Notes of Schoen-Uhlenbeck theory for harmonic maps

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Abstract

Harmonic maps are nonlinear extensions of harmonic functions. They are critical points of natural energy functionals between Riemannian manifolds. There are abundant materials in the study of harmonic maps. One of the most important works on harmonic maps were obtained by R. Schoen and K. Uhlenbeck [16]. However the proofs in this paper are complicated and technical, which is not easy to read. Recently, in [9] F. Lin gave brief proofs for main results in [16]. Due to limited space, [9] is lack of details. In this paper, we will complete the details in it and also give a quick review of some classical and recent results in harmonic maps.

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1 Introduction

Let (M, g) be an n-dimensional Riemannian manifold with or without boundary, where g is a smooth Riemannian metric on it. In local coordinate system (U, x^i), we can write $g = g_{\alpha\beta}dx^i \otimes dx^j$, where $(g_{\alpha\beta})$ is a positive definitive symmetric $n\times n$ matrix and we use the Einstein convention for summation and this convention is assumed throughout this paper. Let $g^{\alpha\beta}$ be the inverse matrix of $(g_{\alpha\beta})$, i.e. $g^{\alpha\zeta}g_{\alpha\beta} = \delta^\zeta_\beta$, where $\delta^\zeta_\beta = 1$ if $\beta = \zeta$ and 0 otherwise. The volume element of (M, g) associated with metric g is $dv_g = \sqrt{\det(g_{\alpha\beta})}dx$. Let (N, γ) be an m-dimensional compact Riemannian manifold without boundary and γ is the smooth metric on it. A harmonic map between (M, g) and (N, γ) is a critical point for the Dirichlet energy

$$\mathcal{E}(u) = \int_M |\nabla u|^2 dv_g,$$

where, for $x \in M$ and chart (U, φ), (V, ψ) such that $x \in U$, $u(x) \in V$ and $\overline{u} = \psi \circ \varphi^{-1}$, we write

$$|\nabla u|^2(x) := \gamma_{ij}(u(x))g^{\alpha\beta}(x)D_\alpha \overline{u}^i(\varphi(x))D_\beta \overline{u}^j(\varphi(x)).$$

Since from the form of $|\nabla u|^2$, the nonlinearity comes from the metric γ_{ij} (if $N = \mathbb{R}^n$, there is no nonlinearity) and the metric (g_{ij}) is not essential in the study of harmonic maps. From this point of view, one can assume that $M = \Omega \subset \mathbb{R}^n$, where Ω is a domain in $\mathbb{R}^n$. For general case, one can admit some assumptions on the curvatures and injective radius on M to get similar results, see §4.8 of [15]. Thanks to an embedding theorem of J. Nash [10], we can further assume that (N, γ) is isometrically embedded into $\mathbb{R}^p$ for some $p \in \mathbb{Z}_+$. Then the Dirichlet energy becomes

$$\mathcal{E}(u) = \int_\Omega |\nabla u|^2 dx, \text{ where } |\nabla u|^2 = \sum_{i=1}^{p} \sum_{\alpha=1}^{n} |D_\alpha u^i|^2 \tag{1.1}$$

and one can study the harmonic maps in the fundamental Sobolev type space

$$W^{1,2}(\Omega, N) := \{u \in W^{1,2}(\Omega, \mathbb{R}^p) : u(x) \in N, \text{ for a.e. } x \in \Omega\}. \tag{1.2}$$

Similarly, $W^{1,2}_{\text{loc}}(\Omega, N)$, $C^r(\Omega, N)$ $(r \geq 0)(r \in \mathbb{Z}_+ \cup \{0\})$, $C^{0,\alpha}(\Omega, N)(0 < \alpha < 1)$ can be defined in the same manners. Now harmonic maps can be defined as follows.

Definition 1.1. A map $u \in C^2(\Omega, N)$ is a harmonic map, if it is a critical point of the Dirichlet energy functional $\mathcal{E}$.

Here $u \in C^2(\Omega, N)$ is a very strong priori assumption and so we need a definition with weaker regularity assumptions on u, which is easy to apply some methods of functional analysis.

Definition 1.2 (Local minimizers)**.** A map $u \in W^{1,2}_{\text{loc}}(\Omega, N)$ is a local minimizer of the Dirichlet energy (1.1) if for every ball $B_\rho(x_0) \subset\subset \Omega$ and every $v \in W^{1,2}(B_\rho(x_0), N)$ with $v = u$ on $\partial B_\rho(x_0)$, we have

$$\int_{B_\rho(x_0)} |\nabla u|^2 dx \leq \int_{B_\rho(x_0)} |\nabla v|^2 dx. \tag{1.3}$$

When it comes to the the problems of variations, the direct method is to find the variational equations of such problems, that is, to find what PDE the "minimizer" satisfies. These can be seen as the necessity of such problems. At first, u may be a very weak solution of some equation and one can used methods of PDE to enhance the regularity of u. For study of harmonic maps, we will also follow this procedure. However it is much more complicated than those for Laplacian operator since the metric γ_{ij} produce the nonlinearity. Let $u \in W^{1,2}_{\text{loc}}(\Omega, N)$ be a minimizer. For $B_\rho(x_0) \subset\subset \Omega$ and $\delta > 0$, suppose that there exists a family of maps $\{u_s\}_{s\in(-\delta,\delta)} \subset W^{1,2}(B_\rho(x_0), N)$ such that $u_0 = u$, $u_s = u$ on $\partial B_\rho(x_0)$ for every $s \in (-\delta, \delta)$. Then by (1.3), we have

$$\frac{d}{ds}\mathcal{E}(u_s, B_\rho(x_0))\bigg|_{s=0} = 0 = \frac{d}{ds}\left(\int_{B_\rho(x_0)} |\nabla u_s|^2 dx\right)\bigg|_{s=0} = 0, \tag{1.4}$$

whenever the derivative exists.

There are two particularly useful ways of choosing families $\{u_s\}_{s\in(-\delta,\delta)}$ above: we shall call them as inner and outer variations. For details on outer variations, one can refer to §2.12.3 of [15] and for inner variations, one can refer to §10.1.1 of [5].

