_j = 0$ and $\delta_{x_0,r_j}(\Lambda(u)) \geq \varepsilon_0$. This implies that such $\sigma(r)$ does not exists. Indeed, $\sigma(r_j) \geq \varepsilon_0$ for any $j \in \mathbb{N}_+$, which is a contradiction to $\sigma(0^+)$. On the other hand, if $\lim_{r\to 0} \delta_{x_0,r}(\Lambda(u)) = 0$, we can define $\sigma(r) = \sup_{0<\rho\leq r} \delta_{x_0,\rho}(\Lambda(u))$. It is easy to show that $\sigma(r)$ is monotone increasing function and $\sigma(0^+) = 0$.

Remark 6.3. If $\lim_{r\to 0} \delta_{x_0,r}(\Lambda(u)) = 0$ is not true, we can define $\varepsilon_0 > 0$ and $\{r_j\}$ as in Remark 6.2. Then $\delta_{x_0,r_j}(\Lambda(u)) \geq \varepsilon_0$. By using methods in Chapter 4, we can see that $\Gamma(u)$ is a C^1 hypersurface inside $B_{\frac{1}{4}}(0)$. Thus $\operatorname{sing} u$ is a relative closed subset of $\Gamma(u)$.

Remark 6.4. Let $w \in P_1(M)$, $0 \in \operatorname{sing}(w) \cap \Gamma(w)$ and $\delta_r(\Lambda(w)) \leq \sigma(r)$ with $\sigma(r)$ defined above, then for $w_\lambda(x) = \lambda^{-2} w(\lambda x)$, with $\lambda \to 0^+$ we have $\delta_r(\Lambda(w_\lambda)) \leq \sigma(\lambda r)$.

The main result of this chapter is the following:

Theorem 6.5. *Given $0 = x_0 \in \Gamma(u)$ and $x_0 \in \operatorname{sing} u$, there is a matrix $M_{x_0} \geq 0$, s.t.*
(a) $Q_{x_0}(x) = \frac{x^T M_{x_0} x}{2} \geq 0$ with $\Delta Q_{x_0} = \operatorname{tr}(M_{x_0}) = 1$;
(b) $|u(x) - Q_{x_0}(x)| \leq |x|^2 \widetilde{\sigma}(|x|)$ for some universal $\widetilde{\sigma}(r)$.
Moreover, M_{x_0} is continuous in x_0 for $x_0 \in \operatorname{sing} u \cap \Gamma$. If $\dim\ker M_{x_0} = k$, then sing u is contained in a k-dimensional C^1-manifold in a neighborhood of x_0. The size of the neighborhood depends on the smallest nonzero eigenvalue of M_{x_0}.

Lemma 6.6. *Let $\sigma(r) : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotone increasing function such that $\sigma(0^+) = 0$. For $\forall \varepsilon > 0$, $\exists N_\varepsilon > 0$, s.t. for any $N \geq N_\varepsilon$ if $w \in P_N(M)$ is a solution in $B_N(0)$, $0 \in \Gamma(w) \cap \operatorname{sing} w$ with $\delta_r(\Lambda(w)) \leq \sigma(r)$ for any $r > 0$, then there is $Q(x) = \frac{x^T A x}{2}$, s.t. $A \geq 0$, $\Delta Q = 1$, $\|\nabla(Q - w)\|_{L^\infty(B_1(0))} + \|Q - w\|_{L^\infty(B_1(0))} \leq \varepsilon$.*

Proof. Suppose not. Then $\exists \{w_k\}$ solutions on $B_k(0)$ with $k \to +\infty$, s.t. $0 \in \Gamma(w_k) \cap \operatorname{sing} w_k$ with $\delta_r(\Lambda(w_k)) \leq \sigma(r)$ for any $r > 0$ and there is no such Q_k that $\|Q_k - w_k\|_{L^\infty(B_1(0))} \leq \varepsilon_0$ for some $\varepsilon_0 > 0$. By compactness, there is an entire solution $w_\infty \in P_\infty(M)$, s.t. $0 \in \Gamma(w_\infty)$ and $w_k \to w_\infty$ in $C^{1,1}_{loc}$. In particular, we have $\|w_k - w_\infty\|_{L^\infty B_1(0)} + \|\nabla(w_k - w_\infty)\|_{L^\infty(B_1(0))} \to 0$ when $k \to \infty$. We claim that w_∞ must be a quadratic polynomial.

We first note that $\Lambda(w_\infty)$ has no interior. If not, since $\Lambda(w_\infty)$ is convex (since $w_k \in R_k(M)$, we can define $u_k(x) = k^{-2} w_k(kx) \in P_1(M)$ and $\lambda_k = k^{-1}$, then the convexity is given by Remark 5.19) for entire solutions, $\Lambda(w_\infty) \cap B_1(0)$ contains a ball $B_{r_0}(a)$ where $w_\infty \equiv 0$. Again by the convexity, $\Lambda(w_\infty)$ would contain a cone C^* (truncated cone) of a fixed size with the vertex at the origin. Precisely speaking

$$C^* = \{x \in \mathbb{R}^n : x_n \leq -\alpha(x_1^2 + x_2^2 + ... + x_{n-1}^2)^{\frac{1}{2}}, |x| \leq R\}.$$

From nondegeneracy (see Lemma 5.6), since $w_k \to w_\infty$ uniformly on $B_{r_0}(a)$, $w_k \equiv 0$ on $B_{\frac{r_0}{2}}(a)$ for sufficiently large k. Indeed, if there eixsts $x_k \in B_{\frac{r_0}{2}}(a)$ such that $w_k(x_k) > 0$, then

$$\sup_{x,y\in B_{r_0}(a)} |w_k(x) - w_k(y)| \geq \sup_{B_{\frac{r_0}{2}}(x_k)} (w_k(x) - w_k(x_k)) \geq \frac{r_0^2}{8n} > 0.$$

Taking $k \to \infty$, it is a contradiction. Consequently, via nondegegeneracy again, $w_k \equiv 0$ on a truncated cone (with $|x| > \frac{r_1}{2}$ and r_1 to be determined) that contains the truncated central one third of the cone (again in the region $|x| > \frac{r_1}{2}$) described above. The statments above actually imply that for some C_n, in

$$C_1^* = \{x \in \mathbb{R}^n : x_n \leq -C_n\alpha(x_1^2 + x_2^2 + ... + x_{n-1}^2)^{\frac{1}{2}}\} \cap \left\{x \in \mathbb{R}^n : \frac{r_1}{2} < |x| < R\right\},$$

$w_k \equiv 0$. Since $0 \in \operatorname{sing} w_k \cap \Gamma(w_k)$, we find $\Lambda(w_k) \cap B_{r_1}(0)$ must be contained in a strip with width $r_0\sigma(r_0)$. In fact, if we define $v_k(x) = k^{-2}w_k(kx) \in P_1(M)$ for $x \in B_1(0)$, $0 \in \operatorname{sing} v_k \cap \Gamma(v_k)$, then

$$\{x \in \mathbb{R}^n : x_n \leq -C_n\alpha(x_1^2 + x_2^2 + ... + x_{n-1}^2)^{\frac{1}{2}}\} \cap \left\{x \in \mathbb{R}^n : \frac{r_0}{2k} < |x| < \frac{R}{k}\right\} \subset \Lambda(v_k).$$

Therefore, $\delta_{r_1/k}(\Lambda(v_k)) \geq c_n$, where c_n is a constant depending only on n. Owing to the fact that $0 \in \operatorname{sing}(v_k) \cap \Gamma(v_k)$ and Remark 6.4, we can obtain a monotone increasing function $\sigma(kr)$, such that $\delta_r(\Lambda(v_k)) \leq \sigma(kr)$ for any $r > 0$. Then $\sigma(r_1) \geq \sigma(kr_1/k) \geq c_n$. Let $r_0 \to 0^+$, it is a contradiction. Hence, $\Lambda(w_\infty)$ has empty interior, which, together with the convexity of w_∞ implies that w_∞ is a quadratic polynomial $\frac{1}{2}x^TBx$ with $B \geq 0$; moreover, $\Delta w_\infty = 1$. Then $\|\nabla(Q_k - w_k)\|_{L^\infty(B_1(0))} + \|Q_k - w_k\|_{L^\infty(B_1(0))} \leq \varepsilon$ must be true if we take $Q_k \equiv w_\infty$ when k is large, which contradicts with our assumption. So we proved the lemma. □

Corollary 6.7. *Let $w \in P_1(M)$ be a solution, with $0 \in \Gamma(w) \cap \operatorname{sing} w$. Then $\forall\varepsilon > 0$, $\exists r_0(\varepsilon)$, s.t. $\forall 0 < r < r_0$, $\exists Q^r = \frac{x^TA_rx}{2}$, satisfying $A_r \geq 0$, $\Delta Q^r \equiv 1$, $\|w - Q^r\|_{L^\infty(B_r(0))} \leq \varepsilon r^2$ and $\|\nabla(w - Q^r)\|_{L^\infty(B_r(0))} \leq \varepsilon r$.*

Proof. Since $0 \in \operatorname{sing}(w) \cap \Gamma(w)$, we can obtain σ as above such that $\delta_r(\Lambda(w)) \leq \sigma(r)$. For any $\varepsilon > 0$, by using Lemma 6.6, we can obtain $N_\varepsilon > 0$ such that for any $N \geq N_\varepsilon$, $v \in P_N(M)$ is a solution in $B_N(0)$, $0 \in \operatorname{sing}(w) \cap \Gamma(v)$ and $\delta_r(\Lambda(w)) \leq \sigma(r)$, then there is $Q(x) = \frac{x^TAx}{2}$, such that $A \geq 0$, $\Delta Q = 1$ and $\|Q - v\|_{L^\infty(B_1(0))} \leq \varepsilon$. Then we can choose $r_0(\varepsilon) = \min\{1, \frac{1}{N_\varepsilon}\}$ and $w_r(x) = r^{-2}w(rx)$ with $0 < r < r_0(\varepsilon)$. It is obvious that $\delta_\rho(\Lambda(w_r)) \leq \sigma(r\rho) \leq \sigma(\rho)$ for any $\rho > 0$. Then by using Lemma 6.6, we can obtain that there is Q such that

$$\begin{aligned}
&\|\nabla(r^{-2}w(rx) - Q(x))\|_{L^\infty(B_1(0))} + \|r^{-2}w(rx) - Q(x)\|_{L^\infty(B_1(0))} \leq \varepsilon, \\
&\Rightarrow \|r^{-2}w(rx) - r^{-2}Q(rx)\|_{L^\infty(B_1(0))} \leq \varepsilon, \quad \|\nabla(r^{-2}w(rx) - r^{-2}Q(rx))\|_{L^\infty(B_1(0))} \leq \varepsilon, \\
&\Rightarrow \|w(x) - Q(x)\|_{L^\infty(B_r(0))} \leq \varepsilon r^2, \quad \|\nabla w(x) - \nabla Q(x)\|_{L^\infty(B_r(0))} \leq \varepsilon r.
\end{aligned}$$

Let $Q^r(x) = Q(x)$, we can complete the proof. □

Remark 6.8. One may write in another way $\varepsilon = \widetilde{\sigma}(r)$, where $\widetilde{\sigma}(r)$ has the same properties as $\sigma(r)$ described above. Indeed, for any $0 < r \leq 1$, we can choose

$$\sigma_1(r) = \inf\{\varepsilon > 0 : \exists Q^r \text{ s.t. } \|w - Q^r\|_{L^\infty(B_r(0))} \leq \varepsilon r^2, \|\nabla(w - Q^r)\|_{L^\infty(B_r(0))} \leq \varepsilon r\},$$

where Q^r is a quadratic polynomial. We need to show that $\widetilde{\sigma}(0^+) = 0$. Firstly, for any $\varepsilon_1 > 0$, we have $r_0(\varepsilon_1) > 0$ such that for any $0 < r \leq r_0(\varepsilon_1)$, there exists a quadratic polynomial Q^r such that $\|w - Q^r\|_{L^\infty(B_r(0))} \leq \varepsilon_1 r^2$ and $\|\nabla(w - Q^r)\|_{L^\infty(B_r(0))} \leq \varepsilon_1 r$. This implies that for any $0 < r \leq r_0(\varepsilon_1)$, $\sigma_1(r) \leq \varepsilon_1$. Then $\sigma_1(0^+) = 0$ and $r^{-2}\|w - Q^r\|_{L^\infty(B_r(0))} \leq \sigma_1(r), r^{-1}\|\nabla(w - Q^r)\|_{L^\infty(B_r(0))} \leq \sigma_1(r)$. Then by choosing $\widetilde{\sigma}(r) = \sup_{0<\rho\leq r}\{\sigma_1(\rho)\}$ and replace $\sigma_1(r)$ by $\widetilde{\sigma}(r)$, we can obtain the following results.

Proposition 6.9. *Let $w \in P_1(M)$ be a solution, with $0 \in \Gamma(w) \cap \operatorname{sing} w$. Then there exists monotone increasing function $\widetilde{\sigma}(r)$ with $\widetilde{\sigma}(0^+) = 0$, such that $\exists Q^r = \frac{x^T A_r x}{2}$, satisfying $A_r \geq 0$, $\Delta Q^r \equiv 1$, $\|w - Q^r\|_{L^\infty(B_r(0))} \leq \widetilde{\sigma}(r) r^2$ and $\|\nabla(w - Q^r)\|_{L^\infty(B_r(0))} \leq \widetilde{\sigma}(r) r$.*

Now the problem of proving Theorem 6.5 is the uniqueness of the limit of such Q^r's as $r \to 0^+$. If, for example, $\|w - Q^r\|_{L^\infty(B_r(0))} \leq C r^\alpha \cdot r^2$ for some $\alpha > 0$, then the limit of $\{Q^r\}$ is unique and we are done! In order to show the uniqueness, we introduce the following monotonicity approach due to Alt, Caffarelli and Friedman.

Theorem 6.10 (Monotonicity)**.** *Let u_1 and u_2 be two continuous functions in $B_1(0) \subset \mathbb{R}^n$, s.t. (a) $u_1 \cdot u_2 = 0$, (b) $u_1(0) = u_2(0) = 0$, (c) $u_i \Delta u_i \geq 0, i = 1, 2$.*
Then
$$I(R) = \left(\frac{1}{R^2} \int_{B_R(0)} \frac{|\nabla u_1|^2(x)}{|x|^{n-2}} dx\right) \left(\frac{1}{R^2} \int_{B_R(0)} \frac{|\nabla u_2|^2(x)}{|x|^{n-2}} dx\right) \triangleq \frac{1}{R^2} I_1(R) \cdot \frac{1}{R^2} I_2(R)$$
is increasing in R.