1. **Outer variations**: For any $\zeta \in C_0^\infty(B_\rho(x_0), \mathbb{R}^p)$, set $u_s := \Pi \circ (u + s\zeta)$, where Π is the nearest point projection onto N. Clearly, for s small enough, the image of $u + s\zeta$ lies in a tubular neighborhood of N so that u_s is well defined. One can verify that (1.4) gives,

$$\int_{B_\rho(x_0)} \left[D_\alpha u^i D_\alpha \zeta^i - \zeta^k A^k_{ij}(u) D_\alpha u^i D_\beta u^j\right] dx = 0, \tag{1.5}$$

where $A(u) = (A^k_{ij}(u))_{1\le i,j,k\le p}$ is the second fundamental form of N at $u(x)$. We can also write (1.5) in the form (the equation is in the weak sense) as

$$\Delta u + A(u)(\nabla u, \nabla u) = 0. \tag{1.6}$$

2. **Inner variations**: For any $\zeta \in C_0^\infty(B_\rho(x_0), \mathbb{R}^n)$, define $u_s(x) := u(x + s\zeta(x))$, which is well defined for s small enough. Then (1.4) implies

$$\int_{B_\rho(x_0)} \left[\frac{1}{2}|\nabla u|^2 \operatorname{div}\zeta - D_\alpha u^i D_\beta u^i D_\alpha \zeta^\beta\right] dx = 0. \tag{1.7}$$

From the discussions above, we can give the following definitions of two types harmonic maps.

Definition 1.3. Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$. Then u is said to be a

1. weakly harmonic map if it satisfies (1.5) for all $B_\rho(x_0) \subset\subset \Omega$ and every $\zeta \in C_0^\infty(B_\rho(x_0, \mathbb{R}^p))$;
2. stationary harmonic map if it is weakly harmonic and satisfies (1.7) for every $B_\rho(x_0) \subset\subset \Omega$ and $\zeta \in C_0^\infty(B_\rho(x_0), \mathbb{R}^n)$.

Remark 1.4. If $u \in C^2(\Omega, N)$, then (1.5) implies (1.7) but this may not true in general for $u \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$.

For $u \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$, we define for $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$ the normalized Dirichlet functional

$$\Theta_u(x, r) = \frac{1}{r^{n-2}} \int_{B_r(x)} |\nabla u|^2 dy. \tag{1.8}$$

By choosing some special test functions ζ in (1.7), we have the following famous monotonicity result for $\Theta_u(x,r)$ by choosing $\zeta = x\eta = (x_i\eta)_{i=1}^n$ for some suitable cut-off functions η.

Proposition 1.5 (Monotonicity formula, [5] Theorem 10.5)**.** *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$ satisfy* (1.7) (*if u us stationary harmonic this is atomically true*)*. Then for any $x_0 \in \Omega$ and almost every $0 < s < r < \mathrm{dist}(x_0, \partial\Omega)$,*

$$\begin{aligned} \Theta_u(x_0,r) - \Theta_u(x_0,s) &= \int_s^r \int_{\partial B_t(x_0)} \frac{2}{t^{n-2}} \left| \frac{\partial u}{\partial t} \right|^2 d\mathcal{H}^{n-1} dt \\ &= \int_{B_r(x_0)\setminus B_s(x_0)} \frac{2|\nabla u \cdot (x-x_0)|^2}{|x-x_0|^n} dx, \end{aligned} \tag{1.9}$$

In particular, $\Theta_u(x,r)$ is increasing with respect to r and $\Theta_u(x,r) = \Theta_u(x,s)$ if and only if u is radially constant on the annulus $A_{s,r}(x) = B_r(x_0)\setminus B_s(x_0)$.

Definition 1.6 (Density)**.** We define the density of the harmonic function u at x_0 as

$$\Theta_u(x_0) := \lim_{\rho\to 0^+} \frac{1}{\rho^{n-2}} \int_{B_\rho(x_0)} |\nabla u|^2 dx, \tag{1.10}$$

where the limit exists thanks to Proposition 1.5.

From the above calculations for variational equations associated to Dirichlet energy, it is easy to see that the condition of $u \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$ has the relations as

$$\text{local minimizers} \Rightarrow \text{stationary harmonic maps} \Rightarrow \text{weakly harmonic maps},$$

which the study of harmonic maps roughly follows.

For the case that $\Omega \subset \mathbb{R}^2$, the results are relatively easy, i.e. any weakly harmonic map is smooth. Such result was given by F. Hélein in [6] and a more general proof was provided by T. Rivière in [12].

If $n \geq 3$, things are much more sophisticated, since then the nonlinearity will produce the singularity of the solution. Therefore, for $u \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$, we need to classify points of Ω into two classes, namely, the regular points and singular points. Precisely speaking, we define

$$\begin{aligned} \operatorname{reg} u &= \{\text{sets of regular points of } u\} = \{x \in \Omega : u \text{ is } C^\infty \text{ in a neighborhood of } x\} \\ \operatorname{sing}(u) &= \{\text{sets of regular points of } u\} = \Omega \setminus \operatorname{reg} u. \end{aligned}$$

The study of harmonic maps aims to show that the singular sets is "small", in the sense of Hausdorff measures. Moreover, researchers also pay much attention to explore fine structures and rectifiability of singular sets.

For local minimizers, if N can be contained in one chart, in [4], M. Giaquinta and E. Giusti used classical regularity theory of nonlinear elliptic equations to get most of results, including regularity and blow-ups. They also used the compactness methods to give the stratifications of singular sets. If N cannot be covered by only one chart, results in [4] were generalized in [16] by R. Schoen and K. Uhlenbeck through some subtle constructions of test functions for local minimizers, this is one of the most important breakthroughs of study in harmonic maps. We will introduce the main results in [16].