Remark 6.11. It is easy to check the following properties of $I(R)$:
(a) If $u_1(x) = \alpha x_1^+, R^{-2} I_1(R) = C(n)\alpha^2$.
(b) If $u_{i,\lambda}(x) \triangleq u_i(\lambda x)/\lambda$, $I(\frac{R}{\lambda}; u_{1,\lambda}, u_{2,\lambda}) = I(R; u_1, u_2)$.
(c) Suppose the supports of u_1 and u_2 are separated by an $(n-1)$-dimensional hypersurface denoted by Γ If Γ is smooth and $\nabla_\nu u_i(x)$ exists, then $R^{-2} I_i(R) \to C(n)|\nabla_\nu u_i(0)|^2$ as $R \to 0^+$. Thus, by Theorem 6.10,
$$C(n)^2 |\nabla_\nu u_1(0)|^2 |\nabla_\nu u_2(0)|^2 \leq I\left(\frac{1}{2}\right).$$

Proof. (a) $|\nabla u_1| = \alpha$ when $x_1 > 0$ and $|\nabla u_1| = 0$ when $x_1 < 0$. Then $R^{-2} I_1(R) = C(n)\alpha^2$.

(b) Trivial.

(c) We only need to show that $R^{-2} I_i(R) \to C(n)|\nabla u_i|^2$ since $u_i \equiv 0$ on Γ, which implies that $|\nabla_{\tan} u_i(0)| = 0$. Let $C(n) = \frac{1}{R^2} \int_{B_R(0)} \frac{1}{|x|^{n-2}} dx$. By simple calculations, we can obtain that
$$\begin{aligned} \left|\frac{I_i(R)}{R^2} - C(n)|\nabla u_i(0)|^2\right| &= \left|\sum_{j=0}^{\infty} \int_{2^{-j+1}R < |x| \leq 2^{-j}R} \frac{\big||\nabla u_i(x)|^2 - |\nabla u_i(0)|^2\big|}{|x|^{n-2}} dx\right| \\ &\leq C_1(n) \sum_{j=0}^{\infty} 2^{-2j} \left\{\frac{1}{|B_{2^{-j}R}(0)|} \int_{B_{2^{-j}R}(0)} \big||\nabla u_i(x)|^2 - |\nabla u_i(0)|^2\big| dx\right\} \end{aligned}$$
Since ∇u_i is continuous, we can see that $|R^{-2} I_i(R) - C(n)|\nabla u_i(0)|^2| \to 0$ when $R \to 0$. □

The following lemma, which is easy to show, would be used in the proof of Theorem 6.10.

Lemma 6.12. *Given a set $\Sigma \subset \mathbb{S}^{n-1}$, let C_Σ be the cone over Σ. Let $h(x) = r^\alpha f(\theta)$, s.t.*
$$\begin{cases} \Delta h(x) = 0 & \text{in } C_\Sigma, \\ h(x)|_{\partial C_\Sigma} = 0, \\ \Delta_\theta f(\theta) + \lambda_1 f(\theta) = 0 & \text{in } \Sigma, \\ f(\theta)|_{\partial \Sigma} = 0. \end{cases} \tag{6.1}$$
Then $\lambda_1 = \alpha(\alpha + n - 2)$.

Proof of Theorem 6.10. By rescaling, we only need to prove $I'(1) \geq 0$, i.e.
$$I'(1) = I_1'(1) I_2(1) + I_1(1) I_2'(1) - 4 I_1(1) I_2(1) \geq 0.$$

So we need to verify

$$\frac{I_1'(1)}{I_1(1)} + \frac{I_2'(1)}{I_2(1)} \geq 4. \tag{6.2}$$

We notice by using the fact that $\Delta(1/|x|^{n-2}) = 0$ for any $n \geq 2$ and $x \in \mathbb{R}^n \backslash \{0\}$,

$$\begin{aligned} I_i(1) &= \int_{B_1(0)} \frac{|\nabla u_i|^2}{|x|^{n-2}} dx \leq \int_{B_1(0)} \frac{\Delta(u_i^2/2)}{|x|^{n-2}} dx = \int_{B_1(0)} \frac{\Delta(u_i^2/2)}{|x|^{n-2}} - \frac{u_i^2}{2}\Delta\left(\frac{1}{|x|^{n-2}}\right) dx \\ &= \int_{\partial B_1(0)} \left[\partial_r\left(\frac{u_i^2}{2}\right)\frac{1}{1^{n-2}} - (n-2)\frac{u_i^2}{2}\right] dA(x) = \int_{\Sigma_i}\left[u_i\partial_r u_i + \frac{n-2}{2}u_i^2\right] dA(x), \end{aligned}$$

where $\Sigma_i = \{x \in \partial B_1(0) : u_i(x) \neq 0\}$. We used Green's identity in the last equation. On the other hand,

$$I_i'(1) = \int_{\Sigma_i} |\nabla u_i|^2 dA(x) = \int_{\Sigma_i} (\partial_r u_i)^2 + |\nabla_{\tan} u_i|^2 dA(x),$$

where $\nabla_{\tan} u_i$ is the gradient along $\partial B_1(0)$. Hence, by Cauchy-Schwarz inequality,

$$\begin{aligned} \frac{I_i'(1)}{I_i(1)} &\geq \frac{\int_{\Sigma_i}(\partial_r u_i)^2 + |\nabla_{\tan} u_i|^2 dA}{\int_{\Sigma_i} u_i\partial_r u_i + \frac{n-2}{2}u_i^2 dA} \geq \frac{\int_{\Sigma_i}(\partial_r u_i)^2 + |\nabla_{\tan} u_i|^2 dA}{\int_{\Sigma_i}\left(\frac{A_i}{2}u_i^2 + \frac{1}{2A_i}(\partial_r u_i)^2\right) + \frac{n-2}{2}u_i^2 dA} \\ &= 2\frac{\int_{\Sigma_i}\partial_r u_i^2 + |\nabla_{\tan} u_i|^2 dA}{\int_{\Sigma_i}\frac{1}{A_i}(\partial_r u_i)^2 + (A_i + n - 2)u_i^2 dA}. \end{aligned}$$

Since

$$\frac{\int_{\Sigma_i} |\nabla_{\tan} u_i|^2 dA}{\int_{\Sigma_i} u_i^2 dA} \geq \lambda_{i,1},$$

where $\lambda_{i,1}$ is the first Dirichlet eigenvalue of $\Delta_{\mathbb{S}^{n-1}}$ on Σ_i, we find

$$\frac{1}{A_i} = \frac{A_i + n - 2}{\lambda_{i,1}}$$

would be optimal choice of A_i's, i.e. $A_i(A_i + n - 2) = \lambda_{i,1}$. In the notation of Lemma 6.12, $A_i = \alpha_i$. This gives $\frac{I_i'(1)}{I_i(1)} \geq 2A_i$. Note that all above become equality when we choose $u_i|_{C_{\Sigma_i}}$ to be the homogeneous extension of degree A_i of $u_i|_{\Sigma_i}$. Now proving (6.2) reduces to showing $A_1 + A_2 \geq 2$. To be more precise, given two disjoint cones $C_i \subset \mathbb{R}^n$, $i = 1, 2$, let h_i be the corresponding homogeneous function of degree A_i, which is harmonic and nonnegative on C_i and vanishes on ∂C_i. Then we want to show $A_1 + A_2 \geq 2$.

The set function A_i as a function of Σ_i was studied by Sperner [23] and by Friedland and Hayman [6]. In [23] it is proved that $A_i(E) \geq A_i(E^*)$ where E, $E^* \subset \partial B_1(0)$ provided E^* is a spherical cap having the same $(n-1)$-dimensional Hausdorff measure as E. In [6] it is proved that $A_i(E) \geq \psi(s)$ where $s = \frac{\mathcal{H}^{n-1}(E)}{\mathcal{H}^{n-1}(\partial B_1(0))}$ and $\psi(s)$ is convex and decreasing:

$$\psi(s) = \begin{cases} \frac{1}{2}\log\frac{1}{4s} + \frac{3}{2} & \text{if } s < \frac{1}{4}, \\ 2(1-s) & \text{if } \frac{1}{4} < s < 1. \end{cases}$$

Setting $s_i = \frac{\mathcal{H}^{n-1}(\Sigma_i)}{\mathcal{H}(\partial B_1(0))}$, we then have $s_1 + s_2 \leq \frac{1}{2}$ and

$$A_1 + A_2 \geq \psi(s_1) + \psi(s_2) \geq 2\psi\left(\frac{s_1 + s_2}{2}\right) \geq 2\psi\left(\frac{1}{2}\right) = 2,$$

where we have used the fact that $s_1 + s_2 = \frac{\mathcal{H}^{n-1}(\Sigma_1)}{\mathcal{H}(\partial B_1(0))} + \frac{\mathcal{H}^{n-1}(\Sigma_2)}{\mathcal{H}(\partial B_2(0))} \leq \frac{1}{2}$. □

Proof of Theorem 6.5. First of all we can choose $\widetilde{\sigma}(r) \geq \sigma(r)$, otherwise, we can consider $\max\{\sigma(r), \widetilde{\sigma}(r)\}$. Let $0 \in \Gamma(u) \cap \operatorname{sing} u$. Then by Corollary 6.7, for $\forall r > 0$,

(a) $\Lambda(u)\cap B_r(0)\subset S_{r\widetilde{\sigma}(r)}$;
(b) $\exists Q^r=\frac{1}{2}x^TM^rx$ with $\Delta Q^r=1$, $M^r\geq 0$ and

$$\|u-Q^r\|_{L^\infty(B_r(0))}\leq r^2\widetilde{\sigma}(r),\quad \|\nabla(u-Q^r)\|_{L^\infty(B_r(0))}\leq r\widetilde{\sigma}(r).$$

Let $u^r(x)=r^{-2}u(rx)$ and $\Lambda^r=\Lambda(u^r)$. Then $\Lambda^r\cap B_1(0)\subset S_{\widetilde{\sigma}(r)}$ and

$$\|u^r-Q^r\|_{L^\infty(B_1(0))}\leq \widetilde{\sigma}(r),\quad \|\nabla(u^r-Q^r)\|_{L^\infty(B_1(0))}\leq \widetilde{\sigma}(r).$$

Hence, outside $S_{\widetilde{\sigma}(r)^{\frac{1}{3}}}$ ($\widetilde{\sigma}(r)\ll 1$), one has $S_{\widetilde{\sigma}(r)}\subset S_{\widetilde{\sigma}(r)^{\frac{1}{3}}}$ and

$$|\partial^2_{ij}(u^r-Q^r)|\leq \frac{C}{\widetilde{\sigma}(r)^{\frac{2}{3}}}\operatorname{osc}(u^r-Q^r)\leq C\widetilde{\sigma}(r)^{\frac{1}{3}},$$

where we have used the fact that $\Delta(u^r-Q^r)=0$ in $B_1(0)\cap\left(S_{\widetilde{\sigma}(r)^{\frac{1}{3}}}\right)^c$. In fact, for any $x\in B_1(0)\cap\left(S_{\widetilde{\sigma}(r)^{\frac{1}{3}}}\right)^c$, one can see that $B_{\frac{1}{2}\left(\widetilde{\sigma}(r)^{\frac{1}{3}}-\widetilde{\sigma}(r)\right)}(x)\subset x\in B_1(0)\cap\left(S_{\widetilde{\sigma}(r)^{\frac{1}{3}}}\right)^c$ and

$$\begin{aligned}|\nabla^2(u^r-Q^r)(x)|&\leq \frac{C}{(\widetilde{\sigma}(r)^{\frac{1}{3}}-\widetilde{\sigma}(r))^2}\|u^r-Q^r\|_{L^\infty(B_1(0))}\\&\leq \frac{C}{\widetilde{\sigma}(r)^{\frac{2}{3}}}\operatorname{osc}_{B_1(0)}(u^r-Q^r)\leq C\widetilde{\sigma}(r)^{\frac{1}{3}},\end{aligned}$$

if $r\ll 1$. Let $u_e=\partial_eu$, where e is a unit vector to be determined later. $\Delta u_e=0$ in $\{u>0\}$ and $u_e\equiv 0$ on $\Lambda(u)$. Define $u_e^+=\max\{u_e,0\}$ and $u_e^-=\min\{u_e,0\}$. We apply monotonicity formula to $u_e^\pm$, i.e.

$$I(R)=\frac{1}{R^2}\int_{B_R(0)}\frac{|\nabla u_e^+|^2}{|x|^{n-2}}dx\cdot\frac{1}{R^2}\int_{B_R(0)}\frac{|\nabla u_e^-|^2}{|x|^{n-2}}dx$$

is increasing in R. We calculate

$$I^+(R,e)=\frac{1}{R^2}\int_{\{\partial_eu>0\}\cap B_R(0)}\frac{|\nabla\partial_eu|^2}{|x|^{n-2}}dx=\int_{\{\partial_eu^R>0\}\cap B_1(0)}\frac{|\nabla\partial_eu^R|^2}{|x|^{n-2}}dx.$$

More precisely,

$$\begin{aligned}I^+(R,e)&=\int_{\{\partial_eu^R>0\}\cap B_1(0)}\frac{|\nabla\partial_eu^R(x)|^2}{|x|^{n-2}}dx=\int_{\{(\partial_eu)(Rx)>0\}\cap B_1(0)}\frac{|(\nabla\partial_eu)(Rx)|^2}{|x|^{n-2}}dx\\&=\frac{1}{R^2}\int_{\{\partial_eu(x)>0\}\cap B_R(0)}\frac{|(\nabla\partial_eu)(x)|^2}{|x|^{n-2}}dx=\frac{1}{R^2}\int_{\{\partial_eu>0\}\cap B_R(0)}\frac{|\nabla\partial_eu|^2}{|x|^{n-2}}dx.\end{aligned}$$

We are going to replace $\nabla\partial_eu^R$ by $\nabla\partial_eQ^R$ and split $I^+(R,e)$ into the integral in $S_{\widetilde{\sigma}(R)^{\frac{1}{3}}}$ and that in its compliment. Noticing that $\partial_j\partial_eQ^R=M^R_je=e_j^TM^Re$, we have

$$I^+(R,e)=\int_{\{\partial_eu^R>0\}\cap B_1(0)}\frac{\|M^Re\|^2}{|x|^{n-2}}dx+O\left(\widetilde{\sigma}(R)^{\frac{1}{3}}\right).$$

More precisely, we use

$$\begin{aligned}I^+(R,e)&=\int_{\{\partial_eu^R>0\}\cap B_1(0)}\frac{\|\nabla\partial_eQ^R\|^2}{|x|^{n-2}}dx+\int_{\{\partial_eu^R>0\}\cap B_1(0)}\frac{2\langle\nabla\partial_eu^R-\nabla\partial_eQ^R,\nabla\partial_eQ^R\rangle}{|x|^{n-2}}dx\\&\quad+\int_{\{\partial_eu^R>0\}\cap B_1(0)}\frac{|\nabla\partial_eu^R-\nabla\partial_eQ^R|^2}{|x|^{n-2}}dx\\&=\int_{\{\partial_eu^R>0\}\cap B_1(0)}\frac{\|M^Re\|^2}{|x|^{n-2}}dx+\int_{\{\partial_eu^R>0\}\cap B_1(0)}\frac{2\langle\nabla\partial_eu^R-\nabla\partial_eQ^R,M^Re\rangle}{|x|^{n-2}}dx\\&\quad+\int_{\{\partial_eu^R>0\}\cap B_1(0)}\frac{|\nabla\partial_eu^R-\nabla\partial_eQ^R|^2}{|x|^{n-2}}dx\end{aligned}$$

and

$$
\begin{aligned}
&\int_{\{\partial_e u^R>0\}\cap B_1(0)} \frac{2\langle \nabla\partial_e u^R-\nabla\partial_e Q^R, M^R e\rangle}{|x|^{n-2}}dx\\
&\quad=\left(\int_{\{\partial_e u^R>0\}\cap B_1(0)\cap\left(S_{\widetilde{\sigma}(R)^{\frac13}}\right)^c}+\int_{\{\partial_e u^R>0\}\cap B_1(0)\cap S_{\widetilde{\sigma}(R)^{\frac13}}}\right)\frac{2\langle \nabla\partial_e u^R-\nabla\partial_e Q^R, M^R e\rangle}{|x|^{n-2}}dx\\
&\quad=O\left(\widetilde{\sigma}(R)^{\frac13}\right)+O\left(\widetilde{\sigma}(R)^{\frac13}\right)=O\left(\widetilde{\sigma}(R)^{\frac13}\right)
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\int_{\{\partial_e u^R>0\}\cap B_1(0)} \frac{|\nabla\partial_e u^R-\nabla\partial_e Q^R|^2}{|x|^{n-2}}dx\\
&\quad=\left(\int_{\{\partial_e u^R>0\}\cap B_1(0)\cap\left(S_{\widetilde{\sigma}(R)^{\frac13}}\right)^c}+\int_{\{\partial_e u^R>0\}\cap B_1(0)\cap S_{\widetilde{\sigma}(R)^{\frac13}}}\right)\frac{|\nabla\partial_e u^R-\nabla\partial_e Q^R|^2}{|x|^{n-2}}dx\\
&\quad=O\left(\widetilde{\sigma}(R)^{\frac13}\right)+O\left(\widetilde{\sigma}(R)^{\frac23}\right)=O\left(\widetilde{\sigma}(R)^{\frac13}\right)
\end{aligned}
$$

Let $D^{\pm}=\{\partial_e Q^R=x^TM^Re>\pm\widetilde{\sigma}(R)\}$ and $D=\{\partial_e Q^R=x^TM^Re>0\}$. Then

$$
\begin{aligned}
&D^+\subset\{\partial_e u^R>0\}\subset D^-(\Leftarrow\|\nabla(u^R-Q^R)\|_{L^\infty(B_1(0))}\le\widetilde{\sigma}(R))\\
&\int_{D^+}\frac{dx}{|x|^{n-2}}\|M^Re\|^2-C\widetilde{\sigma}(R)^{\frac13}\le\int_{\{\partial_e u^R>0\}\cap B_1(0)}\frac{dx}{|x|^{n-2}}\|M^Re\|^2-C\widetilde{\sigma}(R)^{\frac13}\\
&\qquad\le I^+(R,e)\le\int_{D^-}\frac{dx}{|x|^{n-2}}\|M^Re\|^2+C\widetilde{\sigma}(R)^{\frac13},\\
&\left|\int_{D^\pm}\frac{dx}{|x|^{n-2}}-\int_{D}\frac{dx}{|x|^{n-2}}\right|\le\left|\int_{D^-}\frac{dx}{|x|^{n-2}}-\int_{D^+}\frac{dx}{|x|^{n-2}}\right|\le\frac{C\widetilde{\sigma}(R)}{\|M^Re\|}.
\end{aligned}
\tag{6.3}
$$

The last estimate is based on the fact that $D^-\backslash D^+$ is contained in a strip with width $O\left(\frac{\widetilde{\sigma}(R)}{\|M^Re\|}\right)$. Indeed,

$$
\begin{aligned}
x\in D^-\backslash D^+&\Leftrightarrow-\widetilde{\sigma}(R)<x^TM^Re\le\widetilde{\sigma}(R)\\
&\Leftrightarrow x\text{ is in a strip with width }O\left(\frac{\widetilde{\sigma}(R)}{\|M^Re\|}\right).
\end{aligned}
\tag{6.4}
$$

Let $C^*(n)=\int_{B_1^+(0)}\frac{dx}{|x|^{n-2}}$ and we have

$$
\begin{aligned}
I^+(R,e)&=\int_{\{\partial_e u^R>0\}\cap B_1(0)}\frac{\|M^Re\|^2}{|x|^{n-2}}dx+O\left(\widetilde{\sigma}(R)^{\frac13}\right)\\
&=\int_D\frac{\|M^Re\|^2}{|x|^{n-2}}dx+\left(\int_{D^-}-\int_D\right)\frac{\|M^Re\|^2}{|x|^{n-2}}dx+O\left(\widetilde{\sigma}(R)^{\frac13}\right)\\
&=\|M^Re\|^2\left[C^*(n)+O\left(\frac{\widetilde{\sigma}(R)}{\|M^Re\|}\right)\right]+O\left(\widetilde{\sigma}(R)^{\frac13}\right)\\
&=C^*\|M^Re\|^2+O\left(\widetilde{\sigma}(R)^{\frac13}\right).
\end{aligned}
$$

where we have used the fact that $\{\partial_e Q^R=x^TM^Re>0\}\cap B_1(0)$ is a hemisphere and M^R is uniformly bounded. We also use the property that $\|M^R\|$ is uniformly bounded. This is because $M^R\ge0$ and $\mathrm{tr}(M^R)=1$, which gives that the absolute values of eigenvalues of M^R is bounded by 1. Similarly,

$$
I^-(R,e)=C\|M^Re\|^2+O\left(\widetilde{\sigma}(R)^{\frac13}\right).
$$

Then

$$
I(R)^{\frac12}=I^+(R,e)I^-(R,e)=C\|M^Re\|^2+O(\widetilde{\sigma}(R)^{\frac13}),
$$

which complete the proof. □

To this end, we need the following results to complete the proof. As a direct result of the monotonicity (Theorem 6.10) and the calculation in the above proof, we have

Corollary 6.13.

$$\|M^{R_1}e\|^2 \leq \|M^{R_2}e\|^2 + O\left(\widetilde{\sigma}(R_2)^{\frac{1}{3}}\right) \qquad \textit{for } \forall 0 < R_1 < R_2.$$

Lemma 6.14.

$$\|M^{R_1} - M^{R_2}\| \leq O\left(\widetilde{\sigma}(R_2)^{\frac{1}{6}}\right) \qquad \textit{for } \forall 0 < R_1 < R_2.$$

Proof. Let $N = M^{R_2} - M^{R_1}$. Then N is symmetric and $\operatorname{tr}(N) = 0$. By Corollary 6.13,

$$\begin{aligned} &\|(M^{R_2} - N)e\|^2 \leq \|M^{R_2}e\|^2 + O\left(\widetilde{\sigma}(R_2)^{\frac{1}{3}}\right) \\ \Rightarrow & -2\langle M^{R_2}e, Ne\rangle + \|Ne\|^2 \leq O\left(\widetilde{\sigma}(R_2)^{\frac{1}{3}}\right). \end{aligned}$$

Let e be the eigenvector corresponding to the smallest eigenvalue of N, denoted by λ. Note that $\lambda \leq 0$ ($\Leftarrow \operatorname{tr}(N) = 0$). Then

$$-2\lambda e^T M^{R_2} e + \lambda^2 \leq O\left(\widetilde{\sigma}(R)^{\frac{1}{3}}\right).$$

It immediately gives $|\lambda| \leq O\left(\widetilde{\sigma}(R)^{\frac{1}{6}}\right)$, as $e^T M^{R_2} e \geq 0$. Since $\operatorname{tr}(N) = 0$, we find

$$\|N\| \leq \max_{\eta \text{ is the eigenvalue of } N} |\eta| \leq O\left(\widetilde{\sigma}(R)^{\frac{1}{6}}\right),$$

which complete the proof. □

Corollary 6.15. *As a result of Lemma 6.14, we have*
(a) As $R \to 0$, $M^R \to M^0$;
(b) $\|M^R - M^0\| \leq O\left(\widetilde{\sigma}(R)^{\frac{1}{6}}\right)$;
(c) $\|u - \frac{1}{2}x^T M^0 x\|_{L^\infty(B_R)} \leq \|u - Q^R\|_{L^\infty(B_R)} + \|Q^R - Q^0\|_{L^\infty(B_R)} \leq R^2 O\left(\widetilde{\sigma}(R)^{\frac{1}{6}}\right)$.

This readily proves (a) and (b) in Theorem 6.5.

Proof of Theorem 6.5, continued. To show the continuity of M_x on $x \in \Gamma(u) \cap \operatorname{sing} u$, we have

$$\|Q^R_{x_0} - Q^R_{x_1}\|_{B_R(x_0)\cap B_R(x_1)} \leq 2R^2\widetilde{\sigma}(R), \qquad \text{whenever } |x_1 - x_0| \leq \frac{R}{2},$$

where we have used

$$\|Q^R_{x_0} - Q^R_{x_1}\|_{B_R(x_0)\cap B_R(x_1)} \leq \|Q^R_{x_0} - u\|_{L^\infty(B_R(x_0))} + \|Q^R_{x_1} - u\|_{L^\infty(B_R(x_1))}.$$

In particular, if $x_0, x_1 \in \Gamma(u) \cap \operatorname{sing} u$ and $|x_0 - x_1| \leq \delta R$ with some small δ, we will have

$$\|M_{x_1} - M_{x_0}\| \leq C\widetilde{\sigma}(R).$$

The rest of conclusions of the theorem follows from elementary geometric considerations and we leave these to readers. This completes the proof of Theorem 6.5 □

6.2. **Uniqueness of second order blow-up.** In this section following [17], we will prove that the quadratic polynomial Q_{x_0} in the blow ups is unique. This would give an alternative proof of the uniqueness of second order blow ups established in Theorem 6.5. The other conclusions of the Theorem 6.5 thus follow also. Our proof is based on a monotonicity formula due to G.S. Weiss [24].

Remark 6.16. Almgren [1] proved the following monotonicity formula for harmonic functions. Let u be harmonic in $B_1(0) \subset \mathbb{R}^n$. Define for $0 < r < 1$:

$$D(r) = \int_{B_r(0)} |\nabla u|^2 dx, \quad H(r) = \int_{\partial B_r(0)} u^2 dS, \quad N(r) = \frac{rD(r)}{H(r)}.$$

Then $N(r)$ is monotone increasing in r. Indeed, we calculate the logarithmic derivative of $N(r)$:

$$\frac{N'(r)}{N(r)} = \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} = \frac{n-1}{r} + \frac{2\int_{\partial B_r(0)} (\partial_\nu u)^2 dS}{D(r)} - \frac{H'(r)}{H(r)}.$$

from:

$$\begin{aligned}
D'(r) &= \frac{d}{dr}\left(\int_{B_r(0)} |\nabla u|^2 dx\right) = \frac{d}{dr}\left(r^n \int_{B_1(0)} |\nabla u(rx)|^2 dx\right) \\
&= nr^{n-1} \sum_i \int_{B_1(0)} \partial_i u(rx)\partial_i u(rx) dx + 2\sum_{i,j} r^n \int_{B_1(0)} \partial^2_{ij} u(rx)\partial_i u(rx) x_j dx \\
&= \frac{n}{r} \sum_i \int_{B_r(0)} \partial_i u(x)\partial_i u(x) dx - \frac{2}{r}\sum_{i,j} \int_{B_r(0)} \partial_j u(x)\partial^2_{ii} u(x) x_j dx \\
&\quad - \frac{2}{r}\sum_{i,j} \int_{B_r(0)} \partial_j u(x)\partial_i u(x)\delta_{ij} dx + 2\sum_{i,j} \int_{\partial B_r(0)} \partial_j u(x)\partial_i u(x)\frac{x_j}{r}\frac{x_i}{r} dS \\
&= 2\sum_{i,j} \int_{\partial B_r(0)} \partial_j u(x)\partial_i u(x)\frac{x_j}{r}\frac{x_i}{r} dS + \frac{n-2}{r}\int_{B_r(0)} \partial_i u(x)\partial_i u(x) dx, \\
&= 2\int_{\partial B_r(0)} (\partial_\nu u)^2 dS + \frac{(n-2)}{r} D(r),
\end{aligned}$$

where we have used $\Delta u = \sum_i \partial^2_{ii} u = 0$ and integration by parts for the third equality and change of variables in the second equality. Similarly, we have:

$$\begin{aligned}
H'(r) &= 2(n-1)r^{n-2} \sum_i \int_{\partial B_1(0)} \partial_i u(rx) u(rx) x_i dS + (n-1)r^{n-2} \int_{\partial B_1(0)} |u(rx)|^2 dS \\
&= \frac{2(n-1)}{r} \sum_i \int_{\partial B_r(0)} \partial_i u(x) u(x) \frac{x_i}{r} dS + \frac{(n-1)}{r}\int_{\partial B_r(0)} |u(x)|^2 dS \\
&= \frac{n-1}{r} H(r) + 2\int_{\partial B_r(0)} u\partial_\nu u dS.
\end{aligned}$$

Using these expressions for $D'(r)$ and $H'(r)$, we obtain:

$$\frac{N'(r)}{N(r)} = \frac{2\int_{\partial B_r(0)} (\partial_\nu u)^2 dS}{D(r)} - \frac{2\int_{\partial B_r(0)} u\partial_\nu u dS}{\int_{\partial B_r(0)} u^2 dS}.$$

Finally, note that

$$D(r) = \int_{B_r(0)} \Delta\left(\frac{u^2}{2}\right) dx = \int_{\partial B_r(0)} u\partial_\nu u dS,$$

where the first equality is due to the harmonicity of u and the latter due to the divergence theorem. Precisely speaking

$$D(r) = \sum_i \int_{B_r(0)} \partial_i u \partial_i u dx = -\sum_i \int_{B_r(0)} \partial^2_{ii} u u dx + \sum_i \int_{\partial B_r(0)} \partial_i u n_i u dS = \int_{\partial B_r(0)} u\partial_\nu u dS.$$