Theorem 1.7 (Schoen-Uhlenbeck, ε-regularity of minimizers)**.** *Let $u \in W^{1,2}_{\text{loc}}(\Omega, N)$ be a local minimizer of the Dirichlet energy. Assume also that the Riemannian manifold (N, γ) is closed (compact and without boundary). Then there exist $\varepsilon_0 = \varepsilon_0(n, N, \gamma) > 0$, such that if for some ball $B_R(x_0) \subset\subset \Omega$ we have*

$$\frac{1}{R^{n-2}} \int_{B_R(x_0)} |\nabla u|^2 dx \leq \varepsilon_0^2,$$

then $u \in C^\infty(B_{R/2}(x_0), N)$, for some $\sigma \in (0,1)$. Precisely, u satisfies the estimate

$$R^k \|\nabla^k u\|_{L^\infty(B_{R/2}(x_0))} \leq C \left(\frac{1}{R^{n-2}} \int_{B_R(x_0)} |\nabla u|^2 dx \right)^{1/2}, \quad \forall k \in \mathbb{Z}_+ \tag{1.11}$$

where $C = C(n, N, \gamma)$.

This is theorem can be understood that if the normalized energy is sufficiently small, we can establish the interior C^∞ estimates for local minimizers. Theorem 1.7 in [16] were introduced by L. Simon using different methods, namely, by obtaining of Caccioppoli inequality given in §2 of [15] and use Theorem 3.1. In view of the ε-regularity above and Theorem 2.10 of [3], one can provide the following precise characterization of singular sets of harmonic maps.

Theorem 1.8 (Characterization of singular sets for minimizers)**.** *A local minimizer $u \in W^{1,2}_{\text{loc}}(\Omega, N)$ of the Dirichlet energy is smooth, except in the singular set*

$$\operatorname{sing}(u) := \left\{ x \in \Omega : \Theta_u(x) = \lim_{r \to 0^+} \frac{1}{r^{n-2}} \int_{B_r(x)} |\nabla u|^2 dy > 0 \right\},$$

i.e. $u \in C^\infty(M \backslash \operatorname{sing}(u), N)$. Moreover, for every $\Omega_0 \subset\subset M$ one has

$$\operatorname{sing}(u) \cap \Omega_0 := \left\{ x \in \Omega_0 : \Theta_u(x) = \lim_{r \to 0^+} \frac{1}{r^{n-2}} \int_{B_r(x)} |\nabla u|^2 dy > \varepsilon_0^2 \right\}.$$

where $\varepsilon_0 > 0$ depends only on Ω_0 and N. Moreover $\mathcal{H}^{n-2}(\operatorname{sing}(u)) \leq n-2$.

The next is about the compactness of local minimizers. Generally, we see that by using Rellich theorem, i.e. $W^2_{\text{loc}}(\Omega, N) \subset\subset L^2_{\text{loc}}(\Omega, N)$, all bounded sequence in $W^2_{\text{loc}}(\Omega, N)$ is precompact in $L^2_{\text{loc}}(\Omega, N)$. The following theorem shows that if the sequence is local minimizers, bounded sequence in $W^2_{\text{loc}}(\Omega, N)$ is precompact.

Theorem 1.9 (Compactness theorem of local minimizers)**.** *Consider a sequence of energy minimizing harmonic maps $u_j \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$ with locally equibounded energies, i.e. such that for every $B_R(x_0) \subset\subset \Omega$, we have*

$$\sup_{j\in\mathbb{N}} \int_{B_R(x_0)} |\nabla u_j|^2 dx < \infty.$$

Then a subsequence u_{j_k} converges in $W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^p)$ to an energy minimizing harmonic map $u \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$.

For stationary harmonic maps, Theorem 1.7 is also true. Such results were firstly proved by L. C. Evans in [2] for $N = S^{m-1} \subset \mathbb{R}^m$ and then by F. Bethuel in [1] for arbitrary Riemannian manifold N defined at the beginning of this paper. Recently, T. Rivière and M. Struwe gave proof of it by more general methods, see [13]. Proofs in [1, 13] highly relies on the regularity of nonlinear elliptic equations and technical observations for the geometric structures in the second fundamental solutions.

However Theorem 1.9 is not necessarily true for stationary harmonic maps. The compactness results of stationary harmonic maps are completely established by F. Lin in [8]. In this paper the author define a defect measure ν to characterize the compactness results.

Besides the ε-regularity results and compactness results above, mathematicians also care about the fine structures of the singular sets. We use stratifications to study it.

Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$ be a stationary harmonic map, consider a ball $B_R(y) \subset \Omega$ and define the rescaled map $u_{y,\rho} \in W^{1,2}(B_{R/\rho}(0), N)$ by

$$u_{y,\rho}(x) := u(y + \rho x).$$

In view of Proposition 1.9, $u_{y,\rho}$ is uniformly bounded in $W^{1,2}_{\mathrm{loc}}(\Omega, N)$, we can extract a subsequence of u_{y,ρ_j} with $\rho_j \to 0$ and $\varphi \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$ such that $u_{y,\rho_j} \rightharpoonup \varphi$ in $W^{1,2}_{\mathrm{loc}}(\Omega, N)$ and $u_{y,\rho_j} \to \varphi$ in $L^{1,2}_{\mathrm{loc}}(\Omega, N)$. By using 1.9 again, we can get that φ is 0-homogeneous. If we further assume that φ is a local minimizer, we can further assume that the limit $\varphi \in W^{1,2}_{\mathrm{loc}}(\Omega, N)$ is also a minimizer and $u_j \to \varphi$ in $W^{1,2}_{\mathrm{loc}}(\Omega, N)$, namely, we get the following proposition.

Proposition 1.10 (Blow-up for minimizers)**.** *There is a sequence $\rho_j \to 0^+$ and a locally minimizing harmonic map $\varphi \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n, N)$ such that $u_{y,\rho_j} \to \varphi$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^p)$. Moreover φ is positively homogeneous of degree zero:*

$$\varphi(\lambda x) = \varphi(x), \quad \textit{for all } x \in \mathbb{R}^n, \lambda > 0.$$

We call φ the tangent map of u at y.

Here, we call φ be the tangent map of u at the point y. By some basic arguments of semi-continuity, we can further deduce the following properties of the convergence φ.

Proposition 1.11. *Let $\varphi \in W^{1,2}_{\text{loc}}(\mathbb{R}^n, N)$ be defined above. Then, for every $y \in \mathbb{R}^n$ we have $\Theta_\varphi(y) \leq \Theta_\varphi(0)$. Set*

$$S(\varphi) := \{y \in \mathbb{R}^n : \Theta_\varphi(y) = \Theta_\varphi(0)\}$$

Then $S(\varphi)$ is a linear subspace of $\mathbb{R}^n$ and

$$\varphi(x+y) = \varphi(x), \quad \text{for every } x \in \mathbb{R}^n,\ y \in S(\varphi). \tag{1.12}$$

By this, the singular sets of u can be stratified as

$$S_0(u) \subset ... \subset S_k(u) \subset ... \subset S_{n-1}(u) \subset S_n(u) \subset \operatorname{sing}(u) \subset \Omega,$$

where $\{S_i(u)\}_{i=1}^n$ are defined as below.

Definition 1.12. Given a stationary harmonic map $u \in W^{1,2}_{\text{loc}}(\Omega, N)$, define

$$S_j(u) := \{x \in \operatorname{sing}(u) : \dim S(\varphi) \leq j \text{ for every tangent map } \varphi \text{ of } u \text{ at } x\},$$

where $\operatorname{sing}(u)$ is the singular set of u.

By using the famous Federer dimensional reduction methods, one can show that $\dim^{\mathcal{H}} S_j(u) \leq j$, where $\dim^{\mathcal{H}}$ denote the Hausdorff dimension. If u is a local minimizer, one can further get that

$$S_0(u) \subset ... \subset S_k(u) \subset ... \subset S_{n-3}(u) = \operatorname{sing}(u) \subset \Omega,$$

which directly implies that $\dim^{\mathcal{H}}(\operatorname{sing}(u)) \leq n-3$. For the rectifiability of singular sets of local minimizers, one can see §4 of [15]. In this section of the book, L. Simon established the rectifiablity of top stratum S^{n-3} for singular sets assuming that N is an analytic manifold. Also, some further developments of blow-ups for local minimizers can be found in [14].

For stationary harmonic maps F. Lin prove the rectifiablity of the defect measures, which is defined in [8] to measure the convergence of blow-up for stationary harmonic maps. Recently, A. Naber and D. Valtorta used Reifenberg theory to prove that $S^k(u)$ is k-rectifiable for stationary harmonic maps in [11].

In this paper, we will present on the proof of Theorem 1.7 and omit the proof of Theorem 1.9, whose proof needs a lot of dedicated results in [8] that cannot be concluded within 10 pages. We will not present the original proof in [16] but to give the proof of [9], which are much shorter than the former one. We will complete the details in proofs and give some our own intuitions on it. In the proof, we denote by $C(a, b, ...)$ the positive constant depending only on $a, b, ...$ whose value may change line to line.

2 Proof of ε-regularity of local minimizers

In this section, we will prove Theorem 1.7. Since this theorem is a local result and scaling invariant, we can assume that $R = 1$. Also by using Lemma 3.3, we can reduce Theorem 1.7 to the following Proposition.

Proposition 2.1. *Let $u : B_1(0) \subset \mathbb{R}^n \to N$ be a local minimizer. Then there exists $\varepsilon_0 = \varepsilon_0(n, N, \gamma)$ such that if $\Theta_u(0,1) \le \varepsilon_0^2$, then there exists $\alpha = \alpha(n, N, \gamma) \in (0,1)$ such that*

$$[u]_{C^{0,\alpha}(B_{1/2}(0))} \le C \left(\int_{B_1(0)} |\nabla u|^2 dx \right)^{1/2},$$

where $C = C(n, N, \gamma)$.

By standard iterations, Proposition 2.1 is a direct consequence of the following lemma.

Lemma 2.2. *Let $u : B_1(0) \subset \mathbb{R}^n \to N$ be a local minimizer. Then there exists $\varepsilon_0 = \varepsilon_0(n, N, \gamma)$ and $\tau = \tau(N, \gamma) \in (0,1)$ such that if $\Theta_u(0,1) \le \varepsilon_0^2$, we have*

$$\Theta_u(0,\tau) \le \frac{1}{2}\Theta_u(0,1). \tag{2.1}$$

Proof of Proposition 2.1. This proof is from Proposition 10.16 of [5]. Considering $u(\tau\cdot) : B_1(0) \subset \mathbb{R}^n \to N$, we have $\Theta_{u(\tau\cdot)}(0,1) = \Theta_u(0,\tau) \le \Theta_u(0,1) \le \varepsilon_0^2$, (by using the monotonicity). Then by applying Lemma 2.2 to $u(\tau\cdot)$, we get

$$\Theta_u(0,\tau^2) \le \frac{1}{2}\Theta_u(0,\tau) \le \left(\frac{1}{2}\right)^2 \Theta_u(0,1).$$

Repeating this procedure for k times, it follows that

$$\Theta_u(0,\tau^k) \le \left(\frac{1}{2}\right)^k \Theta_u(0,1).$$

If $0 < \rho \le \tau$, we can choose $k_0 \in \mathbb{Z}_+$, such that $\tau^{k_0+1} < \rho \le \tau^{k_0}$. Then by monotonicity,

$$\Theta_u(0,\rho) \le \Theta_u(0,\tau^{k_0}) \le \left(\frac{1}{2}\right)^{k_0} \Theta_u(0,1) \le C(n,N,\gamma)\rho^{2\alpha}\Theta_u(0,1), \tag{2.2}$$

where $\alpha = \min\{1, \frac{\log 2}{2\log(\tau^{-1})}\}$. Since τ depends only on N and τ, we can apply (2.2) to any $x_0 \in B_{1/2}$. Together with the Campanato characterization of Hölder space (see §5.1 of [5]), the proof of Proposition 2.1 is completed as desired. □

To show Lemma 2.2, we need some lemmas as follows. We will prove them as in [9].

Lemma 2.3. *Let $\rho \in (\frac{1}{2}, \frac{3}{4})$, then*

$$g = u|_{\partial B_\rho(0)} \in \mathrm{BMO}(\partial B_\rho(0)) \text{ and } \|g\|_{\mathrm{BMO}} \le C(n)\varepsilon_0,$$

where the space BMO *contains functions of bounded mean oscillations, i.e.*

$$g \in \mathrm{BMO}(\partial B_\rho(0)) \Leftrightarrow \|g\|_{\mathrm{BMO}(\partial B_\rho(0))} = \sup_{Q_r, r \le \rho/4} \frac{1}{\mathcal{H}^{n-1}(Q_r)} \int_{Q_r} |g - g_{Q_r}| d\mathcal{H}^{n-1} < +\infty.$$

Here $g_{Q_r} = \frac{1}{\mathcal{H}^{n-1}(Q_r)} \int_{Q_r} g d\mathcal{H}^{n-1}$ with $Q_r = Q_r(x) \subset \partial B_\rho(0)$ being a ball defined by geodesics with radius r and center x.