Hence, applying the Cauchy-Schwarz inequality to the denominator of the positive term and the numerator of the second term, the claim follows. Indeed,

$$\begin{aligned}\frac{N'(r)}{N(r)} &= \frac{2\int_{\partial B_r(0)}(\partial_\nu u)^2 dS}{D(r)} - \frac{2\int_{\partial B_r(0)} u\partial_\nu u dS}{\int_{\partial B_r(0)} u^2 dS}\\ &= \frac{2\left(\int_{\partial B_r(0)}(\partial_\nu u)^2 dS\right)\left(\int_{\partial B_r(0)} u^2 dS\right) - 2\left(\int_{\partial B_r(0)} u\partial_\nu u dS\right)^2}{D(r)\int_{\partial B_r(0)} u^2} \geq 0.\end{aligned}$$

In particular $k = N(0^+) = \lim_{r\downarrow 0} N(r)$ exists and k is the vanishing order of u at 0. Assuming u vanishes at 0 with order $k \in \{1, 2, ...\}$, Almgren's monotonicity formula implies:

$$A(r) = \frac{D(r)}{r^{n-2+2k}} - k\frac{H(r)}{r^{n-1+2k}}, \quad 0 < r < 1$$

is also a monotone increasing and non-negative function of $r \in (0, 1)$. In fact

$$\begin{aligned}A'(r) &= \frac{D'(r)}{r^{n-2+2k}} - \frac{(n-2+2k)D(r)}{r^{n-1+2k}} - \frac{kH'(r)}{r^{n-1+2k}} + \frac{k(n-1+2k)H(r)}{r^{n+2k}}\\ &= \frac{1}{r^{n+2k}}\left(r^2 D'(r) + k(n-1+2k)H(r) - r(n-2+2k)D(r) - krH'(r)\right).\end{aligned}$$

By using the formula of $D'(r)$ and $H'(r)$, we can obtain that

$$\begin{aligned}&r^2 D'(r) + k(n-1+2k)H(r) - r(n-2+2k)D(r) - krH'(r)\\ &= \left(r^2\left(2\int_{\partial B_r(0)}(\partial_\nu u)^2 dS + \frac{n-2}{r}D(r)\right) - r(n-2+2k)D(r)\right)\\ &\quad + \left(k(n-1+2k)H(r) - kr\left(\frac{n-1}{r}H(r) + 2\int_{\partial B_r(0)} u\partial_\nu u dS\right)\right)\\ &= 2r^2\int_{\partial B_r(0)}(\partial_\nu u)^2 dS - 2krD(r) + 2k^2 H(r) - 2kr\int_{\partial B_r(0)} u\partial_\nu u dS\\ &= 2r^2\int_{\partial B_r(0)}(\partial_\nu u)^2 dS - 4kr\int_{\partial B_r(0)} u\partial_\nu u dS + 2k^2\int_{\partial B_r(0)} u^2 dS.\end{aligned}$$

Then $A'(r) = \frac{2}{r^{n+2k}}\int_{\partial B_r(0)}(r\partial_\nu u - ku)^2 dS \geq 0$. If we assume that u vanishes at 0 with order k, $A'(r)$ makes sense if $r \to 0$.

With a similar proof, Weiss [24] showed that the following monotonicity formula holds in the context of the obstacle problem.

Lemma 6.17 (Weiss Monotonicity)**.** *Let $u \in P_1(M)$, $x_0 \in \Gamma(u)$ such that $B_R(x_0) \subset B_1(0)$ and*

$$\Phi(x_0, r, u) = \frac{\int_{B_r(x_0)}\left(|\nabla u|^2(x) + 2u(x)\right)dx}{r^{n+2}} - 2\frac{\int_{\partial B_r(x_0)} u^2 dS}{r^{n+3}}.$$

Then we have:

$$\frac{d}{dr}\Phi(x_0, r, u) = \frac{2}{r^{n+2}}\int_{\partial B_r(x_0)}\left(\frac{\partial u}{\partial \nu} - \frac{2u}{r}\right)^2 dS \geq 0,$$

for $0 < r \leq R$. Hence it is monotone increasing.

Proof. To simplify the proof, we can assume that $x_0 = 0$. Define $D(r) = \int_{B_r(0)} |\nabla u|^2 ds$, $H(r) = \int_{\partial B_r(0)} u^2 dS$ as above and $B(r) = \int_{B_r(0)} u dx$. Then

$$D'(r) = 2\int_{\partial B_r(0)}(\partial_\nu u)^2 dS + \frac{n-2}{r}D(r) - 2\int_{B_r(0)\cap\{u>0\}} \partial_\nu u dx,$$

$$D(r) = -\int_{B_r(0)\cap\{u>0\}} udx + \int_{\partial B_r(0)} u\partial_\nu udS,$$

$$H'(r) = \frac{n-1}{r}H(r) + 2\int_{\partial B_r(0)} u\partial_\nu udS,$$

and

$$B'(r) = \frac{n}{r}\int_{B_r(0)} udx + \int_{B_r(0)} \partial_\nu udx = \frac{n}{r}B(r) + \int_{B_r(0)} \partial_\nu udx.$$

By simple calculations, we can obtain that

$$\frac{d}{dr}\Phi(0,r,u) = \frac{D'(r)}{r^{n+2}} - \frac{(n+2)D(r)}{r^{n+3}} - \frac{2H'(r)}{r^{n+3}} + \frac{2(n+3)H(r)}{r^{n+4}} + \frac{2B'(r)}{r^{n+2}} - \frac{2(n+2)B(r)}{r^{n+3}}.$$

Moreover, we have

$$\begin{aligned}
\frac{D'(r)}{r^{n+2}} - \frac{(n+2)D(r)}{r^{n+3}} &= \frac{2}{r^{n+2}}\int_{\partial B_r(0)} (\partial_\nu u)^2 dS + \frac{4}{r^{n+3}}\int_{B_r(0)\cap\{u>0\}} udx \\
&\quad - \frac{2}{r^{n+2}}\int_{B_r(0)\cap\{u>0\}} \partial_\nu udx - \frac{4}{r^{n+3}}\int_{\partial B_r(0)} u\partial_\nu udS \\
&= \frac{2}{r^{n+2}}\int_{\partial B_r(0)} (\partial_\nu u)^2 dS + \frac{4}{r^{n+3}}\int_{B_r(0)} udx \\
&\quad - \frac{2}{r^{n+2}}\int_{B_r(0)} \partial_\nu udx - \frac{4}{r^{n+3}}\int_{\partial B_r(0)} u\partial_\nu udS.
\end{aligned}$$

$$\begin{aligned}
-\frac{2H'(r)}{r^{n+3}} + \frac{2(n+3)H(r)}{r^{n+4}} &= \frac{8}{r^{n+4}}H(r) - \frac{4}{r^{n+3}}\int_{\partial B_r(0)} u\partial_\nu udS \\
&= \frac{8}{r^{n+4}}\int_{\partial B_r(0)} u^2 dS - \frac{4}{r^{n+3}}\int_{\partial B_r(0)} u\partial_\nu udS,
\end{aligned}$$

and

$$\begin{aligned}
\frac{2B'(r)}{r^{n+2}} - \frac{2(n+2)B(r)}{r^{n+3}} &= \frac{-4B(r)}{r^{n+3}} + \frac{2}{r^{n+2}}\int_{B_r(0)} \partial_\nu udx \\
&= \frac{-4}{r^{n+3}}\int_{B_r(0)} udx + \frac{2}{r^{n+2}}\int_{B_r(0)} \partial_\nu udx.
\end{aligned}$$

Then

$$\frac{d}{dr}\Phi(0,r,u) = \frac{2}{r^{n+2}}\int_{\partial B_r(x_0)} \left(\frac{\partial u}{\partial \nu} - \frac{2u}{r}\right)^2 dS \geq 0,$$

which complete the proof. □

An easy consequence of this monotonicity formula is the following:

Corollary 6.18 (Existence of Homogeneous Blow-Ups)**.** *Let $u \in P_1(M)$. Then for any sequence λ_m, $\lambda_m \downarrow 0$, there is a subsequence, relabeled as λ_m, such that*

$$u^{\lambda_m}(x) = \frac{u(\lambda_m x)}{\lambda_m^2} \to u_0(x) \in P_\infty^*(M) \text{ in } C^{1,\alpha} \text{ norm}$$

uniformly with any compact set, where:

$$u_0(x) = |x|^2 u_0\left(\frac{x}{|x|}\right).$$

Proof of the Corollary. Since $u^\lambda \in P_1(M)$, we apply Lemma 6.17 to obtain

$$\Phi(0,1,u^\lambda) - \Phi(0,0^+,u^\lambda) = \int_0^1 \frac{2}{r^{n+2}} \int_{\partial B_r(0)} \left(\frac{\partial u^\lambda}{\partial \nu} - \frac{2u^\lambda}{r} \right)^2 dSdr$$
$$= \int_0^\lambda \frac{2}{r^{n+2}} \int_{\partial B_r(0)} \left(\frac{\partial u}{\partial \nu} - \frac{2u}{r} \right)^2 dSdr \to 0^+,$$

as $\lambda \to 0^+$. Indeed

$$\int_0^1 \frac{2}{r^{n+2}} \int_{\partial B_r(0)} \left(\frac{\partial u^\lambda}{\partial \nu} - \frac{2u^\lambda}{r} \right)^2 dSdr = \int_0^\lambda \frac{2\lambda^{n+2}}{r^{n+2}} \int_{\partial B_{r/\lambda}(0)} \left(\frac{1}{\lambda}\frac{\partial u}{\partial \nu}(\lambda x) - \frac{1}{\lambda^2}\frac{2u(\lambda x)}{r} \right)^2 dSdr$$
$$= \int_0^\lambda \frac{2\lambda^{n+2}}{r^{n+2}} \int_{\partial B_r(0)} \left(\frac{1}{\lambda}\frac{\partial u}{\partial \nu}(x) - \frac{2u(x)}{r} \right)^2 dSdr.$$

Thus, for a subsequence, relabeled as u^{λ_m}, such that $u^{\lambda_m} \to u_0(x)$ in $C^{1,\beta}$ for any $0 < \beta < 1$ (due to $u^{\lambda_m} \in P_1(M)$) with $u_0 \in P_1(M)$, we have:

$$\int_0^1 \frac{2}{r^{n+2}} \int_{\partial B_r(0)} \left(\frac{\partial u_0}{\partial \nu} - \frac{2u_0}{r} \right)^2 dr = \lim_{\lambda_m \to 0^+} \int_0^1 \frac{2}{r^{n+2}} \int_{\partial B_r(0)} \left(\frac{\partial u^{\lambda_m}}{\partial \nu} - \frac{2u^{\lambda_m}}{r} \right)^2 dSdr$$
$$= \lim_{\lambda_m \to 0^+} \int_0^{\lambda_m} \frac{2}{r^{n+2}} \int_{\partial B_r(0)} \left(\frac{\partial u}{\partial \nu} - \frac{2u}{r} \right)^2 dSdr = 0.$$

Hence, $\left(\frac{\partial u_0}{\partial \nu} - \frac{2u_0}{r}\right)$ vanishes for almost every $0 < r < 1$. Thus, we conclude $u_0(x) = |x|^2 u_0\left(\frac{x}{|x|}\right)$. □

Remark 6.19. Let $\mathcal{F} = \{\Gamma(u) : u \in P_1(M)\}$. Then:
(1) $\forall E \in \mathcal{F}, a \in E, E_{a,\lambda} \in \mathcal{F}$, where $E_{a,\lambda} = \left(\frac{E-a}{\lambda}\right) \cap B_1, 0 < \lambda \leq 1 - |a|$.
(2) $\forall E \in \mathcal{F}, a \in E, \{\lambda_m\} \downarrow 0$, there is a subsequence, relabeled as $\{\lambda_m\}$ such that

$$E_{a,\lambda_m} \to T \text{ with } T_{0,\lambda} \equiv T \text{ for } 0 < \lambda < 1,$$

i.e. there is a tangent cone of E at each point $a \in E$. The first property is due to the inavariance of $P_1(M)$ under scaling and translation, whereas the latter follows directly from Corollary 6.18. The notation $E_i \to F$ means: for any $\epsilon > 0$ and for all sufficiently large $i, i \geq i(\epsilon), E_i$ is contained in the ϵ-neighborhood of F.

For $E \in \mathcal{F}$ let:

$$S_j = \{a \in E : \text{ the invariant dimension of } T \leq j, \text{ for all tangent cones } T \text{ of } E \text{ at } a\}$$

for $j = 0, 1, 2, ..., n$.

Definition 6.20. For a tangent cone T of E at a, that is: $T_{0,\lambda} \equiv T = \lim_{\lambda_m} E_{a,\lambda_m}$ for a sequence $\lambda_m \downarrow 0$, $0 < \lambda \leq 1$, a linear subspace V of $\mathbb{R}^n$ is called an invariant space of T, if $(T + v) \cap B_1 \subset T$ for all $v \in V$. The maximum dimension of all such invariant spaces V is called the invariant dimension of T.

The following result is a version of the dimension reduction principle of Federer [7] and its improvement due to Almgren [1], [21]. Its proof can be simplified after establishing the uniqueness of the homogeneous degree 2 blow-ups at singular points of $\Gamma(u)$.