Proof. Let $Q_r(x_0) \subset \partial B_\rho(0)$ s.t. $r \leq \frac{\rho}{4}$. Then by using trace theory and Poincaré inequality as well as Proposition 1.9, we can get from $\rho < \frac{3}{4}$ that

$$\frac{1}{\mathcal{H}^{n-1}(Q_r)}\int_{Q_r}|g-g_{Q_r}|d\mathcal{H}^{n-1} \leq \left(\frac{1}{\mathcal{H}^{n-1}(Q_r)}\int_{Q_r}|g-g_{Q_r}|^2 d\mathcal{H}^{n-1}\right)^{1/2} \leq C(n)\Theta_u(x_0,r)$$
$$\leq C(n)\Theta_u\left(x_0,\frac{1}{4}\right) \leq C(n)\Theta_u(0,1) \leq C(n)\varepsilon_0,$$

which completes the proof. □

The next lemma, given as the Caccioppoli-type inequality, is the key to prove Lemma 2.2. This is not the classical Caccioppoli inequality given in [15] but has similar form as it. By using arguments by R. Schoen and K. Uhlenbeck, one can also obtain such result. So we can see that the Caccioppoli inequality is a special property for local minimizers, which is not necessarily true for stationary harmonic maps.

Lemma 2.4. *Let u be defined in Lemma 2.2, then for any $\lambda > 0$,*

$$\Theta_u\left(0,\frac{1}{2}\right) \leq \lambda\Theta_u(0,1) + \frac{C(n,N,\gamma)}{\lambda}\int_{B_1(0)}|u-\overline{u}|^2 dx, \quad \forall \overline{u} \in \mathbb{R}^p.$$

Firstly, we assume that Lemma 2.4 is true and use it to prove Lemma 2.2. This is a basic compactness arguments which is from [7] and not included in the paper [9]. We will complete it here for the sake of completeness.

Proof of Lemma 2.2. We prove it by contradiction. Assume that this is (2.1) is not true, then for any fixed $\tau \in (0,1)$ (to be chosen later), there is a sequence of local minimizers $\{u_i\} \subset W^{1,2}(B_1(0),N)$ with $u_i : B_1(0) \subset \mathbb{R}^n \to N$ and a sequence of positive real number $\{\varepsilon_i\}$ s.t.

$$\Theta_{u_i}(0,\tau) > \frac{1}{2}\varepsilon_i^2,\ \Theta_{u_i}(0,1) = \varepsilon_i^2 \text{ and } \varepsilon_i \to 0 \text{ as } i \to \infty. \tag{2.3}$$

Let $v_i = \varepsilon_i^{-1}(u_i - \overline{u_i})$, where $u_i = \fint_{B_1(0)} u_i$, we see from (1.6) that

$$\Delta v_i + \varepsilon_i A(u_i)(\nabla v_i, \nabla v_i) = 0 \tag{2.4}$$

in weak sense. By (2.3), we get $\|\nabla v_i\|_{L^2(B_1(0))} \leq 1$. Applying Poincaré inequality, $\|v_i\|_{L^2(B_1(0))} \leq C(n)$. Now we can choose a subsequence of $\{v_i\}$ (still denoted by $\{v_i\}$ for simplicity) and $v \in W^{1,2}(B_1(0),\mathbb{R}^p)$ such that $v_i \rightharpoonup v$ weakly in $W^{1,2}$ and $v_i \to v$ strongly in L^2. From this convergence, we can test (2.4) by some $\zeta \in C_0^\infty(B_1(0),\mathbb{R}^p)$ and take the limit $i \to \infty$ to get that $\Delta v = 0$ in $B_1(0)$ in weak sense. Also, by Fatou lemma, the strong L^2 convergence and $\fint_{B_1(0)} v_i = 0$, then

$$\|v\|_{L^2(B_1)} \leq C(n),\ \int_{B_1(0)} v = 0 \text{ and } \|\nabla v\|_{L^2(B_1(0))} \leq 1.$$

Since v is harmonic in $B_1(0)$, we get from the Lipschitz estimate of it that

$$\|\nabla v\|_{L^\infty(B_{3/4}(0))} \leq C(n) \left(\fint_{B_1(0)} |v|^2 dy \right)^{1/2} \leq C(n).$$

Then

$$\fint_{B_r(x)} \left| v - \fint_{B_r(x)} v \right|^2 dy \leq C(n) r^2 \|\nabla v\|_{L^\infty(B_{3/4})} \leq C(n) r^2$$

for any $x \in B_{1/2}(0)$ and $0 < r < \frac{1}{4}$. Then since we can assume that $n > 2$ (if $n = 2$ weakly harmonic maps are smooth), it follows from the equivalence of Campanato space and Morrey space that

$$\fint_{B_r(0)} |v|^2 \leq C(n) r^2, \quad \forall 0 < r \leq 1, \tag{2.5}$$

where we have also used some covering arguments to deal with the case that $\frac{1}{4} < r \leq 1$. Moreover, for $\tau \leq r \leq 1$, and $i = i(\tau, n)$ sufficiently large (depending on τ), we get form the convergence $v_i \to v$ in L^2 that

$$\left| \fint_{B_r(0)} |v_i|^2 dx - \fint_{B_r(0)} |v|^2 dx \right| \leq 4 \left(\fint_{B_r(0)} |v_i - v|^2 dx \right) \leq c(n) \tau^2 \leq c(n) r^2.$$

For such i, it follows from the definition of u_i and (2.5) that

$$\fint_{B_r(0)} |u_i - \overline{u}_i|^2 dx = \varepsilon_i^2 \left(\fint_{B_r} |v_i|^2 dx \right) \leq \varepsilon_i^2 \left(c(n) r^2 + \fint_{B_r(0)} |v|^2 dx \right) \leq C(n) r^2 \varepsilon_i^2, \tag{2.6}$$

when $\tau \leq r \leq 1$. For any $0 < \lambda < 1$, and $0 < \tau < \frac{1}{2}$, since $\lim_{i \to \infty} \varepsilon_i = 0$, we can choose $i \gg 1$ s.t.

$$\Theta_{u_i}(0, 2\tau) \leq \Theta_{u_i}(0, 1) = \varepsilon_i^2 \leq \varepsilon_0^2,$$

where ε_0 is given by Lemma 2.4. For such i, by applying Lemma 2.4 to the function $u_i(2\tau \cdot)$, we can obtain

$$\Theta_{u_i}(0, \tau) \leq \lambda \Theta_{u_i}(0, 2\tau) + \frac{C(n, N, \gamma)}{\lambda} \fint_{B_{2\tau}(0)} |u_i - \overline{u}_i|^2 dx.$$

Choosing the positive integer $j = j(\theta) \in \mathbb{Z}_+$ such that $0 < 2^j \tau \leq 1$, we iterate $j - 1$ more times and apply (2.6), Proposition 1.9 to obtain

$$\begin{aligned}
\Theta_{u_i}(0, \tau) &\leq \lambda^j \Theta_{u_i}(0, 2^j \tau) + C(n, N, \gamma) \sum_{h=1}^{j} \lambda^{h-1} \lambda^{-1} \fint_{B_{2^h \tau}(0)} |u_i - \overline{u}_i|^2 dx \\
&\leq \lambda^j \varepsilon_i^2 + C(n, N, \gamma) \sum_{h=1}^{j} \lambda^{h-1} \lambda^{-1} (2^h \tau)^2 \varepsilon_i^2 \\
&\leq \left(\lambda^j + C(n, N, \gamma)(1 - 4\lambda)^{-1} \lambda^{-1} \tau^2 \right) \varepsilon_i^2,
\end{aligned}$$

for i sufficiently large (depending on τ and λ). Since $j \to \infty$ as $\tau \to 0$, we can let $\lambda = \frac{1}{8}$ and fix $\tau < \frac{1}{4}$ small enough to ensure that

$$\lambda^j + C(n, N, \gamma)(1 - 4\lambda)^{-1}\lambda^{-1}\tau^2 = 2^{-3j} + C(n, N, \gamma)\tau^2 < \frac{1}{4},$$

contradicting the original choice of u_i. $\square$

Now it remains to show Lemma 2.4.

Proof of Lemma 2.4. By Fubini theorem, we get that there exists $\rho \in (\frac{1}{2}, \frac{3}{4})$, such that

$$\begin{aligned}
\int_{\partial B_\rho(0)} |\nabla u|^2 d\mathcal{H}^{n-1} &\leq 16 \left(\int_{B_1(0)} |\nabla u|^2 d\mathcal{H}^{n-1} \right), \\
\int_{\partial B_\rho(0)} |u - \overline{u}| d\mathcal{H}^{n-1} &\leq 16 \left(\int_{B_1(0)} |u - \overline{u}|^2 d\mathcal{H}^{n-1} \right).
\end{aligned} \tag{2.7}$$

Choose v such that $v = u$ on ∂B_ρ and $\Delta u = 0$ in B_ρ, we claim that for any $\delta > 0$, if ε_0 is sufficiently small, $\mathrm{dist}(v, N) < \delta$ and then $\widetilde{v} = \Pi \circ v$ makes sense, where Π is the nearest point projection onto N. The proof of this is given be Lemma 2.5 and Lemma 2.3. Note that $\widetilde{v} = v = u$ on ∂B_ρ, then we can use the definition of local minimizer to obtain that

$$\Theta_u \left(0, \frac{1}{2} \right) \leq \Theta_u(0, \rho) \leq \Theta_{\widetilde{v}}(0, \rho) \leq C(N, \gamma)\Theta_v(0, \rho).$$

Since v is harmonic, we have the following formula by

$$\int_{\partial B_\rho(0)} |\nabla v|^2 d\mathcal{H}^{n-1} \leq C(n, N, \gamma) \left(\int_{\partial B_\rho(0)} |\nabla_{\tan} v| d\mathcal{H}^{n-1} \right), \tag{2.8}$$

(see [17]). Then by using Cauchy inequality and (2.8), we get

$$\begin{aligned}
\Theta_v(0, \rho) = \frac{1}{\rho^{n-2}} |\nabla v|^2 dx &\leq \int_{B_\rho(0)} (v - \overline{u}) \partial_\nu v d\mathcal{H}^{n-1} \\
&\leq \frac{C(n)}{\rho^{n-2}} \left(\int_{\partial B_\rho(0)} |\nabla_{\tan} v|^2 d\mathcal{H}^{n-1} \right)^{1/2} \left(\int_{\partial B_\rho(0)} |v - \overline{u}|^2 d\mathcal{H}^{n-1} \right)^{1/2} \\
&\leq C(n, N, \gamma) \left(\int_{\partial B_\rho(0)} |\nabla v|^2 d\mathcal{H}^{n-1} \right)^{1/2} \left(\int_{\partial B_\rho(0)} |v - \overline{u}|^2 d\mathcal{H}^{n-1} \right)^{1/2} \\
&\leq C(n, N, \gamma) \left(\int_{\partial B_\rho(0)} |\nabla u|^2 d\mathcal{H}^{n-1} \right)^{1/2} \left(\int_{\partial B_\rho(0)} |u - \overline{u}|^2 d\mathcal{H}^{n-1} \right)^{1/2},
\end{aligned}$$

where $\partial_\nu v = \frac{x}{\rho} \cdot \nabla v$ and we have used integration by parts. Then the result follows directly from (2.7) and Young inequality. $\square$

Lemma 2.5. *For any $\delta_0 > 0$, there exists $\varepsilon_0 > 0$ and $r_0 \in (0,1)$ depending only on n, N and γ, such that if $\|g\|_{\mathrm{BMO}(\partial B_1(0))} \leq \varepsilon_0$, then $v(B_1(0) \cap (B_{r_0}(0))^c)$ is contained in a δ_0-neighborhood of N, where v is the harmonic extension of g, i.e. $\Delta v = 0$ in $B_1(0)$, $v = g$ in $\partial B_1(0)$ and g is a measurable map from $\partial B_1(0)$ to N.*