Theorem 6.21 (Reduction and Stratification Principle)**.** (*1*) *For every $E \in \mathcal{F}$, the Hausdorff dimension of $E, \dim_H E \leq n - 1$. Moreover, there is an $(n-1)$-dimensional hyperplane $T \in \mathcal{F}$.*
(*2*) $\dim_H S_j \leq j$, *for $j = 0, 1, 2, ..., n-1$, $S_0 \subset S_1 \subset S_2 \subset ... \subset S_{n-1} = E$. Moreover, S_0 consists of isolated points.*

Instead of pursuing this direction we resume working towards the uniqueness of the second order blow-ups. Let $u \in P_1(M)$ and v be a non-negative quadratic polynomial such that $\Delta v = 1$. Without loss of generality, we assume that v is a homogeneous degree 2 blow-up of u at $0 \in \operatorname{sing}(u) \cap \Gamma(u)$. We define:

$$D(w,r) = \int_{B_r(0)} |\nabla w|^2 dx, \quad H(w,r) = \int_{\partial B_r(0)} w^2.$$

Lemma 6.22 (Generalized Almgren-Weiss Monotonicity)**.**

$$\frac{d}{dr}\left[\frac{D(w,r)}{r^{n+2}} - 2\frac{H(w,r)}{r^{n+3}}\right] = 2\int_{\partial B_r} \frac{\left(r\frac{\partial w}{\partial \nu} - 2w\right)^2}{r^{n+4}} \geq 0$$

Proof. Firstly, we have

$$\begin{aligned}\frac{D'(w,r)}{r^{n+2}} - \frac{(n+2)D(w,r)}{r^{n+3}} &= \frac{2}{r^{n+2}}\int_{\partial B_r(0)} (\partial_\nu w)^2 dS + \frac{4}{r^{n+3}}\int_{B_r(0)} w\Delta w dx \\ &\quad - \frac{2}{r^{n+2}}\int_{B_r(0)} \partial_\nu w \Delta w dx - \frac{4}{r^{n+3}}\int_{\partial B_r(0)} w\partial_\nu w dS \\ &= \frac{2}{r^{n+2}}\int_{\partial B_r(0)} (\partial_\nu w)^2 dS + \frac{4}{r^{n+3}}\int_{B_r(0)} w\Delta w dx \\ &\quad - \frac{2}{r^{n+2}}\int_{B_r(0)} \partial_\nu w \Delta w dx - \frac{4}{r^{n+3}}\int_{\partial B_r(0)} w\partial_\nu w dS.\end{aligned}$$

and

$$\begin{aligned}-\frac{2H'(w,r)}{r^{n+3}} + \frac{2(n+3)H(w,r)}{r^{n+4}} &= \frac{8}{r^{n+4}}H(w,r) - \frac{4}{r^{n+3}}\int_{\partial B_r(0)} w\partial_\nu w dS \\ &= \frac{8}{r^{n+4}}\int_{\partial B_r(0)} w^2 dS - \frac{4}{r^{n+3}}\int_{\partial B_r(0)} w\partial_\nu w dS,\end{aligned}$$

A direct calculation yields the following formula:

$$\frac{d}{dr}\left[\frac{D(w,r)}{r^{n+2}} - 2\frac{H(w,r)}{r^{n+3}}\right] = 2\int_{\partial B_r(0)} \frac{\left(r\frac{\partial w}{\partial \nu} - 2w\right)^2}{r^{n+4}} + \epsilon(r),$$

where the error term $\epsilon(r)$ is:

$$\epsilon(r) = \frac{2}{r^{n+3}}\int_{B_r(0)} \left(2w - |x|\frac{\partial w}{\partial \nu}\right)\Delta w dx = \int_{B_r(0)} (2w - \nabla w \cdot x)\,\Delta w dx.$$

First, we observe that

$$2w - |x|\frac{\partial w}{\partial \nu} = 2u - |x|\frac{\partial u}{\partial \nu},$$

because v is a homogeneous function of order 2. So we consider $\left(2u - r\frac{\partial u}{\partial \nu}\right)(\Delta u - \Delta v)$. If $x \in \Lambda(u)$, then $u \equiv 0$, $\nabla u(x) = 0$. Hence the first factor vanishes. If $x \in N(u)$ instead, then $v(x) > 0$ by the convergence and as a consequence, $\Delta u(x) = \Delta v(x) = 1$. Hence the second factor vanishes. Therefore $\epsilon(r) = 0$ in either case. □

Lemma 6.23 (Convexity)**.**

$$\frac{d}{dr}\left[\frac{\int_{\partial B_r(0)} w^2}{r^{n+3}}\right] \geq \frac{2}{r}\left[\frac{D(w,r)}{r^{n+2}} - \frac{2H(w,r)}{r^{n+3}}\right] \geq 0.$$

Proof. The second inequality is due to the Weiss monotoniticity formula. Hence, proving the first inequality suffices.

$$\frac{d}{dr}\left[\frac{\int_{\partial B_r(0)} w^2}{r^{n+3}}\right] = \frac{2}{r}\left[\frac{D(w,r)}{r^{n+2}} - 2\frac{H(w,r)}{r^{n+3}}\right] + \frac{2}{r^{n+3}}\int_{B_r(0)} w\Delta w dx.$$

Note that $w \in C^{1,1}(B_1(0))$. When $u > 0$, $\Delta w = \Delta u - \Delta v = 0$. If $u = 0$ instead, $\Delta u = 0$ almost everywhere on the set $\{u = 0\}$ and hence $w\Delta w = v\Delta v = v$ (If $v = 0$, then $v\Delta v = 0 = v$. If $v > 0$, then $\Delta v = 1$ and $v\Delta v = v$) almost everywhere on $\{u = 0\}$. Hence $w\Delta w \geq 0$ almost everywhere in $B_1(0)$. The inequality is verified in either case. □

Remark 6.24. Letting $f(t) = \frac{\int_{\partial B_r(0)} w^2}{r^{n+3}}$, $r = e^t$, $-\infty < t < 0$, then $f''(t) \geq 0$ by Lemmas 6.22 and 6.23. This is the reason for the latter being called the convexity lemma. Precisely speacking, we can let $A(r) = \frac{\int_{\partial B_r(0)} w^2}{r^{n+3}}$ and $f(t) = A(e^t)$. Then $f''(t) = A'(e^t)e^t + A''(e^t)e^t \geq 0$.

Theorem 6.25. *For every $x_0 \in \operatorname{sing} u \cap \Gamma(u)$, the matrix M_{x_0} in Theorem 6.5 is unique.*

Proof. Let $u \in P_1(M)$ and v be one of the blow-ups. Then, there exists a sequence λ_m such that $\|u^{\lambda_m} - v\|_{C^{1,\alpha}(B_1(0))} \to 0$. Hence for $w^\lambda = u^\lambda - v^\lambda = u^\lambda - v$ and $w = u - v$, we have

$$\begin{aligned} \frac{D(w,\lambda)}{\lambda^{n+2}} &= \frac{1}{\lambda^{n+2}}\int_{B_\lambda(0)} |\nabla w|^2 dx = \frac{1}{\lambda^{n+2}}\int_{B_\lambda(0)} |\nabla u(x) - \nabla v(x)|^2 dx \\ &= \int_{B_1(0)} |\lambda^{-1}((\nabla u - \nabla v)(\lambda x))|^2 dx = D(w^\lambda, 1) \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} \frac{H(w,\lambda)}{\lambda^{n+3}} &= \frac{2}{\lambda^{n+3}}\int_{\partial B_\lambda(0)} w^2 dS = \frac{2}{\lambda^{n+3}}\int_{\partial B_\lambda(0)} |u(x) - v(x)|^2 dS \\ &= \int_{\partial B_1(0)} |\lambda^{-2}(u - v)(\lambda x)|^2 dS = H(w^\lambda, 1). \end{aligned} \tag{6.6}$$

Set $w_m = u^{\lambda_m} - v$, we have, for any $\varepsilon > 0$, since $\|u^{\lambda_m} - v\|_{C^{1,\alpha}(B_1(0))} \to 0$, there exists N such that for any $m > N$, $|D(w_m, 1)| + 2|H(w_m, 1)| < \varepsilon$. Then for any $0 < \lambda < \lambda_m$, it can be obtained by Lemmas 6.22 and 6.23, (6.5) and (6.6) that

$$\begin{aligned} 0 \leq D(w^\lambda, 1) - 2H(w^\lambda, 1) &= \frac{D(w,\lambda_m)}{\lambda^{n+2}} - 2\frac{H(w,\lambda_m)}{\lambda^{n+3}} \\ &\leq \frac{D(w,\lambda)}{\lambda_m^{n+2}} - 2\frac{H(w,\lambda_m)}{\lambda_m^{n+3}} = D(w_m, 1) - 2H(w_m, 1) < \varepsilon \end{aligned} \tag{6.7}$$

and

$$0 \leq H(w^\lambda, 1) = \frac{H(w,\lambda)}{\lambda^{n+3}} \leq \frac{H(w,\lambda_m)}{\lambda_m^{n+3}} < \varepsilon.$$

Then we can obtain that

$$0 \leq \int_{B_1(0)} |\nabla u^\lambda(x) - \nabla v(x)|^2 dx = D(w^\lambda, 1) < \varepsilon$$

for any $0 < \lambda < \lambda_m$. Then we can see that ∇v is the limit of ∇u^λ in L^2 norm. Then ∇v is unique and v is unique since $v(0) = 0$. Hence, the uniqueness of the second order blow-ups follows by the uniqueness of the limit in L^2 norm. □

The regularty theory for singular sets of free boundaries in higher dimensions was also due to L. Caffarelli. It was presented in his survey article (and lectures): The Obstacle Problem. Lezioni Fermiane. [Fermi Lectures] Academia Nazionale dei Lincei, Rome; Scuola Normale Superiore, Pisa, 1998. ii +54pp. The Obstacle Problem Revisited. J. Fourier Anal. Appl. 4 (1998), no. 4-5, 383402. The second proof of the the uniqueness of the second order blow ups is taken from the author's paper: On Regularity and Singularity of Free Boundaries in Obstacle Problems, Chin. Ann. Math. 30 B(5), 2009, 645 − 652.

7. Higher Regularity of Free Boundary

7.1. **Further regularity of free boundaries.** In this section, we will consider the obstacle problem:

$$\begin{cases} \Delta u = a(x) \text{ in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma, \end{cases} \tag{7.1}$$

under the following assumptions:
(I) Γ is a C^1-hypersurface,
(II) $u \in C^2(\Omega \cup \Gamma)$.

Other problems one can address are the following:

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u = 0, \ g(x, \nabla u) = 0 \text{ on } \Gamma, \end{cases} \tag{7.2}$$

and

$$\begin{cases} \Delta u = 0 \quad \text{ in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, \quad |\nabla u| = f(x) \text{ on } \Gamma, \end{cases} \tag{7.3}$$

or the parabolic problem:

$$\begin{cases} u_t - \Delta u = a(x,t) \text{ in } \Omega, \\ u = |\nabla u| = 0 \text{ on } \Gamma. \end{cases} \tag{7.4}$$

Example 7.1. Consider(7.3), when $n \geq 3$, with $f \equiv 1$. Then:

$$u(x) = x_1, \quad \Gamma = \{x^n = \sigma(x_2, ..., x_{n-1}), \quad \sigma \in C^1\}$$

is a solution.

The main references for the contents of this section are [14] and [15].

Theorem 7.2. *For the obstacle problem* (7.1) *under the assumptions* (*I*), (*II*) *and* $a(x) > 0$, $a \in C^1(\overline{\Omega})$, *the following hold:*
(*1*) Γ *is in* $C^{1,\alpha}$ *for every* $0 < \alpha < 1$,
(*2*) *If* $a \in C^{m+\alpha}$, *then* $\Gamma \in C^{m+1,\alpha}$,
(*3*) *If* $a \in C^\omega$, *then* $\Gamma \in C^\omega$.

Here we say functions or surfaces are C^ω, it means that they are real-analytic. A more general version of the above theorem under the same assumptions can be formulated as below:

Theorem 7.3. *Let* $F(x, u, \nabla u, \nabla^2 u) = 0$ *in* Ω, *where* $F \in C^1$ *in all variables and* $u = \frac{\partial u}{\partial \nu} = 0$ *on* Ω. *Assume* (*I*), (*II*) *as well as* $F(0,0,0,0) \neq 0$. *Then the following hold:*
(*1*) Γ *is in* $C^{1,\alpha}$ *for every* $0 < \alpha < 1$,
(*2*) *if* $F \in C^{m+\alpha}$, *then* $\Gamma \in C^{m+1+\alpha}$,
(*3*) *if* $F \in C^\omega$, *then* $\Gamma \in C^\omega$.

Before we begin the proof of these theorems, we need to introduce the Hodograph and Legendre transformations, as well as their geometric interpolations.

Definition 7.4 (Hodograph and Legendre Transformations)**.** Given $u \in C^2$ with (u_{ij}) non-singular, the hodograph transformation is a local mapping:

$$x \mapsto y = \nabla u,$$

and the Legendre transformation is:

$$u \mapsto v = \sum_i x_i u_{x_i} - u = x \cdot y - u.$$

Definition 7.5 (Geometric interpolation)**.** In (x, y, u) space, consider the graph

$$u(x, y) = \{(x, y, u) : u = u(x, y)\}$$

or the envelope of its tangent planes, i.e. set up the equation which a plane must satisfy in order to be tangent to the surface. If $\overline{x}, \overline{y}, \overline{u}$ are the running coordinates of a plane whose equation is:

$$\overline{u} - \xi\overline{x} - \eta\overline{y} + \omega = 0,$$

then (ξ, η, ω) are called coordinates of this plane. At point $(x, y, u(x, y))$ of the surface, the tangent plane coordinates are:

$$\overline{u} - u - (\overline{x} - x)u_x - (\overline{y} - y)u_y = 0$$

$$\xi = u_x, \quad \eta = u_y, \quad \omega = xu_x + yu_y - u$$

We say that the surface is determined if ω is given as a function of ξ, η:

$$\omega = x\xi + y\eta - u.$$

Then we compute:

$$\begin{cases} w_\xi = x + \xi\dfrac{\partial x}{\partial \xi} + \eta\dfrac{\partial y}{\partial \xi} - u_x\dfrac{\partial x}{\partial \xi} - u_y\dfrac{\partial y}{\partial \xi} = x \\ w_\eta = y \end{cases}$$

$$\omega(\xi, \eta) + u(x, y) = x\xi + y\eta, \quad x = \omega_\xi, \quad y = \omega_\eta, \quad \xi = u_x, \quad \eta = u_y.$$

We have the correspondence:

$$(x, y, u, u_x, u_y) \longleftrightarrow (\xi, \eta, \omega, \omega_\xi, \omega_\eta).$$

When

$$u_{xx}u_{yy} - (u_{xy})^2 = \rho > 0,$$

differentiating

$$\begin{cases} u_x = \xi, \\ u_y = \eta, \end{cases}$$

with respect to ξ, η, where

$$\begin{cases} x = x(\xi, \eta), \\ y = y(\xi, \eta), \end{cases}$$

gives the system:

$$\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} \begin{pmatrix} \omega_{\xi\xi} & \omega_{\xi\eta} \\ \omega_{\xi\eta} & \omega_{\eta\eta} \end{pmatrix} = I.$$

Example 7.6. Consider

$$u(x_1, ..., x_n) + \omega(\xi_1, ..., \xi_n) = x_1\xi_1 + ... + x_n\xi_n,$$

where:

$$u_{x_i} = \xi_i, \quad \omega_{\xi_i} = x_i, \quad i = 1, ..., n.$$

For $F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$, one deduces that

$$0 = G = F(\omega_\xi, \omega_\eta, \xi\omega_\xi + \eta\omega_\eta - \omega, \xi, \eta, \rho\omega_{\eta\eta}, -\rho\omega_{\xi\eta}, \rho\omega_{\eta\eta}),$$

we get:

$$\rho = \frac{1}{\omega_{\xi\xi}\omega_{\eta\eta} - (\omega_{\xi\eta})^2}.$$

In particular, for the minimal surface equation, one has:

$$(1+\xi^2)\omega_{\xi\xi} + 2\xi\eta\omega_{\xi\eta} + (1+\eta^2)\omega_{\eta\eta} = 0.$$

Proof of Theorem. Consider the hodograph transformation:

$$x \mapsto y = (-u_{x_1}, x_2, ..., x_n), \quad \text{assuming} \quad u_{x_1x_1}(0) \neq 0;$$

and the corresponding Legendre transformation gives:

$$\begin{aligned} v &= u - x_1 u_{x_1} = x_1 y_1 + u, \\ dv &= x_1 dy_1 - u_{x_1} dx_1 + du = x_1 dy_1 + \sum_{i\geq 2} u_{x_i} dx^i = x_1 dy^1 + \sum_{i \geq 2} u_{x_i} dy^i. \end{aligned}$$

Denoting $u_{x_i} = u_i$, $u_{x_ix_j} = u_{ij}$, $v_{y_i} = v_i$, $v_{y_iy_j} = v_{ij}$:

$$\begin{cases} u_1 = -y_1, \\ u_\alpha = v_\alpha, \quad \alpha \geq 2. \end{cases}$$

and

$$\begin{cases} v_{y_1} = x_1, \\ v_{y_j} = u_{x_j}, \quad j \geq 2. \end{cases}$$

Denote

$$\left(\frac{\partial Z}{\partial Y}\right) = \begin{pmatrix} v_{11} & v_{1\alpha} \\ 0 & I \end{pmatrix}$$

and

$$\left(\frac{\partial Y}{\partial X}\right) = \begin{pmatrix} \frac{1}{v_{11}} & -\frac{v_{1\alpha}}{v_{11}} \\ 0 & I \end{pmatrix}.$$

As $x_1 = v_{y1}$, $u_{11} = -\frac{1}{v_{11}}$, $u_{1\alpha} = \frac{v_{10}}{v_{11}}$, $\alpha \geq 2$ and

$$\frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial y_\alpha} - \frac{v}{v_{11}}\frac{\partial}{\partial y_1}, \qquad \frac{\partial}{\partial x_1} = \frac{1}{v_{11}}\frac{\partial}{\partial y_{11}},$$

the equation $\Delta u = a(x)$ is transformed to:

$$-\frac{1}{v_{11}} - \frac{1}{v_{11}} \sum_{\alpha\geq 2} (v_{1\alpha})^2 + \sum_{\alpha \geq 2} v_{\alpha\alpha} = \alpha(y_2, ..., y_n, v_{y_1}).$$

Hence, the problem $\Delta u = a(x)$ in Ω with $a \geq 0$ is transformed to:

$$\begin{cases} G(v_{ij}) = a(v_1, y_2, ..., y_n), \\ v|_{\{y_1=0\}} = 0. \end{cases}$$

While the free boundary is flattened out as $\Gamma \mapsto \{y_1 = 0\}$, the domain is transformed also accordingly as $\Omega \mapsto \{y_1 > 0\}$. □

Likewise, in the general case, the system

$$\begin{cases} F(x, u, \nabla u, \nabla^2 u) = 0 \text{ in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma, \end{cases}$$

would be transformed to:

$$\begin{cases} \widetilde{F}(y, v, \nabla v, \nabla^2 v) = 0 \quad \text{in} \quad \{y_1 \geq 0\}, \\ v|_{\{y1=0\}} = 0. \end{cases}$$

At this point we would like to argue that F being elliptic at u implies $\overline{F}$ being elliptic at v.

Definition 7.7. F is said to be elliptic at $u \in C^2$, if
$$(F_{ij}) = \left(\frac{\partial F}{\partial u_{ij}} \left(x, u, \nabla u, \nabla^2 u\right) \right)$$
is positive definite.

One can check in the special case that:
$$-\frac{1}{v_{11}} - \frac{1}{v_{11}} \sum_{\alpha \geq 2} (v_{1\alpha})^2 + \sum_{\alpha \geq 2} v_{\alpha\alpha} = a(v_{y1}, y_2, ..., y_n)$$
is elliptic. Likewise, a direct calculation yields:
$$\tilde{F}_{ij} = \begin{pmatrix} \frac{1+\sum_{\alpha \geq 2}(v_{1\alpha})^2}{(v_{11})^2} & \frac{v_{1\alpha}}{v_{11}} \\ \frac{v_{\alpha 1}}{v_{11}} & 1 \end{pmatrix}.$$
In other words:
$$\widetilde{F}_{ij}\xi_i\xi_j = \sum_{\alpha \geq 2} \xi_\alpha^2 + 2 \sum_{\alpha \geq 2} \frac{v_{1\alpha}}{v_{11}} \xi_\alpha \xi_1 + \frac{1 + \sum_{\alpha \geq 2}(v_{1\alpha})^2}{(v_{11})^2} \xi_1^2.$$
Noting that $v \in C^2$ and $v_{y_1} = x_1$, $v_{y_j} = u_{x_j}$, $j \geq 2$ gives: $v = -x' u_{x_1} + u$, we get:
$$\widetilde{F}_{ij}\xi_i\xi_j = \sum_{\alpha \geq 2} \xi_\alpha^2 - (1 = \delta) \sum_{\alpha \geq 2} \xi_\alpha^2 - \frac{1}{1-\delta} \frac{(v_{1\alpha})^2}{v_{11}^2} \xi_1^2 + \frac{1}{v_{11}^2} \xi_1^2 \geq \delta |\xi|^2$$
for $\delta > 0$ suitably small. In general, choosing $\eta_\alpha = \xi_\alpha - \xi_1 v_{1\alpha}$ we obtain:
$$\widetilde{F}_{ij}\xi_i\xi_j = (F_{11} - 2F_{1\alpha}v_{1\alpha} + F_{\alpha\beta}v_{1\alpha}v_{1\beta})\xi_1^2 + 2\left(F_{1\alpha} - F_{\alpha\beta}v_{1\beta}\right)\xi_1\xi_\alpha + F_{\alpha\beta}\xi_\alpha\xi_\beta = F_{ij}\eta_i\eta_j.$$
Thus, $\widetilde{F}$ is elliptic whenever F is elliptic. Hence, through the Legendre transformation we reduced the original problem to:
$$\begin{cases} F(v_{ij}) = a(v_1, y_2, ..., y_n), \\ v|_{\{y_1 = 0\}} = 0 \end{cases}$$
where $v \in C^2$, $a \in C^1$. Hence by standard elliptic regularity we get: $v \in C^{2,\alpha}(\overline{D}_+)$ for every $0 < \alpha < 1$. Now Γ is the image of $x = X(y)|_{\{y_1=0\}} \in C^{1,\alpha}$, where:
$$\left(\frac{\partial X}{\partial Y} \right) = \begin{pmatrix} v_{11} & v_{1\alpha} \\ 0 & I \end{pmatrix}.$$
Likewise, $a \in C^{m+\alpha}$ implies $v \in C^{m+2+\alpha}$, which in turn gives $\Gamma \in C^{m+1+\alpha}$. The rest of the proof is now standard.

7.2. **Liquid edges.** In this section we study the higher regularity of a multi-phase problem via the Hodograph transformation introduced in the preceding section. Consider the system:
$$\begin{cases} Mu^1 = Mu^2 = 0 \text{ in } \Omega^+, \\ Mu^3 = 0 \text{ in } \Omega^-, \text{where } M(u) = \left(\delta_{ij} - \dfrac{u_i u_j}{1 + |\nabla u|^2} \right) u_{ij} \\ u^1 = u^2 = u^3 \text{ on } \Gamma, \\ \dfrac{\nabla u^j \nabla u^3 + 1}{\sqrt{1 + |\nabla u^j|^2}\sqrt{1 + |\nabla u^3|^2}} = \dfrac{1}{2} \text{ on } \Gamma, \quad j = 1, 2 \\ \nabla u^1 \neq \nabla u^2 \text{ on } \Gamma \end{cases} \tag{7.5}$$
More generally $\frac{1}{2}$ is replaced by $\cos \mu_j < 1$.

Theorem 7.8. *Suppose that Γ is of class $C^{1,\alpha}, u^j \in C^{1,\alpha}(\Omega^+ \cup \Gamma), j = 1, 2$ and $u^3 \in C^{1,\alpha}(\Omega^- \cup \Gamma)$, for some $\alpha, 0 < \alpha \le 1$ satisfy (7.5). If u^1 and u^2 do not meet at at zero angle at $x = 0$, i.e. $\mu_1(0) \neq \mu_2(0)$, then Γ is analytic near $x = 0$.*

Remark 7.9. J. Taylor, J.C.C. Nitsche and others had made some preliminary contributions on the above problem, see [14], [15] and references therein.

We recall the Hodograph transformation. Consider $w \in C^1$, $w_{x_n}(0) > 0$, $y(x) = (x', w(x))$, $x \in B_e(0)$. Suppose $w|_\Gamma = 0$, $0 \in \Gamma$, $U = y(B_e \cap \Omega) \subset \{y_n > 0\}$ and $y(\Gamma) \subset \{y_n = 0\}$. Define $\psi(y) = x_n \cdot g(y) = (y', \psi(y))$, $\Gamma = \{x_n = \psi(x', 0)\}$. Then after a careful computation, one leads to:

$$dx_n = d\psi = \sum_{\alpha \le n-1} [\psi_\alpha dy_\alpha + \psi_n dy_n] = \sum_{\alpha < n} [\psi_\alpha dy_\alpha + \psi_n dw],$$

and

$$\begin{cases} dw = -\displaystyle\sum_{\alpha=1}^{n-1} \frac{\psi_\alpha}{\psi_n} dx_\alpha + \frac{1}{\psi_n} dx_n, \\ \Delta w = -\displaystyle\sum_{\alpha<n} \left(-\frac{\psi_\alpha}{\psi_n}\right)_\alpha + \frac{1}{\psi_n} dx_n, \\ w_\alpha = -\dfrac{\psi_\alpha}{\psi_n}, \quad w_n = \dfrac{1}{\psi_n}. \end{cases}$$

Likewise

$$\begin{cases} w_{nn} = \left(\dfrac{1}{\psi_n}\right)_n \dfrac{\partial y_n}{\partial x_n} = -\dfrac{\psi_{nn}}{\psi_n^3}, \\ w_{\alpha\alpha} = \left(-\dfrac{\psi_\alpha}{\psi_n}\right)_\alpha + \left(\dfrac{\psi_\alpha}{\psi_n}\right)_n. \end{cases}$$

Example 7.10. $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$:

$$\begin{cases} \Delta w = a(x, w), \\ w_\nu = f(x) \text{ on } \quad \Gamma, \\ w = 0 \text{ on } \Gamma, \end{cases}$$

may be transformed into the following:

$$\begin{cases} F\left(D^2\psi, D\psi\right) = a\left(y', \psi, y_n\right) \\ \psi_n - \dfrac{1}{f\left(y', \psi\right)} \sqrt{1 + \displaystyle\sum_{\alpha<n} \psi_\alpha^2} \quad \text{on} \quad \{y_n = 0\} = \Sigma. \end{cases}$$

We leave details to the readers.

Before proceeding with the application of the Hodograph transformation to (7.5), we fix the notation:

$$\begin{cases} y = (x', w(x)) \quad x \in \Omega^+, \\ \psi(y) = x_n, \end{cases}$$

and:

$$\begin{cases} g^+(y) = (y', \psi(y)), \quad \text{for} \quad y \in U \cup \Sigma, \\ g^-(y) = (y', \psi(y) - cy_n), \quad \text{where } C > |\sup \psi_n|, \\ g^-(U) \subset \Omega^-, \quad g^+ = g^- \quad \text{on } y_n = 0. \end{cases}$$

Suppose a, f are analytic near $(x,w)=(0,0), f(0)<0, \nu(0)=-e_n=(0,\dots,0,-1), w_{x_n}(0)=a>0$, $w_\alpha=0$ at $y=0$, for $\alpha<n, a=-f(0)$. Define:

$$L\overline{\psi}=\frac{d}{d\epsilon}F\left(D^2\psi(0)+\epsilon D^2\overline{\psi}, D\psi(0)+\epsilon D\overline{\psi}(0)\right)\Big|_{\epsilon=0}.$$

Computation yields:

$$L\overline{\psi}=-a^3\overline{\psi}_{nn}-a\sum_{\sigma<n}\overline{\psi}_{\sigma\sigma}+\text{ lower order terms.}$$

This is the linearization of the equation after the tranformation. Also define:

$$B\overline{\psi}=\frac{d}{d\epsilon}\left(\psi_n(0)+\epsilon\overline{\psi}_n(0)-\frac{\sqrt{1+\sum_\alpha\left(\psi_\alpha(0)\overline{\psi}_\alpha(0)\right)^2}}{f(0,\psi(0)+\epsilon\overline{\psi})}\right)\Bigg|_{\epsilon=0}.$$

In other words:

$$B\overline{\psi}=\overline{\psi}_n(0)+g\left(\overline{\psi}(0),\overline{\psi}_\alpha(0)\right).$$

This is the linearization of the corresponding nonlinear boundary condition.

Now we are ready to apply the above strategy and the Hodograph transformation to (7.5). Let $a_j-\frac{\theta}{\partial x_n}u^j(0)$ for $j=1,2$. Assume $a_2>a_1$ (and hence $u^2>u^1$) in Ω^+ near $x=0$. Let $\mu_j=\mu_j(0,0)$.