Proof. Let $r \in (0,1)$ and $1-r$ is sufficiently small, we can set $\omega \in \partial B_1(0)$. Define

$$g_{m,r}(\omega) = \fint_{Q_{m(1-r)(\omega)}} g d\mathcal{H}^{n-1},$$

where $Q_R(\omega)$ $(R > 0)$ is a ball with radius R and center ω defined by geodesic lines in $\partial B_1(0)$ and m a positive constant to be determined. By using Poisson formula, we can obtain that

$$\begin{aligned}
|v(r,\omega) - g_{m,r}(\omega)| &\leq \int_{\partial B_1(0)} \frac{1-r^2}{\omega_n |r\omega - y|^n} |g(y) - g_{m,r}(\omega)| d\mathcal{H}^{n-1}(y) \\
&\leq C(n,N,\gamma) \left(\int_{\partial B_1(0) \setminus Q_{m(1-r)}(\omega)} \frac{1-r^2}{\omega_n |r\omega - y|} d\mathcal{H}^{n-1}(y) \right) \\
&\quad + C(n,N,\gamma,m) \left(\fint_{Q_{m(1-r)}(\omega)} |g(y) - g_{m,r}(\omega)| d\mathcal{H}^{n-1}(y) \right) = (\mathrm{I}) + (\mathrm{II}),
\end{aligned}$$

where $\omega_n = \mathcal{H}^n(B_1)$. For any $\delta_0 > 0$, we can choose $m \gg 1$ such that $m(1-r) \sim \frac{1}{8}$ and $1 - r \ll 1$ such that (I) $< \frac{\delta_0}{4}$. If $\|g\|_{\mathrm{BMO}(\partial B_1)}$ is sufficiently small, then (II) $< \frac{\delta_0}{4}$. Since we can choose $\|g\|_{\mathrm{BMO}(\partial B_1)}$ suffiently small, then $\mathrm{dist}(g_{m,r}, N) < \frac{\delta_0}{4}$. Then the proof is completed. ◻

Remark 2.6. To use Lemma 2.5, one also need to show that $v(B_{r_0}(0)$ is also in sufficiently small neighborhood of N if $\|g\|_{\mathrm{BMO}(\partial B_1)} \ll 1$ by simply use the Poisson formula.

3 Appendix: regularity of quasilinear elliptic systems

In this section, we discuss some fundamental results of regularity theory of quasilinear elliptic systems. These materials are mainly from §9 of [5] and §1 of [15]. We will not give proofs for all of them for the sake of simplicity. In the followings, Ω are always defined as a domain (open and connected set).

Theorem 3.1. *Suppose $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ is a weak solution of the systems*

$$-\Delta u^j = f^j(x, u, \nabla u), \quad j = 1, 2, ..., m,$$

such that f is smooth and $|f(x,u,p)| \leq \Lambda |p|^2$ for any $x \in \Omega$, $u \in \mathbb{R}^m$ and $p \in \mathbb{R}^{m \times n}$. Let $B_R(x_0) \subset\subset \Omega$ and assume that u satisfies the Caccioppoli *inequality*

$$\int_{B_{r/2}(y)} |\nabla u|^2 dx \le \frac{C_0}{r^2} \int_{B_r(y)} |u - u_{x_0,r}|^2 dx,$$ [1]

whenever $B_r(y) \subset B_R(x_0)$*. Then for* $\sigma \in (0,1)$*, there exists* $\varepsilon_0 = \varepsilon_0(n, m, \Lambda, \sigma, C_0)$ *such that if*

$$\frac{1}{R^{n-2}} \int_{B_R(x_0)} |\nabla u|^2 dx \le \varepsilon_0^2,$$

then $\nabla u \in C^{0,\sigma}(\overline{B_{R/2}(x_0)}, \mathbb{R}^{m\times n})$ *with the estimate*

$$R\|\nabla u\|_{L^\infty(B_{R/2}(x_0))} + R^{1+\sigma}[\nabla u]_{C^{0,\sigma}(B_{R/2}(x_0))} \le C \left(\frac{1}{R^{n-2}} \int_{B_R(x_0)} |\nabla u|^2 dx \right)^{1/2},$$

where C *depends only on* n, m, Λ, σ *and* C_0*.*

This proof of this theorem can be divided into two steps. We can state them as following lemmas.

Lemma 3.2. *Let* $u, f, \Lambda, C_0, \sigma, \Omega, B_R(x_0), \varepsilon_0$ *be the same as in Theorem 3.1, then*

$$R^\sigma [u]_{C^{0,\sigma}(B_{R/2}(x_0))} \le C \left(\frac{1}{R^{n-2}} \int_{B_R(x_0)} |\nabla u|^2 dx \right)^{1/2},$$

where C *depends only on* n, m, Λ, σ *and* C_0*.*

Lemma 3.3. *Suppose* $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ *is a weak solution of the systems*

$$-\Delta u^j = f^j(x, u, \nabla u), \quad j = 1, 2, ..., m,$$

such that f *is smooth and* $|f(x,u,p)| \le \Lambda |p|^2$ *for any* $x \in \Omega$*,* $u \in \mathbb{R}^m$ *and* $p \in \mathbb{R}^{m\times n}$*. Let* $B_R(x_0) \subset\subset \Omega$ *and assume* $R^\sigma [u]_{C^{0,\sigma}(B_R(x_0))} \le M$ *for some* $\sigma \in (0,1)$*, then* $\nabla u \in C^{0,\sigma}(\overline{B_{R/2}(x_0)}, \mathbb{R}^{m\times n})$ *with*

$$R\|\nabla u\|_{L^\infty(B_{R/2}(x_0))} + R^{1+\sigma}[\nabla u]_{C^{0,\sigma}(B_{R/2}(x_0))} \le C \left(\frac{1}{R^{n-2}} \int_{B_R(x_0)} |\nabla u|^2 dx \right)^{1/2},$$

where C *depends only on* $n, m, \Lambda, \sigma, C_0$ *and* M*.*

From Lemma 3.3, we see the enhancement of the regularity from $C^{0,\sigma}$ to $C^{1,\sigma}$. This is an important method since Caccioppoli inequality may not always true for some types of harmonic maps, especially stationary harmonic maps, but we can always use Lemma 3.3 to enhance the regularity. So we omit the proof of Lemma 3.2 and give a proof of Lemma 3.3.