Assume also $\mu_2\neq\mu_1$ and $\mu_j\neq 0$. $\cos\mu_j=(1+a_j^2)^{-\frac{1}{2}}$. Define $w(x)=u^2(x)-u^1(x)$ for $x\in\Omega^+$. Furthermore:

$$\begin{cases}\phi^+(y)=u^1(x)=u^1(g^+(y)),\\ \phi^-(y)=u^3(x)=u^3(g^-(y)),\\ \psi(y)=x_n \text{ as before.}\end{cases}$$

$$\begin{cases}\nabla w=\left(-\dfrac{\psi_\sigma}{\psi_n},\dfrac{1}{\psi_n}\right), \quad x\in\Omega^+,\\ \nabla u^1=\left(\phi^+_\alpha-\dfrac{\psi_\alpha}{\psi_n}\phi^+_n,\dfrac{\phi^+_n}{\psi_n}\right), \quad x\in\Omega^+,\\ \nabla u^2=\left(\phi^+_\alpha-\dfrac{\psi_\alpha}{\psi_n}\left(\phi^+_n+1\right),\dfrac{1}{\psi_n}\left(\phi^+_n+1\right)\right), \quad x\in\Omega^+,\\ \nabla u^3=\left(\phi^-_\alpha-\dfrac{\psi_\alpha}{\psi_n-C}\phi^-_n,\dfrac{1}{\psi_n-C}\phi^-_n\right), \quad x\in\Omega^-.\end{cases}$$

$$\begin{cases}\partial_{x_\alpha}=\partial_{y_\alpha}-\dfrac{\psi_\alpha}{\psi_n}\partial_{y_n}, \quad \partial_{x_n}=\dfrac{1}{\psi_n}\partial_{y_n} \text{ in } \Omega^+,\\ \partial_{x_\alpha}=\partial_{y_\alpha}-\dfrac{\psi_\alpha}{\psi_n-C}\partial_{y_n}, \quad \partial_{x_n}=\dfrac{1}{\psi_n-C}\partial_{y_n} \text{ in } \Omega^-.\end{cases}$$

The system (7.5) can be expressed as:

$$\begin{cases}0=Mu^1=F_1(D^2\psi,D\psi,D^2\phi^+,D\phi^+),\\ 0=Mu^2=F_2(D^2\psi,D\psi,D^2\phi^+,D\phi^+),\\ 0=Mu^3=F_3(D^2\psi,D\psi,D^2\phi^-,D\phi^-),\end{cases}$$

with boundary conditions:

$$\begin{cases}0=\Phi_1(D\psi,D\phi^+,D\phi^-)=\nabla u^1\nabla u^3+1-\cos\mu_1\sqrt{1+|\nabla u^1|^2}\sqrt{1+|\nabla u^3|^2},\\ 0=\Phi_2(D\psi,D\phi^+,D\phi^-)=\nabla u^2\nabla u^3+1-\cos\mu_2\sqrt{1+|\nabla u^2|^2}\sqrt{1+|\nabla u^3|^2},\\ 0=\Phi_3(\phi^+,\phi^-)=\phi^+-\phi^-=u^3-u^1 \quad \text{on } \Sigma.\end{cases}$$

The linearization of the boundary conditions gives:

$$\begin{cases} \cos^2\mu_1(\overline{\phi_n^+} - a_1\overline{\psi_n}) + \dfrac{1}{C(a_2-a_1)-1}\overline{\phi_n^-} = 0, \\ \cos^2\mu_2(\overline{\phi_n^+} - a_2\overline{\psi_n}) + \dfrac{1}{C(a_2-a_1)-1}\overline{\phi_n^-} = 0, \\ \dfrac{a_2}{a_2-a_1}(\overline{\phi^+} - a_1\overline{\psi}) - \dfrac{a_1}{a_2-a_1}(\overline{\phi^+} - a_2\overline{\psi}) - \overline{\phi^-} = 0. \end{cases}$$

Similarly, the bulk equations can be also linearized to give the following set of linear equations:

$$\begin{cases} \sum\limits_{\alpha<n}(\overline{\phi^+} - a_1\overline{\psi})_{\alpha\alpha} + (a_2-a_1)^2\cos^2\mu_1(\overline{\phi^+} - a_1\overline{\psi})_{nn} = 0 \\ \sum\limits_{\alpha<n}(\overline{\phi^+} - a_2\overline{\psi})_{\alpha\alpha} + (a_2-a_1)^2\cos^2\mu_2(\overline{\phi^+} - a_2\overline{\psi})_{nn} = 0 \\ \sum\limits_{\alpha<n}\overline{\phi^-_{\alpha\alpha}} + \left(\dfrac{a_2-a_1}{1-C(a_2-a_1)}\right)^2\overline{\phi^-_{nn}} = 0 \end{cases}$$

The regularity of solutions of these linear equations with corrsponding linear boundary conditions and the corresponding regularity for the nonlinear equations with nonlinear complementary boundary conditions are contained in classcal theory of Agmon-Douglis-Norenberg, see [8].

8. Appendix

In this Appendix, we will show the famous De Giorgi-Nash estimates for elliptic equations. The result of De Giorgi and Nash regarding the regularity of solutions to equations with bounded measurable coefficients is the following

Theorem 8.1 (De Giorgi-Nash)**.** *Let $v \in H^1(\Omega)$ be any weak solution to*

$$\operatorname{div}(A(x)\nabla v) = 0 \text{ in } \Omega, \tag{8.1}$$

with

$$0 < \lambda I \le A(x) \le \Lambda I. \tag{8.2}$$

Then, there exists some $\alpha > 0$ such that $v \in C^{0,\alpha}(\widetilde{\Omega})$ for any $\widetilde{\Omega} \subset\subset \Omega$, with

$$\|v\|_{C^{0,\alpha}(\widetilde{\Omega})} \le C\|v\|_{L^2(\Omega)}.$$

The constant C depends only on n, λ, Λ, Ω and $\widetilde{\Omega}$. The constant $\alpha > 0$ depends only on n, λ and Λ.

De Giorgi's first step: from L^2 to L^∞. The two main ingredients are the Sobolev inequality

$$\|v\|_{L^p(\mathbb{R}^n)} \le C\|\nabla v\|_{L^2(\mathbb{R}^n)}, \quad p = \frac{2n}{n-2},$$

and the following energy inequality (the Caccioppoli inequality):

Lemma 8.2 (Energy inequality)**.** *Let $v \in H^1(B_1(0))$ with $v \ge 0$ such that $Lv \le 0$ in $B_1(0)$, for some L of the form (8.2). Then, for any $\varphi \in C_0^\infty(B_1(0))$ we have*

$$\int_{B_1(0)} |\nabla(\varphi v)|^2 dx \le C\|\nabla\varphi\|^2_{L^\infty(B_1(0))}\int_{B_1(0)\cap \operatorname{supp}\varphi} v^2 dx,$$

where C depends only on n, λ and Λ.

Proof. Notice that the weak formulation of $-\operatorname{div}(A(x)\nabla v) \le 0$ in $B_1(0)$ is

$$\int_{B_1(0)} A(x)\nabla v(x)\nabla\eta(x)dx \le 0 \quad \text{for all} \quad \eta = \eta(x) \in H_0^1\left(B_1(0)\right), \eta \ge 0.$$

Take $\eta = \varphi^2 v$, to get

$$\int_{B_1(0)} A(x)\nabla v(x)\nabla(\varphi^2 v(x))dx \le 0.$$

Now, we want to "bring one of the φ from the first gradient to the second gradient". Indeed, using

$$\begin{aligned}\nabla(\varphi^2(x)v(x)) &= \varphi(x)\nabla(\varphi(x)v(x)) + (\varphi(x)v(x))\nabla\varphi(x),\\ \nabla(\varphi(x)v(x)) &= \varphi(x)\nabla v(x) + v(x)\nabla\varphi(x),\end{aligned}$$

we get

$$\begin{aligned}0 &\ge \int_{B_1(0)} A\nabla v\nabla(\varphi^2 v)dx\\ &= \int_{B_1(0)} \varphi A\nabla v\nabla(\varphi v)dx + \int_{B_1(0)} \varphi v A\nabla v\nabla\varphi dx\\ &= \int_{B_1(0)} A\nabla(\varphi v)\nabla(\varphi v)dx - \int_{B_1(0)} vA\nabla\varphi\nabla(\varphi v)dx + \int_{B_1(0)} \varphi v A\nabla v\nabla\varphi dx\\ &= \int_{B_1(0)} A\nabla(\varphi v)\nabla(\varphi v)dx - \int_{B_1(0)} v(A-A^T)\nabla\varphi\nabla(\varphi v)dx - \int_{B_1} v^2 A\nabla\varphi\nabla\varphi dx.\end{aligned}$$

Let us first bound the term involving $(A - A^T)$. By Hölder's inequality, using the uniform ellipticity of A and that $(A - A^T)^2 \le 4\Lambda^2 I$, we get

$$\begin{aligned}&\int_{B_1(0)} v(A(x)-A^T(x))\nabla\varphi\nabla(\varphi v)dx\\ &\qquad\le \left(\int_{B_1(0)} |v(A(x)-A^T(x))\nabla\varphi|^2dx\right)^{\frac12}\left(\int_{B_1(0)} |\nabla(\varphi v)|^2dx\right)^{\frac12}\\ &\qquad\le \frac{2\Lambda}{\lambda^{\frac12}}\left(\int_{B_1} |v\nabla\varphi|^2dx\right)^{\frac12}\left(\int_{B_1(0)} A(x)\nabla(\varphi v)\nabla(\varphi v)dx\right)^{\frac12}\\ &\qquad\le \frac12\int_{B_1(0)} A(x)\nabla(\varphi v)\nabla(\varphi v)dx + \frac{2\Lambda^2}{\lambda}\int_{B_1(0)} |v\nabla\varphi|^2dx,\end{aligned}$$

where in the last inequality we are using that $2ab \le a^2 + b^2$. Combining the previous inequalities, we obtain that

$$\frac{2\Lambda^2}{\lambda}\int_{B_1(0)} |v\nabla\varphi|^2dx \ge \frac12\int_{B_1(0)} A(x)\nabla(\varphi v)\nabla(\varphi v)dx - \int_{B_1(0)} v^2 A(x)\nabla\varphi\nabla\varphi dx.$$

Therefore, we deduce

$$\begin{aligned}\lambda\int_{B_1(0)} |\nabla(\varphi v)|^2dx &\le \int_{B_1(0)} A(x)\nabla(\varphi v)\nabla(\varphi v)dx\\ &\le 2\int_{B_1(0)} v^2 A(x)\nabla\varphi\nabla\varphi dx + \frac{4\Lambda^2}{\lambda}\int_{B_1(0)} |v\nabla\varphi|^2dx\\ &\le \left(2\Lambda + \frac{4\Lambda^2}{\lambda}\right)\|\nabla\varphi\|^2_{L^\infty(B_1(0))}\int_{B_1(0)\cap\operatorname{supp}\varphi} v^2dx,\end{aligned}$$

and the lemma is proved. □

We will use the energy inequality (from the previous lemma) applied to the function

$$v_+ := \max\{v, 0\}.$$

Before doing so, let us show that if $Lv \le 0$ (i.e., v is a subsolution), then $Lv_+ \le 0$ (i.e., v_+ is a subsolution as well). (More generally, the maximum of two subsolutions is always a subsolution.)

Lemma 8.3. *Let L be of the form* (8.2)*, let $v \in H^1(B_1(0))$ be such that $Lv \le 0$ in $B_1(0)$. Then, $Lv_+ \le 0$.*

Proof. We proceed by approximation. Let $F \in C^\infty(\mathbb{R})$ be a smooth, nondecreasing, convex function, with globally bounded first derivatives. We start by showing that $L(F(v)) \le 0$ in $B_1(0)$. Notice that if $v \in W^{1,2}(B_1(0))$, then $F(v) \in W^{1,2}(B_1(0))$ as well. We know that $Lv \le 0$, i.e.,

$$\int_{B_1(0)} A\nabla v \nabla \eta dx \le 0 \quad \text{for all} \quad \eta \in H_0^1(B_1(0)), \eta \ge 0.$$

Let us now compute, for any fixed $\eta \in H_0^1(B_1(0))$ satisfying $\eta \ge 0$, $L(F(v))$. Notice that the weak formulation still makes sense.

$$\begin{aligned}\int_{B_1(0)} A\nabla F(v)\nabla \eta dx &= \int_{B_1(0)} F'(v) A\nabla v \nabla \eta dx \\ &= \int_{B_1(0)} A\nabla v \nabla (F'(v)\eta) dx - \int_{B_1(0)} \eta F''(v) A\nabla v \nabla v dx.\end{aligned}$$

The first term is non-positive, since $F'(v)\eta \in H_0^1(B_1(0))$ and $F'(v) \ge 0$ (F is non-decreasing), so that $F'(v)\eta$ is an admissible test function. The second term is also non-positive, since $\eta F''(v) \ge 0$ and $A\nabla v \nabla v \ge 0$ by ellipticity (and the integral is well defined, since $\eta F''(v)$ can be assumed to be bounded by approximation and $\int_{B_1(0)} A\nabla v \nabla v \le \Lambda \|\nabla v\|^2_{L^2(B_1(0))}$). Therefore,

$$\int_{B_1(0)} A\nabla F(v) \nabla \eta dx \le 0,$$

and the proof is complete. We finish by taking smooth approximations of the positive part function, F_ε, converging uniformly in compact sets to $F(x) = \max\{x, 0\}$. Notice that this can be done in such a way that $\|F_\varepsilon(v)\|_{W^{1,2}(B_1(0))} \le C$, for some C independent of $\varepsilon > 0$, which gives the desired result. □

We want to prove the following.

Proposition 8.4 (from L^2 to L^∞)**.** *Let L be of the form* (8.2) *and let $v \in H^1(B_1(0))$ be a solution to $Lv \le 0$ in $B_1(0)$ then*

$$\|v_+\|_{L^\infty(B_{\frac{1}{2}}(0))} \le C \|v_+\|_{L^2(B_1(0))},$$

for some constant C depending only on n, λ and Λ.

We will prove, in fact, the following (which is actually equivalent):

Proposition 8.5 (from L^2 to L^∞)**.** *Let L be of the form* (8.2)*. There exists a constant $\delta > 0$ depending only on n, λ and Λ, such that if $v \in H^1(B_1(0))$ solves $Lv \le 0$ in $B_1(0)$ and*

$$\int_{B_1(0)} v_+^2 \le \delta$$

then

$$v \le 1 \quad \text{in} \quad B_{\frac{1}{2}}(0).$$

Proof. Define, as introduced in the general ideas of the proof, for $k \ge 0$,

$$\widetilde{B}_k(0) := \left\{ x \in \mathbb{R}^n : |x| \le \frac{1}{2} + 2^{-k-1} \right\},$$

$v_k := (v - C_k)_+$ with $C_k = 1 - 2^{-k}$ and let φ_k be a family of shrinking cut-off functions $0 \le \varphi_k \le 1$ that fulfill $\varphi_k \in C_0^\infty(B_1(0))$,

$$\varphi_k = \begin{cases} 1 \text{ in } \widetilde{B}_k(0) \\ 0 \text{ in } \widetilde{B}_{k-1}^c(0) \end{cases}$$

and $|\nabla\varphi_k| \le C2^k$ in $\widetilde{B}_{k-1}(0)\setminus\widetilde{B}_k(0)$, where C here depends only on n. Let

$$V_k := \int_{B_1(0)} \varphi_k^2 v_k^2 dx.$$

Now, the fact that $v_{k+1} \le v_k$ in $B_1(0)$, $\varphi_k = 1$ in $\widetilde{B}_k(0)$, the Sobolev inequality and the energy inequality Lemma 8.2 give

$$\begin{aligned}\left(\int_{B_1(0)} |\varphi_{k+1}v_{k+1}|^p dx\right)^{\frac{2}{p}} &\le C\left(\int_{B_1(0)} |\nabla\left(\varphi_{k+1}v_{k+1}\right)|^2 dx\right)\\ &\le C2^{2k}\int_{\widetilde{B}_k} |v_{k+1}|^2 dx \le C2^{2k}\int_{B_1(0)} (\varphi_k v_k)^2 dx = C2^{2k}V_k,\end{aligned}$$

for $p = \frac{2n}{n-2}$ if $n \ge 3$. If $n = 1$ or 2, we can take $p = 4$. On the other hand, by Hölder's inequality,

$$V_{k+1} = \int_{B_1(0)} \varphi_{k+1}^2 v_{k+1}^2 dx \le \left(\int_{B_1(0)} (\varphi_{k+1}v_{k+1})^p dx\right)^{\frac{2}{p}} |\{\varphi_{k+1}v_{k+1} > 0\}|^\gamma,$$

where $\gamma = \frac{2}{n}$ if $n \ge 3$ and $\gamma = \frac{1}{2}$ if $n = 1$ or 2. Now, from Chebyshev's inequality and the definition of v_k and φ_k,

$$\begin{aligned}|\{x \in B_1(0) : \varphi_{k+1}v_{k+1} > 0\}| &= |\{x \in B_1(0) : \varphi_{k+1}(v - (1 - 2^{-k-1}))_+ > 0\}|\\ &= |\{x \in B_1(0) : \varphi_{k+1}(v - (1 - 2^{-k}) - 2^{-k-1})_+ > 0\}|\\ &= |\{x \in B_1(0) : \varphi_{k+1}(v - (1 - 2^{-k}))_+ > 2^{-k-1}\}|\\ &\le |\{x \in B_1(0) : \varphi_k v_k > 2^{-k-1}\}|\\ &= |\{x \in B_1(0) : \varphi_k^2 v_k^2 > 2^{-2k-2}\}|\\ &\le 2^{2(k+1)}\int_{B_1(0)} \varphi_k^2 v_k^2 dx = 2^{2(k+1)}V_k.\end{aligned}$$

Apart from Chebyshev's inequality, we are using here that if $v_{k+1} > 0$ and $\varphi_{k+1} > 0$, then $v_k > 2^{-k-1}$ and $\varphi_k = 1$. Thus, combining the previous inequalities,

$$\begin{aligned}V_{k+1} &\le \left(\int_{B_1(0)} (\varphi_{k+1}v_{k+1})^p dx\right)^{\frac{2}{p}} |\{\varphi_{k+1}v_{k+1} > 0\}|^\gamma\\ &\le C2^{2k}V_k\left(2^{2(k+1)}V_k\right)^\gamma \le C^{k+1}V_k^{1+\gamma}\end{aligned}$$

where we recall $\gamma = \frac{2}{n}$ if $n \ge 3$ and $\gamma = \frac{1}{2}$ otherwise; and C depends only on n, λ and Λ. Now, we claim that, if $\delta > 0$ is small enough, then

$$\left.\begin{array}{l} 0 \le V_{k+1} \le C^{k+1}V_k^{1+\gamma}\\ 0 \le V_0 \le \delta \end{array}\right\} \Rightarrow V_k \to 0 \quad \text{as} \quad k \to \infty$$

Indeed, in order to see this it is enough to check by induction that if $V_0 \le C^{-1/\gamma - 1/\gamma^2}$ then

$$V_k^\gamma \le \frac{C^{-k-1}}{(2C)^{\frac{1}{\gamma}}},$$

which is a simple computation. Alternatively, one could check by induction that $V_k \le C^{(1+\gamma)^k\left(\sum_{i=1}^k \frac{i}{(1+\gamma)^k}\right)}V_0^{(1+\gamma)^k}$. Hence, we have proved that

$$V_k = \int_{B_1(0)} (\varphi_k v_k)^2 dx \to 0 \quad \text{as} \quad k \to \infty$$

Passing to the limit, we get

$$\int_{B_{\frac{1}{2}}(0)} (v-1)_+^2\, dx = 0$$

and thus, $v \le 1$ in $B_{\frac{1}{2}}(0)$, as wanted. $\square$

Proof of Proposition 8.4 . To deduce the Proposition 8.4 from Proposition 8.5 , just use $\widetilde{v} := \frac{\sqrt{\delta}}{\|v_+\|_{L^2(B_1(0))}} v$ (which solves the same equation). $\square$

This proves the first part of the estimate

$$Lv \le 0 \quad \text{in } B_1(0) \Longrightarrow \|v_+\|_{L^\infty(B_{\frac{1}{2}}(0))} \le C\|v_+\|_{L^2(B_1(0))}. \tag{8.3}$$

Notice that, as a direct consequence, we have the L^2 to L^∞ estimate. Indeed, if $Lv = 0$ then $Lv_+ \le 0$ (see Lemma 8.3) but also $Lv_- \le 0$, where $v_- := \max\{0, -v\}$. Thus, $\|v_-\|_{L^\infty(B_{\frac{1}{2}}(0))} \le C\|v_-\|_{L^2(B_1(0))}$ and since $\|v\|_{L^2(B_1(0))} = \|v_+\|_{L^2(B_1(0))} + \|v_-\|_{L^2(B_1(0))}$, combining the estimate for v_+ and v_- we get

$$Lv = 0 \quad \text{in } B_1(0) \Longrightarrow \|v\|_{L^\infty(B_{\frac{1}{2}}(0))} \le C\|v\|_{L^2(B_1(0))} \tag{8.4}$$

as we wanted to see.

De Giorgi's second step: L^∞ to $C^{0,\alpha}$. We next prove the second step of the De Giorgi's estimate. We want to prove the following:

Proposition 8.6 (Oscillation decay)**.** *Let L be of the form* (8.2)*. Let $v \in H^1(B_2(0))$ be a solution to*

$$Lv = 0 \quad \text{in} \quad B_2(0).$$

Then,

$$\operatorname*{osc}_{B_{\frac{1}{2}}(0)} v \le (1-\theta) \operatorname*{osc}_{B_2(0)} v$$

for some $\theta > 0$ small depending only on n, λ and Λ.

Remark 8.7. This proposition immediately implies $C^{0,\alpha}$ regularity of solutions by standard arguments.

Remark 8.8. If $v \in H^1(B_R(x_0))$ be a solution to $Lv = 0$ in $B_R(x_0)$, we can choose $u(x) = v(R(x + x_0))$ and see that $-\operatorname{div}(A(R(x + x_0))\nabla u(x)) = 0$ in $B_1(0)$. Here $A(Rx)$ satisfies $\lambda I \le A(R(x + x_0)) \le \Lambda I$ for any $x \in B_1(0)$. Then it can be inferred from (8.6) that

$$\operatorname*{osc}_{B_{\frac{R}{2}}(x_0)} v \le (1-\theta) \operatorname*{osc}_{B_R(x_0)} v, \tag{8.5}$$

where $\theta > 0$ small depending only on n, λ and Λ.

As shown next, Proposition 8.6 follows from the following lemma.

Lemma 8.9. *Let L be of the form* (8.2) *and let $v \in H^1(B_2(0))$ be such that $v \le 1$ in B_2 and $Lv \le 0$ in $B_2(0)$. Assume that*

$$|\{v \le 0\} \cap B_1(0)| \ge \mu > 0.$$

Then,

$$\sup_{B_{\frac{1}{2}}(0)} v \le 1 - \gamma,$$

for some small $\gamma > 0$ depending only on n, λ, Λ and μ.

Similarly we can use the scaling arguments to obtain the following results.

Corollary 8.10. *Let L be of the form (8.2) and let $v \in H^1(B_{2R}(x_0))$ be such that and $Lv \le 0$ in $B_{2R}(x_0)$. Assume that*

$$|\{v \le 0\} \cap B_R(x_0)| \ge \mu |B_R(x_0)| > 0.$$

with $\mu > 0$. Then,

$$\sup_{B_{\frac{R}{2}}(0)} v \le (1-\gamma) \sup_{B_{2R}(x_0)} v, \tag{8.6}$$

for some small $\gamma > 0$ depending only on n, λ, Λ and μ.

In other words, if $v \le 1$ and it is "far from 1" in a set of non-zero measure, then v cannot be close to 1 in $B_{\frac{1}{2}}(0)$.

Let us show how this lemma yields the oscillation decay:

Proof of Proposition 8.6. Consider the function

$$w(x) := \frac{2}{\operatorname{osc}_{B_2(0)} v}\left(v(x) - \frac{\sup_{B_2(0)} v + \inf_{B_2(0)} v}{2}\right)$$

and notice that

$$-1 \le w \le 1 \quad \text{in} \quad B_2(0),$$

(in fact, $\operatorname{osc}_{B_2(0)} w = 2$). Let us assume that $|\{w \le 0\} \cap B_1(0)| \ge \frac{1}{2}|B_1(0)|$ (otherwise, we can take $-w$ instead). Then, by Lemma 8.9, we get

$$w \le 1 - \gamma \quad \text{in} \quad B_{1/2}$$

and thus This yields

$$\operatorname*{osc}_{B_{\frac{1}{2}}(0)} w \le 2 - \gamma \Longrightarrow \operatorname*{osc}_{B_{\frac{1}{2}}(0)} v \le \left(1 - \frac{\gamma}{2}\right) \operatorname*{osc}_{B_2(0)} v$$

and thus the proposition is proved. □

To prove Lemma 8.9. we will need the following De Giorgi isoperimetric inequality. It is a kind of quantitative version of the fact that an H^1 function cannot have a jump discontinuity.

Lemma 8.11. *Let $w \in H^1(B_1(0))$ be such that*

$$\int_{B_1(0)} |\nabla w|^2 dx \le C_0.$$

Let

$$A := \{w \le 0\} \cap B_1(0), \quad D := \left\{w \ge \frac{1}{2}\right\} \cap B_1(0), \quad E := \left\{0 < w < \frac{1}{2}\right\} \cap B_1(0).$$

Then, we have

$$C_0|E| \ge c|A|^2|D|^2$$

for some constant c depending only on n.

Proof. We define $\overline{w}$ in $B_1(0)$ as $\overline{w} = w$ in E, $\overline{w} \equiv 0$ in A and $\overline{w} = \frac{1}{2}$ in D. In this way, $\nabla \overline{w} \equiv 0$ in $B_1(0)\backslash E$ and $\int_{B_1(0)} |\nabla \overline{w}|^2 \le C_0$.

Let us denote $\overline{w}_{B_1(0)} := \fint_{B_1(0)} \overline{w}(x)dx$ the average of $\overline{w}$ in $B_1(0)$. Then,

$$\begin{aligned}
|A||D| &\le 2\int_A \int_D |\overline{w}(x) - \overline{w}(y)| dx dy \\
&\le 2\int_{B_1(0)} \int_{B_1(0)} (|\overline{w}(x) - \overline{w}_{B_1(0)}| + |\overline{w}(y) - \overline{w}_{B_1(0)}|) dx dy \\
&= 4|B_1(0)| \int_{B_1(0)} |\overline{w}(x) - \overline{w}_{B_1(0)}| dx \le C \int_E |\nabla \overline{w}(x)| dx,
\end{aligned}$$

where in the last step we have used Poincaré inequality and the fact that $\nabla\overline{w} \equiv 0$ in $B_1(0)\backslash E$. Thus, by Hölder's inequality we reach

$$|A||D| \leq C\int_E |\nabla\overline{w}| \leq C\left(\int_E |\nabla\overline{w}|^2\right)^{\frac{1}{2}} |E|^{\frac{1}{2}} \leq CC_0^{\frac{1}{2}}|E|^{\frac{1}{2}},$$

as we wanted to see. $\square$

Finally, we prove Lemma 8.9:

Proof of Lemma 8.9. Consider the sequence

$$w_k := 2^k[v - (1-2^{-k})]_+.$$

Notice that, since $v \leq 1$ in $B_2(0)$, then $w_k \leq 1$ in $B_2(0)$. Moreover, $Lw_k \leq 0$ in $B_2(0)$.

Using the energy inequality, Lemma 8.2, we easily get that

$$\int_{B_1(0)} |\nabla w_k|^2 \leq C\int_{B_2(0)} w_k^2 \leq C_0 \quad (\text{notice } 0 \leq w_k \leq 1 \text{ in } B_2(0)).$$

We also have

$$|\{w_k \leq 0\} \cap B_1(0)| \geq \mu > 0$$

(by the assumption on v). We now apply the previous lemma, Lemma 8.11, recursively to w_k, as long as

$$\int_{B_1(0)} w_{k+1}^2 \geq \delta^2.$$

We get

$$\left|\left\{w_k \geq \frac{1}{2}\right\} \cap B_1(0)\right| \geq |\{w_{k+1} > 0\} \cap B_1(0)| \geq \int_{B_1(0)} w_{k+1}^2 \geq \delta^2.$$

Thus, from Lemma 8.11,

$$\left|\left\{0 < w_k < \frac{1}{2}\right\} \cap B_1(0)\right| \geq \frac{c}{C_0}\delta^4\mu^2 = \beta > 0,$$

where $\beta > 0$ is independent of k and depends only on n, δ and μ. But notice that the sets $\left\{0 < w_k < \frac{1}{2}\right\}$ are disjoint for all $k \in \mathbb{N}$, therefore we cannot have the previous inequality for every k. This means that, for some $k_0 \in \mathbb{N}$ (depending only on n and β) we have

$$\int_{B_1(0)} w_{k_0}^2 < \delta^2$$

and, hence, by the L^2 to L^∞ estimate from Proposition 8.4

$$\|w_+\|_{L^\infty(B_{\frac{1}{2}}(0))} \leq C\|w_+\|_{L^2(B_1(0))}.$$

We get

$$\|w_{k_0}\|_{L^\infty(B_{\frac{1}{2}}(0))} \leq C\delta \leq \frac{1}{2},$$

provided that $\delta > 0$ is small enough, depending only on n, λ and Λ. This means that $w_{k_0} \leq \frac{1}{2}$ in $B_{\frac{1}{2}}(0)$ and thus

$$v \leq \frac{1}{2}2^{-k_0} + (1-2^{-k_0}) \leq 1 - 2^{-k_0-1} = 1-\gamma$$

as desired, where k_0 (and therefore, γ) depends only on n, λ, Λ and μ. $\square$

Summarizing, we have now proved Lemma 8.9 (by using the L^2 to L^∞ estimate and a lemma). Then, Lemma 8.9 implies the oscillation decay and the oscillation decay implies the Hölder regularity.

Theorem 8.12. *Let L be of the form* (8.2) *and let $v \in H^1(B_1(0))$ solve $Lv = 0$ in $B_1(0)$. Then,*

$$\|v\|_{C^{0,\alpha}(B_{\frac{1}{2}}(0))} \leq C\|v\|_{L^\infty(B_1(0))}$$

for some $\alpha > 0$ and C depending only on n, λ and Λ.

Combining this last result with the L^2 to L^∞ estimate, Proposition 8.4. we finally obtain the theorem of De Giorgi-Nash.

Theorem 8.13. *Let $v \in H^1(B_1(0))$ be a weak solution to* $\operatorname{div}(A(x)\nabla v) = 0$ *in $B_1(0)$, with $0 < \lambda I \leq A(x) \leq \Lambda I$. Then, there exists some $\alpha > 0$ such that $v \in C^{0,\alpha}(B_{\frac{1}{2}})$ and*

$$\|v\|_{C^{0,\alpha}(B_{\frac{1}{2}}(0))} \leq C\|v\|_{L^2(B_1(0))}.$$

The constants C and $\alpha > 0$ depend only on n, λ and Λ.

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