[1] Here $u_{x_0,r} = \fint_{B_r(x_0)} u$ and $\fint_E u = \frac{1}{|E|} \int_E u$ for measurable set E with Lebesgue measure $|E|$.

Proof of Lemma 3.3. For $0 < \rho < r < R$, letting $H \in W^{1,2}(B_r(x_0), \mathbb{R}^m)$ be the solution to $\Delta H = 0$ in $B_r(x_0)$ with $H - u \in W_0^{1,2}(B_r(x_0), \mathbb{R}^m)$. Then by simple calculations, (Proposition 5.8 of [5]),

$$\begin{aligned}
&\int_{B_\rho(x_0)} |\nabla u|^2 dx \leq C\left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |\nabla u|^2 dx + C\int_{B_r(x_0)} |\nabla(u-H)|^2 dx,\\
&\int_{B_\rho(x_0)} |\nabla u - (\nabla u)_{x_0,\rho}|^2 dx\\
&\qquad \leq C\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2 dx + C\int_{B_r(x_0)} |\nabla(u-H)|^2 dx.
\end{aligned} \tag{3.1}$$

By the definition of H, we can deduce that

$$\begin{aligned}
\int_{B_r(x_0)} |\nabla(u-H)|^2 dx &= \int_{B_r(x_0)} (u-H)\cdot f(x,u,\nabla u) dx\\
&\leq C\int_{B_r(x_0)} |u-H||\nabla u|^2 dx \leq C\left(\frac{r}{R}\right)^\sigma \int_{B_r(x_0)} |\nabla u|^2 dx,
\end{aligned} \tag{3.2}$$

where for the third inequality, we have used

$$[u-H]_{C^{0,\sigma}(B_r(x_0))} \leq [u]_{C^{0,\sigma}(B_r(x_0))} + [H]_{C^{0,\sigma}(B_r(x_0))} \leq C[u]_{C^{0,\sigma}(B_r(x_0))}.$$

Combining (3.1), (3.2) and using some simple iterations (Lemma 5.13 of [5]), if ρ is sufficiently small, i.e. $0 < \rho \leq r_0(n, m, \sigma_1, \sigma, M)$,

$$\int_{B_\rho(x_0)} |\nabla u|^2 dx \leq C\left(\frac{\rho}{R}\right)^{n-2+2\sigma_1} \int_{B_R(x_0)} |\nabla u|^2 dx, \text{ for all } \sigma_1,\ 0 < \sigma_1 < 1.$$

Then if $0 < \rho < r \leq r_0$, it follows from (3.1) and (3.2) that

$$\int_{B_\rho(x_0)} |\nabla u-(\nabla u)_{x_0,\rho}|^2 dx \leq C\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |\nabla u-(\nabla u)_{x_0,R}|^2 dx + C\left(\frac{r}{R}\right)^{n+\varepsilon} \int_{B_R(x_0)} |\nabla u|^2 dx,$$

where $0 < \varepsilon = \sigma - 2 + 2\sigma_1 < 1$ is an arbitrary number. Again, by using iterations as above, for $0 < \rho \leq r_0$,

$$\int_{B_\rho(x_0)} |\nabla u - (\nabla u)_{x_0,\rho}|^2 dx \leq C\left(\frac{\rho}{R}\right)^{n+\varepsilon} \int_{B_R(x_0)} |\nabla u|^2 dx.$$

We can change x_0 to any point in $B_{R/2}(x_0)$ and get similar results. By some covering arguments and Campanato characterization of Hölder space (Theorem 5.5 of [5]), these give that

$$R^{1+\frac{\varepsilon}{2}}[\nabla u]_{C^{0,\frac{\varepsilon}{2}}(B_{R/2}(x_0))} \leq C\left(\frac{1}{R^{n-2}}\int_{B_R(x_0)} |\nabla u|^2 dx\right)^{1/2}.$$

By standard arguments in Schauder estimates, (Theorem 5.19 of [5]), we can complete the proof. $\square$

By some modifications, one can generalize Lemma 3.3 to the following form without difficulty.

Lemma 3.4. *Suppose $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ is a weak solution of the systems*

$$-D_\beta(A^{\alpha\beta}_{ij}(x) D_\alpha u^j) = f_i(x, u, \nabla u), \quad j = 1, 2, ..., m,$$

such that $A^{\alpha\beta}_{ij}$ is bounded, satisfies Legendre ellipticity condition, i.e. $\lambda|\xi|^2 \leq A^{\alpha\beta}_{ij}\xi^i_\alpha \xi^j_\beta \leq \Lambda|\xi|^2$ for any $\xi = (\xi^i_\alpha) \in \mathbb{R}^{m\times n}$ with $\lambda, \Lambda > 0$ and f is smooth, $|f(x, u, p)| \leq \Lambda_1|p|^2$ for any $x \in \Omega$, $u \in \mathbb{R}^m$ and $p \in \mathbb{R}^{m\times n}$. Let $B_R(x_0) \subset\subset \Omega$ and assume $R^\sigma[u]_{C^{0,\sigma}(B_R(x_0))} \leq M$ for some $\sigma \in (0, 1)$, then $u \in C^{0,\sigma}(\overline{B_{R/2}(x_0)}, \mathbb{R}^{m\times n})$ with the estimate

$$R\|\nabla u\|_{L^\infty(B_{R/2}(x_0))} + R^{1+\sigma}[\nabla u]_{C^{0,\sigma}(B_{R/2}(x_0))} \leq C\left(\frac{1}{R^{n-2}}\int_{B_R(x_0)} |\nabla u|^2 dx\right)^{1/2},$$

where C depends only on $n, m, \lambda, \Lambda, \Lambda_1, \sigma, C_0$ and M.

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